Wassermann subfactor.

Concrete realization of Rep G in subfactors
(and bimodules, homomorphisms, --)
for compact groups G.

Def. $G \overset{\alpha}{\curvearrowright} M$ action. $\rightsquigarrow$ crossed product is
$M \rtimes_\alpha G = \langle \alpha_\circ(x), 1 \otimes \rho_g : x \in M, g \in G \rangle \subset B(L^2(M) \otimes L^2(G))$

Rem $\alpha_\circ(x)$: as operator on $L^2(G, L^2(M))$
$$(\alpha_\circ(x)\xi)(g) = \alpha_g(x)\xi(g)$$
i.e. $\alpha_\cdot(x) = $ "$\int \alpha_g(x) \otimes \delta_g \, dg$"

so ρ_g becomes $(\rho_g \xi)(h) = \xi(hg)$

$\rightsquigarrow \rho_g \alpha_\circ(x) \rho_g^{-1} = \alpha_\circ(\alpha_g(x))$

Def. $G \overset{\alpha}{\curvearrowright} M$ is free if $M^G = e_{triv}(M \rtimes_\alpha G) e_{triv}$
($e_{triv} \in L(G) \subset M \rtimes_\alpha G$ corresp. to triv rep.)
is a full corner. (central supp. is I).

Rem. free $\Rightarrow \forall \pi \in \operatorname{Rep} G \quad \operatorname{Hom}_G(H_\pi, M) \neq 0$.

Def. $G \overset{\alpha}{\curvearrowright} M$ is minimal if it is free and
$(M^G)' \cap M = \mathbb{C}$

Rem. Freeness $\Rightarrow Z(M^G) \simeq Z(M \rtimes_\alpha G)$ so
$M \rtimes_\alpha G$ is a factor for minimal action.

Example : infinite tensor product (ITPFI)
(π, H_π) (projective) fin dim unitary rep of G
s.t. $\mathrm{Ad}_\pi : G \curvearrowright \operatorname{End}(H_\pi)$ ($\simeq M_{\dim H_\pi}(\mathbb{C})$)
is faithful (no $g \neq e$ satisfies $\pi_g \in \mathbb{C}$.)

$M = (\mathrm{End}(H_\pi)^{\otimes\infty}, \mathrm{tr}^{\otimes\infty})'' = (\underbrace{\varinjlim \mathrm{End}(H_\pi)^{\otimes n}}_{\text{UHF alg.}}, \mathrm{tr})''$

$\alpha_g =$ "$\mathrm{Ad}\, \pi_g^{\otimes\infty}$" defines $\alpha : G \curvearrowright M$

M^G contains img of $\mathrm{End}_G(H_\pi^{\otimes n})$

$= (\mathrm{End}(H_\pi)^{\otimes n})^G$.

in particular "permutation of tensors".

are in M^G ($C^* S_\infty \to M^G$ $*$-hom)

Størmer : $C^* S_\infty{}' \cap M = \mathbb{C}$ ($\Rightarrow (M^G)' \cap M = \mathbb{C}$)

idea : take "very long" cyclic permutation

$\sigma = (1, \cdots, N) \Rightarrow \sigma T \sigma^{-1} \sim 1_{H_\pi}^{\otimes N-1} \otimes T$

T " if $T \in C^* S_\infty{}' \cap M$

Freeness of α : (suppose π is a genuine rep.) $M \rtimes_\alpha G = (M \otimes B(L^2 G))^{\alpha \otimes \mathrm{Ad}_\lambda(G)}$ (general)

$\Rightarrow M \rtimes_\alpha G$ factor iff. $(M \otimes \mathrm{End}(W))^G$ factor

for $\forall W \subset L^2(G)$ $\lambda(G)$-inv

(these are corners of $M \rtimes_\alpha G$).

$\ker \pi = \{e\} \Rightarrow \exists k : W \hookrightarrow H_\pi^{\otimes k}$ as inv. subsp.

$\Rightarrow (M \otimes \mathrm{End}(W))^G$ is a corner of $(M \otimes \mathrm{End}(H_\pi)^{\otimes k})^G$

$\simeq M^G$ factor.

Prop. 1 $G \overset{\alpha}{\curvearrowright} M$ minimal, (σ, W) fin. dim unitary rep of $G \Rightarrow G \overset{\alpha \otimes \mathrm{Ad}_\sigma}{\curvearrowright} M \otimes \mathrm{End}(W)$ minimal

Proof. Step 1 $(M \otimes \mathrm{End}(W))^G$ has trivial rel. commutant.

$\because$ $(M \otimes \mathbb{C})(M \otimes \mathrm{End}(W))^G = M \otimes \mathrm{End}(W)$ by freeness of α

$M \otimes \mathrm{End}(W) \simeq M_k(M)$ $k = \dim W$.

commutant of M^G: scalar entries in this mat. presentation $\rightsquigarrow$ elem of $\mathrm{End}(W)$

$\rightsquigarrow$ commutant of $(M \otimes \mathrm{End})^G$ will commute with $(M \otimes \mathbb{C})(M \otimes \mathrm{End}(W))^G$ (everything).

Step 2 freeness of $\alpha \otimes \mathrm{Ad}_\sigma$

$\rightsquigarrow$ enough to check factoriality of $(M \otimes \mathrm{End}(W)) \rtimes G$ after Step 1.

This crossed prod. is $(M \otimes \mathrm{End}(W) \otimes B(L^2 G))^G$.

Fell absorption principle $W \otimes L^2(G) \simeq L^2(G)^{\oplus k}$

$\rightsquigarrow$ we get amplification of $(M \otimes B(L^2(G)))^G = M \rtimes_\alpha G$ factor $\boxtimes$

Bimodules over M^G.

$G \overset{\alpha}{\curvearrowright} M$, (σ, W) as in Prop

and M II_1 factor

$\rightsquigarrow$ G preserves τ, get unitary rep on $L^2(M)$.

Put $X_\sigma = (L^2(M) \otimes W)^G$ M^G-bimodule.

M^G acts from both sides on the leg of $L^2(M)$.

Prop 2 $\mathrm{Hom}({}_{M^G}(X_{\sigma_1})_{M^G}, {}_{M^G}(X_{\sigma_2})_{M^G}) \simeq \mathrm{Hom}_G(W_1, W_2)$

Proof. $X_{\sigma_1} \otimes_{M^G} X_{\sigma^2} \simeq X_{\sigma^1 \otimes \sigma^2}$.

Proof. Step 1 $X_{\sigma_1} \underset{M^G}{\otimes} X_{\sigma_2} \cong X_{\sigma_1 \otimes \sigma_2}$

$\because$ Write $\xi \in X_{\sigma_1}^{bdd}$ as $\xi = \xi_1 \otimes \xi_2$

$\xi_1 \in M$, $\xi_2 \in W$, similarly for $\eta \in X_{\sigma_2}$

$\rightsquigarrow$ $\xi_1 \eta_1 \otimes \xi_2 \otimes \eta_2 \in X_{\sigma_1 \otimes \sigma_2}$

Step 2 $\mathrm{Hom}(X_{\sigma_1}, X_{\sigma_2})$:

$\because$ Use Frobenius reciprocity & Step 1 to reduce it to $\sigma_1 = \mathrm{triv}$.

$T \in \mathrm{Hom}_{M^G}(L^2(M^G)_{M^G}, X_\sigma)$ is characterized

by $\xi = T(\tau^{1/2}) \in X_\sigma$ $\quad x\xi = \xi x \quad x \in M^G$

this says $\xi \in (\mathbb{C} \otimes W)^G \cong \mathrm{Hom}_G(\mathbb{C}, W)$ $\square$

Cor. $\mathrm{Rep}\, G \to (M^G\text{-bimod})$, $(\sigma, W) \mapsto X_\sigma$

is a full (surj. on mors) C^*-$\otimes$-functor.

i.e. free action $(\mathrm{Rep}\, G) \curvearrowright M^G$.

For each (σ, W), get subfactor $M^G \subset (M \otimes \mathrm{End}(W))^G$.

with rel. comm. $(M^G)' \cap (M \otimes \mathrm{End}(W))^G = \mathrm{End}_G(W)$.

Rem. Jones's basic construction is :

$M^G \subset (M \otimes \mathrm{End}(W))^G \subset (M \otimes \mathrm{End}(W) \otimes \mathrm{End}(\overline{W}))^G$

Rem. $\mathrm{Rep}\, G$ is amenable,

$\rightsquigarrow$ M inj. II_1 $\Rightarrow$ M^G inj II_1

$\Rightarrow$ above action $\mathrm{Rep}\, G \curvearrowright M^G$ does not

depend on $G \overset{\text{minimal}}{\curvearrowright} M$ (rather, this is

essentially unique)