MAT4800 Complex Analysis — Tuyen — tuyentt@math.uio.no — 21.08.2019

- Complex Analysis in 1 variable (summary) — Otto Forster: Lectures on Riemann surfaces
- Divisors
- Sheaves
- Cohomology

Goal: Riemann-Roch theorem $\rightsquigarrow$ upto section 16.

Review Complex Analysis in 1-variable.

Goal: Classification of singularities of analytic/holomorphic functions.

$B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ $\qquad f: B(a,r) \to \mathbb{C}$

Def. f is **holomorphic** if it can be written in the form $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ where the sum is absolute convergent in $|z-a| < s \quad \forall\, 0 < s < r$.

ex ① $f(z) = \sum_{n=0}^{m} c_n (z-a)^n$, $\quad m < \infty$ — a polynomial of degree m $\quad \Rightarrow c_n = 0 \ \forall n \geq m$

$f(z) = z = a + (z-a)$

Holomorphic on all $\{|z-a| < r\}$ (whole $\mathbb{C}$). We call these **entire functions**.

② $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$

③ $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

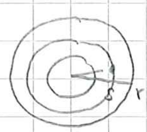
④ $\sin(z) = z - \frac{z^3}{3} + \frac{z^5}{5!} - \dots$

Test to check for absolute convergence:

Criterion: $\quad S(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$

(1) If $S(z)$ converges on $|z-a| \leq s < r \quad \forall s$

$\Rightarrow \lim_{n\to\infty} |c_n| s^n = 0 \quad \forall s < r$



(2) If (1) is satisfied, then $S(z)$ is absolutely convergent $\forall\, |z-a| < s \quad \forall s < r$.

$\lim_{n\to\infty} |c_n| s_1^n = 0 \quad \forall n$ $\qquad |c_n| |z-a|^n \leq |c_n| s^n \leq (|c_n| s_1^n)\left(\frac{s}{s_1}\right)^n \leq A \left(\frac{s}{s_1}\right)^n$ (geometric series)

$0 < \frac{s}{s_1} < 1$

② $\left|\frac{1}{n!} z^n\right| \leq \frac{r^n}{n!} \qquad |z| = r$

Claim: $\forall r > 0 \quad \lim_{n\to\infty} \frac{r^n}{n!} = 0$

Singularities of holomorphic functions

holomorphic

R-R thm: $X \to \mathbb{P}^1$ (projective line)

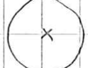
compact complex smooth curve, $S^2 = \mathbb{C} \cup \{\infty\}$

$X \to \mathbb{C}$

ration function $= \frac{P_1(z)}{P_2(z)}$ polynomials

e.g. $\frac{1}{z}$, $\frac{z^2 - z + 1}{z^3 + 2}$

$f: B(a,r) \setminus \{a\} \to \mathbb{C}$ holomorphic



Def. Let $f: U \to \mathbb{C}$ be a function where U is an open set ($\forall z_0 \in U \; \exists B(z_0, r) \subseteq U$, $r>0$)

f is holomorphic on U if $\forall z_0 \in U$, $\exists B(z_0, r) \subseteq U$ so that $f|_{B(z_0,r)} : B(z_0, r) \to \mathbb{C}$ is holomorphic.

$f: B(a,r) \setminus \{a\} \to \mathbb{C}$ holomorphic.
Then we call a a singularity of f.

Theorem Classification of holomorphic/analytic functions

Case ① Riemann removable singularity if f is bounded in $B(a,s) \setminus \{a\}$ for some $0 < s < r$.

$\Rightarrow f$ extends to a holomorphic function $B(a,r) \to \mathbb{C}$.

Case ② Poles. $\lim_{B(a,r)\setminus\{a\} \ni z \to a} |f(z)| = \infty$ $\quad \exists m \in \mathbb{N}$ s.t. $\lim_{z \to a} f(z)(z-a)^m$ exists and non-zero

|| order of the pole.

Case ③ Essential singularity

$\forall\, 0 < s < r$ the image $f(B(a,s) \setminus \{a\})$ is dense in $\mathbb{C}$

(i.e. $\forall w \in \mathbb{C}$, $\exists \{w_n\} \subseteq f(B(a,r)\setminus\{a\})$ & $w_n \to w$)

In RR thm, we only have case 1 and 2.

($\lim_{z \to a} f(z)$ exists and finite.)

ex ① $f(z): B(0,r) \setminus \{0\} \to \mathbb{C}$, $z \mapsto z$ $\quad \lim_{B(0,r)\setminus\{0\} \ni z \to 0} f(z) = 0$

so case 1.

② $f(z): B(0,1) \setminus \{0\} \to \mathbb{C}$, $z \mapsto \frac{1}{z}$ $\quad \lim_{z\to 0} |f(z)| = \lim_{z \to 0} \left|\frac{1}{z}\right| = \infty$

$m=1$: $\lim_{z \to 0} f(z) z^m = \lim_{z \to 0} \frac{1}{z} \cdot z = 1 \neq 0$ exists, so pole of order 1.

③ $f(z) = e^{\frac{1}{z}} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.
has essential singularity at $z=0$, because

~~Case 1~~: $z_n = \frac{1}{2\pi i n} \to 0$

$\lim_{n\to\infty} e^{z_n} = \lim_{n \to \infty} e^{2\pi i n} = 1$

$e^z = e^x(\cos(y) + i\sin(y))$ for $z = x+iy$

two different limits, 1 and -1.

Case 2: $z_n' = \frac{1}{\frac{\pi}{2} + 2\pi i n}$

$\rightsquigarrow \lim e^{z_n'} = 1 \neq \infty$ so not case 2.

f, g holomorphic functions
$\Rightarrow f \pm g$ is holomorphic
$\Rightarrow f \circ g$ — " —
If $g(z_0) \neq 0$ then $\frac{f}{g}$ is holomorphic near z_0.

Theorem (Riemann-Cauchy equations) ⤳ C^1 but more difficult to prove?

$f: U \to \mathbb{C}$ a function and f is C^2 (second derivative exist and continuous)
(U: open set; f = u + iv)

Then f is holomorphic iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

ex $f(z) = z^2 \iff f(x,y) = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xyi}_{i\cdot v(x,y)}$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x \quad \checkmark$$

$$\frac{\partial u}{\partial y} = -2y = -\left(\frac{\partial v}{\partial x}\right) = -(2y) \quad \checkmark$$

} so z^2 is holomorphic.

ex $f(z) = 2z + 3z^2$
$= f_1 + f_2 = (u_1 + iv_1) + (u_2 + iv_2) = \underbrace{(u_1 + u_2)}_{u} + i\underbrace{(v_1 + v_2)}_{v}$

Can check componentwise!

$$\frac{\partial u_j}{\partial x} = \frac{\partial v_j}{\partial y} \wedge \frac{\partial u_j}{\partial y} = -\frac{\partial v_j}{\partial x}. \quad \forall i$$

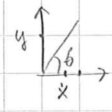
If f is holomorphic on $B(a,r)$, then $\underset{u+iv}{f(z)} = \sum_{n=0}^{\infty} \underbrace{c_n (z-a)^n}_{u_n + iv_n}$

$u = \sum_{n=0}^{\infty} u_n$

$v = \sum_{n=0}^{\infty} v_n \implies$ Only need to check CR equations for

<u>Polar coordinates:</u>

$z = x + iy = r(\cos\theta + i\sin\theta)$



Moivre's formula $z^n = n^r(\cos(n\theta) + i\sin(n\theta))$

Rewrite Cauchy-Riemann in polar coordinates.

$$h(x,y) = g(r,\theta)$$

then
$$\begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial g}{\partial r} \\ \frac{\partial g}{\partial \theta} \end{bmatrix}$$

$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$

$\frac{\partial}{\partial z}(z^2 + 2z\bar{z} - \bar{z}^2) = 2z + 2\bar{z}$

$\frac{\partial}{\partial \bar{z}}(\quad - \text{''} - \quad) = 2z - 2\bar{z}$

If f is holomorphic, then

$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is independent of $\bar{z}$,

that is $\frac{\partial f}{\partial \bar{z}} = 0$

ex $f(x,y) = x^2 + y^2 = \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 = \cdots = z\bar{z}$

$\frac{\partial f}{\partial \bar{z}} = z \neq 0$ so f is <u>not</u> holomorphic.

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f holomorphic $\Rightarrow$ f satisfies CR. (ok/known)

$\Leftarrow$:

Proof: Need to use Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z-\zeta}\, dz \quad \text{for } \zeta \in U = B(a,r)$$

geometric series

Trick: $$\frac{1}{z-\zeta} = \frac{1}{z-a}\cdot\frac{1}{1-\frac{\zeta-a}{z-a}} = \frac{1}{z-a}\sum_{n=0}\frac{(\zeta-a)^n}{(z-a)^n}$$

$$\int_{\partial U} f(z)\frac{(\zeta-a)^n}{(z-a)^n}\, dz$$

$U = B(a,r)$

$z \in \partial U$ if $|z-a| = r \Rightarrow z-a = re^{i\theta}$, $\theta \in [0, 2\pi)$

$$dz = d(a+re^{i\theta}) = rie^{i\theta}\, d\theta$$

$$\int_{\partial U} h(z)\, dz = \int_0^{2\pi} h(a+re^{i\theta})\, r i e^{i\theta}\, d\theta$$

$z = w(\theta)$

$dz = w'(\theta)\, d\theta$

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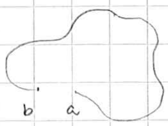
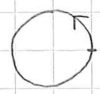
Line integral



- a curve is a smooth function C^1, $\varphi: [0,1] \to \mathbb{C}$
- If $\varphi(a) = \varphi(b)$ we call it a closed curve.
- If φ is not self-intersecting, we call it simple.

$f: \mathbb{C} \to \mathbb{C}$ $\quad \int_D f(z)\, dz := \int_a^b f(\varphi(t))\, \varphi'(t)\, dt$ if φ is a simple, smooth curve.

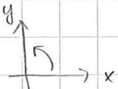
ex $\quad D =$ unit circle $\quad \varphi(t) = e^{it} = z$, $\quad t \in [0, 2\pi]$

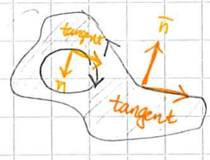


$$\int_D \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it}\, dt = i \int_0^{2\pi} dt = i\, 2\pi \qquad dz = ie^{it}\, dt$$

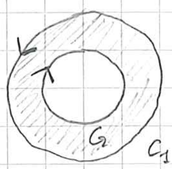
Orientation:

Positive orientation:





positive direction



$\oint f(z)\, dz$ ← positive orientation.

$\mathcal{U}$

$\partial \mathcal{U} = C_1 \cup C_2$

Cauchy's integral formula.

Let $f: \mathcal{U} \to \mathbb{C}$ satisfies C.R. equations. ($\mathcal{U}$ bounded open in $\mathbb{C}$)

Then:
$$f(\xi) = \frac{1}{2\pi i} \oint_{\partial \mathcal{U}} \frac{f(z)}{z - \xi}\, dz \qquad \forall \xi \in \mathcal{U}.$$

Special case: $\mathcal{U} =$ $= S^1$

$$f(\xi) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta} - \xi}\, r i e^{i\theta}\, d\theta \qquad z = re^{i\theta},\ dz = rie^{i\theta}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} r i e^{i\theta} \cdot \frac{1}{1 - \frac{\xi}{re^{i\theta}}}\, d\theta = \frac{1}{2\pi i} i \int_0^{2\pi} f(re^{i\theta}) \sum_{n=0}^{\infty} \left(\frac{\xi}{re^{i\theta}}\right)^n d\theta = \sum_{n=0}^{\infty} c_n \xi^n$$

absolutely convergent power series for $|\xi| < r$

where $\quad c_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^n}\, d\theta$

$$|c_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^n}\, d\theta \right| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{|(re^{i\theta})^n|}\, d\theta = \frac{1}{2\pi} \cdot \frac{1}{r^n} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta$$

MAXIMUM PRINCIPLE: (special case of Elliptic PDE)

$f: \mathcal{U} \to \mathbb{C}$ holomorphic ($\mathcal{U}$ connected, not disjoint). If $\exists z_0 \in \mathcal{U}$ (interior point) so that $|f(z_0)| = \max_{z \in \mathcal{U}} |f(z)|$,

then $f(z) = f(z_0)$ $\quad \forall z \in \mathcal{U}$.

Proof $\quad A = \{z \in \mathcal{U} : |f(z)| = |f(z_0)|\}$. Will first show that $A = \mathcal{U}$.

Topological result: If $\mathcal{U}$ is connected, $A \subseteq \mathcal{U}$ is both closed and open, $A \neq \emptyset \Rightarrow A = \mathcal{U}$.

closed set: $A \subseteq \mathbb{C}$ is closed if $\forall \{z_n\} \subseteq A$ and $\forall z \in \mathbb{C}$ so that $\lim_{n\to\infty} z_n = z \Rightarrow z \in A \subseteq \mathbb{C}$.

$\{z : |z| < 1\}$ is open but not closed.

First: $z_0 \in A \Rightarrow A \neq \emptyset$. Second: A is closed. If $z_n \in A$, $z_n \to z$, then $|f(z_n)| \to |f(z)|$, where $|f(z_n)| = |f(z_0)|$ and so $|f(z)| = |f(z_0)|$.

Third: A is open: By Cauchy's formula, let $B(z_0, r) \subseteq \mathcal{U}$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta \qquad |f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta \le |f(z_0)| \Rightarrow |f(re^{i\theta})| = |f(z_0)| \quad \forall \theta \in [0, 2\pi]$$

(can assume $z_0 = 0$ by change of coordinates; $0 = z_0$; squeezed; $\le$ by assumption)

$\Rightarrow re^{i\theta} \in A \ \forall \theta \Rightarrow z \in A \ \forall z \in B(z_0, r) \Rightarrow A$ is open.

Cauchy-Schwarz inequality

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leq \left(\sum a_n^2\right)\left(\sum b_n^2\right) \quad \text{equality iff } \exists \lambda \text{ s.t. } a_n = \lambda b_n \ \forall n.$$

Same for functions:

$$\left(\frac{1}{2\pi}\int_0^{2\pi} fg\right)^2 \leq \left(\frac{1}{2\pi}\left(\int_0^{2\pi} f^2\right)\frac{1}{2\pi}\int_0^{2\pi} g^2\right)$$

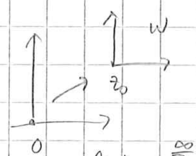
Identity Principle

$f: U \xrightarrow{\subseteq \mathbb{C} \text{ connected, open}} \mathbb{C}$ holomorphic.

If $\exists z_n \in U$, $z_n \to z_0 \in U$ so that $f(z_n) = 0 \ \forall n$ ($z_n \neq z_0 \ \forall n$), then $f(z) = 0 \ \forall z \in U$.

Proof. Assume that we have such z_n and z_0. And so we can assume $z_0 = 0$

Transformation: $z - z_0 = w$



If $f \not\equiv 0$ (identically), then $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $\exists m \geq 0$ (smallest) st.

$$= c_m z^m \left(1 + \sum_{n=m+1}^{\infty} \frac{c_n}{c_m} z^{n-m}\right) \qquad c_m \neq 0$$

(absolutely convergent because $\sum c_n z^n$ is.)

$z \mapsto 1 + \sum_{n=m+1}^{\infty} \frac{c_n}{c_m} z^{n-m}$ is continuous.

$\psi(0) = 1$

$\Rightarrow \psi(z) \neq 0 \quad \forall z \in B(0,r)$ for some small $r > 0$.

Hence in $B(0,r)$, $\{f(z) = 0\}$ has only 1 solution, $z = 0$.

$\Rightarrow$ Contradiction, there is no $z_n \in B(0,r)$ for n big enough.

Singularities:

$f: B(a,r) \setminus \{a\} \to \mathbb{C}$ holomorphic $\qquad$ $f: B(a,r) \to \mathbb{C}$ holomorphic $\to$ Taylor series, otherwise we have

Laurent series for f:

$$f(z) = \underbrace{\sum_{n=-1}^{-\infty} c_n (z-a)^n}_{\text{principal (interesting) part}} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

(Taylor series, if only this, then f is holomorphic.)

which is absolutely convergent on compact subsets of $B(a,r) \setminus \{a\}$.

is unique.

Theorem ① a is a Riemann removable singularity $\iff$ principal part $= 0$.

② (Poles) a is a pole $\iff$ the principal part has only a finite number (& non-empty) of terms

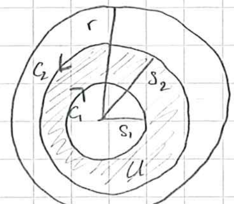
$$\text{principal part} = \sum_{n=-1}^{-m} c_n (z-a)^n$$

③ a is an Essential singularity $\iff$ principal part has infinitely many non-zero terms.

ex $\frac{1}{z} = z^{-1} \Rightarrow$ a pole.

ex $e^{\frac{1}{z}}$ if $z \neq 0$, $w = \frac{1}{z}$. $e^{\frac{1}{z}} = e^{w} = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \dots = 1 + z^{-1} + \frac{1}{2} z^{-2} + \dots$

$\Rightarrow$ essential singularity at $z = 0$.



Proof (Laurent series)

For $s_1 < s_2 < r$: $U = \{ s_1 < |z| < s_2 \}$

Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \forall z \in U$$

- If $\zeta \in C_2$: $|z| < |\zeta|$, then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta} \right)^n$$

so $\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta$ is a power series.

- If $\zeta \in C_1$: $|z| > |\zeta|$

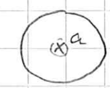
$$\frac{1}{\zeta - z} = \frac{1}{z} \cdot \frac{1}{\frac{\zeta}{z} - 1} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{\zeta}{z}} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z} \right)^n$$

so $-\frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$ is the principal part.

In the Laurent series $f(z) = \sum_{n=-1}^{-\infty} c_n z^n + \sum_{n=0}^{\infty} c_n z^n$

c_{-1} plays an important role (Residue thm)

Proof of Classification of Singularities-thm.



- Case 1: f is bounded near $z_0 = a$.

Assume $|f(z)| \leq M \quad \forall\, 0 < |z - a| < s$.

$$\lim_{\zeta \to 0} \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists, so we can write

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} + 0. \qquad \text{holomorphic.}$$

- Case 2:

$$\lim_{z \to a} |f(z)| = \infty \quad \Rightarrow \quad g(z) = \frac{1}{f(z)} : \quad z \in B(a, s) \setminus \{a\}$$

$$\lim_{z \to a} |g(z)| = 0$$

$\Rightarrow$ By case 1, $g(z)$ has a removeable singularity at a.

$\Rightarrow$ $g(z)$ holomorphic in $B(a, s)$ $g(0) = 0$.

By identity principle, $g(z) = (z-a)^m h(z)$ ($h(a) \neq 0$)

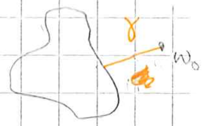
$$f(z) = \frac{1}{(z-a)^m} \frac{1}{h(z)}$$

Hence $\lim_{z \to a} (z-a)^m f(z) = \frac{1}{h(a)}$ finite and non-zero.

Proof (Continued).

- Case 3: Want to show $f(B(a,s)\setminus\{a\})$ is dense in $\mathbb{C}$. $\forall\, 0 < s < r$.
  If not $\Rightarrow$ $\exists\, w_0 \in \mathbb{C}$ and a number $\gamma > 0$ s.t.

  $$|f(z) - w_0| \geq \gamma \qquad \forall z \in B(a,s)\setminus\{a\}.$$



distance between w_0 and $f(B(a,s)\setminus\{a\})$ is $\gamma > 0$

$$g(z) = \frac{1}{f(z) - w_0} = (z-a)^m h(z)$$

$$f(z) - w_0 = \frac{1}{(z-a)^m}\frac{1}{h(z)} \implies \lim_{z\to a} |f(z)| = \infty, \text{ a contradiction.}$$

$$\left(f(z) = \sum_{n=-1}^{-M} c_n (z-a)^n + \text{power series}, \quad \lim_{z\to a} |f(z)| = \infty \quad \text{case 2.} \right)$$

Next time we start Otto Forster's (Riemann surfaces), having finished reviewing complex analysis in 1 variable.

Chapter 1 Covering Spaces

$\{z : |z| < 1\}$ — Riemann sphere

§ 1.1 RIEMANN SURFACES.

Thm. A universal covering space of a Riemann surface must be either $\mathbb{C}$, S^1 or $\mathbb{P}^1$.

Real manifold of dimension 2 $\Longleftrightarrow$ complex dimension 1.
$\mathbb{R}^2 \simeq \mathbb{C}$

Def. A real manifold is some set that locally look like an open set in some $\mathbb{R}^n$.

ex S^2 is compact, Not an open set of $\mathbb{R}^n$, because any open subset of $\mathbb{R}^n$ is not compact.
$S^2 = \mathbb{C} \cup \infty = \mathbb{C} \cup ((\mathbb{C} \setminus \{0\}) \cup \infty)$, where $(\mathbb{C}\setminus\{0\}) \cup \infty \simeq \mathbb{C}$

$X = \bigcup_{i \in I} U_i$, U_i open $\subseteq X$, $U_i \simeq$ open subsets of $\mathbb{R}^n$.

X is a real manifold of dimension n if it locally looks like an open set in $\mathbb{R}^n$. This means

$$X = \bigcup_{i \in I} U_i \qquad I \text{ might be infinite}$$

$U_i \subseteq X$ open, $U_i \simeq V_i$ open $\subseteq \mathbb{R}^n$ (local charts)

V_i and V_j glue together in a good way, that is we get two maps from $U_i \cap U_j$ to V_i and V_j — $U_i \overset{\varphi_i}{\simeq} V_i$, $U_j \overset{\varphi_j}{\simeq} V_j$,

$$U_i \cap U_j$$
$\varphi_i \swarrow \nearrow \varphi_i^{-1}$ $\searrow \varphi_j$

$\varphi_i(U_i \cap U_j) \subseteq V_i$ (in $\mathbb{R}^n$) $\qquad \varphi_j(U_i \cap U_j) \subseteq V_j$ (in $\mathbb{R}^n$)

C^∞ - manifold
C^n - manifold.

We want $\varphi_j \circ \varphi_i : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ (both in $\mathbb{R}^n$)

Depending on how good these maps $\varphi_j \circ \varphi_i$ (transition maps) which the manifold has corresponding good properties.

Complex manifold:

$$U_i \cap U_j$$
$\nearrow \varphi_i^{-1}$ $\searrow \varphi_j$

V_i open in $\mathbb{C}^n \supseteq \varphi_i(U_i \cap U_j)$ $\qquad \varphi_j(U_i \cap U_j) \subseteq \mathbb{C}^n$ $\qquad \varphi_j \circ \varphi_i^{-1}$ is analytic (holomorphic).

if $\boxed{n=1}$: We have Riemann surfaces.

ex ① $\mathbb{C}$ is a Riemann surface because $\mathbb{C} = U_1 \xrightarrow{\varphi_1 = \text{identity map}} \mathbb{C}$,

$$U_1 \cap U_1 = \mathbb{C}$$
$\varphi_1 \swarrow \quad \searrow \varphi_1$
$\mathbb{C} \qquad \mathbb{C}$

$\varphi_1^{-1} \circ \varphi_1 : \mathbb{C} \to \mathbb{C}$, $z \mapsto z$ identity map

ex ② Any open subset U of $\mathbb{C}$ is a Riemann surface, $X = U = U_1$

MAT4800 26.08.2019

§1.1 Riemann surfaces

ex ③ $\mathbb{P}^1$ = Riemann sphere = complex projective line.

$\mathbb{P}^1 = U_1 \cup U_2$ where $U_1 = \mathbb{C}_z$ and $U_2 = \mathbb{C}_w = ((\mathbb{C} \setminus \{0\}) \cup \infty)$

($\mathbb{C}_z$: $\mathbb{C}$ with coordinates z)

$U_1 \cap U_2 = \{z \in \mathbb{C} : z \neq 0\} = \{w \in \mathbb{C} : w \neq 0\}$

$z \in U_1 = \mathbb{C}_z \;\xrightarrow{\varphi_1 = \text{identity}}\; z \in \mathbb{C}$

$w \in U_2 = \mathbb{C}_w \;\xrightarrow{\varphi_2 = \text{identity map}}\; w \in \mathbb{C}_w$

transition map

$\varphi_1(\mathbb{C}) = \{z \neq 0\} \subseteq \mathbb{C}_z \longrightarrow \varphi_2(\mathbb{C}) = \{w \neq 0\} \subseteq \mathbb{C}_w$

$z \longmapsto w = \frac{1}{z}$ holomorphic — Defines $\mathbb{P}^1$

What if the transition map is $w = z$?

Then $X = U_1 \cup U_2 = U_1 = \mathbb{C}$, with $U_1 = \mathbb{C}_z$, $U_2 = \mathbb{C}_w$, $w = z$.

$\varphi(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$

$z \longmapsto w = z$

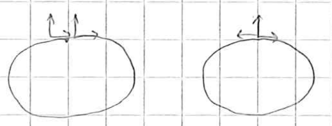
Can be extended to the automorphism $U_1 \to U_2$, $z \longmapsto w = z$

ex ④ NOT a Riemann surface:

Möbius band is not a Riemann-surface because it is not orientable.
We will show later that a R.S. is orientable.

A 2-dimensional real manifold can be turned into a Riemann surface (usually in several inequivalent ways) $\iff$ it is orientable and metrizable.

$\mathbb{P}^2(\mathbb{R})$ does not admit complex structures. (non-orientable)



orientable not orientable

$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \mathbb{D} \to \mathbb{D}$

v_2, v_1:

$d\varphi_{ij}(v_1) = \alpha_1 v_1 + \beta_1 v_2 = w_1$

$d\varphi_{ij}(v_2) = \alpha_2 v_1 + \beta_2 v_2 = w_2$

Linear algebra Lemma:
v_1, v_2 a basis of $\mathbb{R}^2$, and $w_1 = \alpha_1 v_1 + \beta_1 v_2$, $w_2 = \alpha_2 v_1 + \beta_2 v_2$. Then (v_1, v_2) and (w_1, w_2) have the same orientation iff

$$\det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1 > 0$$

If $f = u + iv : \mathbb{D} \to \mathbb{D}$ is holomorphic
$z = (x,y) \mapsto (u(x,y), iv(x,y))$.

Then
$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} > 0 \quad \text{if } f' \neq 0$$

by Cauchy-Riemann equations.

Riemann surfaces

ex ⑤ ELLIPTIC CURVES

$\mathbb{C}$

$\Gamma \subseteq \mathbb{C}$ a lattice, choose v_1, v_2 as a basis for $\mathbb{R}^2 \simeq \mathbb{C}$.

$\Gamma = \{ m v_1 + n v_2 : m, n \in \mathbb{Z} \} \subseteq \mathbb{C}$ (an abelian subgroup of rank 2 of $\mathbb{C}$, thus $\simeq \mathbb{Z}^2$)

Classification of abelian groups: $\Gamma \simeq (\mathbb{Z}/n_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/n_k\mathbb{Z}) \times \mathbb{Z}^m$ m is the rank

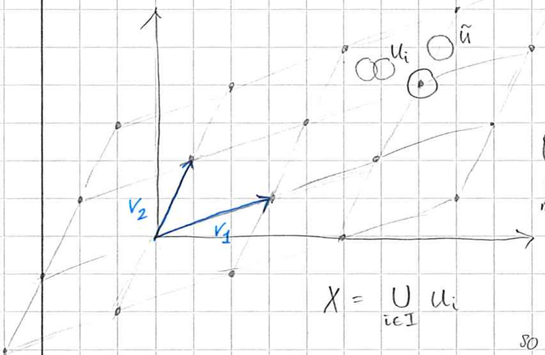
$X = \mathbb{C}/\Gamma$ quotient group meaning:

z_1 is identified with z_2 if $z_1 - z_2 \in \Gamma$

$\Updownarrow$

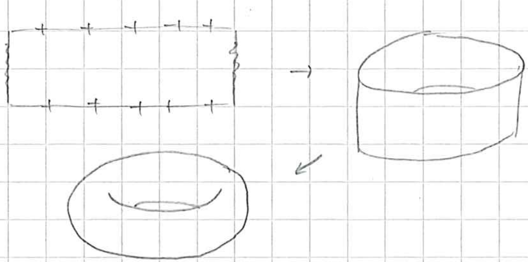
z_1 is identified with z_2 if $z_1 - z_2 = m v_1 + n v_2$ for some $m, n \in \mathbb{Z}$.

X is an abelian group. X is also a Riemann surface, because:



$\tilde{u}$ has no 2 points that are modulo Γ

rectangle → torus.



$X = \bigcup_{i \in I} u_i$

so X is a R.S.

X is also compact (if $X = \bigcup_{i \in I} u_i$ (open) then $\exists$ finite $J \subset I$ s.t. $X = \bigcup_{j \in J} u_j$

X and Y Riemann surfaces.

Know how to define holomorphic map $f : \mathbb{D} = \{ z : |z| < 1 \} \to \mathbb{C}$. (continuous)

We can generalise this to Riemann surfaces. $f: X \to Y$ a map. We say that f is holomorphic if we can write $X = \bigcup_{i \in I} u_i$

$Y = \bigcup_{j \in J} V_j$

$u_i \xrightarrow{\varphi_i} \tilde{u}_i \subseteq \mathbb{C}$ local chart

$V_j \xrightarrow{\psi_j} \tilde{V}_j \subseteq \mathbb{C}$

and for each i, $\exists$ a $j(i)$ s.t. $f(u_i) \subseteq V_{j(i)}$

$$u_i \xrightarrow{f} V_{j(i)}$$

$$\varphi_i : u_i \to \tilde{u}_i \subseteq \mathbb{C}, \qquad \psi_{j(i)} : V_{j(i)} \to \tilde{V}_{j(i)} \subseteq \mathbb{C}$$

$$\mathbb{C} \supseteq \tilde{u}_i \xrightarrow{\varphi_i^{-1}} u_i \xrightarrow{f} V_{j(i)} \xrightarrow{\psi_{j(i)}} \tilde{V}_{j(i)} \subseteq \mathbb{C}$$ holomorphic.

ex $X = \mathbb{P}^1 = U_1 \cup U_2$, $U_1 = \mathbb{C}_z$, $U_2 = \mathbb{C}_w$

$f: X \to X$

If $z \in U_1$, $z \neq 0$ we map $z \mapsto \frac{1}{z}$

If $0 = z \in U_1$, we map $0 \to 0 \in \mathbb{C}_w$.

If $0 = z \in U_2$ we map $0 \in \mathbb{C}_w \mapsto 0 \in \mathbb{C}_z$.

Claim:

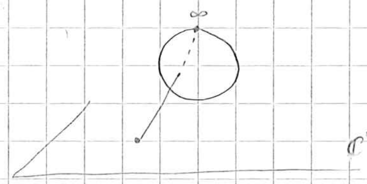
This is a holomorphic map. Actually it is an isomorphism of X.

29.08.2019

$\mathbb{P}^1$: complex projective space of dim 1
= real manifold — " — 2

$\mathbb{RP}^2$ $(\mathbb{P}^2(\mathbb{R}))$ = set of lines through the point $(0,0,0) \in \mathbb{R}^3$. $S^2 \xrightarrow{\pi} \mathbb{RP}^2$ covering map

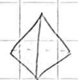
$\mathbb{P}^1 \cong S^2$



$\mathbb{C}^1$

Euler characteristic:

Polyhedra

$X =$  $E(X) = v - e + f = 4 - 6 + 4 = 2$ (vertices, edges, faces) a topological invariant

homeomorphic ≀  S^2

$E(S^2) = 2$, $E(S^2) = \chi(S^2)$

E elliptic curve: $E(E) = 0$, $E(E) = \chi(E)$

Theorem (Hurwitz - Riemann) If X and Y are 2 surfaces and $f: X \to Y$ is a covering map of degree N, then $\chi(X) = N \cdot \chi(Y)$

$\mathbb{P}^1 = \mathbb{C}_z \cup \mathbb{C}_w$, $\varphi_1: \mathbb{C}_z \to \mathbb{C}$, $\varphi_2: \mathbb{C}_w \to \mathbb{C}$

Transition map $\{z \neq 0\} \xrightarrow{\sim} \{w \neq 0\}$, $z \mapsto w = \frac{1}{z}$

back to the example above

$f: \mathbb{P}^1 \to \mathbb{P}^1$, If $z \in \mathbb{C}_z \setminus \{0\} \xrightarrow{f} \frac{1}{z}$

If $z = 0 \in \mathbb{C}_z \xrightarrow{f} w = 0 \in \mathbb{C}_w$

If $w = 0 \xrightarrow{f} z = 0 \in \mathbb{C}_z$

Is this holomorphic?

$\mathbb{P}^1 = \bigcup_{i \in I} U_i$ → local coordinates

$\mathbb{P}^1 = \bigcup_{j \in J} V_j$

$f: U_i \to V_{j(i)}$ is holomorphic

$U_i \xrightarrow{f} V_{j(i)}$, $\varphi_i: U_i \to \mathbb{C} \supseteq \tilde{U}_i$, $\psi_{j(i)}: V_{j(i)} \to \tilde{V}_{j(i)} \subseteq \mathbb{C}$, $\tilde{U}_i \xrightarrow{\text{holomorphic}} \tilde{V}_{j(i)}$

$\mathbb{C}_z \xrightarrow{f} \mathbb{C}_w$

$\mathbb{C}_w \xrightarrow{f} \mathbb{C}_z$

$z \neq 0$

$\mathbb{C}_z \xrightarrow{f} \mathbb{C}_w$, $\varphi_1 \downarrow$, $\downarrow \varphi_2$

$\mathbb{C}_z \ni z \xrightarrow{} w \in \mathbb{C}_w$ ($z \neq 0$)

$\mathbb{C}$? yes

$z \mapsto w$

$0 \mapsto 0$

identity map by change of coordinates

Remark: If X is a R.S. and $p \in X$, then $X \setminus \{p\}$ is also a Riemann surface.

1.8 **Theorem** Riemann classification of singular points

$f: X \setminus \{p\} \to Y$ holomorphic, then we have 3 cases: ($Y = \mathbb{C}$)

(1) Riemann removable singularity
If f is bounded near p we can extend $f: X \to Y$ holomorphic.

$U \setminus \{p\}$ $\odot$ $\xrightarrow{f}$ 

(2) Poles.

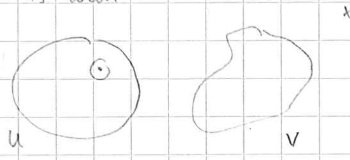
(3) Essential singularity.

X a RS, $U \subseteq X$ open $\Rightarrow$ $\mathcal{O}(U) = \{f: U \to \mathbb{C} \text{ holomorphic}\}$.

Theorem (Identity theorem). $f: X \to Y$ holomorphic (connected) and $\exists \{x_n\} \subseteq X$, $x_n \to x_0 \in X$ with $f(x_n) = p \ \forall n$, $p \in Y$ $\Rightarrow$ f is ~~identically~~ constantly p.

Theorem (Open mapping). $f: X \to Y$ holomorphic, not constant and $U \subseteq X$ is open. $\Rightarrow$ $f(U)$ is open in Y.

Proof The question (??) is local



the Advantage of working with small disk is $U \xrightarrow{f} V$ local coordinates (charts) reduces to chart (?) $U, V \subseteq \mathbb{C}$.

$f: U \subseteq \mathbb{C} \to \mathbb{C}$ (continuous) f not-constant $\Rightarrow$ $f(U) \subseteq \mathbb{C}$ open
can assume that $f(U)$ is bounded. Now if $f(U)$ is not open, then $\exists b \in f(U)$ ($= f(a)$) s.t. $B(b, r) \not\subseteq f(U)$ $\forall r > 0$.

Then $\exists b_j \to b$ s.t. $b_j \notin f(B(a, r_0))$ for some $r_0 > 0$.

$$g_j(z) = \frac{1}{f(z) - b_j}$$ (same trick as when we proved the classification of singularities)

$g_j(z)$ is holomorphic on $U = B(a, r_0)$. By the maximum principle,

$$\max_{z \in B(a, r_0)} |g_j(z)| \leq \max_{\underbrace{z \in \partial B(a, r_0)}_{\text{compact sets}}} |g_j(z)| \leq \frac{1}{\lambda} \quad \forall j.$$

Since ∂U is compact, by compactness $\exists \lambda > 0$ s.t. $\min_{z \in \partial U} |f(z) - a_j| > \lambda$ $\forall j$ (?)

identity theorem.

$\Rightarrow$ Choose $z = a$.

$$\infty \leftarrow \frac{1}{|f(a) - b_j|} = g_j(a) \leq \frac{1}{\lambda} \quad \forall j$$ Contradiction!

Calculations with projective spaces

$\mathbb{P}^1 := \mathbb{C}_z \cup \mathbb{C}_w /_{z \sim \frac{1}{w} \text{ when } z \neq 0}$ = set of complex lines going through $(0,0)$ in $\mathbb{C}^2_{a,b}$

complex line in $\mathbb{C}^2_{a,b}$ going through $(0,0)$ has the equation

$\lambda z a + \lambda z b = za + wb = 0$ (same line, for $\lambda \in \mathbb{C} \setminus \{0\}$; variables z, w) for some $(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

can identify the z, w in the first model with z, w in the second model.

$\mathbb{P}^1 \cong \mathbb{C}^2_{z,w} \setminus \{(0,0)\} /_{(z_1, w_1) = \lambda(z,w),\ \lambda \in \mathbb{C} \setminus \{0\}}$ we write coordinates in $\mathbb{P}^1$ as $[z:w]$ homogeneous coordinates.

use x, y as coordinates in this model now:

Claim $\mathbb{P}^1$ is $\mathbb{C}_z \cup \mathbb{C}_w /_{z \neq 0\ z \sim \frac{1}{w}}$

* $U_1 = \{[x:y] : x, y \in \mathbb{C},\ y \neq 0\}$

If $[x:y] \in U_1 \Rightarrow [x:y] = [\lambda x : \lambda y] = [\frac{x}{y} : 1]$ where $\lambda = \frac{1}{y}$

This representation is unique, therefore π_1 is an isomorphism

Map $\pi_1 : [x:y] \in U_1 \xrightarrow{\sim} \mathbb{C}_z$, $[x:y] \longmapsto z = \frac{x}{y}$ U_1 is open in $\mathbb{P}^1$.

* $U_2 = \{[x:y] : x, y \in \mathbb{C},\ x \neq 0\}$

For $[x:y] \in U_2$ there is an unique representation $[x:y] = [1 : \frac{y}{x}]$

Map $\pi_2 : U_2 \xrightarrow{\sim} \mathbb{C}_w$

$[x:y] \longmapsto w = \frac{y}{x}$

$[1:w] \longleftarrow w$

$\mathbb{P}^1 = U_1 \cup U_2$, $\pi_1 : U_1 \to \mathbb{C}_z$, $\pi_2 : U_2 \to \mathbb{C}_w$

$[z:1] = [1 : \frac{1}{z}]$, $\varphi = \pi$?

$0 \neq z \in \mathbb{C}_z \xrightarrow{\varphi_1^{-1}} [z:1]$, $\xrightarrow{\varphi_2} w = \frac{1}{z} \in \mathbb{C}_w$

$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$\cup$

$\mathbb{C}_z \longrightarrow \mathbb{C}_z$

$z \longmapsto \frac{1}{z}$

$[z:1] \longmapsto [\frac{1}{z} : 1] = [1:z]$

$[0:1] \longmapsto [0:1]$.

So we have a map $\mathbb{C} \to \mathbb{C}$, $z \to w = z$, so this is holomorphic (while $\frac{1}{z}$ is not)

$[1:0]$ slope $1/0 = \infty$ so we call $[1:0]$ for the infinity point

exercise 1 $P(x,y)$ $x^2 + y^2$ Look at the map $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, $[x:y] \mapsto [x^2 + y^2 : y^2]$

Check if it is well-defined, holomorphic, what are the preimages of $[0:1]$ and $[1:0]$?

$[x:y] \mapsto [x^2 + 1 : y]$

$f: \mathbb{P}^1 \to \mathbb{P}^1$

$[x:y] \mapsto [x^2+y^2 : y^2]$ $\qquad \mathbb{P}^1 = \mathbb{C}^2 \setminus \{(0,0)\}/\sim$

Is this well-defined? $\qquad \mathbb{C}^2 \to \mathbb{C}^2$

$F: (x,y) \mapsto (x^2+y^2, y^2)$

Step 1:

Is $\mathbb{C}^2 \setminus \{(0,0)\} \xrightarrow{F} \mathbb{C}^2 \setminus \{(0,0)\}$ a map? $F^{-1}(\{0,0\}) \ni (0,0)$ ← must have this equality. yes.

Does F induce a map $f: \mathbb{P}^1 \to \mathbb{P}^1$? $\quad (x^2+y^2, y^2) = (0,0)$

$\begin{cases} x^2+y^2 = 0 \\ y^2 = 0 \end{cases} \qquad \begin{matrix} x=0 \\ y=0 \end{matrix}$

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{(0,0)\} & \xrightarrow{F} & \mathbb{C}^2 \setminus \{(0,0)\} \\ \pi \downarrow & \circlearrowright & \downarrow \pi \\ \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1 \\ & \exists f? & \end{array}$$

The question becomes:

Is there $f: \mathbb{P}^1 \to \mathbb{P}^1$ s.t. we have a commutative diagram? $\pi \circ F = f \circ \pi$

$\iff \forall (x,y), (x_1,y_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ with $[x:y] = [x_1:y_1]$ then

$[x^2+y^2 : y^2] = \pi \circ F(x,y) = f \circ \pi = f([x:y]) = f([x_1:y_1]) = [x_1^2+y_1^2 : y_1^2]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel \; \pi \circ F(x_1,y_1)$

Now $[x:y] = [x_1:y_1] \iff \exists \lambda \in \mathbb{C} \setminus \{0\}$ s.t. $(x,y) = \lambda(x_1,y_1)$.

$\Rightarrow [x^2+y^2 : y^2] = [(\lambda x_1)^2 + (\lambda y_1)^2 : (\lambda y_1)^2] = [\lambda^2(x_1^2+y_1^2) : \lambda^2 y_1^2] = \lambda^2 [x^2+y^2 : y^2]$

Such an λ exists, so this is well-defined.

Is this holomorphic?

We need to find charts of $\mathbb{P}^1$ where f maps charts to charts, and check whether it is holomorphic.

One convenient way to find local charts is to look at the range $\qquad f: X \to Y$ (f^{-1}), X = domain, Y = range

$\mathbb{P}^1 = U_1 \cup U_2$

$U_1 = \{[x:y], x \neq 0\}$, $\quad U_2 = \{[x:y], y \neq 0\}$ $\qquad V_1 = f^{-1}(U_1)$, $\quad V_2 = f^{-1}(U_2)$

$\mathbb{P}^1 \supseteq V_1 := \{[x:y] \in \mathbb{P}^1 : f[x:y] \in U_1\} \overset{\text{insert our } f}{=} \{[x:y] \in \mathbb{P}^1 : x^2+y^2 \neq 0\}$

$\{[x:y] \in \mathbb{P}^1 : x^2+y^2 \neq 0\} = \mathbb{P}^1 \setminus \{x^2+y^2 = 0\} = \mathbb{P}^1 \setminus \{[i:1], [-i:1]\}$ (two points)

dehomogenisation: $x^2+1 = 0$

$x^2+y^2 = 0 \iff x^2 = -y^2 \iff x = \pm iy$,

$[x:y] \begin{cases} = [iy:y] = [i:1] = [1:\frac{1}{i}] = [1:-i] \\ \text{or} \\ = [-iy:y] = [-i:1] = [1:i] \end{cases}$

$\{x^3+3xy^2+y^3 = 0\} \subseteq \mathbb{P}^1$? If $y=0 \Rightarrow x=0$, but $[0:0] \notin \mathbb{P}^1$, so $y \neq 1$ and hence we can write $x = \lambda y$ for some $\lambda \in \mathbb{C}$:

$\lambda^3 y^3 + 3\lambda y^3 + y^3 = 0 \qquad \lambda^2 + 2\lambda + 1 = 0$

dehomogenisation

$\{xy = 0\} \subseteq \mathbb{P}^1$? degree 2 homogeneous polynomial

If $y = 0 \Rightarrow \forall x \neq 0 : xy = 0$. But the point $[x:y] = [x:0] = [\frac{x}{x} : \frac{0}{x}] = [1:0]$ one point.

If $y \neq 0 \Rightarrow x = 0$, $[x:y] = [0:y] = [\frac{0}{y} : \frac{y}{y}] = [0:1]$

$[x:y] \in U_1 \mapsto z = \frac{y}{x} \in \mathbb{C}_z$ $\qquad$ $[x:y] \in U_2 \mapsto w = \frac{x}{y} \in \mathbb{C}_w$

$V_1 = \mathbb{P}^1 \setminus \{[i:1], [-i:1]\}$ both coordinates of $[i:1]$ are nonzero $\Rightarrow [i:1] \in U_1 \cup U_2$.

$\tilde{U}_1 = U_1 \setminus \{[i:1], [-i:1]\}$ $\quad$ $\tilde{U}_1 \subseteq U_1$ local chart $\Rightarrow \tilde{U}_1$ is also a local chart because $\tilde{U}_1 \subseteq U_1$, $\varphi(\tilde{U}_1) \subseteq \mathbb{C}_z$ via φ_1

If $[x:y] \in \tilde{U}_1$: $z = \frac{y}{x} \Leftrightarrow y = xz \Rightarrow z^2 + 1 \neq 0 \Rightarrow [x:y] \mapsto [x^2+y^2 : y^2]$
$= [x^2 + (xz^2) : (xz)^2]$
$\overset{x \neq 0}{=} [z^2+1 : z^2] \in U_1$
$= [1 : \frac{z^2}{z^2+1}]$ → holomorphic, so f holomorphic

$\tilde{U}_2 = U_2 \setminus \{[i:1], [-i:1]\}$

$w = \frac{x}{y}$ $\quad$ $w^2 + 1 = 0$ $\quad$ $[x:y] \mapsto [1 : \frac{1}{w^2+1}]$.

$V_2 = \{[x:y] \in \mathbb{P}^1 : y^2 \neq 0\} = U_2$

$f : U_2 \longrightarrow U_2$
$[x:y] \longmapsto [x^2+y^2 : y^2]$ $\quad$ $w = \frac{x}{y} \Rightarrow x = wy$
$= [w^2y^2 + y^2 : y^2]$
$= [w^2+1 : 1]$

$w \longmapsto w^2 + 1$ is holomorphic

What is the number of preimages of f?

We show that $\{x^2 + y^2 = 0\} \subseteq \mathbb{P}^1$
$\{[i:1], [-i:1]\} = f^{-1}([0:1])$ " $f^{-1}([0:\overset{\neq 0}{\xi}])$ "

$\Rightarrow f^{-1}([0:1]) = \{[i:1], [-i:1]\}$ has ② points
degree of x^2+y^2

$f^{-1}([1:2]) = ?$

$= \{[x:y] \mid [x^2+y^2 : y^2] = [1:2]\}$

$y \neq 0 \Rightarrow [x:y] \in U_2$: $w = \frac{x}{y}$ $\quad$ $[\frac{x^2+y^2}{y^2} : 1] = [\frac{1}{2} : 1]$

$\frac{x^2}{y^2} + 1 = w^2 + 1 = \frac{1}{2}$
$w^2 = -\frac{1}{2}$
$w = \pm i\sqrt{\frac{1}{2}}$

$[x:y] = [\pm i\sqrt{\frac{1}{2}}y : y] = [\pm i\sqrt{\frac{1}{2}} : 1]$

$f^{-1}([1:0]) = [1:0]$ counted with multiplicity 2

$\frac{y}{x} = w$ $\quad$ $w^2 = 0$

MAT4800 02.09.2019

$$F: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (x^3+y^3,\ x^3-2y^3)$$

holomorphic - Power series, checking the local charts

Does this induce a map $f: \mathbb{P}^1 \to \mathbb{P}^1$? Is it holomorphic? How many preimages do we have? (homogeneous polynomial(s) of degree 3)

Find $f^{-1}([1:0])$, $f^{-1}([0:1])$, $f^{-1}([1:1])$.

$$F: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (x^2+1,\ y)$$

Does F induce $f: \mathbb{P}^1 \to \mathbb{P}^1$?

Step 1: $F^{-1}\{(0,0)\} \ni (0,0)$? No:

$$\begin{cases} x^2+1=0 \\ y=0 \end{cases} \Rightarrow \begin{array}{l} x=\pm i \\ y=0 \end{array}$$

$(0,0) \notin F^{-1}(0,0)$, so this does not induce a map

$$F: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (x^2+y,\ y)$$

① $F^{-1}(0,0) \ni (0,0)$? Yes:

$$\begin{cases} x^2+y=0 \\ y=0 \end{cases} \quad \begin{array}{l} x=0 \\ y=0 \end{array}$$

$[p(x,y) : q(x,y)]$

Step 2: $p(x,y)$, $q(x,y)$ has to be homogeneous and have the same degree.

②
$$\begin{array}{ccc} \mathbb{C}^2\setminus\{(0,0)\} & \xrightarrow{F} & \mathbb{C}^2\setminus\{(0,0)\} \\ \pi\downarrow & \circlearrowright & \downarrow\pi \\ \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1 \end{array}$$
$\exists f$?

$\Updownarrow$

$\pi \circ F = f \circ \pi \iff (x,y), (x_1,y_1) \in \mathbb{C}^2\setminus\{(0,0)\}$ s.t. $[x:y] = [x_1:y_1] \in \mathbb{P}^1$

then

$\Updownarrow$

$\exists \lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ so that $(x,y) = \lambda(x_1,y_1)$

$$[x^2+y : y] = [x_1^2+y_1 : y_1]$$
$$[\lambda^2x^2+\lambda y : \lambda y] \stackrel{?}{=} [x_1^2+y_1 : y_1]$$
$$\parallel$$
$$[\lambda x_1^2 + y_1 : y_1] \stackrel{?}{=} [x_1^2+y_1 : y_1]$$

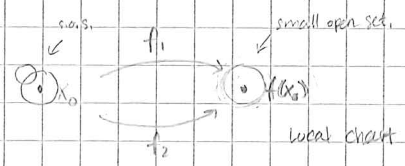
This is not the general $\forall [x_1:y_1] \in \mathbb{P}^1$ and $\lambda \in \mathbb{C}^*$.

Thm (Identity theorem) If $f_1, f_2 : X \to Y$ are holomorphic maps for R.S. and there is a sequence $x_n \in X$, $x_n \to x_0 \in X$ s.t. $f_1(x_n) = f_2(x_n)$ $\forall n$ $\Rightarrow$ $f_1(x) = f_2(x)$ $\forall x \in X$.

Proof If $f_1, f_2 : X \to \mathbb{C}$ then we can subtract $\underset{\text{id}}{f = f_1 - f_2} : X \to \mathbb{C}$

For general Riemann s. Y, we cannot subtract. We can

$\Rightarrow$ $f_1(x_0) = f_2(x_0)$



$A = \{x \in X : f_1(x) = f_2(x)\}$ is closed.

A is open because for a small subset B s.t. B ($\odot$) x_0 $B \cap A \neq \emptyset$, $f(B) = 0$ ...?

Theorem If X is a Riemann surface and $f: X \to \mathbb{C}$ is holomorphic, then f is a constant map

Proof Use maximum principle.

closed unit disk $= \{z \in \mathbb{C} : |z| \leq 1\}$ is a real manifold of dim 2, with boundary $= \{z \in \mathbb{C} : |z| = 1\}$

$f: X \to \mathbb{C}$ is continuous and X is compact $\Rightarrow$ $f(X)$ is compact.

$\Rightarrow$ $f(X)$ is bounded in $\mathbb{C}$. $\Rightarrow$ $\exists\, x_0 \in X$: $|f(x_0)| = \max_{x \in X} |f(x)|$
(f holomorphic)
$\Rightarrow$ $f(x) = f(x_0)$ $\forall x \in X$. □

So we need to look at more general maps than holomorphic maps into $\mathbb{C}$.

MEROMORPHIC MAPS

Remark Meromorphic maps are not maps from X to $\mathbb{C}$, there are some points on X where the map is not defined. They are maps from X into $\mathbb{P}^1$ and maps indeterminacy maps to $\infty = (0,1) \in \mathbb{P}^1 = \mathbb{C} \cup \infty$.

Def ① A meromorphic map on a Riemann surface X is a holomorphic map $f: X \to \mathbb{P}^1$, and $f \not\equiv \infty$ constant map

Def ② A meromorphic map on X includes a pair $(U, f: U \to \mathbb{C})$ (note: U is not X) where

(i) $U \subsetneq X$ open dense set with $X \setminus U$ a countable set and contains only isolated points.

$\Updownarrow$ If $x_n \in X \setminus U$ then no subsequence of $\{x_n\}$ has a limit point in X.

This comes from the identity theorem / Classification of Singularities

(ii) $f: U \to \mathbb{C}$ holomorphic

(iii) Every point of $X \setminus U$ (which is an isolated singularity of f) is either a removable singularity or a pole

$\mathcal{M}(X)$ = set of meromorphic maps on $X \cup \{$constant map $\infty\}$
$\Uparrow$
(Riemann-Roch thm)

MAT4800

05.09.2019

<u>Oral exam</u> last class Thur 21.11.

MAT4510 02.12. 9
MAT4200 05.12. 9
MAT-INF 18.12. 14:30

Oral exam December ~~3~~ 4

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25	26	27	28	29	.	.
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Theorem 1.15 Suppose X is a Riemann surface. $U \subseteq X$ open set, $X \setminus U$ contains only isolated points (at most countable).

If $f : U \to \mathbb{C}$ is holomorphic s.t. every point in $X \setminus U$ is either a removable singularity or a pole. ($\in \mathcal{M}(X)$ meromorphic)

Then f extends to a holomorphic function $\hat{f} : X \to \mathbb{P}^1$

Proof WTS that for every $x_0 \in X \setminus U$, f extends to a holomorphic function in a neighbourhood of x_0

Let $V \subseteq X$ be a local chart so that $x_0 \in V$ and $V \cap (X \setminus U) = x_0$

$f : V \setminus \{x_0\} \longrightarrow \mathbb{C}$ holomorphic.

- If x_0 is a removeable singularity, then f extends to $\tilde{f} : V \to \mathbb{C}$ holomorphic.
- If x_0 is a pole, then by the classification of singularities (CoS)

$$\lim_{U \ni x,\, x \to x_0} |f(x_0)| = \infty \in \mathbb{P}^1 = U_1 \cup U_2^{\ni\infty}$$

So if we choose V small enough, we can assume that $f(V \setminus \{x_0\}) \subseteq U_2$.

In $U_2 \cong \mathbb{C}_w$ the point ∞ is identified with the point $w = 0$. $\left(z = \frac{1}{w},\ \infty = \frac{1}{0}\right)$

$$f : V \setminus \{x_0\} \to \mathbb{C}_w \qquad \lim_{V \setminus \{x_0\} \ni x \to x_0} |f(x)| = 0$$

We can apply Riemann removable singularity like in the case above. □

ex
$$f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \cong U_1 \subseteq \mathbb{P}^1$$
$$z \longmapsto \frac{1}{z^2} = \left[\tfrac{1}{z^2} : 1\right] = [1 : z^2]$$

non-ex $f(z) = e^{\frac{1}{z}} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ has essential singularity at $z=0$, so we cannot extend it to a holomorphic map $\tilde{f} : \mathbb{C} \to$ any Riemann surface

If we let $Y \ni y_0 = \tilde{f}(0)$ then $\exists$ small open set $0 \in V \subseteq \mathbb{C}$ so that $\tilde{f}(V) \subseteq$ in a local chart W around 0.

Then $f : V \setminus \{0\} \to W \subseteq \mathbb{C}$ can be extended to $\tilde{f} : V \to \mathbb{C}$, which is impossible ex If $\frac{P(z)}{Q(z)}$ is a rational function (P, Q are polynomials) then

$$f = \frac{P}{Q} : \mathbb{C} \setminus \underbrace{\{Q(z) = 0\}}_{\text{these are poles}} \to \mathbb{C}$$

$\tilde{f} : \mathbb{C} \to \mathbb{P}^1$ and $F : \mathbb{P}^1 \to \mathbb{P}^1$, because

$\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, $\lim_{z \to \infty} \frac{P(z)}{Q(z)}$ exists in $\mathbb{P}^1$

Exercises §I.1 1, 2, 4, 5

If $\deg P > \deg Q$, $F(\infty) = \infty$,
$\deg P < \deg Q$, $F(\infty) = 0$, (P, Q relatively prime)
$=$, $F(\infty) = z \in \mathbb{C}$ (?)

Another way is to look at the map F:

$\mathbb{P}^1 = [x : y]$ $U_1 \cup U_2$, $U_1 \cong \mathbb{C}_z$, $z = \frac{x}{y}$

ex $f(z) = \frac{z^3 - 1}{z^4 - 1}$ $z - 1$ is the common divisor of P, Q, get rid of it: (polynomial division $P : z-1$, $Q : z-1$):

$$f(z) = \frac{z^2 + z + 1}{z^3 + z^2 + z + 1} = \left[\frac{\left(\frac{x}{y}\right)^2 + \left(\frac{x}{y}\right) + 1}{\left(\frac{x}{y}\right)^3 + \left(\frac{x}{y}\right)^2 + \left(\frac{x}{y}\right) + 1} : 1 \right] = \left[\frac{y(x^2 + xy + y^2)}{(x^3 + x^2y + xy^2 + y^3)} : 1 \right]$$

$$= [\underbrace{y(x^2 + xy + y^2) : x^3 + x^2y + xy^2 + y^3}_{\text{homogeneous polynomials of degree 3}}]$$

MAT4800 §2 Elementary properties of holomorphic mappings. 05.09.2019

Theorem 2.1 (Local Behaviour of Holomorphic maps). Locally, a non-constant holomorphic map between R.S.s $X \to Y$ is of the form

$$\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \longrightarrow \mathbb{D}$$
$$z \longmapsto z^k$$

where $1 \le k :=$ local degree of f.

Corollary: $\forall \zeta \in \mathbb{D}$: $\# f^{-1}(\zeta) = k$ (locally) counted with multiplicity; $\# f^{-1}(\zeta)$ = the number of preimages.

Lemma (Existence of logarithm)

① $\mathbb{C} \supseteq B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$. If $f \in \mathcal{O}(B(a,r))$ (we say holomorphic; ring of regular functions)

$\Rightarrow \exists F \in \mathcal{O}(B(a,r))$ s.t. $F' = f$ (existence of anti-derivative)

Proof $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ where $\lim_{n\to\infty} |c_n| s^n = 0 \quad \forall s < r$.

Define $F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(z-a)^{n+1}$ also absolutely convergent series $\Rightarrow F \in \mathcal{O}(B(a,r))$

$$\Rightarrow F'(z) = \sum_{n=0}^{\infty}\left(\frac{c_n}{n+1}(z-a)^{n+1}\right)' = \sum_{n=0}^{\infty} c_n (z-a)^n = f(z)$$

Lemma cont'd ② If $f \in \mathcal{O}^*(B(a,r)) \Rightarrow \forall k \in \mathbb{N} \ \exists F_k \in \mathcal{O}(B(a,r)) : F_k^k = f$

We can take k-th root. $\exists G \in \mathcal{O}(B(a,r)) : e^G = f$ (we can take logarithm)

Proof Idea: If G is $\ln f$ then $G' = \frac{f'}{f}$.

Now because $f \in \mathcal{O}^*(B(a,r))$ f is nowhere zero (by assumption/def $*$)

$\Rightarrow H = \frac{f'}{f} \in \mathcal{O}(B(a,r))$. By ① $\exists G \in \mathcal{O}(B(a,r))$ so that $G' = \frac{f'}{f}$

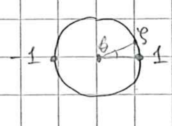
$e^G = Af \quad A = e^B \quad f = e^{G-B}$

Now define
$$F_k = (e^G)^{\frac{1}{k}} := e^{\frac{G}{k}} \in \mathcal{O}(B(a,r))$$
$$\Rightarrow F_k^k = (e^{\frac{G}{k}})^k = e^G = f$$

non-ex $f(z) = z \notin \mathcal{O}^*(B(a,r)) : \mathbb{C} \to \mathbb{C}$ $\quad \nexists F_2 : \mathbb{C} \to \mathbb{C}$ holomorphic s.t. $F^2 = f$.

(monodromy theorem)

$f(z) = z$



$\sqrt{z} = e^{i\theta/2}$

$\sqrt{1} = 1$, $\quad e^{i0/2} = 0$

$\sqrt{1} = -1$, $\quad e^{i2\pi/2} = -1$

not continuous function

Corollary 2.5 If $f: X \to Y$ biholomorphic, non-constant and injective, then $f: X \to f(X)$ is holomorphic.

Proof $f(X)$ open $\subseteq Y \Rightarrow f(X)$ also a R.S. Because f is injective the local degree $= 1$
$\Rightarrow$ locally the map is
$$\mathbb{D} \to \mathbb{D}$$
$$z \mapsto z$$
So it has a local inverse which is also holomorphic □

biholomorphic: the inverse is also holomorphic.

Theorem 2.7 Suppose X, Y Riemann surfaces. Suppose X compact R.S. $f: X \to Y$ holomorphic, non-constant. Then Y is also compact and $f: X \to Y$ is surjective.

Proof X compact $\Rightarrow f(X)$ is compact and hence closed
By open mapping thm, $f(X)$ is open. Y is R.S. $\overset{def}{\Longrightarrow}$ Y is connected $\Rightarrow f(X) = Y$. □

Corollary 2.9 Every meromorphic function on $\mathbb{P}^1$ is rational, that is it can be written as [illegible]
$$f(z) = \frac{P(z)}{Q(z)} \quad P, Q \text{ polynomials.}$$

Proof · If $f: \mathbb{P}^1 \to \mathbb{P}^1$ is constant then it is of the form $[x:y] \mapsto [a:b]$ where $[1:0] = \infty$.

· If f is non-constant, then f has only finite no. poles because if $z_n \in \mathbb{P}^1$ is s.t. $f(z_n) = \infty$ and $z_n \to z_0 \in \mathbb{P}^1$ then $f(z_0) = \infty$. Now look at the coordinate chart $\mathbb{C}_w$ ($\infty = \{w = 0\}$), using the identity thm... (?) ..., because $\mathbb{P}^1$ is a compact metric space, so every sequence will have a converging subsequence.

Locally near each pole z_m f has the form $P_m(z) = \sum_{n=-k}^{-1} c_n (z - z_m)^n + H_m(z)$ (holomorphic function)

$F(z) = f - (p_1(z) + \cdots + p_j(z))$ has no poles, is holomorphic

$\Rightarrow F: \mathbb{P}^1 \to \mathbb{P}^1 \setminus \infty$ ($\mathbb{C}$) $\Rightarrow F: \mathbb{P}^1$ (compact) $\to \mathbb{C} \Rightarrow F$ is constant
(Every holomorphic function on a compact Riemann surface is constant)

$H_w(z)$: $\lim_{z \to \infty} z^n \quad w = \frac{1}{z}$

We proceed $\sum_{p_l = 1}^{l} \sum_{n=-k_m}^{-1} c_{n,m} (z - z_m)^n + H(z)$ holomorphic function on $\mathbb{C}$

H also extends to a holomorphic function on $\mathbb{P}^1$.

Claim: H must be a polynomial

$e^{\frac{1}{z}}: \mathbb{P}^1 \to \mathbb{P}^1$ not holomorphic because the Laurent series is $\sum \frac{1}{n!} \frac{1}{z^n}$ has

$H(z) = \sum_{n=0}^{\infty} c_n z^n$ an infinite power series $\Rightarrow H$ is not holomorphic from $\mathbb{P}^1 \to \mathbb{P}^1$.

Look at $z = \infty$, $w = \frac{1}{z}$ then $w = 0$ at $z = \infty$ $H(w) = \sum_{n=0}^{\infty} c_n \frac{1}{w^n}$.

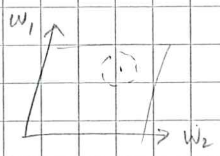
2.12 Doubly Periodic Functions. (Holomorphic function elliptic curves to $\mathbb{P}^1$)

Meromorphic functions from $\mathbb{C}$ to $\mathbb{P}^1$ with two periods ω_1 and ω_2 which means (holomorphic $\to \mathbb{P}^1$)

- $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$ and $\mathbb{R}$-linearly independent
  $\iff \omega_1, \omega_2 \in \mathbb{R}^2$ and ω_1, ω_2 is a basis for $\mathbb{R}^2$.
- $f(z+\omega_1) = f(z) = f(z+\omega_2) \quad \forall z \in \mathbb{C}$

Theorem If $\pi_1 : X_1 \to Y_1$, $\pi_2 : X_2 \to Y_2$ are universal covers of R.S.s, then $\forall f : Y_1 \to Y_2$ holomorphic, $\exists F : X_1 \to X_2$ holomorphic s.t. we have a commutative diagram.

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_2 \\ \pi_1 \downarrow & \circlearrowright & \downarrow \pi_2 \\ Y_1 & \xrightarrow[f]{} & Y_2 \end{array}$$



$\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

(will prove this later)

Applying this to doubly periodic functions:

$$\begin{array}{ccc} \mathbb{C} \text{ (simply connected)} & \longrightarrow & \mathbb{P}^1 \\ \pi_1 \downarrow & \circlearrowright & \downarrow \pi_2 = \text{identity map} \\ Y_1 : \text{Elliptic curve} & \longrightarrow & Y_2 = \mathbb{P}^1 \text{ simply connected} \end{array}$$

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{(0,0)\} & \xrightarrow{F} & \mathbb{C}^2 \setminus \{(0,0)\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

$\exists f$?

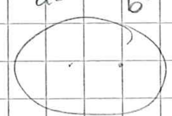
If $\pi \circ F = f \circ \pi$

Def. $\varphi : X \to Y$ a fibre is $\varphi^{-1}(y)$ for some $y \in Y$.

Elliptic curve: $E = \mathbb{C}/\Gamma$. $z_1, z_2 \in \mathbb{C}$ are in the same fibre of $\pi_1 : \mathbb{C} \to E$
$\iff z_1 - z_2 \in \Gamma$
$(z+\omega_1) - z = \omega_1 \in \Gamma$

$\Rightarrow$ Gives holomorphic map $g : E = \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \to \mathbb{P}_1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Periodic function

$f(z) = \sin z$. Period 2π $\quad \sin(z+2\pi) = \sin z \quad \leftarrow$ one period.

Abel & Jacobi found period functions with 2 periods by looking at inverse function of elliptic integral) then Weierstrass.

$\mathbb{C} \supseteq \Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a lattice of rank 2,

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

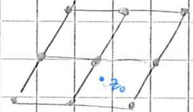
General idea: Start from any function $\varphi(z)$. $\sum_{\omega \in \Gamma} (\varphi(z+\omega) - \varphi(\omega)) = g(z)$.

Proof that $g(z+\omega_0) = g(z) \quad \forall \omega_0 \in \Gamma$:

$$g(z+\omega_0) = \sum_{\omega\in\Gamma} \varphi(z+\omega_0+\omega) - \sum_{\omega\in\Gamma}\varphi(\omega) =$$

$\omega_0 + : \Gamma \to \Gamma$ isomorphism.

$$= \sum_{\omega'\in\Gamma} \varphi(z+\omega') - \sum_{\omega'\in\Gamma}\varphi(\omega')$$



Claim 1: $\wp_\Gamma : \mathbb{C}\setminus\Gamma \to \mathbb{C}$ holomorphic. Need to show absolute convergence

$z_0 \in \mathbb{C}/\Gamma$, $\exists\, \delta > 0$ s.t. $\forall\, \omega \in \Gamma$: $|z-\omega_0| > \delta$

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{\omega^2 - (z-\omega)^2}{(z-\omega)^2\,\omega^2}\right| = \left|\frac{2z\omega - z^2}{(z-\omega)^2\,\omega^2}\right| \le \frac{2|z|}{|z-\omega|^2\,|\omega|} + \frac{|z^2|}{|\omega^2|\,|z-\omega|^2}$$

$|\omega|^2 \approx n^2$ and the series $\sum \frac{1}{n^2}$ converge.

Theorem (Cauchy's) $f(z) = \sum_{n=0}^{\infty} \varphi_n(z)$ (holomorphic) if this is uniformly, absolutely convergent $\Rightarrow$ $f(z)$ is holomorphic)

Def. Discrete set: a set of isolated points.

Claim 2: All $\omega\in\Gamma$ are poles of order 2 of $\wp_\Gamma$.

exerci

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega\in\Gamma\setminus 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

We first show that it has a pole of order 2 at 0. Then because $\wp_\Gamma$ is periodic with periods in Γ $\Rightarrow$ the same is true $\forall \omega\in\Gamma$.

$$\wp_\Gamma(z) = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + \sum_{n=0}^{\infty} c_n z^n, \quad c_{-2} \neq 0$$

(Laurent series; holomorphic)

We will show that

$$\sum_{\omega\in\Gamma\setminus 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$ is holomorphic near $z=0$.

Have already showed that that is uniformly, absolutely convergent, hence

$c_{-2} = 1$, $c_{-1} = 0$.

Claim 3 In Laurent series for $\wp_\Gamma(z)$ near $z=0$ then $c_0 = 0$.

$$\wp_\Gamma(z) = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \sum_{n=1}^{\infty} c_n z^n \implies c_0 + \sum_{n=1}^{\infty} c_n z^n = \left(\wp_\Gamma(z) - \frac{c_{-2}}{z^2} - \frac{c_{-1}}{z}\right)$$

($c_{-1} = 0$; $h(z)$)

$\Rightarrow c_0 = h(0)$ $\qquad h(0) = 0$

$$h(z) = \sum_{\omega\in\Gamma\setminus 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

exer 2.1 b) Let $f: \mathbb{C} \to \mathbb{P}^1$ be a meromorphic function

- f is holomorphic on $\mathbb{C} \setminus \Gamma$.
- f is periodic with periods in Γ.
- f has a pole of order 2 at every point in Γ.
- near $z=0$, f has Laurent series $f(z) = \frac{1}{z^2} + \sum_{n\geq 1}^{\infty} d_n z^n \Rightarrow f = \wp_\Gamma$.

Need to show that $f - \wp_\Gamma = 0$

$h = f - \wp_\Gamma : \mathbb{C} \to \mathbb{C}$ holomorphic, has periods in Γ.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ \pi \downarrow & \circlearrowright & \downarrow id \\ E = \mathbb{C}/\Gamma & \xrightarrow[\hat{h}]{} & \mathbb{C} \end{array}$$

compact (E)

By the maximum principle, a holomorphic function from $X \to \mathbb{C}$ where X is a compact Riemann surface is constant.

$\Rightarrow \hat{h} = \text{constant} \Rightarrow h$ is constant $\Rightarrow h \equiv h(0) = 0$,

so $f = \wp_\Gamma$.

(Proof of theorem 2.13)

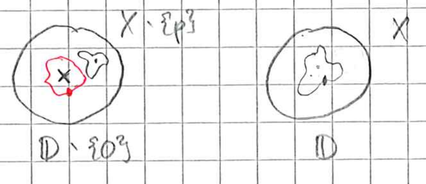
$\wp_\Gamma(z)$.

$$\wp_\Gamma'(z) = -2 \sum_{w \in \Gamma} \frac{1}{(z-w)^3}$$

$[\wp_\Gamma : \wp_\Gamma' : 1] : E = \mathbb{C}/\Gamma \to \mathbb{P}^2$ an embedding

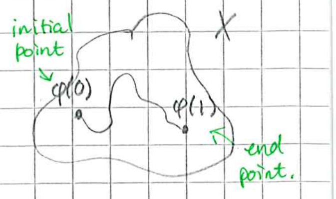
§1.3 Homotopy, Fundamental groups.

$f: X \setminus \{p\} \to \mathbb{C}$ holomorphic, p pole (removable) $\Rightarrow$ f extends to $X \to \mathbb{C}$. (X R.S.)



X is a topological space (a set where we have a notion of open sets)

A curve on X is a continuous function $\varphi : [0,1] \to X$

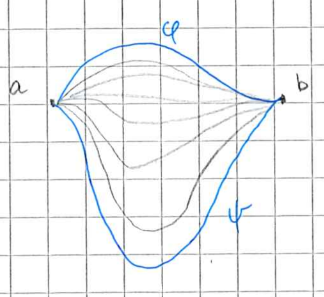


A curve φ is closed if $\varphi(0) = \varphi(1)$

Two curves φ, ψ on X are homotopic if $\exists$ a continuous map $\Omega : [0,1] \times [0,1] \to X$ such that

$$\Omega(t,0) = \varphi(t) \quad \forall t$$
$$\Omega(t,1) = \psi(t) \quad \forall t$$
$$\Omega(0,s) = a \quad \forall s$$
$$\Omega(1,s) = b \quad \forall s$$

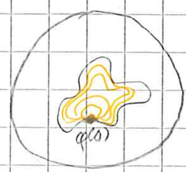
$\varphi(0) = \psi(0) = a$
$\varphi(1) = \psi(1) = b$



$\Omega(t,s) = \lambda_s(t)$ is a family of curves on X

When there is a homotopy between φ and ψ, we write $\varphi \sim \psi$ and say that φ is homotopic to ψ.

ex $X = \mathbb{D}$. Consider a closed curve $\varphi(t)$. Then φ is homotopic to the constant curve
$\psi(t) = \varphi(0) \quad \forall t$



Define $\Omega(t,s) = \varphi(st)$.

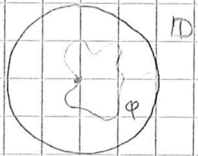
For $s=0$ $\psi(t) = \Omega(t,0) = \varphi(0)$
$s=1$ $\Omega(t,1) = \varphi(1t) = \varphi(t)$

12.09.19

Mandatory oral assignment: 31. october
10 min presentation + answer some questions.
Pick a theorem, present it, give main idea/sketch of proof, give an application.

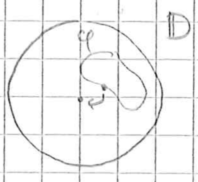
$$P_\Gamma(z) = \frac{1}{z^2} + \sum_{w \in \Gamma \setminus 0}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) = \left(\sum_{w\in\Gamma}\frac{1}{(z-w)^2}\right) - \sum_{w\in\Gamma\setminus 0}\frac{1}{w^2} \qquad c_0 = 0$$

Homotopy



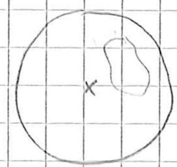
ex. Claim φ is homotopic to the constant curve $\varphi_0 = 0$.

Proof $\Omega(s,t) := s\varphi(t) \in \mathbb{D}$ $|s| \le 1 \Rightarrow |s\varphi(t)| \le |\varphi(t)| < 1$



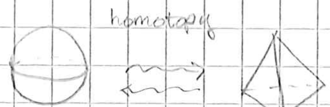
$\varphi_s(t) := \Omega(s,t)$

$\varphi_1(t) = \varphi(t)$, $\varphi_0(t) = 0 \quad \forall t$



ex $\Omega(s,t) = s\varphi(t)$ Everything is fine, except at $s=0$, because $0 \notin X$.

Euler characteristic



Homotopy invariants
Things which are the same for two homotopic spaces.

Examples: Dimension,

Fundamental Group $\pi(X, a)$

X topological space, path-connected For every $x, y \in X$ $\exists \varphi : [0,1] \to X$ so that $\varphi(0) = x$, $\varphi(1) = y$.

ex $\mathbb{D}$, $\mathbb{D} \setminus \{0\}$ are path connected.
$\mathbb{D} - (-1,1)$ not path connected by the Mean Value Theorem

$\mathbb{D} = \{z : |z| < 1\}$

$\pi_1(X, a) = \{$the space of closed curves on X starting and ending at $a\}/\sim$ modulo the homotopy equivalence.

Meaning: We consider only curves $\varphi : [0,1] \to X$ with $\varphi(0) = \varphi(1) = a$

If $\varphi \overset{\text{homotopy}}{\sim} \psi$, then we identify φ and ψ. We write $[\varphi] \in \pi_1(X, a)$ to be the element separated by a curve $\varphi : [0,1] \to X$.

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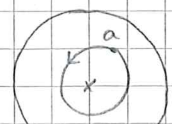
ex $X = \mathbb{D}$, $a \in X$ $\Rightarrow$ $\pi_1(X, a)$ = constant curve $\varphi_a \equiv a$
has only one element.



ex $X = \mathbb{D} \setminus \{0\}$ $\pi_1(X, a) \cong \mathbb{Z}$

$0 \longleftrightarrow \varphi_a \equiv a$

$1 \longleftrightarrow \varphi$ around 0 with positive orientation

$-1 \longleftrightarrow$ negative orientation

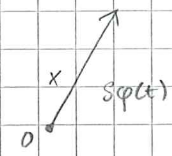
winding numbers

Simply connected

X is simply connected if $\pi_1(X, a) = \{\varphi_a\}$ only one element.

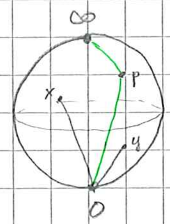
Only these 3 simply-connected R.S.

(1) $X = \mathbb{C}$ is simply-connected (star-shaped) $a = 0$, $\pi_1(X, 0)$



(2) $X = \mathbb{D}$ is simply connected

(3) $X = \mathbb{P}^1$ is simply connected:



Proof. $\mathbb{P}^1$ is path-connected:

Choose $p \in \mathbb{P}^1 \setminus \{0, \infty\}$, then $p \in U_1 \cap U_2$ ($U_1 \cong \mathbb{C}$, $U_2 \cong \mathbb{C}$). There is a curve from $0 \to p$ and a curve from $p \to \infty$

$\mathbb{P}^1$ is simply connected:

$\varphi : [0, 1] \longrightarrow \mathbb{P}^1$ a continuous curve
($[0,1]$: dim = 1, $\mathbb{P}^1$: dim = 2)

Peano curves
$[0,1] \longrightarrow [0,1]^2$ continuous.

Theorem (from topology) If $\dim(X) < \dim(Y)$ and $\varphi : X \to Y$ is smooth (C^1) then $\mathrm{Image}(\varphi) \neq Y$.

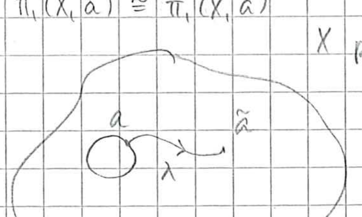
Proof in book doesn't use this, more difficult.

$\exists z_0 \notin \varphi([0,1])$. $\varphi([0,1]) \subseteq \mathbb{P}^1 \setminus z_0 \cong \mathbb{C}$ ← because $\mathbb{C} \ni z_0 \longrightarrow \frac{az+b}{cz+d}$ $ad - bc \neq 0$ automorphism of $\mathbb{P}^1$, $\frac{a}{c} = z_0$ maps $z_0 \mapsto \infty$

$\Omega(s,t) = s\varphi(t)$ $\varphi : [0,1] \to \mathbb{C}$
contracts φ into one point, so (?) $\mathbb{P}^1$ is simply connected.

· $\pi_1(X, a) \cong \pi_1(X, \tilde{a})$

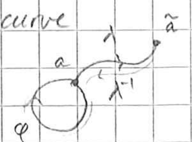
X path connected.



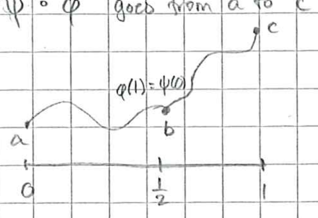
If $\lambda : [0;1] \to X$ is a curve then the reverse is

$\lambda^{-1} : [0,1] \to X$ $\lambda^{-1}(t) = \lambda(1-t)$.

If φ is a closed curve at a $\Rightarrow$ we get a closed curve $\tilde{\varphi}$ at $\tilde{a}$ and vica versa, $\tilde{\varphi} = \lambda \circ \varphi \circ \lambda^{-1}$

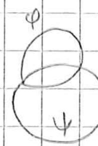


$\psi \circ \varphi$ goes from a to c



$$\psi \circ \varphi = \begin{cases} \varphi(2t) : & t \in [0, \frac{1}{2}] \\ \varphi(2t-1) : & t \in [\frac{1}{2}, 1] \end{cases}$$

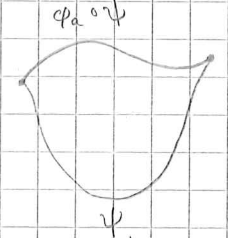
$\pi_1(X, a)$ is a group:



Binary operation $\Phi : \pi_1(X, a) \times \pi_1(X, a) \to \pi_1(X, a)$
$(\varphi, \psi) \mapsto \varphi \circ \psi$

Inverse $\varphi \mapsto \varphi^{-1}$

Unity: $\varphi_a(x) = a \quad \forall t \in [0,1]$



$\varphi_a \circ \psi \overset{\text{want}}{=} \psi$ (homotopy) $\qquad \varphi_a \circ \psi = \begin{cases} \psi(2t) : & t \in [0, \frac{1}{2}] \\ a : & t \in [\frac{1}{2}, 1] \end{cases}$

Then $\Omega(s,t) = \psi \circ h(s,t)$, $\quad h(s,t) = (1-s)t + s v(t)$
is a homotopy. want to prove homotopy.

$v(t) = \begin{cases} 2t : & 0 \le t \le \frac{1}{2} \\ 1 : & \frac{1}{2} \le t \le 1 \end{cases}$

$\lambda \circ \lambda^{-1} = \varphi_a$

$\lambda \circ \lambda^{-1} = \begin{cases} \lambda(2t) : & 0 \le t \le \frac{1}{2} \\ \lambda^{-1}(2t-1) : & \frac{1}{2} \le t \le 1 \end{cases} = \begin{cases} \lambda(2t) & 0 \le t \le \frac{1}{2} \\ \lambda(2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$

Then we have the homotopy: $\quad h(s,t) = \begin{cases} 2t(1-s) : & 0 \le t \le \frac{1}{2} \\ 2(1-t)(1-s) : & \frac{1}{2} \le t \le 1 \end{cases}$

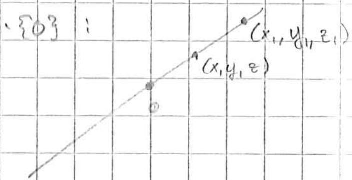
$h(0, t) = \{$

$h(1, t) \equiv 0 \quad \forall t$

$\lambda^{-1} \circ \lambda = \lambda \circ h(0,t) \qquad \lambda \circ h(1,t) = \varphi_a$.

compact space

$\mathbb{P}^2 := \mathbb{C}^3 \setminus \{(0,0,0)\} / \sim \qquad (x,y,z) \sim (x_1, y_1, z_1)$ iff $\exists \lambda \in \mathbb{C} \setminus \{0\}$:
$(x,y,z) = \lambda(x_1, y_1, z_1)$



$= U_1 \cup U_2 \cup U_3$, each $\cong \mathbb{C}^2$

$\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$
$(x,y) \mapsto [x : y : 1]$

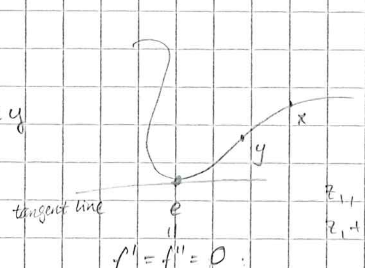
Elliptic curve (homogeneous curves of degree 3) $\quad E = \mathbb{C}/\Gamma \longleftrightarrow \mathbb{P}^2$ (Weierstrass function and derivative)

$\Gamma \ni z \longmapsto [\wp_\Gamma(z) : \wp'_\Gamma(z) : 1]$
$= (\wp_\Gamma(z), \wp'_\Gamma(z)) \in \mathbb{C}^2$

$\simeq \{[x : y : z] \in \mathbb{P}^2 : x^2 z = y^3 + ayz^2\}$ (weierstrass function.) $\quad z = 1$

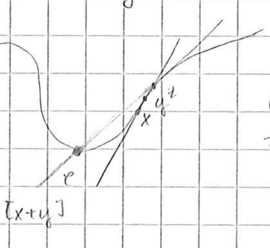
Group structure on Elliptic curves: $\quad x^2 = y^3 + ay$

Reflection point e where $y = x^3$



$z_1, z_2 \in \mathbb{C}$
$z_1 + z_2 \in \mathbb{C}$

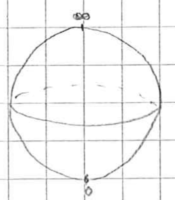
→ Bézout's theorem
3 intersection points



Use the group structure to elliptic encryption.

only compact R.S.s with groups structure

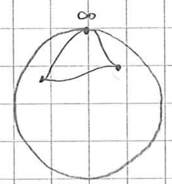
$\mathbb{P}^1$ is simply connected:



$\varphi: [0,1] \to \mathbb{P}^1$ can divide $[0,1]$ into small enough subintervals such that

either $\varphi([t_i, t_{i+1}]) \ni 0$ or $\varphi([t_i, t_{i+1}]) \ni \infty$ and $\varphi(t_i), \varphi(t_{i+1}) \neq 0, \infty$

if $0 \notin \varphi([t_i, t_{i+1}]) \Rightarrow \varphi([t_i, t_{i+1}]) \in \mathbb{P}^1 \setminus 0 \cong \mathbb{C}$



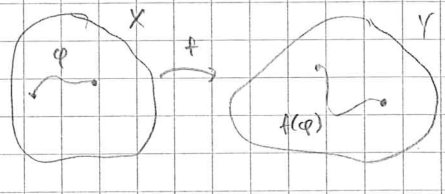
move by homotopy because $\mathbb{P}^1$ simply connected.

We can move $\varphi([t_i, t_{i+1}])$ in such a way that

- the two end points are fixed
- $0, \infty \notin \varphi([t_i, t_{i+1}])$

$f: X \to Y$ continuous map, $\varphi: [0,1] \to X$ a curve

$\Rightarrow f \circ \varphi: [0,1] \xrightarrow{\varphi} X \xrightarrow{f} Y$ is a curve



If φ is closed ($\varphi(0) = \varphi(1)$), then $f \circ \varphi(0) = f \circ \varphi(1)$, so $f \circ \varphi$ is also closed.

If $\Omega: [0,1] \times [0,1] \to X$ is a homotopy, then $f \circ \Omega: [0,1] \times [0,1] \to X$ is also homotopy.

$\Rightarrow$ we have a map called push forward

$f_*: \pi_1(X, a) \to \pi_1(Y, f(a))$ for $f: X \to Y$

If something on X gives rise to something similar on Y, we say that we have a push-forward.

(π_1, T_* = tangent space, H_1 ← homology groups, sheaf ~)

If ___ on Y gives rise to ~ on X, then we say we have a pullback

(differential forms, sheaf, cohomology groups, ~)

TX defined? to $\Omega^1(X)$

So what is $\pi_1(\mathbb{D} \setminus 0)$?

the Excision theorem in homotopy groups

alt 1 • Homotopy theory: $X = Y \cup Z$ then there's a way to relate $\pi_1(X)$ with $\pi_1(Y)$, $\pi_1(Z)$

alt 2 • Theorem: If $f: X \to Y$ is a universal cover (covering space and X simply connected) and $\mathrm{Aut}(X/Y)$ = deck transformation group then $\pi_1(Y) \cong \mathrm{Aut}(X/Y)$

$\mathrm{Aut}(X/Y) \subseteq \mathrm{Aut}(X)$

$$\mathbb{D} \longrightarrow \mathbb{D} \setminus 0$$
$$z \longmapsto e^{-\left(\frac{1+z}{1-z}\right)}$$
and $\mathrm{Aut}(\mathbb{D} / \mathbb{D} \setminus \{0\}) \simeq \mathbb{Z}$.

X, Y R.S (connected)

Theorem If $X \subseteq \mathbb{R}^n$ submanifold and X is connected, then X is path connected ($\exists$ curve).

Def A holomorphic map $f: X \to Y$ is unbranched if $\forall x \in X$, the local represntation of f around x is $z \mapsto z$.

Otherwise, we call it a branched map.

Recall Every holomorphic map between RS locally has the form $z \mapsto z^k$

Lemma $f: X \to Y$. TFAE:

① f is unbranched
② Locally f is biholomorphic map (holomorphic (analytic) property) $\Leftrightarrow$ ④ f is a local homeomorphism (topological property)
③ $f'(z) \neq 0 \quad \forall z \in X$

Proof Locally $f(z) = z^k$. If $k=0$, $f(z) = 1 \Rightarrow f'(z) = 0 \ \forall z$, and f cannot be locally biholomorphic.

generally don't have $z \mapsto z^k$, so more complicated.

If $k=1$: $f(z) = z$, $f'(z) = 1 \quad \forall z$

If $k>1$: $f(z) = z^k$, $f'(z) = kz^{k-1}$ $\quad f'(0) = 0$. $\to$ 0 is a branched points.

③ $f(z) = e^{\left(\frac{1+z}{1-z}\right)} = e^{\frac{-1-z}{(1-z)}}$?

$$f'(z) = \left(-\frac{(1+z)}{(1-z)}\right)' e^{-\left(\frac{1+z}{1-z}\right)} = \ldots = \frac{2}{(1-z)^2} \neq e^{-\left(\frac{1+z}{1-z}\right)}$$

$$f: \mathbb{D} \setminus 0 \to \mathbb{D} \setminus 0$$
$$z \mapsto z^k \qquad k \geq 2 \qquad \text{then } f \text{ is unbranched.}$$

In general if $f: X \to Y$ is holomorphic map between RS then

$$f: X - \{z \in X : f'(z) = 0\} \to Y \quad \text{is unbranched.}$$

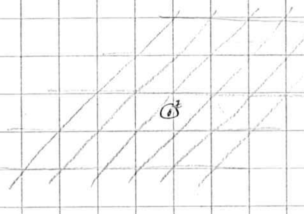
We can use this idea to prove the Riemann-Hurwitz formula:

(Euler characteristic)

$$\chi(X - f^{-1}f\{z \in X : f'(z) = 0\}) = \deg(f) \cdot \chi(Y - f(z \in X : f'(z) = 0\}).$$

ex $\mathbb{C} \to \mathbb{C}/_{\Gamma}$ = elliptic curve

locally just $z \mapsto z$ $\Rightarrow$ unbranched.



Theorem 4.6 Suppose X is RS, Y is Hausdorff and $f: Y \to X$ is a local homeomorphism, then Y is a RS. (Can be made into)

Proof Local homeomorphism: $\forall y \in Y$, $\exists y \in \overset{\text{open}}{U} \subseteq Y$ s.t. $f: U \xrightarrow{\sim} f(U) \subseteq X$. (homeomorphic)

Now if we choose U very small, we can make it so that $f(U)$ is a local chart for X:

The RS structure on Y is pulled back from that on X.

$$\begin{array}{ccc} \overset{\subseteq Y}{U} & \xrightarrow{\sim} & f(U) \subseteq X \\ & \searrow & \downarrow \wr \\ & & W \subseteq \mathbb{C} \end{array}$$

Def 4.11 $f: Y \to X$ (T: X, Y Hausdorff, book: RS) continuous is a covering map if

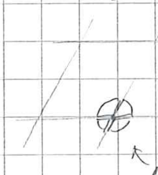
$\forall x \in X$, $\exists x \in \overset{\text{open}}{U} \subseteq X$ s.t. $f^{-1}(U) = \bigsqcup_{i \in I} V_i$ (infinite ok) so that $\overset{\text{open}}{V_i} \subseteq Y$

$f: V_i \to U$ is a homeomorphism

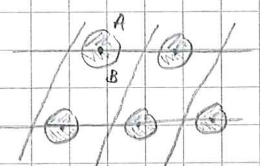
In particular: (1) f must be surjective
(2) f is a local homeomorphism.

Def A covering map $f: Y \to X$ (✓ path connected) is a universal covering if Y is simply connected.

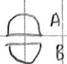
ex $\mathbb{C} \to \mathbb{C}/_{\Gamma}$ is a universal covering
(simply connected; path connected; ψ z)



disjoint unions.



$A \cup B$ is an open neighborhood of z in $\mathbb{C}/_{\Gamma}$.



B not open in $\mathbb{C}$, but $A \cup B$ is.

exercise Check that $\mathbb{D} \to \mathbb{D}\setminus\{0\}$
$z \mapsto e^{-\left(\frac{1+z}{1-z}\right)}$ is a universal covering

non-ex $\mathbb{C}/_{\Gamma}$

Weierstrass $\wp$-function $\wp_\Gamma$: $\mathbb{C} \to \mathbb{C}$ meromorphic

$$z \longmapsto \wp_\Gamma(z) = \left(\sum_{w \in \Gamma} \frac{1}{(z-w)^2}\right) - \left(\sum_{w \in \Gamma - 0} \frac{1}{w^2}\right)$$

periodic with periods in $\Gamma = \{m_1 e_1 + m_2 e_2 : m_1, m_2 \in \mathbb{Z}\}$, $e_1, e_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent.

induces a holomorphic function

$\hat{\wp}: \mathbb{C}/_{\Gamma} \to \mathbb{P}^1$ ← branched map
(elliptic curve)

This is two to one: $\hat{\wp}^{-1}(\infty) = [0] \in \mathbb{C}/_{\Gamma}$ with multiplicity 2,

so $[0]$ is a branched point.

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$$\begin{array}{ccccc} \mathbb{C} & & \mathbb{C}/\Gamma & \longrightarrow & \mathbb{P}^2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ z & \longrightarrow & [z] & \longmapsto & [\underset{x}{\wp_\Gamma(z)} : \underset{y}{\wp_\Gamma'(z)} : \underset{z}{1}] \end{array}$$

$$\mathbb{C}^2 \supseteq \{y^2 = x^3 + ax + b\} \ni (x,y) \mapsto x$$

to find preimage:

(x_0, y) so that $y^2 = x_0^3 + ax_0 + b$ 2:1 map

4.7 Lifting of mappings

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ X & Z \arrow[l, "f"] \arrow[u, dashed, "\exists F?"'] \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[d] \\ X & Z \arrow[l] \arrow[ul, bend right, "\exists ?"'] \end{tikzcd}$$

4.8 Uniqueness of lifting

Theorem If X, Y Hausdorff, p local homeomorphism, f continuous, Z connected then a lifting, if it exists, is unique.

(If $\exists F$, then F is unique (there is at most one lifting))

Proof Locally, $Y \cong X$ because p is a local homeomorphism.

$$\begin{tikzcd} Y \supseteq U \arrow[d, "p"', "?"] \\ X \supseteq V & f^{-1}(V) \arrow[l] \arrow[ul, dashed, "F_1, F_2"'] \end{tikzcd}$$

If $\exists F_1, F_2$ liftings, then

$F_1 \circ p = f = F_2 \circ p$

$F_1 = \underbrace{(F_1 \circ p) \circ p^{-1}}_{\text{because } p \text{ homeomorphism}} = (F_2 \circ p) \circ p^{-1} = F_2$.

Theorem 4.9 X is RS, Y is with pull back also RS structure, f is holomorphic

$\implies$ F (if it exists) is also holomorphic.

HA74800 19.09.19.

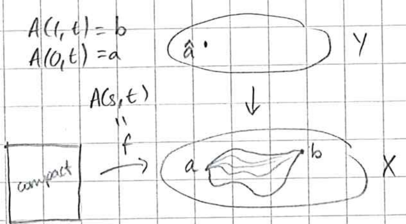
Theorem 4.10

Lifting of Homotopic Curves

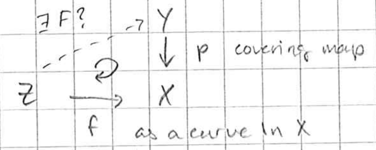
Assume X, Y Hausdorff
$p: Y \to X$ local homeomorphism

(ex: p covering map)

$A(1,t) = b$
$A(0,t) = a$



$Z = [0,1] \times [0,1]$



$f(\hat{a}) = a$

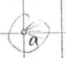
If every curve $u_s \subset X$ can be lifted to a curve $\hat{u}_s$ with $\hat{u}_s(0) = \hat{a}$ $\forall t \in [0,1]$ then $\hat{u}_0(1) = \hat{u}_1(1)$ and are homotopic.

Proof

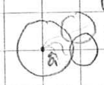
Define $\hat{A} : I \times I \to Y$ where $\hat{A}(s,t) = \hat{u}_s(t)$.
We need to show that $\hat{A}$ is continuous and $\hat{A}(s,1) =$ a constant $\hat{b}$ where $p(\hat{b}) = b$.

By assumption, there is an open set $\hat{U}$ of Y so that
$P|_{\hat{U}} : \hat{U} \xrightarrow{\sim} p(\hat{U}) =: U$
homeomorphism

lift? p^{-1} p

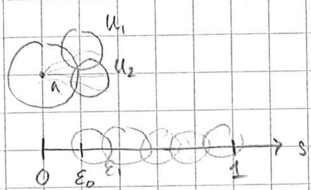


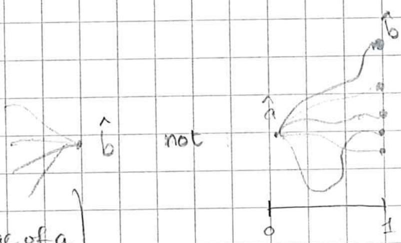
Apply the inverse of p on $\hat{U}$, we get that $\hat{A}$ is continuous on $(s,t) \in [0,1] \times [0,\varepsilon_0)$ for some $\varepsilon_0 > 0$



$\hat{A}(s,t)$ is continuous on $t \in [0, \varepsilon_0$

$\hat{A}(s,t)$ is continuous $\forall s,t \in [0,1]$

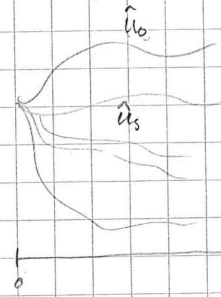




Now need to show

(Can glue them together because $[0,1] \times [0,1]$ is compact and the image of a compact set is compact.)

First, $p^{-1}(b)$ is a discrete set (every point is isolated) because p is a local homeomorphism, $\forall \hat{b} \in p^{-1}(b)$ $\exists \hat{b} \in \hat{U}$ open so that $p|_{\hat{U}} : \hat{U} \xrightarrow{\sim} U$ is homeomorphism.
In particular, $p^{-1}(b) \cap \hat{U} = \{\hat{b}\}$.



$\hat{A}(1 \times [0,1]) \subseteq p^{-1}(b)$
(connected) (discrete)

Theorem: The image of a connected space under a continuous map is connected.
⤳ only one point because

if $\hat{b}_1 \in \hat{A}(1 \times [0,1])$

$\Rightarrow \hat{A}(1 \times [0,1]) = \hat{b}_1 \cup (C \setminus \hat{b}_1) \Longrightarrow C \setminus \hat{b}_1 = \emptyset$

($\hat{b}_1$: open & closed bc. Hausdorff $p^{-1}(b)$; $C \setminus \hat{b}_1$: open)

so $\hat{A}(1 \times [0,1]) = \{\hat{b}_1\}$ one point set

Def A continuous map $p: Y \to X$ satisfies the *lifting of curves property* if:

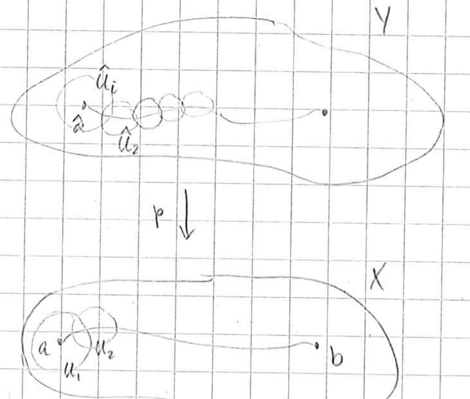
Given $u: [0,1] \to X$ a curve

$\hat{a} \in p^{-1}(u(0))$

then $\exists\, \hat{u}: [0,1] \to Y$ lifting u and $\hat{u}(0) = \hat{a}$.

Remark This property allows us to fulfill the assumption in Thm. 4.10.

Theorem 4.14 Every covering map $p: Y \to X$ of topological Hausdorff (Tuyen, not book) spaces X and Y has the *curve lifting property*.
Moreover the lift is unique.



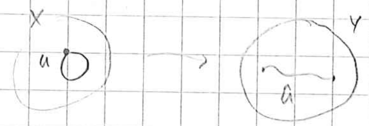
$p^{-1}(u) = \coprod_{i \in I} \hat{u}_i$

with $p|_{\hat{u}_i}: \hat{u}_i \xrightarrow{\sim} u$ $\forall i \in I$

Choose $\hat{u}_1$ to be the unique $a \in \hat{u}_1$.

$[0,1]$ compact so only need finitely many $u_i \overset{p^{-1}}{\mapsto} \hat{u}_i$.

Remark If the curve $u: [0,1] \to X$ is closed ($u(0) = u(1)$), it may happen that $\hat{u}$ is not closed. ($\hat{u}(0) \neq \hat{u}(1)$)



ex $p: \mathbb{C}^* = \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0$

$z \mapsto z^2$

p is unbranched: $p'(z) = 2z \neq 0 \quad \forall z \in \mathbb{C}^*$

p is a covering map

What is the lift of the unit circle φ (closed curve)?

$\varphi(t) = \{z = e^{2\pi i t}\} \quad t \in [0,1]$ $\qquad \hat{\varphi}(0) = 1$
$\qquad \hat{\varphi}(1) \neq 1$ because

$\psi(t): [0,1] \to \mathbb{C}^*$, $\psi(t) = e^{\pi i t}$, then

$\psi(0) = 0$
$p \circ \psi = \varphi$ so ψ is a lift of φ. And we know that a lift with a given starting point, if it exists, is unique.

So $\psi = \hat{\varphi}$ and $\psi(1) = e^{\pi i 1} = -1 \neq 1 = \psi(0)$ ∴ not closed curve

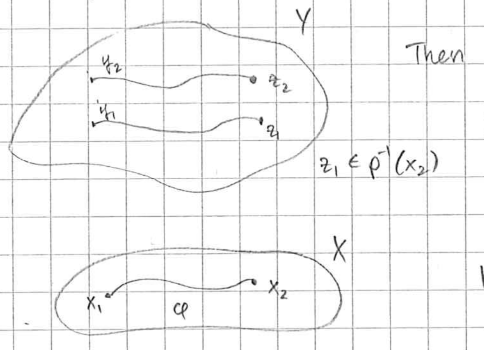
Remark 2 If $p: Y \to X$ is not a covering map, only a local homeomorphism then a lift $\hat{\varphi}$ of φ with $\hat{\varphi}(0)$ given, may not exist.

Proof Let $X \in \mathbb{C}^* \setminus \{1\}$. $X \overset{open}{\subseteq} \mathbb{C} \setminus \{1\}$, then $p: X \to \mathbb{C}^*$, $z \mapsto z^2$ is a local homeomorphism,

but there is no lift $\hat{\varphi}$ $\varphi(t) = e^{2\pi i t}$ with $\hat{\varphi}(0) = 1$.

Thm 4.16 X, Y Hausdorff, X path-connected, $p: Y \to X$ covering map.
Then $\forall x_1, x_2 \in X$ $\exists$ an isomorphism
$$p^{-1}(x_1) \xrightarrow{\sim} p^{-1}(x_2)$$

Proof $p^{-1}(x_1) = y_1, y_2, \ldots$



$\exists \varphi$ because X path-connected.

Then $z_1 \neq z_2$ if $y_1 \neq y_2$ because of uniqueness of lifting.

Define map $p^{-1}(x_1) \to p^{-1}(x_2)$
$y_1 \mapsto z_1$
$y_2 \mapsto z_2$ so injective

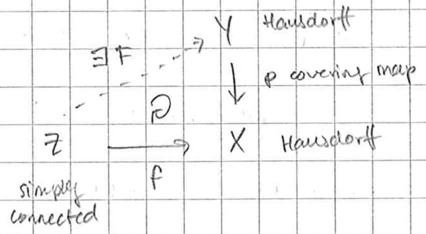
Injective because if $z = p^{-1}(x_2)$ $\exists y = p^{-1}(x_1)$

ex $p: X \to \mathbb{P}^1$ (X: any compact RS)

Thm 4.17 (Generalisation of 4.16, so similar proof). X, Y Hausdorff, $p: Y \to X$ ~~covering map~~ need only p local homeomorphism with curve lifting property.
Replace curves by maps.

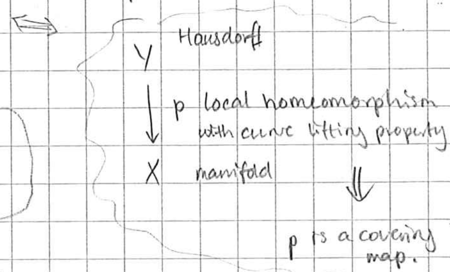
$Z = [0,1]$, $\pi_1(Z) = 0 \Rightarrow Z$ is simply connected.
Let $\varphi : [0,1] \to Z$ a curve

$A(s,t) = s\varphi(t)$.



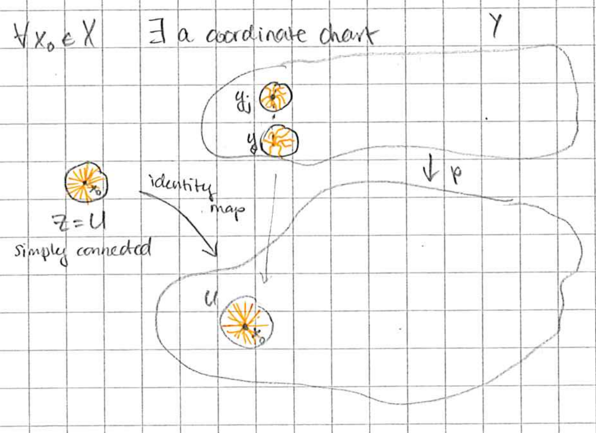
Converse of thm 4.14:

Theorem 4.19 Suppose X is a manifold, Y Hausdorff and $p: Y \to X$ local homeomorphism with the curve lifting property. Then p is a covering map.



X is a manifold, so $\forall x_0 \in X$ $\exists$ a coordinate chart

$x_0 \in U \subseteq X$
$\cong$ ball in $\mathbb{R}^n$, $n = \dim X$



Weierstrass function

$$\wp_\Gamma(z) = \sum_{\omega\in\Gamma} \frac{1}{(z-\omega)^2} - \sum_{\omega\in\Gamma\setminus 0} \frac{1}{\omega^2}$$

$$\wp'_\Gamma(z) = -\sum_{\omega\in\Gamma} \frac{3}{(z-\omega)^3}$$

$\wp_\Gamma : \quad E = \mathbb{C}/\Gamma \to \mathbb{C}$ meromorphic and hence

$\hat{\wp}_\Gamma : \quad \mathbb{C}/\Gamma \to \mathbb{P}^1$ holomorphic

if $z \notin \Gamma$: $\hat{\wp}_\Gamma([z]) := [\wp_\Gamma(z) : 1]$, with $[z] \in \mathbb{C}/\Gamma$, $\wp_\Gamma(z) \in \mathbb{C}$

Note any meromorphic map φ can be made into a holomorphic map $\hat{\varphi}$ onto $\mathbb{P}^1$ by mapping the poles to $[1:0]$? and the rest

If $z_0 \in \Gamma \Rightarrow [z_0]$ is the same point in $\mathbb{C}/\Gamma$.
What is the point $\hat{\wp}_\Gamma([z_0])$ then?
$\hat{\wp}_\Gamma([z_0]) = [\wp_\Gamma(z_0) : 1]$

If $z_0 \in \Gamma \Rightarrow \wp_\Gamma(z_0) = \infty$

Formally, then $\hat{\wp}_\Gamma([z_0]) = [\wp_\Gamma(z_0) : 1] = [\infty : 1]$ "=" $[\frac{\infty}{\infty} : \frac{1}{\infty}] = [1:0]$

↓ formally:

$\exists z_n \in \mathbb{C}/\Gamma$ and $z_n \to z_0$ and $\lim_{n\to\infty} \wp_\Gamma(z_n) = \infty$ (because z_0 pole). Then by def:

$$\hat{\wp}_\Gamma([z_0]) \overset{def}{=} \lim_{n\to\infty} \hat{\wp}_\Gamma([z_n]) \overset{def}{=} \lim [\wp_\Gamma(z_n) : 1] = \lim_{n\to\infty} [1 : \frac{1}{\wp_\Gamma(z_n)}] = [1:0$$

Weierstrass embedding $z \in \mathbb{C}/\Gamma$:

$\varphi_\Gamma([z]) = [\wp_\Gamma(z) : \wp'_\Gamma(z) : 1]$, with $[z] \in \mathbb{C}/\Gamma$

($\wp_\Gamma$: pole of order $n = 2$; $\wp'_\Gamma$: pole of order 3)

$\hat{\wp}_\Gamma : \quad E = \mathbb{C}/\Gamma \hookrightarrow \mathbb{P}^2$. If $z_0 \in \Gamma$, what is $\hat{\wp}_\Gamma([z_0])$?

embedding $\mathbb{C}/\Gamma \subseteq \mathbb{P}^2$

If f has a pole of order n at z_0, then f' has a pole of order $n+1$ at z_0

$$\left[\frac{1}{z^2} : -\frac{3}{z^3} : 1\right] \xrightarrow{z\to 0} \left[\frac{\frac{1}{z^2}}{-\frac{3}{x^3}} : 1 : \frac{1}{-\frac{3}{x^3}}\right] = [0:1:0].$$

4.20 **Proper mappings:**

$f: X \to Y$ continuous is *proper* if $\forall K \subseteq Y$ compact, then $f^{-1}(K)$ is also compact.

ex $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto z^2$ is proper:

$K \subseteq \mathbb{C}$ compact $\iff$ K is closed and bounded.
So there is $R > 0$ s.t. $\forall z \in K$: $|z| \leq R$. K is closed, f continuous $\Rightarrow f^{-1}(K)$ is closed.

$w \in f^{-1}(K) \Rightarrow \exists z \in K$ s.t. $w^2 = f(w) = z \Rightarrow |w|^2 = |z| \leq R$

$\Rightarrow |w| \leq \sqrt{R}$. Hence $f^{-1}(K)$ is bounded $\Rightarrow f^{-1}(K)$ compact.

non-ex. $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto e^z$ is not proper:
↙ infinite, usually not proper

$f^{-1}(\{0\}) = 2\pi i \mathbb{Z}$ — $\{0\}$ compact, $2\pi i \mathbb{Z}$ not compact.

HW If X is a not compact R.S., $f: X \to Y$ (R.S.) is holomorphic, nonconstant, and $\exists y_0 \in Y$ so that no $f^{-1}(\{y_0\})$ is infinite
$\Rightarrow f$ is not proper.

Proof: Hint: Use Identity thm for holomorphic maps.

Lemma 4.21 $p: Y \to X$ proper discrete. X, Y locally compact, then

(i) For every $x \in X$, $p^{-1}(x)$ is finite

(ii) If $x \in X$ and $p^{-1}(x) \subseteq V \subseteq Y$ (V open)

then $\exists x \in U \subseteq X$ (U open) so that $f^{-1}(U) \subseteq Y$.

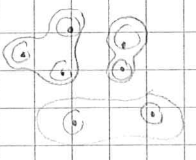
Def X is locally compact if $\forall x \in X$ $\exists x \in U \subseteq X$ (U open) so that $\overline{U}$ (closure of U) is compact

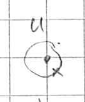
ex: X is a Riemann surface.

Def Discrete map $\forall x$ $f^{-1}(x)$ is discrete.

ex: Non-constant holomorphic maps.

Proof —

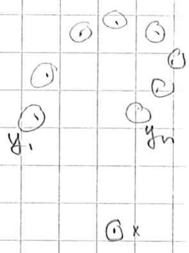

$p^{-1}(x)$ finite points



Theorem 4.22a (Criterion for covering maps)

X, Y locally compact, $p: Y \to X$ is a proper local homeomorphism, then p is a covering map

Proof A local homeomorphism is discrete.
Now consider $x \in X$. Choose small neighborhoods V of $p^{-1}(x)$ as follows:



$V = \bigsqcup_{i=1}^{n} B_i$, ($B_i$ open) $B_i \ni y_i$ where $p^{-1}(x) = \{y_1, \ldots, y_n\}$

B_i small enough so that $p|_{B_i}: B_i \xrightarrow{\simeq} p(B_i)$.

Then choose U by lemma 4.21 s.t. $p^{-1}(U) \subseteq V$.

Let $V_i = p^{-1}(U) \cap B_i \Rightarrow p|_{V_i}: V_i \xrightarrow{\simeq} p(V_i) = C$

important ex X, Y compact R.S. $f: X \to Y$ (proper) holomorphic, non-constant
f is surjective by open mapping property.

Let C(f) be the critical points of f, $= \{x \in X \; f'(x) = 0\}$.

Then $B(f) = f(C(f))$ = branch points of f.

$f: X - f^{-1}(B(f)) \to Y - B(f)$ is proper and local homeomorphism.

By thm 4.22:
$f: X \setminus f^{-1}(B(f)) \to Y \setminus B(f)$ is a covering map.

By lemma 4.21
$\forall y \in Y \setminus B(f)$ the no. $f^{-1}(y)$ is finite.

Actually this is a constant, is call degree of f.

Then for the Euler char
$$\chi(X - f^{-1}(B(f))) = d \cdot \chi(Y - B(f)).$$

Remark $\chi(X \sqcup F) = \# F + \chi(X)$ (F: finite set; #F: no. points in F) ⟶ can use this to prove Riemann-Hurwitz formula.

Theorem 4.24 X, Y R.S. $f: X \to Y$ proper nonconstant holomorphic map. Then $\exists d \in \mathbb{N}$ (degree of f) s.t. $\forall y \in Y$
the number of $f^{-1}(y) = d$ counting multiplicity.

ex $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto z^2$ proper, holomorphic, non-constant. $C(f) = \{x \in X : f'(x) = 0\} = \{0\}$
$B(f) = f(C(f)) = f(0) = 0 \quad f^{-1}(B(f)) = f^{-1}(0) = 0$. So

$f: \mathbb{C} - f^{-1}(B(f)) \to \mathbb{C} - B(f)$ is a covering map.
($f^{-1}(B(f)) = \{0\}$, $B(f) = \{0\}$)

$y \in \mathbb{C} \setminus B(f)$: $f^{-1}(y)$ has exactly 2 points $x_1, x_2 = -x_1$, so $\deg(f) = 2$

$f^{-1}(0) = \{z^2 = 0\} = \{0\}$ with multiplicity 2.

B(f) "branch points" or "critical value"

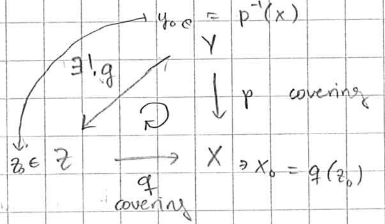
MAT4800 26.09.19

Final exam: 4th December (not 3rd)

Mandatory: Send presentation plan (which theorem) by 15^{th} October.
State theorem, sketch proof, give example 10 min?

§ 5 Universal coverings

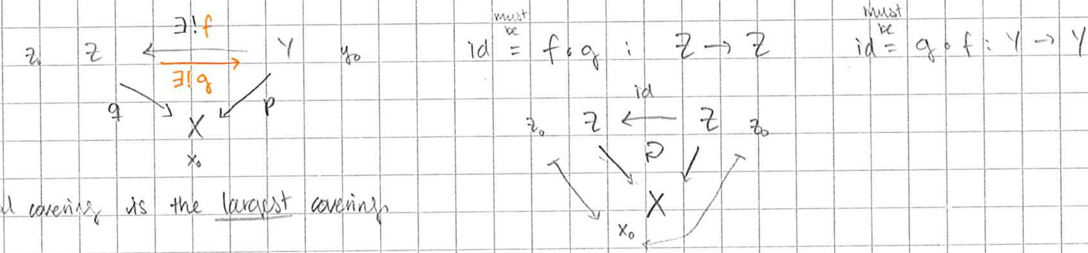
Def A *universal covering* is a covering map which has a certain universal property.



ex $f(z) = e^z$; $\mathbb{C} \to \mathbb{C} \setminus \{0\}$
universal ↖ simply connected

ex $f: \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$, $z \mapsto z^2$ covering,
not simply connected — but not universal

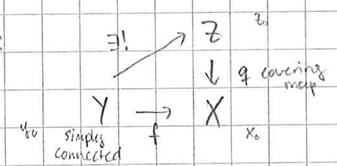
Remark If a universal covering exists, it is unique.



Remark The universal covering is the largest covering.

Theorem 5.2 X, Y connected manifolds, Y simply connected and $p: Y \to X$ covering map,
then p is a universal covering of X.

Proof Thm 4.17:



□

Theorem 5.3 (Another characterisation of universal covering)
X connected manifold $\Rightarrow \exists\, Y$ simply-connected manifold and a universal covering map $p: Y \to X$.
(we only care about manifolds. — path-connected topological spaces)

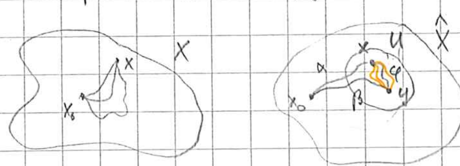
Proof Pick $x_0 \in X$. Path space of (X, x_0). $\hat{X} = \{(x, [\alpha]) : x \in X, \alpha$ curve in X with $\alpha(0) = x_0, \alpha(1) = x\}$ (homotopy class)

$p: \hat{X} \to X$
$(x, [\alpha]) \mapsto x$

To show that p is universal covering, we show (i) $\hat{X}$ simply connected and (ii) that p is a covering map.

(i) Topology on $\hat{X}$ (what sets of $\hat{X}$ are open): If $U \subseteq X$ is open, $x \in U$ and $[\alpha]$ homotopy class of a curve $x_0 \overset{\alpha}{\to} x$ then we define a set $(U, [\alpha]) \subseteq \hat{X}$ as follows:

$[U, \alpha]$ contains points of the form (y, β) where $y \in U$ and β is the homotopy class of the glue of α and any curve φ from x to y, completely contained in U.
$\beta = [\varphi \circ \alpha]$



Remark: If we choose another curve φ' between x and y contained inside U, then $\varphi \simeq \varphi'$ (homotopic) (because U simply connected by def).

$\Rightarrow \beta' = [\varphi' \circ \alpha] = [\varphi \circ \alpha]$, hence $(y, \beta) = (y, \beta')$

We declare that such $[U, \alpha]$ is an open set for $\hat{X}$. cont →

Theorem 5.3

Def: A subset U of $\hat{X}$ is open $\iff$ $U = \bigcup_{i \in I} [U_i, \alpha_i]$.

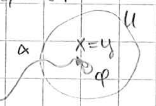
Need to check that this is topology on $\hat{X}$. ($\emptyset$ and $\hat{X}$ are of that form)

Important fact: If X is a manifold, then any open set $\tilde{U} \subseteq X$ has the form $\tilde{U} = \bigcup_{i \in I} U_i$, U_i simply-connected.

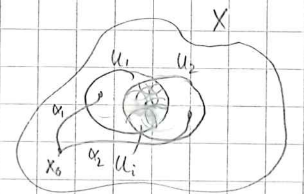
- $\emptyset = [\emptyset, \alpha]$
- $\hat{X} = \bigcup_{i \in I} [U_i, \alpha_i]$ proof:

Choose $(x, \alpha) \in \hat{X}$. choose any $x \in U \subseteq X$ (open, simply connected). Then $(x, \alpha) \in [U, \alpha]$.

If $y = x$, $\alpha = \beta$ since φ is the constant curve



Claim: $[U_1, \alpha_1] \cap [U_2, \alpha_2] = \bigcup_{i \in I} [U_i, \alpha_i]$



$U = U_1 \cap U_2 = \bigcup_{i \in I} U_i$ (simply connected, open.)

Note that $p : [U, \alpha] \to U$ is homeomorphism.

$\hat{U}_1 = p|_{U_1} = p^{-1}(U_i) = [U_i, \beta_i]$

For each U_i pick $x_i \in U_i$
$\rightsquigarrow$ curve α_i from $x_0 \to x_i$

$p : \hat{X} \to X$ is continuous:

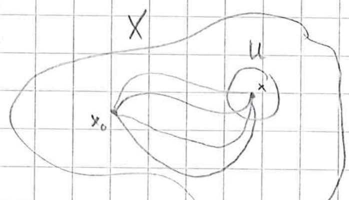
$\Updownarrow$

$\forall\, U \subseteq X$ open then $p^{-1}(U) \subseteq \hat{X}$ is open
$\quad\quad p^{-1}(U) = \bigcup [U_i, \alpha_i]$

Because U = union of simply connected open sets, we can assume that U is simply connected.

Choose a point $x \in U$. $\pi_1(x_0, x)$ = homotopy class of curves in X going from x_0 to x.

$p^{-1}(U) = \bigcup_{\alpha \in \pi_1(x_0, x)} [U, \alpha]$ open, so p is continuous.



p is a covering map:

$p^{-1}(U) = \bigcup_{\alpha \in \pi_1(x_0, x)} [U, \alpha]$

simply connected $\Rightarrow$ only one homotopy class.
$\Downarrow$
$p^{-1}(U)$ is a disjoint union.

Firstly, we saw that $p : [U, \alpha] \xrightarrow{\sim} U$

Secondly, if α, α' are not homotopic, then $[U, \alpha] \cap [U, \alpha'] = \emptyset$

If $(y, \beta) \in [U, \alpha] \cap [U, \alpha']$ $\quad \beta = [(x \to y) \circ \alpha] = [(x \to y) \circ \alpha']$

$\Rightarrow [(x \to y^{-1})] \circ [(x \to y) \circ \alpha] = [(x \to y^{-1})] \circ [(x \to y) \circ \alpha']$

$\alpha = \alpha'$

↑ contradiction.

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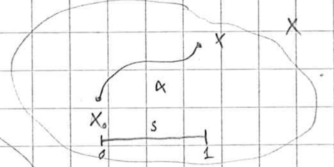
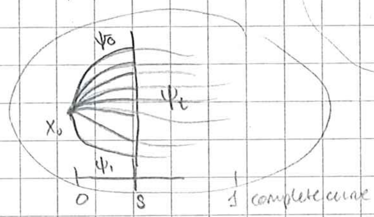
Theorem 5.3 Proof cont. $\hat{X}$ is simply connected:

Let $\psi : [0,1] \to \hat{X}$ be a curve that is homotopic to the constant curve $t \mapsto (x_0, [x_0 \to x_0])$, where $\alpha_0(t) = [x_0 \to x_0]$

α curve on X, then $\hat{\alpha}(s) = (\alpha(s), \alpha_s(t) = \alpha(st))$



$\hat{\psi}(s) = (\psi_t(s), t \mapsto \psi_t(st))$



$\mathbb{P}^2 = \mathbb{C}^3 \setminus \{(0,0,0)\} / \sim$ where $\sim$ is the equivalence relation $(x,y,z) \sim (x',y',z')$ if $\exists \lambda \in \mathbb{C} \setminus \{0\}$ s.t. $(x,y,z) = \lambda(x',y',z')$.

$\mathbb{P}^2$ is the space of lines in $\mathbb{C}^3$ going through $(0,0,0)$.

Homogeneous coordinates

$[x:y:z]$ $\qquad (x,y,z) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$

$\qquad\qquad (x,y,z) \mapsto [x:y:z]$, $\quad \downarrow$ quotient map $\mathbb{P}^2$

Coordinate charts:

$U_1 = \{[x:y:z] : x \neq 0\} = \{[1 : \frac{y}{x} : \frac{z}{x}] : x \neq 0,\ x,y,z \in \mathbb{C}\} \cong \mathbb{C}^2$

$U_2 = \{ \quad y \neq 0\} = \{[\frac{x}{y} : 1 : \frac{z}{y}] : y \neq 0,\ x,y,z \in \mathbb{C}\} \cong \mathbb{C}^2$

$U_3 = \{ \quad z \neq 0\} = \{ \quad \} \cong \mathbb{C}^2$

$\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$

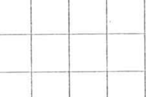
$(x,y) \longmapsto [x:y:1]$

$\mathbb{P}^2 \setminus \mathbb{C}^2 = \{[x:y:z] : z = 0\}$
$= \{[x:y:0] : (x,y) \in \mathbb{C}^2 \setminus \{(0,0)\}\}$
$\cong \mathbb{P}^1$ hyperplane (line) at infinity.
(one less dimension)

a curve in $\mathbb{C}^2$ is of the form:
$\mathcal{C} = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$

ex
(1) $\mathcal{C} = \{x^2 + y^2 - 1 = 0\}$ $\to$ smooth
(2) $\mathcal{C} = \{x^2 + xy + y^3 = 0\}$ $\to$ not smooth

Roughly speaking, a smooth curve is a curve where at every point you can define a unique tangent.



no unique tangent

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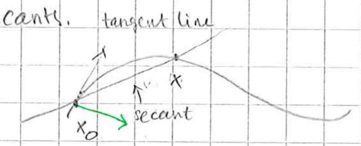
Affine curves ($\mathbb{C}^2$ affine)

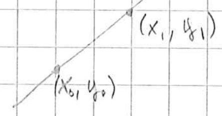
Remark: affine curves never compact.

How to compute tangent? Tangent = limit of secants.



The tangent line, if it exists, has this normal vector

$$\left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)$$



line has equation

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$\mathcal{C} = \{(x, y) : P(x, y) = 0\}$ (polynomial)

$\mathcal{C}$ smooth $\iff$ normal vector nonzero, for all $(x, y) \in \mathcal{C}$.

$\Updownarrow$

$\forall (x_0, y_0)$ $\left(\frac{\partial P}{\partial x}(x_0, y_0), \frac{\partial P}{\partial y}(x_0, y_0)\right) \neq (0,0)$

smooth = non-singular

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Otherwise we say that $\mathcal{C}$ is singular and the set of singular points on $\mathcal{C}$ is

$$\mathcal{C}_{sing} = \{(x, y) \in \mathcal{C} : \vec{n} = \left(\frac{\partial P}{\partial x}(x, y), \frac{\partial P}{\partial y}(x, y)\right) = (0, 0)\}.$$

ex $\mathcal{C} = \{x^2 + y^2 - 1 = 0\}$

On $\mathbb{R}^2$, $\mathcal{C}$ is the unit circle and hence a compact real manifold of dim. 1.
On $\mathbb{C}^2$: $\mathcal{C}$ is not compact because it's unbounded.

$C_{sing} = \{(x_0, y_0) \in \mathbb{C}^2 : (2x_0, 2y_0) = (0, 0)$ and $x^2 + y^2 - 1 = 0\} = \emptyset$, so $\mathcal{C}$ smooth.

ex $\mathcal{C} = \{x^2 + y^2 = 0\}$

$\mathcal{C}_{sing} = \{(x_0, y_0) : (2x_0, 2y_0) = (0, 0)$ and $x^2 + y^2 = 0\} = \{0, 0\}$ not smooth.

ex $\mathcal{C} = \{y^2 = x^3 + 1\}$ $P(x, y) = x^3 + 1 - y^2$

$\mathcal{C}_{sing} = \emptyset$ smooth.

Theorem $\hat{\mathcal{C}} \subseteq \mathbb{P}^2 = U_1 \cup U_2 \cup U_3$
$\Rightarrow C_1 = \hat{\mathcal{C}} \cap U_1$, $C_2 = \hat{\mathcal{C}} \cap U_2$, $C_3 = \hat{\mathcal{C}} \cap U_3$ are open subsets of $\hat{\mathcal{C}}$. $\hat{\mathcal{C}}$ is smooth iff C_1, C_2 and C_3 are smooth.

Projective curves (homogenisation)

ex $C = \{x^2 + y^2 = 1\} \subseteq \mathbb{C}^2$

$\Updownarrow$

$\left(\frac{u}{w}\right)^2 + \left(\frac{v}{w}\right)^2 = 1$

$\Updownarrow$

$\hat{\mathcal{C}} = \{u^2 + v^2 = w^2, w \neq 0\} \subseteq \mathbb{P}^2$,
closed in $\mathbb{P}^2$ which is compact, so also compact.

$\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$
$(x, y) \mapsto [x : y : 1]$
$\left(\frac{u}{w}, \frac{v}{w}\right) \leftarrow [u : v : w]$, $w \neq 0$

$\mathcal{C} = \{y^2 = x^3 + 1\}$

$\left(\frac{v}{w}\right)^2 = \left(\frac{u}{w}\right)^3 + 1$

$\left(\frac{v}{w}\right)^2 - \left(\frac{u}{w}\right)^3 = 1 = \frac{v^2}{w^2} - \frac{u^3}{w^3} = 1 \quad \Big| \cdot w^3$

$\hat{\mathcal{C}} = \{v^2 w - u^3 - w^3 = 0\}$

ex **Dehomogenisation** $\hat{\mathcal{C}} = \{[X : Y : Z] \in \mathbb{P}^2 : X^2 + Y^2 - Z^2 = 0\}$

$\mathbb{P}^2 = U_1 \cup U_2 \cup U_3$

$Z \neq 0$ let $Z = 1$

$\hat{C} \cap U_1$: $\mathcal{C}_1 = \{X^2 + Y^2 - 1 = 0\} = \{x^2 + y^2 - 1 = 0\} \subseteq U_1 \simeq \mathbb{C}^2_{x,y}$
$\hat{C} \cap U_2$: $\mathcal{C}_2 = \{X^2 + 1 - Z^2 = 0\} = \{x^2 + 1 - z^2 = 0\} \subseteq U_2 \simeq \mathbb{C}^2_{x,z}$
$\hat{C} \cap U_3$: $\mathcal{C}_3 = \{1 + Y^2 - Z^2 = 0\} = \{1 + y^2 - z^2 = 0\} \subseteq U_3 \simeq \mathbb{C}^3_{y,z}$

easier to do computations for affine curves.

$\hat{C} \cap U_1$: $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ let $Z = 1$ and simplifies to what done above.

§5 Universal coverings

$\mathrm{Deck}(Y \xrightarrow{p} X)$

5.4 **Def** $p: Y \to X$ covering map. $\mathrm{Deck}(Y/X) = \left\{ \begin{array}{c} Y \xrightarrow{f} Y \\ p_1 \searrow \circlearrowleft \swarrow p_2 \\ X \end{array} \right\}$ deck transformation.

This is the set of homeomorphisms of Y that preserves the fibres of p

$$f(p^{-1}(x)) \subseteq p^{-1}(x) \quad \forall x \in X$$

(fiber: $p^{-1}(x)$)

and since f is a homeomorphism, $f(p^{-1}(x)) = p^{-1}(x)$.

Remarks
- $\mathrm{Deck}(Y/X)$ is a subgroup of homeomorphisms of Y, $\mathrm{Hom}(Y)$.
- If X, Y are complex manifolds, then $\mathrm{Deck}(Y/X)$ is a subgroup of $\mathrm{Aut}(Y)$ holomorphic automorphism.

5.5 **Def** X, Y connected Hausdorff spaces. $p: Y \to X$ covering map is Galois if $\forall y_1, y_2 \in Y$ with $p(y_1) = p(y_2)$ there exists a covering transformation $f: Y \to Y$ s.t. $f(y_1) = y_2$.

ex $\mathbb{C}^* = \mathbb{C} \setminus \{0\} \xrightarrow{p} \mathbb{C}^*$, $z \longmapsto z^k$, $k \in \mathbb{N}$. This is Galois.

If $y_1, y_2 \in Y$ and $p(y_1) = p(y_2)$, $y_1^k = x = y_2^k$

$\Rightarrow \exists\, \omega$ a k-roots of unity $(\omega^k = 1)$ s.t. $\omega y_1 = y_2$.

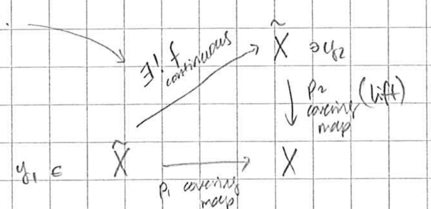
Look at the map $f: \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto \omega z$. $f \in \mathrm{Deck}(\mathbb{C}^* \xrightarrow{p} \mathbb{C}^*)$, $f(y_1) = y_2$.

p is unbranched proper $\xRightarrow{\text{Thm } 4.2.2}$ p is a covering map.

5.6 **Theorem** X connected manifold, $p: \tilde{X} \to X$ (simply connected) is the universal covering $\Longrightarrow$ p is Galois and

$$\pi_1(X) \cong \mathrm{Deck}(\tilde{X} \xrightarrow{p} X).$$

Proof Thm 5.3, 4.17.



Check: $f \in \mathrm{Hom}(\tilde{X}) \Rightarrow p$ Galois.

Now show that

$\pi_1(X, x_0) = \mathrm{Deck}(\tilde{X}/X)$

Now define

$\phi: \mathrm{Deck}(Y \xrightarrow{p} X) \to \pi_1(X, x_0)$ as

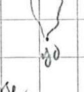
$y_0 \in p^{-1}(x_0)$. For each $y \in p^{-1}(x_0)$ because p is Galois,

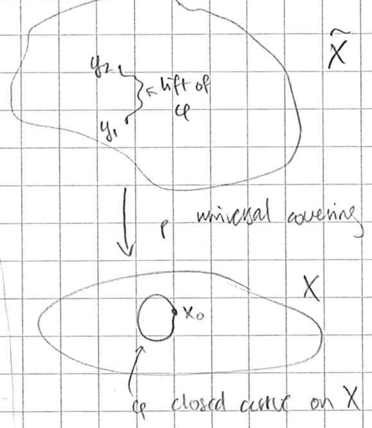
$\exists!\, f_y \in \mathrm{Deck}(Y \xrightarrow{p} X)$ s.t. $f_y(y_0) = y$.

Choose v_{y, y_0} curve in $\tilde{X}$ going through y_0 and $f_y(y_0)$

$[v_{y, y_0}]$ is indepent of the choice of v because

$\tilde{X}$ is simply-connected. $\phi(f_y) = [p \circ v_{y, y_0}]$ closed curve.





So $\pi_1(X, x_0) \cong p^{-1}(y)$

Check ϕ group homomorphism. Now ϕ has inverse

Proof (continued) Need to construct inverse:

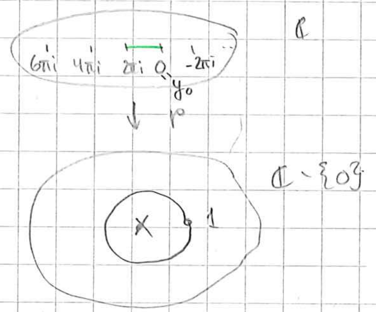
$$\phi^{-1}: \pi_1(X, x_0) \to \mathrm{Deck}(Y \xrightarrow{p} X)$$

Choose closed curve φ. Then $\exists$ a unique lift going through y_0 and some other $y \in p^{-1}(x_0)$.

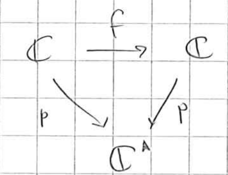
Since p is Galois, $\exists!\, f \in \mathrm{Deck}(X \to X)$ s.t. $f(y_0) = y$.

We define $\phi^{-1}(\gamma) = f$.

ex $p: \mathbb{C} \to \mathbb{C}^*$
$z \mapsto e^z$ unramified $\Rightarrow$ unbranched.



$\mathrm{Deck}(\mathbb{C} \to \mathbb{C}^*)$
$\cap$
$\mathrm{Aut}(\mathbb{C})$
$\cap$
$\{az + b : a, b \in \mathbb{C},\ a \neq 0\}$



$f(z) = az + b$ $\quad p \circ f = p$

$e^{az+b} = e^z \quad \forall z \in \mathbb{C}$

$e^{(a-1)z+b} = 1 \quad \forall z \in \mathbb{C}$

$\Rightarrow a = 1,\ b = 2\pi i n,\ n \in \mathbb{Z}$.

$\mathrm{Deck}(\mathbb{C} \xrightarrow{p} \mathbb{C}^*) = 2\pi i \mathbb{Z}. \Rightarrow \pi_1(\mathbb{C}^*) \simeq 2\pi i \mathbb{Z} \simeq \mathbb{Z}$

Look at $\varphi: [0,1] \to \mathbb{C}^*$ $\quad [\varphi] \in \pi_1(\mathbb{C}^*, 1)$
$t \mapsto e^{2\pi i t}$

Choose $y_0 = 0 \in p^{-1}(y)$.

Claim: $y = 2\pi i$

Consider the curve $\hat{\varphi}(t) = 2\pi i t \in \mathbb{C}$, $t \in [0,1]$

$$\begin{cases} p \circ \hat{\varphi}(t) = p(2\pi i t) = e^{2\pi i t} = \varphi^t \\ \hat{\varphi}(0) = 0. \end{cases} \Longrightarrow y = \hat{y}(1) = 2\pi i.$$

The corresponding map $\in \mathrm{Deck}(\mathbb{C} \xrightarrow{p} \mathbb{C}^*)$ is

$$\begin{cases} f_1(z) = z + b = y = 2\pi i \\ f_1(y_0) = f_1(0) = y = 2\pi i. \end{cases}$$
($f_1(0) = b$)

$f_1(z) = z + 2\pi i$

Easy to check f_1 generates $\mathrm{Deck}(\mathbb{C} \to \mathbb{C}^*) = 2\pi i \mathbb{Z}$.

and so $\varphi = \Phi(f)$ generates $\pi_1(\mathbb{C}^*, 1)$.
$= p(0 \to 2\pi i)$
line

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Theorem 5.9 (Determines all covering maps.) Let X be a connected manifold $\pi: \tilde{X} \to X$ universal covering. Let $p: Y \to X$ be a covering map.

Then $\exists f: \tilde{X} \to Y$ fibre-preserving

$$\begin{array}{ccc} & \overset{\exists f}{\nearrow} & Y \\ & \circlearrowleft & \downarrow p \\ \tilde{X} & \xrightarrow[\pi]{} & X \end{array}$$

and $\exists G \subset Deck(\tilde{X}/X)$ subgroup s.t. $\tilde{x}, \tilde{x}' \in \tilde{X}$, $f(\tilde{x}) = f(\tilde{x}') \iff \exists g \in G$ s.t. $g\tilde{x} = \tilde{x}'$

Hence $\tilde{X} \overset{f}{\downarrow} Y = \tilde{X}/G$ — f is a covering map and $G = Deck(\tilde{X} \xrightarrow[f]{} Y)$.

Remark. If $G \subseteq Deck(\tilde{X}/X)$ satisfies some conditions then $\tilde{X} \to \tilde{X}/G = Y$ is a universal covering and we have

$$\begin{array}{ccc} & \tilde{X} & \\ \swarrow & \circlearrowleft & \searrow \\ X & \xleftarrow[p]{} & Y = \tilde{X}/G \end{array}$$

covering map

So the universal covering is the "largest" covering map from which we can construct every covering map to X.

Proof

- Locally, $f = \pi \circ p^{-1} \Rightarrow f$ is a local homeomorphism since composition of local homeomorphisms.

f is a covering map $\overset{def}{\iff} \forall y_0 \in Y, \exists x_0 \in \underset{open}{U_0} \subseteq Y$ s.t.

$$f^{-1}(U_0) = \bigsqcup_{U_i \text{ open}} U_i$$

$f|_{U_i}: U_i \xrightarrow{\sim} U_0$ is homeomorphism

Choose $y_0 \in Y$, U_0 so that $p: U_0 \xrightarrow{\sim} p(U_0)$ is U_0 is one in the disjoint union of $p^{-1}(p(U_0))$ (eligible because p is a covering map).

$p \circ f = \pi \Rightarrow \pi^{-1}(p(U_0)) = f^{-1} \circ p^{-1}(p(U_0))$

π covering map so can assume that $\pi^{-1}(p(U_0))$ is a disjoint union of open sets which are homeomorphic to $p(U_0)$.

$\Rightarrow f^{-1}(U_0) \subseteq f^{-1} \circ p^{-1} \circ p(U_0)$ is a disjoint union of open sets which are homeomorphic to $p(U_0)$ and hence homeomorphic to U_0.

WTS (want to show)

- Since $\tilde{X}$ is simply-connected $\Rightarrow f: \tilde{X} \to Y$ is a universal covering. If we choose $G = Deck(\tilde{X} \xrightarrow{f} Y)$, we prove the fibre-preserving part.

In particular $\pi_1(Y) \cong G$ is a subgroup of $\pi_1(X) \cong Deck(\tilde{X} \to X)$. To show that $Deck(\tilde{X} \to Y)$ in $Deck(\tilde{X} \to X)$

$$\begin{array}{ccc} & \overset{f}{\nearrow} & Y \\ & \circlearrowleft & \downarrow p \\ \tilde{X} & \xrightarrow[\pi]{} & X \end{array}$$

Let $g \in Deck(\tilde{X} \to Y)$ WTS: $\pi \circ g = \pi$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X} \\ f \searrow & & \swarrow f \\ & Y & \end{array}$$

$\Rightarrow f \circ g = f$. $\pi = p \circ f$

$\Rightarrow p \circ (f \circ g) = p \circ f$

$\Rightarrow (p \circ f) \circ g = p \circ f$

$\Rightarrow \pi \circ g = \pi$

Theorem (Riemann's (Uniformisation)) If $\tilde{X}$ is a simply-connected and connected R.S., then

$$\tilde{X} \simeq \begin{cases} \mathbb{C} & \text{or} \\ \mathbb{P}^1 & \text{or} \\ \mathbb{D} \end{cases}$$

So if X is R.S., $\tilde{X} \simeq \mathbb{C}$, $\mathbb{P}^1$ or $\mathbb{D}$ and $\pi(X) \simeq \mathrm{Deck}(\tilde{X} \to X)$

Theorem

- $\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b : a,b \in \mathbb{C},\ a \neq 0 \}$
- $\mathrm{Aut}(\mathbb{P}^1) = \{ [x:y] \mapsto [ax+by : cx+dy] \}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ (invertible 2x2 matrices) and $ad-bc \neq 0$.
- $\mathrm{Aut}(\mathbb{D}) = \{ z \mapsto \lambda \underbrace{\frac{z-a}{1-\bar{a}z}}_{\text{Baschke map}} : |a|<1 \text{ and } |\lambda| = 1 \}$

Proof (1) Let $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$ be an automorphism. Now show that f extends to a map $F: \mathbb{P}^1 \to \mathbb{P}^1$.

$w = \frac{1}{z}$ then $w = 0$ when $z = \infty$. So want to define $f(z=\infty)$.

$g: \mathbb{C}_w \setminus 0 \to \mathbb{C}$, $w \mapsto f(\frac{1}{w})$ which is holomorphic and 0 is an isolated singularity. WTS the singularity of g at $w=0$ is either bounded or a pole.

If not, $w=0$ is an essential singularity of g. $\overset{\text{classification}}{\Longrightarrow}$ $\forall\, \mathcal{U}$ neighbhood of 0: $g^{-1}(\mathcal{U})$ is dense in $\mathbb{C}_w$.

$\Rightarrow V = \frac{1}{\mathcal{U}} =: \{ z = \frac{1}{w} : w \in \mathcal{U} \}$ is a neighborhood of z in $\mathbb{C}_z$.

Which is impossible because $g^{-1}(\mathcal{U}) = f^{-1}(V)$ cannot be dense in $\mathbb{C}_z$ because f is an automorphism.

$\to$ $F \in \mathrm{Aut}(\mathbb{P}^1)$, $F(\infty) = \infty$. $f: z \mapsto f(z)$

$F([z:1]) = [f(z):1] = [\frac{ax+by}{cx+dy} : 1]$

$z = \frac{x}{y} \Rightarrow = \left[\frac{az+b}{cz+d} : 1 \right]$. $F(z=\infty) = \infty \Leftrightarrow \frac{a}{c} \Rightarrow c = 0$.

Lemma $f: \mathbb{D} \to \mathbb{D}$ holomorphic $f(0) = 0 \Rightarrow |f'(0)| \leq 1$ and $|f'(0) = 1| \Leftrightarrow f(z) = \lambda z$, $|\lambda| = 1$

Proof (3) $f \in \mathrm{Aut}(\mathbb{D})$ $f(0) = a$ for some $a \in \mathbb{D}$, $|a|<1$. Now $z \mapsto \frac{z+a}{1+\bar{a}z} = \varphi_a(z)$ is an automorphism of $\mathbb{D}$.

Take $0 \to a$. So $g = \varphi_a^{-1} \circ f$ then $g \in \mathrm{Aut}(\mathbb{D})$.

$g(0) = 0$.

If $|f'(0)| < 1 \to f \notin \mathrm{Aut}(\mathbb{D})$

If $g(0) = 0 \Rightarrow |g'(0)| = 1$, $g(z) = \lambda z$ for some $|\lambda| = 1$ and $f = \varphi_a(g(z)) = \frac{\lambda z + a}{1 + \bar{a}\lambda z}$

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Remark to theorem 5.9

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\nearrow} \circlearrowleft & \downarrow p \\ \tilde{X} & \xrightarrow[\pi]{} & X \end{array}$$

universal cover

G: $\mathrm{Deck}(\tilde{X} \xrightarrow{f} Y)$

$\Rightarrow \mathrm{Deck}(Y \xrightarrow{p} X) \simeq \mathrm{Deck}(\tilde{X} \to X)/G$

ex $\mathbb{C} \xrightarrow{\pi} E = \mathbb{C}/\Gamma$ (elliptic curve.)

$\mathbb{C}$ simply connected and π covering map $\Rightarrow$ π universal cover, and so

$$\pi_1(E) \simeq \mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} E) \subseteq \mathrm{Aut}(\mathbb{C}) = \{az+b : a \neq 0\}$$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto az+b} & \mathbb{C} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & \mathbb{C}/\Gamma & \end{array}$$

$\pi(z) = \pi(az+b) \quad \forall z \in \mathbb{C}$

$\Leftrightarrow [z] = [az+b] \quad \forall z \in \mathbb{C}$

$\Leftrightarrow \underbrace{(az+b) - z}_{\text{holomorphic}} \in \underbrace{\Gamma}_{\text{discrete set}} \quad \forall z \in \mathbb{C}$

$\mathbb{C} \to \mathbb{C}$

$z \mapsto (az+b-z$ is a constant map $\in \Gamma$

$= \underbrace{(a-1)}_{=0} z + b \quad \Rightarrow a = 1$ and $b \in \Gamma$.

$\Rightarrow \pi_1(E) \simeq \mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} E) \simeq \Gamma \simeq \mathbb{Z}^2$

$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where ω_1, ω_2 basis.

What are the two generators of $\pi_1(E)$? They must be image of ω_1 and ω_2 under

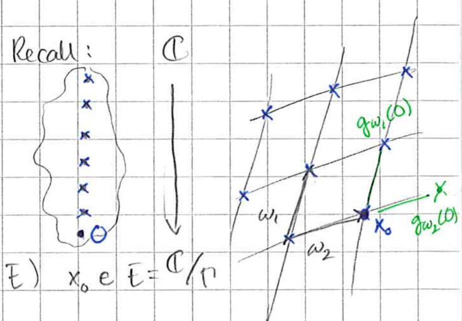
$\mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} E) \xrightarrow{\sim} \pi_1(E)$

$\omega_1 \in \Gamma \Leftrightarrow g_{\omega_1}(z) = z + \omega_1 \in \mathrm{Deck}(\mathbb{C} \to E)$

$\omega_2 \in \Gamma \Leftrightarrow g_{\omega_2}(z) = z + \omega_2 \in \mathrm{Deck}(\mathbb{C} \to E)$

ω_1, ω_2 generate Γ, so g_{ω_1} and g_{ω_2} generate $\mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} E)$

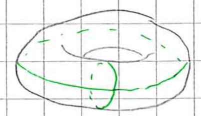
Recall: $\mathbb{C} \to \mathbb{C}/\Gamma$, $x_0 \in E = \mathbb{C}/\Gamma$



$\tilde{X}$ simply connected, so homotopic

Let $C_1 = \pi($any curve from 0 to $g_{\omega_1}(0))$

$C_2 = \pi($ $g_{\omega_2}(0))$



$\Rightarrow C_1, C_2$ generate $\pi_1(E)$

$C = \{X^2 + Y^2 - YZ = 0\} \subseteq \mathbb{P}^2$ Smooth?

$C_1 = C|_{z=1} \quad x^2 + y^2 - y = 0$

$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = (2x, 2y - 1) \quad \left\{\begin{array}{l} x = 0 \\ 2y - 1 = 0 \\ x^2 + y^2 - y = 0 \end{array}\right\} = \emptyset$ C_1 is smooth.

$C_2 = C|_{Y=1} \quad x^2 + 1 - z = 0$

$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = (2x, -1) \neq (0,0)$ C_2 smooth.

$C_3 = C|_{X=1} \quad 1 + y^2 - yz = 0$

$\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (1 - z, -y) \quad \left\{\begin{array}{l} 1 - z = 0 \\ 2y = 0 \\ 1 + y^2 - yz = 0 \end{array}\right\} = \emptyset$ C_3 also smooth

C is smooth

$C \cap \{z = 0\} = \{[x:y:z] \in \mathbb{P}^2 : z = 0,\ x^2 + y^2 = 0\}$ how many points? Two:

$x^2 = -y^2$
$x = \pm i y$ $\implies$ $[\pm iy : y : 0] = [\pm i : 1 : 0] = [1 : \pm i : 0]$

$C \setminus \{[1 : \pm i : 0]\} \longrightarrow \mathbb{C}$

$[x:y:z] \longmapsto \frac{x}{z}$

Q: Can it be extended to a holomorphic map $C \to \mathbb{P}^1$? If so, what is the degree?

MAT4800 § 6 Sheaves 07.10.19

ex Tangent bundle: If X is a real manifold of dimension n, then for each point $x_0 \in X$
$\exists\, x_0 \in U \subseteq X$ (open) s.t. $U \simeq B(0,r) \subseteq \mathbb{R}^n$.

tangent space = {vectors v in $\mathbb{R}^n$ tangent to $B(0,r)$}

$T_y B(0,r) = \mathbb{R}^n$ $\longrightarrow$ $TB(0,r) = B(0,r) \times \mathbb{R}^n$

ex $\longrightarrow$ $\subseteq \mathbb{R}^1$

ex  $\subseteq \mathbb{R}^2$

tangent bundle to X

$\uparrow (U, TX \simeq U \times \mathbb{R}^n$.

TX = real manifold of dim $2n$
(locally $TX \simeq U \times \mathbb{R}^n$). T presheaf:
$T(U) = \{ s : U \to TU \}$ (U open in X, s smooth, TU tangent bundle)
$x \longmapsto s(x) \in T_x X \simeq \mathbb{R}^n$

So locally, a tangent bundle is a product space, but it may not get it.

ex X = Möbius band real manifold of dim 2, non-orientable. So $TX \not\simeq X \times \mathbb{R}^2$.

Def **Presheaf of Abelian groups** (rings, sets) is a pair
$\mathcal{F} : U \longrightarrow \mathcal{F}(U)$ ($U \subseteq X$ open, $\mathcal{F}(U)$ an abelian group).
Whenever $U \hookrightarrow V$ (subset) we define a morphism (of Abelian groups)

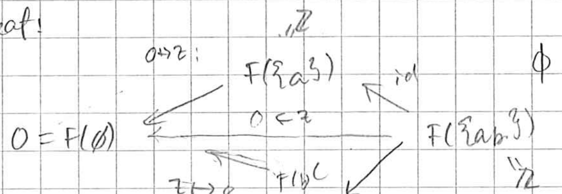
$\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ with

ρ^U_U identity map on $\mathcal{F}(U)$

$\Rightarrow \mathcal{F}(V) \xleftarrow{\rho^U_V} \mathcal{F}(U) \xleftarrow{\rho^W_U} \mathcal{F}(W)$ where $\rho^W_V = \rho^U_V \circ \rho^W_U$ ($V \to U \to W$)

ex $X = \{a, b\}$ with discrete topology. Open sets are $\emptyset$, $\{a\}$, $\{b\}$, $\{b+a\}$)

Inclusion maps: $\emptyset \hookrightarrow \{a\} \hookrightarrow \{a,b\}$, $\{b\}$

Presheaf!


If a presheaf $\mathcal{F}$ has (local) section $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$.
(section of $\mathcal{F}$)

$= A \leftarrow$ a fixed abelian group,

Def **Stalk** at a point $x_0 \in X$ is kind of local information a presheaf has at each point x_0.

ex f holomorphic $f : U \to \mathbb{C}$ (U open) then at any point $x_0 \in U$, f has a unique
series (?) representation:

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

stalk

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Def X top. space $\mathcal{F}$ a presheaf on X. Let $x_0 \in X$.

$$X \supseteq \bigsqcup_{\substack{U \ni a \\ \text{open}}} \mathcal{F}(U)$$ Then we define a relation between these $\mathcal{F}(U)$:

If $V \hookrightarrow U \Rightarrow \mathcal{F}(V) \leftarrow \mathcal{F}(U)$ then we identify $v \in \mathcal{F}(U)$ with its image

$\rho_V^U(v) \in \mathcal{F}(V)$ $\quad v \sim \rho_V^U(v)$

$\mathcal{F}_a := \bigsqcup_{X \supseteq U \ni a} \mathcal{F}(U)/\sim$.

Remark Stalks are only about sections on "small not big" open sets.

Main object of Riemann-Roch.

ex f holomorphic map $f: U \longrightarrow \mathbb{C}$, ($U \subseteq \mathbb{C}$ open) then at any point $x_0 \in U$, then f has a unique series representation

$$f(x) = \sum_{n \geq 0}^{\infty} \underbrace{c_n (x - x_0)^n}_{\text{stalk}}$$

ex X Riemann surface. $\mathcal{F} = \mathcal{O}(X)$ the presheaf where for each $U \subseteq X$ open, we define
↪ structure presheaf of X

$\mathcal{F}(U) = \{f: U \to \mathbb{C} \text{ holomorphic}\}$

$\mathcal{F}(U)$ abelian group:

$(f+g)(z) := f(z) + g(z)$ and the constant 0-map is the identity element

If $V \hookrightarrow U$, $\rho: \mathcal{F}(U) \to \mathcal{F}(V)$

$f: U \to \mathbb{C} \mapsto f|_V: V \to \mathbb{C}$ presheaf.

The stalk of $\mathcal{O}$ at a point x: $\quad U_0 \to B(0,r) \subseteq \mathbb{C}$, $x_0 \mapsto 0$

Choose $U_0 \ni x_0$ small s.t. $U_0 \cong B(0,r) \subseteq \mathbb{C}$. Then $\mathcal{O}(U) = \{f: U \to \mathbb{C} \text{ holomorphic}\}$
$= \{\sum_{n=0}^{\infty} c_n z^n : \text{absolute convergent power series}\}$

important fact: $\mathcal{O}(U)$ stabilises when U is small enough. Specifically if $U \hookrightarrow U_0$ then $\mathcal{O}(U)$ is also isomorphic to the set of absolutely convergent power series.

Claim: $\mathcal{F}_{x_0} = \{\sum_{n=0}^{\infty} c_i z^n \mid \text{absolutely convergent power series in some neighborhood of } \{0\}\}$.
($\subseteq X$ R.S.)

$x_0 \in U$ any open set. $\exists\, x_0 \in U'$ smaller open set for which

$\mathcal{O}(U')$ = set of convergent power series. $U' \hookrightarrow U$

$\mathcal{O}(U)$ is identified with its image in $\mathcal{O}(U')$ = power series □

Def Stalk: Just need to determine sections $\mathcal{O}(U)$ where U is small enough open set.

Remark: X compact R.S. then $\mathcal{O}(X)$ = constant maps from $X \to \mathbb{C} \approx \mathbb{C}$, and

$\forall x \in X$ $\quad \mathcal{O}_x$ = power series $\sum_{n \geq 0} c_n z^n$ a.c. in some nbh. of $0 \Rightarrow \mathcal{O}$ is constant presheaf.

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Recall X R.S. $\to$ $\mathcal{O}$ presheaf. $\mathcal{O}(U) = \{f: U \to \mathbb{C} \text{ holomorphic}\}$

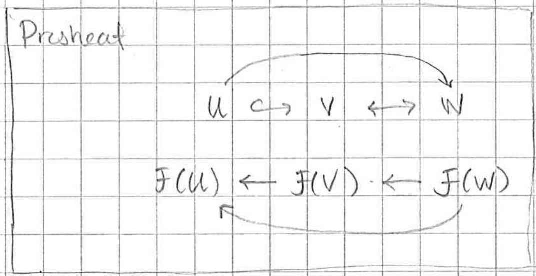
Stalk (only care about small open sets)

$U \hookrightarrow V$
$\mathcal{F}(U) \leftarrow \mathcal{F}(V)$

$\mathcal{O}_x \simeq \{\sum_{n\geq 0} c_n z^n$ power series converging on a nbh of $0\}$

$B(0,r)$ $r>0$

Sheaf = presheaf + some existence + uniqueness property.



Def A presheaf $\mathcal{F}$ is a sheaf $\Longleftrightarrow$ $\forall\, U$ open $\subseteq X$ and an open covering $\{U_i\}_{i\in I}$ of U, we have

① Uniqueness: If $f, g \in \mathcal{F}(U)$ s.t. $f|_{U_i} = g|_{U_i}$ $\forall i \in I$ $\Longrightarrow$ $f = g$

② Existence: If $f_i \in \mathcal{F}(U_i)$ s.t. $f_i|_{U_i\cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i,j$

$\Longrightarrow \exists\, f \in \mathcal{F}(U)$ and $f|_{U_i} = f_i$, where $U = \bigcup_{i\in I} U_i$

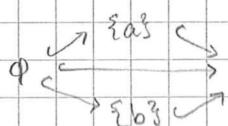
Remarks

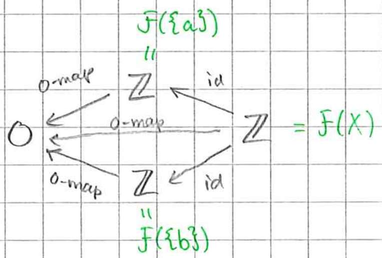
1) $\exists$ presheaves which are not sheaves.

2) $\forall$ presheaf $\mathcal{F}$ $\exists$ a canonical to "make" it bigger to a sheaf $\hat{\mathcal{F}}$ and $\mathcal{F}_x = \hat{\mathcal{F}}_x$ $\forall x$

ex 1

1) $X = \{a,b\}$ discrete topology





Existence property not satisfied:

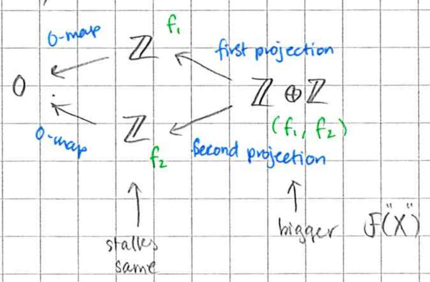
a section is just an integer; pick

$U_1 = \{a\}$ $U = U_1 \cup U_2 = X$ $1 = f_1 \in \mathcal{F}(U_1) = \mathbb{Z}$

$U_2 = \{b\}$ $2 = f_2 \in \mathcal{F}(U_2) = \mathbb{Z}$

$f_1|_{U_1\cap U_2} = f_2|_{U_1 \cap U_2}$ (both $\emptyset$). But $\nexists\, f \in \mathcal{F}(U) = \mathbb{Z}$ s.t. $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$

Making this into a sheaf:



Theorem X Riemann surface $\Rightarrow$ $\mathcal{O}$ is a sheaf.

① By connectedness of X and that R.S. $\Rightarrow$ holomorphic functions.

② Define $f = f_i$ on U_i ...

Constructing sheaves from presheaves:

The topological space associated to a presheaf (e'tale space).

$\mathcal{F}$ is a presheaf over X.

$\mathcal{E} = |\mathcal{F}| \coprod_{x \in X} \mathcal{F}_x$

Now we define a topology on $\mathcal{E}$ (very similar to how we defined the universal covering of a connected manifold):

If $U \subseteq X$ open and $f \in \mathcal{F}(U)$

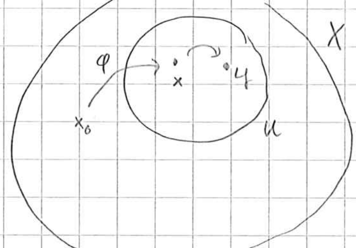
define

$[U, f] := \{ f_x \in \mathcal{F}_x : x \in U \}$ a basis for open sets in $|\mathcal{F}| = \mathcal{E}$

$\mathcal{F}(U) \longrightarrow \mathcal{F}_x$, $f \longmapsto f_x$

ex If $f(\mathcal{O}(U)) \Rightarrow f_x$ = Taylor expansion of f around x. (R.S.)

Claim projection $p: \mathcal{E} \longrightarrow X$, $(x, v) \longmapsto x$ ($v \in \mathcal{F}_x$) is a local homeomorphism:



$[U, f] \longrightarrow U$
$(x, f_x) \longmapsto x$

$\varphi \in \mathcal{E}$ $p(\varphi) = x_0$

$[U, \varphi] = \{ x_0 \xrightarrow{\varphi} x \longrightarrow y$

$\Rightarrow \varphi = f_{x_0}$ for some $f \in \mathcal{F}(U_0)$, $x_0 \in U_0$ open neighbourhood.

$[U_0, f] \longrightarrow U_0$

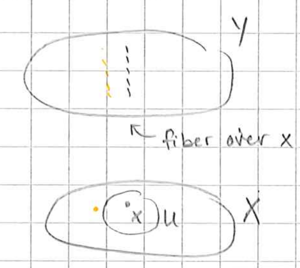
Theorem 6.10 $\mathcal{E} = |\mathcal{F}|$ is Hausdorff $\Leftrightarrow$ $\forall\, U$ open, connected $\subseteq X$ and $f, g \in \mathcal{F}(U)$ with

$f_a = g_a$ for one $a \in U \Rightarrow f = g$

Identity property for $\mathcal{O}$ holds for ex_1.

If $p: Y \to X$ is a continuous map and $U \subseteq X$ open set, then a Section s over U of p is a continuous map $s: U \to Y$ s.t.

$U \xrightarrow{s} s(U) \xrightarrow{p} U$



§6 Sheaves

Define $\hat{\mathcal{F}}$ presheaf of X by $\hat{\mathcal{F}}(U) = \{$ sections $s: U \to \mathcal{E} = |\mathcal{F}|$ of $p: \mathcal{E} \to X \}$

(So we the name section for an element of $\mathcal{F}(U)$)

$\hat{\mathcal{F}}_x$ = Stalk at x = Fiber over x of $p = \mathcal{F}_x$.
Some things that come together

ex $U_1 = \{a\}$, $U_2 = \{b\}$, $U = \{a, b\}$

$\mathcal{E} = \bigsqcup_{x \in X} \mathcal{F}_x = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}_{\{a\}} \sqcup \mathbb{Z}_{\{b\}}$

only two points in X (x = a or b)

stalks of a and b

discrete topology over $\mathcal{E}$ since this is a discrete topology.

$p: \mathcal{E} \to X$ continuous

$z \in \mathbb{Z}_{\{a\}} \to a$

$z \in \mathbb{Z}_{\{b\}} \to b$

If $p: \mathcal{E} \to X$ is a map between discrete topologies then any $s: U \to \mathcal{E}$ which preserves fibres is continuous.

$\hat{\mathcal{F}}(\{a\}) = \{ s: \{a\} \to \mathbb{Z}_{\{a\}} \} \simeq \mathbb{Z}$

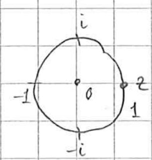
$\hat{\mathcal{F}}(\{b\}) = \{ s: \{b\} \to \mathbb{Z}_{\{b\}} \} \simeq \mathbb{Z}$

$\hat{\mathcal{F}}(X) = \{ s: \{a, b\} \to \mathcal{E} = \mathbb{Z}_{\{a\}} \sqcup \mathbb{Z}_{\{b\}},\ s(a) \in \mathbb{Z}_{\{a\}},\ s(b) \in \mathbb{Z}_{\{b\}}) \simeq \mathbb{Z} \oplus \mathbb{Z}$

(presents fibres).

§7 Analytic continuation.

7.2 was the motivation for Riemann to define R.S.



Can we define $\sqrt{z}$ continuously on $\mathbb{C}$? No.

Claim: $\nexists$ continuous function $f: \mathbb{C} \to \mathbb{C}$ s.t. $f^2(z) = z \quad \forall z \in \mathbb{C}$.

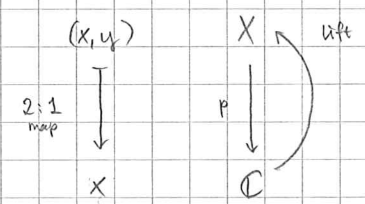
What we can do is to define a map: $F: \mathbb{C} \to R$ = some Riemann surface, by which is "square root" of z

consider $X = \{ (x, y) \in \mathbb{C}^2 : y^2 = x \}$ ((x,y) 1:1 y, $\simeq \mathbb{C}$)

We have a natural projection $p: X \to \mathbb{C}$. A square root function on a curve φ in $\mathbb{C}$ is a section $s: \varphi \to X$ of p

$s(e^{2\pi i t}) = (e^{2\pi i t}, e^{\pi i t})$, $t \in [0, 1]$

closed / not closed



$x^3 + x$

$\{ y^3 + y = x^3 \}$ is the Riemann surface of $x^3 + x$ (inverse)

$p(x)$ polynomial then $\{ p(y) = x \} \cong \mathbb{C}$ (isomorphic) is the Riemann surface of $p(x)$, so for polynomials very easy.

only two branches:

$\sqrt{1} = 1 \Rightarrow \sqrt{-1} = \cdots$

or

$\sqrt{1} = -1 \Rightarrow \sqrt{-1} = e^{-\pi i/2}$ ($-1 = e^{-i\pi}$)

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X R.S. $\mathcal{O}$ = sheaf of holomorphic function, $|\mathcal{O}|$ = étale space of $\mathcal{O} = \{(x, f_x) : f$ holomorphic on nbh of $x\}$

(f_x: germ stalk)

$p: |\mathcal{O}| \to X$ so an inverse of f (defined on some U) is a section $s: U \to |\mathcal{O}|$.

§8

Def $f: X \to Y$ (R.S, R.S) is unbranched holomorphic if $f'(z) \neq 0 \ \forall z \in X$

Def f is proper if $f^{-1}(K)$ is compact $\forall$ K compact $\subseteq Y$.

Theorem 8.4 X R.S. $A \subseteq X$ closed, discrete. $\Rightarrow X' = X \setminus A$ also R.S.

Let $\pi': Y' \to X'$ an unbranched, proper holomorphic covering map. Then $\exists$ Y Riemann surface, $Y \supseteq Y'$, a holomorphic map $\pi: Y \to X$ such that $\pi|_{Y'} = \pi'$.

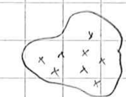
Proof Choose $x_0 \in A \Rightarrow X \supseteq U \ni x_0$ (open) st. $U \cong B(x_0, r)$, $U \cap A = \{x_0\}$,

and $\pi': U \setminus \{x_0\} \xrightarrow{\text{finite}} \pi'(U \setminus \{x_0\})$ is a covering map

$U \setminus \{x_0\} \cong \mathbb{D} \setminus \{0\}$

$\Rightarrow$ theorem 6.10 $\Rightarrow$ π' is isomorphic to the map $\mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$, $z \mapsto z^k$ for some k.

We glue together the map π' on $U \setminus \{x_0\}$ and the map $z \mapsto z^k$ on $\mathbb{D}$.



Y' is $Y \setminus \{$finite set of points$\}$

Thm 8.9 X R.S., $P(T) = T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{M}(X)[T]$ (meromorphic functions: $X \to \mathbb{P}^1$)

Then $\exists$ a R.S. Y, an branched holomorphic n-sheeted covering $\pi: Y \to X$
(π is holomorphic of degree n and $\pi: Y - \pi^{-1}(C) \to X \setminus C$ is a covering map where
C = critical values = image of critical points of π)
and a meromorphic function

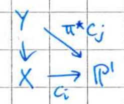
$$f \in \mathcal{M}(Y) \quad \text{such that}$$

$$\underbrace{(\pi^* P)}_{\text{pull back}}(F) = 0 \iff P(\pi(F)) \equiv 0$$

this is uniquely determined
(the other ones are biholomorphically equivalent).

$\pi^* P(T) = T^n + (\pi^* c_1) T^{n-1} + \dots + (\pi^* c_{n-1}) T + \pi^* c_n$

$\pi^* c_i \in \mathcal{M}(Y)$

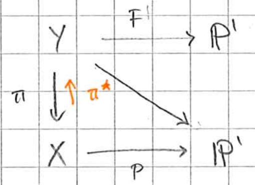


ex $X = \mathbb{C}_z$

$P(T) = T^3 - z \in \mathcal{M}(X)[T]$

$P(T) = 0$ meaning we want to find the 3^{rd} root of z (which is impossible on $\mathbb{C}$).

$Y \xrightarrow{\pi} \mathbb{C}$, $\deg(\pi) = 3$ and a meromorphic function $F: Y \to \mathbb{P}^1$



$\pi^*(P)(F) = 0$

$(\pi^* P) = T^3 - \pi^*(z)$

$\exists F: Y \to \mathbb{P}^1$ so that $F^3 - \pi^*(z) = 0$

So we can find a 3-root of z on a R.S. $Y \neq \mathbb{C}$.

Remark If $P(T)$ is transcendental we may have no associated R.S.

ex: $P(T) = e^T - z$ (If we could we could have defined $\log(z)$)

Because for each fixed z we have infinitely many solutions and so Y will be ∞-sheeted map which is not a nice topology.

Proof (Sketch)

Δ = discriminant of $P(T) := \{x \in X : P(T) = 0$ has multiple solutions in $T\}$
$= \{x \in X : P(T) = P'(T) = 0\} \neq \emptyset \}$

ex: $P(T) = T^3 - z \implies \Delta(P) = \{z = 0\}$.

$X' = X \setminus \Delta$. Assume $P(T)$ is irreducible in $\mathcal{M}(X) \implies \Delta \neq X$.

$\mathcal{O}$ = sheaf of holomorphic function on X

$\pi: |\mathcal{O}| \to X$ $\quad Y' = \{\varphi \in \pi^{-1}(X \setminus \Delta) \subseteq \mathcal{O} : P(\varphi) = 0\}$ (φ: holomorphic germ)

P has degree $n \implies$ at each point $x \in X \setminus \Delta$ there are exactly n solutions $\varphi_1, \dots, \varphi_n$

$\pi = \pi|_{Y'} : Y' \to X \setminus \Delta$ is an n-covering map. By pullback (on the holomorphic structure on X)
Y' is a R.S. and $\pi: Y' \to X' = X - \Delta$ a holomorphic map.
By theorem 8.4 $\implies$ we can find Y. Y is unique because $p: Z \to X$
$Z - p^{-1}(\Delta) \to X - \Delta$

$$|\mathcal{O}| \supseteq Y' \longrightarrow \mathbb{C}$$
$$\varphi \longmapsto \varphi(\pi'(\varphi))$$

$$Y' \supseteq V$$
$$\downarrow \pi'$$
$$X \supseteq U \xrightarrow{\varphi} \mathbb{C}$$

$F(\varphi) = 0$ $\Rightarrow$ $\varphi(f) = 0$, f is bounded near $Y - Y'$.

$\Rightarrow$ can extend to $F: Y \to \mathbb{P}^1$.

§ 9-10: Differential forms   Residue   Integration of differential forms.

Differential forms are dual to tangent vectors.

Residue is an invariant of differential forms, but not of meromorphic functions.

V = real vector space of dim n $\Longrightarrow$ $V^* = \{f : V \to \mathbb{R}$ continuous linear$\}$ (Hamel's basis)
is also an n-dim vector space.

If X = real manifold of dim n

TX = tangent bundle = étale space of a sheaf $\mathcal{F}$ where

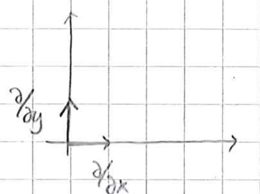
$$\underset{\text{vector space}}{\mathcal{F}(U)} = \{\varphi : U \to TX,\ x \mapsto \varphi(x) \in T_x \simeq \mathbb{R}^n\}$$

$$\mathcal{F}_x = T_x \simeq \mathbb{R}^n$$

Define
$$\Omega^1(U) = (\mathcal{F}(U))^*$$ $\Longrightarrow$ get a sheaf Ω of differential 1-forms

To understand this kind of things, just need to look at small balls in $\mathbb{R}^n$.

$\mathbb{C} \simeq \mathbb{R}^n$



$\partial/\partial x$, $\partial/\partial y$ (tangent vectors) is a (global) basis for $T\mathbb{C} = \mathbb{C} \times \mathbb{R}^2$

Tangent vectors can
give us derivatives: $U \subseteq \mathbb{C}$

$$\frac{\partial}{\partial x} : C^0(U) \to C^0(U)$$
$$f \mapsto \frac{\partial}{\partial x} f$$

$$\frac{\partial}{\partial y} : f \mapsto \frac{\partial}{\partial y} f$$

If V is a vector space with basis $e_1, \dots, e_n$ then V^* has a canonical basis $f_1, \dots, f_n$ (dual basis to $e_1, \dots, e_n$) where

$$f_i : V \to \mathbb{R} \text{ linear}$$
$$f_i(e_j) = 0 \quad \forall j \neq i$$
$$f_i(e_i) = 1$$

We get dual differential 1-forms dx and dy

$$\partial x\left(\frac{\partial}{\partial x}\right) = 1 \qquad \partial y\left(\frac{\partial}{\partial x}\right) = 0$$
$$\partial x\left(\frac{\partial}{\partial y}\right) = 0 \qquad \partial y\left(\frac{\partial}{\partial y}\right) = 1$$

Locally a section $\Omega^1(U)$ has the form: $\alpha = f(x,y)\,dx + g(x,y)\,dy$
usually $f, g \in C^1(U)$.

Then $f\dfrac{\partial}{\partial x} + g\dfrac{\partial}{\partial y}$ is called a vector field.

§9 We have a map

$$d: C^{k+1}(U) \to \Omega^1(U)$$
$$f \mapsto df := \left(\frac{\partial}{\partial x} f\right) dx + \left(\frac{\partial}{\partial y} f\right) dy$$

ex $f(x,y) = x^2 - y^2$

$\Rightarrow \quad df = 2x\,dx - 2y\,dy$

2-forms $\quad \Omega^2$ = sheaf of differential 2-forms:

$\Omega^2(U) = \{\omega = f(x,y)\,dx \wedge dy : f(x,y) \in C^k(U)\}$

$\omega = f(x,y)\,dx \wedge dy$

$dx \wedge dy$: canonical volume form on $\mathbb{R}^2$

In general, if $f(x,y) > 0 \quad \forall (x,y) \in U \subset \mathbb{R}^2$

$\Longrightarrow \quad f(x,y)\,dx \wedge dy$ is a volume form for

$$d: \Omega^1(U) \to \Omega^2(U)$$
$$(f(x,y)dx + g(x,y)dy) \mapsto \left(\frac{\partial f}{\partial x} \partial x \wedge dx + \frac{\partial f}{\partial y} \partial y \wedge dx + \left(\frac{\partial g}{\partial x} \partial x \wedge \partial y + \frac{\partial g}{\partial y} dy \wedge dy\right)\right.$$

Def: a $\mathbb{R}$-bilinear map $\Omega^1 \times \Omega^1 \to \Omega^2$ by

$dx \wedge dx = 0$
$dy \wedge \partial y = 0$

$\partial x \wedge \partial y = -dy \wedge dx \quad$ (change of orientation)

$\Rightarrow$ If $\alpha = \alpha_1 dx + \alpha_2 dy$

$$\alpha \wedge \alpha = (\alpha_1 dx + \alpha_2 dy) \wedge (\alpha_1 \partial x + \alpha_2 dy) = \alpha_1^2 \underbrace{dx \wedge dy}_{0} + \alpha_1 \alpha_2 \underbrace{dy \wedge dy}_{0}$$

Thm: $\alpha \in \Omega^1 \Rightarrow \alpha \wedge \alpha = 0$

On $\mathbb{C}$ we also have coordinates $\underbrace{z = x+iy, \quad \bar{z} = x - iy}_{\text{basis}}$

$\Longrightarrow \quad x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}$

So $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are global basis for $T\mathbb{C}$

$\Longrightarrow$ Get a dual basis dz and $d\bar{z}$

$dz\left(\alpha \frac{\partial}{\partial z}\right) = \alpha \qquad \alpha \in \mathbb{C}. \qquad dz = d(x+iy) = dx + i\,dy$

$d\bar{z} = d(x-iy) = dx - i\,dy$

$dz\left(\alpha \frac{\partial}{\partial \bar{z}}\right) = 0$

What are $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \qquad a,b \in \mathbb{C}.$$

$$\frac{\partial}{\partial \bar{z}} = c\frac{d}{\partial y} + \frac{\partial^2}{\partial y}$$

Finding a, b, c, d:

$$1 = \frac{d}{\partial z} = dx + i\,dy\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a\,dx\left(\frac{\partial}{\partial x}\right)$$

$$0 = d\bar{z}\left(\frac{\partial}{\partial z}\right) = (dx - i\,dy)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a - ib)$$

$$\begin{aligned} a + ib &= 1 \\ a - ib &= 0 \end{aligned} \Rightarrow a = \frac{1}{2},\ b = \frac{1}{2i} = -\frac{i}{2}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

HW $\Rightarrow$ $$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

ex $f(x,y) = x^2 + y^2 = \frac{z^2 + \bar{z}^2}{2}$ $\qquad \frac{\partial g}{\partial z} = z, \quad \frac{\partial g}{\partial \bar{z}} = \bar{z}$

$\quad \| \quad g(z,\bar{z})$

$$\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right) f(x,y) = \frac{1}{2}(2x - \frac{i}{2}(-2y) = x + iy = z = \frac{\partial}{\partial z} g'(z,\bar{z})$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{d}{\partial y}\right) f(x,y) = \frac{1}{2}(2x) + \frac{i}{2}(-2y) = x - iy = \bar{z} = \frac{\partial}{\partial \bar{z}} = g(z$$

"Proof" of Cauchy Riemann: if $f: \mathbb{C} \to \mathbb{C}$ is holomorphic $\overset{def}{\Longrightarrow}$

$f(z) = \sum_{n\geq 0}^{\infty} c_n z^n$ locally independent of $\bar{z}$ $\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$

$$\Rightarrow \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}\right) f(x,y) = 0$$

$$\underset{\mathbb{C}}{\underset{\cap}{f(z)}} = \underset{\mathbb{R}}{\underset{\cap}{u(x,y)}} + i\,\underset{\mathbb{R}}{\underset{\cap}{v(x,y)}}$$

§9 Differential forms

$$d' = \frac{\partial}{\partial z} : \quad C^1(U) \longrightarrow \Omega^1(U)$$
$$f \longmapsto \frac{\partial f}{\partial z} dz$$

$$d'' = \frac{\partial}{\partial \bar{z}} : \quad f \longmapsto \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$d = d' + d''$$

Lemma: $d' \circ d' = 0$, $d'' \circ d'' = 0$, $d \circ d = 0$

Proof $(d' \circ d')(f) = d'\left(\frac{\partial f}{\partial z} dz\right) = \underbrace{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)}_{\frac{\partial^2 f}{\partial z^2}} \underbrace{dz \wedge dz}_{=0}$

$$dz = dx + i\,dy$$
$$dz \wedge dz = (dx + i\,dy) \wedge (dx + i\,dy)$$

Pullback of differential forms

$$f: \underset{\subseteq \mathbb{R}^n}{U} \longrightarrow \underset{\subseteq \mathbb{R}^{n-1}}{V}$$
$$x \longmapsto y = f(x)$$

$$\Rightarrow f^*: \underbrace{\Omega^k(V)}_{k\text{-form}} \longrightarrow \Omega^k(U)$$

$$\underset{\psi}{\sum \alpha(y_1, \dots, y_n) dy_1 \wedge \dots \wedge dy_k} \longmapsto \sum \alpha(f_1(x_1), \dots, f_n(x)) \underbrace{df_1(x)}_{} \wedge \dots \wedge df_k(x)$$

$$df_1(x) = \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} dx_j$$

ex $\overset{\text{holomorphic}}{f}: \mathbb{C}_{z = x+iy} \longrightarrow \mathbb{C}_{w = u+iv}$

$$z \longmapsto z^2 = w$$
$$x + iy \longmapsto \underset{u}{(x^2 - y^2)} + i\underset{v}{(2xy)}$$

$$f^*(du) = d(x^2 - y^2) = 2x\,dx - 2y\,dy$$
$$f^*(v\,du) = (\underset{v}{2xy})(2x\,dx - 2y\,dy)$$
$$f^*(du \wedge dv) = d(x^2 - y^2) \wedge \underbrace{d(2xy)}_{f^*(dv)}$$
$$= (2x\,dx - 2y\,dy) \wedge (2y\,dx + 2x\,dy)$$
$$= 4x^2\, dx \wedge dy + 4y^2\, dx \wedge dy$$
$$= \underbrace{4(x^2 + y^2)}_{>0} dx \wedge dy$$

$$f^*(dw) = f^*(du + i\,dv)$$
$$= (2x\,dx - 2y\,dy) + i(2y\,dx + 2x\,dy)$$
$$= 2(x + iy)\,dx + 2i(x + iy)\,dy$$
$$= 2(x + iy)(dx + i\,dy) = 2z\,dz = d(z^2) = 2z\,dz$$

HW: Compute $f^*(d\bar{w})$ (recall: $d\bar{w} = du - i\,dv$)

Residue of meromorphic 1-forms

A 1-form on X R.S., locally have the form:

$\omega = f_1\, dz + f_2\, d\bar{z} = g_1\, dx + g_2\, dy$ $\quad \left(\begin{array}{l} dz = dx + i dy \\ d\bar{z} = dx - i dy \end{array} \right)$

If $f_2 = 0$ and f_1 holomorphic, then $\omega = f_1\, dz$ is holomorphic 1-form.

If $f_2 = 0$ and f_1 meromorphic, then $\omega = f_1\, dz$ is meromorphic 1-form.

The poles of a meromorphic function are isolated. Let's work around 1 pole of f_1, meaning

$f_1 : \mathbb{D} \setminus \{0\} \longrightarrow \mathbb{C}$ is holomorphic and 0 is not transcendental singularity.

$\omega = f_1\, dz$

<u>Def.</u> $Res_0(\omega) := Res_0(f_1)$

We can write the Laurent series $f_1 = \sum_{n \geq -N_0}^{\infty} c_n z^n \Rightarrow Res_0(f_1) = c_{-1}$.

ex $e^{\frac{1}{z}} = \sum_{n \geq 0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 + \underset{c_{-1}}{1} \cdot \underset{z^{-1}}{\frac{1}{z}} + \frac{1}{2} \frac{1}{z^2} + \cdots$ So $Res(e^{\frac{1}{z}}) = 1$.

ex $f(z) = \frac{z^2 - z + 1}{z^3} = \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z}$ so $c_{-1} = 1$, $Res(f) = 1$.

Change coordinates (to work on R.S.)

$\pi : \mathbb{C}_z \xrightarrow{\sim} \mathbb{C}_w$

$z \longmapsto w = 2z$

$f(z) = \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z}$

$= \frac{1}{\left(\frac{w}{2}\right)^3} - \frac{1}{\left(\frac{w}{2}\right)^2} + \frac{1}{\left(\frac{w}{2}\right)} = \frac{8}{w^3} - \frac{4}{w^2} + \frac{2}{w} = g(w)$

$Res_0(f) \neq Res_0(g) = 2$.

ex cont. $f(z)\,dz = \left(\frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z}\right) dz$

$$(\pi^{-1})^* (f(z)\,dz) = \left(\frac{8}{w^3} - \frac{4}{w^2} + \frac{2}{w}\right) d\left(\frac{w}{2}\right) = \left(\frac{4}{w^3} - \frac{2}{w^2} + \frac{1}{w}\right) dw$$

$$Res_0 (f(z)\,dz) = 1 = Res_0 \left((\pi^{-1})^* (f(z)\,dz)\right)$$

Theorem Let α be a meromorphic 1-form on $\mathbb{D}_z$, with singular point 0.

Let $\pi : \mathbb{D}_w \to \mathbb{D}_z$ be a biholomorphic map $\pi(0) = 0$.

Then $Res_0(\alpha) = Res_0(\pi^*(\alpha))$.

So $Res_0(\alpha)$ is biholomorphic invariant.

§10 Integration of Differential Forms

If ω is a continuous 1-form on U R.S.. Let $\varphi : [0,1] \to U$ be a C^1-curve.

Then $$\int_{\varphi([0,1])} \omega := \int_0^1 \underbrace{(\varphi^* \omega)}_{\text{pull back.}}(t)$$

ex. $\varphi : [0,1] \to \mathbb{D}$

$t \mapsto e^{2\pi i t} = \underbrace{\cos(2\pi t)}_{x} + i\underbrace{\sin(2\pi t)}_{y}$.

$$\omega = dx \implies \varphi^*(\omega) = d(\cos(2\pi t)) = \frac{\partial}{\partial t}(\cos(2\pi t))\,dt = -2\pi \sin(2\pi t)\,dt$$

$$\int_{\varphi([0,1])} \omega = \int_0^1 -2\pi \sin(2\pi t)\,dt$$

HW: $\omega = dy \implies \omega = dz \ldots ?$

Integration over surface

$\psi : \underbrace{[0,1] \times [0,1]}_{\text{2-dim}} \to U$ $\qquad \omega = $ 2-form

$$\int_{Image(\psi)} \omega := \int_0^1\int_0^1 \underbrace{\psi^*(\omega)}_{\text{pull-back}}$$

Riemann integration.

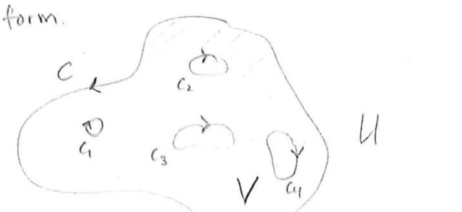
Theorem Green's theorem $\Rightarrow$ Cauchy integration formula

Boundary of a domain $\longleftrightarrow$ d of a differential form.

Riemann surface $\mathcal{U}$. V (connected) bounded domain in $\mathcal{U}$

Let α be a C^1 1-form on $\mathcal{U}$. Then

$$\int_V d\alpha = \int_{\partial V} \alpha$$



boundary $C - C_1 - C_2 - C_3 - C_4$ with orientation

Chapter 2 : Compact Riemann Surfaces

§12 Cohomology Groups

can define for non-compact, but might be trivial or sth. similar.

Remarks Cohomology groups of compact RS are finite dimensional vector spaces.

The dimension is related to the genus of the RS, which is related to the Euler characteristic.

$$\chi(X) = 2 - 2g$$

(X = RS, g = genus)

$\mathbb{P}^1 \Rightarrow g = 0$

elliptic curve $E \Rightarrow g = 1$

$\dim H^{1,0} = g$

$\dim H^{0,1} = g$

Theorem If $X = \{ P(x,y,z) = 0 \} \subseteq \mathbb{P}^2$ (X smooth RS, P homogeneous polynomial of degree d)

recall smooth if $\emptyset = \{P = 0\} \cap \{\frac{\partial P}{\partial x} = 0\} \cap \{\frac{\partial P}{\partial y} = 0\} \cap \{\frac{\partial P}{\partial z} = 0\}$.

then $g(X) = \dfrac{(d-1)(d-2)}{2}$

ex $\mathbb{P}^1 = \{ P(x,y,z) = z = 0 \} \subseteq \mathbb{P}^2$ ($d = 1$) $\Longrightarrow g(\mathbb{P}^1) = \frac{(1-1)(1-2)}{2} = 0$

ex $X = \{ x^2 + y^2 + z^2 = 0 \} \subseteq \mathbb{P}^2$ ($d = 2$) $g(X) = 0$ (actually $X \simeq \mathbb{P}^1$)

ex $X = \{ y^2 z = x^3 - y^3 \} \subseteq \mathbb{P}^2$ ($d = 3$) $g(X) = 1$

$X = \mathbb{C}/\Gamma$, Weierstrass-map $[\wp_\Gamma : \wp_\Gamma'(\wp) : 1]$

Theorem $\exists$ a compact Riemann surface X which cannot be embedded into $\mathbb{P}^2$

(Similar to that Möbius band cannot be embedded in $\mathbb{R}^2$ or S^2, but can in $\mathbb{R}^3$)

Proof $\forall g \in \mathbb{N}_{\geq 0}$, $\exists$ a compact RS. X st. $g(X) = g$. If we choose $g \neq \frac{(d-1)(d-2)}{2}$ $\forall d$

$\Rightarrow X$ ...

ex: We cannot have a RS. X with $g(X) = 2$ embedded in $\mathbb{P}^2$.

§ 12 Cohomology groups ↙ most abstract section in this course.

Def abstractly $\Omega \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \Omega^2 \xrightarrow{\partial} \cdots \longrightarrow 0$

$\Updownarrow$

Def explicitly **Cěch cohomology.** X is a topological Hausdorff space.

$\mathcal{F}$ = a sheaf on X.

$H^i(X, \mathcal{F})$

- Cohomology wrt. an open cover:

Fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ not necessarily finite of X

For $q = 0, 1, 2, 3, \ldots$

$$\underbrace{C^q(\mathcal{U}, \mathcal{F})}_{q^{th} \text{ cochain group of } \mathcal{F}} = \prod_{(i_0, \ldots, i_q) \in \underbrace{I \times \cdots \times I}_{q \text{ times}}} \mathcal{F}(\underbrace{U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}}_{\text{all possible } q \text{ sets in } \mathcal{U}.})$$

For this course, we only consider up to $\mathbb{P}^2$, maybe $\mathbb{P}^3$, so we only need $q = 0, 1, 2$, maybe 3.

$\longrightarrow$ $C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in I} \mathcal{F}(U_i)$

An element $f \in C^0(\mathcal{U}, \mathcal{F})$ is a tuple $\{(f_i)_{i \in I} : f_i \in \mathcal{F}(U_i)\}$

ex. More explicitly, if $X = \mathbb{P}^1 \ni [x:y]$, $\mathcal{F}$ = sheaf of holomorphic functions on X

$$\mathcal{F}(U) = \{f : U \to \text{holomorphic}\}$$

$$\mathcal{U} = \{\underset{x \neq 0}{U_1}, \underset{y \neq 0}{U_2}\} \simeq \{\mathbb{C}_{y/x}, \mathbb{C}_{x/y}\}$$

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_1) \times \mathcal{F}(U_2)$$

$$\ni f = (f_1, f_2)$$

$$\mathbb{C}_{y/x} \to \mathbb{C} \quad \mathbb{C}_{x/y} \to \mathbb{C}$$

$$C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_1 \cap U_2) \times \mathcal{F}(\underset{\mathbb{C}}{U_1} \cap \underset{\mathbb{C} \setminus \{0\} \cup \{\infty\}}{U_2}) \times \mathcal{F}(U_1 \cap U_2)$$

$$\ni f = (f_1, f_2, f_3)$$

$$\mathbb{C} \to \mathbb{C} \quad \mathbb{C} - \{0\} \to \mathbb{C} \quad \mathbb{C} \to \mathbb{C}$$

basically, the same as C^1, just longer tuple

$$C^2(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_1 \cap U_1 \cap U_1) \times \mathcal{F}(U_1 \cap U_1 \cap U_2) \times \mathcal{F}(U_1 \cap U_2 \cap U_2) \times \mathcal{F}(U_2 \cap U_2 \cap U_2)$$

Same for $C^3(\mathcal{U}, \mathcal{F})$

§12. Cohomology
(Homology ⟵ domains)

Coboundary

$\delta : C^i(\mathcal{U}, \mathcal{F}) \longrightarrow C^{i+1}(\mathcal{U}, \mathcal{F})$

Special cases:

$\delta = \delta_0 : C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F})$

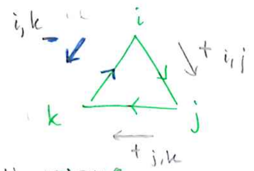
$\{(f_i)_{i \in I}\} \longmapsto \{(g_{i,j}) \in \mathcal{F}(U_i \cap U_j)\}$, $f_i \in \mathcal{F}(U_i)$

$$g_{i,j} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j}$$

$-$ $+$, $i \longmapsto j$

$\delta_1 : C^1(\mathcal{U}, \mathcal{F}) \longrightarrow C^2(\mathcal{U}, \mathcal{F})$

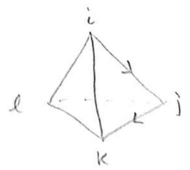
$\{(g_{i,j}) \in \mathcal{F}(U_i \cap U_j)\} \longmapsto \{(h_{i,j,k}) \in \mathcal{F}(U_i \cap U_j \cap U_k)\}$



$h_{i,j,k}$ is given by $g_{i,j}$, $g_{i,k}$, $g_{j,k}$ restricted to $U_i \cap U_j \cap U_k$ with signs

$$h_{i,j,k} = (g_{i,j} + g_{j,k} - g_{i,k})|_{U_i \cap U_j \cap U_k}$$

$\delta : C^2 \longrightarrow C^3$



There is a relation between the signs and combinatorics (orientation)

HW: Compute δ_0 for $X = \mathbb{P}^1$

Recall *Čech cohomology*

X topological Hausdorff space.
$\mathcal{F}$ sheaf on X
$\mathcal{U}$: $(U_i)_{i\in I}$ open covering
I maybe infinite

$$C^i(\mathcal{U}) = \prod_{\substack{n_0,\dots,n_i \in I \\ n_0 \le \dots \le n_i}} \mathcal{F}(U_{n_0} \cap U_{n_1} \cap \dots \cap U_{n_i})$$

- $C^0(\mathcal{U}) = \prod_{i\in I} \mathcal{F}(U_i)$
  Ψ
  $\{ f = (f_i)_{i\in I} : f_i \in \mathcal{F}(U_i) \}$

- $C^1(\mathcal{U}) = \prod_{i_0 \le i_1} \mathcal{F}(U_{i_0} \cap U_{i_1})$
  Ψ
  $\{ g = (g_{i,j})_{i\le j\in I} : g_{i,j} \in \mathcal{F}(U_i \cap U_j) \}$

- $C^2(\mathcal{U}) = \prod_{i\le j\le k\in I} \mathcal{F}(U_i \cap U_j \cap U_k)$
  Ψ
  $\{ h = (h_{i,j,k})_{i\le j\le k\in I} : h_{i,j,k} \in \mathcal{F}(U_i \cap U_j \cap U_k) \}$

For a compact RS, we only need to consider C^0, C^1, C^2 and C^3.

X manifold, $Z \subseteq X$ a submanifold, then Z is called a chain. Z is called a cycle if $\partial Z = \emptyset$

Boundary maps

$$\delta : C^i(\mathcal{U}) \to C^{i+1}(\mathcal{U})$$

Special cases

$\delta_0 : C^0(\mathcal{U}) \to C^1(\mathcal{U})$

$f = (f_i)_{i\in I} \to (g_{ij})_{i,j} = (-f_i|_{U_i\cap U_j} + f_j|_{U_i \cap U_j})$

$\delta_1 : C^1(\mathcal{U}) \to C^2(\mathcal{U})$

$(g_{ij})_{i,j} \longmapsto (h_{i,j,k})_{i,j,k}$

Lemma $\delta \circ \delta = 0$ and δ is linear

Proof For special case:

$$C^0(\mathcal{U}) \longrightarrow C^1(\mathcal{U}) \longrightarrow C^2(\mathcal{U})$$

$$\{f = (f_i)_{i\in I}\} \longmapsto \{g_{ij} = f_j - f_i\} \longmapsto \{h_{i,j,k} = g_{ij} + g_{jk} - g_{ik}$$
$$= (f_j - f_i) + (f_k - f_j) - (f_k - f_i) \equiv 0\}$$

$$\begin{cases} d \\ d' \\ d'' \end{cases} \Rightarrow d^2 = (d')^2 = (d'')^2 = 0$$

$$0 \xrightarrow{\delta} C^0(\mathcal{U}) \xrightarrow{\delta} C^1(\mathcal{U}) \xrightarrow{\delta} C^2(\mathcal{U}) \xrightarrow{\delta} C^3(\mathcal{U}) \longrightarrow \cdots$$

de Rham cohomology

$$0 \xrightarrow{d} C^\infty(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots \xrightarrow{d} \Omega^n(X) \xrightarrow{d} 0 \xrightarrow{d} 0$$

$C^\infty(X) = \{f : X \to \mathbb{C}\}$ smooth

$$0 \xrightarrow{d'} C^\infty(X) \xrightarrow{d'} \Omega^{1,0}(X) \xrightarrow{d'} \Omega^{2,0}(X) \xrightarrow{d'} \cdots \xrightarrow{d'} \Omega^{n,0}(X) \xrightarrow{d'} 0 \xrightarrow{d'} 0$$

$$0 \xrightarrow{d''} C^\infty(X) \xrightarrow{d''} \Omega^{0,1}(X) \xrightarrow{d''} \Omega^{0,2}(X) \xrightarrow{d''} \cdots \xrightarrow{d''} \Omega^{0,n}(X) \xrightarrow{d''} 0 \xrightarrow{d''} 0$$

Dolbreault comohology

$$0 \xrightarrow{d''} \Omega^{p,0}(X) \xrightarrow{d''} \Omega^{p,1}(X) \xrightarrow{d''} \Omega^{p,2}(X) \xrightarrow{d''} \cdots \xrightarrow{d''} \Omega^{p,n}(X) \xrightarrow{d''} 0 \xrightarrow{d''} 0$$

General philosophy of cohomology theory

- If we have a sequence

$$(\mathcal{G}) \qquad 0 \xrightarrow{\delta} G_0 \xrightarrow{\delta} G_1 \xrightarrow{\delta} G_2 \xrightarrow{\delta} \cdots$$

so that $\delta \circ \delta = 0$, then we have a cohomology, $\{H^i(\mathcal{G})\}_{i=0,1,2,3,\dots}$

- If we have a SES (short exact sequence)

$$0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{J} \to 0$$

then we get a long exact sequence (LES)

$$0 \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^0(\mathcal{J}) \to H^1(\mathcal{G}) \to H^1(\mathcal{H}) \to H^1(\mathcal{J}) \to H^2(\mathcal{G}) \to \cdots$$

<u>Lemma</u> Let $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$, V_1, V_2, V_3 vectorspaces, f_1, f_2 linear maps.
Assume that $f_2 \circ f_1 = 0$-map, then $\operatorname{Im}(f_1) \subseteq \operatorname{Ker}(f_2)$
(↑ Snt vector space)

<u>Def</u> $(\mathcal{G})$ $\quad 0 \xrightarrow{\delta} G_0 \xrightarrow{\delta} G_1 \xrightarrow{\delta} G_2 \xrightarrow{\delta} G_3 \xrightarrow{\delta} \cdots$

$$H^i(\mathcal{G}) = \operatorname{Ker}(G_i \xrightarrow{\delta} G_{i+1}) \Big/ \operatorname{Im}(G_{i-1} \xrightarrow{\delta} G_i).$$
("$Z_i(\mathcal{G})$" for the kernel, "$B^i(\mathcal{G})$" for the image)

If $x \in \operatorname{Ker}(G_i \xrightarrow{\delta} G_{i+1})$, we call it a δ-cocycle or δ-closed.

ex $\mathcal{G} = 0 \longrightarrow C^0(\mathcal{U}) \longrightarrow C^1(\mathcal{U}) \longrightarrow \cdots$

$$\begin{aligned} Z^0(\mathcal{G}) &= \operatorname{Ker}(C^0(\mathcal{U}) \xrightarrow{\delta} C^1(\mathcal{U})) := \{ f \in C^0(\mathcal{U}) : \delta f = 0 \} \\ &= \{ (f_i)_{i \in I} \in \mathcal{F}(U_i) : \delta f = (g_{ij} = f_j - f_i) = 0 \} \\ &= \{ (f_i)_{i \in I} : f_j|_{U_i \cap U_j} - f_i|_{U_j \cap U_i} = 0 \ \forall i,j \} \\ &= \{ f \in \mathcal{F}(X) \} = \mathcal{F}(X). \end{aligned}$$

$$B^0(\mathcal{G}) = \operatorname{Im}\{ 0 \xrightarrow{\delta} C^0(\mathcal{U}) \} = 0$$

$$H^0(\mathcal{G}) = Z^0(\mathcal{G}) / B^0(\mathcal{G}) = \mathcal{F}(X)/\{0\} = \mathcal{F}(X).$$

ex X compact Riemann surface. $\mathcal{F} = \mathcal{O} \implies H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) \cong \mathbb{C}$.

Generally $H^p(\mathcal{U}, \mathcal{F})$ for $p \geq 1$ will depend on the choice of the open covering $\mathcal{U}$.

<u>Goal</u> Want to define $H^p(X, \mathcal{F})$ <u>independent</u> of the choice of $\mathcal{U}$.

Idea: Recall: If $\mathcal{F}$ is a sheaf on X and $x \in X$, we define the <u>stalk</u> $\mathcal{F}_x$ by a kind of limits (reverse limit) on $\mathcal{F}(U)$, where U runs on all open nbh of x using that for $U \hookrightarrow V \implies \exists$ an identification $\mathcal{F}(V) \to \mathcal{F}(U)$.

(mimic ↙)

- Define a relation to say when an open covering $\mathcal{U}$ is "smaller" (or "finer") than an open covering $\mathcal{V}$. $\quad \mathcal{U} \longrightarrow \mathcal{V}$
- For each $\mathcal{U} \longrightarrow \mathcal{V}$ define a "restriction" ("identification"):
$$H^p(\mathcal{V}, \mathcal{F}) \longrightarrow H^p(\mathcal{U}, \mathcal{F})$$
- Then define $H^p(X, \mathcal{F})$ as a limit like in the case of stalk.

cont. Want to define $H^p(X, \mathcal{F})$ independent of covering $\mathcal{U}$.

1.step Define a relation (order) between open coverings of X.

X = topological Hausdorff space.

$\mathcal{U} = (U_i)_{i\in I}$, $\mathcal{V} = (V_j)_{j\in J}$ open coverings of X.

We write $\mathcal{U} < \mathcal{V}$ if $\forall i \ \exists j = j(i)$ so that $U_i \hookrightarrow V_{j(i)}$

(Basically, same as when we defined stalks, just family)

Choose one

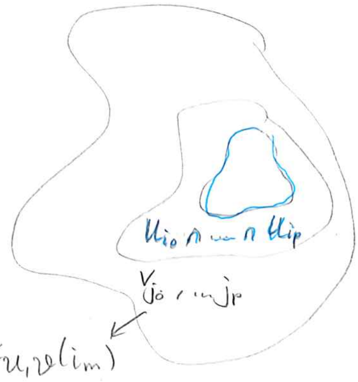
$$\tau_{\mathcal{U},\mathcal{V}} : I \to J \quad \text{so that if } j = \tau_{\mathcal{U},\mathcal{V}}(i) \Rightarrow U_i \hookrightarrow V_j.$$

2.step Let $\mathcal{U} < \mathcal{V}$ and a map $\tau_{\mathcal{U},\mathcal{V}} : I \to J$. Then define a map

$$\tau^p_{\mathcal{U},\mathcal{V}} : Z^p(\mathcal{V}, \mathcal{F}) \longrightarrow Z^p(\mathcal{U}, \mathcal{F})$$

$$f = (f_{j_0,\dots,j_p}) \mapsto g = (g_{i_0,\dots,i_p})$$

$$g_{i_0,\dots,i_p} = f_{\tau_{\mathcal{U},\mathcal{V}}(i_0),\dots,\tau_{\mathcal{U},\mathcal{V}}(i_p)}\big|_{U_{i_0\dots i_p}} \}.$$



Lemma $\delta g_{i_0,\dots,i_p} = \delta f_{\tau(\mathcal{U},\mathcal{V})(i_0),\dots,\tau_{\mathcal{U},\mathcal{V}}(i_p)}\big|_{U_{i_0,\dots,i_p}}$

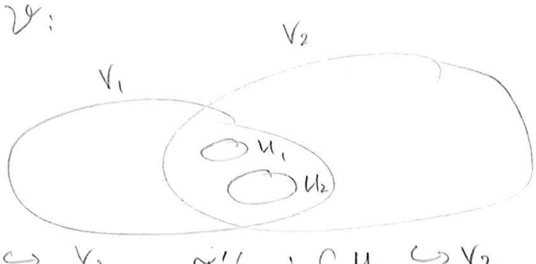
So if f is a p-cocycle $\Rightarrow$ g is also a p-cocycle.

- $\tau^p_{\mathcal{U},\mathcal{V}} : B^p(\mathcal{V}, \mathcal{F}) \longrightarrow B^p(\mathcal{U}, \mathcal{F})$

Lemma There is a map

$$\tau_{\mathcal{U},\mathcal{V},p} : H^p(\mathcal{V}, \mathcal{F}) \longrightarrow H^p(\mathcal{U}, \mathcal{F})$$

This map is better than the counterpart for defining stalks stalks of a sheaf.



$U_3 = V_1$, $V_4 = V_2$

$\mathcal{U} < \mathcal{V}$

$\tau_{\mathcal{U},\mathcal{V}}$: $\begin{cases} U_1 \hookrightarrow V_1 \\ U_2 \hookrightarrow V_2 \\ U_3 \hookrightarrow V_1 \\ U_4 \hookrightarrow V_2 \end{cases}$ $\tau''_{\mathcal{U},\mathcal{V}}$: $\begin{matrix} U_1 \hookrightarrow V_2 \\ U_2 \hookrightarrow V_2 \\ U_3 \hookrightarrow V_1 \\ U_3 \hookrightarrow V_2 \end{matrix}$ $\tau'_{\mathcal{U},\mathcal{F}}$: $\begin{cases} U_1 \hookrightarrow V_2 \\ U_2 \hookrightarrow V_1 \\ U_3 \hookrightarrow V_1 \\ U_4 \hookrightarrow V_2 \end{cases}$

Lemma ~~12.3~~ 12.4 If $\tau_{\mathcal{U},\mathcal{V}}, \tau'_{\mathcal{U},\mathcal{V}} : I \to J$
$(\mathcal{U} \to \mathcal{V})$

then $\tau_{\mathcal{U},\mathcal{V}} = \tau'_{\mathcal{U},\mathcal{V}} : H^p(\mathcal{V}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F})$

$\tau_{\mathcal{U},\mathcal{V}}$ is injective.

HW: Check this lemma (claim) for the previous lemma.

Recall $\mathcal{F}_x = \bigsqcup_{\substack{U \ni x \\ \text{open}}} \mathcal{F}(U)/\sim$ where $f \in \mathcal{F}(V)$ is identified with $\mathrm{im}(f)$ in $\mathcal{F}(V) \to \mathcal{F}(U)$ for each $U \hookrightarrow V$.

Def $H^p(X, \mathcal{F}) = \bigsqcup_{\mathcal{U} \text{ open covering of } X} H^p(\mathcal{U}, \mathcal{F})/\sim$ where $f \in H^p(\mathcal{V}, \mathcal{F})$ is identified with the image of $\mathrm{im}(f)$ in $H^p(\mathcal{V}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F})$ wherever $\mathcal{U} < \mathcal{V}$.

$\mathcal{O}_x$ → If U is isomorphic to $B(a, r) \subset \mathbb{C}$, then $\mathcal{O}(U)$ is easy to compute
→ For every V, there is such a U so that $U \hookrightarrow V$

↓

What is a "good" open covering of X?

Def We say that an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X for $\mathcal{F}$ is "good" if
"Leray covering" of first order: $H^1(U_i, \mathcal{F}) = 0 \quad \forall i \in I$

Theorem 12.8 If $\mathcal{U}$ is a Leray covering of X for $\mathcal{F}$, then $H^1(X, \mathcal{F}) \simeq H^1(\mathcal{U}, \mathcal{F})$.

We look at some cases where $H^1(\mathcal{U}, \mathcal{F}) = 0$

Theorem 12.6 and 12.7

① X RS, $\mathcal{E}$ = sheaf of differentiable functions on X $\Longrightarrow$ $H^1(X, \mathcal{E}) = 0$

② $H^1(X, \mathbb{C}) = 0$ where X is simply-connected RS & $\mathbb{C}$ is the constant sheaf on X and (the sheaf associated with the presheaf)

$H^1(X, \mathbb{Z}) = 0$ if $\pi_1(X) = 0$, $U_{open} \hookrightarrow \mathbb{C}(U) = \mathbb{C}$

24.10.

Proof ① The importance is that we have Partition of Unity for differentiable functions (not true for analytic functions).

So $H^1(X, \mathcal{O})$ may be non-zero

Partition of unity: Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X.

Then $\exists\, \psi_i : X \to \mathbb{R}_{\geq 0}$ smooth $(\forall i \in I)$ s.t:

- $supp(\psi_i) := \{x \in X : \psi_i(x) \neq 0\} \subseteq U_i$
- $\forall x \in X$, $\exists\, U_{open} \ni x$ s.t. the number of $\{j \in I : Supp(\psi_j) \cap U = \phi\}$ is finite.
- $\sum_{i\in I} \psi_i(x) = 1 \quad \forall x$ (need to use the axiom of choice).

Now we show that $H^1(X, \mathcal{E}) = 0$

$$\parallel$$

$$Z^1(\mathcal{U}, \mathcal{E}) / B^1(\mathcal{U}, \mathcal{E}) \Longleftrightarrow \forall g \in Z^1(\mathcal{U}, \mathcal{E}) \Rightarrow g \in B^1(\mathcal{U}, \mathcal{E})$$

$$\Longleftrightarrow \forall g : \delta g = 0 \Rightarrow g = \delta f \text{ for some } f$$

For each $i \in I$; define $h_i = \sum \psi_j g_{ij} \in \mathcal{E}(U_i)$, $(g_{ij})_{i,j\in I} \in \mathcal{E}(U_i \cap U_j)$

$$h_i|_{U_i\cap U_j} - h_j|_{U_i \cap U_j} = \sum_{k\in I} \psi_k g_{ik} - \sum_{k \in I} \psi_k g_{jk} = \sum_{k\in I} \psi_k (g_{ik} - g_{jk}) = \sum_{k\in I} \psi_k g_{ij} = g_{ij} \sum_{k \in I} \psi_k = g_{ij}$$

(independent of k.)

$$0 = (\delta g)_{ij,k} = (g_{ij} + g_{jk} - g_{ik})|_{U_i \cap U_j \cap U_k}$$

So $\delta h = g$

$\Downarrow$

$g \in B^1(\mathcal{U}, \mathcal{E})$

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Proof of Thm. 12.7: X simply connected RS. Then a) $H^1(X, \mathbb{C}_X) = 0$, b) $H^1(X, \mathbb{Z}_X) = 0$.

a) X is top. space, $\mathbb{C}_X$ is the sheaf on X associated to the presheaf $U \mapsto \mathbb{C}$ (U open set)

$\mathbb{C}_X(U) = \bigsqcup_{\#\text{connected components of } U} \mathbb{C}$ $\qquad$ $\mathbb{C} \sqcup \mathbb{C}$.

Want to show that $\forall\, \mathcal{U} = (U_i)_{i\in I}$ open covering of X, then $H^1(\mathcal{U}, \mathbb{C}_X) = 0$

$H^1(\mathcal{U}, \mathbb{C}_X) = Z^1(\mathcal{U}, \mathbb{C}_X)/B^1(\mathcal{U}, \mathbb{C}_X)$

WTS: $f \in Z^1(\mathcal{U}, \mathbb{C}_X) \Rightarrow f \in B^1(\mathcal{U}, \mathbb{C}_X)$

$f = (f_{ij})_{i,j} \in \mathbb{C}$, $f_{ij} \in \mathbb{C}_X(U_i \cap U_j)$

f_{ij} is a complex number on each connected component of $U_i \cap U_j$ (other case is disjoint union of $\mathbb{C}$) and

$$f_{ij} = f_{ik} - f_{jk} \quad \forall i,j,k.$$

We can think about the number f_{ij} as a constant function, so $f_{ij} \in \mathcal{E}(U_i \cap U_j)$.

By Thm 12.6, since $\delta f = 0 \Rightarrow f_{ij} = h_i - h_j$ for some $h_i \in \mathcal{E}(U_i)$.

Since f_{ij} is locally constant on each connected component, $d(f_{ij}) = 0 = dh_i - dh_j \ \forall ij$

$dh_i \in \Omega^1(U_i)$ and $dh_i = dh_j$ on $U_i \cap U_j$.

So $\exists\, \omega \in \Omega^1(X)$ s.t. $\omega|_{U_i} = dh_i$

Now (by a thm in §9 or 10) $\pi_1(X) = 0 \Rightarrow \exists\, h \in \mathcal{E}(X)$ s.t. $d(h) = \omega$, by:

Choose a base-point $x_0 \in X$. Then for every $x \in X$ choose a smooth curve $\ell(x_0, x) \subseteq X$ connecting x_0 and x and define

$$h(x) = \int_{\ell(x_0,x)} \omega$$

independent of the choice of $\ell(x_0, x)$ (homotopic) because $\pi(X) = 0$

Define $g_i = h_i - h \in \mathcal{E}(U_i)$. Now $g_i \in \mathbb{C}_X(U_i) \Leftrightarrow d(g_i) \equiv 0$!

Then $f_{ij} = h_i - h_j = (h_i - h) - (h_j - h) = g_i - g_j = \delta g$

(with $h_i \in \mathcal{E}(U_i)$, $h_j \in \mathcal{E}(U_j)$, $g_i \in \mathbb{C}_X(U_i)$, $g_j \in \mathbb{C}_X(U_j)$)

The main point of the proof is

$$\mathbb{C}_X(U) = \{ f \in \mathcal{E}(U) : df = 0 \}$$

So we can make use of $\mathcal{E}(U)$ in computing with $\mathbb{C}_X(U)$

b) Similar to a) but need to make sure that $g_i = h_i - h$ is actually in $\mathbb{Z}$, not only in $\mathbb{C}$.

$z \in \mathbb{Z} \Leftrightarrow z \in \mathbb{C} \wedge \exp(2\pi i z) = 1$

$\mathbb{Z}_X \hookrightarrow \mathbb{C}_X$

$$0 \to \mathbb{Z}_X \to \mathbb{C}_X \to \mathbb{C}_X \setminus \{0\} \to 0$$

$z \mapsto e^{2\pi i z}$

Proof Thm 12.7 b cont.

If $f := (f_{ij}) \in Z^1(\mathcal{U}, \mathbb{Z}_X)$. By 12.7 a) $\exists\ c_i \in \mathcal{C}_X(U_i)$ s.t. $f_{ij} = c_i - c_j$
$\quad \cap Z^1(\mathcal{U}, \mathcal{C}_X)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\cap}{\mathbb{Z}}$ locally

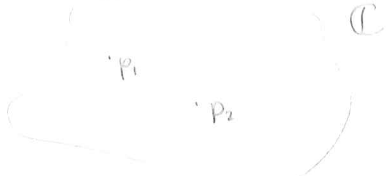
$$1 = e^{2\pi i f_{ij}} = \frac{e^{2\pi i c_i} \in \mathcal{C}_X(U_i)}{e^{2\pi i c_j} \in \mathcal{C}_X(U_j)}$$

Since $\mathcal{C}_X$ is a sheaf, $\exists\ c \in \mathcal{C}_X(X)$ s.t. $c|_{U_i} = e^{2\pi i c_i}$ by the sheaf-axioms.

X is connected, so $c \in \mathbb{C}$ must be a constant. Choose any d s.t. $e^{2\pi i d} = c$,

then define $h_i = c_i - d \in \mathbb{Z} \implies f = dh$.

$$\mathcal{E}(U) \supseteq \mathcal{C}_X(U) \supseteq \mathbb{Z}_X(U)$$
$\{d(f) = 0\}$ $\qquad$ $\{e^{2\pi i f} = 1\}$ $\leftarrow$ useful technique for sheaf cohomology.



exer 12.1. $H^1(X, \mathbb{Z}) = \mathbb{Z}^n$ where $X = \mathbb{C} \setminus \{p_1, \dots, p_n\}$

Hint: $\mathbb{C} \setminus \{p_1, \dots, p_n\} = U_1 \cup U_2$ where U_1, U_2 are simply connected and $U_1 \cap U_2$ = $n+1$-connected components

What is $H^1(\mathbb{P}^1, \mathbb{C})$?

$\mathbb{P}^1 = U_1 \cap U_2$ where $U_i \cong \mathbb{C}$ simply connected $(\pi_1(U_1) = \pi_1(U_2) = 0$

and $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ is connected.

By Thm 12.6 and 7, $H^1(U_1, \mathbb{C}) = H^1(U_2, \mathbb{C}) = 0$

so $\mathcal{U} = \{U_1, U_2\}$ is a Leray covering.

By Leray's thm $H^1(\mathbb{P}^1, \mathbb{C}) = H^1(\mathcal{U}, \mathbb{C})$. Now we compute $H^1(\mathcal{U}, \mathbb{C})$.

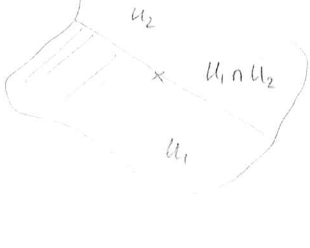
$C^0(\mathcal{U}, \mathbb{C}) = \{(f_1, f_2) : f_1 \in \mathbb{C}(U_1), f_2 \in \mathbb{C}(U_2)\}$. (with $\mathbb{C}(U_1) = \mathbb{C}$, $\mathbb{C}(U_2) = \mathbb{C}$)

$C^1(\mathcal{U}, \mathbb{C}) = \{g : g \in \mathbb{C}(U_1 \cap U_2)\}$ (with $\mathbb{C}(U_1 \cap U_2) = \mathbb{C}$)

$Z^1(\mathcal{U}, \mathbb{C}) = \{g \in C^1(\mathcal{U}, \mathbb{C}) : \delta g = 0\} = \mathbb{C}$ $\qquad B^1(\mathcal{U}, \mathbb{C}) = \{g \in C^1(\mathcal{U}, \mathbb{C}) : g \in \delta f,\ f \in C^0(\mathcal{U}, \mathbb{C})\}$

$\delta f = f_1 - f_2$

$\mathbb{C}$ $\qquad \mathbb{C} = Z^1(\mathcal{U}, \mathbb{C})$



exer 12.1 cont.

$\mathcal{U}$ is a Leray covering.

$H^1(\mathbb{C} \setminus \{p_1, \dots, p_n\}, \mathbb{Z}) = H^1(\mathcal{U}, \mathbb{Z})$

$C^0(\mathcal{U}, \mathbb{Z}) = \{(f_1, f_2) : f_1 \in \mathbb{Z}(U_1) = \mathbb{Z} \quad f_2 \in \mathbb{Z}(U_2) = \mathbb{Z}\}$

$C^1(\mathcal{U}, \mathbb{Z}) = \{g : g \in \mathbb{Z}(U_1 \cap U_2)\}$
$\qquad = (g_0, g_1, \dots, g_n) : g_i \in \mathbb{Z}$

$\mathbb{Z}^{n+1} \simeq Z^1(\mathcal{U}, \mathbb{Z}) = C^1(\mathcal{U}, \mathbb{Z})$

$\mathbb{Z} \simeq B^1(\mathcal{U}, \mathbb{Z}) = \{g : (g_0, \dots, g_n) : g_j = (f_1 - f_2)|_{U_j}\}$
$\qquad = \{g = (g_0, \dots, g_0) : g_0 \in \mathbb{Z}\}$

Examples of meromorphic 1-form RS

$\mathbb{P}^1 = U_1 \cup U_2$, $U_1 = \mathbb{C}_z$, $U_2 = \mathbb{C}_w$ $\qquad [x:y]$, $z = \frac{x}{y}$ $\qquad$ Relation $w = \frac{1}{z}$

$\mathbb{C}_z \hookrightarrow \mathbb{P}^1 \qquad \alpha = dz$ on $\mathbb{C}_z = \mathbb{P}^1 \setminus \{z = \infty\}$ $\qquad$ holomorphic 1-form
$\qquad \{z=\infty\} \Updownarrow w = 0$

Want to extend α to a meromorphic 1-form on $\mathbb{P}^1$:

$\alpha = dz = d(\frac{1}{w})$ on $\mathbb{C}_z \cap \mathbb{C}_w$

$= \frac{-dw}{w^2} = f(w)\,dw$ $\qquad$ so α is a holomorphic 1-form on $\mathbb{C}_w \setminus \{w=0\}$ and has a pole at $w=0$. $\quad (f(w) = -\frac{1}{w^2})$

So α extends to a meromorphic 1-form on $\mathbb{P}^1$. $\quad \mathrm{Res}_{\{w=0\}} \alpha = 0$.

Is there a non-zero holomorphic 1-form on $\mathbb{P}^1$?

[Goal: Riemann-Roch: the vector space of holomorphic 1-forms on X = compact RS is isomorphic to $H^1(X, \mathcal{O}) \subseteq H^1(X, \mathbb{C})$. For $\mathbb{P}^1$: $H^1(\mathbb{P}^1, \mathbb{C}) = 0$.

Assume α is a non-zero holomorphic 1-form on $\mathbb{P}^1$.

$\alpha|_{\mathbb{C}_z} = P(z)\,dz$, $\qquad$ P a holomorphic polynomial with no zero.
$\Rightarrow P$ is a nonzero constant

$\Rightarrow \alpha = -\frac{P\,dw}{w^2}$ is not holomorphic ex $C = \{[x:y:z] \in \mathbb{P}^2 : y^2 z = x^3 + z^3\}$ is an elliptic curve.

Look at $\mathbb{C}^2_{X,Y} \subseteq \mathbb{P}^2$ and the map

$\| \quad \{[x:y:z] \in \mathbb{P}^2 : z \neq 0\}, \; X = \frac{x}{z}, \; Y = \frac{y}{z}.$

$C \cap \mathbb{C}^2_{X,Y} \longrightarrow \mathbb{C}$

$\{Y^2 = X^3 + 1\} \mapsto X$

1) Can we extend the map as a holomorphic function C to $\mathbb{P}^1$? 2) What are the branch points?

$y^2 z = x^3 + z^3, \; z = 0 \Rightarrow x = 0$

1) We need to see whether $\lim_{[x:y:z] \in C \to [0:1:0]} \left|\frac{x}{z}\right|$ is bounded (can extend) or ∞ (pole).

$[0:1:0] \in \{y \neq 0\}$ so we can work on $\{y = 1\}$ $\quad X = \frac{x}{y} \quad Z = \frac{z}{y}$

so we get $C: \{Z = X^3 + Z^3\}$

Want to compute $\lim \frac{x}{z} = \lim \frac{(\frac{x}{y})}{(\frac{z}{y})} = \lim_{Z \to 0} \frac{X}{Z}$ $\quad Z = X^3 + Z^3$

$\overset{\text{l'H}}{=} \lim_{Z \to 0} \frac{X(Z)}{Z} = X'(0)$ $\quad \frac{X(Z) - X(0)}{Z - 0}$

Implicit derivation:

$Z = X^3 + Z^3 \xrightarrow{\text{take derivative of } Z} 1 = 3X^2 \cdot X'(Z) + 3Z^2$ $\quad Z = 0 \Rightarrow X = 0$

$1 = 3 \cdot 0 \cdot X'(0)$ $\quad \lim \frac{X}{Z} = \infty$ $\rightsquigarrow$ so pole.

28.10.19

$C \setminus \{z = 0\} \longrightarrow \mathbb{C}$

$[x:y:z] \longmapsto x/z$ $\quad C \cap \{z = 0\} = [0:1:0] \in \{y \neq 0\}$

So $U = C \cap \{y \neq 0\}$ is an open neighborhood of $[0:1:0]$ inside C. On U we have convenient coordinate system to work with:
$$\begin{cases} X = x = x/y \\ y = 1 \\ Z = z = z/y \end{cases}$$
Here $C_U = \{(X,Z) \in \mathbb{C}^2 : Z = X^3 + Z^3\}$ and map $[0:1:0] = (0,0)$ in this coordinate system.

$C_U \setminus \{(0,0)\} \longrightarrow \mathbb{C}$

$(X,Z) \longmapsto \frac{x}{z} = X/Z$ $\quad$ want to see what happens to $\lim_{Z \to 0} X/Z$

$C_U \setminus \{(0,0)\} \ni (X,Z)$

Implicit derivation: Say we have point (X,Z) that satisfy an equation $f(X,Z) = 0$ and when $X = a \Rightarrow Z = b$. If $\frac{\partial f}{\partial Z}(a,b) \neq 0$ then near (a,b) Z is a function of X, say $Z = g(X)$, but in general we cannot write the explicit formula. Now <u>how</u> do we compute $g'(a)$?

We have $h(X) = f(X, g(X)) = 0$

$$h'(X) = \frac{\partial f}{\partial X}(X, g(X)) + \frac{\partial f}{\partial Z}(X, g(X)) \cdot g'(X)$$

In particular, if $X = a \Rightarrow g(X) = Z = b$ and $\frac{\partial f}{\partial Z}(a,b) \neq 0$ by assumption,

so we obtain $0 = \frac{\partial f}{\partial X}(a,b) + \frac{\partial f}{\partial Z}(a,b) \cdot g'(a)$

$$\Longrightarrow \quad g'(a) = \frac{-\frac{\partial f}{\partial X}(a,b)}{\frac{\partial f}{\partial Z}(a,b)}$$

ex cont. applying implicit derivation, $f(X,Z) = X^3 + Z^3 - Z$ $\quad X=0 \Rightarrow Z=0$.

$$\frac{\partial f}{\partial X}(0,0) = 3X^2\Big|_{(0,0)} = 0, \quad \frac{\partial f}{\partial Z}(0,0) = 3Z^2 - 1\Big|_{(0,0)} = -1.$$

$$\frac{dZ}{dX}\Big|_{(0,0)} = \frac{-0}{-1} = 0.$$

<u>l'Hospital's rule</u>

If $Z = g(X)$ is so that $g(0) = 0$ and $g'(0)$ exists, then $\lim_{X\to 0} \frac{g'(X)}{X} = g'(0)$.

$Z = g(X)$ satisfies $\begin{cases} g(0) = 0 \\ g'(0) = 0 \end{cases}$ $\quad \lim_{\substack{X\to 0 \\ Z \to 0}} \frac{X}{Z} = \lim_{X\to 0} \frac{1}{\frac{Z}{X}} = \frac{1}{g'(0)} = \frac{1}{0} = \infty$ so pole.

Hence this map has a pole at $(X,Z) = (0,0)$ and thus can be extended to a map $C \to \mathbb{P}^1$.

$$\{y^2 z = x^3 + z^3\} = C \longrightarrow \mathbb{P}^1$$

$[0:1:0] \neq [x:y:z] \mapsto [x/z : 1] = [x:z]$

$[0:1:0] \mapsto [1:0]$

What is the degree of this map? (no points in the preimage of a point)

$z \neq 0$ Work in coordinates $X = \frac{x}{z}$, $Y = \frac{y}{z}$, $z = 1$

$Y^2 = X^3 + 1$

$(X,Y) \to X$

For each fixed X there are <u>two</u> values of Y $\Rightarrow$ degree $= 2$.

<u>Critical points</u> are points $[x_0 : y_0 : z_0] \in C$ which appear with multiplicity at least 2 in $\varphi^{-1}(\varphi([x_0:y_0:z_0]))$ (same as $\varphi'([x_0:y_0:z_0]) = 0$ in any local chart around $[x_0:y_0:z_0]$).

<u>Corollary</u>: If $\deg(\varphi) = 2$ then $[x_0 : y_0 : z_0]$ is a critical point of φ

$\iff \varphi^{-1}(\varphi[x_0:y_0:z_0]) = \{[x_0:y_0:z_0]\}$

If $z_0 \neq 0$ $\quad Y^2 = X^3 + 1$

$Y_1, Y_2 = \pm\sqrt{X^3+1}$ $\quad$ so $Y_1 = Y_2$, $\varphi^{-1}(\varphi[x_0 : y_0 : z_0]) = \{\text{one point}\} \iff \left(\frac{x_0}{z_0}\right)^3 + 1 = 0$

If $z_0 = 0 \Rightarrow$ the point is $[0:1:0]$ and $\varphi([0:1:0]) = [1:0]$ so

$\varphi^{-1}([1:0]) = [0:1:0]$ with multiplicity 2 (by applying theorem 8...)

HW: $C = \{[x:y:z] \in \mathbb{P}^2 : y^2 z = x^3 + z^3\}$

$C \setminus \{z=0\} \longrightarrow \mathbb{C}$

$[x:y:z] \mapsto \frac{x^3 - y^3}{z^3}$

$H^1(\cdot, \mathcal{E})$
↳ differentiable functions. We can use partition of unity

For $H^1(\cdot, \mathcal{O})$
↳ holomorphic functions we (need) use Dolbeault's lemma.

We use that $\mathcal{O} \hookrightarrow \mathcal{E}$.
$\mathcal{O} = \{f : \bar\partial f = 0\}$

$\mathbb{C} \hookrightarrow \mathcal{E}$, $\mathbb{C} = \{f : df = 0\}$

Lemma 13.1 If $g \in \mathcal{E}(\mathbb{C})$ has compact support (i.e. $\{x \in \mathbb{C} : g(x) \neq 0\}$ is bounded) then $\exists f \in \mathcal{E}(\mathbb{C})$ so that

$$\bar\partial f := \frac{\partial f}{\partial \bar z} = g.$$

Proof (Sketch) Define $f(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z-\zeta}\, dz \wedge d\bar z$

$z = x + iy$
$dz = dx + i\,dy$
$d\bar z = dx - i\,dy$
$\Rightarrow dz \wedge d\bar z = -2i\, dx \wedge dy$ (because of this, $\iint$.)

Green's theorem.

Theorem 13.2 Suppose $X := \{z \in \mathbb{C} : |z| < R\}$. $0 < R \leq \infty$. $g \in \mathcal{E}(X)$. $\Rightarrow \exists f \in \mathcal{E}(X)$ such that $\frac{\partial f}{\partial \bar z} = g$.

ex $u \in Z^1(\mathcal{U}, \mathbb{C})$, $u = h_1 - h_2$, $h_1, h_2 \in Z^1(\mathcal{U}, \mathcal{E})$ (because we know true in $\mathcal{E}$). Now h_1, h_2 may not be in $Z^1(\mathcal{U}, \mathbb{C})$, but we can try to find $h \in Z^1(\mathcal{U}, \mathcal{E})$ s.t. $u = (h_1 - h) - (h_2 - h)$ and $d(h_1 - h) = d(h_2 - h) = 0$

Remark Lemma 13.1 is a special case of Theorem 13.2 where we choose $R > 0$ large enough so that $\mathrm{supp}(g) \subseteq \{z \in \mathbb{C} : |z| < R\}$.

On the other hand we can solve thm 13.2 by applying a trick to Lemma 13.1

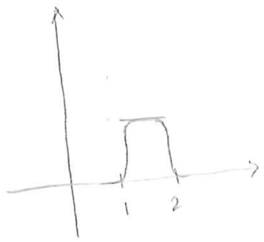
Trick (cut-off function) Every function is a uniform limit of functions with compact support. (increase to)

$\psi_n \nearrow 1$

For example we can find a sequence $\psi_n \in \mathcal{E}(\mathbb{C})$ s.t. ψ_n has compact support and in $X = \{z \in \mathbb{C} : |z| < R\}$.

Can assume $\psi_n = 1$ on $\{|z| < R_n\} = X_n$ where $R_n \nearrow R$.

$g_n = \psi_n g$ has compact support.



So by lemma 13.1 we have an explicit formula:

$$f_n(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g_n(z)}{z - \zeta}\, dz \wedge d\bar z \quad \text{such that}$$

$$\frac{\partial f_n}{\partial \bar z} = g_n = \psi_n g$$

if f_n converges uniformly to f we are done, because then $\frac{\partial f}{\partial \bar z} = \lim_{n\to\infty} \frac{\partial f_n}{\partial \bar z} = \lim_{n\to\infty} \psi_n g = g$.

In general, what we can do is to change to

$\tilde f_{n+1} = f_{n+1} - \underbrace{p_n(z)}_{\text{polynomial}}$ (good thing is $\bar\partial p_n = 0$ since p_n polynomial).

We choose p_n so that $\| \tilde f_n - f_{n+1} - p_n \|_{X_{n-1}} < \underbrace{2^{-n}}_{\text{geometric series}}$, $X_{n-1} = \{|z| < R_n\}$

So $\| \tilde f_n - f_{n+1} \|_{X_{n-1}} \leq \frac{1}{2^n}$

Trick telescope series: If $\sum_{n=0}^{\infty} (f_{n+1} - f_n)$ is uniformly convergent, then $\{f_n\}$ converges.

$f_{n+1} = (f_{n+1} - f_n) + (f_n - f_{n-1}) + \dots + (f_2 - f_1) + (f_1 + f_0) + f_0$

Why does such a polynomial p_n exist?

The point is that $f_{n+1} - f_n$ is holomorphic on X_n because

$$\bar\partial(f_{n+1} - f_n) = \bar\partial f_{n+1} - \bar\partial f_n = \psi_{n+1} g - \psi_n g = 0 \quad \text{on } X_n \gg X_{n-1}$$

So we have a Taylor expansion

$$f_{n+1} - f_n = p_n(z) + \text{higher order term of small value on } X_{n-1}.$$

Theorem 13.4 $H^1(X, \mathcal{O}) = 0$ if $X = \{z \in \mathbb{C} : |z| < R\}$.

Proof Repeat the proof for $H^1(X, \mathbb{C})$, $H^1(X, \mathbb{Z})$

$$\mathbb{C} \longrightarrow \mathcal{E}$$
$$\mathbb{C} = \{f : df = 0\} \subset \mathcal{E}$$
$$\mathcal{O} \hookrightarrow \mathcal{E}$$
$$\mathcal{O} = \{f \in \mathcal{E} : \bar\partial f = 0\}$$

$\mathcal{U}$ an open covering of X. $H^1(\mathcal{U}, \mathcal{E}) = 0$ if $f \in Z^1(\mathcal{U}, \mathcal{O}) \Rightarrow f \in Z^1(\mathcal{U}, \mathcal{E})$

So $f = g_1 - g_2$ where $g_1, g_2 \in Z^0(\mathcal{U}, \mathcal{E})$

Want to find $g \in \mathcal{E}(X)$ s.t. $g_1 - g, \; g_2 - g \in Z^0(\mathcal{U}, \mathcal{O})$.

We have $0 = \bar\partial f = \bar\partial g_1 - \bar\partial g_2$

$\Downarrow$

$\{\bar\partial g : \ldots\}$ this gives us a global $(0,1)$-form on X

g_1, g_2 locally defined, g globally

If $g \in \mathcal{E}(X)$ is s.t.

$$\frac{\partial g}{\partial \bar z} = \mathcal{K} \Rightarrow \bar\partial g = \frac{\partial g}{\partial \bar z} d\bar z = \mathcal{K}\, d\bar z$$

we can solve for differential 1-forms, similarly can solve for $(0,1)$-forms.

$\mathbb{C} = \{f \in \mathcal{E} : df = 0\}$ (a differential equation)

$\downarrow$

$\mathcal{E} \hookleftarrow \mathcal{O}$

$\mathcal{O} = \{f \in \mathcal{E} : \bar\partial f = 0\}$ (differential equation)

<u>Remark</u> Knowing existence of solutions to differentiable equations helps knowing about cohomology and vice versa.