27 September 2023

MAT 4800

Thm: X is path connected + ...

Then $\exists$ a universal covering

$$p : \underset{\underset{\tilde{X}}{\parallel}}{Y} \to X. \quad \pi_1(Y) = 0$$

Proof: $\pi(x_0, x)$ = homotopy classes of paths from x_0 to x.

$$\tilde{X} = \{ (x, \alpha) : \; x \in X, \; \alpha \in \pi(x_0, x) \}$$

$$p : \tilde{X} \to X$$
$$(x, \alpha) \mapsto x$$

First: Define open sets in $\tilde{X}$.

Any open set on $\tilde{X}$ will be of the form $U = \bigcup_{i \in I} U_i[\;\alpha_i]$

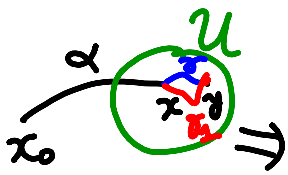
Here, for $x \in X$, & $\alpha \in \pi(x_0, x)$,

$x \in \mathcal{U}$, we define.

$\downarrow$ simply connected

$$\mathcal{U}[\alpha] = \{ (y, \beta) : y \in \mathcal{U},$$

β is the homotopy class of a path $x_0 \xrightarrow{\alpha} x \xrightarrow{\gamma} y$, where γ is any curve from x to y & $\gamma \subseteq \mathcal{U}$ }



$\Rightarrow \alpha \circ \gamma \underset{\text{homotopic}}{=} \alpha \circ \gamma_1$ (Because $\mathcal{U}$ simply connected $\Rightarrow \gamma \underset{\text{homotopic}}{\sim} \gamma_1$)

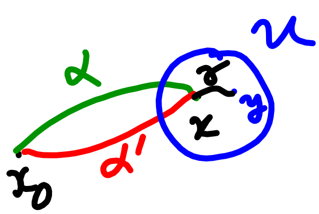
Claim: $\mathcal{U}[\alpha] \overset{\text{homeomorphic to}}{\sim} \mathcal{U}$.

$$(y, \beta) \longmapsto y$$

$\downarrow$ is uniquely defined from α & y

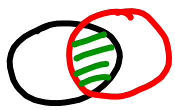
Claim: If α & α' not homotopic, then

$$[\mathcal{U}, \alpha] \cap [\mathcal{U}, \alpha'] = \phi$$



· Basis for a topology on a space X: is a collection of subsets $\{U_i\}_{i \in I}$ such that $\forall\, i, j$: $\boxed{U_i \cap U_j}$ = union of U_k's

basic open set



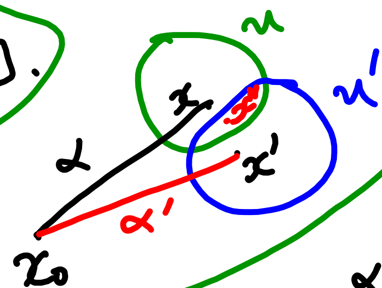
To show that $\mathcal{U}[\alpha]$ defines a topology, we need the above condition:

$$\mathcal{U}[\alpha] \cap \mathcal{U}'[\alpha'] = \bigcup_i \mathcal{U}_i[\alpha_i]$$

Idea for proof: We show that $\mathcal{U}[\alpha] \underset{\text{homeomorphic}}{\sim} \mathcal{U}$,

So $\boxed{\mathcal{U}[\alpha] \cap \mathcal{U}'[\alpha']}$ is like $\mathcal{U} \cap \mathcal{U}'$, but we need to write it in terms of $\mathcal{U}_i[\alpha_i]$.

a union of such



$\boxed{\mathcal{U}''[\alpha'']}$

$x'' \in \mathcal{U}''$ → simply connected

α'' = a path from x_0 to x'', & we can choose for example $x_0 \xrightarrow{\alpha} x \to x''$

(Want α'' also in $\mathcal{U}'[\alpha']$)

* Check that p is continuous.

$\mathcal{U} \subseteq X$ is a set. What is $p^{-1}(\mathcal{U})$?

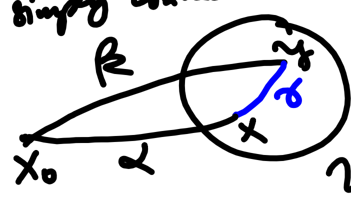
$$p^{-1}(\mathcal{U}) = \bigcup_{\alpha \in \pi_1(x_0, x)} \mathcal{U}[\alpha]$$

Fix $x \in \mathcal{U}$

By definition $p^{-1}(\mathcal{U}) = \{ (x, \alpha) : x \in \mathcal{U}, \alpha \in \pi_1(x_0, x) \}$

If $\mathcal{U}$ is simply connected:

Assume $\mathcal{U}$ simply connected:



$\alpha := \beta \circ \gamma$ then by definition of $\mathcal{U}[\alpha]$:

$(y, \beta) \in \mathcal{U}[\alpha]$.

Actually, $\bigcup_{\alpha \in \pi_1(x_0, x)} \mathcal{U}[\alpha]$ is a disjoint union.

So we proved that p is a covering map.

Now need to show that $\tilde{X}$ is simply connected.

$\psi : [0,1] \to \tilde{X}$ be a curve.

Let $u = p(\psi) : [0,1] \to X$

The point is to show that u is homotopic to a point.

To see this, use:

$$u_t : [0,1] \to X$$

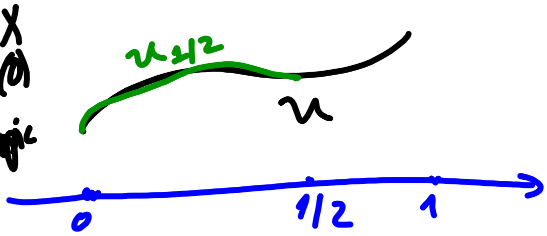
defined by $u_t(s) = u(ts)$.

$u_0 : [0,1] \to X$

$s \mapsto u(0)$

$\downarrow$

null-homotopic



What is this to do with $\tilde{X}$?

$\tilde{X}$ set of curves upto homotopy on X

So we use this to:

If u is null-homotopic, then $\tilde{u}$ which is a lift of u, is also null-homotopic.

p is covering map, so p can lift curves from X.

Example: If ~~the~~ $p : \tilde{X} \to X$ is universal covering, then $\pi_1(X) \simeq \mathrm{Deck}(\tilde{X}/X)$.

$$p : \mathbb{D} \to \mathbb{D}^* = \mathbb{D} \setminus 0$$
$$z \mapsto e^{-\left(\frac{1+z}{1-z}\right)}$$

is a covering map.

First, check that $e^{-\left(\frac{1+z}{1-z}\right)} \in \mathbb{D}^*$ if $z \in \mathbb{D}$.

$(z \in \mathbb{D}:$ $1-z \neq 0 \Rightarrow \frac{1+z}{1-z}$ is holomorphic,
$\Rightarrow e^{-\left(\frac{1+z}{1-z}\right)}$ holomorphic

$z = x + iy$

$$\frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \frac{(1+x+iy)(1-x+iy)}{(1-x-iy)(1-x+iy)}$$

$$= \frac{(1-x^2-y^2) + i\,g(x,y)}{(1-x)^2 + y^2}$$

$$\Rightarrow \left|e^{-\left(\frac{1+z}{1-z}\right)}\right| = e^{+\,\mathrm{Re}\left(-\frac{1+z}{1-z}\right)} < 1.$$

$$|e^{\zeta}| = e^{Re(\zeta)}$$

If $Re(\zeta) < 0 \Rightarrow |e^{\zeta}| < 1$.

HW: Check that it is covering map.

$\pi_1(\mathbb{D}) = 0 \Rightarrow p : \mathbb{D} \to \mathbb{D}^*$ is universal covering. So

$$\pi_1(\mathbb{D}^*) \simeq Deck(\mathbb{D}/\mathbb{D}^*).$$

<u>$Deck(\tilde{X}/X)$</u>: $p : \tilde{X} \to X$ a universal covering. A deck transformation is a map F:

$$\begin{tikzcd} \tilde{X} \arrow[r, "F"] \arrow[dr, "p"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

F a homeomorphism.

(Uniqueness of liftings)

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \arrow[dl, dashed, bend right, "G"'] \\ \tilde{X} \arrow[r, "p"] \arrow[ur, dashed, "F"] & X \end{tikzcd}$$

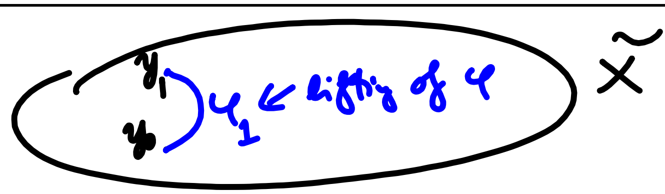
F is lift of p to $\tilde{X}$.

Why is it a group?

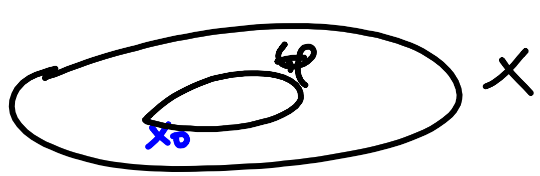
$$\begin{tikzcd}
\tilde X \arrow[rr, "F"] \arrow[dr, "p"'] \arrow[rrrr, bend left] \arrow[ddrr, bend right] & & \tilde X \arrow[rr, "G"] \arrow[dl, "p"] \arrow[dr, "p"] & & \tilde X \arrow[dl, "p"] \arrow[ddll, bend left] \\
& X \arrow[dr, "id"'] & & X \arrow[dl, "id"] & \\
& & X & &
\end{tikzcd}$$

Relation between homotopy group & deck transformations

<u>1st step:</u> Homotopy group $\Rightarrow$ Deck transformation.



$y_0 \in p^{-1}(x_0)$



$\varphi \in \pi_1(x_0, X)$

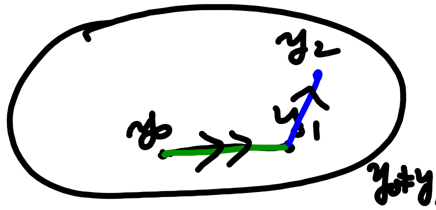
Define a map $F_\varphi: p^{-1}(x_0) \to p^{-1}(x_0)$

$y_0 \mapsto$ the end point of the lifting of φ with start point y_0

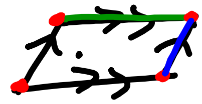
(Continue next time)

Example:

$$\begin{array}{c}\mathbb{C}\\ \downarrow \\ E = \mathbb{C}/\Lambda\end{array}$$



E



$$\pi_1(E) = \mathbb{Z} \oplus \mathbb{Z}$$

$$E = \{ Y^2 Z = X^3 + Z^3 \}$$

$$\pi : E \longrightarrow \mathbb{P}^1 \qquad \text{Weierstrass map}$$

$$[X:Y:Z] \mapsto [X:Z]$$

Image of $[0:1:0]$ by π?

$$[0:1:0] \in \mathcal{U}_2 = \{ [X:Y:Z] : Y \neq 0 \}$$

$$\mathcal{U}_2 = \mathbb{C}^2_{x_2, z_2}, \quad x_2 = X/Y, \quad z_2 = Z/Y$$

$Y = 1$

$$E_2 = E \cap \mathcal{U}_2 = \{ z_2 = x_2^3 + z_2^3 \}$$

$$\pi [X:Y:Z] = [X:Z] = [x_2 : z_2]$$

$$x_2 = X/Y, \quad z_2 = Z/Y \Rightarrow \frac{x_2}{z_2} = \frac{X}{Z}$$

What is $\lim\limits_{(x_2, z_2) \to (0,0)} \frac{x_2}{z_2}$?

$$[0:1:0] = (x_2, z_2) = (0,0)$$

$$z_2 = x_2^3 + z_2^3$$

$$\lim_{(x_2, z_2) \to (0,0)} \frac{x_2}{z_2} = \infty \ ?$$

So $\pi[0:1:0] = [x_2 : z_2] = [1:0]$ $\in \mathbb{P}^1$

$E_2 = \{f_2 = 0\}$ $\quad f_2 = z_2 - x_2^3 - z_2^3$

$$\nabla f_2 = (-3x_2^2, \ 1 - 3z_2^2)$$

$$\nabla f_2(0,0) = (0, \ 1)$$

$1 \neq 0$

So around $(0,0)$, E_2 is smooth &

we can use ~~z_2~~ x_2 as a coordinate

Which means:

$z_2 = g(x_2)$ where g is holomorphic function. $g(0) = 0$

l' Hospital rule:

$$\lim_{(x_2, z_2) \to (0,0)} \frac{x_2}{z_2} = \lim_{x_2 \to 0} \frac{x_2'}{z_2'} = \lim_{x_2 \to 0} \frac{1}{g'(x_2)}$$

$$= \frac{1}{0} = \infty.$$

Implicit differentiation:

$$z_2 = x_2^3 + z_2^3 \quad , \quad z_2 = g(x_2)$$

Take derivative w.r.t x_2

$$\Rightarrow \frac{dz_2}{dx_2} = \frac{d(x_2^3)}{dx_2} + \frac{d(z_2^3)}{dx_2}$$

$$g'(x_2) = 3x_2^2 + 3z_2^2 \cdot \frac{dz_2}{dx_2}$$

$$= 3x_2^2 + 3z_2^2 \cdot g'(x_2)$$

When $x_2 = 0 \Rightarrow z_2 = 0$:

$$g'(0) = 3 \times 0 + 3 \times 0 \times g'(0)$$

$$\Rightarrow g'(0) = 0 .$$

HW: What are the branch points of

$$\pi : E_2 \to \mathbb{P}^1 \; ?$$

$$(x_2, z_2) \mapsto \frac{x_2}{z_2} = [x_2 : z_2]$$