7 November 2023
MAT4800

A notion of "good open covering" in view of cohomology theory:

Def (Leray covering for 1st cohomology)
An open covering $\mathcal{U} = (U_i)_{i \in I}$ of X is a Leray covering for 1st cohomology is one such that:
$H^1(U_i, \mathcal{F}) = 0 \ \forall i \in I$.

Thm 12.8: If $\mathcal{U}$ is a Leray covering (for 1st cohomology) then
$$H^1(X, \mathcal{F}) = H^1(\mathcal{U}, \mathcal{F}). \quad \square$$

$\mathcal{E} =$ the sheaf of differentiable functions on X
(So $\mathcal{E}(U) = \{ f : U \to \mathbb{R}$ or $\mathbb{C}$, f is differentiable)

$\mathbb{C}_X$ = constant sheaf with stalk $\mathbb{C}$

$\mathbb{Z}_X$ = constant sheaf with stalk $\mathbb{Z}$.

Thm 12.6 & 12.7:

1. $X = RS$, then $H^1(X, \mathcal{E}) = 0$.
2. X is simply-connected RS then $H^1(X, \mathbb{C}) = H^1(X, \mathbb{C}_X) = 0$
3. X is simply connected RS $\Rightarrow H^1(X, \mathbb{Z}) = 0$.

Corollary: If $\mathcal{U} : (U_i)_{i \in I}$ is an open covering of $X = RS$, so that U_i is simply connected for ~~...~~ $\forall i \Rightarrow$ $\mathcal{U}$ is Leray-covering for 1st cohomology for $\mathcal{F} = \mathbb{C}_X$ or $\mathcal{F} = \mathbb{Z}_X$.

(Example: $U_i \cong \mathbb{D}(0, r)$)

Proof:

1. We use Partition of unity (exists for $\mathcal{E}$ but not for $\mathcal{O}$).

<u>Partition of unity</u> : Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering for X. Then $\exists$ $\psi_i : X \to [0,1]$ smooth functions such that :

- $\text{supp}(\psi_i) \overset{\text{def}}{=} \overline{\{x \in X : \psi_i(x) \neq 0\}} \subseteq U_i$.
- $\forall x \in X$: $\exists\, x \in U$ open nbd $\subseteq X$, so that $\#\{j \in I : \text{supp}(\psi_j) \cap U \neq \emptyset\} < +\infty$.

- $\sum_{i \in I} \psi_i(x) = 1$, $\forall x$.

Remark: this ~~theorem~~ Partition of unity needs Axiom of Choice.

By definition $H^1(X, \mathcal{E}) = \varinjlim_{\substack{\mathcal{U} : \text{open} \\ \text{covering of } X}} H^1(\mathcal{U}, \mathcal{E})$.

We only need to show :

$H^1(\mathcal{U}, \mathcal{E}) = 0$ $\forall$ open covering $\mathcal{U}$ of X.

$H^1(\mathcal{U}, \mathcal{F}) = Z^1(\mathcal{U}, \mathcal{E}) / B^1(\mathcal{U}, \mathcal{E})$

$g \in Z^1(\mathcal{U}, \Sigma)$

$\Rightarrow \delta g = 0$

$g \in Z^1(\mathcal{U}, \Sigma) \Rightarrow g = (g_{i,j})_{i,j \in I}$

$g_{i,j} \in U_i \cap U_j$.

$\delta g = \left(g_{i,j} + g_{j,k} - g_{i,j} \big|_{U_i \cap U_j \cap U_k} \right)$

$\delta g = 0 \Leftrightarrow g_{i,j} + g_{j,k} - g_{i,k} \big|_{U_i \cap U_j \cap U_k} = 0 \quad \forall i, j, k$.

$\Downarrow$

$\boxed{g_{i,k} - g_{j,k} = g_{i,j}}$

$\forall i \in I$ define

$h_i = \sum_{j \in I} \psi_j \, g_{i,j} \in C^0(\mathcal{U}_i, \Sigma) = \Sigma(U_i)$

Claim:

$\delta h = g \Rightarrow g$ is in $B^1(\mathcal{U}, \Sigma)$ as wanted.

$\delta h : h_i - h_j \big|_{U_{i,j} = U_i \cap U_j}$

$h_i - h_j = \sum_{k \in I} \psi_k \, g_{ik} - \sum_{k \in I} \psi_k \, g_{jk}$

$\delta g = 0 \Rightarrow g_{ik} - g_{jk} = g_{i,j} \quad \forall i, j, k$.

$$h_i - h_j = \sum_{k \in I} \psi_k (g_{i,k} - g_{j,k})$$

$$= \underbrace{\sum_{k \in I} \psi_k}_{1} \; \underbrace{g_{i,j}}$$

$$= g_{i,j} \, . \quad \square$$

2 Show that X is simply connected

$\Rightarrow \quad H^1(X, \mathbb{C}_X) = 0.$

$\mathbb{C}_X \hookrightarrow \mathcal{E}$, as the set of locally constant functions.

$f \in Z^1(\mathcal{U}, \mathbb{C}_X)$, w.T.S.

$f \in B^1(\mathcal{U}, \mathbb{C}_X)$.

$\delta f = 0$ in $C^2(\mathcal{U}, \mathbb{C}_X) \Rightarrow$

$\delta f = 0$ in $C^2(\mathcal{U}, \mathcal{E})$.

By part 1, $\exists \; h \in C^0(\mathcal{U}, \underline{\mathcal{E}})$

so that $\underbrace{h_i}_{\text{function}} - \underbrace{h_j}_{\text{function}} = \underbrace{f_{i,j}}_{\text{constant}} \quad \forall i, j.$

Take d: $\quad d(h_i) - d(h_j) = d(f_{i,j}) = 0$

$d(h_i)$ ~~is~~ is a differentiable 1-form on U_i & $d(h_i) - d(h_j) = 0$ on $U_i \cap U_j$. We have the sheaf of & ~~the set of differ~~ differentiable 1-forms $\Rightarrow$ ω a global 1-form on X so that $\omega|_{U_i} = dh_i$.

$$\Rightarrow d\omega|_{U_i} = d \circ d(h_i) = 0.$$

$\Rightarrow$ ω is an exact 1-form. Since X is simply-connected $\Rightarrow \exists\, h \in \mathcal{E}(X)$. $dh = \omega$.

[How? Just fix a point $x_0 \in X$. For any $x \in X$, choose a curve C from x_0 to x & define:

$$h(x) = \int_C \omega.$$

Show that g is independent of the choice of C, because X is simply connected.]

Define: $g_i = h_i - h \in C^0(U_i, \mathcal{E})$

So $d(g_i) = d(h_i) - d(h) = 0$ on U_i

$\Rightarrow g_i$ a locally constant function on U_i $\Rightarrow g_i \in C^0(U_i, \mathbb{C}_X)$

$$f_{ij} = \underset{C^0(U_i,\mathcal{E})}{h_i} - \underset{C^0(U_j,\mathcal{E})}{h_j} = \underset{C^0(U_i,\mathbb{C}_X)}{(h_i - h)} - \underset{C^0(U_j,\mathbb{C}_X)}{(h_j - h)}$$

$\Rightarrow (f_{ij}) \in B^1(\mathcal{U}, \mathbb{C}_X)$, as wanted.

3 WTS: $H^1(X, \mathbb{Z}_X) = 0$
if X is simply-connected.

In the proof of 2, if we can assume that for $f \in Z^1(\mathcal{U}, \mathbb{Z}_X)$, then $h_i - h$ & $h_j - h$ are $\in \mathbb{Z}$, then we are done. But this can be difficult.

Now we think about

$$\mathbb{Z}_X \hookrightarrow \mathbb{C}_X$$

SES

$$0 \to \mathbb{Z}_X \to \mathbb{C}_X \to \mathbb{C}_X^* \to 0$$
$$z \to e^{2\pi i z}$$

(Same proof as:
$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0)$$

Let $f = (f_{ij}) \in Z^1(\mathcal{U}, \mathbb{Z}_X)$.
WTS: $\exists$ $g = (g_i) \in C^0(\mathcal{U}, \mathbb{Z}_X)$
so that $f_{ij} = g_i - g_j|_{U_i \cap U_j}$.

By ②, $\exists$

$$\hat{g} \in C^0(\mathcal{U}, \underline{\mathbb{C}_X})$$

so that $f_{ij} = \hat{g}_i - \hat{g}_j \ \forall i, j.$

$$\Rightarrow \quad 1 = e^{2\pi i f_{ij}} = \frac{e^{2\pi i \hat{g}_i}}{e^{2\pi i \hat{g}_j}}$$

$$e^{2\pi i \hat{g}_i} \in C^0(U_i, \mathbb{C}_X)$$

$$e^{2\pi i \hat{g}_i} = e^{2\pi i \hat{g}_j} \text{ on } U_i \cap U_j$$

$\Rightarrow \quad \exists \ \hat{c} \in C^0(X, \mathbb{C}_X):$

$$\hat{c}|_{U_i} = e^{2\pi i \hat{g}_i}.$$

X is simply-connected $\Rightarrow$

$\hat{c} =$ constant $\neq 0$.

Choose $\hat{g}$ to be any complex number so that $e^{2\pi i \hat{g}} = \hat{c}$.

$\Rightarrow \quad \hat{g}$ is a global section of $\mathbb{C}_X$ & $e^{2\pi i \hat{g}} = e^{2\pi i \hat{g}_i} \ \forall i$ on U_i.

$$\Rightarrow \quad \hat{g}_i - \hat{g} \in \mathbb{Z}_X(U) \ \forall i$$

Then we can do like before:

$$\underset{C^0(U_i \cap U_j, \mathbb{Z}_X)}{f_{ij}} = \underset{C^0(U_i, \mathbb{C}_X)}{\hat{g}_i} - \underset{C^0(U_j, \mathbb{C}_X)}{\hat{g}_j}$$

$$= \underset{C^0(U_i, \mathbb{Z}_X)}{(\hat{g}_i - \hat{g})} - \underset{C^0(U_j, \mathbb{Z}_X)}{(\hat{g}_j - \hat{g})}$$

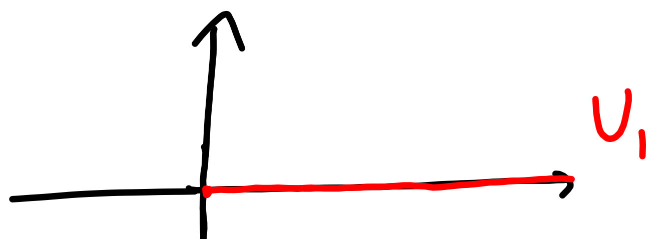
$$\Rightarrow f_{ij} \in B^1(\mathcal{U}, \mathbb{Z}_X). \ \square$$

<u>Ex</u>: $H^1(X, \mathbb{Z}) = \mathbb{Z}$ if

$$X = \mathbb{C} \setminus \{0\}.$$

$$U_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$$

$$U_2 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$$



U_1, U_2 : simply connected, open subsets of $\mathbb{C}$

$U_1 \cup U_2 = \mathbb{C}$

$U_1 \cap U_2 = \mathbb{C} \setminus \mathbb{R}$

So $H^1(U_1, \mathbb{Z}) = H^1(U_2, \mathbb{Z})$

$= 0$ by Thm 12.6 & 12.7.

So $\mathcal{U} = (U_1, U_2)$ is a Leray covering of X.

Hence $H^1(X, \mathbb{Z}_X) = H^1(\mathcal{U}, \mathbb{Z}_X)$

Need to compute $H^1(\mathcal{U}, \mathbb{Z}_X)$:

$= Z^1(\mathcal{U}, \mathbb{Z}_X) / B^1(\mathcal{U}, \mathbb{Z}_X)$.

$Z^1(\mathcal{U}, \mathbb{Z}_X) = \{ g \in C^1(\mathcal{U}, \mathbb{Z}_X) : \delta g = 0 \}$

$g = (g_{1,1}, g_{1,2}, g_{2,2})$

where $g_{1,1} \in C^0(U_{1,1} = \underbrace{U_1 \cap U_1}_{U_1 = \text{connected}}, \mathbb{Z}_X)$

$= a \in \mathbb{Z}$.

Similarly $g_{2,2} \in C^0(U_2, \mathbb{Z}_X) = \mathbb{Z}$

$g_{1,2} \in C^0(\overbrace{U_1 \cap U_2}^{2 \text{ components}}, \mathbb{Z}_X) = \mathbb{Z}$

$= \mathbb{Z} \oplus \mathbb{Z}$

$$\delta f = \left(f_{i,j} + f_{j,k} - f_{i,k} \big|_{U_i \cap U_j \cap U_k} \right)$$
$$= 0.$$

$i=1,\ j=1,\ k=1 \Rightarrow f_{1,1} + f_{1,1} - f_{1,1} \big|_{U_1} = 0$

$\Rightarrow f_{1,1} = 0.$

Similarly: $f_{2,2} = 0.$

$i=1,\ j=1,\ k=2: \quad 0 = 0$

$\dots$

$$\Rightarrow Z^1(\mathcal{U}, \mathbb{Z}_X) = \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\|}{}} \Big(\underset{0 \in \mathbb{Z}}{\underset{\|}{f_{1,1}}}, \underbrace{f_{1,2}}_{\mathbb{Z}\oplus\mathbb{Z}\ \ni}, \underset{0 \in \mathbb{Z}}{\underset{\|}{f_{2,2}}} \Big)$$

$B^1(\mathcal{U}, \mathbb{Z}_X) = \{ g = (g_{i,j}) \in C^0(U_i \cap U_j, \mathbb{Z}_X) : g = \delta h$, for $h \in C^0(\mathcal{U}, \mathbb{Z}_X) \}$

$h = (h_1, h_2)$, $h_1 \in C^0(U_1, \mathbb{Z}_X) = \mathbb{Z}$, $h_2 \in C^0(U_2, \mathbb{Z}_X) = \mathbb{Z}$

$g_{1,1} = h_1 - h_1 = 0.$

$g_{2,2} = h_2 - h_2 = 0$

$g_{1,2} = h_1 - h_2 \in \mathbb{Z}.$

$g \in B^1(\mathcal{U}, \mathbb{Z}_X)$

$\Leftrightarrow \; g = (g_{1,1}, g_{1,2}, g_{2,2})$

$g_{1,1} = 0$, $g_{1,2}$ = constant on $U_1 \cap U_2$, $g_{2,2} = 0$

$$B^1(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}.$$

$$H^1(X, \mathbb{Z}_X) \cong \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z}$$

$X = \mathbb{C} \setminus \{0\}$

$\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ as the diagonal embedding

Remark: To compute $H^*(X, \mathcal{F})$

need only to consider

$$U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_j}$$

where $i_\ell \neq i_{\ell'}$ if $\ell \neq \ell'$.

For example, don't need to consider $U_{1,1} = U_1 \cap U_1 = U_1 \ldots$

Ex: $H^1(\mathbb{P}^1, \mathbb{C}) = ?$

$\mathbb{P}^1 = U_1 \cup U_2$, where

$U_1 = \{[x : y] \in \mathbb{P}^1 : y \neq 0\}$,

$U_2 = \{[x:y] \in \mathbb{P}^1 : x \neq 0\}$

$U_1, U_2 \simeq \mathbb{C} \rightarrow$ simply connected

$U_1 \cap U_2 \cong \mathbb{C} \setminus \{0\}$

So $\mathcal{U} = (U_1, U_2)$ is a Leray covering for $\mathbb{C}_X$, for $X = \mathbb{P}^1$.

$\Rightarrow H^1(\mathbb{P}^1, \mathbb{C}) = H^1(\mathcal{U}, \mathbb{C})$.

Claim $Z^1(\mathcal{U}, \mathbb{C}) \simeq \mathbb{C}$, $B^1(\mathcal{U}, \mathbb{C}) \simeq \mathbb{C}$. $\Rightarrow H^1(\mathcal{U}, \mathbb{C}) = 0$.

By the remark, need to only look at $U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_k}$ where $i_1, \dots, i_k$ are pairwise distinct.

So for $Z^1(\mathcal{U}, \mathbb{C})$, need to tell only $g = g_{1,2} \in C^0(\underbrace{U_1 \cap U_2}_{\text{connected}} = \mathbb{C} \setminus \{0\}, \mathbb{C})$ $'' \mathbb{C}$

δg will be in C^2 which is in $C^0(U_i \cap U_j \cap U_k, \mathbb{C})$ & we only consider $i \neq j \neq k \Rightarrow$ empty since $i, j, k \in \{1, 2\}$.

$\Rightarrow \cancel{Z}\ Z^1(\mathcal{U}, \mathbb{C}) \simeq \mathbb{C}$.

For $B^1(\mathcal{U}, \mathbb{C})$ then the same like before:

$\Rightarrow B^1(\mathcal{U}, \mathbb{C}) \simeq \mathbb{C}$.

So $H^1(\mathbb{P}^1, \mathbb{C}) = 0$.

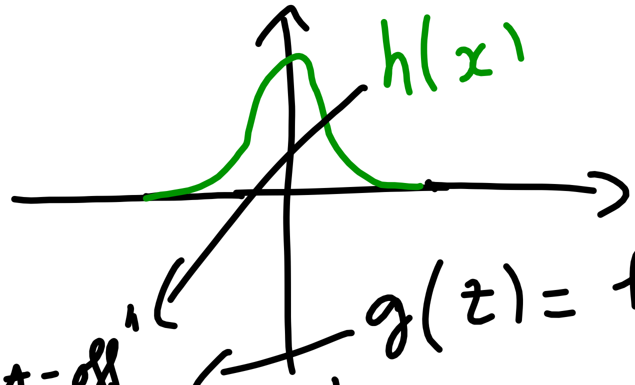
(Rmk: Also, can use directly that $\mathbb{P}^1$ is simply connected.) The above proof assumes that we don't know that $\mathbb{P}^1$ is simply connected)

§13. Dolbeault's Lemma.

Lemma 13.1:

Suppose $g \in \mathcal{E}(\mathbb{C})$ has compact support (i.e. $\mathrm{Support}(g) = \{z \in \mathbb{C} : g(z) \neq 0\}$ is contained in a compact set). Then $\exists$ $f \in \mathcal{E}(\mathbb{C})$ such that:

$$\frac{\partial f}{\partial \bar{z}} = g.$$



"cut-off" function

$g(z) = h(|z|^2) \in \mathcal{E}(\mathbb{C})$ has compact support

Proof: $f(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$

$\square$

in other words, $\bar{\partial}$-equation has solution if the RHS has compact support.

Thm 13.2 (a generalisation):

$X = \{ z \in \mathbb{C} : |z| < R \}$ $(0 < R \leq \infty,\ R = \infty \Rightarrow X = \mathbb{C})$

$g \in \mathcal{E}(X) \Rightarrow \exists f \in \mathcal{E}(X)$: $\frac{\partial f}{\partial \bar{z}} = g$.

Proof: The idea is to approximate g by g_n, g_n has compact support, reduce to lemma 13.1, solve $\frac{\partial f_n}{\partial \bar{z}} = g_n$, & show that f_n converges.