(cont.) outline of prev. Thm

Look at the family

$J = \{ (e_x)_{x \in K} :$ satisfies cond. 1, 2, 3, 6 $\}$.

Want : $J \supset I$ inductive ordered sets s.t.

maximal elem $(e_x)_{x \in K} \in I$ satisfies 4 & 5.

For $E = (e_x)_{x \in K} \in J$ put

$$a_E = \frac{1}{|F|_0} \sum_{x \in F} \sum_{Y \in \mathrm{Irr}(e_i)} d(xY)^2 \| \alpha_x(e_Y) - u_{x,Y}^* e_{xY} u_{x,Y} \|_{L^1(\alpha_x \kappa_Y^\omega \varphi^\omega)}$$

(see 5)

$$b_E = \varphi^\omega\left(\sum_{x \in K} f_x\right) = \varphi^\omega\left(\sum_{x \in K} \underbrace{d(x)^2}\, e_x\right) \text{ in the notn of 3}$$

(see 4) from $\tau^\omega(e_x) \in C$

$\tau^\omega(e_x x) = \tau^\omega(e_x)\tau^\omega(x)$.

Key Lem : if $b_E < 1 - \sqrt{\delta}$, then $\exists\, E' = (e'_x)_x \in J$

s.t. $a_{E'} - a_E \le 6\sqrt{\delta}\,(b_{E'} - b_E)$

$$0 < \frac{\sqrt{\delta}}{2} \sum_{x \in K} d(x)^2 \varphi^\omega(|e'_x - e_x|) \le b_{E'} - b_E$$

Order on J : $E < E'$ means the above ineqs

hold, $I = \{ E \in J,\ a_E \le 6\sqrt{\delta}\, b_E \}$ (0 is minimal)

Maximal $E \in I$ will satisfy : $b_E \ge 1 - \sqrt{\delta}$ (4)

and $a_E = a_E - a_0 \le 6\sqrt{\delta}(b_E - b_0) \le 6\sqrt{\delta}$.

Key technique behind Key Lem :

<u>Local</u> <u>quantization</u> (Popa) $Q_1 \subset M^\omega$ ctbly gen

from central freeness, $\forall \delta, \exists X$ part. unity

$(q_r)_{r=0}^{n} \subset Q_1' \cap M_\omega$ s.t. $\varphi^\omega(q_0) < \delta$,

$$\sum_{x \in F} \| q_r \alpha_x(q_r) \|_{L^1(\varphi^\omega)} < \delta\, \varphi^\omega(q_r) \quad \text{for } r \ge 1$$

$\rightsquigarrow$ we can improve second cond to $q_r \alpha_X(q_r) = 0$

<u>Fast reindexing</u> (Ocneanu) we can achieve

$$\tau^\omega(x\alpha_X(q_r)) = \tau^\omega(x)\,\tau^\omega(\alpha_X(q_r))$$

for $X \in \mathrm{Irr}\, \mathcal{C}_{ij}$, $x \in Q_1 = Q \vee (\alpha_X(e_Y) : Y \in \mathcal{K}, X \in \mathrm{Irr}\,\mathcal{C}_{ij})''$
(countably gen.)

$\because$ choose ctbl generators $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots)$

of Q_1, set $y = \alpha_X(q_r) = (y_1, y_2, \dots)$

$\exists\, i_n$ $\forall \varphi \in (M_*)_1$: $\varphi(x_{i_n}^{(k)} y_{i_n}) \sim_{1/n} \varphi(x_{i_n}^{(k)} \tau^\omega(y))$

$\Rightarrow z = (y_{i_1}, y_{i_2}, \dots)$ satisfies $\tau^\omega(xz) = \tau^\omega(x)\tau^\omega(z)$

($= \alpha_X(q'_r)$ for "reindexed" q'_r)

Then one of $q_1, \dots, q_n$ (call it e) satisfies

$$\sum_{X,Y\in\mathcal{K}} d(X)^2 d(Y)^2 \varphi^\omega(e_X\, \alpha_{\bar{Y}}^{-1} E^{M^\omega}_{\alpha_{\bar{Y}}(M^\omega)}(e)) < (1-\sqrt{\delta})|\mathcal{K}|_\sigma \varphi^\omega(e)$$

$\rightsquigarrow$ $f = |\mathcal{K}|_\sigma \sum_{X\in\mathcal{K}} d(X)^2 \alpha_{\bar{X}}^{-1} E_{\alpha_{\bar{X}}(M^\omega)}(e)$ is a proj.
in $Q_1' \cap M_\omega$

$e'_X = e_X f^\perp + \alpha_X(e)$ will do. (for Key Lem)

How to use Rokhlin towers for (approximate) cohomology vanishing.

- 2-cohom. vanishing : want $u_X \in \mathcal{U}(M_1^\omega)$ for $X \in \mathcal{C}_{ij}$ "natural" in X s.t. 1) $\beta_X = \mathrm{Ad}\, u_X^* \circ \alpha_X$

2) $v_{X,Y} = u_{XY}\, u_{X,Y}\, \alpha_X(u_Y)^* u_X^*$

Approx unitary equiv. between α_X & β_X
$(\Rightarrow)$ $\exists\, u_X^0 \in \mathcal{U}(M_1^\omega)$ satisfying 1

$\rightsquigarrow$ we want to correct the error for 2.

by setting $m_X = m'_X m^0_X$.

Put $c_{X,Y} = m^0_{XY} u_{X,Y} \alpha_X(m^0_Y)^* m^{0*}_X$ so want

$$\gamma_X = \mathrm{Ad}\, m^*_X \circ \alpha_X \quad (\text{as } M^\omega_j \to M^\omega_j)$$

$(e_X \mid X \in \mathcal{K}$ (F,δ)-inv. prjs for α

$\Rightarrow$ $e'_X = m^*_X e_X m_X$ (F,δ)-inv. for γ

We want $\gamma_X(m'^*_Y)\, m'^*_X = v^*_{X,Y}\, m'^*_{XY}\, c_{X,Y}$ ---(#)

$$m^{(\delta)}_X = \sum_{Y\in\mathcal{K}} d(Y)^2\, \gamma(\bar{R}_Y)^*\, c_{Y,\bar{Y}}\, \underbrace{\gamma_Y(v^*_{\bar{Y},X}\, c_{\bar{Y},X})\, e'_Y}\, c^*_{Y,\bar{Y}}\, \gamma(\bar{R}_Y)$$

$$+ \Big(\sum_{Y\in\mathcal{K}} d(Y)^2 \text{ above w/o } \downarrow \Big)^\perp$$

This satisfies $\|\text{left side} - \text{right side of } \#\|_{L^2(\gamma_X\gamma_Y\psi^\omega)}$

$\leq 8\sqrt{\delta}\, |F|\, \sigma$ from almost invariance.

Choose $\delta_1 > \delta_2 > \cdots \to 0$, set $m^{(\delta_n)}_X$ $n = 1, 2, \ldots$

$\rightsquigarrow$ <u>diagonal argument</u>:

$m'_X = (m^{(\delta_1)}_1, m^{(\delta_2)}_2, \ldots)$ satisfies (#).