Def. $N \subset M$ vN algs. A <u>conditional</u> <u>expectation</u> is a normal compl. pos. map $E : M \to N$ s.t. $E(x) = x$ for $x \in N$

$C(M, N) = \{ E : M \to N \text{ cond. exp.} \}$

Recall: M II_1 factor $\rightsquigarrow$ $\exists!$ tr.-preserving cond. exp. $E : M \to N$, and Pimsner–Popa inequality $E(x) \geq [M:N]^{-1} x$ $(x \geq 0)$

Def. $E \in C(M, N)$ The <u>index</u> of E is $\mathrm{Ind}\, E = \delta$, $\delta^{-1} = \sup\{ 0 \leq t \leq 1 : \forall x \geq 0 \; E(x) \geq t x \}$.

Fact • (Next time) $\exists$ fin ind. $E \in C(M, N)$ $\iff$ $\forall E \in C(M, N)$ (nonempty) is fin ind.

• (Next?) $N \subset M \subset B(H)$

$\exists$ bijective map $C(M, N) \to C(N', M')$ index-preserving $E \mapsto \overline{E}$

If $e \in N'$ is proj. s.t. $e J^M x J^M e = J^M E(x) J^M e$ for $x \in M$ $(\Rightarrow N' = \langle M', e \rangle = J^M M_1 J^M)$

then $\overline{E}(e) = (\mathrm{Ind}\, E)^{-1}$

So $\overline{E}(x e y) = (\mathrm{Ind}\, E)^{-1} x y$ $\quad x, y \in M'$

Th'm (Longo) $N \subset M$ prop. infin. subfactor

$\exists E \in C(M, N)$ fin ind $\iff$ incl. hom $\iota : N \to M$ has a dual in the 2-cat of homs, given by "the" canonical end. $\gamma : M \to N$

Rem ${}_N L^2 M_M \xrightarrow[\text{conj. mod}]{} {}_M L^2 M_N \cong {}_{\gamma(M)} L^2(N)_N$

Proof of "γ dual of ι" $\Rightarrow$ $\exists$ exp. E fin ind.

By assumption $R \in (\mathrm{id}_N, \gamma\iota)$, $\bar{R} \in (\mathrm{id}_M, \iota\gamma)$

s.t. $1_{(\iota)} = (1_\iota \otimes R^*)(\bar{R} \otimes 1_\iota) = R^*\bar{R}$

$1_{(\gamma)} = (1_\gamma \otimes \bar{R}^*)(R \otimes 1_\gamma) = \gamma(\bar{R})^* R$

normalize so that $R^*R = \bar{R}^*\bar{R} = \delta$

Put $E(x) = \delta^{-1} R^* \gamma(x) R \in N$ for $x \in M$.

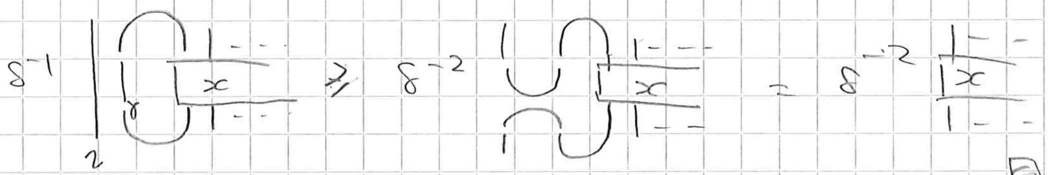
$x \in N \Rightarrow E(x) = \delta^{-1} R^* \gamma\iota(x) R = x$

(• $e = \delta^{-1}\bar{R}\bar{R}^*$ is a proj. commuting with $\gamma(x)$ for $x \in M$

• $y \mapsto \delta^{-1} R^* y R$ is ucp

$\Rightarrow$ for $x \geq 0$ $\gamma(x) \geq e\gamma(x)$

$E(x) = \delta^{-1} R^* \gamma(x) R \geq \delta^{-1} R^* e\gamma(x) R = \delta^{-2} x$



Before exp $\Rightarrow$ dual,

Lem. $S = \{ v \in (\mathrm{id}_N, \gamma\iota),\ v^*v = 1 \}$
$(\|v\| = 1)$

$S \to \mathcal{C}(M, N)$, $v \mapsto E_v(x) = v^*\gamma(x)v$ is bijective (homeo)

cf. $(\mathrm{id}_N, \gamma\iota) \simeq (\iota, \iota) = N' \cap M$.
Frobenius recip.

Proof (surjectivity) Fix φ s.t. $L^2(M, \varphi) = L^2(N, \varphi)$

Step 1 Take $E \in \mathcal{C}(M, N)$, put $\psi = \varphi \circ E$.

$\xi_\psi \in L^2(M, \varphi)$ implementing ψ

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Step 2 $v_0 : x\varphi^{1/2} \to x\xi_\psi$ $(x \in N)$

is an isomet. in N'

$\because$ $(x\xi_\psi, y\xi_\psi) = \psi(y^*x) = \varphi(E(y^*x))$

$\underset{E|_N = id_N}{=} \varphi(y^*x) = (x\varphi^{1/2}, y\varphi^{1/2})$

Step 3 $p = v_0 v_0^*$ satisfies $pxp = E(x)p$ $(x \in M)$

$(y, z \in N)$

$\because$ $(p x \underset{\text{absorve}}{p\, y}\xi_\psi, \underset{\text{abs.}}{z}\xi_\psi) = \psi(z^*xy)$

$= \varphi(E(z^*xy)) = \psi(z^*E(x)y)$

$= (E(x)py\xi_\psi, z\xi_\psi)$

Step 4 $u = J_\varphi^N J_\varphi^M$, $v = J_\varphi^N v_0 J_\varphi^N$ satisfy

$u^* v = v_0$

$\because$ $u^*v = J_\varphi^M \underbrace{J_\varphi^N J_\varphi^N}_{\text{cancel}} v_0 J_\varphi^N = J_\varphi^M J_{\xi_\psi}^{N_p} v_0$

N-mod hom v_0 intertw. $\varphi^{1/2}$ & ξ_ψ

$J_{\xi_\psi}^{N_p} = J_\varphi^M |_{\overline{N\xi_\psi}}$ $\Rightarrow J_\varphi^M J_{\xi_\psi}^{N_p} v_0 = v_0$

$\text{ran}(v_0) = \text{ran}(p)$

we took ξ_ψ from positive cone (clos. of $aJa\varphi^{1/2}$, $a \in M$)

do rel. mod. theory const. for this ξ_ψ

$u_{\psi,\varphi} : L^2(M,\varphi) \to L^2(M,\psi)$ (std. form)

M-mod hom $\quad \xi_\psi \longmapsto \psi^{1/2}$

by this $N\xi_\psi$ conjugated to $L^2(N,\psi)$

J_φ^M conj. to J_ψ^M

on the right side $J_\psi^M |_{L^2(N,\psi)} = J_\psi^N$

Step 5 $v^* \gamma(x) v = E(x)$ $\quad \longleftarrow v_0 \in N'$

$\because$ Left hand side $\underset{\text{Step4}}{=} v_0^* x v_0 = v_0^* p x p v_0 \underset{\text{Step3}}{=} v_0^* E(x) v_0$

Step 6 $v \in (\mathrm{id}_N, \gamma\iota)$

$\because$ $\gamma(x) v = u x u^* v \underset{\text{Step 4}}{=} u x v_0 \underset{v_0 \in N'}{=} u v_0 x \underset{\text{Step 4}}{=} v x$

$x \in N$

$\square$

Proof of Th'm ; exp $\Rightarrow$ dual

Step 1 $E_1 \in C(M, N)$ fin ind

$\rightsquigarrow$ $\overline{E_1} \in C(N', M')$ corr. to $E_2 \in C(M_1, M)$

$\xrightarrow{\mathrm{Ad}\, J_\varphi^M}$ $M_1 = J_\varphi^M N' J_\varphi^M$

& canonical end for $M \subset M_1$ is $\mathrm{Ad}\, J_\varphi^N J_\varphi^M = $ "γ"

$\underset{\text{Lem}}{\Rightarrow}$ $\exists\ \bar v \in (\mathrm{id}_M, \gamma|_{M_1}\iota) = (\mathrm{id}_M, \iota\gamma)$ isomet

implementing E_2 : $E_2(x) = \bar v^* \gamma(x) \bar v$

We'll also use $\tilde{E_2} : N = J_\varphi^N N' J_\varphi^N \to \gamma(M) = J_\varphi^N M' J_\varphi^N$

corresp. to $\tilde{E_2}$; $\bar v \bar v^* x \bar v \bar v^* = \bar v E_2(\gamma^{-1}(x)) \bar v^*$
$= \underbrace{\gamma E_2(\gamma^{-1}(x))}_{\tilde{E_2}(x)} \bar v \bar v^*$) $\bar v \bar v^*$ is Jones proj. for $\tilde{E_2}$

Step 2 $E_1(\bar v) \in (\mathrm{id}_N, \gamma\iota)$

$\because$ apply E_1 to $\bar v x = \gamma(x) \bar v$

Step 3 $v = \delta^{1/2} E_1(\bar v)$ for $\delta^{-1} = E_1(\bar v)^* E_1(\bar v)$

Goal : $\bar v \in (\mathrm{id}_M, \iota\gamma) \longleftrightarrow \bar R$ ~~up to scalar~~

$v \in (\mathrm{id}_N, \gamma\iota) \longleftrightarrow R$

want "$(1_\gamma \otimes \bar R^*)(R \otimes 1_\gamma) = 1_\gamma$"

or $\gamma(\bar v)^* v$ = scalar,

and $v^* \bar v$ = scalar.

Step 4 $\gamma(\bar v)^* v = (\mathrm{Ind}\, E_1)^{-1/2}$

$\because$ $\gamma(\bar v^*) v = \delta^{1/2} \gamma(\bar v^*) E_1(\bar v) = \delta^{1/2} E_1(\gamma(\bar v^*) \bar v)$

$= \delta^{1/2} E_1(\bar v \bar v^*)$

$\bar v \bar v^* \in M$ is the Jones proj. for $N \overset{\tilde{E_2}}{\supset} \gamma(M)$

$\rightsquigarrow$ $E_1(\bar v \bar v^*) = (\mathrm{Ind}\, E_2)^{-1} = (\mathrm{Ind}\, E_1)^{-1}$

$\delta = \mathrm{Ind}\, E_1^{1/2}$ (Step 3.9)

Step 5 v implements E_1

$\because$ $E'(x) = v^* \gamma(x) v$ is a cond. exp. $M \to N$ by $v \in (\mathrm{id}_N, \gamma 2)$

By step 4 this sends $\bar{v}\bar{v}^*$ to $(\mathrm{Ind}\, E)^{-1}$

$\rightsquigarrow$ eq. to the dual exp. of E_2.

Step B.9 $\delta \; (= (E_1(\bar{v})^* E_1(\bar{v}))^{-1})$ is $\mathrm{Ind}\, E_1$

$\because$ $\bar{v} = \bar{v}\bar{v}^*\bar{v} = \bar{v}\bar{v}^* (\mathrm{Ind}\, E_1) E_1(\underbrace{\bar{v}\bar{v}^*\bar{v}})$

(Approx. $\bar{v} \in M = \langle N, \bar{v}\bar{v}^* \rangle$ by $\bar{v}$ again.

$\sum x_i \bar{v}\bar{v}^* y_i$, with $x_i, y_i \in N$, observe

$\bar{v}\bar{v}^* x_i \bar{v}\bar{v}^* y_i = \bar{v}\bar{v}^* \tilde{E}_2(x_i) y_i$

apply E_1 to get $\tilde{E}_2(x_i) y_i = \underset{(\mathrm{Ind}\, E_1)}{=} E_1(\bar{v}\bar{v}^* \cdots))$

Then $E_1(\bar{v})^* E_1(\bar{v}) = (\mathrm{Ind}\, E_1)\, E_1(E_1(\bar{v})^* \underbrace{\bar{v}\bar{v}^*}_{\text{proj.}} E_1(\bar{v}))$

$= (\mathrm{Ind}\, E_1)^{-1}\, E_1(\underbrace{\bar{v}^*\bar{v}}_{1})$

Step 6 $v^*\bar{v} = (\mathrm{Ind}\, E_1)^{-1/2}$

$\because$ $v = \gamma(a)$ for $a \in M_1 = \gamma^{-1}(N)$

a implements $\tilde{E}_1 = \gamma \circ E_1 \circ \gamma^{-1}$

$\xrightarrow{\text{same arg}}$ $\gamma(a)^* \bar{v} = (\mathrm{Ind}\, E_1)^{-1}$. ∎