13 September 2023

MAT 4800

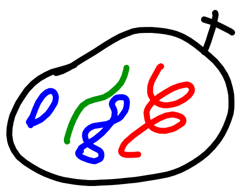
§3. Homotopy

(π_1 : fundamental group)

Study about closed curves on a space.

<u>Curve:</u> A curve on a space X is a continuous function:

$$\varphi : [0,1] \to X$$



<u>Closed curve:</u> It is a curve $\varphi : [0,1] \to X$ so that $\varphi(0) = \varphi(1)$. □

Roughly speaking a homotopy between 2 curves φ_0, φ_1 is a family of curves connecting φ_0 & φ_1, and keep the end points fixed.

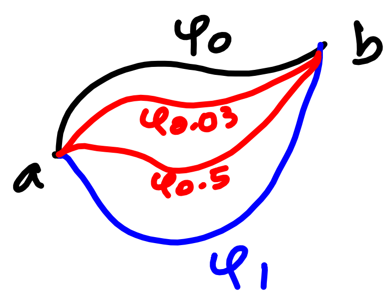
Def: Two curves $\varphi_0, \varphi_1 : [0,1] \to X$ are homotopic if $\varphi_0(0) = \varphi_1(0) = a$, $\varphi_0(1) = \varphi_1(1) = b$ there is a continuous map

$$\Omega : [0,1] \times [0,1] \to X$$

such that:

$$\varphi(0,\cdot) = \varphi_0, \quad \varphi(1,\cdot) = \varphi_1$$
$$\varphi(\cdot,0) = a, \quad \varphi(\cdot,1) = b$$

Ω is called a homotopy between φ_0 & φ_1.



$\underline{\varphi_t}(s) := \varphi(t,s)$

Example 1: $X = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

Every closed curve is homotopic to the constant curve.

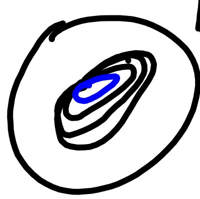
A constant curve:

$$\varphi_c : [0,1] \to X$$
$$s \mapsto c$$

Proof: Let $\varphi : [0,1] \to \mathbb{D}$ be any curve.
$(\varphi(0) = \varphi(1) = 0)$

Define $\Omega : [0,1] \times [0,1] \to \mathbb{D}$ to be
$$(t, s) \mapsto t \cdot \varphi(s) \quad 0$$

HW: $\varphi_1 = \varphi$, $\varphi_0 =$ constant curve $\cancel{\varphi(0)}$



$\mathbb{D}$

Ex 2: $X = \mathbb{D} \setminus \{0\}$

$$\varphi_0 : [0,1] \to X$$
$$s \mapsto \tfrac{1}{2} e^{4\pi i s}$$
$$\varphi_1 : [0,1] \to X$$
$$s \mapsto \tfrac{1}{2} e^{6\pi i s}$$

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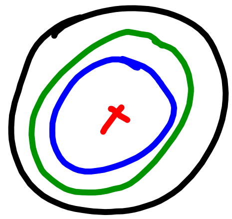
As sets φ_0 & φ_1 images of are the same.

$$\{ z \in \mathbb{C} : |z| = \tfrac{1}{2} \}$$



But They are not homotopic.

$\pi_1(X) = \mathbb{Z}$

$\underline{\pi_1(X)}$: Assume X is path-connected.

(so $\pi_1(X)$ is independent of choice of base-point $a \in X$).

$\pi_1(X,a) = \{$ closed curves on X with end point $a \} / \sim_{\text{homotopic relation}}$

$\underline{\text{Thm:}}$ $\pi_1(X)$ is a group.

(G is a group if it has a binary operation $\cdot : G \times G \to G$

$(a, b) \mapsto a.b$

with the following properties:

associative $(a.b).c = a(b.c)$

identity: $\exists$ element 1 so that: $a \cdot 1 = 1 \cdot a = a \ \forall a$

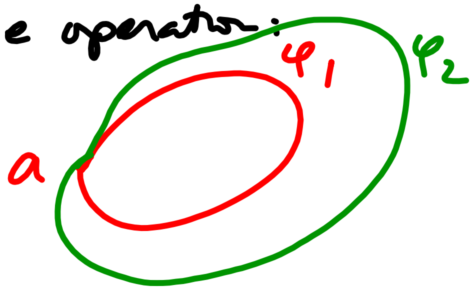
Inverse: $\forall a$, $\exists$ an element a^{-1}: $a.a^{-1} = a^{-1}.a = 1$.)

We need to show:

- $\exists$ a Binary operator on π_1
- Identity curve.
- Inverse of a curve.

We construct first on the level of closed curves with endpoint a, & show that it respects homotopy.

The operation:



$\varphi_1 \circ \varphi_2$ = first do φ_2, then do φ_1.

$$\varphi : [0,2] \to X$$

$$\varphi(t) = \begin{cases} \varphi_2(t) & \text{if } 0 \le t \le 1 \\ \varphi_1(2-t) & \text{if } 1 \le t \le 2 \end{cases}$$

Can rescale $[0,2] \to [0,1]$

$t \mapsto \frac{t}{2}$

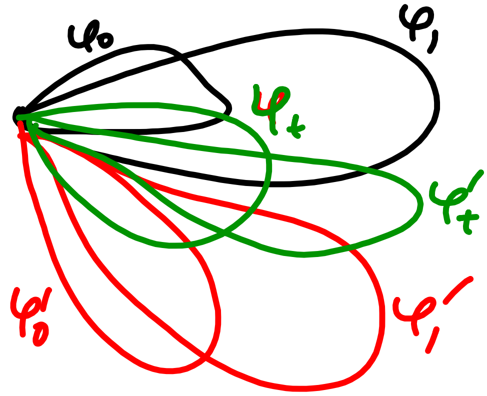
Associative is Ok.

Now show that it respects homotopy.

If $\varphi_0 \underset{\text{homotopic}}{\sim} \varphi_0'$, $\varphi_1 \underset{\text{homotopic}}{\sim} \varphi_1'$

then $\varphi_0\varphi_1 \underset{\text{homotopic}}{\sim} \varphi_0'\varphi_1'$

Homotopy family between $\varphi_0\varphi_1$ & $\varphi_0'\varphi_1'$:



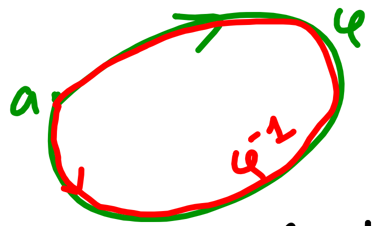
$\bar{\Omega}(s,t) = \Omega(s,t) \circ \Omega'(s,t)$

composition of 2 curves

HW: Check $\bar{\Omega}$ continuous.

Identity curve: $\varphi_a : [0,1] \to X$, $s \mapsto a$

Inverse curve to $\varphi: [0,1] \to X$
is the same curve, but we go backward.



$$\varphi^{-1}(s) = \varphi(1-s)$$

Need to check $\varphi_a \varphi \underset{\text{homotopic}}{=} \varphi \varphi_a \underset{\text{homotopic}}{=} \varphi$

$\varphi \varphi^{-1} \underset{\text{homotopic}}{=} \varphi^{-1}\varphi = \varphi_a$.

<u>Rmk</u>: These two identities cannot be satisfied on the level of curves. Only satisfied on the level of homotopy. Which means that we need to cook up a homotopy to show the identities.

HW: Check the proof in the book.

<u>Important tool</u>: (Reparametrization) $\varphi: [0,1] \to X$ a closed curve.

$\psi: [0,1] \to [0,1]$ continuous function with $\psi(0) = 0$, $\psi(1) = 1$

Then φ homotopic $\varphi \circ \psi$.

$[0,1]$ is simply connected. So

$$\boxed{\psi \underset{\text{homotopic}}{\sim} id_{[0,1]}}$$

$\Omega(s,t) = s\psi(t) + (1-s)\, id_{[0,1]} t$

$$\varphi = \varphi \circ id_{[0,1]} \underset{\text{homotopic}}{\sim} \varphi \circ \psi$$

<u>Q1</u>: What is $\pi_1(\mathbb{P}^1)$?

<u>Q2</u>: What is $\pi_1(E)$? (next time)

elliptic curve

<u>Van Kampen theorem</u>: Let $X = U \cup V$.

path connected

open sets

Then we can compute $\pi_1(X)$ from $\pi_1(U)$, $\pi_1(V)$ & $\pi_1(U \cap V)$ in the following way.

Let

$\pi_1(U) = \langle u_1, \ldots, u_k \mid \alpha_1, \ldots, \alpha_\ell \rangle$

generators

relations between generators

$\mathbb{Z} = \langle 1 \rangle$

$\mathbb{Z}[w] \quad (w^2 = -1)$

$$= \langle \underbrace{1, w}_{\text{generators}} \mid \underbrace{w^2+1=0}_{\text{relations}} \rangle$$

$$\pi_1(V) = \langle \underbrace{v_1, \ldots, v_m}_{\text{generators}} \mid \underbrace{\beta_1, \ldots, \beta_n}_{\text{relations}} \rangle$$

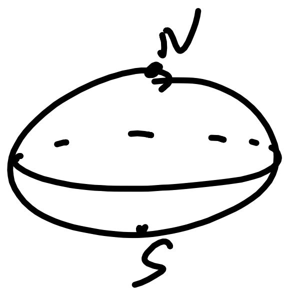
$$\pi_1(U \cap V) = \langle w_1, \ldots, w_p \mid \gamma_1, \ldots, \gamma_q \rangle$$

$I: \quad \pi_1(U \cap V) \to \pi_1(U)$

$J: \quad \pi_1(U \cap V) \to \pi_1(V)$

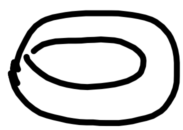
$$\pi_1(X) = \langle u_1, \ldots, u_k, v_1, \ldots, v_m \mid \alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_n, I(w_1)J(w_1)^{-1}, \ldots, I(w_p)J(w_p)^{-1} \rangle$$

$\mathbb{P}^1 = S^2$



$U = S^2 \setminus \{N\}$
$\approx \mathbb{C}^* \approx \mathbb{R}^2$

$V = S^2 \setminus \{S\}$
$\approx \mathbb{C} \approx \mathbb{R}^2$

$U \cap V \approx$  cylinder

$\pi_1(U) = \pi_1(V) = 0.$

$\Rightarrow \pi_1(X) = 0$ by van Kampen. $\square$

Come back to $\mathbb{P}^2$:

$[x : y : z]$, $(x, y, z) \in \mathbb{C}^3$
$(x, y, z) \neq (0,0,0)$

$\mathbb{C}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbb{P}^2$

$(x, y, z) \mapsto [x : y : z]$

Q: When is an equation $\{f(x, y, z) = 0\}$, $(x,y,z) \in \mathbb{C}^3$, defines a subset of $\mathbb{P}^2$.

Answer: It defines some subset $\mathbb{P}^2$ iff the following condition is satisfied:
Whenever $f(x_0, y_0, z_0) = 0 \Rightarrow$

continuous

$f(\alpha x_0, \alpha y_0, \alpha z_0) = 0 \quad \forall \alpha \neq 0$
$\in \mathbb{C}$

HW: Show that if this condition is satisfied then $f(0,0,0) = 0$.

Example: $f(x,y,z) = e^x + 2y + 3z$
(No: $f(0,0,0) = 1 \neq 0$)

Ex: $f(x,y,z) = y^2 z - x^3 - z^3$
(Yes: If $y_0^2 z_0 - x_0^3 - z_0^3 = 0$,
& $\alpha \in \mathbb{C}$: $(\alpha y_0)^2 (\alpha z_0) - (\alpha x_0)^3 - (\alpha z_0)^3 = \alpha^3 [y_0^2 z_0 - x_0^3 - z_0^3] = \alpha^3 . 0 = 0$)

This ~~fig~~ the elliptic curve corresponding to the affine elliptic curve $y^2 = x^3 + 1$.

Homogenization: If $f(x,y)$ is a polynomial, then the homogenization is $z^d . f\left(\frac{x}{z}, \frac{y}{z}\right) = F(x,y,z)$ where d is the least number so that

$F(x, y, z)$ is a polynomial.

Ex: $f(x, y) = y^2 - x^3 - 1$

$$z^d f\left(\frac{x}{z}, \frac{y}{z}\right) = z^d\left(\left(\frac{y}{z}\right)^2 - \left(\frac{x}{z}\right)^3 - 1\right)$$

$\Rightarrow$ $d = 3$ & $F(x, y, z) = y^2 z - x^3 - z^3$.

$$\left(\frac{x}{z}, \frac{y}{z}\right) = \left[\frac{x}{z} : \frac{y}{z} : 1\right]$$