31 October 2023

MAT 4800

2-forms: (in dim 2) Locally has the form

$$\underbrace{f}\ \underbrace{dx}_{1\text{-form}} \wedge \underbrace{dy}_{1\text{-form}}$$

wedge product

, f: a function

(x, y local coordinates)

Rule: $dx \wedge dx = 0$

$$dx \wedge dy = - dy \wedge dx$$

$$(\alpha_1 dx + \alpha_2 dy) \wedge (\beta_1 dx + \beta_2 dy)$$
$$= \alpha_1 \beta_1 \underbrace{dx \wedge dx}_{0} + \alpha_1 \beta_2 dx \wedge dy$$

$$+ \alpha_2 \beta_1 \underbrace{dy \wedge dx}_{-dx \wedge dy} + \alpha_2 \beta_2 \underbrace{dy \wedge dy}_{0}.$$

$$= (\alpha_1 \beta_2 - \alpha_2 \beta_1) dx \wedge dy$$

*

$$d(\alpha_1 dx + \alpha_2 dy)$$
$$= \left(\frac{\partial \alpha_1}{\partial x} dx + \frac{\partial \alpha_1}{\partial y} dy\right) \wedge dx$$
$$+ \left(\frac{\partial \alpha_2}{\partial x} dx + \frac{\partial \alpha_2}{\partial y} dy\right) \wedge dy$$

$$= \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y} \right) dx \wedge dy$$

* Integration of 1-form on a curve $(x(t), y(t))$ $(a \le t \le b)$

$x = x(t) \Rightarrow dx = x'(t)dt$, $dy = y'(t)dt$

$$\int_C \left(\alpha_1(x,y)\,dx + \alpha_2(x,y)\,dy \right)$$

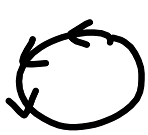
$$= \int_a^b \left[\alpha_1(x(t),y(t))\,x'(t) + \alpha_2(x(t),y(t))\,y'(t) \right] dt$$

C = circle of radius r

$\Rightarrow (x(t), y(t)) = (r\cos(t), r\sin(t))$

$0 \le t \le 2\pi$.

Positive orientation: Ω, $C = \partial\Omega$ when we move on C, then Ω is always on the left.



<u>Integration of 2-form</u>:

Let $\Omega \subseteq \mathbb{C}$ (or a RS)

be an open set.

$f(x,y)\, dx \wedge dy$ be a 2-form. Then

$$\int_\Omega f(x,y)\, dx \wedge dy = \int_\Omega f(x,y)\, dx\, dy$$

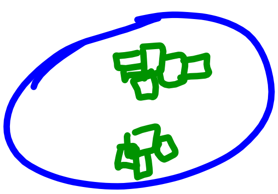
Def: If Ω is a rectangle $a \leq x \leq b$, $c \leq y \leq d$

then:

$$\int_\Omega f(x,y)\, dx\, dy := \int_a^b \int_c^d f(x,y)\, dx\, dy$$

In general, Ω can be divided ~~into~~ into many rectangles & then

$$\int_\Omega f(x,y)\, dx\, dy = \lim \sum \int_{\text{small rectangles}} f(x,y)\, dx\, dy$$



Ω

Stokes theorem:

If ω is a 1-form

$C = \partial \Omega$ has positive orientation

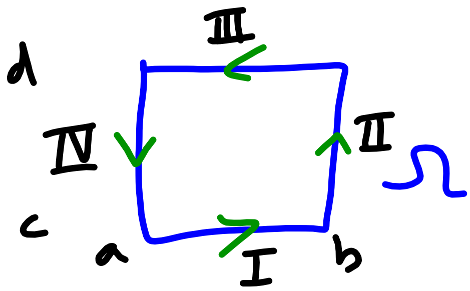
then
$$\int_{\Omega} d\omega = \oint_{C} \omega .$$

Proof: Need only to do in the case Ω = rectangle $= [a,b] \times [c,d]$.

Also suffices to show for

$$\omega = \alpha \, dx$$

$$d\omega = \left(\frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy \right) \wedge dx$$
$$= - \frac{\partial \alpha}{\partial y} dx \wedge dy$$



$$\int_{\Omega} d\omega = - \int_a^b \int_c^d \frac{\partial \alpha}{\partial y} dx \wedge dy$$
$$= - \int_a^b dx \left[\int_c^d \frac{\partial \alpha}{\partial y} dy \right]$$

$$= -\int_a^b \left(\alpha(x,d) - \alpha(x,c)\right) dx$$

$$\oint_C \omega = \int_{\mathrm{I}} \omega + \int_{\mathrm{II}} \omega + \int_{\mathrm{III}} \omega + \int_{\mathrm{IV}} \omega$$

On I, $y = c$, & x goes from a to b

$$\int_{\mathrm{I}} \omega = \int_a^b \alpha(x,c)\, dx$$

On III: $y = d$ & x goes from b to a.

$$\int_{\mathrm{III}} \omega = \int_b^a \alpha(x,d)\, dx$$

$$= -\int_a^b \alpha(x,d)\, dx$$

For II: $x = b$, y goes from c to d:

$$\int_{\mathrm{II}} \omega = \int_b^b \alpha(x,y)\, dx = 0$$

$$\int_{\mathrm{IV}} \omega = 0.$$

$\square$

Corollary: If $d\omega = 0$

$$\Rightarrow \int_{\partial\Omega} \omega = 0. \quad \square$$

d-closed form: 1-form ω is d-closed if $d\omega = 0$.

∂-closed, $\bar{\partial}$-closed: $\partial\omega = 0$ (or $\bar{\partial}\omega = 0$)

$$\omega = \alpha(z, \bar{z})\, dz + \beta(z, \bar{z})\, d\bar{z}$$

$(z = x + iy, \bar{z} = x - iy)$

$$\partial\omega = \frac{\partial}{\partial z}(\alpha\, dz + \beta\, d\bar{z})$$

$$= \frac{\partial\alpha}{\partial z} dz \wedge dz + \frac{\partial\beta}{\partial z} dz \wedge d\bar{z}$$

$$= \frac{\partial\beta}{\partial z} dz \wedge d\bar{z}$$

Check: $dz \wedge dz = 0$?

$dz = (dx + idy) \Rightarrow dz \wedge dz$

$= (dx + idy) \wedge (dx + idy) = 0$

$$\bar{\partial}\omega = \frac{\partial}{\partial \bar{z}}(\alpha\, dz + \beta\, d\bar{z})$$

$$= -\frac{\partial\alpha}{\partial \bar{z}} dz \wedge d\bar{z}$$

Q: What is $dz \wedge d\bar{z}$?

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy)$$

$$= -2i \, dx \wedge dy$$

$$\Rightarrow \boxed{\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy.}$$

HW: if ω is a holomorphic 1-form then what is $d\omega$?

$$\omega = f(z)\, dz$$

$$\Rightarrow d\omega = \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{0} + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0} d\bar{z} \wedge dz$$

$$= 0.$$

Corollary: if ω holomorphic 1-form in Ω & on the boundary $\partial\Omega$, then $\int_{\partial\Omega} \omega = 0$!

Sum of residues of a meromorphic 1-form on a compact RS:

Let $X = \mathbb{P}^1 = \mathbb{C}_z \cup \mathbb{C}_w$

($w = \frac{1}{z}$ on $\mathbb{C}_z \cap \mathbb{C}_w$).

$\varphi(z) = \frac{dz}{z}$ on $\mathbb{C}_z$.

Extend it to $\mathbb{C}_w$ also:

$$\varphi(w) = \frac{d(\frac{1}{w})}{\frac{1}{w}} = \frac{-\frac{1}{w^2}\,dw}{\frac{1}{w}} = -\frac{dw}{w}.$$

φ has 2 poles:

one at $z = 0$

& one at $w = 0$ ($\Leftrightarrow z = \infty$)

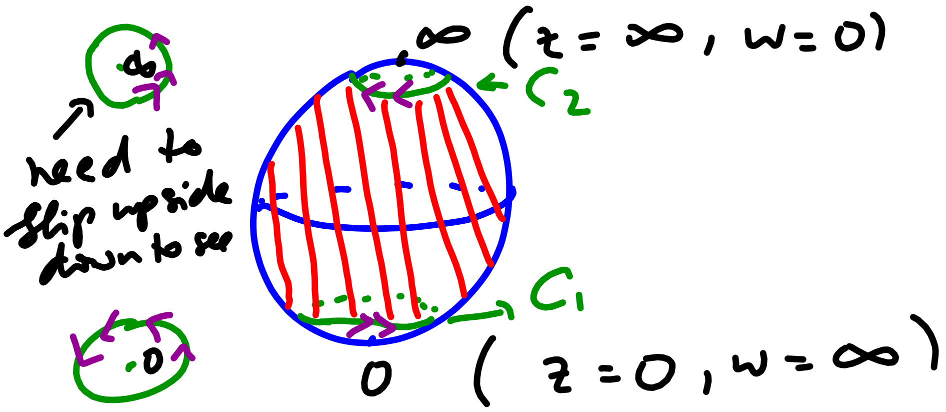
$\text{Res}(\varphi, z=0) = 1$

$\text{Res}(\varphi, w=0) = -1$

$\Rightarrow \text{Res}(\varphi, z=0) + \text{Res}(\varphi, w=0)$

$= 1 + (-1) = 0$!

Can we see another way?



Ω = be the red domain.

Boundary of Ω consists of C_1 & C_2.

φ is holomorphic on Ω

$$\Rightarrow \oint_{\partial\Omega} \varphi = 0.$$

$$0 = \oint_{\partial\Omega} \varphi = \oint_{C_1} \varphi + \oint_{C_2} \varphi$$

By Residue theorem

$$\oint_{C_1} \varphi = 2\pi i \operatorname{Res}[\varphi, z=0]$$

$$\oint_{C_2} \varphi = 2\pi i \operatorname{Res}[\varphi, z=\infty]$$

$$\Rightarrow \operatorname{Res}[\varphi, z=0] + \operatorname{Res}[\varphi, z=\infty] = 0!$$

We can generalise:

Thm (Sum of Residues):

Let X be a compact RS & ω a non-zero meromorphic 1-form on X. Let $p_1, \dots, p_m$ be all the poles of ω. Then:

$$\sum_{i=1}^{m} \operatorname{Res}[\omega, p_i] = 0.$$

Similar proof.

Chapter 2: Cohomology

Summary: If X is compact RS then we have 3 cohomology groups $H^0(X)$, $H^1(X)$, $H^2(X)$.

$H^0(X)$, $H^2(X)$ $\longrightarrow$ Kind of trivial.

The study of H^1 is Riemann-Roch theorem.

§ 12. Cohomology groups of a sheaf.

X = topological space

$\mathcal{F}$ = a sheaf over X.
(of Abelian groups)

$\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X

q-cochain: $C^q(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$

$\mathcal{F}(U)$ = global sections of $\mathcal{F}$ over U.

$C^0(\mathcal{U}, \mathcal{F})$

$= \{ f \in \prod_{i \in I} \mathcal{F}(U_i)$

$= \{ (f_i)_{i \in I} : f_i \in \mathcal{F}(U_i) \}$

$C^1(\mathcal{U}, \mathcal{F}) = \{ g \in \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \}$

$= \{ (g_{i,j})_{i,j \in I} : g_{i,j} \in \mathcal{F}(U_i \cap U_j) \}$

$$C^2(\mathcal{U}, \mathcal{F}) = \{ (f_{i,j,k})_{i,j,k \in I} : f_{i,j,k} \in \mathcal{F}(U_i \cap U_j \cap U_k) \}$$

<u>Coboundary operator</u>:

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$
$$\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$$

$f \in C^0(\mathcal{U}, \mathcal{F}) \Rightarrow f = (f_i)_{i \in I}$, $f_i \in \mathcal{F}(U_i)$

$\delta f = g \in C^1(\mathcal{U}, \mathcal{F})$

$g = (g_{i,j})_{i,j \in I}$, $g_{i,j} \in \mathcal{F}(U_i \cap U_j)$

& $g_{i,j} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}$

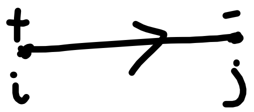
- If $g = (g_{i,j})_{i,j \in I} \in C^1(\mathcal{U}, \mathcal{F})$

& $h = \delta g \Rightarrow h = (h_{i,j,k})_{i,j,k \in I}$

$h_{i,j,k} \in \mathcal{F}(U_i \cap U_j \cap U_k)$

$$h_{i,j,k} := g_{j,k} - g_{i,k} + g_{i,j}$$

To remember for $\delta : C^0 \to C^1$

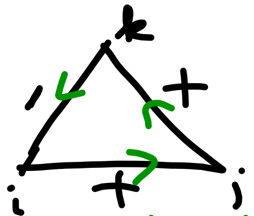


To remember for $\delta : C^1 \to C^2$

$ki \to +$

hence $ik \to -$.

positive orientation of the boundary to the triangle



lemma: Let $f \in C^0(\mathcal{U}, \mathcal{F})$

$$\Rightarrow \quad \delta\delta(f) = 0. \quad \square$$

HW: Check if $\mathcal{U} = (U_1, U_2, U_3, U_4)$

Whenever we see some linear map δ which satisfies $\delta \circ \delta = 0$, you can define cohomology.

(Example, will see later:

$d \circ d = 0$, $\partial \circ \partial = 0$,

$\bar{\partial} \circ \bar{\partial} = 0$. $d = \partial + \bar{\partial}$)

cocycles $\rightarrow Z^1(\mathcal{U}, \mathcal{F}) = \{ g \in C^1(\mathcal{U}, \mathcal{F}) : \delta(g) = 0 \}$

$\cap$

$C^2(\mathcal{U}, \mathcal{F})$ = kernel of $\delta : C^1 \to C^2$.

$B^1(\mathcal{U}, \mathcal{F}) = \{ g \in C^1(\mathcal{U}, \mathcal{F}). \ \exists f \in C^0(\mathcal{U}, \mathcal{F}) : g = \delta f \}$

= image of $\delta : C^0 \to C^1$.

If $g \in B^1(\mathcal{U}, \mathcal{F}) \Rightarrow \delta g = \delta(\delta f) = \delta \circ \delta(f) = 0$

$\Rightarrow g \in Z^1(\mathcal{U}, \mathcal{F})$.

So $B^1(\mathcal{U}, \mathcal{F})$ is a Abelian subgroup of $Z^1(\mathcal{U}, \mathcal{F})$.

1st cohomology of $\mathcal{F}$ w.r.t the cover $\mathcal{U}$ $\rightarrow$ $H^1(\mathcal{U}, \mathcal{F}) := Z^1(\mathcal{U}, \mathcal{F}) / B^1(\mathcal{U}, \mathcal{F})$

= "the difference" between Z^1 & B^1.

$H^1(\mathcal{U}, \mathcal{F})$ depends on $\mathcal{U}$.

<u>Remark</u>:

Can we define something $H^1(\mathcal{F})$ independent of $\mathcal{U}$? The idea is to mimic the way we defined stalks of a sheaf. For stalks we identify $F(\mathcal{U})$ to $F(\mathcal{V})$ if $\mathcal{U} \supseteq \mathcal{V}$ are open sets

So for two open coverings $\mathcal{U}$ & $\mathcal{V}$ so that $\mathcal{V} < \mathcal{U}$ we can identify $H^1(\mathcal{U}, F)$ to $H^1(\mathcal{V}, F)$.

First need to define $\mathcal{V} < \mathcal{U}$ for open coverings:

$$\mathcal{U} = (U_i)_{i \in I}, \quad \mathcal{V} = (V_j)_{j \in J}.$$

$\mathcal{V} < \mathcal{U}$ if $\forall j$, $\exists i$ so that $V_j \subseteq U_i$.

Define "restriction maps"

Let $\mathcal{V} < \mathcal{U}$ be open coverings

Define restriction map

$$\tau^{\mathcal{U}}_{\mathcal{V}} : Z^1(\mathcal{U}, F) \to Z^1(\mathcal{V}, F)$$

as follows:

Let $(f_{i,j})_{i,j} \in Z^1(\mathcal{U}, F)$

$f_{i,j} \in F(U_i \cap U_j) \quad \forall i, j \in I$

Let $k, l \in J$

& U_i, U_j be such that

$$V_k \subseteq U_i, \quad V_l \subseteq U_j$$

Define $g_{k,l} = f_{i,j}|_{V_k \cap V_l}$.

$g = (g_{k,l}) = \tau^{\mathcal{U}}_{\mathcal{V}}(f)$.

Q: What if we choose another

$V_k \subseteq U_{i_1}$, $U_{j_1} \supseteq V_l$? Is $f_{i,j}|_{V_{k,l}} = f_{i_1,j_1}|_{V_{kl}}$?

No: We may have

$$f_{i,j}|_{V_k \cap V_l} \neq f_{i_1,j_1}|_{V_k \cap V_l},$$

but $f_{i,j}|_{V_k \cap V_l} - f_{i_1,j_1}|_{V_k \cap V_l}$

is in $B^1(\mathcal{V}, \mathcal{F})$.

So $(f_{i,j}) = (f_{i_1,j_1})$

in $H^1(\mathcal{V}, \mathcal{F})$!!

This is how we define: $\tau^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$.

Def:

$$H^1(\mathcal{F}) = \prod_{\substack{\mathcal{U}:\text{ an}\\ \text{open}\\ \text{covering}\\ \text{of } X}} H^1(\mathcal{U},\mathcal{F}) \Big/ \sim$$

where $g \in H^1(\mathcal{U},\mathcal{F})$ & $g' \in H^1(\mathcal{U}',\mathcal{F})$ are identified if $\mathcal{U}' < \mathcal{U}$ &

$$\tau^{\mathcal{U}}_{\mathcal{U}'}(g) = g'.$$

[Basically: we look at open coverings by smaller & smaller open sets.]

$X =$ "the limit" of smaller & smaller open coverings of X $= \mathbb{C}$

(Example: $\mathcal{U}_r := \{$covering X by open subsets of radius $= r\}$

If $r \to 0$ then $\mathcal{U}_r$ "$\to$" X.

HW: What is $H^1(X, \mathcal{F})$?

$X = \mathbb{C}$, $\mathcal{F}$ = sheaf of holomorphic functions.