8 November 2023
MAT 4800

* If g has compact support
$\Rightarrow \exists f$ so that $\frac{\partial f}{\partial \bar{z}} = g$.

* **Thm**: If $g \in \mathcal{E}(X)$, $X = \{z \in \mathbb{C} : |z| < R\}$,
($\mathcal{E}$: smooth functions)
then $\exists f \in \mathcal{E}(X)$: $\frac{\partial f}{\partial \bar{z}} = g$.

Idea for proof: Choose a cut off function:

$$\varphi_n(z) = \begin{cases} 1 & \text{if } |z| < R - \frac{1}{n} \\ 0 & \text{if } |z| > R - \frac{2}{n} \end{cases}$$

$\varphi_n : \mathbb{C} \to [0,1]$ smooth

$g_n = \varphi_n g \Rightarrow g_n$ is defined on the whole $\mathbb{C}$

g_n has compact support:

$$\Rightarrow \exists f_n : \frac{\partial f_n}{\partial \bar{z}} = g_n.$$

WTS that f_n converges, but unfortunately it does not converge!

What we do is to modify f_n by a polynomial $p_n(z)$. $\frac{\partial p_n}{\partial \bar{z}} = 0$

$$\Rightarrow \quad \frac{\partial f_{n+1}}{\partial \bar{z}} = \varphi_{n+1} g \Rightarrow$$

$$\frac{\partial (f_{n+1} + p_n)}{\partial \bar{z}} = \varphi_{n+1} g \,!$$

What's good is we can approximate any function by a polynomial! (Newton's expansion)

So we define

$$f_{n+1} - p_n(z)$$

& choose $\tilde{f}_n$ so that

$$\frac{\partial \tilde{f}_n}{\partial \bar{z}} = g \text{ on } \{ |z| < R - \tfrac{2}{n}$$

& choose P_n so that

$$\boxed{(*)} \quad | f_{n+1} - \tilde{f}_n - P_n | \leq \frac{1}{2^n} \text{ on}$$

$(P_n = \underbrace{\text{Taylor's expansion of } f_{n+1} - \tilde{f}_n}_{\text{to order } n}$ $\{|z| < R - \frac{2}{n}\}$

Choose $\tilde{f}_{n+1} = f_{n+1} - P_n$

Then $\sum \| \tilde{f}_{n+1} - \tilde{f}_n \| < \infty$ so $\tilde{f}_n$ converges

If $\tilde{f}_n \to \tilde{f}$ then:

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = g \text{ on } \{ |z| < R - \tfrac{2}{n} \}$$

for all n

$$\Rightarrow \frac{\partial \tilde{f}}{\partial \bar{z}} = g \text{ on } X \, !$$

Why such a polynomial p_n exists?

In general, for a smooth function f on $\mathbb{C}$, then Taylor's expansion depends on both z & $\bar{z}$.

$$f = f(0) + \left(\frac{\partial f}{\partial z}(0) z + \frac{\partial f}{\partial \bar{z}} \bar{z} \right) + \dots$$

$$+ \left(\frac{\partial^2 f}{\partial z^2} z^2 + 2 \frac{\partial^2 f}{\partial z \partial \bar{z}} z\bar{z} + \frac{\partial^2 f}{\partial \bar{z}^2} (\bar{z})^2 \right)$$

$$+ \dots$$

The point is:

$$\frac{\partial}{\partial \bar{z}}(f_{n+1} - f_n) = \varphi_{n+1} g - \varphi_n g = 0 \quad \text{on } \{ |z| < R - \tfrac{1}{n} \}$$

$\Rightarrow f_{n+1} - f_n$ is <u>holomorphic</u> on $\{|z| < R - \frac{1}{n}\}$

$\Rightarrow$ So it has convergent power series in z.

$\square$

<u>Thm 13.4</u>: If $X = \{ z \in \mathbb{C} : |z| < R \}$,

then $H^1(X, \mathcal{O}) = 0$.

<u>Corollary</u>: If X is RS, & $\mathcal{U} = (U_i)_{i \in I}$ is an open covering so that U_i is isomorphic to $\{ z \in \mathbb{C} : |z| < R_i \}$, then

$\mathcal{U}$ is a Leray-covering for 1-st cohomology group ~~wrt~~ the $\mathcal{O}$ sheaf. So,

$$H^1(X, \mathcal{O}) = H^1(\mathcal{U}, \mathcal{O}).$$

<u>Proof</u>: Repeat the proof for $H^1(X, \mathbb{C})$, $H^1(X, \mathbb{Z})$:

$$\mathcal{O} \hookrightarrow \mathcal{E} \quad (\text{sub-sheaf})$$

let $\mathcal{U}$ = an open covering of X

Hence if $f \in Z^1(\mathcal{U}, \mathcal{O}) \Rightarrow f \in Z^1(\mathcal{U}, \mathcal{E})$.

By Theorem .... (§12), $H^1(X, \mathcal{E}) = 0 \Rightarrow$

$f \in B^1(\mathcal{U}, \mathcal{E})$.

$$f = (f_{ij}), \quad f_{ij} \in \mathcal{O}(U_i \cap U_j)$$

$$f = \delta g, \quad g \in C^0(\mathcal{U}, \mathcal{E})$$

$$g = (g_i), \quad g_i \in \mathcal{E}(U_i)$$

$$f_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j$$

Want to write

$f = \delta \tilde{g}$, where $\tilde{g} \in C^0(\mathcal{U}, \mathcal{O})$

(Remember how we did for $\mathbb{C}_X$, $\mathbb{Z}_X$?)

So want to find $h \in C^0(\mathcal{U}, \mathcal{E})$ so that

$g - h \in \underline{\underline{C^0(\mathcal{U}, \mathcal{O})}}$!

$$f_{ij} = g_i - g_j, \quad f_{ij} \in \mathcal{O}(U_i \cap U_j)$$

$$g_i, g_j \in \mathcal{E}(U_i), \mathcal{E}(U_j)$$

$$\Rightarrow 0 = \bar{\partial} f_{ij} = \bar{\partial} g_i - \bar{\partial} g_j \text{ on } U_i \cap U_j.$$

Now $\bar{\partial} g_i \in \mathcal{E}(U_i)$ & compatible with restriction maps (to $U_i \cap U_j$) $\Rightarrow$ $\{\bar{\partial} g_i\}_i$ defines a <u>global smooth 1-form</u> on X.

ω

So ~~by~~ we can write $\omega = h \, d\bar{z}$, on $X = \{|z| < R\}$

$h \in \mathcal{E}(X)$.

By Dolbeault's lemma:

$\exists\ \tilde{g} \in \mathcal{E}(X)$ so that:

$$\frac{\partial \tilde{g}}{\partial \bar{z}} = h. \quad \text{on } X$$

$$\Rightarrow \quad \bar{\partial} \tilde{g} = \omega ! \quad \text{on } X.$$

Consider $\underset{\substack{\uparrow \\ C^0(\mathcal{U},\mathcal{E})}}{g} - \underset{\substack{\uparrow \\ \mathcal{E}(X)}}{\tilde{g}} \in C^0(\mathcal{U},\mathcal{E})$

On U_i: $\bar{\partial}(g - \tilde{g}) = \bar{\partial} g - \bar{\partial}\tilde{g} = \bar{\partial} g_i - \bar{\partial}\tilde{g} = \omega|_{U_i} - \omega|_{U_i} = 0.$

So $g - \tilde{g} \in C^0(\mathcal{U}, \mathcal{O})$, & we are done!

& $\delta(g-\tilde{g}) = \delta g = f!$

Trick: look at $\{\bar{\partial} g\}$, $\Rightarrow$ define a global 1-form then solve

even though g is not globally defined!

equation $\bar{\partial} h = \bar{\partial} g$

& now h is globally defined.

$$\delta h = 0.$$

So $\delta(g-h) = \delta g$ represents the same element in $Z^1(\mathcal{U}, \mathcal{F})$.

Also $g-h$ is better than g because $\bar{\partial}(g-h) = 0$ on U_i. $\square$

§14.. Finiteness theorem

(Cohomology of Compact RS has finite dimension).

Functional analysis preliminary:

L^1 & L^2 - spaces .

Motivation: We want function spaces to have the Cauchy property (completeness).

Example: Consider $f_n(t) = t^n$, $t \in [0,1]$, $n = 1, 2, 3, \dots$ smooth function

We see that

$$f_n(t) \to \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t = 1. \end{cases}$$

not a smooth function.

$\Rightarrow$ The space of smooth function with pointwise convergence is not good.

If we consider the absolute norm

$$|f| = \max_{x \in X} |f(x)|,$$

then we have Cauchy property, but we miss several good properties of $\mathbb{R}^2, \mathbb{R}^3, \dots$

L^2-norm gives us something similar to these (Hilbert space). Many good things.

What about $\mathbb{R}^k$:

We have so called an inner product:

$$\langle x,y\rangle = \sum_{i=1}^{k} x_i y_i, \quad x=(x_1,\dots,x_k), \quad y=(y_1,\dots,y_k)$$

Norm is $\|x\| = \sqrt{\langle x,x\rangle}$.

We can define the notion of orthogonality:

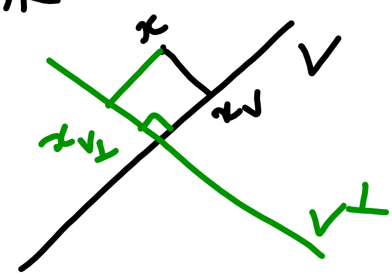
$$\langle x,y\rangle = 0 \iff x \perp y.$$

If $V \subseteq \mathbb{R}^k$ is a sub-vector space, then can decompose any x in

$$x = x_V + x_{V^\perp},$$

where $V^\perp$ = orthogonal space to V, $= \{y\in\mathbb{R}^k : \langle x,y\rangle = 0 \ \forall x\in V\}$

Example $\mathbb{R}^2$



Now, want to do the same for function space.
The absolute norm $|f| = \max|f|$ does not have this property.
The appropriate one is L^2.

Now look at the space of (continuous, or smooth) functions $f : \mathbb{C} \to \mathbb{C}$.

For 2 complex numbers z & w, we can define an inner product:

$$\langle z, w \rangle_{\mathbb{C}} = z\overline{w} \in \mathbb{C}.$$

$z \perp w$ if $z\overline{w} = 0$ (quite silly).

(Make sense in 2-dim:

$$z = (z_1, z_2),\ w = (w_1, w_2):$$

$$z \perp w \iff z_1\overline{w_1} + z_2\overline{w_2} = 0 \ )$$

Come back to functions: let f & $g : \mathbb{C} \to \mathbb{C}$

So we can mimic to get $f(z)\overline{g(z)}$, but this is not a <u>number</u>, only a function!

How to get a number from a function?

We do integration!

$$\langle f, g \rangle_{L^2} \overset{def}{=} \iint_{\mathbb{C}} f(z)\overline{g(z)}\, dx\,dy \qquad z = x + iy$$

$$\Rightarrow \quad \|f\|_{L^2} = \left(\langle f, f \rangle \right)^{1/2} = \left(\iint f \cdot \overline{f} \right)^{1/2}.$$

The space $(\mathcal{E}(\mathbb{D}), L^2)$ is <u>not complete</u>!

But here is a trick by Cauchy to get a complete space: for any Cauchy sequence, we define an element. $(0,1)$

(Lebesgue integration allows to get completion of (Σ, L^2).) $L^2(\Sigma) = \{ f \in \Sigma(X) : \|f\|_{L^2} < \infty \}$.

Ex: $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

Then $f(z) = \dfrac{1}{1-z} \notin L^2(\mathbb{D})$.

$(\|f\|_{L^2} = \infty)$

(Theorem: if $f(x) \approx \dfrac{1}{|x|^\alpha}$, $x \in \mathbb{R}^k$, near 0 & $\alpha \geq k$ then $f \notin L^1$.

$f(z) = \dfrac{1}{1-z}$, $f \in L^2 \iff \int_{\mathbb{D}} \dfrac{1}{|1-z|^2} dx\,dy = \infty$ $\alpha = 2$, $\dim = 2 = k$)

$f(z) = \dfrac{1}{|1-z|^{1/3}} \Rightarrow f \in L^2(\mathbb{D})$

Claim. $L^2(D)$ is a Hilbert space. $\square$
(Just describe above).

Now, define:

$$L^2(D, \mathcal{O}) = L^2(D) \cap \mathcal{O}(D)$$

(so we don't look at all holomorphic functions, but only holomorphic functions with bounded L^2-norm.)

Good thing is: (Thm 14.2)

If $D' \subset\subset D$, $f \in L^2(D, \mathcal{O})$

then $\|f\|_{L^2(D')} \simeq \|f\|_{L^\infty(D')}$

absolute value norm.

This is because of Cauchy integral formula.
holomorphic $f(z)$ can be computed through integrals.
& Jensen's inequality $\int |f| \leq C \, \|f\|_{L^2}$.
boundary D
bound $|f(z)|$

(Proof can be proven by looking at finite sums like: $\sum_{i=1}^{N} |x_i| \leq C \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2}$
& integral is Riemann limit of finite sums).

<u>Corollary</u>: $L^2(D, \mathcal{O})$ is a <u>closed</u> subspace of $L^2(D)$.

<u>Proof</u>: The only thing is to show that

if $f_n \in L^2(D, \mathcal{O})$

& f_n is Cauchy

$\Rightarrow f_n \to f \in L^2(D, \mathcal{O})$.

First off, $L^2(D)$ is Hilbert space & f_n is Cauchy in $L^2(D) \Rightarrow \exists f \in L^2(D)$:

$f_n \to f$.

Only need to show that $f \in \mathcal{O}(D)$.

$(\Leftrightarrow)$ $\forall D' \subset\subset D \Rightarrow f \in \mathcal{O}(D')$.

Since $D' \subset\subset D$

$\Rightarrow \quad \|f_n - f_m\|_{L^\infty(D')} \sim \|f_n - f_m\|_{L^2(D')}$

Since

$\|f_n - f_m\|_{L^2(D')} \to 0$ as $n, m \to \infty$

$\Rightarrow \|f_n - f_m\|_{L^\infty(D')}$ as $n, m \to \infty$.

$\Rightarrow$ if $f_n \to f$ in D' then

f ~~D'~~ is also in $\mathcal{O}(D')$!! $\square$

Corollary: $L^2(D, \mathcal{O})$ is also a Hilbert space.

Lemma 14.3: . Let $D' \subset\subset D$ & $\varepsilon > 0$.

$\exists \; A_\varepsilon \subseteq L^2(D, \mathcal{O})$ a closed vector subspace

so that:

- $\|f\|_{L^2(D')} \leq \varepsilon \|f\|_{L^2(D)} \; \forall f \in A_\varepsilon$

finiteness $\to$. $\operatorname{Dim}\left(L^2(D, \mathcal{O}) / A_\varepsilon \right) < \infty$.

Proof: $L^2(D, \mathcal{O}) / A_\varepsilon$ is the space of Taylor's ~~exp~~ series upto an appropriate order

A_ε = the space of remaining term.

Assume $f \in \mathcal{O}(\mathbb{D})$

$\Rightarrow$ $f(z) = \underbrace{a_0 + a_1 z + \dots + a_n z^n}_{\text{A Taylor's series upto order } n} + \underbrace{\text{error}}_{\downarrow \ A_\varepsilon}$ L^ε

$L^2(D, \mathcal{O}) / A_\varepsilon$.

For every f, this is OK, with n depends on both ε & f. Now we only need to show that we can choose n independent of f.

~~Work for~~ $D' \subset\subset D$, $\overline{D'}$ compact,

So we can cover $\overline{D'}$ by a finite number of disks $\mathbb{D}_i$. So only need to prove for one $\mathbb{D}_i$.

$\mathbb{D}_i = \{ |z| < r \} \Rightarrow f \in \mathcal{O}(2\mathbb{D}_i) \Rightarrow$

f has a power series.

Cauchy's estimate: (ML-estimate)

if $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$

$\Rightarrow |a_n| = |f^{(n)}(0)| \le C \dfrac{M}{n!}$,

$(C = 2\pi r)$ $\quad M = \|f\|_{L^\infty(\mathbb{D}_i)}$

So ~~also~~ divide $\dfrac{f}{\|f\|_{L^2(D)}}$, can assume that $\|f\|_{L^2(D)} \leq 1$ & so

$\forall f \in \mathcal{O}(D)$ with $\|f\|_{L^2(D)} \leq 1$

$\Rightarrow |a_n| \leq \dfrac{C}{n!}$ (independent of f).

Only need to choose n so that the remaining term is $< \varepsilon$:

$$|a_{n+1}| r^{n+1} + |a_{n+2}| r^{n+2} + \cdots < \varepsilon$$

$$\Leftarrow \quad C\left(\frac{r^{n+1}}{(n+1)!} + \frac{r^{n+2}}{(n+2)!} + \cdots\right) < \varepsilon.$$

$\square$