MAT 4800

30 August 2023

Applications of Cauchy integral formula.

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z-\zeta}\, dz$$

$$\frac{1}{z-\zeta} = (z-\zeta)^{-1}$$

$$\frac{d^k}{d\zeta^k}\left((z-\zeta)^{-1}\right) = (k\ )!\ (z-\zeta)^{-(k+1)}$$

$$\frac{d}{d\zeta}\left((z-\zeta)^{-1}\right) = (-1)\times(-1)(z-\zeta)^{-2}$$
$$= (z-\zeta)^{-2}$$

$$\frac{d^k f}{d\zeta^k} = \frac{1}{2\pi i} \times k! \oint \frac{f(\zeta)}{(z-\zeta)^{k+1}}$$

$\underline{\text{Maximum principle}}$: Let $f: \Omega \to \mathbb{C}$ be holomorphic. Assume that $\exists\, z_0 \in \Omega$ so that $|f(z_0)| = \max_{z \in \Omega} |f(z)| \Rightarrow f(z) =$ constant.

(Ω connected, open)

$\underline{\text{Step 1}}$: $A = \{z \in \Omega : |f(z)| = |f(z_0)|\}$.

We show that $A = \Omega$.

We show that A is $\neq \emptyset$, A is open & A is closed. (Then $A = \Omega$.)

. $A \neq \emptyset$: $z_0 \in A$

. A is closed: Show that if $\{z_n\} \subseteq A$, $z_n \to z^*$, then $z^* \in A$.

$$|f(z_n)| = |f(z_0)| \quad \forall n$$

$$z_n \to z^* \Rightarrow f(z_n) \to f(z^*)$$

$$\Rightarrow \underbrace{|f(z_n)|}_{= |f(z_0)|} \to \underbrace{|f(z^*)|}_{= |f(z_0)|}$$

· A is open: If $z^* \in A$, show that $\exists \ B(z^*, r) \subseteq \Omega$ so that $B(z^*, r) \subseteq A$.

$$f(z^*) = \frac{1}{2\pi i} \oint_{\partial B(z^*, r)} \frac{f(z)}{z - z^*} \, dz$$

(Like: if $a = \sum b_n \Rightarrow |a| \le \sum |b_n|$)

$$|f(z^*)| \le \frac{1}{2\pi} \oint_{\partial B} \frac{|f(z)|}{|z - z^*|} \, |dz|$$

what is this?

$\partial B(z^*, r) = \{ z \in \mathbb{C} : z - z^* = r e^{i\theta}, \ \theta \in [0, 2\pi] \}$

$$dz = r i e^{i\theta} \, d\theta$$

$$|dz| = r \, d\theta$$

$$|f(z^*)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{r} \, r \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |f(z)| \, d\theta$$

But $|f(z)| \le |f(z^*)| \ \forall z$

$$\Rightarrow \quad |f(z)| = |f(z^*)| \quad \forall \ z \in \partial B(z^*, r)$$

i.e. $(|z - z^*| = r)$

We can reduce r & hence get

$$|f(z)| = |f(z^*)| \quad \forall \ z \in B(z^*, r)$$

$A =$ open $\Rightarrow \ldots \quad A = \Omega.$

<u>HW</u>: $f: \Omega \to \mathbb{C}$ holomorphic &

$$|f(z)| = \text{const} \Rightarrow f(z) = \text{const}$$

<u>Proof</u>: Cauchy - Riemann equations + Existence of Logarithm

Later.

$\Rightarrow \quad v_x^2 + v_y^2 = 0$

$u_{xx}^2 =$

Liouville's theorem:

If $f: \mathbb{C} \to \mathbb{C}$ is holomorphic & f is bounded then f is a constant.

$$f(\zeta) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-\zeta}\, dz$$

If $f(z)$ is bounded $\Rightarrow \exists\ M \in \mathbb{C}$:
$|f(z) - M| \geq 1 \quad \forall z \in \mathbb{C}$.

$\dfrac{1}{f(z)-M}$ is holomorphic & also bounded.

Identity principle:

If $f: \Omega \to \mathbb{C}$ is holomorphic

& if $\exists \{z_n\}$ different, $z_n \to z^*$,
$f(z_n) = 0 \ \forall n \Rightarrow f \equiv 0$.

Proof: If f is not $\equiv 0$,
$f(z) = a_0 + a_1 (z - z^*) + a_2 (z - z^*)^2 + \cdots$

Choose k the first index so that $a_k \neq 0$. Then we can write.

$$f(z) = a_k (z - z^*)^k \underbrace{\left[1 + c_1 (z - z^*) + c_2 (\ ) + \cdots\ \right)}_{\text{holomorphic function}}$$

(it has the same $\limsup_{n \to \infty} |a_n|^{1/n}$)

Claim: near z^*, there is no other root. $\square$

<u>Liouville's theorem:</u>

$f: \mathbb{C} \to \mathbb{C}$ holomorphic & f is bounded $\Rightarrow$ f is constant.

<u>Proof:</u>

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\partial B(0,r)} \frac{f(z)}{z - \zeta} dz$$

$$f^{(k)}(\zeta) = \frac{1}{2\pi i} k! \oint_{\partial B(0,r)} \frac{f(z)}{(z - \zeta)^{k+1}} dz$$

$\zeta = 0$:

$$f^{(k)}(0) = \frac{1}{2\pi i} k! \oint_{\partial B(0,r)} \frac{f(z)}{z^{k+1}} dz$$

$\forall r$

$$|f^{(k)}(0)| \le \frac{1}{2\pi} k! \oint_{\partial B(0,r)}^{2\pi} \frac{\overbrace{|f(z)|}^{\le M}}{r^{k+1}} r\,d\theta$$

$k \geq 1$:

Take $r \to \infty$:

$\Rightarrow f^{(k)}(0) = 0$. $\quad \frac{r}{r^{k+1}} \to 0$ if $r \to \infty$ $\forall k \geq 1$.

$$f(\zeta) = f(0) + \underbrace{\frac{f'(0)}{1!}}_{0}\zeta + \underbrace{\frac{f''(0)}{2!}}_{0}\zeta^2 + \dots$$

$$= f(0). \qquad \forall \zeta. \qquad \square$$

<u>HW</u>: If $|f(z)| = |f(0)|$ $\forall z \in \Omega$ (open, connected)

$\Rightarrow f(z) = f(0)$.

<u>Proof</u>: First, just need to show that the set $A = \{z : f(z) = f(0)\} \neq \emptyset$, closed, open.

We can assume $\Omega = B(0,r)$

If $f(0) = 0 \Rightarrow |f(z)| = |f(0)| = 0$

$\Rightarrow f(z) = 0 \cdot \forall z.$

We can assume $f(0) \neq 0$.

$g(z) = \dfrac{f(z)}{f(0)}$, $|g(z)| = 1 \ \forall$ $z \in B(0,r)$.

Existence of Logarithm $\Rightarrow \exists\, G : \Omega \to \mathbb{C}$ holomorphic so that $g(z) = e^{G(z)}$.

$$G(z) = u(x,y) + i v(x,y)$$

$$\Rightarrow g(z) = e^{u(x,y) + i v(x,y)}$$

$$1 = |g(z)| = e^{u(x,y)} \quad \forall z$$

$$\Rightarrow u(x,y) \equiv 0$$

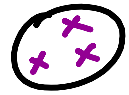
CR $\Rightarrow$ $v_x = 0$, $v_y = 0$.

$\Rightarrow v = \text{const}.$

$G(z) = i\alpha \Rightarrow \dfrac{f(z)}{f(0)} = e^{i\alpha}$ $\forall z$.

Choose $z = 0 \Rightarrow e^{i\alpha} = 1 \Rightarrow f(z) = f(0)$ $\forall z$. $\square$

Existence of logarithm:



→ Not simply connected

$\Omega = B(0,r)$ (more generally, Ω is simply connected)

$\mathcal{O}(\Omega) = \{f : \Omega \to \mathbb{C} \text{ holomorphic}\}$

① If $f \in \mathcal{O}(\Omega) \Rightarrow \exists\, F \in \mathcal{O}(\Omega)$ such that: $F' = f$.

② If $f \in \mathcal{O}^*(\Omega) = \{f \in \Omega : f \text{ is never } 0\}$
$\Rightarrow \exists\, F \in \mathcal{O}(\Omega) : f = e^F$.

Proof: Because Ω is $B(0,r)$ every holomorphic function on Ω has a power series.

① $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in B(0,r)$

$\Rightarrow F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$

f & F have the same radius of convergence

② $f \in \mathcal{O}^*(\Omega)$

$\Rightarrow \dfrac{f'}{f} \in \mathcal{O}(\Omega)$

$\Rightarrow \exists\, G \in \mathcal{O}(\Omega)$: $G' = \dfrac{f'}{f}$.

$\Rightarrow \exists\, \alpha$ constant: $\cancel{fe^{-G}}$ $\underbrace{fe^{-G}}_{\text{derivative } = 0} = \alpha = e^{\beta}$

$\Rightarrow f = e^{G+\beta}$. $\square$

Open Mapping Theorem: Ω connected, open

Let $f : \Omega \to \mathbb{C}$ be holomorphic, $\Omega = B(0,r)$

~~Be~~ f not a constant. Then $f(\Omega)$ open.

Proof: Assume $f(\Omega)$ is not open. $\exists$.

$a_j \notin f(B(0,r))$ so that

$a_j \to f(0) = 0$.

Consider $g_j(z) = \dfrac{1}{f(z) - a_j}$.

$$\sup_{z \in \partial B(a,r)} |g_j(z)| \leq \frac{1}{\left(\min_{\partial B(a,r)} |f(z)|\right) - |a_j|}$$

Maximum principle for $g_j(z)$:

$$\Rightarrow \quad |g_j(0)| \leq \sup_{z \in \partial B(a,r)} |g_j(z)| \leq \frac{1}{\min_{\partial B(a,r)} |f(z)| - |a_j|}$$

$$|g_j(0)| = \frac{1}{|a_j|}$$

$j \to \infty$: $\frac{1}{|a_j|} \to \infty$ /

$$\frac{1}{\min_{\partial B(a,r)} |f(z)| - |a_j|} \to \frac{1}{\min_{\partial B(a,r)} |f(z)|} < \infty$$

Contradiction! $\square$

<u>Singularities</u>: $f: B(a,r) \setminus \{a\} \to \mathbb{C}$ holomorphic.

a is called a singularity of f.

Classification of singularities:

<u>Case 1</u>: (Riemann removable theorem):

If $f(z)$ is bounded $\Rightarrow$ f extends to a holomorphic function $f: B(a,r) \to \mathbb{C}$.

Case 2: (Poles)

$$\lim_{z \to a} |f(z)| = \infty .$$

Case 3: (Essential singularities) Casorati-Weierstrass theorem:

$\forall 0 < s < r$: $f(B(a,s))$ is dense in $\mathbb{C}$.

Example:

(Removable) $f(z) = z^k$, $z \in B(0,r) \setminus \{0\}$

(Pole) $f(z) = \dfrac{1}{z^2}$, $z \in B(0,r) \setminus \{0\}$

(Essential singularity): $f(z) = e^{-\frac{1}{z}}$, $B(0,r) \setminus \{0\}$

not bounded: e.g. $z \to 0$, for $z \in (-\infty, 0)$

Not pole: e.g. $z \to 0$, for $z \in (0, \infty)$

$\Rightarrow$ must be essential singularity.

Proof:

(Removable case): For any $\zeta \in B(a,r) \setminus \{a\}$

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\partial(B(a,r)\setminus B(a,s))} \frac{f(z)}{z-\zeta}\,dz$$

Choose $r > s > 0$ such that $\zeta \in B(a,r)\setminus B(a,s)$

Now, the thing is that since ~~ζ is~~ f is bounded, $\lim_{s\to 0}$ exists, & this integral is well-defined: $\oint_{\partial B(a,r)} \frac{f(z)}{z-\zeta}\,dz$

& defines a holomorphic function.

(2. Poles): $\lim_{z\to a} |f(z)| = \infty$

$g(z) = \frac{1}{f(z)}$ is holomorphic near a.

($\lim_{z\to a} |f(z)| = \infty \Rightarrow$ can assume that $f(z) \neq 0 \;\; \forall z \in B(a,r)$.

$g(z) = \frac{1}{f(z)}$ is holomorphic on $B(a,r)\setminus\{a\}$ & bounded. $g(z)$ extends to a holomorphic function in $B(a, r)$ & $g(a) = 0$.

$$g(z) = \alpha (z-a)^k \cdot h(z)$$

$h(z) \neq 0$ at a

$$f(z) = \frac{1}{g(z)} = \frac{1}{\alpha (z-a)^k} \cdot \frac{1}{h(z)}.$$

k = the order of pole at a.

(3. Essential singularities):

Assume that Casorati – Weierstrass theorem is not satisfied.

$\exists$ $s > 0$: such that $f(B(a,s) \setminus \{a\})$ is not dense in $\mathbb{C}$.

$\Rightarrow$ $\lambda \in \mathbb{C}$ so that $|f(z) - \lambda| \geq \varepsilon > 0$ for all $z \in B(a,s) \setminus \{a\}$.

$$g(z) = \frac{1}{f(z) - \lambda}$$ holomorphic on $B(a,s) \setminus \{a\}$

& bounded $\left(|g(z)| \leq \frac{1}{\varepsilon} \right)$

Removable singularity $\Rightarrow$ g extends to holomorphic function on $B(a,s)$.

If $g(a) \neq 0 \Rightarrow f$ has removable singularity at a.

If $g(a) = 0 \Rightarrow \lim_{z \to a} |f(z)| = \infty$

$\Rightarrow f$ has a pole at a. $\square$

<u>HW</u>: e^z has any singularity at $z=\infty$?

Is it bounded near $z=\infty$? No: $z_n = n$, $e^{z_n} = e^n \to \infty$.

Does it have a pole near ∞? No, $z_n = 2\pi i n \Rightarrow e^{z_n} = 1$ bounded.

<u>Laurent expansion:</u>

$$f \in \mathcal{O}(B(0,s) \setminus \{0\})$$

$$\Rightarrow f(z) = \sum_{n=-1}^{-\infty} a_n z^n + \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{\text{power series}}$$

<u>HW</u>:
- f has a pole at 0 iff $\exists N : \forall n \leq N : a_n = 0$, $\exists n < 0 : a_n \neq 0$
- f is removable at 0 iff $a_n = 0 \ \forall n < 0$.

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{z^2}\frac{1}{2!} - \frac{1}{z^3}\cdot\frac{1}{3!}$$

$$+ \cdots$$