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Operator valued weights

Def. $N \subset M$ vN algs

An <u>operator valued weight</u> from M to N is a map $T: M_+ \to \hat{N}_+$ extended positive part

- $T(rx) = rT(x)$ $r \geq 0$
- $T(x+y) = T(x) + T(y)$
- $T(a^*xa) = a^*T(x)a$ $a \in N$
- $x_i \nearrow x \Rightarrow T(x_i) \nearrow T(x)$.

Ex. weights : op-val. weights for $N = \mathbb{C}$

cond. exp

Rem $T(1) \in [0, \infty]$ if N is a factor

$P(M, N) = \{$ op. val. wghts $M \xrightarrow{T} N \}$

Spatial derivative & dual op. val. wghts.

Suppose $M \subset B(\mathcal{H})$, fix normal states φ on M, ψ on M'

Put : • $D(\mathcal{H}, \psi) = \{ \xi \in \mathcal{H} : \exists C > 0 \ \forall T \in M'$
$\|T\xi\| \leq C \|T\psi^{1/2}\| \}$

"right bounded vectors" for M', ψ

• $R^\psi(\xi) : L^2(M', \psi) \to \mathcal{H}$, $T\psi^{1/2} \mapsto T\xi$

$\xi \in D(\mathcal{H}, \psi)$ commutes w/ M'

• $\Theta^\psi(\xi, \eta) = R^\psi(\xi) R^\psi(\eta)^*$

Obs / def : $\xi \mapsto \varphi(\Theta^\psi(\xi, \xi))$ is a quadratic form (defs Hermitian inner prod.) on $D(\mathcal{H}, \psi)$

$\rightsquigarrow$ defines an unbdd pos. op. $\frac{d\varphi}{d\psi}$ char'd by

$\left(\frac{d\varphi}{d\psi} \xi, \xi \right) = \varphi(\Theta^\psi(\xi, \xi))$.

Ex. $M = L^\infty(X, \mu)$, $\mathcal{H} = L^2(X, \mu) \rightsquigarrow$ Radon-Nikodym deriv.

Fact (Haagerup, Connes) $\exists$ bijective map

$P(M,N) \to P(N',M')$, $T \mapsto T^{-1}$ s.t.

$$\frac{d\varphi\circ T}{d\psi} = \frac{d\varphi}{d\psi\circ T^{-1}} \quad \text{for any } \varphi \text{ on } N, \ \psi \text{ on } M'$$

$\rightsquigarrow$ want: use this to understand

$$C(M,N) \to C(N',M') \quad \left(\begin{array}{l} \simeq C(N,\gamma(M)) \\ \simeq C(M_1, M) \end{array}\right.$$

(Rem Combes - Delaroche
$C(M,N) \simeq$ states on $N' \cap M$.

Jones projection. for $E \in C(M,N)$

φ: ~~faithful~~ normal state on M

$\psi = \varphi\circ E$ ($\psi = \psi\circ E$, faithful)

Def. $e_N x \psi^{1/2} = E(x)\psi^{1/2}$ on $L^2(M,\psi)$.

(this is bounded by $(e_N x\psi^{1/2}, e_N x \psi^{1/2})$

$\leq \psi(E(x^*)E(x)) \leq \underbrace{\psi(E}_{\psi}(x^*x)) = (x\psi^{1/2}, x\psi^{1/2})$

Rem. $\bullet\ e_N \in N'$, $N' = \langle M', e_N\rangle$

$(\Leftarrow N = M \cap \{e_N\}')$

$\bullet\ e_N x e_N = E(x) e_N$.

$\bullet\ e_N$ comm. w/ J^M_ψ, $J^M_\psi e_N = J^N_\psi$.

Prop. Suppose $E^{-1}(1) < \infty$. Then $\bar{E}(x) = (E^{-1}(1))^{-1} E^{-1}(x)$

defs a cond. exp. $N' \to M'$

Proof. We need to check $\bar{E}(x) = x$ for

$x \in M'$. Enough to check for $x \geq 0$

$$\bar{E}(x) = x^{\frac{1}{2}} E^{-1}(1)^{-1} E(1) x^{1/2} = x. \quad \Box$$

op. val. wght

Cor. $E^{-1}(1) < \infty \Rightarrow C(M,N) \simeq C(N',M')$ by $E \mapsto \bar{E}$.

Lem $E^{-1}(e_N) = 1$

Proof Step 1 ω st on M

$\Rightarrow$ $\theta^{\omega}(\eta, \eta) = \omega^{-1}(|\eta\rangle\langle\eta|)$

op. val wght $B(H) \to M'$

$\because$ ξ : state implementing ω

ω' : vector state on M' for ξ

$$\frac{d\, id_C \circ \omega}{d\, \omega'} = \frac{d\, id_C}{d(\omega' \circ \omega^{-1})}$$ should be Tr.

$\hookrightarrow id_H$ by $R^{\omega}(J^M_\xi \eta) = R^{\omega'}(\eta) J^M_\xi$, etc.

$\omega'(R^{\omega'}(\eta) R^{\omega'}(\eta)^*) = (\eta, \eta)$ by this.

Step 2 $e_N = \theta^{\varphi|_N (= \psi|_N)}(\psi^{1/2}, \psi^{1/2})$

$\because$ $R^{\varphi|_N}(\psi^{1/2})^* : L^2(M, \psi) \to L^2(N, \psi)$

is $x\psi^{1/2} \mapsto E(x)\psi^{1/2}$ by

$(x\psi^{1/2}, y\psi^{1/2}) = \psi(y^* x) = \psi(y^* E(x))$

$y \in N$

Step 3 $E^{-1}(e_N) = E^{-1} \varphi|_N^{-1}(|\psi^{1/2}\rangle\langle\psi^{1/2}|)$

$= (\varphi|_N \circ E)^{-1}(|\psi^{1/2}\rangle\langle\psi^{1/2}|) = \theta^{\psi}(\psi^{1/2}, \psi^{1/2}) = 1$ ∎

Prop. $E^{-1}(1) < \infty \Rightarrow E^{-1}(1)^{-1}$ is the best const λ s.t. $E(x) \geq \lambda x$ $x \in N'_+$

Cor. $\mathrm{Ind}\, E = E^{-1}(1)$

Proof. (cf. 01.22)

Step 1 We may assume $x = aa^*$,

$a = \sum_{j=1}^{n} y_j e_N z_j$ $y_j, z_j \in M'$

$\because$ density

Step 2 $e_N z_j z_k^* e_N = E^J(z_j z_k^*) e_N$ for

$E^J(z) = \mathrm{Ad}_{J^M_\psi} E(\mathrm{Ad}_{J^M_\psi} z))$

$\because$ J^M_ψ commutes with e_N.

Step 3 $\bar{E}(x) = E^{-1}(1)^{-1} \sum y_j E^J(z_j z_k^*) y_k^*$

$\because$ Lem & M'-bimodularity of E^{-1}

Step 4 $Z = (E^J(z_j z_k^*))_{j,k} \in M_n(J_\varphi^M N J_\varphi^M) \subset M_n(M')$

is positive, $Z \geq Z^{1/2} \begin{bmatrix} e_N & & \\ & \ddots & \\ & & e_N \end{bmatrix} Z^{1/2} = Z \begin{bmatrix} e_N & & \\ & \ddots & \\ & & e_N \end{bmatrix}$

sandwitch by $\sum y_j (\ )_{jk} y_k^*$ gives

$$E^{-1}(1)\bar{E}(x) \geq x$$ ☐

Outline for $T \mapsto T^{-1}$:

Fact $(u_t)_{t\in\mathbb{R}} \subset \mathcal{U}(M)$ satisfies

$u_{s+t} = u_s \sigma_s^\varphi(u_t) \iff \exists \psi$ fnf weight ; $u_t = \Delta_\psi^{it}\Delta_\varphi^{-it}$

(write $u_t = (D\psi : D\varphi)_t$ Connes cocycle)

Fix φ on N, ψ on M', ω on N'

$\left(\frac{d\varphi\circ T}{d\psi}\right)^{-it}\left(\frac{d\varphi}{d\omega}\right)^{it}$ is a cocycle for σ_t^ω

$\rightsquigarrow$ there should be $\tilde{\psi} = \psi\circ T^{-1}$ on N'

Key property : $\mathrm{Ad}\left(\frac{d\varphi}{d\psi}\right)^{it}(x) = \sigma_t^\varphi(x)$ $x \in M$

$\mathrm{Ad}\left(\frac{d\varphi}{d\psi}\right)^{it}(y) = \sigma_{-t}^\psi(y)$ $y \in M'$

for φ on M, ψ on M'

$(D\varphi_1 : D\varphi_2)_t = \left(\frac{d\varphi_1}{d\psi}\right)^{it}\left(\frac{d\varphi_2}{d\psi}\right)^{-it}$, etc.