Continuous core

M : von Neumann alg, φ : faithful normal state (or weight)

Recall: modular aut. group $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$
from $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$: closure of $x\varphi^{1/2} \mapsto x^*\varphi^{1/2}$

Def The (<u>continuous</u>) <u>core</u> $C(M)$ of M is
$M \rtimes_{\sigma^\varphi} \mathbb{R} = \langle x \in M, \underbrace{(\lambda_\varphi^t)_{t\in\mathbb{R}}}_{\text{unitaries}} : \lambda_\varphi^t x \lambda_\varphi^{-t} = \sigma_t^\varphi(x) \rangle$
realized on $L^2(M,\varphi) \otimes L^2(\mathbb{R})$

Rem • this construction is independent on φ
$\because$ we may assume $\lambda_\varphi^t = \underbrace{(D\varphi : D\psi)_t}_{\Delta_\varphi^{it}\Delta_\psi^{-it} \in M} \lambda_\psi^t$

• $C(M) \simeq M \otimes \underbrace{L(\mathbb{R})}_{\text{group alg}} \iff \sigma^\varphi$ is inner
$\iff M$ is <u>semifinite</u> ($\exists$ tracial weight; type I or II)

Def The <u>dual</u> <u>action</u> $\theta : \mathbb{R} \curvearrowright C(M)$ is defined by $\theta_s(x\lambda_\varphi^t) = e^{-ist} x \lambda_\varphi^t$

$\rightsquigarrow$ again indep. of φ, $C(M)^\theta = M$.

$\theta : \mathbb{R} \curvearrowright Z(C(M))$ is called the <u>flow of weights</u>.

Prop. $C(M)$ is semifinite.

Proof. We need to construct a tracial weight
$\tau : C(M)_+ \to [0,\infty]$

$C_c(\mathbb{R}, M)$ compactly supported cont. funcs.
convolution product twisted by σ :
$$(f * f')(t) = \int_{-\infty}^{\infty} f(s)\, \sigma_s^\varphi(f'(t-s))\, ds$$
$\rightsquigarrow C_c(\mathbb{R}, M) \hookrightarrow C(M)$ by $f \mapsto \int_{-\infty}^{\infty} f(t) \lambda_\varphi^t\, dt$.

Put $\hat\varphi(f) = \varphi(f(0))$ for $f \in C_c(\mathbb{R}, M)$,
$\tau(f) = \hat\varphi(h^{-\frac12} f h^{-\frac12})$ for h pos. $h^{it} = \lambda_\varphi^t$

1(cont.)

this "extends" to a tracial weight

- regularity as weight :

$\hat{\varphi}$ on $C_c(\mathbb{R}) \subset L(\mathbb{R}) \subset C(M)$ is $\hat{\varphi}(f) = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi$

up to Fourier transform $L(\mathbb{R}) \xrightarrow{\sim} L^\infty(\mathbb{R})$

$f \mapsto \hat{f}$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i t \xi} f(t)\, dt$

so $\hat{f} \in L^1(\mathbb{R}) \Rightarrow \hat{\varphi}(f)$ is well def'ed in $\mathbb{C}$

$x \in C(M)_+ \Rightarrow \hat{\varphi}(f^* x f) < \infty$ for such f

ex. $x \in M \Rightarrow \hat{\varphi}(f^* x f) = \varphi\left(\int \sigma_s^\varphi(x) \bar{f}(s) f(-s) ds\right)$

$= \varphi(x) \int \bar{f}(s) f(-s) ds$

- by σ^φ-invariance of φ

$\rightsquigarrow \tau$ is also well def'ed as weight.

- trace property : enough to check on $C_c(\mathbb{R}, M)$

σ^φ has analytic extension σ_z^φ

(on a dense subset of M)

$\varphi(a\, \sigma_i^\varphi(b)) = \varphi(ba)$ KMS property

$h^{-1} x h = "\lambda_\varphi^{i}" x "\lambda_\varphi^{-i}" = \sigma_i^\varphi(x)$.

We have $\tau(f) = \varphi(f(i))$ for $f \in C_c^\infty(\mathbb{R}, M)$

s.t. $f(i) = \int_{-\infty}^{\infty} e^{-2\pi \xi} \hat{f}(\xi)\, d\xi$ makes sense

$\varphi((f * f')(i)) = \int_{-\infty}^{\infty} \varphi(f(s)\, \sigma_s^\varphi(f'(i-s)))\, ds$

and by "σ_i^φ-invariance" of φ

we can replace the range of s to

$i + \mathbb{R}$. Put $s = i + s'$ $s' \in \mathbb{R}$

$\rightsquigarrow \int_{-\infty}^{\infty} \underbrace{\varphi(f(i+s')\, \sigma_{i+s'}^\varphi(f'(-s')))}_{\varphi(\sigma_{s'}^\varphi(f'(-s'))\, f(i+s'))} ds'$

again by $\sigma^{\varphi}_{s'}$ - invariance of φ

this is $\int_{-\infty}^{\infty} \varphi(f'(-s'))\, \sigma^{\varphi}_{-s'}(f(i+s)))\, ds'$

$= \varphi((f' * f)(i)))$ i.e. $\tau(f * f') = \tau(f' * f)$.

$\blacksquare$

Rem. on $L(\mathbb{R}) \subset C(M)$

$$\tau(f) = \int_{-\infty}^{\infty} e^{-2\pi\xi} \hat{f}(\xi)\, d\xi$$

i.e. $\tau|_{L(\mathbb{R})} \leftrightarrow$ measure with density $e^{-2\pi\xi}$

on $L^{\infty}(\mathbb{R})$

Prop. $\tau(\theta_s(x)) = e^{-2\pi s}\, \tau(x)$.

(Morally : θ_s on $L(\mathbb{R}) \leftrightarrow$ translation by s on $L^{\infty}(\mathbb{R})$)

Prop. $M \otimes B(L^2(\mathbb{R})) \simeq C(M) \rtimes_{\theta} \mathbb{R}$

$\uparrow$

can be M if properly infinite.

$\because$ Takesaki duality.

Extension of homomorphisms.

$\rho : N \to M$ unital $*$-hom, with dual

in the C^*-2-cat.

i.e. $\exists$ fin. ind. cond. exp. $E : M \to \rho(N)$

Take a faithful normal state φ on N

$\psi = \varphi \circ E$ on M.

Th'm (Izumi) $\tilde{\rho} : C(N) = N \rtimes_{\sigma^{\varphi}} \mathbb{R} \to C(M) = M \rtimes_{\sigma^{\psi}} \mathbb{R}$

$x \lambda^t_{\varphi} \mapsto d(\rho)^{it} \rho(x) \lambda^t_{\psi}$

has the same index as ρ

Outline • $\sigma^{\psi}|_N = \sigma^{\varphi}$ from $\psi = \varphi \circ E$.

• canonical endomorphism for $\tilde{\rho}(C(N)) \subset C(M)$

can be taken as $\tilde{\gamma}(x\lambda_{\psi}^{t}) = \gamma(x)\lambda_{\varphi}^{t}$

$\rightsquigarrow$ compare dual weights $\hat{\varphi}$ & $\hat{\psi}$

Consequently $\mathcal{C} = (I, (\mathcal{C}_{ij})_{i,j\in I})$ C^*-2-cat

action $\mathcal{C} \curvearrowright (M_i)_{i\in I}$ on vN alg's

$\rightsquigarrow$ $\mathcal{C} \curvearrowright (C(M_i))_{i\in I}$

But $C(M_i)$ might not be a factor

($C(M)$ factor $\Leftrightarrow$ M is type III_1 factor)

Freeness in nonfactorial case

$(M_i)_{i\in I}$ vN algs.

Def $\mathcal{C} \overset{\alpha}{\curvearrowright} (M_i)_{i\in I}$ is free if

$\forall X, Y \in \mathcal{C}_{ij}$, $(\alpha_X, \alpha_Y) = \alpha(\mathrm{Mor}(X,Y))Z(M_i)$

$\supset$ is automatic.