MAT4800 §14 Finiteness Theorem 04.11.19

Roughly speaking $H^i(X, \mathcal{O})$ has finite dimension if X is compact.

In this course, X is a compact RS. $\mathcal{F} = \mathcal{O}$

(General idea: If X is compact, then any open cover has a finite subcover. We can in most cases construct Leray's covering for $\mathcal{F}$. So we can construct a finite open covering $\mathcal{U} = (U_i)_{i\in I}$ I finite which is also a Leray's covering. Then by Thm 12.8,

$$H^1(X, \mathcal{F}) = \underbrace{H^1(\mathcal{U}, \mathcal{F})}_{\text{a quotient of } \underbrace{\prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)}_{\text{finite dimension}}}$$

For X compact RS. $\mathcal{O}$ sheaf of holomorphic functions, we relate $H^1(X, \mathcal{O})$ to harmonic forms.

Functional analysis preliminary:

① L^2-norm (a Sololev's space)

If $f: D \to \mathbb{C}$ is continuous, or holomorphic in our case, then we define $\|f\|_{L^2(D)} = \left(\iint_D |f(x+iy)|^2 \, dx\,dy \right)^{1/2}$

$L^2(D) = \{ f: D \to \mathbb{C},\ \|f\|_{L^2} < \infty \}$

ex D = unit disk $= \{ z \in \mathbb{C} : |z| < 1 \}$

$f(z) = \frac{1}{1-z}$ $\quad f \notin L^2(D)$

$f(z) = \frac{1}{|1-z|^{1/3}}$ $\quad f \in L^2(D)$

Buniakovski-Cauchy-Schwarz

proof: $(\Sigma |x_i + y_i|^2)^{1/2} \leq (\Sigma |x_i|^2)^{1/2} + (\Sigma |y_i|^2)^{1/2}$ for $x_i, y_i \in \mathbb{C}$ and approximation by Riemann sums.

Claim $L^2(D)$ is a normal space, that is a vectorspace and $\|f+g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$,
$\|\lambda f\|_{L^2} = |\lambda| \, \|f\|_{L^2}$ for $\lambda \in \mathbb{C}$.

$L^2(D)$ is also a Banach space (normed space), in fact it is a Hilbert space, meaning we have an inner product $\langle f, g \rangle := \iint_D f(x+iy)\, g(x+iy) \, dx\,dy$ such that $\langle f, f \rangle = \|f\|^2$.

$L^2(D, \mathcal{O}) = L^2(D) \cap \mathcal{O}(D)$

If we define $f: D \to \mathbb{C}$ continuous, and $D' \subset\subset D$ ($\overline{D'}$ is compact in D), then

$\|f\|_{D'} = \max_{z \in D'} |f(z)|$ $\quad$ (L^∞-norm)

A good thing about $L^2(D, \mathcal{O})$ is that $\|\cdot\|_{L^2(D')}$ and $\|\cdot\|_{L^\infty(D')}$ are comparable for $D' \subset\subset D$.

Theorem 14.2 $D_r = \{z \in \mathbb{C} : B(z,r) \subseteq D\}$ = points $z \in D$ with distance at least r to ∂D.

Then $\|f\|_{D_r} \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)}$

$\|f\|_{L^2(D_r)} \leq \mathrm{Vol}(D_r) \|f\|_{D_r}$

Proof If $z_0 \in D_r$ then $B(z_0, r) \subseteq D$ and by Cauchy's formula

$$\cancel{f(z_0)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \frac{|f(z_0) + se^{i\theta}|}{|se^{i\theta}|} d\theta$$

easier using Taylor extension $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$

$$\Rightarrow \left(\int_D |f(z)|^2\right)^{1/2} = \left(\int \sum_{n=0}^{\infty} c_n (z - z_0)^n\right) \left(\sum_{n=0}^{\infty} \bar{c}_n \overline{(z - z_0)^n}\right)^{1/2}$$

Use polar coordinates $z - z_0 = se^{i\theta}$ $\int_0^{2\pi} e^{ni\theta} \cdot e^{-mi\theta} d\theta = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$

$$\Rightarrow \|f\|^2_{L^2(B(z_0, z))} = \sum_{n=0}^{\infty} \pi r^{2n+2} |c_n|^2 \underset{n=0}{\geq} \pi r^2 |c_0|^2 = \pi r^2 \|f(z_0)\|^2$$

Lemma 14.3 $D' \subset\subset D \Rightarrow \forall \varepsilon > 0 \quad \exists A_\varepsilon \subseteq L^2(D, \mathcal{O})$ closed vector space s.t.

- $\mathrm{Dim}\,(L^2(D, \mathcal{O})/A_\varepsilon) < \infty$
- $\|f\|_{L^2(D')} \leq \varepsilon \|f\|_{L^2(D)} \quad \forall f \in A_\varepsilon$.

Proof $D^2 \subset D \Rightarrow D^2 \subset \bigcup_{i \in I} B(z_i, \delta/2)$ $|I| < \infty$ $\delta = \mathrm{dist}(D', \partial D)$

Let $A_\varepsilon = \{f \in L^2(D, \mathcal{O})$ s.t. f vanishes to order n_ε at all z_i, $i \in I\}$

We need to determine n_ε: Locally, near z_i:

$$\|f\|^2_{B(z_0, r)} \geq \sum_{n \geq n_\varepsilon}^{\infty} \frac{\pi r^{2n+2}}{n+1} |c_n|^2 \geq \frac{\pi r^{2n_\varepsilon + 2}}{n_\varepsilon + 1}.$$

Locally $f(z) = \sum_{n \geq n_\varepsilon + 1}^{\infty} (z - z_i)^n c_n$

$$\Rightarrow \|f\|_{L^2(B(z_i, \delta/2))} = \sum_{n \geq n_\varepsilon + 1}^{\infty} \left(\frac{\delta}{2}\right)^n |c_n|^2$$

$$\|f\|_{L^2(D)} \geq \underset{\substack{\uparrow \\ \text{Fourier analysis}}}{\|f\|_{L^2(B(z_i, \delta))}} = \sum_{n \geq n_\varepsilon + 1}^{\infty} \delta^n |c_n|^2 \geq \sum_{n \geq n_\varepsilon + 1}^{\infty} 2^{n+1} \left(\frac{\delta}{2}\right)^n |c_n|^2 \geq \|f\|_{L^2(B(z_i, \delta/2)}$$

Choose n_ε s.t.

$$\frac{\# I_\varepsilon}{2^{n_\varepsilon + 1}} \leq \varepsilon$$

no. I

$L^2(D, \mathcal{O})/A_\varepsilon = \bigoplus_{z \in I}$ (polynomials of degree up to n_ε) $\# I < \infty$

Good thing about $L^2(D, \mathcal{O})$ compared to $L^\infty(D, \mathcal{O})$ and $\mathcal{O}(D)$:

- $\mathcal{O}(D)$ to big to work with
- $L^\infty(D, \mathcal{O})$ good, but may be too small
- $L^2(D, \mathcal{O})$ is in between, and in particular a Hilbert space.

Čech cohomology but with L^2- cochains:
when defining $H^1(X, \mathcal{O})$ we used $\mathcal{O}$- cochains. Now we will use $L^2(\ , \mathcal{O})$ - cochains which are better and will relate to the 2 Čech cohomology (equal if X compact).

Define like before, but replace $\mathcal{O}(U_i)$ by $L^2(U_i, \mathcal{O})$ for an open covering $\mathcal{U} = (U_i)_{i\in I}$.

Only need to take care when defining the L^2-norm for this in $C^i(L^2(U_i, \mathcal{O}))$.

ex $f \in C^0(L^2(\mathcal{U}, \mathcal{O})) = \{f = (f_i) : f_i \in L^2(U_i, \mathcal{O})\}$

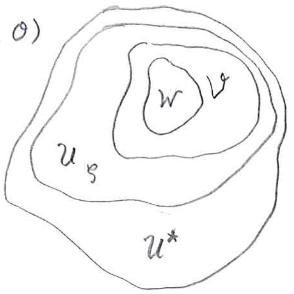
$$\Rightarrow \|f\|_{L^2(\mathcal{U})} = \sum_{i\in I} \|f_i\|_{L^2(U_i)}$$

Recall $\delta : C^0 \to C^1$

$$f = (f_i) \mapsto g = (g_{ij}) \qquad g_{ij} = (f_i - f_j)\big|_{U_i \cap U_j}$$

Lemma 14.6 (giving some finiteness) Assume $\mathcal{U} = (U_i)_{i\in I}$ where $\# I < \infty$ (can get this if X is compact)

Then for each covering $\mathcal{W} < \mathcal{V} < \mathcal{U} < \mathcal{U}^*$ so that each $\xi \in Z^1_{L^2}(\mathcal{V}, \mathcal{O})$
better covers



$U_i \subset\subset U_j^*$

has $\zeta \in Z^1_{L^2}(\mathcal{U}, \mathcal{O})$ and $\eta \in C^0_{L^2}(\mathcal{W}, \mathcal{O})$

s.t. $\max\{\|\zeta\|^2_{L^2(\mathcal{U})}, \|\eta\|_{L^2(\mathcal{W})}\} \leq C\|\xi\|_{L^2(\mathcal{V})}$

and $\underset{\substack{\uparrow \\ \text{boundary}}}{\zeta = \xi + \delta\eta} \Rightarrow [\zeta] = [\xi] \in H^1(X, \mathcal{O})$

MAT4800 §14 04.11.19

Lemma 14.7 $\exists$ finite dimensional vector subspace $S \subseteq Z^1(\mathcal{U}, \mathcal{O})$ s.t.

$\forall\ \xi \in Z^1(\mathcal{U}, \mathcal{O})$, $\exists\ \sigma \in S$ and $\eta \in C^0(\mathcal{W}, \mathcal{O})$: $\sigma = \xi + \delta_n$.

Proof Choose $\varepsilon = \frac{1}{2c}$ (s.t. $\varepsilon \cdot C < 1$ where C is the one in the previous proof

$\zeta_0 = \xi + \delta_{\eta_0}$ $\|\zeta_0\|_{L^2}, \|\eta_0\|_{L^2} < C\|\xi\|_{L^2(\mathcal{V})}$. Then by Lemma 14.3, we can write

$\swarrow$ induction Choose $A \subseteq Z^1(L^2(\mathcal{U}, \mathcal{O}))$ s.t. $S_\varepsilon = Z^1(L^2(\mathcal{U}, \mathcal{V}))/A_\varepsilon$ finite dim.

and if $f \in A_\varepsilon \Rightarrow \|f\|_{L^2(\mathcal{V})} \le \|\varepsilon\|_{L^2(\mathcal{U})}$

Write $\zeta_0 = \underset{A_\varepsilon}{\underset{\cap}{\xi_0}} + \underset{S_\varepsilon}{\underset{\cap}{\sigma_0}}$ orthogonal decomposition $\nearrow$ why L^2 good.

If H is a Hilbert space and $H = A \oplus S$ where

$S = \{f \in H : \langle f, x\rangle = 0 \quad \forall x \in V\}$

then $H = A \oplus S$ is an orthogonal decomposition,

$\|\xi_1\|^2_{L^2(\mathcal{U})} \le \|\zeta_0\|^2_{L^2} - \|\sigma_0\|^2_{L^2} \Longrightarrow \|\xi_0\|^2_{L^2(\mathcal{U})} \le \|\zeta_0\|^2_{L^2(\mathcal{U})}$ for orthogonal decomposition.

by 14.6, also. $\|\xi_0\|_{L^2(\mathcal{V})} \le \varepsilon\|\xi_0\|_{L^2(\mathcal{U})} \le \varepsilon\|\zeta_0\|^2_{L^2(\mathcal{U})}$

By induction and 14.6, we can write

$\zeta_v = \xi_{v-1} + \delta\eta_v$ 14.6 (ζ_v and ξ_{v-1} represent same cohomology) $\Rightarrow \|\zeta_v\| \le c\|\xi_{v-1}\|$

$\zeta_v = \underset{\in A_\varepsilon}{\xi_v} + \underset{\in S_\varepsilon}{\sigma_v}$ orthogonal decomposition $\Rightarrow \|\zeta_v\| \ge \|\xi_v\|$

and $\|\zeta_v\|_{L^2(\mathcal{U})} \le 2^{-v} c\|\xi\|_{L^2(\mathcal{V})}$

$$\left.\begin{aligned} \zeta_0 &= \xi + \delta\eta_0 \\ \zeta_0 &= \xi_0 + \sigma_0 \\ \zeta_1 &= \xi_0 + \delta\eta_1 \\ \zeta_1 &= \xi_1 + \sigma_1 \\ &\vdots \end{aligned}\right\} \Longrightarrow \begin{aligned} H^1(X, \mathcal{O}) \ni [\xi] &= [\zeta_0] = [\xi_0 + \sigma_0] \\ &= [\zeta_1 + \sigma_0] \\ &= [\xi_1 + \sigma_1 + \sigma_0] \\ &= [\zeta_2 + \sigma_1 + \sigma_0] \\ &= [\xi_2 + \sigma_2 + \sigma_1 + \sigma_0] \\ &\vdots \end{aligned}$$

$[\xi] = [\xi_n + \sigma_n + \cdots + \sigma_0] \Longrightarrow [\xi] = [\underset{n \ge n}{\Sigma}\sigma_n]$ $\forall n$ $\in_S$

$\|\xi_n\| \le \|\zeta_n\| \le 2^{-n} CM \longrightarrow 0$ $\qquad \|\sigma_n\| \le \|\xi_n\| \le 2^{-n} CM$

so $\lim\limits_{n \to \infty} \xi_n = 0$ and $\sum\limits_{n \ge 0} \sigma_n$ converges $\qquad \sigma_i \in S\ \forall i \Rightarrow \Sigma\sigma_n \in S$ = finite

$\uparrow$ because L^2 Hilbert space.

MAT4800 §14 Finiteness Theorem 07.11.

Pull back on cohomology

$f: Y \to X$ f holomorphic

$\Rightarrow f^*: H^i(\underset{\mathcal{O} \text{ on } X}{X, \mathcal{O}_X}) \longrightarrow H^i(\underset{\mathcal{O} \text{ on } Y}{Y, \mathcal{O}_Y})$ in general $f^*: H^i(X, \mathcal{F}) \to H^i(Y, f^*\mathcal{F})$

$\underset{\text{open}}{X \supseteq U} \hookrightarrow V$ can define for open $W \subseteq Y$, $f^*\mathcal{F}(W) = \lim\limits_{U,\, f(W) \subseteq U} \mathcal{F}(U)$

$\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right] & \underset{\text{open}}{U} \arrow[r, "\overset{f}{\text{holomorphic}}"] & \mathbb{C} \end{tikzcd}$$

$H^1(X, \mathcal{O}) = \{\omega \in \Omega^{1,0}\}$ $\Omega^{1,0}_X \longrightarrow \Omega^{1,0}_Y$

$\omega \longmapsto f(\omega)$

$f: \mathbb{C}_z \to \mathbb{C}_w$

$z \mapsto z^k$

$\alpha = (w^3 + e^w)\,dw$

$f^*(\alpha) = ((z^k)^3 + e^{z^k})\, \underbrace{d(z^k)}_{\overset{\shortparallel}{kz^{k-1}dz}}$

$\underset{\text{open map}}{Y \hookrightarrow X}$

if $\mathcal{U} = (U_i)_{i \in I}$ is an open covering of X, then $\underset{\overset{\shortparallel}{\mathcal{U} \cap Y}}{\hat{\mathcal{U}}} := (U_i \cap Y)_{i \in I}$ is an open covering of Y.

$f \in C^i(\mathcal{U}, \mathcal{O}_X) \xmapsto{\text{restict to } \mathcal{U} \cap Y} C^i(\mathcal{U} \cap Y, \mathcal{O}_Y).$

cocycle $\Rightarrow$ cocycle

$\Rightarrow$ (pull back) restriction map

$H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y)$

Theorem 14.9 X R.S. $\overset{\text{compact subset}}{Y_1 \subset\subset} \underset{\text{open}}{Y_2} \subset X \Rightarrow H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$ has a finite dimensional image.

Proof Recall Lemma 14.7: If $\mathcal{W} \ll \mathcal{U}$, then $H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{W}, \mathcal{O})$ has finite dim. image.

Choose $\mathcal{U}$ to be a Leray covering of Y_2 and $\mathcal{W}$ a Leray covering of Y_1.

Genus of X $g = \dim_{\mathbb{C}} H^1(X, \mathcal{O})$

ex $g(\mathbb{P}^1) = 0$ smooth curve in $\mathbb{P}^2$ of deg 4

$g(E) = 1$ $g = \frac{(4-1)(4-2)}{2} = 3$

elliptic curve

genus $\updownarrow$ no. holes.

14.12
Theorem (Existence of meromorphic function with poles) precise poles $\overset{\wedge}{\Rightarrow}$ Riemann Roch

X RS. $Y \subset\subset X$. Then $\forall a \in Y$ $\exists f \in \mathcal{O}(Y \setminus a)$ with pole at a,
open

and the multiplicity is $\leq 1 + \dim(H^1(X,\mathcal{O}) \to H^1(Y,\mathcal{O}))$ ← used to construct maps $Y \to \mathbb{P}^1$ with a pole only at a.

Proof $U_1 \subseteq Y$ an open coordinate neighborhood of a.

$z(a) = 0$

$U_2 = X \setminus \{a\}$ $\quad \mathcal{U} = \{U_1, U_2\}$ is an open covering of X.

Then by Theorem 14.9:

$$\dim(\underbrace{H^1(X,\mathcal{O}) \to H^1(Y,\mathcal{O})}_{\text{the map}}) = k < \infty$$

Look at functions

$$C^1(\mathcal{U},\mathcal{O}) \ni \zeta_j : U_1 \setminus a \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{z^j}$$

$U_1 \setminus a = U_1 \cap U_2$.

$$\dim(H^1(\mathcal{U},\mathcal{O}_X) \longrightarrow H^1(\mathcal{U},\mathcal{O}_Y)) \leq k.$$

Lemma 12.4:

$$\begin{tikzcd} H^1(\mathcal{U},\mathcal{O}_X) \arrow[r] \arrow[d, hook, "\text{injective}"] & H^1(\mathcal{U},\mathcal{O}_Y) \arrow[d, hook] \\ H^1(X,\mathcal{O}) & H^1(Y,\mathcal{O}) \end{tikzcd}$$

$$H^1(\mathcal{U},\mathcal{O}_X) \ni \overbrace{\zeta_1, \dots, \zeta_{k+1}}^{\text{might be lin. indep}} \longrightarrow \zeta_1|_{\mathcal{U}\cap Y}, \dots, \zeta_{k+1}|_{\mathcal{U}\cap Y} \quad \text{not linearly indep.}$$

$\Rightarrow \exists\, c_1, \dots, c_{k+1} \in \mathbb{C}$ not all 0 s.t. $c_1\zeta_1 + \dots + c_{k+1}\zeta_{k+1} = 0$ in $H^1(\mathcal{U}\cap Y, \mathcal{O})$

$\mathcal{U} \cap Y = (U_1 \cap Y, U_2 \cap Y) = (U_1, Y \setminus a)$

$\Rightarrow c_1\zeta_1 + \dots + c_{k+1}\zeta_{k+1}$ is a coboundary in $H^1(\mathcal{U}\cap Y, \mathcal{O}_Y)$

$\Rightarrow \exists\; h_1 \in \mathcal{O}(U_1)$
$\quad f \in \mathcal{O}(U_2 \cap Y = Y \setminus a)$ s.t.

$$\underbrace{c_1\zeta_1 + \dots + c_{k+1}\zeta_{k+1}}_{\text{singularity of } f} = h_1 - f\,\big|_{U_1 \cap (Y \setminus a) = U_1 \setminus a}$$

$= $ singularity of $c_1\zeta_1 + \dots + c_{k+1}\zeta_{k+1}$

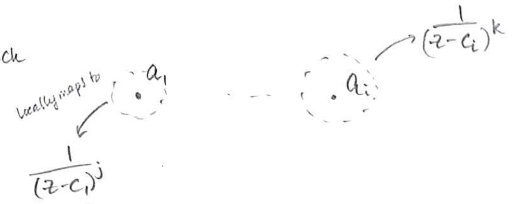
MAT4800 07.11.

14.13
Corollary / Existence of maps with precise interpolation

X compact RS, distinct $a_1, \dots, a_n \in X$, $c_1, \dots, c_n \in \mathbb{C}$, then $\exists f \in \mathcal{M}(X)$ ($f: X \to \mathbb{P}^1$ holomorphic)

s.t. $f(a_i) = c_i \quad \forall i = 1, \dots, n$.

Trick



cohomologies are global, but only need to define locally.

$$\overset{\frac{1}{(z-c_1)^j}}{\parallel} \quad f_1(a_1) = \infty, \qquad \frac{1}{f_1} + c_1 \quad \text{then } a_1 \mapsto c_1$$

14.16
Corollary (Relation between cohomology and differential anti-holomorphic 1-forms)

X non-compact RS and $Y \subset\subset Y' \subset X$ open subsets. Then $\forall$ differential forms

$\omega \in \mathcal{E}^{0,1}(Y')$ (locally $\omega = h d\bar{z}$), $\exists f \in \mathcal{E}(Y)$ s.t. $\omega|_Y = d''f = \bar{\partial} f$.

Proof Choose $\mathcal{U} = (U_i)_{i \in I}$ open covering of Y, s.t. U_i are simply-connected.

So for each U_i $\exists f_i \in \mathcal{E}(U_i)$: $\bar{\partial} f_i = \omega|_{f_i}$

Then $g_{ij} = (f_i - f_j)|_{U_i \cap U_j} \Rightarrow \bar{\partial} g_{ij} = \bar{\partial} f_i|_{U_i \cap U_j} - \bar{\partial} f_j|_{U_i \cap U_j} = 0$

$\Rightarrow g_{ij} \in C^1(\mathcal{U}, \mathcal{O}_Y)$.

Theorem 14.15: X non-compact RS, $Y \subset\subset Y' \subset X$ open subsets. Then
$\operatorname{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0$

$\Downarrow$

$$(g_{ij}|_{\mathcal{U} \cap Y}) = 0 \quad \text{in} \quad \begin{array}{c} H^1(\mathcal{U} \cap Y, \mathcal{O}_Y) \\ \uparrow \\ H^1(\mathcal{U}, \mathcal{O}_{Y'}) \end{array}$$

$\Rightarrow \exists h_i \in \mathcal{O}(U_i \cap Y)$ s.t. $h_i - h_j = g_{ij} = f_i - f_j \Rightarrow f_i - f_j = h_i - h_j$ on $U_i \cap U_j$

and $\bar{\partial}(f_i - h_i) = \bar{\partial} f_i - \underset{\substack{\parallel \\ 0 \text{ since } h_i \text{ holomorphic}}}{\bar{\partial} h_i} = \omega|_{U_i}$

$\underline{f = f_i - h_i|_{U_i} \quad \text{and} \quad \bar{\partial} f = \omega}$.

ex Elliptic curve $E \overset{\text{thm}}{\Longrightarrow} \exists f : E \to \mathbb{P}^1$ holomorphic s.t. $f^{-1}(\infty)$ = a point of multiplicity at most $1 + \dim \underbrace{H^1(E, \mathcal{O})}_{\overset{\|}{g} = 1}$

Observation 1: The pole must be of multiplicity 2:

Proof: The $\underset{\substack{\| \\ \deg(f)}}{\text{multiplicity}}$ of the pole is 1 or 2

If mult = 1 $\Rightarrow \deg(f) = 1 \Rightarrow f$ is an isomorphism, contradiction because $E \not\cong \mathbb{P}^1$

$\Rightarrow$ mult = 2.

$$\mathbb{C}/\Gamma = E \hookrightarrow \mathbb{P}^2$$

$$z \longmapsto [\wp_\Gamma(z) : \wp_\Gamma'(z) : 1]$$

$$\Gamma \ni z \longmapsto [\infty^2 : \infty^3 : 1] = \left[\frac{1}{\infty} : 1 : \frac{1}{\infty^3}\right] = [0 : 1 : 0]$$

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{w \in \Gamma} \left(\frac{1}{(z-w)^2} - \frac{1}{w^3} \right)$$

has a pole of order 2 at 0

$$\wp_\Gamma'(z) = -\frac{1}{z^3} + \sum_{w \in \Gamma \setminus 0} \frac{w}{(z-w)^3}$$

has a pole of order 3 at 0

$$\{[x:y:z] \in \mathbb{P}^2 : y^2 z = x^3 + axz^2 + bz^3\}$$

$$[x:y:z] \downarrow [x:z]$$

$E = \{y^2 z = x^3 + z^3\}$

if $z \neq 0 \Rightarrow [x:z] \in \mathbb{C} \subseteq \mathbb{P}^1$, $[x:z] = [x' : 1]$

$z = 0$ $\left(\frac{x}{z}\right) \Rightarrow [x:y:z] = [0:1:0] \downarrow [1:0]$

So $f^{-1}(\underset{\substack{\| \\ \infty}}{[1:0]}) = [0:1:0]$, and it has multiplicity $= \deg(f) = 2$

X RS, so only interested in $k=0,1$.

ex A way to construct differential forms on a RS.

$$\begin{tikzcd} \tilde{X} = \text{universal cover} \arrow[d, "\pi"'] \\ X = RS \end{tikzcd}$$

If $\omega \in \Omega^k(X) \xRightarrow{\text{pull back}} \tilde{\omega} = \pi^*(\omega) \in \Omega^k(\tilde{X})$,

which is invariant by $\sigma \in \mathrm{Deck}(\tilde{X}/X)$; proof:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & \circlearrowleft & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

$$\pi \circ \sigma = \pi$$

$$(\pi \circ \sigma)^*(\omega) = \pi^*(\omega)$$

$$\shortparallel$$

$$\sigma^* \circ \underbrace{\pi^*(\omega)}_{\tilde{\omega}} = \underbrace{\pi^*(\omega)}_{\tilde{\omega}}$$

$$\Rightarrow \sigma^* \tilde{\omega} = \tilde{\omega} \quad \forall \tilde{\omega} \in \Omega^k(\tilde{X}).$$

Theorem (converse)

$$\begin{tikzcd} \hat{X} = \text{universal cover} \arrow[d, "\pi"'] \\ X = RS \end{tikzcd}$$

Let $\tilde{\omega} \in \Omega^k(\hat{X})$. Then TFAE:

① $\sigma^* \tilde{\omega} = \hat{\omega} \quad \forall \sigma \in \mathrm{Deck}(\tilde{X}/X)$

② $\exists\, \omega \in \Omega^k(X)$ s.t. $\tilde{\omega} = \pi^*(\omega)$

Proof $2 \Rightarrow 1$ above. $1 \Rightarrow 2$:

For each $x \in X$. Choose $X \supseteq \underset{\text{open}}{U} \ni x$ s.t. $\pi^{-1}(U) = \bigsqcup_{i \in I} U_i$ $\quad U_i \overset{\pi}{\simeq} U$

Then if $\sigma \in \mathrm{Deck}(\tilde{X}/X)$, $\sigma(U_i) = U_j$ for some j.

Define $\omega|_{\underset{\uparrow}{U}} := \pi^*(\tilde{\omega}|_{U_i})$

can choose 1, from many → which? Show that $\pi_*(\tilde{\omega}|_{U_i}) = \pi_*(\tilde{\omega}|_{U_j}) \ \forall i,j$.

$\exists\, \sigma \in \mathrm{Deck}(\tilde{X}/X)$

$$\begin{tikzcd} U_i \arrow[rr, "\sigma"] & & U_j \arrow[dl, "\pi_j"] \\ & U \arrow[ul, "\pi_i^{-1}"] & \end{tikzcd}$$

$$\sigma \circ \pi_i^{-1} = \pi_j^{-1}$$

$$\shortparallel$$

$$(\sigma \circ \pi_i^{-1})^*(\tilde{\omega}|_{U_i} = (\pi_j^{-1})^*(\hat{\omega}|_{U_j})$$

$$\shortparallel$$

$$(\pi_i^{-1})^* \underbrace{\sigma^*(\tilde{\omega}|_{U_i}}_{\substack{\shortparallel \\ \tilde{\omega}|_{U_j}}}$$

07.11.

ex
$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"'] \\ \mathbb{C}/\Gamma = E \end{tikzcd}$$

$\Gamma = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$

$\mathrm{Deck}(\mathbb{C}/E) = \{ z \mapsto z+b,\ b \in \Gamma \}$

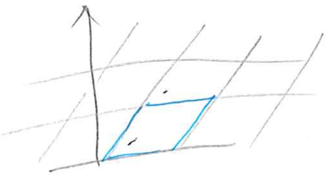
What are all holomorphic 1-forms on E?

They are the holomorphic 1-forms for $\mathbb{C}$ invariant by $\mathrm{Deck}(\mathbb{C}/E)$.

$\tilde{\omega} = h$ 1-form on $\mathbb{C}$ $\Rightarrow$ $\tilde{\omega} = f(z)\,dz$ $f: \mathbb{C} \to \mathbb{C}$ holomorphic.

$$\sigma: \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z+b \end{array}$$

$\Rightarrow \sigma^* \tilde{\omega} = f(z+b)\,dz = f(z)\,dz \Rightarrow f(z) = f(a+b)\ \forall z \in \Gamma.$



$\Rightarrow$ f has bounded image (= image of any parallelogram and hence a constant.

$H^1(X, \Omega) \cong$ the space of holomorphic 1-forms on X.

11.11.19

§13 The exact Cohomology Sequence (important tool for calculating cohomology groups)

X topological space. $\mathcal{F}, \mathcal{G}$ two sheaves of abelian groups over X.
A morphism from $\mathcal{F}$ to $\mathcal{G}$ is a collection of (abelian) group homomorphisms

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open $U \subseteq X$ so that we have commutative diagrams for each embedding $V \hookrightarrow U$:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "h_U"] \arrow[d, "\text{restriction}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d, "\text{restriction}"] \\ \mathcal{F}(V) \arrow[r, "h_V"'] & \mathcal{G}(V) \end{tikzcd}$$

Lemma (Homomorphisms between sheaves induce homomorphisms between stalks.)

$h: \mathcal{F} \to \mathcal{G}$ homomorphism of sheaves. Let $x \in X$ and $f_u \in \mathcal{F}(U)$, $f_v \in \mathcal{F}(V)$ s.t. $V \hookrightarrow U$

$$\begin{tikzcd} f_u \in \mathcal{F}(U) \arrow[d] \arrow[d, shift left=4ex, "\text{restriction}"] \\ f_v \in \mathcal{F}(V) \end{tikzcd}$$

(i.e. f_u and f_v determine the same element in the stalk $\mathcal{F}_x$.) Then

$$\begin{tikzcd} \mathcal{F}(U) \ni f_u \arrow[r, "h_u"] \arrow[d, "\text{restr.}"'] \arrow[dr, phantom, "\circlearrowleft"] & h_U(f_u) \in \mathcal{G}(U) \arrow[d, "\text{restr.}"] \\ \mathcal{F}(V) \ni f_v \arrow[dr, "h_v"'] & \text{restriction}(h_U(f_u)) \arrow[d, equal] \\ & h_v(f_v) \in \mathcal{G}(V) \end{tikzcd}$$

So $h_u(f_u)$ and $h_v(f_u)$ represent the same point in $\mathcal{G}_x$. So we have a map $h_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks which is a homomorphism of abelian groups $\mathcal{F}_x$ and $\mathcal{G}_x$.

Def. $h: \mathcal{F} \to \mathcal{G}$ homomorphism of sheaves of abelian groups. Then

① h is a monomorphism (injective) if $\forall x \in X$ $h_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.

② h is an epimorphisms (surjective) if $\forall x \in X$ $h_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective

(Category Theory) an abelian group is the same as a sheaf of abelian groups over a point.

so if sth. not true for abelian groups, it can't be true for sheaves.

Def $h: \mathcal{F} \to \mathcal{G}$ a homomorphism of sheaves. $\ker f = f^{-1}(0)$

① $\mathrm{Ker}(h)(U) = \mathrm{Kernel}(h_u : \mathcal{F}(U) \to \mathcal{G}(U))$ for open $U \subseteq X$.

Can check that $U \to \ker(h)(U)$ gives a sheaf over X, called $\ker(h)$. Kernel

② Image: $U \subseteq X$ open $\Rightarrow$ define $\mathrm{Im}(h)(U) = \mathrm{Image}(h_U : \mathcal{F}(U) \to \mathcal{G}(U))$.

Can check that $U \to \mathrm{Im}(h)(U)$ is a presheaf. (doesn't generally satisfy axiom II for sheaves).

Define $\mathrm{Im}(h)$ the sheaf associated to the presheaf $U \to \mathrm{Im}(h)(U)$

MAT4800 11.11.

Lemma $h: \mathcal{F} \to \mathcal{G}$ is a monomorphism $\iff$ $\forall$ open $U \subseteq X$, $h_U : \mathcal{F}(U) \to \mathcal{G}(U)$ injective.

$\iff$ $\forall x \in X$: $h_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.

Remark a) if wts h is monomorphism instead of checking with stalks, we can work with local section $\mathcal{F}(U) \to \mathcal{G}(U)$ which are easier to compute.

b) There is no analogue for epimorphisms, so it's easier to check monomorphism.

Def · Given morphisms $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ between sheaves. This is exact if $\forall x \in X$, the maps on stalks

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

are exact; i.e

$$\text{Image}(\varphi_x) = \text{Kernel}(\psi_x)$$

· In general, given $\mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_3 \xrightarrow{\varphi_3} \cdots$

then this is exact if $\forall n$ $\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{F}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{F}_{n+2}$ is exact.

· Special cases

1) $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ φ injective

2) $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$ ψ surjective.

3) $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ short exact sequence (SES).

15.8 Lemma $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact $\iff$ $\forall$ open $U \subseteq X$. $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is an exact sequence of abelian groups.

Proof $0 \to \mathcal{F} \to \mathcal{G}$ $\mathcal{F} \to \mathcal{G}$ is a monomorphism, so by previous Lemma $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective $\forall U$.

$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ Choose U so that $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U) \to \mathcal{H}(U)$. Choose $x \in U$, $\mathcal{G}(U) \ni g(x) \mapsto 0$ $\mathcal{G}(U) \to \mathcal{H}(U)$

15.11 (Connecting Homomorphism)

If

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence on sheaves on the topological space X.

Then $\exists$ a naturally constructed

$$\delta^* : \underset{\mathcal{H}(X)}{\underset{\|}{H^0(X, \mathcal{H})}} \longrightarrow H^1(X, \mathcal{F})$$

Proof Let $h \in H^0(X, \mathcal{H}) = \mathcal{H}(X)$. Then $\exists \overset{\mathcal{U}}{\overset{\|}{(U_i)}}_{i \in I}$ open covering of X so that $\exists g_i \in C^0(\mathcal{U}, \mathcal{G})$ and

$\beta(g_i) = h|_{U_i}$ $\forall i \in I$, because

$\forall x \in X$: $\mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \to 0$ is exact. So $\exists g_x \in \mathcal{G}_x$: $\beta_x(g_x) = h_x$.

$\exists x \in U_x$ open $\subseteq X$ with

g_x is defined by an element in $\mathcal{G}(U_x)$.

Choose $I = X'$ and $U_i = U_x$. Since h is defined globally, $h_x := h|_{U_x}$.

$\beta(g_i) = h|_{U_i} := \beta(g_i - g_j) = \beta(g_i) - \beta(g_j) = h|_{U_i} - h|_{U_j} = 0$ on $U_i \cap U_j$, so

$g_{ij} = g_i - g_j|_{U_i \cap U_j}$ is in the kernel of $\beta_{U_i \cap U_j} : \mathcal{G}(U_i \cap U_j) \to \mathcal{H}(U_i \cap U_j)$.

Apply Lemma 15.8, $\forall i, j$: $\exists (f_{ij}) \in C^0(\mathcal{U}, \mathcal{F})$ s.t. $\alpha(f_{i,j}) = g_{i,j} = g_i - g_j|_{U_i \cap U_j}$.

Then $\alpha(f_{i,j} + f_{j,k} + f_{k,i}) = (g_i - g_j) + (g_j - g_k) + (g_k - g_i) = 0$ on $U_i \cap U_j \cap U_k$

$$\mathcal{F}(U_i \cap U_j \cap U_k) \xrightarrow{\alpha} \mathcal{G}(U_i \cap U_j \cap U_k) \text{ is injective}$$

$\Rightarrow f_{i,j} + f_{j,k} + f_{k,i} = 0 \longrightarrow \underset{\mathcal{F}(U_i \cap U_j)}{\overset{\cap}{(f_{ij})}} \in C^1(\mathcal{U}, \mathcal{F})$ is a cocycle and hence define an element in $H^1(\mathcal{U}, \mathcal{F}) \hookrightarrow H^1(X, \mathcal{F})$.

Then we define

$$\delta^*(h) = (f_{i,j}) \in H^1(X, \mathcal{F})$$

Remark In general we have the connecting morphisms For $i = 0$ what we did!
For $i \geq 0$ more difficult but we can do similarly.

15.12 **Theorem** Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be a SES of sheaves over X.
Then we have an exact sequence (vlog exact sequence VLEG).

↑
one of the main cohomology theorems

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "\alpha"] & H^0(X,\mathcal{G}) \arrow[r, "\beta"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta^*"] \\
& H^1(X,\mathcal{F}) \arrow[r, "\alpha"] & H^1(X,\mathcal{G}) \arrow[r, "\beta"] & H^1(X,\mathcal{H}) \arrow[dll, "\delta^*"] \\
& H^2(X,\mathcal{F}) \arrow[r] & H^2(X,\mathcal{G}) \arrow[r] & H^2(X,\mathcal{H}) \arrow[dll, "\delta^*"] \\
& H^3(X,\mathcal{F}) & &
\end{tikzcd}$$

ex $X = RS$. $\mathcal{E}$ = sheaf of smooth functions on X. $\mathcal{E} \xrightarrow{d'' = \bar\partial} \mathcal{E}^{0,1}$ $\quad U \subseteq X$ open $\quad \mathcal{E}(U) \ni f \mapsto \bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z$

$\mathcal{E}^{0,1}$ = sheaf of (0,1)-forms on X = locally $f(z,\bar z)\, d\bar z$

So $\mathcal{O} = Ker(\mathcal{E} \to \mathcal{E}^{0,1})$.

⇓

$0 \to \mathcal{O} \hookrightarrow \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1}$ is an exact sequence

ex $\mathcal{E}^{1,0} \to \mathcal{E}^2$ = 2-forms on X

$\mathcal{E}(U) \ni f(z,\bar z)dz \xrightarrow{d = d' + d''} \frac{\partial f}{\partial \bar z} d\bar z \wedge dz$

so $Ker(d)(U) = \{ f\, dz \in \mathcal{E}(U) : \frac{\partial f}{\partial \bar z} = 0 \}$
$=: \Omega(U)$ = sheaf of holomorphic 1-forms.

$\Longrightarrow 0 \to \Omega \hookrightarrow \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^2$ is an exact sequence

Remark General construction:
If $\mathcal{G} \xrightarrow{\alpha} \mathcal{H}$ is a homomorphism of sheaves and $\mathcal{F} = Ker(\alpha)$,
then we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

by applying the theorem.

MAT4800

14.11.19

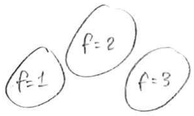
SES
Short exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

Long
implies LES of cohomology :

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{F}) \arrow[r, "\alpha"] & H^0(\mathcal{G}) \arrow[r, "\beta"] & H^0(\mathcal{H}) \arrow[dll, "\delta"'] \\ & H^1(\mathcal{F}) \arrow[r, "\alpha"] & H^1(\mathcal{G}) \arrow[r, "\beta"] & H^1(\mathcal{H}) \arrow[dlll, "\delta"'] \\ \cdots \end{tikzcd}$$

ex $\quad 0 \to \mathbb{C} \hookrightarrow \mathcal{O} \xrightarrow{d} \Omega \to 0$



$U \subseteq X$ R.S
open

$\mathbb{C}(U) = \{f : U \to \mathbb{C}, \text{ locally constant}\}$

$\mathcal{O}(U) = \{f : U \to \mathbb{C} \text{ holomorphic} = \text{smooth and } \bar\partial f = 0\}$

$\Omega(U) = \{\text{holomorphic 1-forms on } U\}$

If $f \in \mathcal{O}(U) \Rightarrow df = d'f + d''f = \frac{\partial f}{\partial z} dz + \underset{\overset{\|}{0}}{\frac{\partial f}{\partial \bar z}} dz$

① Check that $0 \to \mathbb{C} \to \mathcal{O} \to \Omega$ is exact, using Lemma 15.8 :

($0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact $\iff \forall x : 0 \to \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \to \mathcal{H}_x \to 0$ is exact,
meaning α_x injective, β_x surjective, $\operatorname{Im}(\alpha_x) = \ker(\beta_x)$)

For each open $U \subseteq X$ we check that $\mathbb{C}(U) \hookrightarrow \mathcal{O}(U)$ is injective :

$f \in \ker(\mathcal{O}(U) \xrightarrow{d} \Omega(U)) \iff df = 0 = \frac{\partial f}{\partial z} dz$

$\Rightarrow \frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \bar z} = 0$ because f is holomorphic

$\Rightarrow f$ is locally constant $\to f \in \mathbb{C}(U)$

② Check that

$\mathbb{C} \to \mathcal{O} \to \Omega \to 0$ is exact,

$\Updownarrow$

$\forall x \in X : \mathbb{C}_x \xrightarrow{i_x} \mathcal{O}_x \xrightarrow{d_x} \Omega_x \to 0$ is exact. So we need to show that $\operatorname{Im}(i_x) = \operatorname{Ker}(d_x)$ and d_x surjective :

Let $\omega_x \in \Omega_x \iff$ an open set U so that ω is representated by a holomorphic 1-form ω defined in U.

We can shrink U and assume that U is simply connected, $U = B(0,r) \subseteq \mathbb{C}$

ex cont.

Write $\omega = g\, dz$. Then g is holomorphic, $g(z) = \sum_{n \geq 0}^{\infty} c_n z^n$. Define

$$f(z) := \sum_{n\geq 0}^{\infty} \frac{c_n z^{n+1}}{n+1} \in \mathcal{O}(U) \quad \text{and} \quad df = \frac{\partial f}{\partial z} dz + \underset{0}{\underset{\shortparallel}{\frac{\partial f}{\partial \bar{z}}}} d\bar{z} = g(z)\, dz,$$

choose $f_x \in \mathcal{O}_x$ represented by f, then $d_x f_x = \omega_x$

and so d_x is surjective.

$\operatorname{Im}(i_x) = \operatorname{Ker}(d_x)$: showed when $0 \to \mathbb{C} \to \mathcal{O} \to \Omega$ is exact.

ex $0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\overset{\bar{\partial}}{\overset{\shortparallel}{d''}}} \mathcal{E}^{0,1} \to 0$ is SES

ex $\underset{\substack{\text{smooth 1-forms} \\ \text{which are } d\text{-closed}}}{\underset{\shortparallel}{\mathcal{Z}}} = \operatorname{Ker}(\underset{\substack{\text{smooth} \\ \text{1-forms}}}{\mathcal{E}^{(1)}} \xrightarrow{d} \underset{\substack{\text{smooth} \\ \text{2-forms}}}{\mathcal{E}^{(2)}})$, then $0 \to \mathbb{C} \to \mathcal{E} \to \mathcal{Z} \to 0$ is SES.

ex $0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$ is SES.

$$g\, d\bar{z} \longmapsto \frac{\partial g}{\partial \bar{z}} d\bar{z} \wedge dz$$

ex $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \underset{\{f \in \mathcal{O} \,:\, f \text{ nowhere } 0\}}{\underset{\shortparallel}{\mathcal{O}^*}} \to 0$ is SES.

$$g \mapsto e^g$$

use existence of logarithms on simply-connected sets.

used it in alg. geo. to compute Picard groups

ex Compute $H^1(X, \mathbb{C})$:

$0 \to \mathbb{C} \to \mathcal{O} \to \Omega \to 0$ is SES, so for X $\overset{\text{compact}}{\text{RS}}$,

$$\Rightarrow 0 \to \underset{\mathbb{C}}{\underset{\shortparallel}{H^0(X, \mathbb{C})}} \xrightarrow{\text{id}} \underset{\mathbb{C}}{\underset{\shortparallel}{H^0(X, \mathcal{O})}} \xrightarrow{0\text{-map}} \underset{0}{\underset{\shortparallel}{H^0(X, \Omega)}}$$

$$\hookrightarrow H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}) \to H^1(X, \Omega)$$

$$\hookrightarrow H^2(X, \mathbb{C} \to H^2(X, \mathcal{O}) \to H^2(X, \Omega)$$

$$\hookrightarrow \underset{0}{\underset{\shortparallel}{H^3(X, \mathbb{C})}}$$

↖ because X has real dimension 2

$\Rightarrow 0 \to H^0(X, \Omega) \underset{\text{injective}}{\hookrightarrow} H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$

$H^0(X, \Omega) \simeq \operatorname{Ker}(H^1(X, \mathbb{C}) \to H^1(X, \Omega))$

15.13 Dolbeault's thm: $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ ses

$\Rightarrow H^1(X, \mathcal{F}) = H^1(X, \mathcal{H}) / \operatorname{Im}(\gamma_\beta)$

15.13
Theorem $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ ses s.t. $H^1(X, \mathcal{G}) = 0$,

$\Longrightarrow H^1(X, \mathcal{F}) \cong \mathcal{H}(X) / \beta\mathcal{G}(X)$

Proof $0 \to H^0(\mathcal{F}) \xrightarrow{\alpha} H^0(\mathcal{G}) \xrightarrow{\beta} H^0(\mathcal{H}) \xrightarrow{\delta} H^1(\mathcal{F}) \to H^1(\mathcal{G}) = 0$

$\mathrm{Im}(\beta) = \mathrm{Ker}(\delta)$ and $H^0(\mathcal{H}) \to H^1(\mathcal{F})$ is surjective.

So $H^1(\mathcal{F}) \cong H^0(\mathcal{H}) / \mathrm{Ker}(H^0(\mathcal{H}) \xrightarrow{\delta} H^1(\mathcal{F}))$

15.14
Theorem (Dolbeault's) X RS

ⓐ $H^1(X, \mathcal{O}) \cong \mathcal{E}^{0,1}(X) / d''\mathcal{E}(X)$

ⓑ $H^1(X, \Omega) \cong \mathcal{E}^{(2)}(X) / d\mathcal{E}^{1,0}(X)$

Proof ⓐ use SES $0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$ and that $H^k(\mathcal{E}^{i,j}) = 0$
because of the existence of partition of unity for smooth functions

ⓑ use $0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$ and $H^1(\mathcal{E}^{1,0}) \overset{\uparrow}{=} 0$

15.15
Theorem (de Rham) X RS. Then

$$H^1(X, \mathbb{C}) \cong Rh^1(X) := \frac{\mathrm{Ker}(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X))}{\mathrm{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X))}$$

Proof $\mathcal{Z} = \mathrm{ker}(\mathcal{E}^{(1)} \to \mathcal{E}^{(2)}) \subseteq \mathcal{E}^{(1)}$ then $0 \to \mathbb{C} \to \mathcal{E} \to \mathcal{Z} \to 0$ $H^1(\mathcal{E}) = 0$

$$\Longrightarrow H^1(X, \mathbb{C}) = H^0(\mathcal{Z}) / \underbrace{\mathrm{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{Z}(X))}_{\overset{\shortparallel}{\mathrm{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X)}}$$

Claim $X = \mathbb{C}/\Gamma$ elliptic curve. then $H^1(X, \mathbb{C}) \cong \mathbb{C}\,dz \oplus \mathbb{C}\,d\bar{z} \simeq \mathbb{C}^2$

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Gamma \arrow[r] & \mathbb{C}/\Gamma \end{tikzcd}$$

Proof $\omega \in \mathrm{ker}(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X)) \longrightarrow \omega$ comes from a d-closed 1-form on $\mathbb{C}$: $\hat{\omega}$

Then $\hat{\omega}(z) = f(z,\bar{z})\,dz + g(z,\bar{z})\,dz$. $\hat{\omega}$ is d-closed $\Rightarrow d(\hat{\omega}) = 0$

$$\Rightarrow 0 = d(f\,dz + g\,d\bar{z}) = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial g}{\partial z} dz + d\bar{z} = \underbrace{\left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)}_{\overset{\shortparallel}{0}} dz \wedge d\bar{z}$$

$\Longrightarrow \frac{\partial g}{\partial z} = \frac{\partial f}{\partial \bar{z}}$. f, g invariant by $\Gamma \Rightarrow$ $f(z+\gamma) = f(z)$, $g(z+\gamma) = g(z)$ $\forall \gamma \in \Gamma$ Deck$(\mathbb{C}/\Gamma)$ = translations by Γ

$\mathrm{Im}(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X)$

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi\ \text{universal cover}"] \\ X \end{tikzcd}$$

$h \in \mathcal{E}(X) \Longrightarrow \hat{h} \in \mathcal{E}(\mathbb{C})$

$dh = \pi_*(d\hat{h}) = \pi_*\left(\frac{\partial h}{\partial z} dz + \frac{\partial h}{\partial \bar{z}} d\bar{z}\right)$

MAT4800 14.11.19

Proof cont. Claim: $\exists c_1, c_2 \in \mathbb{C}$ and $h \in \mathcal{E}(\mathbb{C})$ s.t. $\begin{cases} f = \frac{\partial h}{\partial z} + c_1 \\ g = \frac{\partial h}{\partial \bar{z}} + c_2 \end{cases}$

$$\Rightarrow f\,dz + g\,d\bar{z} \equiv c_1\,dz + c_2\,d\bar{z} \text{ in } H^1(X, \mathbb{C})$$

$\downarrow$

these lin. indep, just lift to $\mathbb{C}$.

ex $g = z + c_1$, $f = \bar{z}$ $\quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial z} = 1 \quad h = z\bar{z} \quad h = \int_a^z f\,d\xi$

§16 Riemann-Roch's theorem

16.1 **Divisors** X RS. A divisor D on X is a mapping $D: X \to \mathbb{Z}$, that is a formal finite sum

$$D = n_1 x_1 + \cdots + n_p x_p \quad \text{for some } p \in \mathbb{N},\ n_j \in \mathbb{Z},\ x_j \in X.$$

ex $X = \mathbb{P}^1$, then $D = [0:1]$ is a divisor as well as $D = [0:1] - 2[1:0] + 3[1:1]$

Remark Divisors give us an easy way to encode the zeros and poles of meromorphic functions and 1-forms.

Remark Another way to define a divisor is if $D = n_1 x_1 + \cdots n_p x_p$, define a map $D: X \to \mathbb{Z}$ by

$$D(x) = \begin{cases} n_j & \text{if } x = x_j \\ 0 & \text{otherwise} \end{cases}$$

ex $D = [0:1] - 2[1:0] + 3[1:1]$ on $\mathbb{P}^1$

$$\Rightarrow D(x) = \begin{cases} 1 & \text{if } x = [0:1] \\ -2 & \text{if } x = [1:0] \\ 3 & \text{if } x = [1:1] \\ 0 & \text{otherwise} \end{cases}$$

Remark The advantage of defining divisors as functions is that we can add 2 divisors and compare them.

We say $D_1 \leq D_2 \iff \forall x \in X: D_1(x) \leq D_2(x)$.

16.2 Divisors of Meromorphic functions

$$f: X \longrightarrow \mathbb{P}^1 \quad \text{meromorphic}$$

$a \in X$

$$\operatorname{ord}_a(f) = \begin{cases} 0 & \text{if } f \text{ holomorphic and nonzero at } a \\ k & \text{if } f \text{ has a zero of order } k \text{ at } a \\ -k & \text{— " — pole — " —} \\ \infty & \text{if } f \text{ is identically zero in a neighborhood of } a \end{cases}$$

ex $f: \mathbb{P}^1 \to \mathbb{P}^1$

$[x:y] \mapsto [x^3:y^3]$ $\quad 0 = [0:1] \in \mathbb{P}^1$ $\quad f^{-1}([0:1]) = [0:1]$ with multiplicity 3

$\infty = [1:0] \in \mathbb{P}^1$ $\quad f^{-1}([1:0]) = [1:0]$ — " —

$$\operatorname{ord}_a(f) = \begin{cases} 3 & \text{if } a = [0:1] \\ -3 & \text{if } a = [1:0] \\ 0 & \text{otherwise} \end{cases}$$

Thus for any meromorphic function $f \in \mathcal{M}(X) \setminus \{0\}$ $\quad x \mapsto \operatorname{ord}_x(f)$ is a divisor on X.

Divisors of a meromorphic 1-forms:

Let $a \in X$ $\rightarrow$ choose a nbh $U \ni x$ s.t. $U \cong B(0,1)$.

ω meromorphic 1-form on X, then in U

$$\underset{\substack{\uparrow \\ \text{local repr. of } \omega \text{ at } a}}{\omega = f(z)\,dz} \qquad f: U \to \mathbb{P}^1 \text{ meromorphic}$$

Define $\operatorname{ord}_a(\omega) = \operatorname{ord}_a(f)$

ex $\omega: \mathbb{C} \to \mathbb{C}$ $\qquad [x:y] \quad z = \frac{x}{y} \Rightarrow w = \frac{y}{x}$

$\omega(z) = dz$ holomorphic 1-form on $\mathbb{C}$

ω can be extended to a meromorphic 1-form on $\mathbb{P}^1$. Near the point ∞ we choose coordinate $w = \frac{1}{z}$ so $\infty \longleftrightarrow \{w=0\}$

$$z = \frac{1}{w} \qquad dz = d\left(\frac{1}{w}\right) = -\frac{dw}{w^2} = \underset{\substack{\shortparallel \\ -\frac{1}{w^2}}}{f(w)}\,dw$$

so at $w=0$ f has a pole of order 2

$$\Rightarrow \operatorname{ord}_a(\omega) = \begin{cases} 0 & \text{if } a \in \mathbb{C} \\ -2 & \text{if } a = \infty = \{w=0\} \end{cases}$$

$\operatorname{Div}(\omega) = -2[1:0]$.

Degree of Divisor:

If D is a divisor on $X = RS$, $D = \sum_{i\in I} n_i p_i$. $n_i \in \mathbb{Z}$, $p_i \in X$

then $\deg(D) = \sum_{i \in I} n_i$

ex X compact RS. then $\forall f: X \to \mathbb{P}^1$ holomorphic, nonconstant, we have only a finite number of poles and zeros (by the identity theorem). Then

$$\mathrm{Div}(f) = \underset{\substack{p_i \text{ a zero} \\ n_i \text{ multiplicity}}}{\sum n_i p_i} - \underset{\substack{q_j \text{ a pole} \\ m_j \text{ multiplicity}}}{\sum m_j q_j}$$

is well defined and

$$\mathrm{Deg}(\mathrm{Div}(f)) = \sum n_i - \sum m_j = 0$$

because.

$\sum n_i$ = no. $f^{-1}(\{0\})$ = $\deg(f)$

$\sum m_j$ = no. $f^{-1}(\{\infty\})$ = $\deg(f)$

However if ω is a meromorphic 1-form on X, $\deg(\omega)$ can be nonzero.

ex $X = \mathbb{P}^1$, ω = extension to ∞ of dz on $\mathbb{C}$.

$\omega = -\frac{1}{w^2} dw$ near $\infty = \{w = 0\}$

So $\mathrm{Div}(\omega) = -2(\infty)$ and $\deg(\mathrm{Div}(\omega)) = -2 \neq 0$

Def Principal divisor

$\omega \neq 0$ is a principal divisor if $\exists f \in \mathcal{M}(X)$ s.t. $\omega = \mathrm{Div}(f)$ $\Rightarrow \deg(\omega) = 0$.

Remark If ω_1, ω_2 are meromorphic 1-forms on X, then $\mathrm{Div}(\omega_1) - \mathrm{Div}(\omega_2)$ is principal.

Proof Locally, $\omega_1 = f_1 dz$, $\omega_2 = f_2 dz$ $\Longrightarrow$ $\mathrm{Div}(\omega_1) - \mathrm{Div}(\omega_2) = \mathrm{Div}(f_1) - \mathrm{Div}(f_2$
$= \mathrm{Div}(f_1/f_2)$,

and f_1 and f_2 can be glued together to a global $h \in \mathcal{M}(X)$.

$\rightsquigarrow$

The sheaf of meromorphic 1-forms on X is a lie bundle.

Def X RS. D a divisor on X. Then $\mathcal{O}_D$ is the sheaf on X, where for every $U \subseteq X$ open the local sections are

$$\underbrace{\mathcal{O}_D(U)} = \{ f \in \mathcal{M}(U) : \mathrm{Div}(f) \geq 0 \}$$

abelian group under usual addition of functions,

$(f+g)(z) = f(z) + g(z)$

best to

$\mathcal{O}_D$ are sheaves:

Assume z_0 is a pole of $-D$:

$$-D = -n(z_0) + \sum_i n_i z_i \qquad \text{for } n>0$$

$$f, g \in \mathcal{O}_D(U) \Rightarrow f(z) = \sum_{k=-n}^{\infty} c_k z^k$$

$$g(z) = \sum_{k=-n}^{\infty} d_k z^k$$

$$\Rightarrow \underset{\mathcal{O}_D(U)}{\underset{\cap}{f(z)+g(z)}} = \sum_{k=-n}^{\infty} (c_k + d_k) z^k$$

Remark If $D - D' = \mathrm{Div}(f)$, $\underset{\text{nonconstant}}{f \in \mathcal{M}(X)}$ then we say $\underset{\text{equivalent}}{D \sim D'}$

Then we define an isomorphism

$$\begin{tikzcd}
\mathcal{O}_D \arrow[r] & \mathcal{O}_{D'} \\
\mathcal{O}_{D'}(U) \ni h \arrow[r, mapsto] & hf \in \mathcal{O}_{D'}(U) \\
h/f & g \arrow[l, mapsto]
\end{tikzcd}$$

easy & important!)

Theorem 6.3 X compact RS, D divisor, $\deg(D)<0$. Then $H^0(X, \mathcal{O}_D) = 0$.

Proof $\Leftrightarrow$ If $f \in \mathcal{M}(X)$ s.t. $\mathrm{Div}(f) \geq D$, then $f = 0$.

Otherwise, since $\deg(-D) > 0$ $\Rightarrow$ f is nonconstant had has some zos.

$\Rightarrow P_N(f)$ cannot be $k \geq 0$.

RR

Want to compute $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O})$, which is very difficult to compute in general.

But there is a relation between $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$

In order to prove this we need to use induction, and at one step it is important to compute the cohomology groups of $\mathcal{O}_D$ and $\mathcal{O}_{D+p}$ where p is a point.

The difference between $\mathcal{O}_D$ and $\mathcal{O}_{D+p}$ is called a skyscraper sheaf λ

Def Let $p \in X$. The skyscraper sheaf $\mathbb{C}_p$ on X

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

$$U \hookrightarrow V \qquad \mathbb{C}_p(V) \to \mathbb{C}_p(U) = \begin{cases} \mathrm{id} & \text{if } p \in V \\ 0 & \text{if } p \notin V \end{cases}$$

Theorem i) $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$

ii) $H^1(X, \mathbb{C}_p) = 0$

Proof i) $H^0(X, \mathbb{C}_p) = \underset{p \in X}{\mathbb{C}_p(X)} = \mathbb{C}$.

ii) Choose any leray covering of $\mathbb{C}_p$ $\mathcal{U} = (U_i)_{i \in I}$ where only $p \in U_1$ and $p \notin U_j$ for $j \neq 1$.

$$C^1(\mathbb{C}_p(\mathcal{U})) = \prod_{i \neq j} \mathbb{C}_p(\underbrace{U_i \cap U_j}_{p \notin}) = 0$$

Idea $\mathbb{C}_p(U)$ is kind of quotient of $\mathcal{O}_{D+p}$ by $\mathcal{O}_D$ for any divisor D and any point $p \in X$.

In general, given two sheaves $\mathcal{F}$ and $\mathcal{G}$ and want to compare them, we may try to define a "quotient $\mathcal{F}/\mathcal{G}$".

Important techinique

Let D be a divisor on X, $p \in X$ a point, $U \subseteq X$ open subset.

$$\mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \mathrm{Div}(f) \geq -D \}$$

$$\mathcal{O}_{D+p}(U) = \{ f \in \mathcal{M}(U) : \mathrm{Div}(f) \geq -D-p \}$$

Since $-D \geq -D-p \Rightarrow \mathcal{O}_D(U) \hookrightarrow \mathcal{O}_{D+p}(U)$.

ex What is the quotient of

$$\mathcal{O}_D(U) \hookrightarrow \mathcal{O}_{D+p}(U)$$

If $p \notin U$ then

$$\mathcal{O}_D(U) = \mathcal{O}_{D+p}(U).$$

If $p \in U$ then

$$D = np + \sum_i n_i p_i$$

$$D + p = (n+1) + \sum_i n_i p_i$$

If $f \in \mathcal{O}_{D+p}(U)$, then locally near U we have Laurent series for f:
(Choose coordinates s.t. $p = 0$)

$$f(z) = \sum_{k=-n-1}^{\infty} c_k z^k = \underline{c_{-n-1} z^{-n-1}} + \underset{\in \mathcal{O}_D(U)}{\left(\sum_{k=-n}^{\infty} c_k z^k\right)}$$

So provided we know the coefficient $c_{-n-1} \in \mathbb{C} = \mathbb{C}_p(U)$
we know the difference between f and $\mathcal{O}_D(U)$

$U \subseteq X$,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_D(U) \arrow[r, hook] & \mathcal{O}_{D+p}(U) \arrow[r] & \mathbb{C}_p(U) \arrow[r] & 0 \\ & f \arrow[r, mapsto] & f \arrow[r, mapsto] & \begin{matrix} 0 \text{ if } p \notin U \\ c_{-n-1} \text{ if } p \in U \end{matrix} & \end{tikzcd}$$

So

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+p} \to \mathbb{C}_p \text{ is exact.}$$

To show that

$$\mathcal{O}_{D+p} \to \mathbb{C}_p \to 0$$ is exact, we can choose U to be isomorphic

to a disk in $\mathbb{C}$, so that f has no zero or pole in $U \setminus \{p\}$.

In summary, we have the short exact sequence (SES)

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+p} \to \mathbb{C}_p \to 0$$

and hence by §15, we have a LES

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}_D) \arrow[r] & H^0(\mathcal{O}_{D+p}) \arrow[r] & \overset{\overset{\mathbb{C}}{\cong}}{H^0(\mathbb{C}_p)} \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Z) [near end]\tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & H^1(\mathcal{O}_D) \arrow[r] & H^1(\mathcal{O}_{D+p}) \arrow[r] & \underset{\underset{0}{\parallel}}{H^1(\mathbb{C}_p)} \arrow[r] & \cdots \end{tikzcd}$$

Lemma Let

$$0 \to W_1 \to W_2 \to W_3 \to W_4 \to \cdots \to 0$$

be a LES of linear maps between vector spaces.

Then $\dim(W_1) + \dim(W_3) + \cdots = \dim(W_2) + \dim(W_4) + \cdots$

Applying this to the previous LES, we get

$$\dim_{\mathbb{C}} H^0(\mathcal{O}_D) + \underbrace{\dim_{\mathbb{C}} H^0(\mathbb{C}_p)}_{\overset{\shortparallel}{1}} + \dim_{\mathbb{C}} H^1(\mathcal{O}_{D+p})$$

$$= \dim_{\mathbb{C}}(H^0(\mathcal{O}_{D+p})) + \dim_{\mathbb{C}}(H^1(\mathcal{O}_D))$$

$$\Rightarrow \dim(H^0(\mathcal{O}_{D+p})) - \dim H^1(\mathcal{O}_{D+p}) = 1 + [\dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D)]$$

So we can use induction and we get

Theorem (Riemann Roch) If X compact RS., D is a divisor on X, then

$$\dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D) = 1 - g + \deg D$$

$$= [\dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O})] + \deg D$$

$g = \dim H^1(X, \mathcal{O}) = $ genus of X.

X compact RS $\underset{\text{maximum principle}}{\Longrightarrow} \mathcal{O}(X) \cong \mathbb{C} \Longrightarrow \dim H^0(\mathcal{O}) = 1.$

ex How to show this for $D = 2p_1 + p_2 - p_3$.

$\deg(D) = 2 + 1 - 1 = 2$ $\qquad D_1 = p_1 + p_2 - p_3 \Rightarrow D = D_1 + p_1$

$\dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D) = 1 + [\dim H^0(\mathcal{O}_{D_1}) - \dim H^1(\mathcal{O}_{D_1})]$

$D_2 = p_2 - p_3 \Rightarrow D_1 = D_2 + p_1$

$\dim(H(\mathcal{O}_{D_1}) - \dim(H^1(\mathcal{O}_{D_1}) = 1 + [\dim H^0(\mathcal{O}_{D_1}) - \dim H^1(\mathcal{O}_{D_2})]$

$D_3 = -p_3 \Rightarrow D_2 = D_3 + D_2$

$\dim(H^0(\mathcal{O}_{D_2}) - \dim(H(\mathcal{O}_{D_2}) = 1 + [\dim H^0(\mathcal{O}_{D_3}) - \dim H^1(\mathcal{O}_{D_3})]$

so can't do like before,

$\mathcal{O} = D_3 + p_3$ $\qquad \underset{\overset{\shortparallel}{1}}{\dim H^0(\mathcal{O})} - \underset{\overset{\shortparallel}{g}}{\dim H^1(\mathcal{O})} = 1 + (\dim H^0(\mathcal{O}_{D_3}) - \dim H^1(\mathcal{O}_{D_3}))$

RR is good because

1) deg D is easy to compute

2) g also easy to compute

ex If $X \overset{\substack{\text{smooth}\\ \downarrow}}{=} \{P(x,y,z)=0\} \subset \mathbb{P}^2$ then $g = \frac{(d-1)(d-2)}{2}$ where $d = \deg(P)$

or we can have a surjective holomorphic map $X \to Y$ and can use Hurwitz-Riemann thm to compute the genus of X when we know exact branch points of this map (with multiplicity) and genus of Y.

3) In some cases $H^0(\mathcal{O}_D)$ or $H^1(\mathcal{O}_D)$ may be easy to compute and hence we can decide the other.

ex If $\deg(D) < 0$, we know that $H^0(\mathcal{O}_D) = 0$

$\Rightarrow$ $\dim(H^1(\mathcal{O}_D)) = 1 - g + \deg(D)$

can be nonzero only if $1 - g - \deg(D) < 0 \iff \deg(D) < g-1$

Corollary If $X = \mathbb{P}^1$, D divisor on X, $\deg(D) < 0$ then $H^1(\mathcal{O}_D) = 0$.

Theorem X compact RS genus $= g$ $a \in X$. Then $\exists$ a non-constant meromorphic map $f: X \to \mathbb{P}^1$ which has a pole of order $\leq g+1$ at a and which is holomorphic on $X \setminus \{a\}$

Proof $D = (g+1)a$ then $\dim H^0(\mathcal{O}_D) - \dim H^1(\overset{0}{\overset{\shortparallel}{\mathcal{O}_D}}) = 1 - g + \overset{g+1}{\overset{\shortparallel}{\deg(D)}} = 2$

$\dim H^1(\mathcal{O}_D) \geq 0$ so $\dim H^0(\mathcal{O}_D) \geq 2$

$$\underset{\substack{\shortparallel\\ \text{constant functions}\\ \text{on } X}}{\overset{\dim = 1}{\mathbb{C}}} \hookrightarrow \underset{\substack{\shortparallel\\ \{f: X \to \mathbb{P}^1 \;:\; \operatorname{div}(f) \geq -(g+1)a\}}}{\overset{\dim \geq 2}{\mathcal{O}_D(X)}}$$

So $\exists$ $f \in \mathcal{O}_D(X) \setminus \mathbb{C}$ and $\operatorname{div}(f) \geq -(g+1)a$.

HA74800 §17

(just mention)

Theorem (Serre Duality)

$$\dim H^0(X, \mathcal{O}_{-D}) = \dim H^1(X, \Omega_D)$$

Here $\Omega =$ sheaf of holomorphic 1-forms.

ex For $X =$ elliptic curve $\Rightarrow \Omega \cong \mathcal{O}$, $g(X) = 1$. So we can apply this to exer. 16.2:

$p \in X$, $n \geq 0$

$$\dim H^0(\mathcal{O}_{np}) - \dim H^1(\mathcal{O}_{np}) = 1 - g + \deg(np) = \deg(np) = n$$

$$\Omega \cong \mathcal{O} \Rightarrow \mathcal{O}_{np} \cong \Omega_{np}$$

$$\dim H^1(\mathcal{O}_{np}) = \deg H^1(\Omega_{np})$$

$$\overset{SO}{=} \dim H^0(X, \mathcal{O}_{-np}) = 0$$

$\Rightarrow \dim H^0(\mathcal{O}_{np}) = n$ if $n \geq 0$, $p \in X$ X elliptic curve

HW: $X = \mathbb{P}^1 \Rightarrow \Omega \cong \mathcal{O}_{-2p}$ for every $p \in X$