Subfactor theory is about:

- inclusion of factors $N \subset M$

  M, N: vN alg. triv. center
- relation to tensor categories $\mathcal{C}, \otimes$

  monoidal op. $X \otimes Y$ for objs, morphs.

  $\rightsquigarrow$ connects to QFT, quantum groups, ...

Motivating examples.

1. P factor, G discr. grp, $\alpha: G \curvearrowright P$ s.t.

   $g \neq e \Rightarrow \nexists\, u \in P$ unitary s.t. $\alpha_g(x) = uxu^*$.

   (α is an <u>outer</u> action)

   ex. $P = \bigotimes_G (P_0, \omega)$ for some factor P_0, α: perm. tensor comp.

   fns ω

   $\Rightarrow P \subset P \rtimes_\alpha G$ is a subfactor. ($P' \cap P \rtimes_\alpha G = \mathbb{C}$) <u>irreducible</u> subf.

   G finite $\Rightarrow P^G \subset P$ also.

2. G compact $\alpha: G \curvearrowright P$ s.t. $(P^G)' \cap P = \mathbb{C}$ <u>minimal</u>

   $\Rightarrow (\pi, H_\pi)$ f.d. unitary rep. of G.

   $P^G \subset (P \otimes B(H_\pi))^G$ for $\alpha \otimes \mathrm{Ad}_\pi$.

   $P = (B(H_{\pi_0}), \mathrm{tr}.)^{\otimes\infty}$ $\overset{\mathrm{Ad}_{\pi_0}^{\otimes\infty}}{\curvearrowleft} G$

Goal:

- basic concepts, e.g.
  - standard form $P \curvearrowright L^2(P, \omega)$, ...
  - index of subfactors $[M:N]$ Jones.

    "relative size of M over N"

    $\rightsquigarrow [M:N] \in \{4\cos^2\frac{\pi}{n} : n \geq 3\} \cup [4, \infty)$

    $= \{(q + q^{-1})^2 > 0 : q \text{ root of unity or } > 0\}$

- standard invariant of subfactor.

  invar. of $N \subset M$ as a rigid C^*-2-cat.

  - $\exists$ tracial st on M $\rightsquigarrow$ bimodules over M & N, $\otimes$.
    "finite case"
  - $\nexists$ tr. st. $\rightsquigarrow$ endomorphisms, compos.
    "(properly) infinite case"

- amenability of standard invars

  - amen $\Rightarrow$ classification Popa, Tomatsu.

    $N \subset M$ & $N \subset L$ same std inv.

    $\Rightarrow \exists\ \theta : M \xrightarrow{\sim} L$, $\theta|_N = \mathrm{id}$.

- beyond usual subfactor theory

  ? algebraic quantum (conformal) field theory ?

§ II_1 factors (mostly after Jones–Sunder §2)

Rec. P is a II_1 factor iff

1. C^*-alg s.t. $P = (\underbrace{P_*}_{\text{predual}})^*$ as a Banach sp.

   $\exists!$ tracial st τ : $\tau(xy) = \tau(yx)$

2. $\underset{\infty\text{-dim.}}{P} \subset B(H)$, $P = P''$, $Z(P)\ (= P \cap P') = \mathbb{C}$

   e, f prjs in P, $e \le f$, $e \underset{\text{MvN}}{\sim} f \Rightarrow e = f$.

Rem. $\{\tau(e) : e \in P \text{ prj}\} = [0, 1]$.

$\tau(e) = \tau(f) \iff e \sim f$.

Ex. 1. $P = (M_2(\mathbb{C}), \mathrm{tr})^{\otimes\infty}$ (as vN alg)

i.e. take UHF alg $A = M_2(\mathbb{C})^{\otimes\infty}$

GNS rep for $\tau = \mathrm{tr}^{\otimes\infty}$, w^*-clos.

2. (Connes) Q : II_1 factor, inj. op. sys.

$\iff P \simeq Q$.

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Def. P vN alg. a <u>left</u> (Hilbert) <u>P-module</u>: is given by
- Hilb. sp. H
- normal unital $*$-hom $P \xrightarrow{\pi} B(H)$

write ${}_P H$

<u>right</u> P-mod: π is <u>anti</u> $*$-hom. (H_P)

$\pi(ab) = \pi(b)\pi(a)$

Q: another vN alg., P-Q-mod. is given by

H, $\pi^P : P \to B(H)$ normal $*$-hom, $\pi^Q : Q \to B(H)$ normal anti $*$-hom.

$\pi^P(a)\pi^Q(b) = \pi^Q(b)\pi^P(a)$

Often write $a\xi b$ for $\pi^P(a)\pi^Q(b)\xi$.

Ex. τ: tr. st. on P. $H = L^2(P, \tau)$.

write $H = \langle a\tau^{1/2} : a \in P \rangle^{-\text{compl.}}$

$a(b\tau^{1/2})c = abc\tau^{1/2}$

$\leadsto$ How do we characterize $P\tau^{1/2} \subset L^2(P,\tau)$?

Def. ${}_P H \ni \xi$ is right P-bdd (w.r.t. τ)

if $\exists c > 0$: $\|a\xi\|_H \le c\|a\tau^{1/2}\|$

In this case $R_\xi : L^2(P,\tau) \to H$, $a\tau^{1/2} \mapsto a\xi$

Parallelly: $H_P \ni \xi$ is left P-bdd w.r.t. τ..

$L_\xi : L^2(P,\tau) \to H_P$, $a\tau^{1/2} \mapsto \xi a$.

Thm-def: ${}_P H^0 = \{\xi \in H : \text{right bdd}\}$

is dense in H, and

${}_P\langle \xi, \eta \rangle = R_\eta^* R_\xi$ is a P-valued inn. prod.

for $\xi, \eta \in {}_P H^0$

Prop. $_P H \overset{\text{isomet}}{\hookrightarrow} L^2(P,\tau) \otimes \ell_2 \mathbb{N}$ by P-hom.
separable

Proof. Step 1 reduction to cyclic rep.

Step 2 $H \ni \xi$ cyclic, $\varphi(a) = \langle a\xi, \xi \rangle$.
normal st on $P \rightsquigarrow \exists (\eta_k)_{k=1}^{\infty} \subset L^2(M,\tau)$
s.t. $\varphi(a) = \sum_k (a\eta_k, \eta_k)$

Step 3 $H \to L^2(P,\tau) \otimes \ell_2 \mathbb{N}$, $a\xi \mapsto \sum_k a\eta_k \otimes \delta_k$

is an isometric emb.

Rem. • on $L^2(P,\tau)$; commutant of left action of P $= P^{op}$ = right action of P.
τ tr. $\Rightarrow$ $a\tau^{1/2} \mapsto ab\tau^{1/2}$ has op. norm $\|b\|$.

$\Rightarrow$ on $L^2(P,\tau) \otimes \ell_2 \mathbb{N}$; $P' = P^{op} \bar{\otimes} B(\ell_2 \mathbb{N})$
II_∞ factor.

P-mod. $\subset L^2(P,\tau) \otimes \ell_2 \mathbb{N} \leftrightarrow$ prj. $e \in P'$
isom of P-submod $\leftrightarrow$ M v N equiv.

P: II_1 factor
Thm-Def. if $_P H \simeq e(L^2(P,\tau) \otimes \ell_2(\mathbb{N}))$
for prj $e \in P^{op} \bar{\otimes} B(\ell_2 \mathbb{N})$

put $\dim_P H = (\tau^{op} \otimes \mathrm{Tr})(e) = \sum_{i=1}^{\infty} \tau(e_{ii})$
semifin. tr.

- this is well-defined $\in [0, \infty]$
- $_P H \simeq {}_P K \Leftrightarrow \dim_P H = \dim_P K$

Rem same for H_P

Def. $N \subset M$ II_1 subf. $[M:N] = \dim_N L^2(M,\tau)$
<u>index</u>