Math 4800
5 September 2023

§1. Riemann surfaces

$f: A \to B$ continuous map
$A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$

f is homeomorphism if it is bijective & the inverse $f^{-1}: B \to A$ is also continuous.

Recall about real manifolds: X
"Locally" it looks like ~~$\mathbb{R}$~~ a ball in $\mathbb{R}^n$.

We need 2 properties:

① For every $x \in X$, $\exists\, x \in U \subseteq X$ open
& a homeomorphism
$f: U \to$ ~~$\mathbb{R}^n$~~ $f(U) \subseteq \mathbb{R}^n$, $f(U) = V$
where V is an open set in $\mathbb{R}^n$.

(2) Transition maps are smooth:

$$
\begin{tikzcd}
X \supseteq U_1 \arrow[d, "f_1"'] & U_2 \subseteq X \arrow[d, "f_2"] \\
\mathbb{R}^n \supseteq V_1 & V_2 \subseteq \mathbb{R}^n
\end{tikzcd}
$$

$$U_{1,2} = U_1 \cap U_2$$

We have 2 different maps from $U_{1,2}$ to $\mathbb{R}^n$: $f_1 : U_{1,2} \to f_1(U_{1,2}) = V_{1,2} \subseteq \mathbb{R}^n$

$$f_2 : U_{1,2} \longrightarrow f_2(U_{1,2}) \subseteq \mathbb{R}^n, \quad f_2(U_{1,2}) = V_{2,1}$$

They may be different maps but we want them to be smoothly transited:

$$
\begin{tikzcd}
& U_{1,2} \arrow[dl, "f_1"', "\sim"] \arrow[dr, "f_2", "\sim"'] & \\
\mathbb{R}^n \supseteq V_{1,2} & & V_{2,1} \subseteq \mathbb{R}^n
\end{tikzcd}
$$

Transition map : $f_{1,2} : V_{1,2} \xrightarrow[\sim]{f_1^{-1}} U_{1,2} \xrightarrow[\sim]{f_2} V_{2,1}$

Want $f_{1,2}$ to be C^∞!!

(Need some technical conditions on the topology of X to avoid pathological properties.)

Riemann surfaces: X Very similar

(1) $\forall x \in X$, $\exists x \in U \subseteq X$

homeomorphic open subsets:

$f: U \xrightarrow{\sim} f(U) \subseteq \mathbb{C}$

$f(U) = V \to$ open

(2) Transition maps are holomorphic!

Trivial examples: Every open set X of $\mathbb{C}$ is a RS. ($f: X \to \mathbb{C}$ identity map.

Transition maps are identity map.)

$f(z) = z$

Nontrivial example:

Example 1: Riemann sphere

$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$

$X =$ Riemann sphere $= \mathbb{P}^1$

$\Downarrow$ projective line

$\mathbb{C} \cup \infty$

$\parallel$

$X_1 = \mathbb{C}$, $X_2 = X \setminus \{0\}$

$\downarrow$ south pole

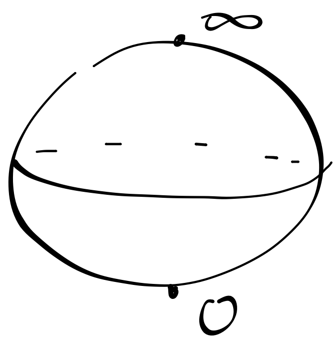
$= X \setminus \{\infty\}$

$\downarrow$ north pole

$X = X_1 \cup X_2$

$X_1 = \mathbb{C}$ with coordinate z

$X_2 = \mathbb{C}$ with coordinate w



$f_1 : X_1 \xrightarrow{\sim} \mathbb{C}_z$

$z \longrightarrow z$

$\boxed{w = \frac{1}{z}}$

$f_2 : X_2 \xrightarrow{\sim} \mathbb{C}_w$

$w \longrightarrow w$

$$X_1 \cap X_2 = \mathbb{C} \setminus \{0, \infty\}$$

$$\begin{array}{ccc} & X_1 \cap X_2 & \\ \swarrow^{\varphi_1} & & \searrow \\ z & & \boxed{w = \frac{1}{z}} \end{array}$$

Transition map

$$\mathbb{C}_z \setminus \{0\} \longrightarrow \mathbb{C}_w \setminus \{0\}$$
$$z \longmapsto w = \frac{1}{z}$$

So here we are not given embedding maps but actually transition map.

Remark: Since z & w coordinates are different, the 0 in z-coordinate is not the same as the 0 in w-coordinate. Actually, because $w = \frac{1}{z}$, the 0 in z-coordinate is the ∞ in w-coordinate.

<u>Holomorphic map from RS to $\mathbb{C}$:</u>

$g : X^{\to RS} \longrightarrow \mathbb{C}$ is holomorphic if $\forall$ coordinate open set $U \subseteq X$

$$\begin{array}{c} U \subseteq X \\ \varphi \downarrow \\ V \subseteq \mathbb{C} \end{array}$$

$$\begin{tikzcd} \mathbb{C} & X & \\ V \arrow[u, phantom, "\supseteq" description] \arrow[r, "f^{-1}"] \arrow[rr, bend right, "h"'] & U \arrow[u, phantom, "\subseteq" description] \arrow[r, "g|_U"] & \mathbb{C} \end{tikzcd}$$

then h is <u>holomorphic</u>!

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "f_1"', "\sim"] \arrow[dr, "f_2", "\sim"'] & \\ \mathbb{C} \supseteq V_{1,2} & & V_{2,1} \subseteq \mathbb{C} \end{tikzcd}$$

If $V_{1,2} \to \mathbb{C}$ is holomorphic
then $V_{2,1} \to \mathbb{C}$ is holomorphic.

<u>Proof:</u>

$$V_{2,1} \to \mathbb{C} \;=\; V_{2,1} \to V_{1,2} \to \mathbb{C}$$

holomorphic by assumption

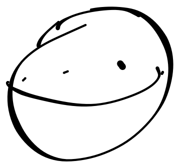
transition $\Downarrow$ holomorphic

Q: Is $X = \mathbb{P}^1$ a subset of $\mathbb{C}$?
(i.e. biholomorphic to a subset of $\mathbb{C}$?)

<u>Answer:</u> If X were biholomorphic to a subset of $\mathbb{C}$, then there would be a non-constant holomorphic map from X to $\mathbb{C}$.

X is compact.

$\Rightarrow f(X)$ is compact. Will violate the maximum principle.

 $\xrightarrow[\sim]{f}$ $W \subseteq \mathbb{C}$

$\downarrow$

compact

Let $w^* \in W$ so that $|w^*| = \max_{w \in W} |w|$.

Look at $x^* = f^{-1}(w^*)$ & a small coordinate set U^* around x^*.

$$X \supseteq U^* \xrightarrow{f} W \subseteq \mathbb{C}$$

$$g \downarrow \wr$$

$$\mathbb{C} \supseteq g(U^*) = V^* \subseteq \mathbb{C}$$

Look at $V^* \to U^* \to W \subseteq \mathbb{C}$ f

apply maximum principle for this map.

X : RS from now on assume to be connected.

<u>Another way to look at</u> $\mathbb{P}^1$:

look at the set of Λ complex lines in $\mathbb{C}^2$ going through $(0,0)$. $\mathbb{C}^2_{z_1,z_2}$

$$L_{a,b} = \{ az_1 + bz_2 = 0 \}$$
$$(a,b) \neq (0,0)$$

$$L_{a,b} = L_{\alpha a, \alpha b}, \text{ for } \alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$\Lambda = \mathbb{P}^1 \ !$$

$$\Lambda = \{ (a,b) \in \mathbb{C}^2 \setminus \{(0,0)\} \} / \sim$$

has quotient topology.

Q: What are open sets in Λ?

$(a,b) \sim (c,d)$ if $\exists$ $\alpha \in \mathbb{C}^*$ so that

$$\begin{cases} a = \alpha c \\ b = \alpha d \end{cases}$$

To remember this we use the notation:

$[a:b]$ instead of (a,b)

$$[a:b] = [\alpha a : \alpha b] \quad \forall \alpha \in \mathbb{C}^*$$

Claim $\Lambda = \mathbb{P}^1$.

$$\Lambda_1 = \{ [a:b] \in \Lambda : b \neq 0 \}$$

$$\Lambda_2 = \{ [a:b] \in \Lambda : a \neq 0 \}$$

$\boxed{\text{Claim}}$ $f_1 : \Lambda_1 \simeq \mathbb{C}$

Define a map

$$\Lambda_1 \longrightarrow \mathbb{C}$$

$$[\tfrac{a}{b} : 1] = [a:b] \mapsto z = \frac{a}{b}$$

This is a homeomorphism

$$[z : 1] \leftarrow z$$ Inverse map

$$\Lambda_2 = \{ [a:b], a \neq 0 \}$$

$$\begin{gathered} [a:b] \\ \| \\ [\underbrace{1 : \tfrac{b}{a}}] \end{gathered}$$

$$f_2 \quad \Lambda_2 \longrightarrow \mathbb{C}_w$$

$$[a:b] \mapsto w = \frac{b}{a}$$

$$[1 : w] \leftarrow w$$

What is the relation between z & w?

Look at a point $[a:b] \in \Lambda_1 \cap \Lambda_2$

$$= \{ [a:b] \in : a, b \in \mathbb{C}^* \}$$

$$z = \frac{a}{b} \Rightarrow \boxed{w = \frac{1}{z}}$$
$$w = \frac{b}{a}$$

Transition map between coordinates z & w

Example 3: $X = \{(z_1, z_2) \in \mathbb{C}^2 : f(z_1, z_2) = 0\}$

f: holomorphic map $\mathbb{C}^2 \to \mathbb{C}$
(For this course, just need to care about f = polynomial)

Assume that $\forall\, (z_1^*, z_2^*) \in X$:
$$\left(\frac{\partial f(z_1^*, z_2^*)}{\partial z_1}, \frac{\partial f}{\partial z_2}(z_1^*, z_2^*) \right) \neq (0,0).$$

Then X is a RS.
(Proof later)

Example: $f(z_1, z_2) = z_1 - z_2^2$
$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \right) = (1, -2z_2) \neq (0,0)\ \forall z_1, z_2$$
$$\Rightarrow X = \{ z_1 - z_2^2 = 0\} \text{ is a RS.}$$

$X \subseteq \mathbb{C}^2$

$\parallel$ graph of the function $f(z) = z^2$

We can think about z_2 like $\sqrt{z_1}$.

If $\sqrt{1} = 1 \Rightarrow \sqrt{e^{2\pi\theta i}} = e^{\pi\theta i}$

(Riemann's idea) $X = \{z_1 - z_2^2 = 0\}$ is the RS of the $z \mapsto \sqrt{z}$:

$w = \sqrt{z} \Rightarrow w^2 = z$

$X \subseteq \mathbb{C}_{z_2}$

(z_1, z_2)

$\mathbb{C}_{z_2} \quad z_2$

$X \to \mathbb{C}_{z_1}$ RS of $\sqrt{z}$

$(z_1, z_2) \mapsto z_1$

Example: $f(z_1, z_2) = z_1^2 - 3z_2^3 + 1$

gradient

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right) = (2z_1, -9z_2^2)$$

it is $(0,0)$ at $(z_1, z_2) = (0,0)$.

But $(0,0) \notin X = \{f(z_1, z_2) = 0\}$

$\Rightarrow X$ is RS (affine part of elliptic curve)

Proof of Theorem:

Let $z^* = (z_1^*, z_2^*) \in X$ so that

$$\left(\frac{\partial f}{\partial z_1}(z_1^*, z_2^*), \frac{\partial f}{\partial z_2}(z_1^*, z_2^*) \right) \neq (0,0).$$

Then $\exists$ an open set $z^* \in U \subseteq X$ so that

$$g : U \xrightarrow[\text{homeomorphic}]{\sim} g(U) \subseteq \mathbb{C}$$

& also transition function is holomorphic.

Inverse function theorem for complex functions:

$$f(z_1, z_2) = 0 \quad \text{on } X$$

$$\Rightarrow \quad df = 0 \quad \text{on } X$$

Near (z_1^*, z_2^*) we can write:

$$0 = df = \frac{\partial f}{\partial z_1}(z_1^*, z_2^*) \cdot \cancel{(z_1 - z_1^*)}\, dz_1 + \frac{\partial f}{\partial z_2}(z_1^*, z_2^*) \cdot \cancel{(z_2 - z_2^*)}\, dz_2 + \dots$$

Just assume that linear approximation is precise & assume $\frac{\partial f}{\partial z_1}(z_1^*, z_2^*) \neq 0$

$\Rightarrow$ we can divide it.

$$dz_1 = -\frac{\frac{\partial f}{\partial z_2}(z_1^*, z_2^*)}{\frac{\partial f}{\partial z_1}(z_1^*, z_2^*)}\, dz_2$$

$\Rightarrow z_1$ is a holomorphic function in z_2.
(in a neighborhood U).

So. Meaning $U \to \mathbb{C}_{z_2}$
$(z_1, z_2) \mapsto z_2$ is . $\square$

HW: $f(z_1, z_2) = z_1^2 - 3z_2^3 + 1$

Look at $(i, 0)$

What is your coordinate system near this point?