ALGEBRAIC TOPOLOGY III – MAT 9580 – SPRING 2015
INTRODUCTION TO THE ADAMS SPECTRAL SEQUENCE

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Contents

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1. The long exact sequence of a pair

Let (X, A) be a pair of spaces. The relationship between the homology groups $H_*(A)$, $H_*(X)$ and $H_*(X, A)$ is expressed by the long exact sequence

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(X, A) \xrightarrow{i_*} \dots$$

Exactness at $H_n(X)$ amounts to the condition that

$$\operatorname{im}(i_*\colon H_n(A) \to H_n(X)) = \ker(j_*\colon H_n(X) \to H_n(X, A))\,.$$

The homomorphism i_* induces a canonical isomorphism from

$$\begin{aligned}\operatorname{cok}(\partial\colon H_{n+1}(X,A) \to H_n(A)) &= \frac{H_n(A)}{\operatorname{im}(\partial\colon H_{n+1}(X,A) \to H_n(A))} \\ &= \frac{H_n(A)}{\ker(i_*\colon H_n(A) \to H_n(X))}\end{aligned}$$

to $\operatorname{im}(i_*\colon H_n(A) \to H_n(X))$. Exactness at $H_n(A)$ and at $H_n(X,A)$ amounts to similar conditions, and ∂ and j_* induce similar isomorphisms.

We can use the long exact sequence to get information about $H_*(X)$ from information about $H_*(A)$ and $H_*(X,A)$, if we can compute the kernel $\ker(\partial) = \operatorname{im}(j_*) \cong \operatorname{cok}(i_*)$ and the cokernel $\operatorname{cok}(\partial) \cong \operatorname{im}(i_*)$ of the boundary homomorphism ∂, and determine the extension

$$0 \to \operatorname{im}(i_*) \longrightarrow H_*(X) \longrightarrow \operatorname{cok}(i_*) \to 0$$

of graded abelian groups.

Let us carefully spell this out in a manner that generalizes from long exact sequences to spectral sequences.

We are interested in the graded abelian group $H_*(X)$. The map $i\colon A \to X$ induces the homomorphism $i_*\colon H_*(A) \to H_*(X)$, and we may consider the subgroup of $H_*(X)$ given by its image, $\operatorname{im}(i_*)$. We get a short increasing filtration

$$0 \subset \operatorname{im}(i_*) \subset H_*(X)\,.$$

More elaborately, we can let

$$F_s = \begin{cases} 0 & \text{for } s \le -1 \\ \operatorname{im}(i_*) & \text{for } s = 0 \\ H_*(X) & \text{for } s \ge 1 \end{cases}$$

for all integers s. We call s the *filtration degree.*

The possibly nontrivial filtration quotients are

$$\frac{\operatorname{im}(i_*)}{0} = \operatorname{im}(i_*) \qquad \text{and} \qquad \frac{H_*(X)}{\operatorname{im}(i_*)} = \operatorname{cok}(i_*)\,.$$

We find

$$frac F_s F_{s-1} = \begin{cases} 0 & \text{for } s \le -1 \\ \operatorname{im}(i_*) & \text{for } s = 0 \\ \operatorname{cok}(i_*) & \text{for } s = 1 \\ 0 & \text{for } s \ge 2. \end{cases}$$

The short exact sequence

$$0 \to \operatorname{im}(i_*) \longrightarrow H_*(X) \longrightarrow \operatorname{cok}(i_*) \to 0$$

expresses $H_*(X)$ as an extension of two graded abelian groups. This does not in general suffice to determine the group structure of $H_*(X)$, but it is often a tractable problem. More generally we have short exact sequences

$$0 \to F_{s-1} \longrightarrow F_s \longrightarrow F_s/F_{s-1} \to 0$$

for each integer s. If we can determine the previous filtration group F_{s-1}, say by induction on s, and if we know the filtration quotient F_s/F_{s-1}, then the short exact sequence above determines the next filtration group F_s, up to an extension problem.

In the present example $F_{-1} = 0$, $F_0 = \operatorname{im}(i_*)$ and $F_1 = H_*(X)$, so there is only one extension problem, from F_0 to F_1, given the quotient $F_1/F_0 = \operatorname{cok}(i_*)$.

We therefore need to understand $\operatorname{im}(i_*)$ and $\operatorname{cok}(i_*)$. By definition and exactness

$$\operatorname{im}(i_*) \cong \operatorname{cok}(\partial) \qquad \text{and} \qquad \operatorname{cok}(i_*) \cong \operatorname{im}(j_*) = \ker(\partial)\,,$$

so both of these graded abelian groups are determined by the connecting homomorphism

$$\partial\colon H_{*+1}(X,A) \longrightarrow H_*(A)\,.$$

If we assume that we know $H_*(A)$ and $H_{*+1}(X,A)$, we must therefore determine this homomorphism ∂, and compute its cokernel $\operatorname{cok}(\partial) = H_*(A)/\operatorname{im}(\partial)$ and its kernel $\ker(\partial) \subset H_{*+1}(X,A)$.

In view of the short exact sequences

$$0 \to \operatorname{im}(\partial) \longrightarrow H_*(A) \longrightarrow \operatorname{cok}(\partial) \to 0$$

and

$$0 \to \ker(\partial) \longrightarrow H_{*+1}(X, A) \longrightarrow \mathrm{im}(\partial) \to 0$$

we can say that the original groups $H_*(A)$ and $H_{*+1}(X, A)$ have been reduced to the subquotient groups $\mathrm{cok}(\partial)$ and $\ker(\partial)$, respectively, and that both groups have been reduced by the same factor, namely by $\mathrm{im}(\partial)$. This makes sense in terms of orders of groups if all of these groups are finite, but must be more carefully interpreted in general. The change between the old groups and the new groups is in each case created by the non-triviality of the homomorphism ∂.

1.1. **E^r-terms and d^r-differentials.** We can present the steps in this approach to calculating $H_*(X)$ using the following chart. First we place the known groups $H_*(A)$ and $H_*(X, A)$ in two columns of the (s, t)-plane:

$$\begin{array}{ccc}
 & \vdots & \vdots \\
t = 2 & H_2(A) & H_3(X, A) \\
t = 1 & H_1(A) & H_2(X, A) \\
t = 0 & H_0(A) & H_1(X, A) \\
t = -1 & 0 & H_0(X, A) \\
 & s = 0 & s = 1
\end{array}$$

We call t the *internal degree*, even if this is not particularly meaningful in this example. The sum $s+t$ is called the *total degree*, and corresponds to the usual homological grading of $H_*(A)$, $H_*(X)$ and $H_*(X, A)$. This first page is called the E^1-term. It is a bigraded abelian group $E^1_{*,*}$, with

$$E^1_{0,t} = H_t(A) \qquad \text{and} \qquad E^1_{1,t} = H_{1+t}(X, A)$$

for all integers t. We extend the notation by setting $E^1_{s,t} = 0$ for $s \leq -1$ and for $s \geq 2$. This appears as follows

$$\begin{array}{cccccc}
 & & \vdots & & & \\
 & E^1_{-1,2} & E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & \\
 & E^1_{-1,1} & E^1_{0,1} & E^1_{1,1} & E^1_{2,1} & \\
\cdots & E^1_{-1,0} & E^1_{0,0} & E^1_{1,0} & E^1_{2,0} & \cdots \\
 & E^1_{-1,-1} & E^1_{0,-1} & E^1_{1,-1} & E^1_{2,-1} & \\
 & & \vdots & & &
\end{array}$$

with nonzero groups only in the two central columns.

We next introduce the boundary homomorphism ∂. In the (s,t)-plane it has bidegree $(-1,0)$, i.e., maps one unit to the left. We can display it as follows:

$$\begin{array}{ccc}
 & \vdots & \vdots \\
t=2 & H_2(A) \xleftarrow{\partial} & H_3(X,A) \\
t=1 & H_1(A) \xleftarrow{\partial} & H_2(X,A) \\
t=0 & H_0(A) \xleftarrow{\partial} & H_1(X,A) \\
t=-1 & 0 & H_0(X,A) \\
 & s=0 & s=1
\end{array}$$

In spectral sequence parlance, this homomorphism is called the d^1-differential. It extends trivially to a homomorphism

$$d^1_{s,t} \colon E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

for all integers s and t. In all other cases than those displayed above, this homomorphism is zero, since for $s \le 0$ the target is zero, for $s \ge 2$ the source is zero, and for $s = 1$ and $t \le -1$ the target is also zero.

We now replace each group $E^1_{0,t} = H_t(A)$ by its quotient group

$$\operatorname{cok}(\partial \colon H_{1+t}(X,A) \to H_t(A)) = \operatorname{cok}(d^1_{1,t})$$

and replace each group $E^1_{1,t} = H_{1+t}(X,A)$ by its subgroup

$$\ker(\partial \colon H_{1+t}(X,A) \to H_t(A)) = \ker(d^1_{1,t}) \,.$$

This leaves the following diagram

$$\begin{array}{ccc}
 & \vdots & \vdots \\
t=2 & \operatorname{cok}(d^1_{1,2}) & \ker(d^1_{1,2}) \\
t=1 & \operatorname{cok}(d^1_{1,1}) & \ker(d^1_{1,1}) \\
t=0 & \operatorname{cok}(d^1_{1,0}) & \ker(d^1_{1,0}) \\
t=-1 & 0 & H_0(X,A) \\
 & s=0 & s=1
\end{array}$$

We call this second page the E^2-term. It is a bigraded abelian group $E^2_{*,*}$, with

$$E^2_{0,t} = \operatorname{cok}(d^1_{1,t}) \qquad \text{and} \qquad E^2_{1,t} = \ker(d^1_{1,t})$$

for all integers t. As before, we extend the notation by setting $E^2_{s,t} = 0$ for $s \le -1$ and for $s \ge 2$.

What is the relation between the E^1-term and the E^2-term? This may be easier to see if we expand the diagram consisting of the E^1-term and the d^1-differential to also include the trivial groups surrounding the two interesting columns.

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
0 & H_2(A) \arrow[l] & H_3(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & H_1(A) \arrow[l] & H_2(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & H_0(A) \arrow[l] & H_1(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & 0 \arrow[l] & H_0(X,A) \arrow[l, "\partial"'] & 0 \arrow[l]
\end{tikzcd}
$$

In the other notation, this appears as follows:

$$
\begin{tikzcd}
 & & \vdots & & & \\
 & 0 & E^1_{0,2} \arrow[l, "d^1_{0,2}"'] & E^1_{1,2} \arrow[l, "d^1_{1,2}"'] & 0 \arrow[l, "d^1_{2,2}"'] & \\
 & 0 & E^1_{0,1} \arrow[l, "d^1_{0,1}"'] & E^1_{1,1} \arrow[l, "d^1_{1,1}"'] & 0 \arrow[l, "d^1_{2,1}"'] & \\
\cdots & 0 & E^1_{0,0} \arrow[l, "d^1_{0,0}"'] & E^1_{1,0} \arrow[l, "d^1_{1,0}"'] & 0 \arrow[l, "d^1_{2,0}"'] & \cdots \\
 & 0 & E^1_{0,-1} \arrow[l, "d^1_{0,-1}"'] & E^1_{1,-1} \arrow[l, "d^1_{1,-1}"'] & 0 \arrow[l, "d^1_{2,-1}"'] & \\
 & & \vdots & & &
\end{tikzcd}
$$

Now notice that each row $(E^1_{*,t}, d^1_{*,t})$ of the E^1-term with the d^1-differentials forms a chain complex, and the E^2-term is the homology of that chain complex:

$$E^2_{s,t} = \frac{\ker(d^1_{s,t})}{\mathrm{im}(d^1_{s+1,t})} = H_s(E^1_{*,t}, d^1_{*,t})$$

for all integers s and t. For $s = 0$ this is clear because

$$\ker(d^1_{0,t}) = E^1_{0,t} = H_t(A) \qquad \text{and} \qquad \mathrm{im}(d^1_{1,t}) = \mathrm{im}(\partial\colon H_{1+t}(X,A) \to H_t(A))\,.$$

For $s = 1$ it is also clear, because

$$\ker(d^1_{1,t}) = \ker(\partial\colon H_{1+t}(X,A) \to H_t(A)) \qquad \text{and} \qquad \mathrm{im}(d^1_{2,t}) = 0\,.$$

For the remaining values of s, all groups are trivial.

Having obtained the E^2-term as the homology of the E^1-term with respect to the d^1-differentials, we can now locate the short exact sequence

$$0 \to \mathrm{cok}(d^1_{1,n}) \longrightarrow H_n(X) \longrightarrow \ker(d^1_{1,n-1}) \to 0$$

within the diagram, for each n. This is nothing but the degree n part of the short exact sequences previously denoted

$$0 \to \mathrm{cok}(\partial) \longrightarrow H_*(X) \longrightarrow \ker(\partial) \to 0$$

and

$$0 \to \mathrm{im}(i_*) \longrightarrow H_*(X) \longrightarrow \mathrm{cok}(i_*) \to 0\,,$$

and is now written

$$0 \to E^2_{0,n} \longrightarrow H_n(X) \longrightarrow E^2_{1,n-1} \to 0\,.$$

These extensions appear along anti-diagonals in the E^2-term, or equivalently, along lines of slope -1:

$$\begin{tikzcd}
\vdots & & \vdots \\
E^2_{0,2} \arrow[dr, tail] & & E^2_{1,2} \\
& H_2(X) \arrow[dr, two heads] & \\
E^2_{0,1} \arrow[dr, tail] & & E^2_{1,1} \\
& H_1(X) \arrow[dr, two heads] & \\
E^2_{0,0} \arrow[dr, tail] & & E^2_{1,0} \\
& H_0(X) \arrow[dr, two heads] & \\
0 & & E^2_{1,-1}
\end{tikzcd}$$

In other words, the filtration quotients $(F_s/F_{s-1})_n$ associated to the increasing filtration

$$0 \subset \mathrm{im}(i_*) \subset H_n(X)$$

appear along the line in the (s,t)-plane where the total degree is $s+t=n$, starting with $(F_0)_n = \mathrm{im}(i_*)$ at $E^2_{0,n}$, and continuing with the filtration quotient $(F_1/F_0)_n = \mathrm{cok}(i_*)$ at $E^2_{1,n-1}$. The group we are interested in, $H_n(X)$, is realized as an extension of the two parts of the E^2-term in bidegrees $(0,n)$ and $(1,n-1)$.

This indexing system is standard for the Serre spectral sequence.

1.2. **Adams indexing.** In some cases it is more convenient to collect the terms contributing to a single degree in the answer, in our case the terms $E^2_{0,n}$ and $E^2_{1,n-1}$ contributing to $H_n(X)$, in a single column. This means that the terms $E^1_{0,n}$ and $E^1_{1,n-1}$ are also placed in a single column, and the d^1-differential will map diagonally to the left and upwards. The E^1-term is then displayed as follows, in the (n,s)-plane:

$$\begin{array}{cccccc}
s=1 & H_0(A) & H_1(A) & H_2(A) & H_3(A) & \dots \\
 & & & & & \\
s=0 & H_0(X,A) & H_1(X,A) & H_2(X,A) & H_3(X,A) & \dots \\
 & & & & & \\
 & n=0 & n=1 & n=2 & n=3 &
\end{array}$$

The orientation of the s-axis has also been switched, so that $H_0(X,A)$ rather than $H_0(A)$ sits at the origin, and the total degree n is related to the filtration degree s and the internal degree t by $n = t - s$ instead of $n = s + t$. We will discuss this more precisely later. The d^1-differential is still the connecting homomorphism ∂:

$$\begin{tikzcd}
H_0(A) & H_1(A) & H_2(A) & H_3(A) & \dots \\
H_0(X,A) & H_1(X,A) \arrow[ul, "\partial"'] & H_2(X,A) \arrow[ul, "\partial"'] & H_3(X,A) \arrow[ul, "\partial"'] & \dots \arrow[ul, "\partial"']
\end{tikzcd}$$

The E^2-term is the homology of the E^1-term with respect to the d^1-differential:

$$\begin{tikzcd}
\mathrm{cok}(\partial)_0 & \mathrm{cok}(\partial)_1 & \mathrm{cok}(\partial)_2 & \mathrm{cok}(\partial)_3 & \dots \\
H_0(X,A) & \ker(\partial)_1 & \ker(\partial)_2 & \ker(\partial)_3 & \dots
\end{tikzcd}$$

The end product, known as the abutment, of the spectral sequence, is now determined up to an extension problem, by the following vertical short exact sequences:

$$\begin{tikzcd}
\mathrm{cok}(\partial)_0 \arrow[d, tail] & \mathrm{cok}(\partial)_1 \arrow[d, tail] & \mathrm{cok}(\partial)_2 \arrow[d, tail] & \mathrm{cok}(\partial)_3 \arrow[d, tail] & \dots \\
H_0(X) \arrow[d, two heads] & H_1(X) \arrow[d, two heads] & H_2(X) \arrow[d, two heads] & H_3(X) \arrow[d, two heads] & \\
H_0(X,A) & \ker(\partial)_1 & \ker(\partial)_2 & \ker(\partial)_3 & \dots
\end{tikzcd}$$

This indexing system is standard for the Adams spectral sequence, and we refer to it as Adams indexing.

2. Spectral sequences

Definition 2.1. A *homological spectral sequence* is a sequence $(E^r, d^r)_r$ of bigraded abelian groups and differentials, together with isomorphisms

$$E^{r+1} \cong H(E^r, d^r) = \frac{\ker(d^r)}{\mathrm{im}(d^r)},$$

for all natural numbers r. Each $E^r = E^r_{*,*} = (E^r_{s,t})_{s,t}$ is a bigraded abelian group, called the E^r-term of the spectral sequence. The r-th differential is a homomorphism $d^r \colon E^r_{*,*} \to E^r_{*,*}$ of bidegree $(-r, r-1)$, satisfying $d^r \circ d^r = 0$. We write

$$d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$$

for the component of d^r starting in bidegree (s,t). The isomorphism

$$E^{r+1}_{s,t} \cong H_{s,t}(E^r, d^r) = \frac{\ker(d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t+r-1})}{\mathrm{im}(d^r_{s+r,t-r+1} \colon E^r_{s+r,t-r+1} \to E^r_{s,t})}$$

is part of the data.

Here is the typical E^1-term and d^1-differential, depicted in the (s,t)-plane:

$$\begin{tikzcd}
\cdots & E^1_{-1,2} \arrow[l] & E^1_{0,2} \arrow[l] & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & E^1_{3,2} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,1} \arrow[l] & E^1_{0,1} \arrow[l] & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & E^1_{3,1} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,0} \arrow[l] & E^1_{0,0} \arrow[l] & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & E^1_{3,0} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,-1} \arrow[l] & E^1_{0,-1} \arrow[l] & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & E^1_{3,-1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Each row is a chain complex, and the homology of this chain complex is isomorphic to the E^2-term. That E^2-term, together with the d^2-differentials, appears as follows:

$$\begin{tikzcd}
\cdots & E^2_{-1,2} & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & E^2_{3,2} & \cdots \\
\cdots & E^2_{-1,1} & E^2_{0,1} \arrow[ull] & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] & E^2_{3,1} \arrow[ull] & \cdots \arrow[ull] \\
\cdots & E^2_{-1,0} & E^2_{0,0} \arrow[ull] & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull] & \cdots \arrow[ull] \\
\cdots & E^2_{-1,-1} & E^2_{0,-1} \arrow[ull] & E^2_{1,-1} \arrow[ull] & E^2_{2,-1} \arrow[ull] & E^2_{3,-1} \arrow[ull] & \cdots \arrow[ull]
\end{tikzcd}$$

(The differentials entering or leaving the displayed part are not shown.) Each line of slope $-1/2$ is a chain complex, with homology isomorphic to the E^3-term. That E^3-term, together with the d^3-differentials, appears as follows:

$$\begin{tikzcd}
\cdots & E^3_{-1,2} & E^3_{0,2} & E^3_{1,2} & E^3_{2,2} & E^3_{3,2} & \cdots \\
\cdots & E^3_{-1,1} & E^3_{0,1} & E^3_{1,1} & E^3_{2,1} & E^3_{3,1} & \cdots \\
\cdots & E^3_{-1,0} & E^3_{0,0} & E^3_{1,0} \arrow[uulll] & E^3_{2,0} \arrow[uulll] & E^3_{3,0} \arrow[uulll] & \cdots \arrow[uulll] \\
\cdots & E^3_{-1,-1} & E^3_{0,-1} & E^3_{1,-1} \arrow[uulll] & E^3_{2,-1} \arrow[uulll] & E^3_{3,-1} \arrow[uulll] & \cdots \arrow[uulll]
\end{tikzcd}$$

Each line of slope $-2/3$ is a chain complex, with homology isomorphic to the E^4-term:

$$\begin{tikzcd}
\cdots & E^4_{-1,2} & E^4_{0,2} & E^4_{1,2} & E^4_{2,2} & E^4_{3,2} & \cdots \\
\cdots & E^4_{-1,1} & E^4_{0,1} & E^4_{1,1} & E^4_{2,1} & E^4_{3,1} & \cdots \\
\cdots & E^4_{-1,0} & E^4_{0,0} & E^4_{1,0} & E^4_{2,0} & E^4_{3,0} & \cdots \\
\cdots & E^4_{-1,-1} & E^4_{0,-1} & E^4_{1,-1} & E^4_{2,-1} \arrow[uuullll] & E^4_{3,-1} \arrow[uuullll] & \cdots \arrow[uuullll]
\end{tikzcd}$$

Each line of slope $-3/4$ is a chain complex, with homology isomorphic to the E^5-term:

$$\begin{array}{ccccccc}
\dots & E^5_{-1,2} & E^5_{0,2} & E^5_{1,2} & E^5_{2,2} & E^5_{3,2} & \dots \\
\\
\dots & E^5_{-1,1} & E^5_{0,1} & E^5_{1,1} & E^5_{2,1} & E^5_{3,1} & \dots \\
\\
\dots & E^5_{-1,0} & E^5_{0,0} & E^5_{1,0} & E^5_{2,0} & E^5_{3,0} & \dots \\
\\
\dots & E^5_{-1,-1} & E^5_{0,-1} & E^5_{1,-1} & E^5_{2,-1} & E^5_{3,-1} & \dots
\end{array}$$

At this point there is not room for any further differentials within the finite part of the spectral sequence that is displayed. There may of course always be longer differentials that enter or leave the displayed region.

2.1. **E^∞-terms.** We now want to give sense to the limiting term, the E^∞-term $E^\infty = E^\infty_{*,*}$, of a spectral sequence. This is a bigraded abelian group, and we would like to make sense of $E^\infty_{s,t}$ as an algebraic limit of the abelian groups $E^r_{s,t}$ as $r \to \infty$.

In many cases the spectral sequence is *locally eventually constant*, in the sense that for each fixed bidegree (s,t) there is a natural number $m(s,t)$ such that the homomorphisms

$$d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1} \qquad \text{and} \qquad d^r \colon E^r_{s+r,t-r+1} \to E^r_{s,t}$$

are zero for all $r \geq m(s,t)$. Then $E^{r+1}_{s,t} \cong E^r_{s,t}$ for all $r \geq m(s,t)$, and we define

$$E^\infty_{s,t} = E^{m(s,t)}_{s,t}$$

to be this common value. If there is a fixed bound m that works for each bidegree (s,t), so that $d^r = 0$ for all $r \geq m$ and $E^{r+1} \cong E^r$ for all $r \geq m$, we say that the spectral sequence *collapses* at the E^m-term. In this case $E^\infty = E^m$.

In general, a spectral sequence determines a descending sequence of *r-cycles*

$$\dots \subset Z^{r+1} \subset Z^r \subset \dots \subset Z^2 = \ker(d^1) \subset Z^1 = E^1$$

and an increasing sequence of *r-boundaries*

$$0 = B^1 \subset \operatorname{im}(d^1) = B^2 \subset \dots \subset B^r \subset B^{r+1} \subset \dots \subset E^1 \,,$$

with $B^r \subset Z^r$ and $E^r \cong Z^r/B^r$ for all $r \geq 1$. (This is Boardman's indexing convention. Other authors like Mac Lane (1963) have $E^{r+1} \cong Z^r/B^r$.) We then define the bigraded abelian groups of infinite cycles and infinite boundaries to be

$$Z^\infty = \bigcap_r Z^r = \lim_r Z^r \qquad \text{and} \qquad B^\infty = \bigcup_r B^r = \operatorname*{colim}_r B^r \,,$$

respectively, and set $E^\infty = Z^\infty/B^\infty$. This definition is reasonable if the limit system of r-cycles is well-behaved, i.e., if the left derived limit $\operatorname{Rlim}_r Z^r$ vanishes. In the case of a locally eventually constant spectral sequence, the general definition agrees with the previous definition, since $Z^\infty_{s,t} = Z^r_{s,t}$ and $B^r_{s,t} = B^\infty_{s,t}$ for all $r \geq m(s,t) - 1$. [[More about this later.]]

2.2. **Filtrations.**

Definition 2.2. An *increasing filtration* of an abelian group G is a sequence $\{F_s\}_s$ of subgroups

$$\dots \subset F_{s-1} \subset F_s \subset \dots \subset G \,.$$

The filtration is *exhaustive* if the canonical map

$$\operatorname*{colim}_s F_s \longrightarrow G$$

is an isomorphism. The filtration is *Hausdorff* if

$$\lim_s F_s = 0 \,,$$

and it is *complete* if

$$\operatorname*{Rlim}_s F_s = 0\,.$$

Here $\operatorname{colim}_s F_s \cong \bigcup_s F_s$, so the filtration is exhaustive precisely if each element in G lies in some F_s. We can think of the F_s as specifying neighborhoods of 0 in a (linear) topology on G. Since $\bigcap_s F_s = \lim_s F_s$, this topology is Hausdorff if and only if the filtration is Hausdorff. An Cauchy sequence is an element in $\lim_s G/F_s$, so the topology is complete exactly when the canonical map $G \to \lim_s G/F_s$ is surjective, i.e., when $\operatorname{Rlim}_s F_s = 0$. [[More about this later.]]

In the case of a *finite filtration*, these conditions are easily verified. If there are integers $a \le b$ such that $F_s = 0$ for $s < a$ and $F_s = G$ for $s \ge b$, then the filtration has the form

$$0 \subset F_a \subset \cdots \subset F_b = G\,.$$

Clearly $\operatorname{colim}_s F_s = G$, $\lim_s F_s = 0$ and $\operatorname{Rlim}_s F_s = 0$. In this case, the only nontrivial filtration quotients are the F_s/F_{s-1} for integers s in the finite interval $[a, b]$.

In the case of a finite filtration, the group G appears as the filtration subquotient F_b/F_{a-1}. Under the three conditions above, G is also algebraically determined by the finite filtration subquotients F_s/F_{s-r}.

Lemma 2.3. *If $\{F_s\}_s$ is an exhaustive complete Hausdorff filtration of G, then*

$$G \cong \operatorname*{colim}_s \lim_r F_s/F_{s-r}$$

so that G can be recovered from the subquotients F_s/F_{s-r} of a filtration.

Proof. For each s, there is a tower of short exact sequences

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_{s-r} \arrow[r] \arrow[d] & F_s \arrow[r] \arrow[d] & F_s/F_{s-r} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_{s-1} \arrow[r] & F_s \arrow[r] & F_s/F_{s-1} \arrow[r] & 0
\end{tikzcd}$$

giving rise to the six-term exact sequence

$$0 \to \lim_r F_{s-r} \to F_s \to \lim_s F_s/F_{s-r} \longrightarrow \operatorname*{Rlim}_r F_{s-r} \to 0 \to \operatorname*{Rlim}_r F_s/F_{s-r} \to 0\,.$$

By the complete Hausdorff assumption, $\lim_r F_{s-r} = 0$ and $\operatorname{Rlim}_r F_{s-r} = 0$, so

$$F_s \xrightarrow{\cong} \lim_s F_s/F_{s-r}$$

is an isomorphism. Passing to the colimit over s, and using the exhaustive assumption, we get the asserted formula. □

A filtration of a graded abelian group is a filtration in each degree.

Definition 2.4. A homological spectral sequence $(E^r, d^r)_r$ *converges* to a (graded) abelian group G if there is an increasing exhaustive Hausdorff filtration $\{F_s\}_s$ of G, and isomorphisms of (graded) abelian groups

$$E^\infty_{s,*} \cong F_s/F_{s-1}$$

for all integers s. The spectral sequence *converges strongly* if the filtration is also complete. In these cases we write

$$E^r \Longrightarrow G\,.$$

We call G the *target*, or the *abutment*, of the spectral sequence.

If it is necessary to emphasize the filtration degree s, we write $E^r \Longrightarrow_s G$. We may also make the bigrading explicit, as in $E^r_{*,*} \Longrightarrow G_*$ or $E^r_{s,t} \Longrightarrow G_{s+t}$.

A strongly convergent spectral sequence determines its abutment, up to questions about differentials and extensions. If we know the E^m-term for some $m \geq 1$, and can determine the d^r-differentials for all $r \geq m$, then we know the E^r-terms for all $r \geq m$, and can pass to the limit to determine the E^∞-term. [[Elaborate on how the Z^r and B^r are found, and how they specify Z^∞ and B^∞.]]

By convergence, this determines the filtration quotients $E^\infty_{s,*} \cong F_s/F_{s-1}$ for each s. There are short exact sequences

$$0 \to F_{s-r}/F_{s-r-1} \longrightarrow F_s/F_{s-r-1} \longrightarrow F_s/F_{s-r} \to 0$$

for all $r \geq 1$ and integers s, so if we inductively have determined F_s/F_{s-r}, and know $F_{s-r}/F_{s-r-1} = E^\infty_{s-r,*}$, then only an extension problem of abelian groups remains in our quest to determine F_s/F_{s-r-1}. This gives the input for the next inductive step, over r.

In the case of a finite filtration, this process gives us G after a finite number of steps. In the general case, assuming strong convergence, passing to limits over r and colimits over s recovers the abutment G.

3. The spectral sequence of a triple

To illustrate the general definitions in the first case that does not reduce to a long exact sequence, let us consider a triple (X, B, A) of spaces, and aim to understand the relationship between the homology groups $H_*(A)$, $H_*(B, A)$, $H_*(X, B)$ and $H_*(X)$. The essential pairs and maps appear in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] & B \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "j"] \\
& (B, A) & (X, B)\,,
\end{tikzcd}$$

but can be more systematically embedded in the larger diagram

$$\begin{tikzcd}
\dots \arrow[r, "i", "="'] & \emptyset \arrow[r, "i"] \arrow[d, "j"] & A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[r, "i"] \arrow[d, "j"] & X \arrow[r, "i", "="'] \arrow[d, "j"] & X \arrow[r, "i", "="'] \arrow[d, "j"] & \dots \\
& (\emptyset, \emptyset) & (A, \emptyset) & (B, A) & (X, B) & (X, X)\,.
\end{tikzcd}$$

We have two long exact sequences, associated to the pairs (B, A) and (X, A), respectively:

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(B, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \dots$$

and

$$\dots \xrightarrow{\partial} H_n(B) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{i_*} \dots\,.$$

We can also display these two long exact sequences together, as follows, where ∂ has degree -1 and each triangle is exact.

$$\begin{tikzcd}
H_*(A) \arrow[r, "i_*"] & H_*(B) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X) \arrow[d, "j_*"] \\
& H_*(B, A) \arrow[ul, "\partial"] & H_*(X, B) \arrow[ul, "\partial"]
\end{tikzcd}$$

Again, this is the essential part of the bigger diagram

$$\begin{tikzcd}
\dots \arrow[r, "i_*"] & 0 \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(A) \arrow[r, "i_*"] \arrow[d, "j_*", "="'] & H_*(B) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X) \arrow[r, "i_*", "="'] \arrow[d, "j_*"] & H_*(X) \arrow[r, "i_*", "="'] \arrow[d, "j_*"] & \dots \\
& 0 \arrow[ul, "\partial"] & H_*(A) \arrow[ul, "\partial"] & H_*(B, A) \arrow[ul, "\partial"] & H_*(X, B) \arrow[ul, "\partial"] & 0 \arrow[ul, "\partial"] & \dots \arrow[ul, "\partial"]
\end{tikzcd}$$

Our aim is to construct a spectral sequence starting with an E^1-term given by the homology groups in the lower row of this diagram, namely, $H_*(A)$, $H_*(B, A)$ and $H_*(X, B)$, and converging to the homology group $G = H_*(X)$, equipped with the finite filtration

$$0 \subset F_0 \subset F_1 \subset F_2 = G$$

where

$$\begin{aligned}
F_0 &= \operatorname{im}(i_*^2 \colon H_*(A) \to H_*(X)) \\
F_1 &= \operatorname{im}(i_* \colon H_*(B) \to H_*(X)) \\
F_2 &= H_*(X)\,.
\end{aligned}$$

To emphasize the grading we may write $(F_s)_n$ for the part of F_s in degree n. Convergence means that there should be isomorphisms

$$E^\infty_{0,t} \cong (F_0)_t \qquad , \qquad E^\infty_{1,t} \cong (F_1/F_0)_{1+t} \qquad \text{and} \qquad E^\infty_{2,t} \cong (F_2/F_1)_{2+t}\,.$$

In fact, this spectral sequence will collapse at the E^3-term, so that there are nonzero d^1- and d^2-differentials, but $d^r = 0$ for $r \ge 3$ and $E^3 = E^\infty$.

First, the E^1-term is given by

$$E^1_{0,t} = H_t(A) \qquad , \qquad E^1_{1,t} = H_{1+t}(B,A) \qquad \text{and} \qquad E^1_{2,t} = H_{2+t}(X,B)\,,$$

and the d^1-differential is defined to be the composite $d^1 = j_* \circ \partial$. More explicitly,

$$d^1_{1,t} = \partial \colon H_{1+t}(B,A) \longrightarrow H_t(A) \qquad \text{and} \qquad d^1_{2,t} = j_*\partial \colon H_{2+t}(X,B) \longrightarrow H_{1+t}(B,A)$$

are visible in the diagrams above. Note that $d^1_{s-1,t} \circ d^1_{s,t} = 0$ for all s and t, since $\partial j_* = 0$.

The (E^1, d^1)-chart appears as follows:

$$\begin{array}{cccccc}
\vdots & & & & & \\
 & & & & & \\
H_2(A) & \xleftarrow{\partial} & H_3(B,A) & \xleftarrow{j_*\partial} & H_4(X,B) & \\
 & & & & & \\
H_1(A) & \xleftarrow{\partial} & H_2(B,A) & \xleftarrow{j_*\partial} & H_3(X,B) & \\
 & & & & & \\
H_0(A) & \xleftarrow{\partial} & H_1(B,A) & \xleftarrow{j_*\partial} & H_2(X,B) & \dots \\
 & & & & & \\
0 & & H_0(B,A) & \xleftarrow{j_*\partial} & H_1(X,B) & \\
 & & & & & \\
0 & & 0 & & H_0(X,B) &
\end{array}$$

Passing to homology, we get to the E^2-term.

In column $s = 0$, we compute

$$\begin{aligned}
E^2_{0,t} = \frac{\ker(d^1_{0,t})}{\operatorname{im}(d^1_{1,t})} &= \frac{H_t(A)}{\operatorname{im}(\partial \colon H_{1+t}(B,A) \to H_t(A))} \\
&= \frac{H_t(A)}{\ker(i_* \colon H_t(A) \to H_t(B))} \cong \operatorname{im}(i_* \colon H_t(A) \to H_t(B)) = \operatorname{im}(i_*)_t
\end{aligned}$$

using exactness at $H_t(A)$. The right hand isomorphism is induced by i_*.

In column $s = 1$, we find

$$\begin{aligned}
E^2_{1,t} = \frac{\ker(d^1_{1,t})}{\operatorname{im}(d^1_{2,t})} &= \frac{\ker(\partial \colon H_{1+t}(B,A) \to H_t(A))}{\operatorname{im}(j_*\partial \colon H_{2+t}(X,B) \to H_{1+t}(B,A))} \\
&= \frac{\operatorname{im}(j_* \colon H_{1+t}(B) \to H_{1+t}(B,A))}{\operatorname{im}(j_*\partial \colon H_{2+t}(X,B) \to H_{1+t}(B,A))} \\
&\cong \frac{H_{1+t}(B)}{\ker(j_* \colon H_{1+t}(B) \to H_{1+t}(B,A)) + \operatorname{im}(\partial \colon H_{2+t}(X,B) \to H_{1+t}(B))}
\end{aligned}$$

using exactness at $H_{1+t}(B,A)$. The last isomorphism is induced by j_*. To verify it, it is clear that j_* induces a surjection from $H_{1+t}(B)$ to the quotient $\operatorname{im}(j_*)/\operatorname{im}(j_*\partial)$ (on the preceding line). Its kernel consists of the elements that map under j_* to elements in the image of $j_*\partial$. These differ by elements in $\ker(j_*)$ from elements in $\operatorname{im}(\partial)$, hence are in the sum $\ker(j_*) + \operatorname{im}(\partial)$. This is an internal sum of subgroups of $H_{1+t}(B)$, not necessarily a direct sum. Using exactness at $H_{1+t}(B)$ in two different exact sequences, we can rewrite this as follows:

$$
\begin{aligned}
\dots &= \frac{H_{1+t}(B)}{\operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B)) + \ker(i_*\colon H_{1+t}(B) \to H_{1+t}(X))} \\
&\cong \frac{\operatorname{im}(i_*\colon H_{1+t}(B) \to H_{1+t}(X))}{\operatorname{im}(i_*^2\colon H_{1+t}(A) \to H_{1+t}(X))} = (F_1/F_0)_{1+t}
\end{aligned}
$$

The second isomorphism is induced by i_*, and is formally of the same type as the one we just discussed: The homomorphism i_* induces a surjection from $H_{1+t}(B)$ to $\operatorname{im}(i_*)/\operatorname{im}(i_*^2)$, with kernel given by the internal sum of $\ker(i_*\colon H_{1+t}(B) \to H_{1+t}(X))$ and $\operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B))$.

In column $s = 2$, we calculate

$$
\begin{aligned}
E^2_{2,t} = \frac{\ker(d^1_{2,t})}{\operatorname{im}(d^1_{3,t})} &= \frac{\ker(j_*\partial\colon H_{2+t}(X,B) \to H_{1+t}(B,A))}{0} \\
&= \partial^{-1}\ker(j_*\colon H_{1+t}(B) \to H_{1+t}(B,A)) \\
&= \partial^{-1}\operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B)) = \partial^{-1}(\operatorname{im}(i_*)_{1+t})
\end{aligned}
$$

using exactness at $H_{1+t}(B)$. This is the subgroup of $H_{2+t}(X,B)$ consisting of elements x with $\partial(x) \in H_{1+t}(B)$ lying in the image of $i_*\colon H_{1+t}(A) \to H_{1+t}(B)$.

The d^2-differential acting on the E^2-term is now defined to be the homomorphism

$$
d^2_{2,t}\colon E^2_{2,t} = \partial^{-1}(\operatorname{im}(i_*)_{1+t}) \longrightarrow \operatorname{im}(i_*)_{1+t} = E^2_{0,t+1}
$$

induced by ∂, mapping a class $x \in E^2_{2,t}$ with $\partial(x) \in \operatorname{im}(i_*)_{1+t}$ to the class $d^2(x) = \partial(x) \in E^2_{0,1+t}$.

The (E^2, d^2)-chart appears as follows:

$$
\begin{tikzcd}
\vdots & & & \\
\operatorname{im}(i_*)_2 & (F_1/F_0)_3 & \partial^{-1}(\operatorname{im}(i_*)_3) \arrow[llu, "d^2_{2,2}"'] & \\
\operatorname{im}(i_*)_1 & (F_1/F_0)_2 & \partial^{-1}(\operatorname{im}(i_*)_2) \arrow[llu, "d^2_{2,1}"'] & \\
\operatorname{im}(i_*)_0 & (F_1/F_0)_1 & \partial^{-1}(\operatorname{im}(i_*)_1) \arrow[llu, "d^2_{2,0}"'] & \dots \\
0 & (F_1/F_0)_0 & \partial^{-1}(\operatorname{im}(i_*)_0) \arrow[llu, "d^2_{2,-1}"'] & \\
0 & 0 & H_0(X,B) &
\end{tikzcd}
$$

Passing to homology once more, we get to the E^3-term.

In column $s = 0$, the E^3-term is

$$
\begin{aligned}
E^3_{0,t} = \frac{\ker(d^2_{0,t})}{\operatorname{im}(d^2_{2,t-1})} &\cong \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\operatorname{im}(\partial\colon \partial^{-1}(\operatorname{im} i_*)_t \to \operatorname{im}(i_*)_t)} \\
&= \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\operatorname{im}(\partial\colon H_{1+t}(X,B) \to H_t(B)) \cap \operatorname{im}(i_*\colon H_t(A) \to H_t(B))} \\
&= \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\ker(i_*\colon H_t(B) \to H_t(X)) \cap \operatorname{im}(i_*\colon H_t(A) \to H_t(B))} \\
&\cong \operatorname{im}(i_*^2\colon H_t(A) \to H_t(X)) = (F_0)_t
\end{aligned}
$$

using the definition of $d^2_{2,t-1}$ and exactness at $H_t(B)$. The last isomorphism is induced by $i_*\colon H_t(B) \to H_t(X)$.

Column $s = 1$ is not affected by the d^2-differentials, so

$$E^3_{1,t} = \frac{\ker(d^2_{1,t})}{\operatorname{im}(d^2_{3,t-1})} = \frac{E^2_{1,t}}{0} \cong (F_1/F_0)_{1+t}\,.$$

In column $s = 2$, the E^3-term is

$$\begin{aligned} E^3_{2,t} = \frac{\ker(d^2_{2,t})}{\operatorname{im}(d^2_{4,t-1})} &= \frac{\ker(d^2_{2,t}\colon \partial^{-1}(\operatorname{im}(i_*)_{1+t}) \to \operatorname{im}(i_*)_{1+t})}{0} \\ &= \ker(\partial\colon H_{2+t}(X,B) \to H_{1+t}(B)) \\ &= \operatorname{im}(j_*\colon H_{2+t}(X) \to H_{2+t}(X,B)) \\ &\cong \frac{H_{2+t}(X)}{\operatorname{im}(i_*\colon H_{2+t}(B) \to H_{2+t}(X))} = (F_2/F_1)_{2+t} \end{aligned}$$

by the definition of the d^2-differential, and exactness at $H_{2+t}(X,B)$ and at $H_{2+t}(X)$.

The E^3-term appears as follows:

$$\begin{array}{cccc} \vdots & & & \\ & & & \\ (F_0)_2 & (F_1/F_0)_3 & (F_2/F_1)_4 & \\ (F_0)_1 & (F_1/F_0)_2 & (F_2/F_1)_3 & \\ (F_0)_0 & (F_1/F_0)_1 & (F_2/F_1)_2 & \dots \\ 0 & (F_1/F_0)_0 & (F_2/F_1)_1 & \\ 0 & 0 & (F_2/F_1)_0 & \end{array}$$

There is no room for further nonzero differentials, since d^r for $r \geq 3$ must involve columns three or more units apart. Hence this spectral sequence collapses at the E^3-term, and $E^\infty = E^3$ is as displayed above.

In view of our calculations, we have isomorphisms

$$E^\infty_{s,t} \cong (F_s/F_{s-1})_{s+t}$$

in all bidegrees (s,t), which proves that the spectral sequence we have constructed, with the given E^1-term, d^1-differential and d^2-differential, indeed converges strongly to the abutment $H_*(X)$, with the finite filtration given by

$$(F_s)_n = \begin{cases} 0 & \text{for } s \leq -1 \\ \operatorname{im}(i^2_*\colon H_n(A) \to H_n(X)) & \text{for } s = 0 \\ \operatorname{im}(i_*\colon H_n(B) \to H_n(X)) & \text{for } s = 1 \\ H_n(X) & \text{for } s \geq 2. \end{cases}$$

Hence we can conclude that there is a strongly convergent spectral sequence

$$E^1_{s,t} \Longrightarrow_s H_{s+t}(X)$$

with three nonzero columns

$$E^1_{s,t} = \begin{cases} 0 & \text{for } s \le -1 \\ H_t(A) & \text{for } s = 0 \\ H_{1+t}(B, A) & \text{for } s = 1 \\ H_{2+t}(X, B) & \text{for } s = 2 \\ 0 & \text{for } s \ge 3. \end{cases}$$

[[Illustrate with an example?]]

[[The K-theory based Adams spectral sequence is an interesting three-line spectral sequence (Adams–Baird, Bousfield, Dwyer–Mitchell).]]

4. Cohomological spectral sequences

So far we have focused on so-called *homological* spectral sequences, where the differentials reduce total degrees and filtration indices. If one applies cohomology to the same diagrams of spaces, one instead obtains a *cohomological* spectral sequence.

Definition 4.1. A *cohomological spectral sequence* is a sequence $(E_r, d_r)_r$ of bigraded abelian groups and differentials, together with isomorphisms

$$E_{r+1} \cong H(E_r, d_r)$$

for all $r \ge 1$. Each E_r-term is a bigraded abelian group $E_r = E_r^{*,*} = (E_r^{s,t})_{s,t}$, and each d^r-differential is a homomorphism $d_r \colon E_r^{*,*} \to E_r^{*,*}$ of bidegree $(r, 1-r)$, satisfying $d_r \circ d_r = 0$.

Definition 4.2. A *decreasing filtration* of an abelian group G is a sequence $\{F^s\}_s$ of subgroups

$$G \supset \cdots \supset F^s \supset F^{s+1} \supset \dots .$$

It is *exhaustive* if $\operatorname{colim}_s F^s \cong G$, *Hausdorff* if $\lim_s F^s = 0$ and *complete* if $\operatorname{Rlim}_s F^s = 0$.

Definition 4.3. A cohomological spectral sequence $(E^r, d^r)_r$ *converges* to a graded abelian group G if there is a decreasing exhaustive Hausdorff filtration $\{F^s\}_s$ of G, and isomorphisms of (graded) abelian groups

$$E_\infty^{s,*} \cong F^s/F^{s+1}$$

for all integers s. The spectral sequence *converges strongly* if the filtration is also complete.

The algebraic structure in a cohomological spectral sequence is really the same as in a homological spectral sequence; the difference only lies in the sign conventions for the grading. To each homological spectral sequence $(E^r, d^r)_r$ there is an associated cohomological spectral sequence $(E_r, d_r)_r$ with

$$E_r^{s,t} = E^r_{-s,-t}$$

for all integers s and t, and with

$$d_r^{s,t} = d^r_{-s,-t} \,.$$

To each increasing filtration $\{F_s\}_s$ of an abelian group G there is an associated decreasing filtration $\{F^s\}_s$ of the same group, with

$$F^s = F_{-s} \,.$$

The spectral sequence $(E^r, d^r)_r$ converges (strongly) to the abutment G, filtered by $\{F_s\}_s$, if and only if the associated cohomological spectra sequence $(E_r, d_r)_r$ converges (strongly) to the abutment G, filtered by $\{F^s\}_s$.

The sign change in the bidegree of the spectral sequence differentials implies that the direction of the arrows in an (E_r, d_r)-chart is reversed in comparison with the direction in an (E^r, d^r)-chart. For instance, an (E_2, d_2)-term typically appears as follows. (Compare with the (E^2, d^2)-term displayed earlier.)

$$\begin{tikzcd}
\cdots \arrow[drr] & E_2^{-1,2} \arrow[drr] & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} & \cdots \\
\cdots \arrow[drr] & E_2^{-1,1} \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & \cdots \\
\cdots \arrow[drr] & E_2^{-1,0} \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & E_2^{3,0} & \cdots \\
\cdots & E_2^{-1,-1} & E_2^{0,-1} & E_2^{1,-1} & E_2^{2,-1} & E_2^{3,-1} & \cdots
\end{tikzcd}$$

One reason for switching from a homological to a cohomological indexing occurs when the spectral sequence occupies a quadrant, or a half-plane. If the homological spectral sequence $E^r_{s,t}$ is nonzero only for $s \le 0$ and $t \le 0$ (or for $s \le 0$), then the associated cohomological spectral sequence $E_r^{s,t}$ is nonzero only for $s \ge 0$ and $t \ge 0$ (or for $s \ge 0$). It tends to be notationally easier to work with the latter conventions. We refer to such a spectral sequence as a *first quadrant* (or *right half-plane*) spectral sequence. [[Formalize this definition?]]

[[Another reason for working with cohomology has to do with product structures. The cup product in cohomology can be well respected by the spectral sequence.]]

[[Also mention Adams indexing. Since this will be our main focus, once we get the basic formalism for spectral sequences in place, we will return to this in more detail later.]]

5. Example: The Serre spectral sequence

5.1. **Serre fibrations.** A *Serre fibration* is a map $p\colon E \to B$ with the homotopy lifting property for CW complexes (or, equivalently, for polyhedra), cf. Serre (1951). This means that for any CW complex X, map $f\colon X \to E$ and homotopy $H\colon X \times I \to B$ such that $H(x, 0) = pf(x)$, there exists a homotopy $\tilde{H}\colon X \times I \to E$ with $\tilde{H}(x, 0) = f(x)$ and $p\tilde{H} = H$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

Any fiber bundle over a paracompact base space is a Serre fibration, cf. Spanier (1981, Theorem 2.7.13). Suppose that B is a connected CW complex, and choose a base point $b_0 \in B$. Let $F = p^{-1}(b_0)$ be the fiber above that base point. The fundamental group $\pi_1(B, b_0)$ acts on the homology $H_*(F)$ of that fiber.

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\{b_0\} \arrow[r] & B
\end{tikzcd}$$

5.2. **The homological Serre spectral sequence.** The homological Serre spectral sequence for $F \to E \to B$ is a spectral sequence converging to the homology $H_*(E)$ of the total space. It has E^1-term

$$E^1_{s,t} = C_s(B; \mathscr{H}_t(F))$$

given in bidegree (s, t) by the cellular s-chains of B with coefficients in the local coefficient system $\mathscr{H}_t(F)$ associated to the action of the fundamental group on the homology of the fiber, and

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F))$$

is given by the cellular homology of B with these local coefficients. If B is simply-connected, or more generally, if the action is trivial, then this is the ordinary cellular homology of B with coefficients in the abelian group $H_t(F)$. Notice that the E^2-term, unlike the E^1-term, does not depend on the chosen CW structure on B. Hence the remaining terms of the spectral sequence are topological invariants of the Serre fibration $p\colon E \to B$. Notice also that $E^2_{s,t}$ can only be nonzero for $s \ge 0$ and $t \ge 0$, hence the same holds for every later term $E^r_{s,t}$. It follows that the Serre spectral sequence, like any other first quadrant spectral sequence, is locally eventually constant, because $d^r_{s,t} = 0$ when $s - r < 0$ and $d^r_{s+r,t-r+1} = 0$ when $t - r + 1 < 0$, so both of these differentials vanish whenever $r \ge m(s,t) = \max\{s+1, t+2\}$. The Serre spectral sequence converges strongly to the homology of the total space:

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E)\,.$$

5.3. **The cohomological Serre spectral sequence.** There is also a cohomological Serre spectral sequence, with E_1-term

$$E_1^{s,t} = C^s(B; \mathscr{H}^t(F))$$

given by the cellular s-cochains on B with coefficients in the local coefficient system $\mathscr{H}^t(F)$. The E_2-term

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F))$$

is given by the cellular cohomology with the same coefficients, and the spectral sequence converges strongly to the cohomology of the total space:

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E)\,.$$

5.4. **Killing homotopy groups.** We illustrate by an example, based on the method of "killing homotopy groups", which was used by Serre [[and others?]] to determine several of the first nontrivial homotopy groups of spheres, i.e., the homotopy groups $\pi_i(S^j)$ for varying i and j. It is quite easy to show that $\pi_i(S^j) = 0$ for $i < j$. In the case $i = j$ the Hurewicz theorem shows that $\pi_i(S^i) \cong H_i(S^i) \cong \mathbb{Z}$ for $i \ge 1$. The cases $i > j$ remain. When $j = 1$ we know that the contractible space $\mathbb{R}$ is the universal covering space of S^1, so $\pi_i(\mathbb{R}) \cong \pi_i(S^1)$ for all $i \ge 2$, hence $\pi_i(S^1) = 0$ for $i \ge 2$. The cases $j \ge 2$ are significantly harder. There is a fiber sequence

$$S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2\,,$$

where η is the complex Hopf fibration, and the associated long exact sequence of homotopy groups tells us that $\eta_*\colon \pi_i(S^3) \to \pi_i(S^2)$ is an isomorphism for $i \ge 3$. Hence the cases $j = 2$ and $j = 3$ are practically equivalent.

It turns out to be most convenient to start the analysis with the space S^3. As already mentioned, the first homotopy groups of S^3 are $\pi_i(S^3) = 0$ for $i < 3$ and $\pi_3(S^3) \cong H_3(S^3) \cong \mathbb{Z}$, by the Hurewicz theorem. To calculate $\pi_4(S^3)$, we shall construct the 3-connected cover E of S^3, i.e., a map $g\colon E \to S^3$ such that $\pi_i(E) = 0$ for $i \le 3$ and $g_*\colon \pi_i(E) \to \pi_i(S^3)$ is an isomorphism for $i > 3$, in such a way that we can calculate the homology $H_*(E)$ using a Serre spectral sequence. First we construct a map $h\colon S^3 \to K$, where $h_*\colon \pi_i(S^3) \to \pi_i(K)$ is an isomorphism for $i \le 3$ and $\pi_i(K) = 0$ for $i > 3$. The space K will then be an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, 3)$.

To construct K, start with $K^{(4)} = S^3$ and attach 5-cells to kill the non-zero classes in $\pi_4(S^3)$. Then attach 6-cells to kill the non-zero classes in π_5 of the resulting CW complex $K^{(5)}$. Inductively, suppose we have constructed a CW pair $(K^{(n)}, S^3)$, such that $\pi_i(S^3) \cong \pi_i(K^{(n)})$ for $i \le 3$ and $\pi_i(K^{(n)}) = 0$ for $3 < i < n$. Attach $(n+1)$-cells to $K^{(n)}$ to kill the non-zero classes in $\pi_n(K^{(n)})$, and call the result $K^{(n+1)}$. Continuing, we can let $K = \bigcup_n K^{(n)} = \operatorname{colim}_n K^{(n)}$, and the inclusion $h\colon S^3 \to K$ has the properties described above. To prove that this works, use the homotopy excision theorem. [[Reference in Hatcher?]]

[[Also comment on Pontryagin and Whitehead's early work using framed bordism, and its pitfalls.]]

5.5. **The 3-connected cover of S^3.** Let $g\colon E \to S^3$ be the homotopy fiber of $h\colon S^3 \to K$. By the long exact sequence in homotopy, E is the 3-connected cover of S^3. Furthermore, g is a Serre fibration. Let $f\colon F \to E$ be the homotopy fiber of $g\colon E \to S^3$.

$$F \xrightarrow{f} E \xrightarrow{g} S^3 \xrightarrow{h} K\,.$$

By a general result for such Puppe fiber sequences, we know that $F \simeq \Omega K$, so F is an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, 2)$. In other words, $F \simeq \mathbb{C}P^\infty$. Hence we have a homotopy fiber sequence

$$\mathbb{C}P^\infty \longrightarrow E \xrightarrow{g} S^3\,.$$

The associated homological Serre spectral sequence has E^2-term

$$E^2_{s,t} = H_s(S^3; H_t(\mathbb{C}P^\infty)) \cong \begin{cases} \mathbb{Z} & \text{for } s \in \{0,3\} \text{ and } t \ge 0 \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and converges strongly to $H_{s+t}(E)$. We can display the E^2-term as in Figure 1. Notice that there is

$$
\begin{array}{cccccccc}
 & \vdots & & & & & \\
t=5 & 0 & 0 & 0 & 0 & 0 & \\
t=4 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
t=3 & 0 & 0 & 0 & 0 & 0 & \\
t=2 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
t=1 & 0 & 0 & 0 & 0 & 0 & \\
t=0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \dots \\
 & s=0 & s=1 & s=2 & s=3 & s=4 &
\end{array}
$$

FIGURE 1. Serre E^2-term for $H_*(E)$

no room for nonzero d^2-differentials, since $d^2_{s,t}$ can only originate from a nonzero group $E^2_{s,t}$ if $s = 0$ or $s = 3$, and in either case the target group $E^2_{s-2,t+1}$ is zero.

Hence $d^2 = 0$ in this spectral sequence, which implies that $E^3 = E^2$. There is, however, room for d^3-differentials, which we display in the E^3-term as in Figure 2. The difficulty now is to determine these homomorphisms $d^3_{3,t}$ for $t \ge 0$ even. At this point we can already deduce that each group $H_n(E)$ is a finitely generated abelian group (of rank 0 or 1), since whatever the d^r-differentials are, only a trivial, finite cyclic or infinite cyclic group will be left at the E^∞-term in each bidegree (s, t) with $s \in \{0, 3\}$ and $t \ge 0$ even. Since there is at most one nontrivial group in each total degree of $E^\infty_{*,*}$, we can conclude that the abutment $H_*(E)$ is also either trivial, finite cyclic or infinite cyclic in each degree.

5.6. **The first differential.** By looking a bit ahead and working backwards, we can prove that the first differential, $d^3_{3,0}$, is an isomorphism. This is because at the E^4-term the only possibly nonzero groups in total degree $s + t \le 3$ will be

$$E^4_{0,2} = \mathrm{cok}(d^2_{3,0}) \qquad \text{and} \qquad E^4_{3,0} = \ker(d^2_{3,0})\,.$$

Since the spectral sequence is concentrated in the two columns $s = 0$ and $s = 3$, i.e., is zero for all other s, there is no room for any longer differentials than the d^3-differentials. Hence $d^r = 0$ for $r \ge 4$, and $E^4 = E^\infty$. So if the cokernel or kernel of $d^2_{3,0}$ is nonzero, then it will survive to the E^∞-term of the spectral sequence, in total degree 2 or 3, respectively. The spectral sequence converges to $H_*(E)$, where E by construction is 3-connected. Hence, by the Hurewicz theorem, $H_n(E) = 0$ for $0 < n \le 3$. It follows that $(F_s)_n = 0$ and $(F_s/F_{s-1})_n = 0$ for all s and all $0 < n \le 3$. Convergence of the spectral sequence thus implies that $E^\infty_{s,t} = 0$ for $0 < s+t \le 3$. In particular, $E^\infty_{0,2} = \mathrm{cok}(d^2_{3,0}) = 0$ and $E^4_{3,0} = \ker(d^2_{3,0}) = 0$. This is equivalent to the assertion that $d^2_{3,0}$ is an isomorphism.

5.7. **The cohomological version.** How do we proceed from here to determine the second differential, $d^3_{3,2}$? It is not clear how to do this using only the additive structure in homology. Instead, we will pass to cohomology, and use the multiplicative structure in the cohomological Serre spectral sequence, related to the cup product in cohomology, to calculate all the later cohomological d_3-differentials from the first d_3-differential, dual to the homological d^3-differential that we just identified.

Let us use the notations $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$ and $H^*(S^3) = \mathbb{Z}[z]/(z^2)$, with algebra generators y in degree $|y| = 2$, and z in degree $|z| = 3$. The cohomological Serre spectral sequence for $\mathbb{C}P^\infty \to E \to S^3$

$$
\begin{tikzcd}
 & \vdots & & & & & \\
t=5 & 0 & 0 & 0 & 0 & 0 & \\
t=4 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "d^3_{3,4}"'] & 0 & \\
t=3 & 0 & 0 & 0 & 0 & 0 & \\
t=2 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "d^3_{3,2}"'] & 0 & \\
t=1 & 0 & 0 & 0 & 0 & 0 & \\
t=0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "d^3_{3,0}"'] & 0 & \cdots \\
 & s=0 & s=1 & s=2 & s=3 & s=4 &
\end{tikzcd}
$$

FIGURE 2. Serre E^3-term for $H_*(E)$

has E_2-term

$$E_2^{s,t} = H^s(S^3; H^t(\mathbb{C}P^\infty)) = \begin{cases} \mathbb{Z} & \text{for } s \in \{0,3\} \text{ and } t \geq 0 \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and converges strongly to $H^{s+t}(E)$. So far, this looks just like in homology. However, the cohomological Serre spectral sequence has the additional property of being an *algebra spectral sequence*, meaning that each E_r-term is a graded algebra, and each d_r-differential is a derivation with respect to this algebra structure. This means that d_r satisfies a Leibniz rule, of the form

$$d_r(ab) = d_r(a)b + (-1)^{|a|} a d_r(b)\,,$$

for classes $a, b \in E_r$ and their (cup) product $ab = a \cup b$. We shall return to the precise definition and interpretation of multiplicative structures in spectral sequences, later.

For now, we just observe that the algebra structure of the cohomological Serre spectral sequence E_2-term can be written as

$$E_2^{*,*} = H^*(S^3; H^*(\mathbb{C}P^\infty)) \cong \mathbb{Z}[z]/(z^2) \otimes \mathbb{Z}[y]\,.$$

Since the spectral sequence is concentrated in the columns $s = 0$ and $s = 3$, there is only room for d_3-differentials, so $E_2 = E_3$ and $E_4 = E_\infty$. We now display the cohomological E_3-term and the d_3-differentials, in Figure 3. Note that the direction of the differentials is reversed, compared to the homological case. Note also that we can now give names to the additive generators in the various bidegrees, as products of powers of y and z.

We can now argue as before, that $\ker(d_3^{0,2}) = E_4^{0,2} = E_\infty^{0,2}$ and $\mathrm{cok}(d_3^{0,2}) = E_4^{3,0} = E_\infty^{3,0}$ must contribute to $H^2(E)$ and $H^3(E)$, respectively, and since the latter two groups are trivial, hence so is the kernel and cokernel of $d_3^{0,2}$. Alternatively, one can appeal to a Kronecker pairing of spectral sequences, evaluating the cohomological spectral sequence on the homological one, to deduce that

$$d_3^{0,2} \colon H^0(S^3; H^2(\mathbb{C}P^\infty)) \longrightarrow H^3(S^3; H^0(\mathbb{C}P^\infty))$$

is dual to

$$d^3_{3,0} \colon H_3(S^3; H_0(\mathbb{C}P^\infty)) \longrightarrow H_0(S^3; H_2(\mathbb{C}P^\infty))\,,$$

$$
\begin{tikzcd}
 & \vdots \arrow[rrrdd, "d_3^{0,6}"] & & & & & \\
t=5 & 0 & 0 & 0 & 0 & 0 & \\
t=4 & \mathbb{Z}\{y^2\} \arrow[rrrdd, "d_3^{0,4}"] & 0 & 0 & \mathbb{Z}\{zy^2\} & 0 & \\
t=3 & 0 & 0 & 0 & 0 & 0 & \\
t=2 & \mathbb{Z}\{y\} \arrow[rrrdd, "d_3^{0,2}"] & 0 & 0 & \mathbb{Z}\{zy\} & 0 & \\
t=1 & 0 & 0 & 0 & 0 & 0 & \\
t=0 & \mathbb{Z}\{1\} & 0 & 0 & \mathbb{Z}\{z\} & 0 & \cdots \\
 & s=0 & s=1 & s=2 & s=3 & s=4 &
\end{tikzcd}
$$

FIGURE 3. Serre E_3-term for $H^*(E)$

in the sense of the universal coefficient theorem. This leads to the same conclusion. Hence we find that

$$d_3(y) = z\,,$$

up to a possible sign. If necessary we replace y or z by its negative, to make sure that the formula above holds.

5.8. **The remaining differentials.** At this point, the algebra structure comes to our aid. The Leibniz rule for d_3 applied with $a = y$ and $b = y$ asserts that

$$d_3(y^2) = d_3(y)y + yd_3(y) = zy + yz = 2zy\,.$$

By induction, it follows that

$$d_3(y^k) = kzy^{k-1}$$

for all $k \geq 1$. Hence the homomorphism

$$d_3^{0,2k} \colon \mathbb{Z}\{y^k\} \longrightarrow \mathbb{Z}\{zy^{k-1}\}$$

is given by multiplication by k, with respect to this basis. This lets us calculate the $E_4 = E_\infty$-term

$$E_\infty^{s,t} \cong \begin{cases} \mathbb{Z} & \text{for } s = t = 0, \\ \mathbb{Z}/k & \text{for } s = 3 \text{ and } t = 2k-2, \\ 0 & \text{otherwise.} \end{cases}$$

It appears as in Figure 4. The group $\mathbb{Z}/1\{z\}$ in bidegree $(3, 0)$ is of course trivial. This leads to the conclusion

$$H^n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k & \text{for } n = 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, in total degree $n = 5$, we have a finite descending filtration

$$H^5(E) = (F^0)^5 \supset (F^1)^5 \supset (F^2)^5 \supset (F^3)^5 \supset (F^4)^5 \supset (F^5)^5 \supset (F^6)^5 = 0\,,$$

	$\vdots$					
$t=5$	0	0	0	0	0	
$t=4$	0	0	0	$\mathbb{Z}/3\{zy^2\}$	0	
$t=3$	0	0	0	0	0	
$t=2$	0	0	0	$\mathbb{Z}/2\{zy\}$	0	
$t=1$	0	0	0	0	0	
$t=0$	$\mathbb{Z}\{1\}$	0	0	0	0	$\dots$
	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	

FIGURE 4. Serre E_∞-term for $H^*(E)$

with filtration quotients

$$(F^s/F^{s+1})^5 = (F^s)^5/(F^{s+1})^5 \cong E_\infty^{s,5-s}$$

for all $0 \le s \le 5$. Hence

$$H^5(E) = (F^0)^5 = (F^1)^5 = (F^2)^5 = (F^3)^5 \qquad \text{and} \qquad (F^4)^5 = (F^5)^5 = (F^6)^5 = 0$$

while $(F^3)^5/(F^4)^5 \cong E_\infty^{3,2} = \mathbb{Z}/2\{zy\}$. Thus $H^5(E) \cong \mathbb{Z}/2$. In this case there were no (non-obvious) extension questions, since there was at most one nontrivial group in each total degree of the E_∞-term.

5.9. **Conclusions about homotopy groups.** We observed from the homological Serre spectral sequence that $H_*(E)$ is of *finite type*, i.e., a finitely generated abelian group in each degree, so the universal coefficient theorem allows us to determine these homology groups from the corresponding cohomology groups. We obtain

$$H_n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n=0, \\ \mathbb{Z}/k & \text{for } n=2k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the first nontrivial homology group of the 3-connected cover E of S^3 is $H_4(E) \cong \mathbb{Z}/2$. By the Hurewicz theorem, $\pi_4(E) \cong H_4(E)$, and by the defining property of a 3-connected cover, $\pi_4(S^3) \cong \pi_4(E)$.

Theorem 5.1. $\pi_4(S^3) \cong \mathbb{Z}/2$.

By a refinement of these methods, it is possible to concentrate on a prime p, such as 2, 3 or 5, and to calculate the p-localized homology and homotopy groups of all the spaces involved. For instance, we write $H_n(E)_{(p)}$ for the localization of $H_n(E)$ at p, which means the result of making multiplication by each prime other than p invertible in $H_n(E)$. More explicitly,

$$H_n(E)_{(p)} \cong H_n(E) \otimes \mathbb{Z}_{(p)}$$

where $\mathbb{Z}_{(p)}$ is the ring of p-local integers, i.e., the ring of rational numbers a/b where p does not divide b. We find that $(\mathbb{Z}/k)_{(p)} = \mathbb{Z}/(k,p)$ for $k \ge 1$, where (k,p) denotes the greatest common divisor of k and

p. This equals the p-Sylow subgroup of $\mathbb{Z}/k$, which is only nontrivial if p divides k. Hence

$$H_n(E)_{(p)} \cong \begin{cases} \mathbb{Z}_{(p)} & \text{for } n = 0, \\ \mathbb{Z}/(k,p) & \text{for } n = 2k, \\ 0 & \text{otherwise,} \end{cases}$$

and the first nontrivial p-local homology group of E is $H_{2p}(E)_{(p)} \cong \mathbb{Z}/p$. By the p-local Hurewicz theorem, $\pi_{2p}(E)_{(p)} \cong H_{2p}(E)_{(p)}$, and by the defining property of E, $\pi_{2p}(S^3)_{(p)} \cong \pi_{2p}(E)_{(p)}$.

Theorem 5.2. *$\pi_i(S^3)_{(p)} = 0$ for $3 < i < 2p$, and $\pi_{2p}(S^3)_{(p)} \cong \mathbb{Z}/p$.*

The formalism for working with p-local homology and homotopy groups is a special case of a more general theory of localizations. Its first incarnation, which suffices for the computation above, is known as the theory of "Serre classes", cf. Spanier (1981, Section 9.6). [[References for later work: Sullivan, Bousfield–Kan (1972).]]

5.10. **Stable homotopy groups.** There are suspension homomorphisms

$$\pi_i(S^j) \xrightarrow{\Sigma} \pi_{i+1}(S^{j+1})$$

taking the homotopy class of a map $\alpha\colon S^i \to S^j$ to the homotopy class of its suspension, $\Sigma\alpha\colon S^{i+1} = \Sigma S^i \to \Sigma S^j = S^{j+1}$. The homomorphism Σ is often denoted E, for the German "Einhängung". By Freudenthal's suspension theorem, Σ is an isomorphism if $i \le 2j-2$, and it is surjective if $i = 2j-1$. Iterating, and passing to the colimit, we come to the stable homotopy groups of spheres, also known as the *stable stems*:

$$\pi_n^S = \operatorname*{colim}_j \pi_{j+n}(S^j)\,.$$

By Freudenthal's theorem, the colimit system consists of isomorphisms for $j \ge n+2$, so we have isomorphisms $\pi_{j+n}(S^j) \cong \pi_n^S$ for all $j \ge n+2$, and a surjection $\Sigma\colon \pi_{2n+1}(S^{n+1}) \to \pi_n^S$.

In particular, $\pi_3(S^2) \cong \mathbb{Z}\{\eta\}$ surjects onto $\pi_4(S^3) \cong \pi_1^S$, so the first stable stem $\pi_1^S \cong \mathbb{Z}/2$ is generated by (the suspensions of) the Hopf map η. The Freudenthal theorem does not suffice to prove that $\pi_{2p}(S^3) \to \pi_{2p-3}^S$ becomes an isomorphism after p-localization, but this is true:

Theorem 5.3. *$(\pi_n^S)_{(p)} = 0$ for $0 < n < 2p-3$ and $(\pi_{2p-3}^S)_{(p)} \cong \mathbb{Z}/p$.*

For each odd prime p, the generator of $(\pi_{2p-3}^S)_{(p)}$ given by the suspensions of the generator of $\pi_{2p}(S^3)_{(p)}$ is usually denoted α_1. It is the first class in the first of the so-called *Greek letter families* in the stable homotopy groups of spheres. [[Reference to Ravenel.]]

6. Example: The Adams spectral sequence

6.1. **Eilenberg–Mac Lane spectra and the Steenrod algebra.** Let X and Y be spectra, i.e., objects in one of the categories modeling stable homotopy theory. The Adams spectral sequence is a tool for analyzing the homotopy classes $[X,Y]_n$ of spectrum maps $\Sigma^n X \to Y$, for all integers n, starting with the mod p cohomology groups $H^*(X;\mathbb{F}_p)$ and $H^*(Y;\mathbb{F}_p)$ as modules over the mod p Steenrod algebra $\mathscr{A}$. For instance, if $X = Y = S$ are both equal to the sphere spectrum, with k-th space S^k, then the group

$$\pi_n(S) = [S,S]_n$$

equals the n-th stable stem π_n^S.

Let $H = H\mathbb{F}_p$ denote the mod p Eilenberg–Mac Lane spectrum, representing mod p cohomology. There is a natural isomorphism

$$[X,H]_{-*} \cong H^*(X;\mathbb{F}_p)$$

for all spectra X. By the Yoneda lemma, the natural graded transformations

$$H^*(X;\mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p)\,,$$

i.e., the mod p cohomology operations for spectra, are in one-to-one correspondence with the elements of

$$\mathscr{A} = [H,H]_{-*} \cong H^*(H;\mathbb{F}_p)\,.$$

This graded endomorphism algebra of the spectrum H is the mod p Steenrod algebra. It is concentrated in non-negative cohomological degrees, i.e., is only nonzero for $* \ge 0$ in the notation above.

6.2. **The d-invariant.** Each spectrum map $f\colon X \to Y$ induces a homomorphism $f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$, which by the discussion above is a homomorphism of $\mathscr{A}$-modules. Hence the rule that takes the homotopy class $[f]$ to the $\mathscr{A}$-module homomorphism f^* is a homomorphism

$$\begin{aligned} d\colon [X,Y]_* &\longrightarrow \operatorname{Hom}^*_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \\ [f] &\longmapsto f^* \,. \end{aligned}$$

This is sometimes called the *d-invariant*, by analogy with the name "degree" for the integer $\deg(f)$ such that $f_*[M] = \deg(f)[N]$, where $f\colon M \to N$ is a map of oriented closed n-manifolds with fundamental classes $[M] \in H_n(M)$ and $[N] \in H_n(N)$.

The (cohomological) grading of $\operatorname{Hom}_{\mathscr{A}}$-groups works as follows: An element $[f] \in [X,Y]_t$ is the homotopy class of a spectrum map $f\colon \Sigma^t X \to Y$, which induces a homomorphism

$$f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(\Sigma^t X;\mathbb{F}_p) \cong H^{*-t}(X;\mathbb{F}_p)\,.$$

By definition this is an $\mathscr{A}$-module homomorphism of degree t, i.e., an element of

$$\operatorname{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,$$

taking elements in $H^i(Y;\mathbb{F}_p)$ to elements of $H^{i-t}(X;\mathbb{F}_p)$, for all integers i.

6.3. **Wedge sums of suspended Eilenberg–Mac Lane spectra.** In the special case $Y = H$ we have $H^*(Y;\mathbb{F}_p) = \mathscr{A}$, and the d-invariant

$$d\colon [X,H]_t \longrightarrow \operatorname{Hom}^t_{\mathscr{A}}(\mathscr{A}, H^*(X;\mathbb{F}_p))$$

is an isomorphism for each t, since both sides are naturally isomorphic to $H^{-t}(X;\mathbb{F}_p)$. More generally, suppose that

$$Y \simeq \bigvee_u \Sigma^{n_u} H$$

is a wedge sum of suspensions of mod p Eilenberg–Mac Lane spectra. If $\pi_*(Y) = \bigoplus_u \Sigma^{n_u}\mathbb{F}_p$ is bounded below and of finite type, or equivalently, if $n_u \to \infty$ as $u \to \infty$, then the canonical map

$$\bigvee_u \Sigma^{n_u} H \longrightarrow \prod_u \Sigma^{n_u} H$$

is an equivalence. In this case the d-invariant is also an isomorphism, since

$$[X,Y]_t \cong [X, \prod_u \Sigma^{n_u} H]_t \cong \prod_u H^{n_u - t}(X;\mathbb{F}_p)$$

is naturally isomorphic to

$$\operatorname{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \cong \operatorname{Hom}^t_{\mathscr{A}}(\bigoplus_u \Sigma^{n_u}\mathscr{A}, H^*(X;\mathbb{F}_p)) \cong \prod_u \operatorname{Hom}^t_{\mathscr{A}}(\Sigma^{n_u}\mathscr{A}, H^*(X;\mathbb{F}_p))$$

for each integer t.

In general d is not an isomorphism. For instance, the Hopf map $\eta\colon S^3 \to S^2$ induces the zero homomorphism in (reduced) cohomology, but stabilizes to a nontrivial homotopy class of maps $S^1 = \Sigma S \to S$. Furthermore, the target of d is always a graded $\mathbb{F}_p$-vector space, while the source may be any graded abelian group.

6.4. **Two-stage extensions.** If Y is an extension of two wedge sums of suspended Eilenberg–Mac Lane spectra, so that there is a cofiber sequence of spectra

$$K_1 \xrightarrow{i} Y \xrightarrow{j} K_0$$

with

$$K_0 \simeq \bigvee_u \Sigma^{n_u} H \qquad \text{and} \qquad K_1 \simeq \bigvee_v \Sigma^{n_v} H\,,$$

then there are long exact sequences

$$\dots \longrightarrow [X,K_1]_* \xrightarrow{i_*} [X,Y]_* \xrightarrow{j_*} [X,K_0]_* \xrightarrow{\partial} [X,K_1]_{*-1} \to \dots$$

and

$$\dots \to H^*(K_0;\mathbb{F}_p) \xrightarrow{j^*} H^*(Y;\mathbb{F}_p) \xrightarrow{i^*} H^*(K_1;\mathbb{F}_p) \xrightarrow{\delta} H^{*+1}(K_0;\mathbb{F}_p) \to \dots\,,$$

but the complex

$$\dots \longrightarrow \mathrm{Hom}^*_{\mathscr{A}}(H^*(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{(i^*)^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\xrightarrow{(j^*)^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{\delta^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^{*-1}(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow \dots$$

is typically not exact. Here $\delta^\#$ denotes the value of the contravariant functor $\mathrm{Hom}^*_{\mathscr{A}}(-, H^*(X;\mathbb{F}_p))$ applied to the homomorphism δ, and likewise for $(i^*)^\#$ and $(j^*)^\#$.

Now suppose that j^* is surjective, which is equivalent to asking that i^* is zero. Then there is instead a short exact sequence of $\mathscr{A}$-modules

$$0 \to H^{*-1}(K_1;\mathbb{F}_p) \xrightarrow{\delta} H^*(K_0;\mathbb{F}_p) \xrightarrow{j^*} H^*(Y;\mathbb{F}_p) \to 0\,.$$

If $\pi_*(K_0)$ and $\pi_*(K_1)$ are bounded below and of finite type, as before, then the left hand and middle $\mathscr{A}$-modules are free, so there is an associated exact sequence

$$0 \to \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{(j^*)^\#} \mathrm{Hom}^t_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\xrightarrow{\delta^\#} \mathrm{Hom}^t_{\mathscr{A}}(H^{*-1}(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to 0\,.$$

Here $\mathrm{Ext}^1_{\mathscr{A}}$ denotes the first right derived functor of $\mathrm{Hom}_{\mathscr{A}}$. More generally we write $\mathrm{Ext}^{s,t}_{\mathscr{A}}$ for the internal degree t part of the s-th derived functor of $\mathrm{Hom}^*_{\mathscr{A}}$, for each $s \ge 0$. Recall that $\mathrm{Ext}^0_{\mathscr{A}} = \mathrm{Hom}_{\mathscr{A}}$. The groups

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

for $s \ge 2$ are zero for the Y that we are presently considering, due to the existence of the short exact sequence of $\mathscr{A}$-modules above.

Under the d-invariant isomorphisms associated to K_0 and K_1, the homomorphism $\partial\colon [X, K_0]_* \to [X, K_1]_{*-1}$ corresponds to the homomorphism $\delta^\#$ above:

$$\begin{array}{ccc}
[X, K_0]_t & \xrightarrow{\quad\partial\quad} & [X, K_1]_{t-1} \\
{\scriptstyle d}\downarrow{\scriptstyle\cong} & & {\scriptstyle d}\downarrow{\scriptstyle\cong} \\
\mathrm{Hom}^t_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) & \xrightarrow{\delta^\#} & \mathrm{Hom}^{t-1}_{\mathscr{A}}(H^*(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,
\end{array}$$

so there are isomorphisms

$$\ker(\partial)_t \cong \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\mathrm{cok}(\partial)_{t-1} \cong \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

for all integers t. Hence the short exact sequence

$$0 \to \mathrm{cok}(\partial)_t \longrightarrow [X, Y]_t \longrightarrow \ker(\partial)_t \to 0$$

can be rewritten as

$$0 \to \mathrm{Ext}^{1,t+1}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow [X, Y]_t \xrightarrow{d} \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to 0\,,$$

for each integer t. In particular d is surjective in these cases. The homomorphism

$$e\colon \ker(d)_t \longrightarrow \mathrm{Ext}^{1,t+1}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,$$

which in this case is an isomorphism, is often called the *e-invariant*. Here e refers to "extension", and goes well with d.

This extension can be presented as a spectral sequence with E_2-term

$$E_2^{s,t} = \begin{cases} \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) & \text{for } s = 0, \\ \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

that collapses at the $E_2 = E_\infty$-term, and which converges to a finite decreasing filtration

$$[X, Y]_t = F^0 \supset F^1 \supset F^2 = 0$$

in the sense that

$$(F^0/F^1)_t = E_\infty^{0,t} \qquad \text{and} \qquad (F^1)_t = E_\infty^{1,t+1}\,.$$

Thus for such Y the d-invariant is surjective, and $F^1 = \ker(d)$ is its kernel.

This discussion suggests that in order to get a better approximation to the graded abelian group $[X,Y]_*$, it is necessary to take the derived functors of $\mathrm{Hom}_{\mathscr{A}}$ into account.

6.5. **The mod p Adams spectral sequence.** The mod p Adams spectral sequence for X and Y has E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) .$$

If X is a finite CW spectrum, and Y is bounded below and of finite type, then it converges strongly to the p-completion

$$([X,Y]_{t-s})^\wedge_p$$

of the abelian group $[X,Y]_{t-s}$, equipped with a decreasing filtration

$$([X,Y]_{t-s})^\wedge_p = F^0 \supset F^1 \supset \dots \supset F^s \supset \dots$$

called the Adams filtration. For a finitely generated abelian group G the p-completion can be defined as the limit

$$G^\wedge_p = \lim_n G/p^n G .$$

If $G = \mathbb{Z}$, this equals the p-adic integers $\mathbb{Z}_p$. If G is finite, this is a quotient group of G that is isomorphic to the p-Sylow subgroup of G.

In the special case $X = S$ we get the E_2-term of the mod p Adams spectral sequence for Y, namely

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y;\mathbb{F}_p), \mathbb{F}_p) .$$

When Y is bounded below and of finite type it converges strongly to the p-completed homotopy groups

$$\pi_{t-s}(Y)^\wedge_p = ([S,Y]_{t-s})^\wedge_p$$

of Y. In particular, the mod p Adams spectral sequence for the sphere spectrum itself has E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) ,$$

and converges strongly to

$$(\pi_{t-s}^S)^\wedge_p = \pi_{t-s}(S)^\wedge_p = ([S,S]_{t-s})^\wedge_p ,$$

i.e., the p-completed stable stems.

By the Hurewicz theorem, $\pi_n^S = 0$ for $n < 0$ and $\pi_0^S \cong \mathbb{Z}$, via the isomorphisms $\pi_j(S^j) \cong H_j(S^j)$ for all $j \geq 1$. Using the theory of Serre classes, one can prove that each stable stem π_n^S for $n \geq 1$ is a finite abelian group. Hence it is the product of its p-Sylow subgroups, or equivalently, of the groups $(\pi_n^S)^\wedge_p$, which we can hope to calculate using the corresponding mod p Adams spectral sequence. This will be the aim of much of the remainder of these lectures.

6.6. **Endomorphism ring spectra and their modules.** Working at the spectrum level, without passing to homotopy classes of maps, we can instead consider the function spectra $F(X,Y)$, $X^H = F(X,H)$, $Y^H = F(Y,H)$ and $R = F(H,H)$, with $\pi_* F(X,Y) = [X,Y]_*$, $\pi_{-*}F(X,H) = H^*(X;\mathbb{F}_p)$, $\pi_{-*}F(Y,H) = H^*(Y;\mathbb{F}_p)$ and $\pi_{-*}F(H,H) = \mathscr{A}$. The endomorphism spectrum $R = F(H,H)$ is a ring spectrum, with product corresponding to the composition of cohomology operations. The map $F(X,Y) \to F(Y^H, X^H)$ factors through the spectrum of R-module maps, so that there is a spectrum level degree map

$$d \colon F(X,Y) \longrightarrow F_R(Y^H, X^H) .$$

This turns out to be an equivalence in a wider range of cases than that for which the group level degree map is an isomorphism, and to amount to a p-completion map of the source in an even wider range of cases. Passing to homotopy groups, there is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

converging [[conditionally? strongly?]] to π_{t-s} of the target of the spectrum level degree map. [[This is the Adams spectral sequence converging to $\pi_{t-s}F(X,Y)^\wedge_p$.]]

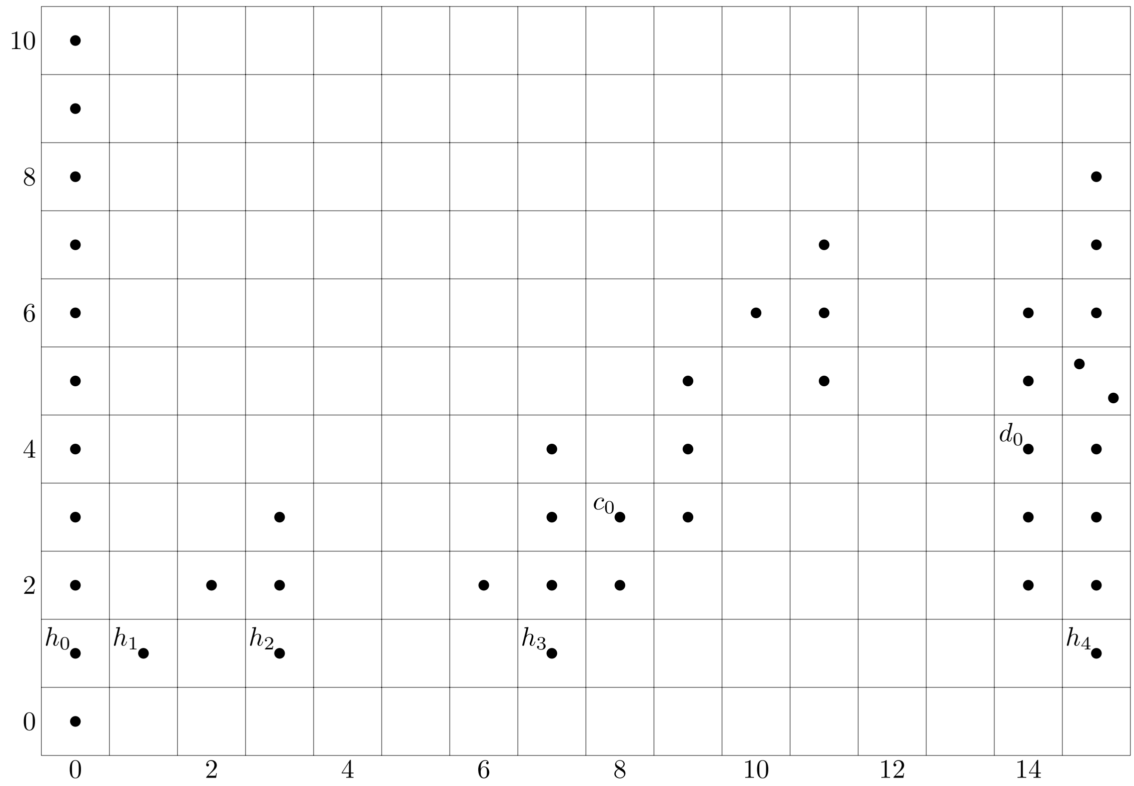


FIGURE 5. Adams E_2-term for $t-s \le 15$

6.7. **The mod 2 Adams spectral sequence for the sphere.** Let us look more closely at the mod 2 Adams spectral sequence for the sphere:

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s}(S)_2^\wedge \,.$$

The E_2-term is an $\mathbb{F}_2$-vector space in each bidegree, concentrated in the region $0 \le s \le t$, or equivalently, in the region $t-s \ge 0$ and $s \ge 0$. We display the part where $0 \le t-s \le 15$ and $0 \le s \le 10$ of this E_2-term in Figure 5, using the Adams indexing with the topological degree $t-s$ on the horizontal axis and the filtration degree s on the vertical axis. This picture is usually called an *Adams chart*, and we refer to $(t-s, s)$ as the *Adams bidegree*.

The dots in a square corresponding to a given $(t-s, s)$-bidegree represent the elements of a basis for the $\mathbb{F}_2$-vector space in that bidegree. Empty bidegrees correspond to 0-dimensional vector spaces, bidegrees with a single dot correspond to 1-dimensional vector spaces, and so on. In this range of bidegrees the only 2-dimensional vector space is $E_2^{5,20}$, located at $(t-s, s) = (15, 5)$. Some of the generators are labeled with their standard names. We will explain later how these $\mathrm{Ext}_{\mathscr{A}}$-groups can be calculated, at least in a limited range.

The chart continues upward and to the right. In the upward direction, only the groups in the zeroth column are nonzero, while the groups in columns $1 \le t-s \le 15$ for $s \ge 9$ are all zero. There is much more structure present in this E_2-term, and in the subsequent terms of the spectral sequence, than that of a bigraded $\mathbb{F}_2$-vector space, but let us introduce these structures one by one.

The d_2-differentials in the Adams spectral sequence are homomorphisms

$$d_2^{s,t} \colon E_2^{s,t} \longrightarrow E_2^{s+2,t+1} \,,$$

mapping bidegree $(t-s, s)$ to bidegree $(t-s-1, s+2)$, i.e., one unit to the left and two units upwards in the $(t-s, s)$-plane. Looking at the chart, the d_2-differentials that could possibly be nonzero are those originating in bidegrees $(t-s, s) = (1, 1)$, $(8, 2)$, $(15, 1)$, $(15, 2)$, $(15, 3)$ and $(15, 4)$. See Figure 6.

More generally, the d_r-differentials in the Adams spectral sequence are homomorphisms

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1} \,,$$

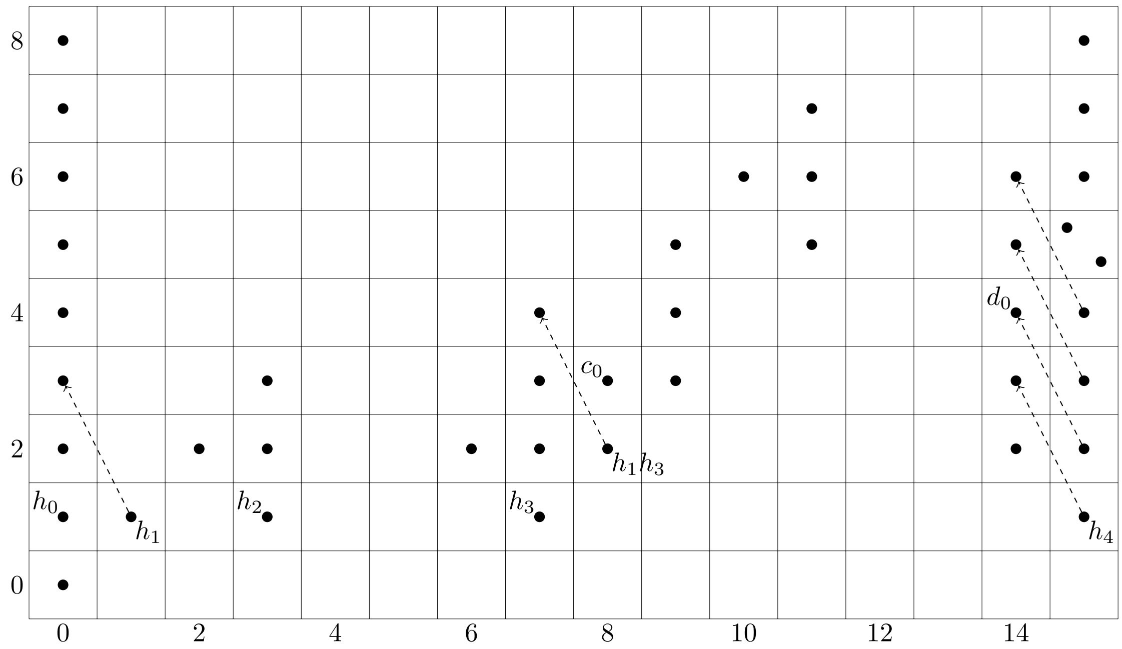


FIGURE 6. Possible d_2-differentials

mapping bidegree $(t-s,s)$ to bidegree $(t-s-1,s+r)$, i.e., one unit to the left and r units upwards in the $(t-s,s)$-plane. For $r \ge 3$, the only possible d_r-differentials are those originating in bidegrees $(t-s,s) = (1,1)$, $(15,1)$, $(15,2)$ and $(15,3)$.

The Adams E_∞-term is a subquotient of the E_2-term, hence is zero in all the bidegrees where $E_2^{s,t} = 0$. By strong convergence, there is a descending complete Hausdorff filtration

$$\pi_n(S)^\wedge_2 = (F^0)_n \supset (F^1)_n \supset \cdots \supset (F^s)_n \supset \dots ,$$

the *Adams filtration*, and isomorphisms

$$(F^s/F^{s+1})_{t-s} \cong E_\infty^{s,t}$$

for all s and t. For each integer n, the groups in the E_∞-term that contribute as filtration quotients to the filtration of $\pi_n(S)^\wedge_2$ are the groups $E_\infty^{s,t}$ with $t-s=n$, i.e., the groups in the n-th column. Thus the Adams indexing has the feature that all of the terms that contribute to the same topological degree are aligned in the same column in the $(t-s,s)$-plane.

In fact the d_r-differentials originating on the class $h_1 \in E_2^{1,2}$ in Adams bidegree $(1,1)$ are all zero. In other words, h_1 is an infinite cycle, and *survives* to the E_∞-term. To see this, we might start from our knowledge that $\pi_1(S) = \pi_1^S \cong \mathbb{Z}/2$. If some d_r-differential on h_1 were nonzero, then h_1 would not survive to $E_{r+1}^{1,2}$, so $E_{r+1}^{1,2} = 0$ and $E_\infty^{1,2} = 0$. Hence every group $E_\infty^{s,s+1}$ for $s \ge 0$ would be zero, and the filtration of $\pi_1(S)^\wedge_2$ would have to be constant:

$$\pi_1(S)^\wedge_2 = (F^0)_1 = (F^1)_1 = \cdots = (F^s)_1 = \dots .$$

Since the filtration is Hausdorff, $\lim_s (F^s)_1 = 0$, so this implies that each $(F^s)_1 = 0$. In particular $\pi_1(S)^\wedge_2$ would be zero, contradicting our earlier calculation.

Theorem 6.1. *$d_r(h_1) = 0$ for all $r \ge 2$.*

Hence the Adams E_2-term equals the Adams E_∞-term in topological degrees $t-s \le 6$. In degree 0, the Adams filtration

$$\pi_0(S)^\wedge_2 = (F^0)_0 \supset (F^1)_0 \supset \cdots \supset (F^s)_0 \supset \dots$$

has $(F^s/F^{s+1})_0 \cong E_\infty^{s,s} \cong \mathbb{Z}/2$, so $(F^{s+1})_0$ has index 2 in $(F^s)_0$, for each $s \ge 0$. Hence this complete Hausdorff filtration is equal to the 2-adic filtration

$$\mathbb{Z}^\wedge_2 = \mathbb{Z}_2 \supset 2\mathbb{Z}_2 \supset \cdots \supset 2^s\mathbb{Z}_2 \supset \dots$$

of the 2-adic integers. Note that $\mathbb{Z}_2/2^s\mathbb{Z}_2 \cong \mathbb{Z}/2^s$, and $\mathbb{Z}_2 \cong \lim_s \mathbb{Z}/2^s$.

In degree 1, the Adams filtration has

$$\pi_1(S)^\wedge_2 = (F^0)_1 = (F^1)_1 \qquad \text{and} \qquad (F^2)_1 = \dots = (F^s)_1 = 0$$

for $s \ge 2$, so

$$\pi_1(S)^\wedge_2 \cong (F^1/F^2)_1 \cong E_\infty^{1,2} \cong \mathbb{Z}/2\{h_1\}\,.$$

The generator of $\pi_1(S) \cong \mathbb{Z}/2$, represented by the Hopf map η, has d-invariant $d(\eta) = 0$, hence lifts to Adams filtration 1 and is represented in $(F^1/F^2)_1 \cong E_\infty^{1,2}$ by the infinite cycle h_1 in $E_2^{1,2}$.

We can now go beyond what we already knew. In degree 2, the only nonzero class in the E_∞-term, i.e., in the groups $E_\infty^{s,s+2}$ for $s \ge 0$, is the generator of $E_2^{2,4} = E_\infty^{2,4}$ in Adams bidegree $(2,2)$. Foreshadowing the existence of a multiplicative structure on the Adams E_2-term, this generator is usually called h_1^2. The Adams filtration has

$$\pi_2(S)^\wedge_2 = (F^0)_2 = (F^1)_2 = (F^2)_2 \qquad \text{and} \qquad (F^3)_2 = \dots = (F^s)_2 = 0$$

for $s \ge 3$, so

$$\pi_2(S)^\wedge_2 \cong (F^2/F^3)_1 \cong E_\infty^{2,4} \cong \mathbb{Z}/2\{h_1^2\}\,.$$

The generator of $\pi_2(S) \cong \mathbb{Z}/2$, represented by the square η^2 of the Hopf map, lifts to Adams filtration 2 and is represented in $(F^2/F^3)_2 \cong E_\infty^{2,4}$ by the infinite cycle h_1^2.

In degree 3, there are three generators of the $E_2 = E_\infty$-term, namely h_2 generating $E_2^{1,4}$, a class we call h_0h_2 generating $E_2^{2,5}$, and a class we call $h_0^2h_2$ generating $E_2^{3,6}$. The Adams filtration has

$$\begin{aligned} \pi_3(S)^\wedge_2 &= (F^0)_3 = (F^1)_3 \\ (F^1/F^2)_3 &\cong \mathbb{Z}/2\{h_2\} \\ (F^2/F^3)_3 &\cong \mathbb{Z}/2\{h_0h_2\} \\ (F^3/F^4)_3 &\cong \mathbb{Z}/2\{h_0^2h_2\} \\ (F^4)_3 &= \dots = (F^s)_3 = 0 \end{aligned}$$

for $s \ge 4$. This proves that $(F^3)_3 \cong \mathbb{Z}/2\{h_0^2h_2\}$, but without further information we have two ambiguous extension problems

$$0 \to (F^3)_3 \longrightarrow (F^2)_3 \longrightarrow \mathbb{Z}/2\{h_0h_2\} \to 0 \tag{1}$$

and

$$0 \to (F^2)_3 \longrightarrow (F^1)_3 \longrightarrow \mathbb{Z}/2\{h_2\} \to 0\,. \tag{2}$$

It is clear that $(F^2)_3$ is an abelian group of order four, and that $\pi_3(S)^\wedge_2 = (F^1)_3$ is an abelian group of order eight. In fact both of these extensions are nontrivial, and $\pi_3(S)^\wedge_2$ is cyclic of order eight, but we will need to refer to the multiplicative structure in the spectral sequence to deduce this. [[Relate h_2 to the quaternionic Hopf fibration ν?]]

In degrees 4 and 5 the $E_2 = E_\infty$-term only contains trivial groups, so $\pi_4(S)^\wedge_2 = 0$ and $\pi_5(S)^\wedge_2$ are both trivial.

In degree 6, the only nonzero class in the E_∞-term is the generator of $E_2^{2,8}$ in Adams bidegree $(2,6)$, which is usually called h_2^2. The Adams filtration has

$$\pi_6(S)^\wedge_2 = (F^0)_6 = (F^1)_6 = (F^2)_6 \qquad \text{and} \qquad (F^3)_6 = \dots = (F^s)_6 = 0$$

for $s \ge 3$, so

$$\pi_6(S)^\wedge_2 \cong (F^2/F^3)_6 \cong E_\infty^{2,8} \cong \mathbb{Z}/2\{h_2^2\}\,.$$

The generator of $\pi_6(S) \cong \mathbb{Z}/2$, represented by the square lifts to Adams filtration 2 and is represented in $(F^2/F^3)_6 \cong E_\infty^{2,8}$ by the infinite cycle h_2^2.

In degree 7, we can see that $\pi_7(S)^\wedge_2$ has order $2^3 = 8$ or $2^4 = 16$, but in order to decide between these two cases, we need to determine the possibly nonzero differential $d_2^{2,10} \colon E_2^{2,10} \to E_2^{4,11}$.

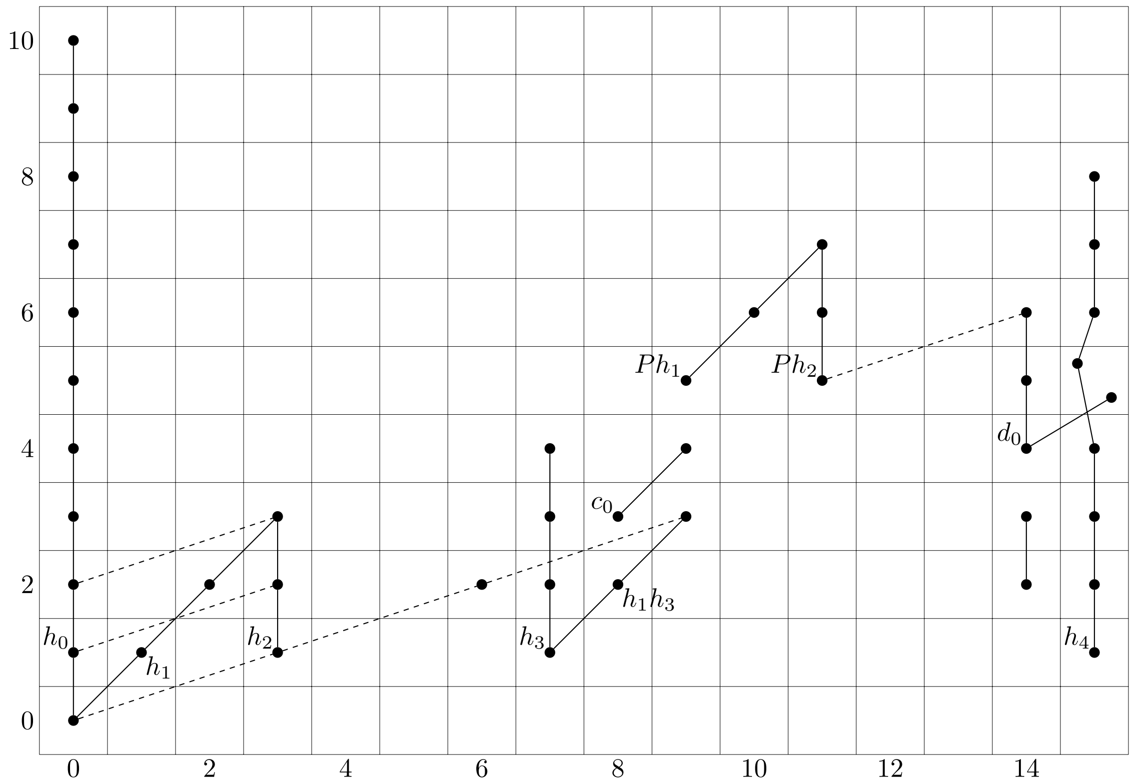


FIGURE 7. Adams E_2-term, with h_i-multiplications

6.8. **Multiplicative structure.** The sphere spectrum S is a homotopy commutative ring spectrum, or in fact a "homotopy everything ring spectrum", more technically known as an E_∞ ring spectrum, or as a commutative structured ring spectrum. This implies that $\pi_*(S)$ is a graded commutative ring, with the pairing

$$\wedge\colon \pi_m(S) \otimes \pi_n(S) \longrightarrow \pi_{m+n}(S)$$

mapping $[f] \otimes [g]$ to $[f \wedge g]$, where $f\colon S^m \to S$ and $g\colon S^n \to S$ are spectrum maps, with smash product $f \wedge g\colon S^{m+n} \cong S^m \wedge S^n \to S \wedge S = S$.

This graded commutative ring structure is reflected in the Adams spectral sequence. There is a Yoneda pairing

$$\circ\colon \mathrm{Ext}_{\mathscr{A}}^{s_1,t_1}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Ext}_{\mathscr{A}}^{s_2,t_2}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s_1+s_2,t_1+t_2}(\mathbb{F}_p, \mathbb{F}_p)$$

making the Adams E_2-term a graded commutative $\mathbb{F}_p$-algebra, and in fact the Adams spectral sequence is an algebra spectral sequence, in the sense that each E_r-term is a graded (commutative) algebra, each d_r-differential is a derivation, and these multiplicative structures are compatible under the isomorphism $E_{r+1} \cong H(E_r, d_r)$. Furthermore, the convergence of the Adams spectral sequence is compatible with the multiplicative structure, in the sense that the Adams filtration $\{F^s\}_s$ of $\pi_*(S)^\wedge_p$ is such that the smash pairing takes $F^{s_1} \otimes F^{s_2}$ into $F^{s_1+s_2}$, and the induced pairing

$$\wedge\colon F^{s_1}/F^{s_1+1} \otimes F^{s_2}/F^{s_2+1} \longrightarrow F^{s_1+s_2}/F^{s_1+s_2+1}$$

agrees with the pairing

$$\circ\colon E_\infty^{s_1,*} \otimes E_\infty^{s_2,*} \longrightarrow E_\infty^{s_1+s_2,*}$$

under the isomorphisms $F^s/F^{s+1} \cong E_\infty^s$.

The class h_0 generating $E_2^{1,1} = E_\infty^{1,1}$, in Adams bidegree $(t-s, s) = (0, 1)$, represents 2 times the generator ι in $\pi_0(S)^\wedge_2 \cong \mathbb{Z}_2$. Thus 2ι has Adams filtration 1. If an infinite cycle $x \in E_\infty^{s,t}$ represents a class $[f] \in \pi_{t-s}(S)^\wedge_2$, in Adams filtration s, then the product $h_0 \circ x = h_0 x \in E_\infty^{s+1,t+1}$ represents the product $2\iota \wedge [f] = 2[f] \in \pi_{t-s}(S)^\wedge_2$, in Adams filtration $s+1$, modulo classes in Adams filtration $s+2$ (or greater).

We can use this to determine much of the additive structure of the groups $\pi_n(S)^\wedge_2$ in this range. In Figure 7, a vertical line of length 1, from a class x in Adams bidegree $(t-s, s)$ to a class y in Adams bidegree $(t-s, s+1)$, indicates that $h_0 \circ x = y$, i.e., the line connects x to $h_0 x$. We say that these vertical lines show the h_0-multiplications. If $h_0 x = 0$, no line is drawn.

In the same way, the class h_1 generating $E_2^{1,2} = E_\infty^{1,2}$, in Adams bidegree $(t-s, s) = (1, 1)$, represents the generator η of $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2$. If $x \in E_\infty^{s,t}$ represents a class $[f] \in \pi_{t-s}(S)^\wedge_2$ in filtration s, then $h_1 x$ represents $\eta[f] \in \pi_{t-s+1}(S)^\wedge_2$ in filtration $s+1$, modulo Adams filtration $s+2$. This is indicated in Figure 7 by a line of slope 1, from x in Adams bidegree $(t-s, s)$ to $h_1 x$ in Adams bidegree $(t-s+1, s+1)$. If $h_1 x = 0$, no line is drawn.

The dashed lines of slope 1/3 correspond to multiplications by h_2, the class generating $E_2^{1,4} = E_\infty^{1,4}$. [[Relate to ν?]] We could add lines of slope 1/7 corresponding to multiplications by h_3, the class generating $E_2^{1,8} = E_\infty^{1,8}$, but these tend to clutter the diagram too much. [[Relate to σ?]]

Using the multiplicative structure, we can now deduce that $d_r^{s,t} = 0$ for all $r \geq 2$ and $t - s \leq 13$, so that $E_2^{s,t} = E_\infty^{s,t}$ for all $t - s \leq 13$.

It is clear that $d_r(h_0) = 0$ for all $r \geq 2$, since these differentials land in trivial groups. The product $h_0 h_1 = 0$ vanishes for the same reason. Hence if h_1 survives to the E_r-term, we have

$$0 = d_r(0) = d_r(h_0 h_1) = d_r(h_0) h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$$

by the Leibniz rule. But $d_r(h_1)$ lies in the bidegree $(0, r+1)$ generated by h_0^{r+1}, and multiplication by h_0 acts injectively on this bidegree. Hence $d_r(h_1) = 0$, also for all $r \geq 2$.

We can also use the multiplicative structure to deduce that $d_2^{2,10} = 0$. The group $E_2^{2,10}$ is generated by the product $h_1 h_3$. We know that $d_2(h_3) = 0$, for bidegree reasons, so by the Leibniz rule $d_2(h_1 h_3) = d_2(h_1) h_3 + h_1 d_2(h_3) = 0$, as claimed.

6.9. **The first 13 stems.** The h_i-multiplications seen at the E_2-term, allow us to determine the group structures of $\pi_n(S)^\wedge_2$ for $0 \leq n \leq 13$.

Theorem 6.2. (1) $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2$ *generated by* η *represented by* h_1.

(2) $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2$ *generated by* η^2 *represented by* h_1^2.
(3) $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/8$ *generated by* ν *represented by* h_2. *Here* 2ν *is represented by* $h_0 h_2$, *and* $4\nu = \eta^3$ *is represented by* $h_0^2 h_2 = h_1^3$.
(4) $\pi_4(S)^\wedge_2 = 0$.
(5) $\pi_5(S)^\wedge_2 = 0$.
(6) $\pi_6(S)^\wedge_2 \cong \mathbb{Z}/2$ *generated by* ν^2 *represented by* h_2^2.
(7) $\pi_7(S)^\wedge_2 \cong \mathbb{Z}/16$ *generated by* σ *represented by* h_3. *Here* 2σ *is represented by* $h_0 h_3$, 4σ *is represented by* $h_0^2 h_3$, *and* 8σ *is represented by* $h_0^3 h_3$.
(8) $\pi_8(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *generated by* $\eta\sigma$ *and* ϵ, *represented by* $h_1 h_3$ *and* c_0, *respectively.*
(9) $\pi_9(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *generated by* $\eta^2\sigma$, $\eta\epsilon$ *and* μ, *represented by* $h_1^2 h_3$, $h_1 c_0$ *and* Ph_1, *respectively. [[Explain* Ph_1. *No hidden additive extensions.]]*
(10) $\pi_{10}(S)^\wedge_2 \cong \mathbb{Z}/2$ *generated by* $\eta\mu$ *represented by* $h_1 Ph_1$.
(11) $\pi_{11}(S)^\wedge_2 \cong \mathbb{Z}/8$ *generated by* ζ *represented by* Ph_2. *Here* 2ζ *is represented by* $h_0 Ph_2$, *and* $4\zeta = \eta^2\mu$ *is represented by* $h_0^2 Ph_2 = h_1^2 Ph_1$.
(12) $\pi_{12}(S)^\wedge_2 = 0$.
(13) $\pi_{13}(S)^\wedge_2 = 0$.

Remark 6.3. To remember the nomenclature in $\pi_*(S)^\wedge_2$, as used by Toda in [Tod62], one may note that h_1, h_2 and h_3 represent classes η, ν and σ, which are the Greek letters expressing the beginning sounds in 'ichi', 'ni' and 'san', the Japanese words for 'one', 'two' and 'three'. The identity map of S corresponds to the unit class ι.

Proof. Let $\nu \in \pi_3(S)^\wedge_2$ be a class represented by h_2 in $E_2^{1,4} = E_\infty^{1,4}$. [[We may prove later that any class in $\pi_3(S)$ of Hopf invariant 1 mod 2 has this property, for instance, the stable class $S^3 \to S$ of the quaternionic Hopf fibration $S^7 \to S^4$. The product $2\nu = 2\iota \wedge \nu$ is then represented by $h_0 h_3$, and 4ν is represented by $h_0^2 h_3$. Hence both extensions in (1) and (2) are nontrivial, with $(F^2)_3 \cong \mathbb{Z}/4$ generated by 2ν and $(F^1)_3 \cong \mathbb{Z}/8$ generated by ν.

Let $\sigma \in \pi_7(S)^\wedge_2$ be a class represented by h_3 in $E_2^{1,8} = E_\infty^{1,8}$. [[We may prove later that any class in $\pi_7(S)$ of Hopf invariant 1 mod 2 has this property, for instance, the stable class $S^7 \to S$ of the octonionic Hopf fibration $S^{15} \to S^8$. The product $2\sigma = 2\iota \wedge \sigma$ is then represented by $h_0 h_4$, 4σ is represented by $h_0^2 h_4$, and 8σ is represented by $h_0^3 h_4$. Hence $(F^4)_7 = \mathbb{Z}/2$ is generated by 8σ, $(F^3)_7 = \mathbb{Z}/4$ is generated by 4σ, $(F^2)_7 = \mathbb{Z}/8$ is generated by 2σ, and $\pi_7(S)^\wedge_2 = (F^1)_7 = \mathbb{Z}/16$ is generated by σ.

In the 8-stem, we have an extension

$$0 \to \mathbb{Z}/2\{c_0\} \longrightarrow \pi_8(S)^\wedge_2 \longrightarrow \mathbb{Z}/2\{h_1h_3\} \to 0\,.$$

The element ϵ in $\pi_8(S)^\wedge_2$ that is represented by c_0 in Adams filtration 3 is uniquely defined by this property. The product $\eta\sigma = \eta \wedge \sigma$, represented by h_1h_3 in Adams filtration 2, modulo Adams filtration 3, is also well defined, since the ambiguity in the definition of σ is given by the even multiples of σ, and $\eta \wedge 2\sigma = 0$ since $2\eta = 0$. The latter relation also implies that the extension above is split, so $\pi_8(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is a Klein four-group, not a cyclic group of order four.

In the 9-stem, we have a well-defined element $\mu \in \pi_9(S)^\wedge_2$ that is represented by the generator $Ph_1 \in E_\infty^{5,14}$. The notation refers to an operator P called the *Adams periodicity operator*, which is defined in part of the E_2-term, and which takes h_1 to Ph_1 and h_2 to Ph_2. The product classes $\eta\epsilon$ and $\eta^2\sigma$ are well defined, and are represented by h_1c_0 and $h_1^2h_3$, modulo the Adams filtration. Hence $(F^5)_7 = \mathbb{Z}/2$ is generated by μ, the extension

$$0 \to (F^5)_7 \longrightarrow (F^4)_7 \longrightarrow \mathbb{Z}/2\{\eta\epsilon\} \to 0$$

splits, and so does the extension

$$0 \to (F^4)_7 \longrightarrow \pi_9(S)^\wedge_2 \longrightarrow \mathbb{Z}/2\{\eta^2\sigma\} \to 0\,.$$

The additive extensions in the 11-stem are all nontrivial, just like in the 3-stem. The generator ζ is only defined up to an odd multiple, much like the case of ν. □

We can also deduce most of the product structure on $\pi_*(S)^\wedge_2$ in this range.

Theorem 6.4. *Multiplication by η satisfies the relations $\eta\nu = 0$, $\eta^3\sigma = 0$, $\eta^2\epsilon = 0$ (!), $\eta^3\mu = 0$, $\eta\zeta = 0$.*

Multiplication by ν satisfies the relations $\nu\sigma = 0$ (!), $\nu^3 = \eta^2\sigma + \eta\epsilon$ (!), $\nu\epsilon = 0$ (!) and $\nu\mu = 0$.

Proof. [[Why is $\nu^3 = \eta^2\sigma + \eta\epsilon$? Use $e\colon S \to j$ to deduce that $\eta^2\epsilon = 0$ and $\nu\epsilon = 0$. How about $\nu\sigma$?]] □

6.10. **The first Adams differential.** Recall that $\sigma \in \pi_7(S)^\wedge_2$ denotes a class represented by h_3 in $E_2^{1,8} = E_\infty^{1,8}$, e.g. the stable octonionic Hopf fibration. By graded commutativity of $\pi_*(S)^\wedge_2$ we know that $\sigma \wedge \sigma = -\sigma \wedge \sigma$, since σ is in an odd degree, so $2\sigma^2 = 0$ in $\pi_{14}(S)^\wedge_2$. Here σ^2 is represented by h_3^2 in $E_2^{2,16} = E_\infty^{2,16}$, so $2\sigma^2$ is represented by $h_0h_3^2$ in $E_\infty^{3,17}$, modulo Adams filtration 4. Since $2\sigma^2 = 0$, it follows that $h_0h_3^2$ must be equal to 0 at the E_∞-term. Since this product is not 0 at the E_2-term (and $d_r(h_0h_3^2) = 0$ for all $r \geq 2$ by the Leibniz rule), the only way to explain this is that $h_0h_3^2$ is a boundary, i.e., is hit by a differential. For bidegree reasons, the only possibility candidate is the d_2-differential originating at h_4 in $E_2^{1,16}$. Hence the "first" nonzero differential in the mod 2 Adams spectral sequence is

$$d_2(h_4) = h_0h_3^2\,.$$

There are in fact also nonzero d_3-differentials on h_0h_4 and $h_0^2h_4$, from Adams bidegrees $(15,2)$ and $(15,3)$, but these are harder to establish.

7. Exact couples

Following Massey (1952, 1953) and Boardman (1981 preprint, 1999), we introduce the notion of an *exact couple*, and show how to use it to construct a spectral sequence. [[First additive, then convergence, then perhaps products.]]

7.1. The spectral sequence associated to an unrolled exact couple.

Definition 7.1. An *unrolled exact couple* of homological type is a diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_{s-2} \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] \arrow[d, "j"] & A_s \arrow[r, "i"] \arrow[d, "j"] & A_{s+1} \arrow[r, "i"] \arrow[d, "j"] & \dots \\
\dots & & E_{s-1} \arrow[ul, "k"] & E_s \arrow[ul, "k"] & E_{s+1} \arrow[ul, "k"] & \dots
\end{tikzcd}$$

of graded abelian groups and homomorphisms, in which each triangle

$$\dots \to A_{s-1} \xrightarrow{i} A_s \xrightarrow{j} E_s \xrightarrow{k} A_{s-1} \to \dots$$

is a long exact sequence.

Usually i is of internal degree 0, while j and k are of internal degree 0 and -1, in one order or the other.

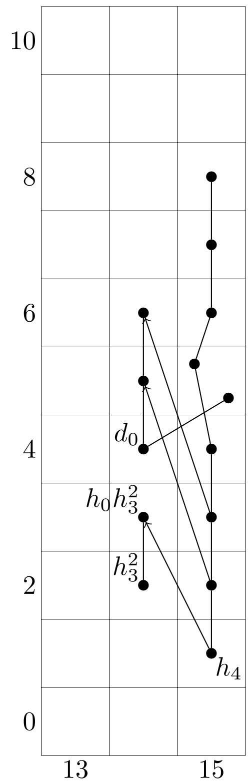


FIGURE 8. Adams E_2-term, differentials near $t-s=14$

Definition 7.2. For $r \geq 1$, let
$$Z_s^r = k^{-1}(\mathrm{im}(i^{r-1}\colon A_{s-r} \to A_{s-1}))$$
be the *r-th cycle subgroup* of E_s, and let
$$B_s^r = j(\ker(i^{r-1}\colon A_s \to A_{s+r-1}))$$
be the *r-th boundary subgroup*. We have inclusions
$$0 = B_s^1 \subset \cdots \subset B_s^r \subset B_s^{r+1} \subset \cdots \subset \mathrm{im}(j) = \ker(k) \subset \cdots \subset Z_s^{r+1} \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s$$
of graded abelian groups, for each filtration index s. Let
$$E_s^r = Z_s^r / B_s^r$$
be the *E^r-term* of the spectral sequence, and let the *d^r-differential*
$$d_s^r\colon E_s^r \longrightarrow E_{s-r}^r$$
be defined by $d_s^r([x]) = [j(y)]$, where $x \in Z_s^r$, $y \in A_{s-r}$ and $k(x) = i^{r-1}(y)$.

To see that the definition of the d^r-differential makes sense, note that for each $x \in Z_s^r$, $k(x)$ lies in the image of i^{r-1}, so there exists a $y \in A_{s-r}$ with $k(x) = i^{r-1}(y)$. If y' is another class with $k(x) = i^{r-1}(y')$, then $y' - y \in \ker(i^{r-1})$, so $j(y') - j(y)$ lies in B_s^r, so the class of $j(y)$ in E_{s-r}^r is well-defined. If $x \in B_s^r$, then $x \in \mathrm{im}(j) = \ker(k)$, so $k(x) = 0$ and we may take $y = 0$ in this case, with $[j(y)] = 0$. In general it follows that $[j(y)]$ only depends on the class $[x]$ of x in E_s^r. To see that d^r is a differential, i.e., that $d_{s-r}^r \circ d_s^r = 0$, just note that with notation as above, $kj(y) = 0$.

For $r = 1$ we identify $E_s^1 = Z_s^1/B_s^1 = E_s/0$ with E_s, and note that $d_s^1\colon E_s^1 \to E_{s-1}^1$ equals the composite $jk\colon E_s \to E_{s-1}$. Hence the E^2-term is the homology of the chain complex
$$\cdots \leftarrow E_{s-1} \xleftarrow{jk} E_s \xleftarrow{jk} E_{s+1} \leftarrow \ldots .$$

Proposition 7.3. $\ker(d_s^r) = Z_s^{r+1}/B_s^r$ *and* $\mathrm{im}(d_{s+r}^r) = B_s^{r+1}/B_s^r$, *so there is a canonical isomorphism*
$$H_s(E^r, d^r) = \frac{\ker(d_s^r)}{\mathrm{im}(d_{s+r}^r)} = \frac{Z_s^{r+1}/B_s^r}{B_s^{r+1}/B_s^r} \cong \frac{Z_s^{r+1}}{B_s^{r+1}} = E_s^{r+1}\,,$$
for each $r \geq 1$ and each s.

We call $(E^r, d^r)_r$ the spectral sequence associated to the unrolled exact couple in Definition 7.1.

Proof. If $x \in Z_s^r$ satisfies $d_s^r([x]) = 0$, then $k(x) = i^{r-1}(y)$ for and $y \in A_{s-r}$ with $j(y) \in B_{s-r}^r$. Hence $j(y) = j(y')$ for some $y' \in A_{s-r}$ with $i^{r-1}(y') = 0$. Thus $j(y - y') = 0$, so $y - y' = i(z)$ for some $z \in A_{s-r-1}$, and $k(x) = i^{r-1}(y) = i^{r-1}(y - y') = i^r(z)$ is in $\operatorname{im}(i^r)$. Hence $x \in Z_s^{r+1}$.

Conversely, if $x \in Z_s^{r+1}$, then $k(x) = i^r(z)$ for some $z \in A_{s-r-1}$, so $k(x) = i^{r-1}(y)$ with $y = i(z)$, and $j(y) = ji(z) = 0$. Thus $d_s^r([x]) = [j(y)] = 0$.

$$\begin{tikzcd}
A_{s-r-1} \arrow[r, "i"] & A_{s-r} \arrow[r, "i^{r-1}"] \arrow[d, "j"] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r, "i^{r-1}"] \arrow[d, "j"] & A_{s+r-1} \arrow[r, "i"] & A_{s+r} \arrow[d, "j"] \\
& E_{s-r} \arrow[ul, "k"] & & E_s \arrow[ul, "k"] & & E_{s+r} \arrow[ul, "k"]
\end{tikzcd}$$

If $u \in Z_s^r$ satisfies $[u] = d_s^r([v])$ for some $v \in Z_{s+r}^r$, then $[u] = [j(w)]$ for some $w \in A_{s+1}$ with $k(v) = i^{r-1}(w)$. Then $i^r(w) = ik(v) = 0$, so $j(w) \in B_s^{r+1}$. Hence $u \in B_s^{r+1}$.

Conversely, if $u \in B_s^{r+1}$, then $u = j(w)$ for some $w \in A_s$ with $i^r(w) = 0$. Then $i^{r-1}(w) \in A_{s+r-1}$ lies in $\ker(i) = \operatorname{im}(k)$, so $i^{r-1}(w) = k(v)$ for some $v \in E_{s+r}$. This relation shows that $v \in Z_{s+r}^r$, and by definition, $d_{s+r}^r([v]) = [j(w)] = [u]$, so $[u] \in \operatorname{im}(d_{s+r}^r)$. $\square$

[[Define maps of exact couples. Maybe derived exact couples. When do two maps of exact couples induce the same homomorphism of E_r-terms, for some $r \geq 2$?]]

7.2. E^∞-terms and target groups.

Definition 7.4. Let

$$Z_s^\infty = \lim_r Z_s^r = \bigcap_r Z_s^r$$

be the subgroup of *infinite cycles* in E_s, and let

$$B_s^\infty = \operatorname*{colim}_r B_s^r = \bigcup_r B_s^r$$

be the subgroup of *infinite boundaries.* Let

$$E_s^\infty = Z_s^\infty / B_s^\infty$$

be the E^∞*-term* of the spectral sequence. For later use, let

$$RE_s^\infty = \operatorname*{Rlim}_r Z_s^r$$

denote the *derived* E^∞*-term.*

To justify the notation RE_s^∞ in place of RZ_s^∞, note that if the boundary group $B_{s,t}^r$ in a fixed bidegree (s,t) is independent of r for $r \geq m = m(s,t)$, then $\operatorname{Rlim}_r Z_{s,t}^r \cong \operatorname{Rlim}_r Z_{s,t}^r / B_{s,t}^m = \operatorname{Rlim}_r E_{s,t}^r$. If the spectral sequence collapses at a finite stage, or is locally eventually constant, then $RE_s^\infty = 0$ for all s.

In particular, we have inclusions

$$B_s^\infty \subset \operatorname{im}(j) = \ker(k) \subset Z_s^\infty$$

of (graded) subgroups of E_s, and an associated short exact sequence

$$0 \to \frac{\operatorname{im}(j)}{B_s^\infty} \longrightarrow \frac{Z_s^\infty}{B_s^\infty} \longrightarrow \frac{Z_s^\infty}{\ker(k)} \to 0 \tag{3}$$

expressing the E^∞-term as an extension.

If $A_{s-r} = 0$ for r sufficiently large, then $Z_s^r = \ker(k)$ for all these r, so that $Z_s^\infty / \ker(k) = 0$ and $\operatorname{im}(j)/B_s^\infty \cong E_s^\infty$. We shall give other sufficient conditions for the vanishing of this group in the next subsection. On the other hand, if $A_{s+r-1} = 0$ for r sufficiently large, then $B_s^r = \operatorname{im}(j)$ for all these r, so that $\operatorname{im}(j)/B_s^\infty = 0$ and $E_s^\infty \cong Z_s^\infty / \ker(k)$.

Definition 7.5. Let

$$\begin{aligned}
A_{-\infty} &= \lim_s A_s \\
RA_{-\infty} &= \operatorname*{Rlim}_s A_s \\
A_\infty &= \operatorname*{colim}_s A_s
\end{aligned}$$

be the limit, derived limit and colimit of the bi-infinite sequence $(A_s)_s$.

We consider two possible target groups for the spectral sequence; the colimit A_∞ and the limit $A_{-\infty}$. Each comes with a natural increasing filtration.

Definition 7.6. Let $F_s A_\infty = \mathrm{im}(A_s \to A_\infty)$ and $F_s A_{-\infty} = \ker(A_{-\infty} \to A_s)$, for each integer s.

Lemma 7.7. *The filtration $\{F_s A_\infty\}_s$ of A_∞ is exhaustive, and the filtration $\{F_s A_{-\infty}\}_s$ of $A_{-\infty}$ is complete Hausdorff.*

Proof. The first claim is clear. For the second claim, use the lim-Rlim exact sequences for

$$0 \to F_s A_{-\infty} \longrightarrow A_{-\infty} \longrightarrow \mathrm{im}(A_{-\infty} \to A_s) \to 0$$

and

$$0 \to \mathrm{im}(A_{-\infty} \to A_s) \longrightarrow A_s \longrightarrow \mathrm{cok}(A_{-\infty} \to A_s) \to 0\,.$$

□

Proposition 7.8. *There are natural isomorphisms*

$$\frac{F_s A_\infty}{F_{s-1} A_\infty} \cong \frac{\mathrm{im}(j)}{B_s^\infty} \qquad \text{and} \qquad \frac{F_s A_{-\infty}}{F_{s-1} A_{-\infty}} \cong [[ETC]]\,.$$

Proof. Consider the diagram

$$\begin{tikzcd}
A_{-\infty} \arrow[r] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r] \arrow[d, "j"] & A_\infty \\
 & & E_s\,. \arrow[ul, "k"] &
\end{tikzcd}$$

The homomorphisms $A_s \to A_\infty$ and $j \colon A_s \to E_s$ induce isomorphisms

$$\frac{F_s A_\infty}{F_{s-1} A_\infty} \cong \frac{A_s}{\ker(A_s \to A_\infty) + \mathrm{im}(i \colon A_{s-1} \to A_s)}$$

and

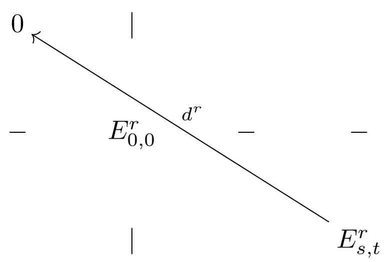
$$\frac{\mathrm{im}(j \colon A_s \to E_s)}{j(\ker(A_s \to A_\infty))} \cong \frac{A_s}{\ker(A_s \to A_\infty) + \ker(j \colon A_s \to E_s)}\,,$$

respectively, and the right hand sides are equal. Finally, $j(\ker(A_s \to A_\infty)) = B_s^\infty$ by passage to colimits over r from the definition $j(\ker(A_s \to A_{s+r-1})) = B_s^r$.

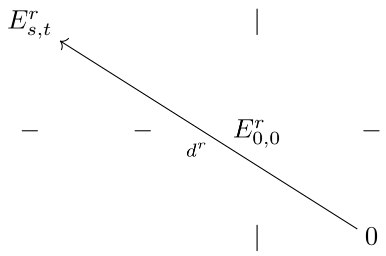
[[ETC, limit case]] □

7.3. **Conditional convergence.**

Definition 7.9. A *homological right half-plane spectral sequence* is a spectral sequence such that $E^r_{s,t} = 0$ for all $s < 0$. More generally, a *spectral sequence with exiting differentials* is a spectral sequence such that in each bidegree (s,t) only finitely many of the differentials starting in that bidegree map to nonzero groups.



A *homological left half-plane spectral sequence* is a spectral sequence such that $E^r_{s,t} = 0$ for all $s > 0$. More generally, a *spectral sequence with entering differentials* is a spectral sequence such that in each bidegree (s,t) only finitely many of the differentials ending in that bidegree map from nonzero groups.



Let $(E^r, d^r)_r$ be the homological spectral sequence associated to an unrolled exact couple, as in Definition 7.1 and Proposition 7.3. In the right half-plane case, the following classical theorem suffices.

Theorem 7.10 (Cartan–Eilenberg(?)). *Suppose that $E_s = 0$ for all $s < 0$, so that $A_{-\infty} \cong A_s$ for all $s < 0$ and $(E^r, d^r)_r$ is a spectral sequence with exiting differentials.*

(1) *If $A_{-\infty} = 0$ then the spectral sequence converges strongly to the colimit A_∞.*
(2) *If $A_\infty = 0$ then the spectral sequence converges strongly to the limit $A_{-\infty} \cong A_{-1}$.*

In the homological left half-plane case, as well as in the case of whole-plane spectral sequences, the utility of the following definition was explained by Boardman (1981 preprint, 1999). [[Check what Adams writes in the Chicago lecture notes, or his 1971 survey of 1960s algebraic topology.]]

Definition 7.11. The spectral sequence $(E^r, d^r)_r$ associated to an unrolled exact couple *converges conditionally to the colimit A_∞* if $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. It *converges conditionally to the limit $A_{-\infty}$* if $A_\infty = 0$.

Theorem 7.12 (Adams(?), Boardman). *Suppose that $E_s = 0$ for all $s > 0$, so that $A_s \cong A_\infty$ for all $s \ge 0$ and $(E^r, d^r)_r$ is a spectral sequence with entering differentials.*

(1) *If the spectral sequence converges conditionally to the colimit A_∞, and if $RE^\infty = 0$, then the spectral sequence converges strongly to that colimit.*
(2) *If the spectral sequence converges conditionally to the limit $A_{-\infty}$, and if $RE^\infty = 0$, then the spectral sequence converges strongly to that limit.*

Conditional convergence is a property of the unrolled exact couple, which can often be verified in terms of its construction. The additional assumption that $RE^\infty = 0$ can often be verified in concrete cases, e.g., in the presence of finiteness assumptions. The theorem asserts that it combination, these two properties suffice to ensure strong convergence. We shall only discuss a minimal path towards this result. The reader should consult Boardman (1999) for a much more complete story, including comparison theorems and results about when the sufficient conditions are also necessary.

Proof. In view of equation (3) and Proposition 7.8, in order to prove that $F_s A_\infty / F_{s-1} A_\infty \cong E^\infty_s$ it suffices to prove that $Z^\infty_s / \ker(k) = 0$. To establish strong convergence, we also need to prove that the filtration $\{F_s A_\infty\}_s$ is complete and Hausdorff.

Definition 7.13. Let $Q_s = \lim_r \operatorname{im}(i^r \colon A_{s-r} \to A_s)$ and $RQ_s = \operatorname{Rlim}_r \operatorname{im}(i^r \colon A_{s-r} \to A_s)$ be the limit and the derived limit, respectively, of the image filtration

$$\cdots \subset \operatorname{im}(i^r \colon A_{s-r} \to A_s) \subset \cdots \subset \operatorname{im}(i \colon A_{s-1} \to A_s) \subset A_s \,.$$

Lemma 7.14. *There is a six term exact sequence*

$$0 \to \frac{Z^\infty_s}{\ker(k)} \xrightarrow{k} Q_{s-1} \xrightarrow{i} Q_s \longrightarrow RE^\infty_s \xrightarrow{k} RQ_{s-1} \xrightarrow{i} RQ_s \to 0 \,.$$

Proof. For each r and s, there is a short exact sequence

$$0 \to \frac{Z^r_s}{\ker(k)} \xrightarrow{k} \operatorname{im}(i^{r-1} \colon A_{s-r} \to A_{s-1}) \xrightarrow{i} \operatorname{im}(i^r \colon A_{s-r} \to A_s) \to 0 \,.$$

Passing to limits over r, we get the asserted six term exact sequence. □

Corollary 7.15. *If $RE^\infty = 0$, then each $i \colon Q_{s-1} \to Q_s$ is surjective and each $i \colon RQ_{s-1} \to RQ_s$ is an isomorphism. Hence $\lim_s Q_s \to Q_m$ is surjective and $\lim_s RQ_s \to RQ_m$ is an isomorphism, for each m.*

By assumption $A_0 = A_\infty$, so we have

$$Q_0 = \lim_r \operatorname{im}(A_{-r} \to A_0) = \lim_s F_s A_\infty$$

and

$$RQ_0 = \operatorname{Rlim}_r \operatorname{im}(A_{-r} \to A_0) = \operatorname{Rlim}_s F_s A_\infty \,.$$

Hence proving that $Q_0 = 0$ and $RQ_0 = 0$ is equivalent to proving that $\{F_s A_\infty\}_s$ is complete Hausdorff. By the corollary, when $RE^\infty = 0$ it will suffice to prove that $\lim_s Q_s = 0$ and $\lim_s RQ_s = 0$. This will then also imply that each $Q_{s-1} = 0$, so $Z_s^\infty / \ker(k) = 0$, as desired. By the following lemma, these properties follow from the assumptions $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. □

Lemma 7.16. *If $A_{-\infty} = 0$ and $RA_{-\infty} = 0$ then $\lim_s Q_s = 0$ and $\lim_s RQ_s = 0$.*

Proof. Consider the double limit system

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r, "i"] & \operatorname{im}(i^r \colon A_{s-1-r} \to A_{s-1}) \arrow[r, "i"] \arrow[d] & \operatorname{im}(i^r \colon A_{s-r} \to A_s) \arrow[r, "i"] \arrow[d] & \dots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r, "i"] & \dots
\end{tikzcd}$$

where the vertical maps are inclusions. The limit of the s-th column is by definition Q_s. The limit of the r-th row maps identically to the limit of the bottom row, i.e., to $A_{-\infty}$. Hence

$$\lim_s Q_s = \lim_s \lim_r \operatorname{im}(i^r \colon A_{s-r} \to A_s) \cong \lim_r \lim_s \operatorname{im}(i^r \colon A_{s-r} \to A_s) \cong \lim_r A_{-\infty} \cong A_{-\infty} \,.$$

For each s let

$$\widehat{A}_s = \lim_r \frac{A_s}{\operatorname{im}(i^r \colon A_{s-r} \to A_s)}$$

be the completion of A_s with respect to the image filtration. The lim-Rlim sequence of the r-indexed system of short exact sequences

$$0 \to \operatorname{im}(i^r \colon A_{s-r} \to A_s) \longrightarrow A_s \longrightarrow \frac{A_s}{\operatorname{im}(i^r \colon A_{s-r} \to A_s)} \to 0$$

contains the exact sequence

$$0 \to Q_s \longrightarrow A_s \longrightarrow \widehat{A}_s \longrightarrow RQ_s \to 0 \,,$$

which breaks into the two s-indexed systems of short exact sequences

$$0 \to Q_s \longrightarrow A_s \longrightarrow A_s/Q_s \to 0$$

and

$$0 \to A_s/Q_s \longrightarrow \widehat{A}_s \longrightarrow RQ_s \to 0 \,.$$

These in turn give rise to the exact lim-Rlim sequences

$$0 \to \lim_s Q_s \longrightarrow A_{-\infty} \longrightarrow \lim_s A_s/Q_s \longrightarrow \operatorname{Rlim}_s Q_s \longrightarrow RA_{-\infty} \longrightarrow \operatorname{Rlim}_s A_s/Q_s \to 0$$

and

$$0 \to \lim_s A_s/Q_s \longrightarrow \lim_s \widehat{A}_s \longrightarrow \lim_s RQ_s \longrightarrow \operatorname{Rlim}_s A_s/Q_s \longrightarrow \operatorname{Rlim}_s \widehat{A}_s \longrightarrow \operatorname{Rlim}_s RQ_s \to 0 \,.$$

Here

$$\lim_s \widehat{A}_s = \lim_s \lim_r \frac{A_s}{\operatorname{im}(i^r \colon A_{s-r} \to A_s)} \cong \lim_r \lim_s \frac{A_s}{\operatorname{im}(i^r \colon A_{s-r} \to A_s)} = \lim_r 0 = 0 \,,$$

since for each fixed r, the r-fold composite

$$i^r \colon \frac{A_{s-r}}{\operatorname{im}(i^r \colon A_{s-2r} \to A_{s-r})} \longrightarrow \frac{A_s}{\operatorname{im}(i^r \colon A_{s-r} \to A_s)}$$

is zero.

The assumptions $A_{-\infty} = 0$ and $RA_{-\infty} = 0$ now yield $\lim_s Q_s = 0$, $\lim_s A_s/Q_s \cong \mathrm{Rlim}_s Q_s$ and $\mathrm{Rlim}_s A_s/Q_s = 0$. Combined with the vanishing of $\lim_s \widehat{A}_s$, this implies $\lim_s A_s/Q_s = 0$, $\lim_s RQ_s = 0$ and $\mathrm{Rlim}_s \widehat{A}_s \cong \mathrm{Rlim}_s RQ_s$. In fact $\mathrm{Rlim}_s RQ_s = 0$, since $i\colon RQ_{s-1} \to RQ_s$ is surjective for each s. □

Boardman in fact proves the following more precise result, the middle part of which he refers to as the Mittag–Leffler exact sequence. This is a special case of the Grothendieck spectral sequence for the composite of two limit functors, which was first (?) analyzed by Roos.

Proposition 7.17. $\lim_s Q_s \cong A_{-\infty}$, *there is a short exact sequence*

$$0 \to \operatorname*{Rlim}_s Q_s \longrightarrow RA_{-\infty} \longrightarrow \lim_s RQ_s \to 0\,,$$

and $\mathrm{Rlim}_s RQ_s = 0$.

8. Examples of exact couples

8.1. **Homology of sequences of cofibrations.** Generalizing the examples from Section 1 and Section 3, consider a sequence of spaces

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

where each inclusion $i\colon X_{s-1} \to X_s$ is a cofibration and $X = \mathrm{colim}_s X_s \simeq \mathrm{hocolim}_s X_s$ has the weak (colimit) topology. For instance, X might be a CW complex and X_s its s-skeleton. Applying homology, we obtain an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & H_*(X_{s-2}) \arrow[rr, "i_*"] & & H_*(X_{s-1}) \arrow[rr, "i_*"] \arrow[d, "j_*"] & & H_*(X_s) \arrow[rr, "i_*"] \arrow[d, "j_*"] & & H_*(X_{s+1}) \arrow[r, "i_*"] \arrow[d, "j_*"] & \dots \\
\dots & & & H_*(X_{s-1}, X_{s-2}) \arrow[ull, "\partial"] & & H_*(X_s, X_{s-1}) \arrow[ull, "\partial"] & & H_*(X_{s+1}, X_s) \arrow[ull, "\partial"] & \dots
\end{tikzcd}$$

with $A_s = H_*(X_s)$ and $E_s = H_*(X_s, X_{s-1})$. Each triangle is the long exact sequence of a pair, hence is exact. The homomorphisms $i = i_*$ and $j = j_*$ preserve the internal grading, while $k = \partial$ has degree -1. The E^1-term is

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$$

and the d^1-differential is

$$d^1_{s,t} = j_* \circ \partial\colon H_{s+t}(X_s, X_{s-1}) \longrightarrow H_{s+t-1}(X_{s-1}, X_{s-2})\,,$$

i.e., the connecting homomorphism in the long exact sequence in homology for the triple (X_s, X_{s-1}, X_{s-2}). Here $A_s = 0$ for $s < 0$, so we have a homological right half-plane spectral sequence, with exiting differentials. By Theorem 7.10, it converges strongly to

$$A_\infty = \operatorname*{colim}_s H_*(X_s) \cong H_*(X)\,.$$

In the special case when $X_s = X^{(s)}$ is the s-skeleton of a CW complex X, $E^1_{s,0} = H_s(X^{(s)}, X^{(s-1)}) = C_s(X)$ and $E^1_{s,t} = 0$ for $t \neq 0$, so (E^1, d^1) equals the cellular chain complex of X, concentrated on the horizontal axis. The E^2-term equals the cellular homology, and the spectral sequence collapses at this stage. These observations give a spectral sequence proof of the fact that cellular homology is isomorphic to singular homology for CW complexes.

8.2. **Cohomology of sequences of cofibrations.** Applying cohomology to the same sequence of spaces, we get another unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & H^*(X_{s+1}) \arrow[rr, "i^*"] & & H^*(X_s) \arrow[rr, "i^*"] \arrow[d, "\delta"] & & H^*(X_{s-1}) \arrow[rr, "i^*"] \arrow[d, "\delta"] & & H^*(X_{s-2}) \arrow[r, "i^*"] \arrow[d, "\delta"] & \dots \\
\dots & & & H^*(X_{s+1}, X_s) \arrow[ull, "j^*"] & & H^*(X_s, X_{s-1}) \arrow[ull, "j^*"] & & H^*(X_{s-1}, X_{s-2}) \arrow[ull, "j^*"] & \dots,
\end{tikzcd}$$

now with $A^s = A_{-s} = H^*(X_{s-1})$ and $E^s = E_{-s} = H^*(X_s, X_{s-1})$. In this case $i = i^*$ and $k = j^*$ preserve degrees, and $j = \delta$ has degree $+1$. The associated spectral sequence is a left half-plane spectral sequence with entering differentials, and converges conditionally to the limit

$$A_{-\infty} = \lim_s H^*(X_s)$$

since $A_\infty = \operatorname{colim}_s H^*(X_s) = 0$. By Theorem 7.12, the spectral sequence converges strongly to this limit if $RE^\infty = 0$. In general, the homomorphism $H^*(X) \to \lim_s H^*(X_s)$ is not an isomorphism, so this spectral sequence is not always useful for the computation of $H^*(X)$.

Instead, one can consider the sequence of pairs of spaces

$$(X, \emptyset) = (X, X_{-1}) \subset (X, X_0) \subset \dots \subset (X, X_{s-1}) \subset (X, X_s) \subset \dots \subset (X, X)$$

and apply relative cohomology. The result is an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & H^*(X, X_{s+1}) \arrow[r, "j^*"] & H^*(X, X_s) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*(X, X_{s-1}) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*(X, X_{s-2}) \arrow[r, "j^*"] \arrow[d, "i^*"] & \dots \\
\dots & & H^*(X_{s+1}, X_s) \arrow[ul, "\delta"] & H^*(X_s, X_{s-1}) \arrow[ul, "\delta"] & H^*(X_{s-1}, X_{s-2}) \arrow[ul, "\delta"] & \dots,
\end{tikzcd}$$

where $A^s = A_{-s} = H^*(X, X_{s-1})$ and $E^s = E_{-s} = H^*(X_s, X_{s-1})$. In this case $i = j^*$ and $j = i^*$ preserve degrees, and $k = \delta$ has degree $+1$. The associated spectral sequence has

$$E_1^{s,t} = E^1_{-s,-t} = H^{s+t}(X_s, X_{s-1})$$

and $d^1 = i^* \circ \delta$. In homological indexing it is concentrated in the left half-plane, hence has entering differentials, and converges conditionally to the colimit

$$A_\infty = \operatorname*{colim}_s H^*(X, X_s) \cong H^*(X)$$

whenever $A_{-\infty} = \lim_s H^*(X, X_s) = 0$ and $RA_{-\infty} = \operatorname{Rlim}_s H^*(X, X_s) = 0$. In view of the Milnor lim-Rlim short exact sequence

$$0 \to \operatorname*{Rlim}_s H^{*-1}(X, X_s) \longrightarrow H^*(X, \operatorname*{hocolim}_s X_s) \longrightarrow \lim_s H^*(X, X_s) \to 0\,,$$

where we use the equivalence $X \simeq \operatorname{hocolim}_s X_s$, these conditions are always satisfied. By Theorem 7.12, the spectral sequence therefore converges strongly to $H^*(X)$ whenever $RE^\infty = 0$, e.g., if the spectral sequence collapses at a finite stage.

8.3. **The Atiyah–Hirzebruch spectral sequence.** Replacing singular homology with a generalized homology theory E_*, such as stable homotopy, topological K-homology or complex bordism, we instead obtain an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & E_*(X_{s-2}) \arrow[r, "i"] & E_*(X_{s-1}) \arrow[r, "i"] \arrow[d, "j"] & E_*(X_s) \arrow[r, "i"] \arrow[d, "j"] & E_*(X_{s+1}) \arrow[r, "i"] \arrow[d, "j"] & \dots \\
\dots & & E_*(X_{s-1}, X_{s-2}) \arrow[ul, "\partial"] & E_*(X_s, X_{s-1}) \arrow[ul, "\partial"] & E_*(X_{s+1}, X_s) \arrow[ul, "\partial"] & \dots
\end{tikzcd}$$

with associated spectral sequence having $A_s = E_*(X_s)$ and $E_s = E_*(X_s, X_{s-1})$. The E^1-term is

$$E^1_{s,t} = E_{s+t}(X_s, X_{s-1})$$

and the d^1-differential is $j\partial$, as before. This is now the connecting homomorphism in the long exact sequence in E_*-theory, for the triple (X_s, X_{s-1}, X_{s-2}). Again this is a right half-plane spectral sequence, converging strongly to the colimit

$$A_\infty = \operatorname*{colim}_s E_*(X_s) \cong E_*(X)\,.$$

In this generality the special case $X_s = X^{(s)}$ is interesting, since

$$E^1_{s,t} = E_{s+t}(X^{(s)}, X^{(s-1)}) = C_s(X; E_t)$$

is the group of cellular s-chains of X with coefficients in the coefficient group $E_t = E_t(*) = \pi_t(E)$ of the generalized homology theory E_*. The d^1-differential is the boundary homomorphism in the cellular chain complex $C_*(X; E_t)$, so the E^2-term

$$E^2_{s,t} = H_s(X; E_t)$$

is the s-th cellular homology group of X with coefficients in E_t. This example is the E_*-theory Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; E_t) \Longrightarrow_s E_{s+t}(X)$$

converging strongly to $E_*(X)$. The target group is filtered by the images

$$F_s E_*(X) = \operatorname{im}(E_*(X_s) \longrightarrow E_*(X))$$

and there are isomorphisms $F_sE_*(X)/F_{s-1}E_*(X) \cong (E^\infty_s)_*$, for all integers s. If $H_*(X)$ and $E_* = E_*(*)$ are concentrated in even degrees (meaning that $H_s(X) = 0$ for s odd and $E_t = 0$ for t odd), and at least one of these graded groups are torsion-free, then

$$E^2_{*,*} = H_*(X; E_*) \cong H_*(X) \otimes_{\mathbb{Z}} E_*$$

is concentrated in bidegrees (s, t) with both s and t even. It follows that each differential

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

must be zero for bidegree reasons, so that the spectral sequence collapses at the E^2-term, with $E^2 = E^\infty$. This happens frequently enough to be worthy of note, for instance if $E = KU$ or MU represents complex K-theory or complex (co-)bordism.

The Atiyah–Hirzebruch spectral sequence for stable homotopy theory

$$E^2_{s,t} = H_s(X; \pi^S_t) \Longrightarrow_s \pi^S_{s+t}(X)$$

is sometimes useful in conjunction with the Adams spectral sequence.

The cohomological version of the Atiyah–Hirzebruch spectral sequence is the spectral sequence

$$E^{s,t}_2 = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

with entering differentials, where $E^t = E^t(*) = \pi_{-t}(E)$, associated to the unrolled exact couple with

$$A^{s,t} = E^{s+t}(X, X^{(s-1)})$$

and

$$E^{s,t} = E^{s+t}(X^{(s)}, X^{(s-1)}) = C^s(X; E^t)\,.$$

It converges conditionally to the colimit, and converges strongly if $RE_\infty = 0$.

The original paper of Atiyah and Hirzebruch (1961) concerned the generalized cohomology theory given by topological K-theory, with $K^t = K_{-t} = \mathbb{Z}$ for t even and $K^t = K_{-t} = 0$ for t odd, so the K-cohomology Atiyah–Hirzebruch spectral sequence

$$E^{s,t}_2 = H^s(X; K^t) \Longrightarrow K^{s+t}(X)$$

collapses at the E_2-term for each space X whose cohomology $H^*(X)$ is concentrated in even degrees. [[Describe d_3-differential in terms of cohomology operations.]]

8.4. **The Serre spectral sequence.** Consider a Serre fibration $p \colon E \to B$, with B path-connected. Suppose that the base space B is a CW complex, with skeleton filtration $\{B^{(s)}\}_s$. Define a filtration

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E$$

of the total space E by taking the preimages of this skeleton filtration:

$$E_s = p^{-1}(B^{(s)})\,.$$

We get an unrolled exact couple with $A_{s,t} = H_{s+t}(E_s)$ and $E_{s,t} = H_{s+t}(E_s, E_{s-1})$, and an associated spectral sequence

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1})$$

converging strongly to $H_{s+t}(E)$. We use the hypothesis that $p \colon E \to B$ is a Serre fibration to rewrite the E^1-term in terms of the cellular chains on B. Let

$$\Phi = \coprod_\alpha \Phi_\alpha \colon \coprod_\alpha D^s \longrightarrow B^{(s)}$$

be the combined characteristic maps of the s-cells of B, and let $\phi_\alpha \colon \partial D^s \to B^{(s-1)}$ be the attaching map of the α-indexed s-cell, i.e., the restriction of Φ_α to $\partial D^s \subset D^s$, viewed as a map into $B^{(s-1)} \subset B^{(s)}$.

Let $\Phi_\alpha^* E = D^s \times_B E$ be the pullback of E along Φ_α, and let $\phi_\alpha^* E = \partial D^s \times_B E$ be its restriction to ∂D^s.

$$\begin{tikzcd}
\phi_\alpha^* E \arrow[rr] \arrow[dr] \arrow[dd] & & \Phi_\alpha^* E \arrow[dr] \arrow[dd] & & \\
& E_{s-1} \arrow[rr] \arrow[dd] & & E_s \arrow[r] \arrow[dd] & E \arrow[dd, "p"] \\
\partial D^s \arrow[rr] \arrow[dr, "\phi_\alpha"'] & & D^s \arrow[dr, "\Phi_\alpha"] & & \\
& B^{(s-1)} \arrow[rr] & & B^{(s)} \arrow[r] & B
\end{tikzcd}$$

By excision, the sum of homomorphisms

$$\bigoplus_\alpha H_*(\Phi_\alpha^* E, \phi_\alpha^* E) \xrightarrow{\cong} H_*(E_s, E_{s-1})$$

is an isomorphism. For each α, the map

$$(\Phi_\alpha^* E, \phi_\alpha^* E) \longrightarrow (D^s \times \Phi_\alpha^* E, \partial D^s \times \Phi_\alpha^* E)$$

is a homotopy equivalence of pairs, since D^s is contractible. For any fixed choice of base point $d_0 \in D^s$, mapping to $b_\alpha = \Phi_\alpha(d_0) \in B$, the inclusion

$$F_{b_\alpha} = p^{-1}(b_\alpha) = \{b_\alpha\} \times_B E \cong \{d_0\} \times_{D^s} \Phi_\alpha^* E \subset \Phi_\alpha^* E$$

is a (weak) homotopy equivalence, in view of the long exact sequence in homotopy for the Serre fibration $\Phi_\alpha^* E \to D^s$, again using that D^s is contractible. Hence there are preferred isomorphisms

$$H_*(\Phi_\alpha^* E, \phi_\alpha^* E) \cong H_*(D^s \times \Phi_\alpha^* E, \partial D^s \times \Phi_\alpha^* E) \cong H_*(D^s, \partial D^s) \otimes H_*(\Phi_\alpha^* E) \cong H_*(D^s, \partial D^s) \otimes H_*(F_{b_\alpha}) \,.$$

Thus

$$E^1_{s,t} \cong \bigoplus_\alpha H_t(F_{b_\alpha})$$

with $b_\alpha = \Phi_\alpha(d_0)$, varying with α. By definition this is the group of cellular s-chains $C_s(B; \mathscr{H}_t(F))$ of B with local coefficients in the system $\mathscr{H}_t(F)$, taking $b \in B$ to $H_t(F_b)$.

A *local coefficient system* on B can be defined as a functor from the fundamental groupoid $\Pi_1(B)$ of B to the category of abelian groups. The objects of $\Pi_1(B)$ are the points of B, and a morphism from b_0 to b_1 is a homotopy class $[f]$, relative to the endpoints, of paths $f \colon I \to B$ from b_1 to b_0. With this convention, the composite of $[f]$ and the class $[g]$ of a path $g \colon I \to B$ from b_2 to b_1 is the class $[g] \circ [f] = [g * f]$ of the path $g * f$ from b_2 to b_0. When B is path connected, all objects of $\Pi_1(B)$ are isomorphic, and for any choice of base point $b_0 \in B$, the inclusion $\pi_1(B, b_0) \subset \Pi_1(B)$ of the fundamental group of B based at b_0, viewed as a groupoid with one object, is an equivalence of categories.

The local coefficient system $\mathscr{H}_t(F)$ takes $b \in B$ to $H_t(F_b)$, where $F_b = p^{-1}(b)$ is the fiber of $p \colon E \to B$ over b. To the homotopy class $[f]$ of a path f from b_1 to b_0, as above, we associate the composite isomorphism

$$[f]_* \colon H_t(F_{b_1}) \xrightarrow{\cong} H_t(I \times_B E) \xleftarrow{\cong} H_t(F_{b_0}) \,.$$

Here each inclusion $F_{b_t} \to I \times_B E$ is a (weak) homotopy equivalence, since $I \times_B E \to I$ is a Serre fibration, and the interval I is contractible. Exercise: Prove that if $H \colon I \times I \to B$ is a homotopy, relative to the endpoints, from f to $f' \colon I \to B$, then $[f]_* = [f']_*$.

A boundary homomorphism

$$\partial \colon C_s(B; \mathscr{H}_t(F)) \longrightarrow C_{s-1}(B; \mathscr{H}_t(F))$$

can be defined so as to agree with $d^1_{s,t}$ under the identifications above. [[ETC]]

In particular, $(C_*(B; \mathscr{H}_t(F), \partial)$ is a chain complex, and its homology $H_*(B; \mathscr{H}_t(F))$ is the cellular homology of B with local coefficients in $\mathscr{H}_t(F)$. This then computes the E^2-term of the homological Serre spectral sequence

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E) \,.$$

If B is simply-connected, then $\mathscr{H}_t(F)$ is isomorphic (as a coefficient system) to the constant system at $H_t(F_{b_0})$, for any fixed choice of base point $b_0 \in B$, so in this case we can write the E^2-term as $H_s(B; H_t(F))$, with ordinary coefficients.

[[Relate to π-equivariant homology for the universal covering space $\tilde{B}$, with $\pi = \pi_1(B, b_0)$.]]

The cohomological version of the Serre spectral sequence is associated to the unrolled exact couple with

$$A^{s,t} = H^{s+t}(E, E_{s-1})$$

and

$$E^{s,t} = H^{s+t}(E_s, E_{s-1})\,.$$

It has

$$E_1^{s,t} = C^s(B; \mathscr{H}^t(F))$$

and

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E)\,.$$

It is concentrated in the first quadrant, in the cohomological indexing, and converges strongly to the colimit $H^*(E)$.

[[There are many examples of calculations with Serre spectral sequences in the literature, e.g., for loop-path fibrations $\Omega X \to PX \to X$, or for homogeneous spaces $H \to G \to G/H$ or $G/H \to BH \to BG$.]]

8.5. **Homotopy of towers of fibrations.** Turning in a different direction, consider a tower of spaces

$$Y \to \dots \to Y^s \to Y^{s-1} \to \dots \to Y^0 \to Y^{-1} = *$$

where each map $p\colon Y^s \to Y^{s-1}$ is a Serre fibration, and $Y = \lim_s Y^s \simeq \operatorname{holim}_s Y^s$.

We assume that Y is not empty, so that we can choose a base point $y_0 \in Y$, and take its image y_s under $Y \to Y^s$ as the base point for Y^s, for each integer s. Let

$$F^s = p^{-1}(y_{s-1}) = \{y_{s-1}\} \times_{Y^{s-1}} Y^s$$

be the fiber of $p\colon Y^s \to Y^{s-1}$ at y_{s-1}, based at y_s, so that there is a long exact sequence of homotopy groups

$$\dots \to \pi_n(F^s, y_s) \longrightarrow \pi_n(Y^s, y_s) \longrightarrow \pi_n(Y^{s-1}, y_{s-1}) \xrightarrow{\partial} \pi_{n-1}(F^s, y_s) \to \dots\,.$$

We would like to link these together to an unrolled exact couple, but note that in general the end

$$\begin{aligned} \dots \to \pi_1(F^s, y_s) \longrightarrow \pi_1(Y^s, y_s) \longrightarrow \pi_1(Y^{s-1}, y_{s-1}) & \\ \xrightarrow{\partial} \pi_0(F^s, y_s) \longrightarrow \pi_0(Y^s, y_s) \longrightarrow \pi_0(Y^{s-1}, y_{s-1}) & \end{aligned}$$

of this sequence is not a diagram of abelian groups, and we might not be able to extend the sequence to the right with trivial groups.

Bousfield–Kan (1972, Section IX.4) address this problem by considering "extended" spectral sequences, which consist of possibly non-abelian groups and pointed sets near the edge.

Another solution is to assume that each Y^s is a homotopy commutative H-space, with y_s as neutral element, and that each map $p\colon Y^s \to Y^{s-1}$ is strictly compatible with this H-space structure. Then each fiber F^s is also a homotopy commutative H-space, and the diagram above is one of abelian groups and group homomorphisms. It is still not necessarily exact at $\pi_0(Y^{s-1}, y_{s-1})$, since $\pi_0(p)$ does not need to be surjective. We must therefore make this additional assumption. It is satisfied, for instance, if each space Y^s is path-connected.

Under these additional hypotheses, we get an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y^{s+1}, y_{s+1}) \arrow[r, "p_*"] & \pi_*(Y^s, y_s) \arrow[r, "p_*"] \arrow[d, "\partial"] & \pi_*(Y^{s-1}, y_{s-1}) \arrow[r, "p_*"] \arrow[d, "\partial"] & \pi_*(Y^{s-2}, y_{s-2}) \arrow[r] \arrow[d, "\partial"] & \dots \\
 & \dots & \pi_*(F^{s+1}, y_{s+1}) \arrow[ul] & \pi_*(F^s, y_s) \arrow[ul] & \pi_*(F^{s-1}, y_{s-1}) \arrow[ul] &
\end{tikzcd}$$

with $i = p_*$ of degree 0, $j = \partial$ of degree -1, and k of degree 0. The associated spectral sequence

$$E_1^{s,*} = \pi_*(F^s, y_s) \Longrightarrow_s \pi_*(Y, y_0)$$

has entering differentials and converges conditionally to the limit $\lim_s \pi_*(Y^s, y_s)$. [[Claim: If $RE_\infty = 0$, then $\operatorname{Rlim}_s \pi_*(Y^s, y_s) = 0$ and the spectral sequence converges strongly to $\pi_*(Y, y_0)$.]]

8.6. **Homotopy of towers of spectra.** The difficulty with the lack of abelian group structures, and lack of surjectivity at π_0, is not present when we consider towers of spectra. Consider a diagram of spectra

$$\dots \to Y^s \xrightarrow{i} Y^{s-1} \to \dots \to Y^0 = Y$$

and let $Y^\infty = \operatorname{holim}_s Y^s$, so that there is a Milnor lim-Rlim short exact sequence

$$0 \to \operatorname{Rlim}_s \pi_{n+1}(Y^s) \longrightarrow \pi_n(Y^\infty) \longrightarrow \lim_s \pi_n(Y^s) \to 0\,.$$

Let K^s be the homotopy cofiber of the map $i\colon Y^{s+1} \to Y^s$, so that there is a Puppe cofiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}\,.$$

We let $Y^s = Y$ and $K^s = *$ for all $s < 0$. Applying homotopy to these spectra, we get an unrolled exact couple of graded abelian groups

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y^{s+2}) \arrow[r, "i"] & \pi_*(Y^{s+1}) \arrow[r, "i"] \arrow[d, "j"] & \pi_*(Y^s) \arrow[r, "i"] \arrow[d, "j"] & \pi_*(Y^{s-1}) \arrow[r] \arrow[d, "j"] & \dots \\
& \dots & \pi_*(K^{s+1}) \arrow[ul, "k"] & \pi_*(K^s) \arrow[ul, "k"] & \pi_*(K^{s-1}) \arrow[ul, "k"] &
\end{tikzcd}$$

with $i = i_*$ and $j = j_*$ of degree 0, and $k = \partial_*$ of degree -1.

In homological indexing, we would write $A_{s,t} = \pi_{s+t}(Y^{-s})$ and $E_{s,t} = \pi_{s+t}(K^{-s})$, for $s \le 0$, but we switch to Adams indexing $A^{s,t} = A_{-s,t}$ and $E^{s,t} = E_{-s,t}$ so that

$$\begin{aligned} A^{s,t} &= \pi_{t-s}(Y^s) \\ E^{s,t} &= \pi_{t-s}(K^s)\,. \end{aligned}$$

The associated spectral sequence

$$E_1^{s,t} = \pi_{t-s}(K^s) \Longrightarrow_s \pi_{t-s}(Y)$$

has entering differentials. By definition, it converges conditionally to the colimit $A_\infty (= A^{-\infty}) = \pi_*(Y)$ if the two groups

$$A_{-\infty}(= A^\infty) = \lim_s \pi_*(Y^s) \qquad \text{and} \qquad RA_{-\infty}(= RA^\infty) = \operatorname{Rlim}_s \pi_*(Y^s)$$

both vanish. By the lim-Rlim exact sequence recalled above, this is equivalent to the condition that $\pi_*(Y^\infty) = 0$, i.e., that $\operatorname{holim}_s Y^s \simeq *$.

Proposition 8.1. *The spectral sequence*

$$E_1^{s,t} = \pi_{t-s}(K^s) \Longrightarrow_s \pi_{t-s}(Y)$$

associated to the tower $\dots \to Y^s \to Y^{s-1} \to \dots \to Y^0 = Y$ converges conditionally to the colimit $\pi_*(Y)$ if (and only if) $\operatorname{holim}_s Y^s \simeq *$. If $RE_\infty = 0$ then the spectral sequence converges strongly to that colimit, equipped with the descending filtration by the image subgroups $F^s = \operatorname{im}(\pi_*(Y^s) \to \pi_*(Y))$.

The mod p Adams spectral sequence converging to $\pi_*(Y)^\wedge_p$ will be constructed as a special case of this spectral sequence, where we make special assumptions about the Puppe cofiber sequence displayed above, so as to be able to express the E_2-term of the spectral sequence in purely algebraic terms.

9. The Steenrod algebra

9.1. Steenrod's reduced squares and powers.

Theorem 9.1. *There are natural transformations*

$$Sq^i\colon \tilde H^n(X;\mathbb{F}_2) \longrightarrow \tilde H^{n+i}(X;\mathbb{F}_2)$$

for $i \ge 0$, of contravariant functors from based spaces to abelian groups, called Steenrod's reduced squares. These satisfy $Sq^0(x) = x$, $Sq^1(x) = \beta(x)$ (the Bockstein homomorphism associated to the extension $\mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2$), $Sq^i(x) = x^2$ for $i = |x|$, and $Sq^i(x) = 0$ for $i > |x|$. They also satisfy the internal Cartan formula

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x)Sq^j(y)$$

$$
\begin{array}{ll}
Sq^1Sq^1 = 0 & Sq^1Sq^2 = Sq^3 \\
Sq^1Sq^3 = 0 & Sq^2Sq^2 = Sq^3Sq^1 \\
Sq^1Sq^4 = Sq^5 & Sq^2Sq^3 = Sq^5 + Sq^4Sq^1 \\
Sq^3Sq^2 = 0 & Sq^1Sq^5 = 0 \\
Sq^2Sq^4 = Sq^6 + Sq^5Sq^1 & Sq^3Sq^3 = Sq^5Sq^1 \\
Sq^1Sq^6 = Sq^7 & Sq^2Sq^5 = Sq^6Sq^1 \\
Sq^3Sq^4 = Sq^7 & Sq^4Sq^3 = Sq^5Sq^2 \\
Sq^1Sq^7 = 0 & Sq^2Sq^6 = Sq^7Sq^1 \\
Sq^3Sq^5 = Sq^7Sq^1 & Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2 \\
Sq^5Sq^3 = 0 & Sq^1Sq^8 = Sq^9 \\
Sq^2Sq^7 = Sq^9 + Sq^8Sq^1 & Sq^3Sq^6 = 0 \\
Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 & Sq^5Sq^4 = Sq^7Sq^2 \\
Sq^1Sq^9 = 0 & Sq^2Sq^8 = Sq^{10} + Sq^9Sq^1 \\
Sq^3Sq^7 = Sq^9Sq^1 & Sq^4Sq^6 = Sq^{10} + Sq^8Sq^2 \\
Sq^5Sq^5 = Sq^9Sq^1 & Sq^6Sq^4 = Sq^7Sq^3 \\
Sq^1Sq^{10} = Sq^{11} & Sq^2Sq^9 = Sq^{10}Sq^1 \\
Sq^3Sq^8 = Sq^{11} & Sq^4Sq^7 = Sq^{11} + Sq^9Sq^2 \\
Sq^5Sq^6 = Sq^{11} + Sq^9Sq^2 & Sq^6Sq^5 = Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^7Sq^4 = 0 &
\end{array}
$$

FIGURE 9. The Adem relations at $p = 2$ in degrees ≤ 11

and the Adem relations

$$Sq^aSq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j$$

for $0 < a < 2b$.

Proofs can be found in Steenrod and Epstein (1962).

By naturality, the internal Cartan formula for the cup product $xy = x \cup y$ is equivalent to an external Cartan formula for the smash product $x \wedge y$. See Figure 9 for the Adem relations in degrees ≤ 11.

Example 9.2. The squaring operations for $X = \mathbb{R}P^\infty_+$ can be calculated as follows: Consider the total squaring operation $Sq(x) = \sum_{i\geq 0} Sq^i(x)$. Then $Sq(xy) = Sq(x)Sq(y)$. In $\tilde{H}^*(X;\mathbb{F}_2) = H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[u]$ with $|u| = 1$ we have $Sq(u) = u + u^2$, so $Sq(u^n) = (u+u^2)^n = \sum_{i=0}^n \binom{n}{i} u^{n+i}$. Hence $Sq^i(u^n) = \binom{n}{i} u^{n+i}$.

We outline one possible construction of the squaring operations. Let $H_n = K(\mathbb{F}_2, n)$ be an Eilenberg–Mac Lane complex of type $(\mathbb{F}_2, n)$, i.e., a space with $\pi_i(H_n) = \mathbb{F}_2$ for $i = n$ and 0 otherwise. For $n = 0$ we may take $H_0 = \mathbb{F}_2$. For $n \geq 1$ we may construct H_n from the Moore space $S^n \cup_2 e^{n+1}$ by the method of killing homotopy groups. Note that $H_1 \simeq \mathbb{R}P^\infty$.

There is a universal class $\iota_n \in \tilde{H}^n(H_n;\mathbb{F}_2)$ that corresponds to the identity homomorphism under the isomorphisms $\tilde{H}^n(H_n;\mathbb{F}_2) \cong \mathrm{Hom}(H_n(\mathbb{F}_2),\mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{F}_2,\mathbb{F}_2)$. By a theorem of Eilenberg and Mac Lane, there is a natural isomorphism

$$[X, H_n] \simeq H^n(X;\mathbb{F}_2)$$

that maps the homotopy class of $f\colon X \to H_n$ to $f^*(\iota_n)$. See Hatcher (2002, Theorem 4.57).

The smash product $\iota_n \wedge \iota_n \in \tilde{H}^{2n}(H_n \wedge H_n;\mathbb{F}_2)$ is represented by a map

$$\phi\colon H_n \wedge H_n \longrightarrow H_{2n}\,.$$

The composite $\phi\gamma$, where $\gamma\colon H_n \wedge H_n \to H_n \wedge H_n$ denotes the twist map, represents the same cohomology class, hence there is a homotopy $I_+ \wedge H_n \wedge H_n \longrightarrow H_{2n}$ from ϕ to $\phi\gamma$. We identify the interval I with the upper semicircle in S^1, and reinterpret this homotopy as a C_2-equivariant map $S^1_+ \wedge H_n \wedge H_n \longrightarrow H_{2n}$ where the generator of C_2 takes (s,x,y) to $(-s,y,x)$, and acts trivially on the target. Equivalently, it provides a map

$$\phi_1\colon S^1_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}\,,$$

which expresses the homotopy commutativity of the cup product ϕ. There exists unique extensions, up to homotopy, $\phi_k\colon S^k_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}$ of this map, for all $k \ge 2$, where C_2 acts antipodally on S^k. In the limit, these define a homotopy class of maps

$$\Phi\colon S^\infty_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}\,,$$

where S^∞ is a contractible space with free C_2-action. We call $D_2(H_n) = S^\infty_+ \wedge_{C_2} H_n \wedge H_n$ the *quadratic construction* on H_n. The structure map $\Phi\colon D_2(H_n) \to H_{2n}$ is part of the E_∞ ring spectrum structure on the Eilenberg–Mac Lane spectrum $H = \{n \mapsto H_n\}$. Let

$$\nabla\colon \mathbb{R}P^\infty_+ \wedge H_n \longrightarrow S^\infty_+ \wedge_{C_2} H_n \wedge H_n$$

be given by $([s],x) = [s,x,x]$, for $s \in S^\infty$ with image $[s] \in \mathbb{R}P^\infty = S^\infty/C_2$. The composite map $\Phi\nabla$ induces a homomorphism

$$(\Phi\nabla)^*\colon \tilde H^*(H_{2n}\,\mathbb{F}_2) \longrightarrow H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \otimes \tilde H^*(H_n;\mathbb{F}_2)\,.$$

Here $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[u]$ with $|u| = 1$, so we can write $(\Phi\nabla)^*(\iota_{2n})$ in a unique way as a sum

$$(\Phi\nabla)^*(\iota_{2n}) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i(\iota_n)\,,$$

for some well-defined classes $Sq^i(\iota_n) \in \tilde H^{n+i}(H_n;\mathbb{F}_2)$, with $0 \le i \le n$. We define $Sq^i(\iota_n) = 0$ for $i < 0$ and for $i > n$. For a general class $x \in \tilde H^n(X;\mathbb{F}_2)$, write $x = f^*(\iota_n)$ for a map $f\colon X \to H_n$, and define

$$Sq^i(x) = f^*(Sq^i(\iota_n)) \in \tilde H^{n+i}(X;\mathbb{F}_2)\,.$$

This defines an operation

$$Sq^i\colon \tilde H^n(X;\mathbb{F}_2) \longrightarrow \tilde H^{n+i}(X;\mathbb{F}_2)$$

which is obviously natural in X.

It is easy to see that $Sq^n(x) = \phi^*(x \wedge x) = x^2$ for $|x| = n$, while checking that $Sq^0(x) = x$ and $Sq^1(x) = \beta(x)$ requires more work. [[Relate $Sq^{n-1}(x)$ to $x \cup_1 x$ derived from the commuting homotopy.]]

The situation at an odd prime p is similar.

Theorem 9.3. *There are natural transformations*

$$P^i\colon \tilde H^n(X;\mathbb{F}_p) \longrightarrow \tilde H^{n+2i(p-1)}(X;\mathbb{F}_p)$$

for $i \ge 0$, of contravariant functors from based spaces to abelian groups, called Steenrod's reduced powers. These satisfy $P^0(x) = x$, $P^i(x) = x^p$ for $2i = |x|$, and $P^i(x) = 0$ for $2i > |x|$. They also satisfy the Cartan formula

$$P^k(xy) = \sum_{i+j=k} P^i(x)P^j(y)$$

and the Adem relations

$$P^aP^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j$$

for $0 < a < pb$, and

$$P^a\beta P^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j}P^j - \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j}\beta P^j$$

for $0 < a \le pb$. Here $\beta\colon \tilde H^n(X;\mathbb{F}_p) \to \tilde H^{n+1}(X;\mathbb{F}_p)$ is the Bockstein homomorphism associated to the extension $\mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$, which satisfies $\beta^2 = 0$.

n	admissible Sq^I of degree n
0	$Sq^{()}$
1	Sq^1
2	Sq^2
3	Sq^3, Sq^2Sq^1
4	Sq^4, Sq^3Sq^1
5	Sq^5, Sq^4Sq^1
6	Sq^6, Sq^5Sq^1, Sq^4Sq^2
7	$Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1$
8	$Sq^8, Sq^7Sq^1, Sq^6Sq^2, Sq^5Sq^2Sq^1$
9	$Sq^9, Sq^8Sq^1, Sq^7Sq^2, Sq^6Sq^3, Sq^6Sq^2Sq^1$
10	$Sq^{10}, Sq^9Sq^1, Sq^8Sq^2, Sq^7Sq^3, Sq^7Sq^2Sq^1, Sq^6Sq^3Sq^1$
11	$Sq^{11}, Sq^{10}Sq^1, Sq^9Sq^2, Sq^8Sq^3, Sq^8Sq^2Sq^1, Sq^7Sq^3Sq^1$

FIGURE 10. The admissible monomials at $p = 2$ in degrees ≤ 11

The first few p-primary Adem relations (for $0 < a < p$ and $b = 1$) are

$$P^aP^1 = (-1)^a \binom{p-2}{a} P^{a+1}$$

and

$$P^a\beta P^1 = (-1)^a \binom{p-1}{a} \beta P^{a+1} - (-1)^a \binom{p-2}{a-1} P^{a+1}\beta\,.$$

They imply that $(P^1)^a$ is a unit in $\mathbb{F}_p$ times P^a, for all $0 < a < p$, that $(P^1)^p = 0$, and that $P^{p-1}\beta P^1 = \beta P^p - P^p\beta$.

9.2. **The Steenrod algebra.**

Definition 9.4. Let the mod 2 Steenrod algebra $\mathscr{A} = \mathscr{A}(2)$ be the graded (associative, unital) $\mathbb{F}_2$-algebra generated by the symbols Sq^i of degree i for $i \ge 0$, subject to the relations $Sq^0 = 1$ and $Sq^aSq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j$ for $0 < a < 2b$.

For each finite sequence $I = (i_1, \dots, i_\ell)$ of non-negative integers we write

$$Sq^I = Sq^{i_1} \cdot \ldots \cdot Sq^{i_\ell}$$

for the product in $\mathscr{A}$. We say that I has length ℓ and degree $i_1 + \dots + i_\ell$.

For any based space X, the reduced mod 2 cohomology $\tilde{H}^*(X; \mathbb{F}_2)$ is naturally a left $\mathscr{A}$-module, with the action given by

$$Sq^I(x) = Sq^{i_1}(\dots(Sq^{i_\ell}(x))\dots)\,.$$

We write

$$\lambda\colon \mathscr{A} \otimes \tilde{H}^*(X; \mathbb{F}_2) \longrightarrow \tilde{H}^*(X; \mathbb{F}_2)$$

for the left module action map.

If $i_s < 2i_{s+1}$ for some $1 \le s < \ell$, then the product Sq^I can be rewritten in terms of other products Sq^J with lower moment $\sum_s sj_s < \sum_s si_s$. Likewise, if some $i_s = 0$, then the product Sq^I can be rewritten as a Sq^J of shorter length. Hence only the monomials Sq^I with I admissible, in the sense of the following definition, are needed to generate $\mathscr{A}$ additively.

Definition 9.5. $I = (i_1, \dots, i_\ell)$ is *admissible* if $i_s \ge 2i_{s+1}$ for all $1 \le s < \ell$, and if each $i_s \ge 1$. The empty sequence $I = ()$ is admissible of length 0, and $Sq^{()} = 1$.

See Figure 10 for the admissible monomials in degrees ≤ 11.

Theorem 9.6. *The admissible monomials Sq^I are linearly independent, hence form a vector space basis for the Steenrod algebra:*

$$\mathscr{A} = \mathbb{F}_2\{Sq^I \mid I \text{ admissible}\}\,.$$

This can be proved by evaluating the action of the Sq^I on the cohomology of products $X = (\mathbb{R}P^\infty)^n$ of many copies of $\mathbb{R}P^\infty$, see Steenrod and Epstein (1972, Theorem I.3.1).

n	admissible P^I of degree n
0	$P^{()}$
1	β
4	P^1
5	$\beta P^1, P^1\beta$
6	$\beta P^1\beta$
8	P^2
9	$\beta P^2, P^2\beta$
10	$\beta P^2\beta$
12	P^p
13	$\beta P^p, P^p\beta$
14	$\beta P^p\beta$
16	P^{p+1}, P^pP^1
17	$\beta P^{p+1}, P^{p+1}\beta, \beta P^pP^1, P^pP^1\beta$
18	$\beta P^{p+1}\beta, \beta P^pP^1\beta$

FIGURE 11. The admissible monomials at $p = 3$ in degrees ≤ 19

Definition 9.7. For each odd prime p, let the mod p Steenrod algebra $\mathscr{A} = \mathscr{A}(p)$ be the graded $\mathbb{F}_p$-algebra generated by the symbols P^i of degree $2i(p-1)$ for $i \geq 0$ and β of degree 1, subject to the relations $P^0 = 1$, the Adem relation for P^aP^b when $0 < a < pb$, the Adem relation for $P^a\beta P^b$ when $0 < a \leq pb$, and $\beta^2 = 0$.

For each finite sequence $I = (\epsilon_0, i_1, \epsilon_1, \dots, i_\ell, \epsilon_\ell)$, with $i_s \geq 0$ and $\epsilon_s \in \{0, 1\}$ for each s, we write

$$P^I = \beta^{\epsilon_0} P^{i_1} \beta^{\epsilon_1} \cdot \ldots \cdot P^{i_\ell} \beta^{\epsilon_\ell}$$

for the product in $\mathscr{A}$. Here $\beta^0 = 1$. The degree of I is $\epsilon_0 + 2i_1(p-1) + \epsilon_1 + \cdots + 2i_\ell(p-1) + \epsilon_\ell$.

For any based space X, the reduced mod p cohomology $\tilde{H}^*(X; \mathbb{F}_p)$ is naturally a left $\mathscr{A}$-module, with the action given by

$$P^I(x) = \beta^{\epsilon_0}(P^{i_1}(\beta^{\epsilon_1}(\dots(P^{i_\ell}(\beta^{\epsilon_\ell}(x)))\dots))) .$$

We write

$$\lambda \colon \mathscr{A} \otimes \tilde{H}^*(X; \mathbb{F}_p) \longrightarrow \tilde{H}^*(X; \mathbb{F}_p)$$

for the left module action map.

Definition 9.8. $I = (\epsilon_0, i_1, \epsilon_1 \dots, i_\ell, \epsilon_\ell)$ is *admissible* if $i_s \geq \epsilon_s + p i_{s+1}$ for all $1 \leq s < \ell$, and if each $i_s \geq 1$. The empty sequence $I = ()$ is admissible of length 0, and $P^{()} = 1$.

See Figure 11 for the admissible monomials for $p = 3$ in degrees ≤ 19.

Theorem 9.9. *The admissible monomials P^I are linearly independent, hence form a vector space basis for the Steenrod algebra:*

$$\mathscr{A} = \mathbb{F}_p\{P^I \mid I \text{ admissible}\} .$$

See Steenrod and Epstein (1972, Theorem VI.2.5).

9.3. **Indecomposables and subalgebras.**

Definition 9.10. For each prime p, let $\phi \colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ be the algebra multiplication map, let $\eta \colon \mathbb{F}_p \to \mathscr{A}$ be the unit map, and let $\epsilon \colon \mathscr{A} \to \mathbb{F}_p$ be the counit map, so that $\epsilon\eta = 1$. Let $I(\mathscr{A}) = \ker(\epsilon)$ be the *augmentation ideal*. It equals the ideal of elements of positive degree in $\mathscr{A}$. The decomposable part of $\mathscr{A}$ is the image

$$I(\mathscr{A})^2 = \operatorname{im}(\phi \colon I(\mathscr{A}) \otimes I(\mathscr{A}) \longrightarrow I(\mathscr{A}))$$

and the *indecomposable* part of $\mathscr{A}$ is the $\mathbb{F}_p$-vector space

$$Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 .$$

Theorem 9.11. *The squaring operation Sq^k is decomposable if and only if $k = 2^i$ for some $i \geq 0$, so*

$$Q(\mathscr{A}) = \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\} .$$

Hence the elements Sq^{2^i} for $i \geq 0$ form a minimal set of algebra generators for $\mathscr{A} = \mathscr{A}(2)$.

Proof. To prove that Sq^{2^i} is indecomposable, consider its action on u^{2^i} in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. We have $Sq^j(u^{2^i}) = \binom{2^i}{j} u^{2^i+j}$, which is 0 for $0 < j < 2^i$ and equals 2^{i+1} for $j = 2^i$. It follows that Sq^{2^i} cannot be written as a sum of products of positive-degree operations.

The Adem relation for $Sq^a Sq^b$ with $0 < a < 2b$ shows that Sq^{a+b} is decomposable if $\binom{b-1}{a} \not\equiv 0 \mod 2$. If k is not a power of 2, then $k = a + b$ with $0 < a < b$ and $b = 2^i$, for some i. Then $\binom{b-1}{a} \equiv 1 \mod 2$ by the case $p = 2$ of the following lemma, since $b - 1 = 1 + 2 + \cdots + 2^{i-1}$ and $\binom{1}{0} = \binom{1}{1} = 1$. □

Lemma 9.12. *Let $n = n_0 + n_1 p + \cdots + n_\ell p^\ell$ and $k = k_0 + k_1 p + \cdots + k_\ell p^\ell$, with $0 \le n_s, k_s < p$ for all s. Then*

$$\binom{n}{k} \equiv \prod_{s=0}^{\ell} \binom{n_s}{k_s} \mod p\,.$$

Proof. The coefficient of $x^k = \prod_s x^{k_s p^s}$ in

$$(1+x)^n = \prod_s (1+x)^{n_s p^s} \equiv \prod_s (1 + x^{p^s})^{n_s} \mod p$$

is the product over s of the coefficient of $x^{k_s p^s}$ in $(1 + x^{p^s})^{n_s}$. □

Theorem 9.13. *The power operation P^k is decomposable if and only if $k = p^i$ for some $i \ge 0$, so*

$$Q(\mathscr{A}) = \mathbb{F}_p\{\beta, P^{p^i} \mid i \ge 0\}\,.$$

Hence the elements β and P^{p^i} for $i \ge 0$ form a minimal set of algebra generators for $\mathscr{A} = \mathscr{A}(p)$.

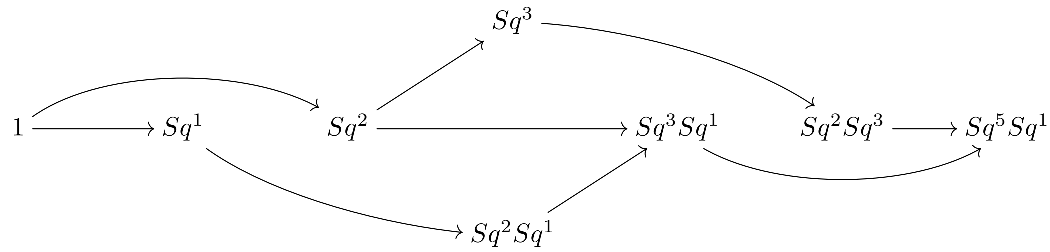
Example 9.14. For $p = 2$, the subalgebra of $\mathscr{A}$ generated by Sq^1 is the exterior algebra

$$A(0) = E(0) = \mathbb{F}_2\{1, Sq^1\}\,.$$

The subalgebra of $\mathscr{A}$ generated by Sq^1 and Sq^2 is the 8-dimensional algebra

$$A(1) = \mathbb{F}_2\{1, Sq^1, Sq^2, Sq^3, Sq^2Sq^1, Sq^3Sq^1, Sq^5 + Sq^4Sq^1, Sq^5Sq^1\}\,.$$

It can be displayed as follows, where for typographical reasons we write Sq^2Sq^3 for $Sq^5 + Sq^4Sq^1$.



The arrows of length 1 connect θ to $Sq^1\theta$, and the arrows of length 2 connect θ to $Sq^2\theta$, for $\theta \in A(1) \subset \mathscr{A}$.

[[Define $A(n)$ for general n?]]

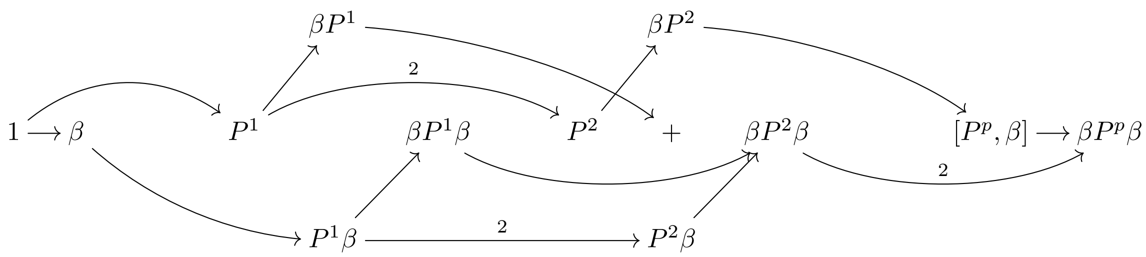
Example 9.15. For p odd, the subalgebra of $\mathscr{A}$ generated by β is the exterior algebra

$$A(0) = E(0) = \mathbb{F}_p\{1, \beta\}\,.$$

The subalgebra of $\mathscr{A}$ generated by β and P^1 is the $4p$-dimensional algebra

$$A(1) = \mathbb{F}_p\{1, \beta, P^1, \beta P^1, P^1\beta, \beta P^1\beta, \dots, P^{p-1}, \beta P^{p-1}, P^{p-1}\beta, \beta P^{p-1}\beta, P^p\beta - \beta P^p, \beta P^p\beta\}\,.$$

For $p = 3$ it can be displayed as follows, where we use the notation $[P^p, \beta] = P^p\beta - \beta P^p$.



The arrows of length 1 connect θ to $\beta\theta$, and the arrows of length 4 connect θ to $P^1\theta$, except the arrows labeled '2', which connect θ to $2P^1\theta = -P^1\theta$. The arrow from βP^1 to the symbol '+' is meant to express that P^1 applied to βP^1 is the sum $\beta P^2 + P^2\beta$.

[[Define $A(n)$ for general n?]]

9.4. Eilenberg–Mac Lane spectra.

Definition 9.16. Let $H = H\mathbb{F}_p$ denote the mod p Eilenberg–Mac Lane spectrum, with n-th space H_n an Eilenberg–Mac Lane complex of type $(\mathbb{F}_p, n)$, for each $n \geq 0$. The structure map $\sigma\colon \Sigma H_n \to H_{n+1}$ is left adjoint to a homotopy equivalence $\tilde{\sigma}\colon H_n \xrightarrow{\simeq} \Omega H_{n+1}$, for each $n \geq 0$.

There are maps $\eta_n\colon S^n \to H_n$ and pairings $\phi_{m,n}\colon H_m \wedge H_n \to H_{m+n}$, suitably compatible with the spectrum structure maps, which define a unit map $\eta\colon S \to H$ and a pairing $\phi\colon H \wedge H \to H$ that make H a homotopy commutative ring spectrum. In particular, $\phi(\eta \wedge 1) \simeq 1 \simeq \phi(1 \wedge \eta)$ and $\phi(\phi \wedge 1) \simeq \phi(1 \wedge \phi)$.

[[This multiplication can be refined to that of an E_∞ ring spectrum, or a commutative structured ring spectrum.]]

Proposition 9.17 (Whitehead)**.** *There are natural isomorphisms*

$$H_n(Y; \mathbb{F}_p) \cong \pi_n(H \wedge Y) = [S^n, H \wedge Y]$$

and

$$H^n(Y; \mathbb{F}_p) \cong \pi_{-n} F(Y, H) = [Y, \Sigma^n H]$$

for all spectra Y and integers n.

The unit map η induces the mod p Hurewicz homomorphism

$$h_1 = \eta_*\colon \pi_n(Y) \longrightarrow H_n(Y; \mathbb{F}_p)\,.$$

The multiplication ϕ induces the smash product pairing

$$\wedge = \phi_*\colon H^m(X; \mathbb{F}_p) \otimes H^n(Y; \mathbb{F}_p) \longrightarrow H^{m+n}(X \wedge Y; \mathbb{F}_p)\,.$$

Using the Serre spectral sequence for the loop-path fibration over H_n, Serre and Cartan were able to calculate $H^*(H_n; \mathbb{F}_p)$ for $p = 2$ and for p odd, respectively. Recall the fundamental class $\iota_n \in \tilde{H}^n(H_n; \mathbb{F}_p)$.

Proposition 9.18 (Serre (1953), Cartan (1954))**.** *The homomorphism*

$$\Sigma^n \mathscr{A} \longrightarrow \tilde{H}^*(H_n; \mathbb{F}_p)\,,$$

mapping $\Sigma^n\theta$ to $\theta(\iota_n)$ for each $\theta \in \mathscr{A}$, induces an isomorphism in degrees $* \leq 2n$. Hence there is an isomorphism

$$\mathscr{A} \xrightarrow{\cong} H^*(H; \mathbb{F}_p) = [H, H]_{-*}$$

of graded algebras, taking each class $\theta \in \mathscr{A}$ to its representing homotopy class of maps $H \to \Sigma^i H$, where $i = |\theta|$.

The second claim follows from the first, because of the exact sequence

$$0 \to \operatorname*{Rlim}_n H^{n+i-1}(H_n; \mathbb{F}_p) \longrightarrow H^i(H; \mathbb{F}_p) \longrightarrow \lim_n H^{n+i}(H_n; \mathbb{F}_p) \to 0\,.$$

The limit system stabilizes for $n \geq i$, so the derived limit is zero.

We collect a few lemmas relating maps of spectra to homomorphisms of cohomology groups.

Lemma 9.19. *Let*

$$K = \bigvee_u \Sigma^{n_u} H$$

be a wedge sum of suspensions of H, and suppose that K is bounded below and of finite type. Then the canonical map

$$K \xrightarrow{\simeq} \prod_u \Sigma^{n_u} H$$

is an equivalence, and the d-invariant

$$d\colon [X, K] \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(H^*(K; \mathbb{F}_p), H^*(X; \mathbb{F}_p))$$

is an isomorphism, for any spectrum X. In particular,

$$d\colon \pi_t(K) \xrightarrow{\cong} \operatorname{Hom}^t_{\mathscr{A}}(H^*(K; \mathbb{F}_p), \mathbb{F}_p)\,.$$

We discussed this earlier, in subsection 6.3.

Lemma 9.20. *Let $\pi_*(Y)$ be bounded below, with $H_*(Y;\mathbb{F}_p) = \mathbb{F}_p\{\alpha_u\}_u$ of finite type. Let $\{a_u\}_u$ be the dual basis for $H^*(Y;\mathbb{F}_p)$, with $|a_u| = |\alpha_u| = n_u$. Let $\alpha_u\colon S^{n_u} \to H \wedge Y$ and $a_u\colon Y \to \Sigma^{n_u} H$ also denote the representing homotopy classes of maps. Then the sum of the composites*

$$\Sigma^{n_u} H = H \wedge S^{n_u} \xrightarrow{1\wedge\alpha_u} H \wedge H \wedge Y \xrightarrow{\phi\wedge 1} H \wedge Y$$

is an equivalence

$$\bigvee_u \Sigma^{n_u} H \xrightarrow{\simeq} H \wedge Y$$

and the product of the composites

$$H \wedge Y \xrightarrow{1\wedge a_u} H \wedge \Sigma^{n_u} H \xrightarrow{\phi\wedge 1} H \wedge S^{n_u} = \Sigma^{n_u} H$$

is an equivalence

$$H \wedge Y \xrightarrow{\simeq} \prod_n \Sigma^{n_u} H\,.$$

Proof. The two maps induce the isomorphisms

$$\bigoplus_u \Sigma^{n_u}\mathbb{F}_p \xrightarrow{\cong} H_*(Y;\mathbb{F}_p) \xrightarrow{\cong} \prod_u \Sigma^{n_u}\mathbb{F}_p$$

at the level of homotopy groups. □

Lemma 9.21. *Let $j\colon Y \to K$ be a map of spectra, with $K \simeq \bigvee_u \Sigma^{n_u} H$ bounded below and of finite type, and suppose that $j^*\colon H^*(K;\mathbb{F}_p) \to H^*(Y;\mathbb{F}_p)$ is surjective. Then a map $f\colon X \to Y$ induces the zero homomorphism $f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ if and only if the composite $jf\colon X \to K$ is null-homotopic.*

Proof. By Lemma 9.19,

$$[X, K] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(K;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

is an isomorphism. Hence jf is null-homotopic if and only if f^*j^* is zero. By assumption j^* is surjective, so this holds if and only if f^* is zero. □

Corollary 9.22. *Let Y be bounded below, with $H_*(Y;\mathbb{F}_p)$ of finite type. Then a map $f\colon X \to Y$ induces the zero homomorphism $f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ if and only if the composite*

$$X \xrightarrow{f} Y \xrightarrow{j} H \wedge Y$$

is null-homotopic, where

$$j = \eta \wedge 1\colon Y \cong S \wedge Y \longrightarrow H \wedge Y$$

induces the mod p Hurewicz homomorphism.

Proof. We only need to verify that j^* is surjective. It is dual to the homomorphism $j_*\colon H_*(Y;\mathbb{F}_p) \longrightarrow H_*(H \wedge Y;\mathbb{F}_p)$ induced by the map

$$1 \wedge \eta \wedge 1\colon H \wedge Y \cong H \wedge S \wedge Y \longrightarrow H \wedge H \wedge Y\,.$$

The ring spectrum multiplication

$$\phi \wedge 1\colon H \wedge H \wedge Y \longrightarrow H \wedge Y$$

induces a right inverse $H_*(H \wedge Y;\mathbb{F}_p) \to H_*(Y;\mathbb{F}_p)$ to j_*, showing that j_* is (split) injective and j^* is (split) surjective. □

10. The Adams spectral sequence

10.1. **Adams resolutions.**

Definition 10.1. A spectrum Y is *bounded below* if there exists an integer N such that $\pi_*(Y) = 0$ for all $* < N$. It is of *finite type* if $\pi_*(Y)$ is finitely generated in each degree. If Y is bounded below, then it is of finite type if and only if $H_*(Y;\mathbb{Z})$ is finitely generated in each degree [[explain this?]], and we say that it is *of finite type mod p* if $H_*(Y;\mathbb{F}_p)$ is finitely generated in each degree. This is equivalent to asking that $H_*(Y;\mathbb{F}_p)$ is finite in each degree.

Hereafter we fix a prime p, and briefly write $H_*Y = H_*(Y;\mathbb{F}_p)$ and $H^*Y = H^*(Y;\mathbb{F}_p)$ for mod p homology and cohomology.

Definition 10.2. An *mod p Adams resolution* of a spectrum Y is a diagram of spectra

$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
& K^2 \arrow[ul, dashed] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where $\partial\colon K^s \to \Sigma Y^{s+1}$ for each $s \ge 0$, such that (a) each diagram

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}$$

is a homotopy cofiber sequence, (b) each spectrum K^s is a wedge sum of suspension of mod p Eilenberg–Mac Lane spectra, that is bounded below and of finite type, and (c) each map $j\colon Y^s \to K^s$ induces a surjection $j^*\colon H^*K^s \to H^*Y^s$ in mod p cohomology.

Writing $K^s = \bigvee_u \Sigma^{n_u} H$, the finiteness condition in (b) is equivalent to asking that $\{u \mid n_u \le N\}$ is finite for each integer N. By induction on s it implies that each Y^s is bounded below and of finite type mod p. In view of the long exact sequence

$$\cdots \to H^*(\Sigma Y^{s+1}) \xrightarrow{\partial^*} H^*K^s \xrightarrow{j^*} H^*Y^s \xrightarrow{i^*} H^*Y^{s+1} \to \dots,$$

condition (c) is equivalent to asking that $i^*\colon H^*Y^s \to H^*Y^{s+1}$ is zero, or equivalently, that $\partial^*\colon H^*(\Sigma Y^{s+1}) \to H^*K^s$ is injective, for each $s \ge 0$. By the universal coefficient theorem, these conditions are also equivalent to asking that $i_*\colon H_*Y^{s+1} \to H_*Y^s$ is zero, that $j_*\colon H_*Y^s \to H_*K^s$ is injective, or that $\partial_*\colon H_*K_s \to H_*(\Sigma Y^{s+1})$ is surjective, for each $s \ge 0$.

Lemma 10.3. *If Y is bounded below and of finite type mod p, then it admits an Adams resolution.*

Proof. Starting with $Y^0 = Y$, suppose that Y^s has been constructed, for some $s \ge 0$, as a bounded below spectrum of finite type mod p. Let $K^s = H \wedge Y^s$, and let $j = \eta \wedge 1$ be the map

$$Y^s = S \wedge Y^s \xrightarrow{\eta \wedge 1} H \wedge Y^s = K^s .$$

By Lemma 9.20, $K^s \simeq \bigvee_u \Sigma^{n_u} H$ is a wedge sum of suspensions of H. Here

$$\pi_* K^s = H_* Y^s = \bigoplus_u \Sigma^{n_u} \mathbb{F}_p$$

is bounded below, and

$$H_*K^s \cong H_*H \otimes H_*Y^s$$

is a tensor product of bounded below $\mathbb{F}_p$-vector spaces of finite type, hence is again bounded below and of finite type. The map j induces a surjection $j^*\colon H^*K^s \to H^*Y^s$ by the proof of Corollary 9.22: It suffices to prove that $j_*\colon H_*Y^s \to H_*K^s$ is injective, but this is the homomorphism induced on homotopy by the map

$$1 \wedge j = 1 \wedge \eta \wedge 1\colon H \wedge Y^s = H \wedge S \wedge Y^s \longrightarrow H \wedge H \wedge Y^s = H \wedge K^s$$

which is split by the map

$$\phi \wedge 1\colon H \wedge H \wedge Y^s \longrightarrow H \wedge Y^s .$$

Let Y^{s+1} be the homotopy fiber of $j\colon Y^s \to K^s$, and let $i\colon Y^{s+1} \to Y^s$ be the canonical map from the homotopy fiber. Then there is a homotopy (co)fiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma K^{s+1} ,$$

which identifies the homotopy cofiber of j with the suspension of the homotopy fiber of j. By the long exact sequences in homotopy and mod p homology, it follows that Y^{s+1} is bounded below and of finite type mod p. Now continue the construction by induction. $\square$

Let $\bar{H}$ be the homotopy cofiber of the unit map $\eta\colon S \to H$, so that there is a homotopy cofiber sequence

$$\Sigma^{-1}\bar{H} \longrightarrow S \xrightarrow{\eta} H \longrightarrow \bar{H} . \tag{4}$$

The homotopy cofiber constructed in the proof above can then be written as

$$\Sigma^{-1}\bar{H} \wedge Y^s \longrightarrow S \wedge Y^s \xrightarrow{\eta \wedge 1} H \wedge Y^s \longrightarrow \bar{H} \wedge Y^s .$$

By induction, we therefore have

$$
\begin{aligned}
Y^s &= (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y \\
K^s &= H \wedge (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y
\end{aligned}
$$

for all $s \geq 0$, with $j = \eta \wedge 1$.

Definition 10.4. The *canonical Adams resolution* of Y is the diagram

$$
\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\bar{H} \wedge Y \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[r, equal] \arrow[d, "j"] & Y \\
& H \wedge (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[ul, dashed] & H \wedge \Sigma^{-1}\bar{H} \wedge Y \arrow[ul, dashed, "\partial"] & H \wedge Y \arrow[ul, dashed, "\partial"] &
\end{tikzcd}
$$

constructed in the proof above.

By the Künneth theorem,

$$
\begin{aligned}
H^*Y^s &\cong H^*(\Sigma^{-1}\bar{H})^{\otimes s} \otimes H^*Y = (\Sigma^{-1}I(\mathscr{A}))^{\otimes s} \otimes H^*Y \\
H^*K^s &\cong H^*H \otimes H^*(\Sigma^{-1}\bar{H})^{\otimes s} \otimes H^*Y = \mathscr{A} \otimes (\Sigma^{-1}I(\mathscr{A}))^{\otimes s} \otimes H^*Y
\end{aligned}
$$

for each $s \geq 0$, with

$$
j^* \colon H^*K^s \cong \mathscr{A} \otimes H^*Y^s \xrightarrow{\epsilon \otimes 1} \mathbb{F}_p \otimes H^*Y^s \cong H^*Y^s
$$

induced by the augmentation $\epsilon \colon \mathscr{A} \to \mathbb{F}_p$ of the Steenrod algebra, and

$$
\partial^* \colon H^*\Sigma Y^{s+1} = I(\mathscr{A}) \otimes H^*Y^s \longrightarrow \mathscr{A} \otimes H^*Y^s = H^*K^s
$$

is induced by the inclusion $I(\mathscr{A}) \subset \mathscr{A}$. Note also that the canonical Adams resolution in natural in the spectrum Y.

The added generality of arbitrary Adams resolutions, as opposed to the canonical ones, will be useful when we consider convergence questions and multiplicative structure.

Lemma 10.5. *For any Adams resolution of Y, let*

$$
\begin{aligned}
P_s &= H^*(\Sigma^s K^s) \\
\partial_s &= \partial^* j^* \colon H^*(\Sigma^s K^s) \to H^*(\Sigma^{s-1} K^{s-1})
\end{aligned}
$$

and $\epsilon = j^* \colon H^*K^0 \to H^*Y$. Then

$$
\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*Y \to 0
$$

is a resolution of H^*Y by free $\mathscr{A}$-modules, each of which is bounded below and of finite type.

With this indexing, the homomorphisms ∂_s and ϵ all preserve the cohomological grading of P_s and H^*Y, which we call the *internal degree* and usually denote by t.

Proof. By assumption (b), $K^s \simeq \bigvee_u \Sigma^{n_u} H$ with $\{u \mid n_u \leq N\}$ finite for each N, so

$$
H^*(K^s) \cong \prod_u \Sigma^{n_u} H^*(H) = \prod_u \Sigma^{n_u} \mathscr{A} = \bigoplus_u \Sigma^{n_u} \mathscr{A}
$$

is a bounded below free $\mathscr{A}$-module of finite type, for each $s \geq 0$. Hence each $P_s = H^*(\Sigma^s K^s)$ is a bounded below free $\mathscr{A}$-module of finite type.

By assumption (c), the long exact sequence in cohomology of each cofiber sequence

$$
Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}
$$

breaks up into short exact sequences of $\mathscr{A}$-modules

$$
0 \to H^*(\Sigma Y^{s+1}) \xrightarrow{\partial^*} H^*(K^s) \xrightarrow{j^*} H^*(Y^s) \to 0 \,.
$$

These splice together to a long exact sequence

$$
\begin{tikzcd}
H^*(\Sigma^3 Y^3) \arrow[dr, hook] & & H^*(\Sigma^2 Y^2) \arrow[dr, hook, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, hook, "\partial^*"] & & H^*Y \\
\dots \arrow[u, two heads] \arrow[r] & H^*(\Sigma^2 K^2) \arrow[ur, two heads, "j^*"] \arrow[rr, "\partial_2"'] & & H^*(\Sigma K^1) \arrow[ur, two heads, "j^*"] \arrow[rr, "\partial_1"'] & & H^*K^0 \arrow[ur, two heads, "j^*"] &
\end{tikzcd}
$$

along the lower edge of this commutative diagram of graded $\mathscr{A}$-modules and degree-preserving homomorphisms. Alternatively, this diagram may be displayed as follows:

$$\begin{tikzcd}
\cdots \arrow[dr] & & H^*(\Sigma^2 Y^2) \arrow[dr, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, "\partial^*"] & & H^*Y \\
\cdots \arrow[r] & H^*(\Sigma^2 K^2) \arrow[ur, "j^*"] \arrow[rr, "\partial_2"'] & & H^*(\Sigma K^1) \arrow[ur, "j^*"] \arrow[rr, "\partial_1"'] & & H^*K^0 \arrow[ur, "j^*"'] &
\end{tikzcd}$$

Here $\mathrm{im}(\partial_{s+1}) = \mathrm{im}(\partial^*) = \ker(j^*) = \ker(\partial_s)$ as subgroups of $H^*(\Sigma^s K^s)$, for $s \geq 1$, since j^* is surjective and ∂^* is injective, $\mathrm{im}(\partial_1) = \mathrm{im}(\partial^*) = \ker(j^*)$ as subgroups of H^*K^0, since j^* is surjective, and $j^* \colon H^*K^0 \longrightarrow H^*Y$ is already known to be surjective. Hence $\epsilon \colon P_* \to H^*Y$ is a free $\mathscr{A}$-module resolution of H^*Y. □

We say that the Adams resolution $\{Y^{s+1} \to Y^s \to K^s \to \Sigma Y^{s+1}\}_s$ of Y is a *realization* of the free $\mathscr{A}$-module resolution $P_* = (P_s, \partial_s)$ of H^*Y. It is induced by passage to cohomology from the diagram

$$\cdots \leftarrow \Sigma^2 K^2 \xleftarrow{j\partial} \Sigma K^1 \xleftarrow{j\partial} K^0 \xleftarrow{j} Y\,,$$

where each composite of two maps is null-homotopic. In the case of the canonical resolution, this diagram appears as follows:

$$\cdots \leftarrow H \wedge (\bar{H})^{\wedge 2} \wedge Y \xleftarrow{j\partial} H \wedge \bar{H} \wedge Y \xleftarrow{j\partial} H \wedge Y \xleftarrow{j} Y\,.$$

The associated free resolution has the form

$$\cdots \to \mathscr{A} \otimes I(\mathscr{A})^{\otimes 2} \otimes H^*Y \xrightarrow{\partial_2} \mathscr{A} \otimes I(\mathscr{A}) \otimes H^*Y \xrightarrow{\partial_1} \mathscr{A} \otimes H^*Y \xrightarrow{\epsilon} H^*Y \to 0\,.$$

Here

$$\begin{aligned}
\epsilon(\theta_0 \otimes y) &= \epsilon(\theta_0)y \\
\partial_1(\theta_0 \otimes \theta_1 \otimes y) &= \epsilon(\theta_0)\theta_1 \otimes y \\
\partial_2(\theta_0 \otimes \theta_1 \otimes \theta_2 \otimes y) &= \epsilon(\theta_0)\theta_1 \otimes \theta_2 \otimes y
\end{aligned}$$

and so on, where $\theta_0 \in \mathscr{A}$, $\theta_1, \dots, \theta_s \in I(\mathscr{A})$, $y \in H^*Y$, and $\epsilon(\theta_0)\theta_1$ is viewed as an element of $\mathscr{A}$.

We shall return to this complex later, in the context of the bar resolution of the $\mathscr{A}$-module H^*Y. The complex above is isomorphic to the bar resolution, but not equal to it. Note that each term $\mathscr{A} \otimes I(\mathscr{A})^{\otimes} \otimes H^*Y$ has the "diagonal" $\mathscr{A}$-module structure, prescribed by the Künneth theorem, which is not the same as the "induced" $\mathscr{A}$-module structure where $\mathscr{A}$ only acts on the leftmost tensor factor. Nonetheless, each term is free as an $\mathscr{A}$-module, by the argument given in Lemma 9.20 for the existence of a wedge sum decomposition of $K^s = H \wedge Y^s$.

10.2. **The Adams E_2-term.** We follows Adams (1958), using the spectrum level reformulation that appears in Moss (1968).

Let Y be a spectrum such that $\pi_*(Y)$ is bounded below and H_*Y is of finite type. Consider any Adams resolution

$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
& K^2 \arrow[ul, dashed] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

of Y. Applying homotopy groups, we get an unrolled exact couple of Adams type

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y^2) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^1) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^0) \arrow[r, equal] \arrow[d, "j_*"] & \pi_*(Y) \\
& \pi_*(K^2) \arrow[ul] & \pi_*(K^1) \arrow[ul, "\partial_*"] & \pi_*(K^0) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where

$$\begin{aligned}
A^s &= \pi_*(Y^s) \\
E^s &= \pi_*(K^s)
\end{aligned}$$

are graded abelian groups, for each filtration degree $s \geq 0$, with components

$$A^{s,t} = \pi_{t-s}(Y^s)$$
$$E^{s,t} = \pi_{t-s}(K^s)$$

in each internal degree t. By convention, $A^s = A^0$ and $E^s = 0$ for $s < 0$. The homomorphisms $i = i_*$ and $j = j_*$ have degree 0, and $k = \partial_*$ has degree -1. There is an associated spectral sequence $(E_r, d_r)_r = (E_r^{*,*}, d_r^{*,*})_r$ of Adams type, with

$$E_1^{s,t} = \pi_{t-s}(K^s)$$

and

$$d_1^{s,t} = (j\partial)_* \colon \pi_{t-s}(K^s) \longrightarrow \pi_{t-s}(\Sigma K^{s+1}) = \pi_{t-s-1}(K^{s+1})$$

for $s \geq 0$. The d_r-differentials have bidegree $(r, r-1)$: If $x \in \pi_{t-s}(K^s) = E_1^{s,t}$ is such that $\partial_*(x) = i_*^{r-1}(y)$ for some $y \in \pi_{t-s-1}(Y^{s+r})$, then $d_r([x]) = [j_*(y)]$ is the class of $j_*(y) \in \pi_{t-s-1}(K^{s+r}) = E_1^{s+r,t+r-1}$.

This is the *Adams spectral sequence for* Y, sometimes denoted $E_r(Y) = E_r^{*,*}(Y)$. We shall be interested in the possible convergence of this spectral sequence to the achieved colimit

$$G = \pi_*(Y) = \operatorname*{colim}_s \pi_*(Y^s)\,,$$

filtered by the image groups

$$F^s = F^s\pi_*(Y) = \operatorname{im}(i_*^s \colon \pi_*(Y^s) \to \pi_*(Y))\,.$$

This is an exhaustive and descending filtration:

$$\cdots \subset F^{s+1} \subset F^s \subset \cdots \subset F^1 \subset F^0 = \pi_*(Y)\,.$$

We recall that, by definition, the spectral sequence is conditionally convergent to $\pi_*(Y)$ if $\lim_s \pi_*(Y^s) = 0$ and $\operatorname{Rlim}_s \pi_*(Y^s) = 0$.

Definition 10.6. An element in $E_r^{s,t}$ is said to be of *filtration* s, *total degree* $t-s$ and *internal degree* t. An element in $F^s \subset \pi_*(Y)$ is said to be of *Adams filtration* $\geq s$.

[[EDIT FROM HERE]]

A class in $\pi_*(Y)$ has Adams filtration 0 if it is detected by the d-invariant in $\pi_*(K^0)$, i.e., if it has non-zero mod 2 Hurewicz image. If the Hurewicz image is zero, then the class lifts to $\pi_*(Y^1)$. Then it has Adams filtration 1 if the lift is detected in $\pi_*(K^1)$, i.e., if the lift has non-zero mod 2 Hurewicz image. If also that Hurewicz image is zero, then the class lifts to $\pi_*(Y^2)$. And so on.

Theorem 10.7. *The E_2-term of the Adams spectral sequence of Y is*

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2)\,.$$

In particular, it is independent of the choice of Adams resolution.

Proof. The Adams E_1-term and d_1-differential is the complex

$$\ldots \longleftarrow \pi_*(\Sigma^2 K^2) \xleftarrow{(j\partial)_*} \pi_*(\Sigma K^1) \xleftarrow{(j\partial)_*} \pi_*(K^0) \longleftarrow 0$$

of graded abelian groups. It maps isomorphically, under the d-invariant $\pi_*(K) \to \operatorname{Hom}_{\mathscr{A}}(H^*(K), \mathbb{F}_2)$, to the complex

$$\ldots \longleftarrow \operatorname{Hom}_{\mathscr{A}}(H^*(\Sigma^2 K^2), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \operatorname{Hom}_{\mathscr{A}}(H^*(\Sigma K^1), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \operatorname{Hom}_{\mathscr{A}}(H^*(K^0), \mathbb{F}_2) \longleftarrow 0$$

where $((j\partial)^*)^* = \operatorname{Hom}_{\mathscr{A}}((j\partial)^*, 1)$. With the notation of the previous subsection, this complex can be rewritten as

$$\ldots \longleftarrow \operatorname{Hom}_{\mathscr{A}}(P_2, \mathbb{F}_2) \xleftarrow{\partial_2^*} \operatorname{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \xleftarrow{\partial_1^*} \operatorname{Hom}_{\mathscr{A}}(P_0, \mathbb{F}_2) \longleftarrow 0\,.$$

This is the complex $\operatorname{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ obtained by applying the functor $\operatorname{Hom}_{\mathscr{A}}(-, \mathbb{F}_2)$ to the resolution $\epsilon \colon P_* \to H^*(Y)$ of $H^*(Y)$ by free $\mathscr{A}$-modules. Its cohomology groups are by definition, the Ext-groups

$$\operatorname{Ext}_{\mathscr{A}}^s(H^*(Y), \mathbb{F}_2) = H^s(\operatorname{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2))\,.$$

At the same time, the cohomology of the E_1-term of a spectral sequence is the E_2-term. Hence

$$E_2^s \cong \operatorname{Ext}_{\mathscr{A}}^s(H^*(Y), \mathbb{F}_2)\,.$$

As regards the internal grading, $E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s)$ corresponds to the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \Sigma^t \mathbb{F}_2$. This is the same as the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \mathbb{F}_2$ that lower the cohomological degrees by t. We denote the group of these homomorphisms by $\mathrm{Hom}^t_{\mathscr{A}}(H^*(\Sigma^s K^s), \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}}(P_s, \mathbb{F}_2)$, and similarly for the derived functors. With these conventions, $E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y), \mathbb{F}_2)$, as asserted. □

We are particularly interested in the special case $Y = S$, with $H^*(S) = \mathbb{F}_2$ and $\pi_*(S) = \pi_*^S$ equal to the stable homotopy groups of spheres.

Theorem 10.8. *The Adams spectral sequence for S has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)\,.$$

On the other hand, we can also generalize (following Brinkmann (1968)). Let X be any spectrum and apply the functor $[X, -]_*$ to an Adams resolution of Y. This yields an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & {[X,Y^2]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X,Y^1]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X,Y^0]_*} \arrow[r, equal] \arrow[d, "j_*"] & {[X,Y]_*} \\
 & {[X,K^2]_*} \arrow[ul, "\partial_*"] & {[X,K^1]_*} \arrow[ul, "\partial_*"] & {[X,K^0]_*} \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where $A^s = [X, Y^s]_*$, $E^s = [X, K^s]_*$ are graded abelian groups, i_* and j_* have degree 0, and ∂_* has degree -1. There is an associated spectral sequence with

$$E_1^{s,t} = [X, K^s]_{t-s}$$

and

$$d_1^{s,t} = (j\partial)_* \colon [X, K^s]_{t-s} \to [X, K^{s+1}]_{t-s-1}\,.$$

The d_r-differentials have bidegree $(r, r-1)$. The expected abutment is the graded abelian group $G = [X, Y]_*$, filtered by the image groups $F^s = \mathrm{im}(i^s_* \colon [X, Y^s]_* \to [X, Y]_*)$.

Theorem 10.9. *The Adams spectral sequence $\{E_r(X,Y) = E_r^{*,*}(X,Y)\}_r$ of maps $X \to Y$, with expected abutment $[X,Y]_*$, has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y), H^*(X))\,.$$

The proof is the same as for $X = S$, replacing $\mathbb{F}_2$ by $H^*(X)$ in the right hand argument of all $\mathrm{Hom}_{\mathscr{A}}$- and $\mathrm{Ext}^s_{\mathscr{A}}$-groups. [[ETC]]
[[EDIT TO HERE]]

10.3. **A minimal resolution at $p = 2$.** To compute the Adams E_2-term for the sphere spectrum, at $p = 2$, we need to compute

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2), \delta)$$

for any free $\mathscr{A}$-module resolution

$$\dots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \to \dots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2$, where $\delta = \mathrm{Hom}_{\mathscr{A}}(\partial, 1)$. We now construct such a free resolution P_* by hand, in a small range of degrees. We start in filtration degree $s = 0$, and calculate up to some internal degree t. Then we proceed with filtration degree $s = 1$, calculate up to the same internal degree t, and then repeat for larger s.

We need a surjection $\epsilon \colon P_0 \to \mathbb{F}_2$, so we let

$$P_0 = \mathscr{A}\{g_{0,0}\} = \mathscr{A}$$

be the free $\mathscr{A}$-module on a generator $g_{0,0} = 1$ in internal degree 0.

In this filtration a single generator suffices, but in higher filtrations, infinitely many generators will be needed. We will denote the generators in filtration s by $g_{s,0}$, $g_{s,1}$, $g_{s,2}$ and so on, in order of increasing (or more precisely, non-decreasing) internal degrees.

10.3.1. *Filtration* $s = 1$. Next, we need a surjection $\partial_1 \colon P_1 \to \ker(\epsilon)$, where $\ker(\epsilon) = I(\mathscr{A})$. An additive basis for $\ker(\epsilon)$ is given by the classes Sq^I for I admissible of length ≥ 1. We listed these monomials in internal degree $t \leq 11$ in Figure 10.

Starting in the lowest degree, we first need a generator $g_{1,0} = [Sq^1]$ in internal degree 1 that is mapped by ∂_1 to Sq^1. The free summand $\mathscr{A}\{g_{1,0}\}$ of P_1 will then map by ∂_1 to all sums of classes of the form $Sq^I \circ Sq^1$ with I admissible. In view of the Adem relation $Sq^1 \circ Sq^1 = 0$, this image consists of all sums of classes of the form Sq^J, where $J = (j_1, \dots, j_\ell)$ is admissible and $j_\ell = 1$. See the left hand column of Figure 12.

The lowest-degree class in $\ker(\epsilon)$ that is not in the image from $\mathscr{A}\{g_{1,0}\}$ is Sq^2, in internal degree 2, so we must add a second generator $g_{1,1} = [Sq^2]$ to P_1, mapping under ∂_1 to Sq^2. Using the Adem relations, we can compute the image $Sq^I \circ Sq^2$ of each basis element $Sq^I g_{1,1}$ of $\mathscr{A}\{g_{1,1}\}$. See the right hand column of Figure 12.

The images of $Sq^2 g_{1,0}$ and $Sq^1 g_{1,1}$, namely $Sq^2 Sq^1$ and Sq^3, generate $\ker(\epsilon)$ in internal degree 3, but Sq^4 is not in the image from $\mathscr{A}\{g_{1,0}, g_{1,1}\}$, so we must add a third generator $g_{1,2} = [Sq^4]$ to P_1, mapping to $Sq^4 g_{0,0}$ under ∂_1. See the left hand column of Figure 13.

All classes in degree $t \leq 7$ of $\ker(\epsilon)$ are then hit by ∂_1 on $\mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}\}$, but $Sq^8 g_{0,0}$ is not in that image. We must therefore add a fourth generator $g_{1,3} = [Sq^8]$ to P_1, mapping to Sq^8. See the right hand column of Figure 13.

In general, we must add enough $\mathscr{A}$-module generators $g_{1,i}$ to P_1 so that their images $\partial_1(g_{1,i})$ generate the $\mathbb{F}_2$-vector space $Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2$ of algebra indecomposables in the augmented algebra $\mathscr{A}$. This is necessary, since if $\partial_1 \colon P_1 \to \ker(\epsilon)$ is surjective, then so is its composite with the canonical map $\ker(\epsilon) = I(\mathscr{A}) \to Q(\mathscr{A})$. It is also a sufficient condition, because of $\partial_1 \colon P_1 \to \ker(\epsilon)$ is surjective below degree t and $P_1 \to Q(\mathscr{A})$ is surjective in degree t, then any chosen class in $I(\mathscr{A})$ of degree t is congruent modulo $I(\mathscr{A})^2$ to a class in the image of ∂_1. Any class in $I(\mathscr{A})^2$ is a sum of products of classes of degree less than t, hence is also in the image of ∂_1, by the assumption that ∂_1 is surjective below degree t. Thus the chosen class in $I(\mathscr{A})$ is also in the image of ∂_1. [[State this as a lemma?]]

By Theorem 9.11, the Sq^{2^i} for $i \geq 0$ give a basis for $Q(\mathscr{A})$, hence the minimal choice of a free $\mathscr{A}$-module mapping onto $\ker(\epsilon)$ is

$$P_1 = \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, \dots\}$$

with $g_{1,i} = [Sq^{2^i}]$ in internal degree 2^i, for each $i \geq 0$. Here $g_{1,4}$ is in degree 16, hence the first four generators suffice in our smaller range of degrees.

[[Comment on how $\partial_1([\theta]) = \theta$, and how P_* relates to the bar resolution.]]

10.3.2. *Filtration* $s = 2$. To continue, we need a surjection $\partial_2 \colon P_2 \to \ker(\partial_1)$. First we go through the linear algebra exercise of computing an additive basis for $\ker(\partial_1)$. The result is shown in Figure 14.

The class in lowest degree in $\ker(\partial_1)$ is $Sq^1[Sq^1]$, which corresponds to the Adem relation $Sq^1 Sq^1 = 0$. We put a first generator $g_{2,0}$ of degree 2 in P_2, with $\partial_2(g_{2,0}) = Sq^1[Sq^1]$. See the left hand column of Figure 15.

The first class in $\ker(\partial_1)$ that is not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}\}$ is $Sq^3[Sq^1] + Sq^2[Sq^2]$, which corresponds to the Adem relation $Sq^2 Sq^2 = Sq^3 Sq^1$. We add a second generator $g_{2,1}$ to P_2, in degree 4, with $\partial_2(g_{2,1}) = Sq^3[Sq^1] + Sq^2[Sq^2]$, and compute the value of $\partial_2(Sq^I g_{2,1}) = Sq^I(Sq^3[Sq^1] + Sq^2[Sq^2])$ in $\ker(\partial_1) \subset P_1$ for each admissible I, using the Adem relations. See the right hand column of Figure 15.

The lowest degree class not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$ is $Sq^4[Sq^1] + Sq^2 Sq^1[Sq^2] + Sq^1[Sq^4]$, in degree 5. It corresponds to the Adem relation $Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1$, in view of the identities $Sq^1 Sq^2 = Sq^3$ and $Sq^1 Sq^4 = Sq^5$. We add a third generator $g_{2,2}$ to P_2, with $\partial_2(g_{2,2}) = Sq^4[Sq^1] + Sq^2 Sq^1[Sq^2] + Sq^1[Sq^4]$, and compute $\partial_2(Sq^I g_{2,2})$, as before. See Figure 16. [[At this point we deviate from the minimal resolution chosen by `ext`, which replaces $Sq^2 Sq^1[Sq^2]$ with $Sq^{(0,1)}[Sq^2] = (Sq^3 + Sq^2 Sq^1)[Sq^2]$.]]

The first class in $\ker(\partial_1)$ not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}, g_{2,2}\}$ is $Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$. We add a fourth generator $g_{2,3}$ to P_2 in degree 8, corresponding to the Adem relation $Sq^4 Sq^4 = Sq^7 Sq^1 +$

$$
\begin{aligned}
g_{1,0} = [Sq^1] &\xmapsto{\partial_1} Sq^1 \\
Sq^1[Sq^1] &\longmapsto 0 \\
Sq^2[Sq^1] &\longmapsto Sq^2Sq^1 \\
Sq^3[Sq^1] &\longmapsto Sq^3Sq^1 \\
Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^4[Sq^1] &\longmapsto Sq^4Sq^1 \\
Sq^3Sq^1[Sq^1] &\longmapsto 0 \\
Sq^5[Sq^1] &\longmapsto Sq^5Sq^1 \\
Sq^4Sq^1[Sq^1] &\longmapsto 0 \\
Sq^6[Sq^1] &\longmapsto Sq^6Sq^1 \\
Sq^5Sq^1[Sq^1] &\longmapsto 0 \\
Sq^4Sq^2[Sq^1] &\longmapsto Sq^4Sq^2Sq^1 \\
Sq^7[Sq^1] &\longmapsto Sq^7Sq^1 \\
Sq^6Sq^1[Sq^1] &\longmapsto 0 \\
Sq^5Sq^2[Sq^1] &\longmapsto Sq^5Sq^2Sq^1 \\
Sq^4Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^8[Sq^1] &\longmapsto Sq^8Sq^1 \\
Sq^7Sq^1[Sq^1] &\longmapsto 0 \\
Sq^6Sq^2[Sq^1] &\longmapsto Sq^6Sq^2Sq^1 \\
Sq^5Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^9[Sq^1] &\longmapsto Sq^9Sq^1 \\
Sq^8Sq^1[Sq^1] &\longmapsto 0 \\
Sq^7Sq^2[Sq^1] &\longmapsto Sq^7Sq^2Sq^1 \\
Sq^6Sq^3[Sq^1] &\longmapsto Sq^6Sq^3Sq^1 \\
Sq^6Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^{10}[Sq^1] &\longmapsto Sq^{10}Sq^1 \\
Sq^9Sq^1[Sq^1] &\longmapsto 0 \\
Sq^8Sq^2[Sq^1] &\longmapsto Sq^8Sq^2Sq^1 \\
Sq^7Sq^3[Sq^1] &\longmapsto Sq^7Sq^3Sq^1 \\
Sq^7Sq^2Sq^1[Sq^1] &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{1,1} = [Sq^2] &\xmapsto{\partial_1} Sq^2 \\
Sq^1[Sq^2] &\longmapsto Sq^3 \\
Sq^2[Sq^2] &\longmapsto Sq^3Sq^1 \\
Sq^3[Sq^2] &\longmapsto 0 \\
Sq^2Sq^1[Sq^2] &\longmapsto Sq^5 + Sq^4Sq^1 \\
Sq^4[Sq^2] &\longmapsto Sq^4Sq^2 \\
Sq^3Sq^1[Sq^2] &\longmapsto Sq^5Sq^1 \\
Sq^5[Sq^2] &\longmapsto Sq^5Sq^2 \\
Sq^4Sq^1[Sq^2] &\longmapsto Sq^5Sq^2 \\
Sq^6[Sq^2] &\longmapsto Sq^6Sq^2 \\
Sq^5Sq^1[Sq^2] &\longmapsto 0 \\
Sq^4Sq^2[Sq^2] &\longmapsto Sq^5Sq^2Sq^1 \\
Sq^7[Sq^2] &\longmapsto Sq^7Sq^2 \\
Sq^6Sq^1[Sq^2] &\longmapsto Sq^6Sq^3 \\
Sq^5Sq^2[Sq^2] &\longmapsto 0 \\
Sq^4Sq^2Sq^1[Sq^2] &\longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1 \\
Sq^8[Sq^2] &\longmapsto Sq^8Sq^2 \\
Sq^7Sq^1[Sq^2] &\longmapsto Sq^7Sq^3 \\
Sq^6Sq^2[Sq^2] &\longmapsto Sq^6Sq^3Sq^1 \\
Sq^5Sq^2Sq^1[Sq^2] &\longmapsto Sq^9Sq^1 + Sq^7Sq^2Sq^1 \\
Sq^9[Sq^2] &\longmapsto Sq^9Sq^2 \\
Sq^8Sq^1[Sq^2] &\longmapsto Sq^8Sq^3 \\
Sq^7Sq^2[Sq^2] &\longmapsto Sq^7Sq^3Sq^1 \\
Sq^6Sq^3[Sq^2] &\longmapsto 0 \\
Sq^6Sq^2Sq^1[Sq^2] &\longmapsto Sq^9Sq^2 + Sq^8Sq^3 + Sq^7Sq^3Sq^1
\end{aligned}
$$

FIGURE 12. ∂_1 on $\mathscr{A}\{g_{1,0}, g_{1,1}\} \subset P_1$

Sq^6Sq^2, and let $\partial_2(g_{2,3}) = Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$.

$$
\begin{aligned}
g_{2,3} &\xmapsto{\partial_2} Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4] \\
Sq^1 g_{2,3} &\longmapsto Sq^7[Sq^2] + Sq^5[Sq^4] \\
Sq^2 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^1] + Sq^7Sq^1[Sq^2] + (Sq^6 + Sq^5Sq^1)[Sq^4] \\
Sq^3 g_{2,3} &\longmapsto Sq^9Sq^1[Sq^1] + Sq^7[Sq^4] \\
Sq^2Sq^1 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]
\end{aligned}
$$

This still leaves $Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$ not in the image of ∂_2, so we add a fifth generator $g_{2,4}$ in degree 9, corresponding to the Adem relation $Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2$,

$$
\begin{aligned}
g_{1,2} = [Sq^4] &\xmapsto{\partial_1} Sq^4 \\
Sq^1[Sq^4] &\longmapsto Sq^5 \\
Sq^2[Sq^4] &\longmapsto Sq^6 + Sq^5Sq^1 \\
Sq^3[Sq^4] &\longmapsto Sq^7 \\
Sq^2Sq^1[Sq^4] &\longmapsto Sq^6Sq^1 \\
Sq^4[Sq^4] &\longmapsto Sq^7Sq^1 + Sq^6Sq^2 \\
Sq^3Sq^1[Sq^4] &\longmapsto Sq^7Sq^1 \\
Sq^5[Sq^4] &\longmapsto Sq^7Sq^2 \\
Sq^4Sq^1[Sq^4] &\longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 \\
Sq^6[Sq^4] &\longmapsto Sq^7Sq^3 \\
Sq^5Sq^1[Sq^4] &\longmapsto Sq^9Sq^1 \\
Sq^4Sq^2[Sq^4] &\longmapsto Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1 \\
Sq^7[Sq^4] &\longmapsto 0 \\
Sq^6Sq^1[Sq^4] &\longmapsto Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^5Sq^2[Sq^4] &\longmapsto Sq^{11} + Sq^9Sq^2 \\
Sq^4Sq^2Sq^1[Sq^4] &\longmapsto Sq^{10}Sq^1 + Sq^8Sq^2Sq^1
\end{aligned}
$$

$$
\begin{aligned}
g_{1,3} = [Sq^8] &\xmapsto{\partial_1} Sq^8 \\
Sq^1[Sq^8] &\longmapsto Sq^9 \\
Sq^2[Sq^8] &\longmapsto Sq^{10} + Sq^9Sq^1 \\
Sq^3[Sq^8] &\longmapsto Sq^{11} \\
Sq^2Sq^1[Sq^8] &\longmapsto Sq^{10}Sq^1
\end{aligned}
$$

FIGURE 13. ∂_1 on $\mathscr{A}\{g_{1,2}, g_{1,3}\} \subset P_1$

and let $\partial_2(g_{2,4}) = Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$.

$$
\begin{aligned}
g_{2,4} &\xmapsto{\partial_2} Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8] \\
Sq^1g_{2,4} &\longmapsto Sq^9[Sq^1] + Sq^5Sq^1[Sq^4] \\
Sq^2g_{2,4} &\longmapsto (Sq^{10} + Sq^9Sq^1)[Sq^1] + (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4] + Sq^2Sq^1[Sq^8]
\end{aligned}
$$

Finally we need a sixth generator, $g_{2,5}$ in degree 10, mapping to $Sq^7Sq^2[Sq^1]+Sq^8[Sq^2]+Sq^4Sq^2[Sq^4]+Sq^2[Sq^8]$. It derives from the Adem relations for Sq^2Sq^8 and for Sq^4Sq^6, using the Adem relation for Sq^2Sq^4. [[Can we pick a different generator that corresponds to just a single Adem relation?]]

$$
\begin{aligned}
g_{2,5} &\xmapsto{\partial_2} Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8] \\
Sq^1g_{2,5} &\longmapsto Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]
\end{aligned}
$$

Now $\partial_2\colon \mathscr{A}\{g_{2,0}, \dots, g_{2,5}\} \to \ker(\partial_1)$ is surjective in degrees $t \le 11$. In fact, it is surjective for $t \le 15$.

10.3.3. *Filtration $s = 3$.* We carry on to filtration degree $s = 3$, looking for a surjection $\partial_3\colon P_3 \to \ker(\partial_2)$. First we must compute a basis for $\ker(\partial_2) \subset P_2$, in our range of degrees. The result is displayed in Figure 17.

As usual, the lowest degree class is $Sq^1g_{2,0}$, so we first put a generator $g_{3,0}$ of degree 3 in P_3 with $\partial_3(g_{3,0}) = Sq^1g_{2,0}$. The extension to $\mathscr{A}\{g_{3,0}\}$ is given in the left hand column of Figure 18.

The lowest class not in the image of this extension is $\partial_3(g_{3,1}) = Sq^4g_{2,0}+Sq^2g_{2,1}+Sq^1g_{2,2}$ in degree 6. See the right hand column of Figure 18.

After this, the next class not in the image of ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\}$ is $\partial_3(g_{3,2}) = Sq^8g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1g_{2,4}$ in degree 10:

$$
\begin{aligned}
g_{3,2} &\xmapsto{\partial_3} Sq^8g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1g_{2,4} \\
Sq^1g_{3,2} &\longmapsto Sq^9g_{2,0} + Sq^5Sq^1g_{2,2}
\end{aligned}
$$

Finally, we need a fourth generator, $g_{3,3}$ in degree 11, with

$$
g_{3,3} \xmapsto{\partial_3} Sq^4Sq^2Sq^1g_{2,0} + Sq^6g_{2,2} + Sq^2Sq^1g_{2,3}\,.
$$

$Sq^1[Sq^1]$
$Sq^2Sq^1[Sq^1]$
$Sq^3[Sq^1] + Sq^2[Sq^2]$
$Sq^3Sq^1[Sq^1]$
$Sq^3[Sq^2]$
$Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$
$Sq^4Sq^1[Sq^1]$
$Sq^5[Sq^1] + Sq^3Sq^1[Sq^2]$
$Sq^5Sq^1[Sq^1]$
$(Sq^5 + Sq^4Sq^1)[Sq^2]$
$Sq^6[Sq^1] + Sq^2Sq^1[Sq^4]$
$Sq^6Sq^1[Sq^1]$
$Sq^4Sq^2Sq^1[Sq^1]$
$Sq^5Sq^1[Sq^2]$
$Sq^5Sq^2[Sq^1] + Sq^4Sq^2[Sq^2]$
$Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$
$Sq^7[Sq^1] + Sq^3Sq^1[Sq^4]$
$Sq^7Sq^1[Sq^1]$
$Sq^5Sq^2Sq^1[Sq^1]$
$Sq^5Sq^2[Sq^2]$
$Sq^7[Sq^2] + Sq^5[Sq^4]$
$Sq^6Sq^2[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4]$
$Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$
$Sq^8Sq^1[Sq^1]$
$Sq^6Sq^2Sq^1[Sq^1]$
$Sq^6Sq^3[Sq^1] + Sq^6Sq^2[Sq^2]$
$(Sq^9 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2]$
$Sq^7Sq^1[Sq^2] + Sq^6[Sq^4]$
$Sq^9[Sq^1] + Sq^5Sq^1[Sq^4]$
$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$
$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$
$Sq^9Sq^1[Sq^1]$
$Sq^7Sq^2Sq^1[Sq^1]$
$Sq^6Sq^3Sq^1[Sq^1]$
$Sq^7Sq^3[Sq^1] + Sq^7Sq^2[Sq^2]$
$Sq^6Sq^3[Sq^2]$
$Sq^7Sq^3[Sq^1] + (Sq^9 + Sq^8Sq^1 + Sq^6Sq^2Sq^1)[Sq^2]$
$Sq^7[Sq^4]$
$(Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]$
$(Sq^{10} + Sq^8Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]$
$Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]$
$Sq^{10}[Sq^1] + Sq^2Sq^1[Sq^8]$

FIGURE 14. A basis for $\ker(\partial_1)$ in degrees ≤ 11

(This generator will be particularly interesting when we get to the multiplicative structure in the Adams E_2-term, since it is dual to the indecomposable class c_0 in $\mathrm{Ext}_{\mathscr{A}}^{3,11}(\mathbb{F}_2, \mathbb{F}_2)$.) Then $\partial_3 \colon \mathscr{A}\{g_{3,0}, \dots, g_{3,3}\} \to \ker(\partial_2)$ is surjective in degrees $t \leq 11$.

10.3.4. *Filtration $s = 4$.* In degrees ≤ 11 we have an additive basis

$Sq^1 g_{3,0}$
$Sq^2Sq^1 g_{3,0}$
$Sq^3Sq^1 g_{3,0}$
$Sq^4Sq^1 g_{3,0}$
$Sq^5Sq^1 g_{3,0}$
$Sq^6Sq^1 g_{3,0}$
$Sq^4Sq^2Sq^1 g_{3,0}$
$Sq^7Sq^1 g_{3,0}$
$Sq^5Sq^2Sq^1 g_{3,0}$
$Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}$

for $\ker(\partial_3)$, and a surjection $\partial_4 \colon P_4 = \mathscr{A}\{g_{4,0}, g_{4,1}\} \to \ker(\partial_3)$ where

$$\partial_4(g_{4,0}) = Sq^1 g_{3,0}$$

in degree 4, and

$$\partial_4(g_{4,1}) = Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}$$

in degree 11.

$$
\begin{aligned}
g_{2,0} &\xmapsto{\partial_2} Sq^1[Sq^1] \\
Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^2 g_{2,0} &\longmapsto Sq^2 Sq^1[Sq^1] \\
Sq^3 g_{2,0} &\longmapsto Sq^3 Sq^1[Sq^1] \\
Sq^2 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^4 g_{2,0} &\longmapsto Sq^4 Sq^1[Sq^1] \\
Sq^3 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^5 g_{2,0} &\longmapsto Sq^5 Sq^1[Sq^1] \\
Sq^4 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^6 g_{2,0} &\longmapsto Sq^6 Sq^1[Sq^1] \\
Sq^5 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^4 Sq^2 g_{2,0} &\longmapsto Sq^4 Sq^2 Sq^1[Sq^1] \\
Sq^7 g_{2,0} &\longmapsto Sq^7 Sq^1[Sq^1] \\
Sq^6 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^5 Sq^2 g_{2,0} &\longmapsto Sq^5 Sq^2 Sq^1[Sq^1] \\
Sq^4 Sq^2 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^8 g_{2,0} &\longmapsto Sq^8 Sq^1[Sq^1] \\
Sq^7 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^6 Sq^2 g_{2,0} &\longmapsto Sq^6 Sq^2 Sq^1[Sq^1] \\
Sq^5 Sq^2 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^9 g_{2,0} &\longmapsto Sq^9 Sq^1[Sq^1] \\
Sq^8 Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^7 Sq^2 g_{2,0} &\longmapsto Sq^7 Sq^2 Sq^1[Sq^1] \\
Sq^6 Sq^3 g_{2,0} &\longmapsto Sq^6 Sq^3 Sq^1[Sq^1] \\
Sq^6 Sq^2 Sq^1 g_{2,0} &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{2,1} &\xmapsto{\partial_2} Sq^3[Sq^1] + Sq^2[Sq^2] \\
Sq^1 g_{2,1} &\longmapsto Sq^3[Sq^2] \\
Sq^2 g_{2,1} &\longmapsto (Sq^5 + Sq^4 Sq^1)[Sq^1] + Sq^3 Sq^1[Sq^2] \\
Sq^3 g_{2,1} &\longmapsto Sq^5 Sq^1[Sq^1] \\
Sq^2 Sq^1 g_{2,1} &\longmapsto (Sq^5 + Sq^4 Sq^1)[Sq^2] \\
Sq^4 g_{2,1} &\longmapsto Sq^5 Sq^2[Sq^1] + Sq^4 Sq^2[Sq^2] \\
Sq^3 Sq^1 g_{2,1} &\longmapsto Sq^5 Sq^1[Sq^2] \\
Sq^5 g_{2,1} &\longmapsto Sq^5 Sq^2[Sq^2] \\
Sq^4 Sq^1 g_{2,1} &\longmapsto Sq^5 Sq^2[Sq^2] \\
Sq^6 g_{2,1} &\longmapsto Sq^6 Sq^3[Sq^1] + Sq^6 Sq^2[Sq^2] \\
Sq^5 Sq^1 g_{2,1} &\longmapsto 0 \\
Sq^4 Sq^2 g_{2,1} &\longmapsto (Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2 + Sq^6 Sq^2 Sq^1)[Sq^1] + \\
&\qquad + Sq^5 Sq^2 Sq^1[Sq^2] \\
Sq^7 g_{2,1} &\longmapsto Sq^7 Sq^3[Sq^1] + Sq^7 Sq^2[Sq^2] \\
Sq^6 Sq^1 g_{2,1} &\longmapsto Sq^6 Sq^3[Sq^2] \\
Sq^5 Sq^2 g_{2,1} &\longmapsto (Sq^9 Sq^1 + Sq^7 Sq^2 Sq^1)[Sq^1] \\
Sq^4 Sq^2 Sq^1 g_{2,1} &\longmapsto (Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2 + Sq^6 Sq^2 Sq^1)[Sq^2]
\end{aligned}
$$

FIGURE 15. ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$

10.3.5. *Filtration $s \geq 5$.* Things become quite simple from filtration degree $s = 5$ and onwards. In degrees ≤ 11 we have an additive basis

$$
\begin{array}{ll}
Sq^1 g_{4,0} & Sq^5 Sq^1 g_{4,0} \\
Sq^2 Sq^1 g_{4,0} & Sq^6 Sq^1 g_{4,0} \\
Sq^3 Sq^1 g_{4,0} & Sq^4 Sq^2 Sq^1 g_{4,0} \\
Sq^4 Sq^1 g_{4,0} &
\end{array}
$$

for $\ker(\partial_4)$, and a surjection $\partial_5 \colon P_5 = \mathscr{A}\{g_{5,0}\} \to \ker(\partial_4)$ where $\partial_5(g_{5,0}) = Sq^1 g_{4,0}$ in degree 5. Continuing, we have a surjection $\partial_s \colon P_s = \mathscr{A}\{g_{s,0}\} \to \ker(\partial_{s-1})$ in degrees ≤ 11, where $\partial_s(g_{s,0}) = Sq^1 g_{s-1,0}$ in degree s, for all $5 \leq s \leq 11$.

Definition 10.10. We say that P_* is a *minimal resolution* when $\mathrm{im}(\partial_{s+1}) \subset I(\mathscr{A}) \cdot P_s$ for all $s \geq 0$. Then $1 \otimes \partial_{s+1} \colon \mathbb{F}_2 \otimes_{\mathscr{A}} P_{s+1} \to \mathbb{F}_2 \otimes_{\mathscr{A}} P_s$ and $\mathrm{Hom}(\partial_{s+1}, 1) \colon \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \to \mathrm{Hom}_{\mathscr{A}}(P_{s+1}, \mathbb{F}_2)$ are the zero homomorphisms, so that

$$\mathrm{Tor}_s^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2 \otimes_{\mathscr{A}} P_s = \mathbb{F}_2\{g_{s,i}\}_i$$

and

$$\mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) = \mathbb{F}_2\{g_{s,i}\}_i^*$$

$$
\begin{aligned}
g_{2,2} &\xmapsto{\partial_2} Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4] \\
Sq^1 g_{2,2} &\longmapsto Sq^5[Sq^1] + Sq^3Sq^1[Sq^2] \\
Sq^2 g_{2,2} &\longmapsto (Sq^6 + Sq^5Sq^1)[Sq^1] + Sq^2Sq^1[Sq^4] \\
Sq^3 g_{2,2} &\longmapsto Sq^7[Sq^1] + Sq^3Sq^1[Sq^4] \\
Sq^2Sq^1 g_{2,2} &\longmapsto Sq^6Sq^1[Sq^1] + Sq^5Sq^1[Sq^2] \\
Sq^4 g_{2,2} &\longmapsto (Sq^7Sq^1 + Sq^6Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4] \\
Sq^3Sq^1 g_{2,2} &\longmapsto Sq^7Sq^1[Sq^1] \\
Sq^5 g_{2,2} &\longmapsto Sq^7Sq^2[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] + Sq^5Sq^1[Sq^4] \\
Sq^4Sq^1 g_{2,2} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] \\
Sq^6 g_{2,2} &\longmapsto Sq^7Sq^3[Sq^1] + Sq^6Sq^2Sq^1[Sq^2] + Sq^6Sq^1[Sq^4] \\
Sq^5Sq^1 g_{2,2} &\longmapsto Sq^9Sq^1[Sq^1] \\
Sq^4Sq^2 g_{2,2} &\longmapsto (Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]
\end{aligned}
$$

FIGURE 16. ∂_2 on $\mathscr{A}\{g_{2,2}\} \subset P_2$

$Sq^1 g_{2,0}$
$Sq^2Sq^1 g_{2,0}$
$Sq^3Sq^1 g_{2,0}$
$Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}$
$Sq^4Sq^1 g_{2,0}$
$Sq^5 g_{2,0} + Sq^3 g_{2,1}$
$Sq^5Sq^1 g_{2,0}$
$Sq^6 g_{2,0} + Sq^3Sq^1 g_{2,1} + Sq^2Sq^1 g_{2,2}$
$Sq^6Sq^1 g_{2,0}$
$Sq^4Sq^2Sq^1 g_{2,0}$
$(Sq^5 + Sq^4Sq^1) g_{2,1}$
$Sq^7 g_{2,0} + Sq^3Sq^1 g_{2,2}$
$Sq^7Sq^1 g_{2,0}$
$Sq^5Sq^2Sq^1 g_{2,0}$
$Sq^5Sq^1 g_{2,1}$
$Sq^6Sq^2 g_{2,0} + Sq^4Sq^2 g_{2,1} + Sq^4Sq^1 g_{2,2}$
$Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1) g_{2,2} + Sq^1 g_{2,4}$
$Sq^8Sq^1 g_{2,0}$
$Sq^6Sq^2Sq^1 g_{2,0}$
$(Sq^9 + Sq^7Sq^2) g_{2,0} + Sq^5Sq^2 g_{2,1}$
$Sq^9 g_{2,0} + Sq^5Sq^1 g_{2,2}$
$Sq^4Sq^2Sq^1 g_{2,0} + Sq^6 g_{2,2} + Sq^2Sq^1 g_{2,3}$

FIGURE 17. A basis for $\ker(\partial_2)$ in degrees ≤ 11

for each $s \geq 0$, where $P_s = \mathscr{A}\{g_{s,i}\}_i$. Equivalently, the number of generators of P_s is minimal in each internal degree. (This number is finite, since $\mathscr{A}$ is of finite type.)

$$
\begin{aligned}
g_{3,0} &\overset{\partial_3}{\longmapsto} Sq^1 g_{2,0}\\
Sq^1 g_{3,0} &\longmapsto 0\\
Sq^2 g_{3,0} &\longmapsto Sq^2Sq^1 g_{2,0}\\
Sq^3 g_{3,0} &\longmapsto Sq^3Sq^1 g_{2,0}\\
Sq^2Sq^1 g_{3,0} &\longmapsto 0\\
Sq^4 g_{3,0} &\longmapsto Sq^4Sq^1 g_{2,0}\\
Sq^3Sq^1 g_{3,0} &\longmapsto 0\\
Sq^5 g_{3,0} &\longmapsto Sq^5Sq^1 g_{2,0}\\
Sq^4Sq^1 g_{3,0} &\longmapsto 0\\
Sq^6 g_{3,0} &\longmapsto Sq^6Sq^1 g_{2,0}\\
Sq^5Sq^1 g_{3,0} &\longmapsto 0\\
Sq^4Sq^2 g_{3,0} &\longmapsto Sq^4Sq^2Sq^1 g_{2,0}\\
Sq^7 g_{3,0} &\longmapsto Sq^7Sq^1 g_{2,0}\\
Sq^6Sq^1 g_{3,0} &\longmapsto 0\\
Sq^5Sq^2 g_{3,0} &\longmapsto Sq^5Sq^2Sq^1 g_{2,0}\\
Sq^4Sq^2Sq^1 g_{3,0} &\longmapsto 0\\
Sq^8 g_{3,0} &\longmapsto Sq^8Sq^1 g_{2,0}\\
Sq^7Sq^1 g_{3,0} &\longmapsto 0\\
Sq^6Sq^2 g_{3,0} &\longmapsto Sq^6Sq^2Sq^1 g_{2,0}\\
Sq^5Sq^2Sq^1 g_{3,0} &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{3,1} &\overset{\partial_3}{\longmapsto} Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}\\
Sq^1 g_{3,1} &\longmapsto Sq^5 g_{2,0} + Sq^3 g_{2,1}\\
Sq^2 g_{3,1} &\longmapsto (Sq^6 + Sq^5Sq^1) g_{2,0} + Sq^3Sq^1 g_{2,1} + Sq^2Sq^1 g_{2,2}\\
Sq^3 g_{3,1} &\longmapsto Sq^7 g_{2,0} + Sq^3Sq^1 g_{2,2}\\
Sq^2Sq^1 g_{3,1} &\longmapsto Sq^6Sq^1 g_{2,0} + (Sq^5 + Sq^4Sq^1) g_{2,1}\\
Sq^4 g_{3,1} &\longmapsto (Sq^7Sq^1 + Sq^6Sq^2) g_{2,0} + Sq^4Sq^2 g_{2,1} + Sq^4Sq^1 g_{2,2}\\
Sq^3Sq^1 g_{3,1} &\longmapsto Sq^7Sq^1 g_{2,0} + Sq^5Sq^1 g_{2,1}\\
Sq^5 g_{3,1} &\longmapsto Sq^7Sq^2 g_{2,0} + Sq^5Sq^2 g_{2,1} + Sq^5Sq^1 g_{2,2}\\
Sq^4Sq^1 g_{3,1} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2) g_{2,0} + Sq^5Sq^2 g_{2,1}
\end{aligned}
$$

FIGURE 18. ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\} \subset P_3$

Theorem 10.11. *There is a minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$ with $P_0 = \mathscr{A}\{g_{0,0}\}$ and $P_s = \mathscr{A}\{g_{s,i} \mid i \geq 0\}$, where $\partial_s\colon P_s \to P_{s-1}$ is given in internal degrees $t \leq 11$ by*

$$
\begin{aligned}
\partial_1(g_{1,0}) &= Sq^1 g_{0,0}\\
\partial_1(g_{1,1}) &= Sq^2 g_{0,0}\\
\partial_1(g_{1,2}) &= Sq^4 g_{0,0}\\
\partial_1(g_{1,3}) &= Sq^8 g_{0,0}\\
\partial_2(g_{2,0}) &= Sq^1 g_{1,0}\\
\partial_2(g_{2,1}) &= Sq^3 g_{1,0} + Sq^2 g_{1,1}\\
\partial_2(g_{2,2}) &= Sq^4 g_{1,0} + Sq^2Sq^1 g_{1,1} + Sq^1 g_{1,2}\\
\partial_2(g_{2,3}) &= Sq^7 g_{1,0} + Sq^6 g_{1,1} + Sq^4 g_{1,2}\\
\partial_2(g_{2,4}) &= Sq^8 g_{1,0} + Sq^7 g_{1,1} + Sq^4Sq^1 g_{1,2} + Sq^1 g_{1,3}\\
\partial_2(g_{2,5}) &= Sq^7Sq^2 g_{1,0} + Sq^8 g_{1,1} + Sq^4Sq^2 g_{1,2} + Sq^2 g_{1,3}\\
\partial_3(g_{3,0}) &= Sq^1 g_{2,0}\\
\partial_3(g_{3,1}) &= Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}\\
\partial_3(g_{3,2}) &= Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1) g_{2,2} + Sq^1 g_{2,4}\\
\partial_3(g_{3,3}) &= (Sq^7 + Sq^4Sq^2Sq^1) g_{2,1} + Sq^6 g_{2,2} + Sq^2Sq^1 g_{2,3}\\
\partial_4(g_{4,0}) &= Sq^1 g_{3,0}\\
\partial_4(g_{4,1}) &= Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}\\
\partial_5(g_{5,0}) &= Sq^1 g_{4,0}\\
&\ \ \dots\\
\partial_{11}(g_{11,0}) &= Sq^1 g_{10,0}\,.
\end{aligned}
$$

Proof. This summarizes the calculations above. The resolution is minimal, since we only added generators $g_{s,i}$ with $\partial_s(g_{s,i}) \in I(\mathscr{A}) \cdot P_{s-1} = I(\mathscr{A})\{g_{s-1,j}\}_j$. It should be clear that we can continue that way, since $\mathscr{A}$ is connected. If any sum involving $1 \cdot g_{s,i}$ occurs in $\ker(\partial_s)$, then $g_{s,i}$ could be omitted from the basis for P_s and $\partial_s \colon P_s \to \ker(\partial_{s-1})$ would still be surjective. □

Theorem 10.12. $\mathrm{Ext}^{s,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$ *where* $\gamma_{s,i} \colon P_s \to \mathbb{F}_2$ *is the* $\mathscr{A}$*-module homomorphism dual to* $g_{s,i}$*, for each* $s \geq 0$*. The bidegrees of the generators in internal degrees* $t \leq 11$ *are as displayed in the following chart. The horizontal coordinate is the topological degree* $t-s$*, the vertical coordinate is the cohomological degree* s*, and the sum of these coordinates is the internal degree* t*.*

	$\gamma_{11,0}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
10	$\gamma_{10,0}$		$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\gamma_{9,0}$			$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
8	$\gamma_{8,0}$				$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\gamma_{7,0}$					$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	?
6	$\gamma_{6,0}$						$\cdot$	$\cdot$	$\cdot$	$\cdot$	?	?
	$\gamma_{5,0}$							$\cdot$	$\cdot$	?	?	?
4	$\gamma_{4,0}$							$\gamma_{4,1}$	?	?	?	?
	$\gamma_{3,0}$			$\gamma_{3,1}$				$\gamma_{3,2}$	$\gamma_{3,3}$	?	?	?
2	$\gamma_{2,0}$		$\gamma_{2,1}$	$\gamma_{2,2}$			$\gamma_{2,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$		?	?
	$\gamma_{1,0}$	$\gamma_{1,1}$		$\gamma_{1,2}$				$\gamma_{1,3}$				?
0	$\gamma_{0,0}$											
	0		2		4		6		8		10	

We have not yet computed the groups labeled $\cdot$ or ?, but we will prove below that the groups labeled $\cdot$ are 0. (This is the Adams (1966) vanishing theorem.) In fact, many of the groups labeled ? are also zero.

Proof. For each $s \geq 0$ we have $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}\{g_{s,i}\}_i, \mathbb{F}_2) \cong \prod_i \mathbb{F}_2\{\gamma_{s,i}\}$, where $\gamma_{s,i}(g_{s,j}) = \delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. It will be clear later that there are at most finitely many $g_{s,i}$ in a given bidegree, so this product is finite in each degree. Then $\gamma_{s,i} \circ \partial_{s+1} = 0$, so the cocomplex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ has trivial coboundary. Hence $\mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$, as claimed. □

Lemma 10.13. *Let* $\epsilon \colon P_* \to \mathbb{F}_2$ *be a free* $\mathscr{A}$*-module resolution. Then* $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{F}_2 \otimes_{\mathscr{A}} P_s, \mathbb{F}_2)$*, so there is an isomorphism* $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2)$.

10.4. **A minimal resolution at $p = 3$.** Now consider the case of an odd prime p. The mod p Adams E_2-term for the sphere spectrum is

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) = H^{*,*}(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_p), \delta)\,,$$

where

$$\dots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \to \dots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{F}_p \to 0$$

is any free $\mathscr{A}$-module resolution of $\mathbb{F}_p$ and $\delta = \mathrm{Hom}_{\mathscr{A}}(\partial, 1)$.

We calculate a minimal such resolution for $p = 3$ in internal degrees $t < 2p^2 - 2 = 16$. To begin, let $P_0 = \mathscr{A}\{g_{0,0}\} \cong \mathscr{A}$, with $g_{0,0}$ in degree 0 and $\epsilon(g_{0,0}) = 1$. The admissible monomials

$$\beta, P^1, \beta P^1, P^1\beta, \beta P^1\beta, P^2, \beta P^2, P^2\beta, \beta P^2\beta, P^p, \beta P^p, P^p\beta, \beta P^p\beta$$

form a basis for $\ker(\epsilon) = I(\mathscr{A})$ in degrees $t < 16$.

$$
\begin{aligned}
g_{1,0} &\overset{\partial_1}{\longmapsto} \beta g_{0,0} \\
\beta g_{1,0} &\longmapsto 0 \\
g_{1,1} &\longmapsto P^1 g_{0,0} \\
P^1 g_{1,0} &\longmapsto P^1 \beta g_{0,0} \\
\beta g_{1,1} &\longmapsto \beta P^1 g_{0,0} \\
\beta P^1 g_{1,0} &\longmapsto \beta P^1 \beta g_{0,0} \\
P^1 \beta g_{1,0} &\longmapsto 0 \\
\beta P^1 \beta g_{1,0} &\longmapsto 0 \\
P^1 g_{1,1} &\longmapsto P^1 P^1 g_{0,0} = 2P^2 g_{0,0} \\
P^2 g_{1,0} &\longmapsto P^2 \beta g_{0,0} \\
\beta P^1 g_{1,1} &\longmapsto 2\beta P^2 g_{0,0} \\
P^1 \beta g_{1,1} &\longmapsto P^1 \beta P^1 g_{0,0} = (\beta P^2 + P^2 \beta) g_{0,0} \\
\beta P^2 g_{1,0} &\longmapsto \beta P^2 \beta g_{0,0} \\
P^2 \beta g_{1,0} &\longmapsto 0 \\
\beta P^1 \beta g_{1,1} &\longmapsto \beta P^2 \beta g_{0,0} \\
\beta P^2 \beta g_{1,0} &\longmapsto 0 \\
P^2 g_{1,1} &\longmapsto 0 \\
g_{1,2} &\longmapsto P^p g_{0,0} \\
P^p g_{1,0} &\longmapsto P^p \beta g_{0,0} \\
\beta P^2 g_{1,1} &\longmapsto 0 \\
P^2 \beta g_{1,1} &\longmapsto P^2 \beta P^1 g_{0,0} = (\beta P^p - P^p \beta) g_{0,0} \\
\beta g_{1,2} &\longmapsto \beta P^p g_{0,0} \\
\beta P^p g_{1,0} &\longmapsto \beta P^p \beta g_{0,0} \\
P^p \beta g_{1,0} &\longmapsto 0 \\
\beta P^2 \beta g_{1,1} &\longmapsto -\beta P^p \beta g_{0,0} \\
\beta P^p \beta g_{1,0} &\longmapsto 0
\end{aligned}
$$

FIGURE 19. $\partial_1 \colon P_1 \to P_0$ for $p = 3$

10.4.1. *Filtration $s = 1$.* To define a surjection $\partial_1 \colon P_1 \to \ker(\epsilon)$, it suffices to add generators to P_1 that map to a basis for the algebra indecomposables

$$Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 = \mathbb{F}_p\{\beta, P^1, P^p, \dots\}\,.$$

Let

$$P_1 = \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, \dots\}$$

be generated by $g_{1,0}$ in degree $t = 1$ with $\partial_1(g_{1,0}) = \beta g_{0,0}$, $g_{1,1}$ in degree $t = 2p - 2 = 4$ with $\partial_1(g_{1,1}) = P^1 g_{0,0}$, $g_{1,2}$ in degree $t = 2p^2 - 2p = 12$ with $\partial_1(g_{1,2}) = P^p g_{0,0}$, and so on. In general, $g_{1,i+1}$ in degree $t = 2p^i(p-1)$ maps to $P^{p^i} g_{0,0}$ for each $i \ge 0$. The boundary ∂_1 is given in Figure 19, in internal degrees $t \le 15$. A basis for its kernel is shown in Figure 20, in the same range of degrees.

10.4.2. *Filtration $s = 2$.* Next we define a surjection $\partial_2 \colon P_2 \to \ker(\partial_1)$. Let

$$P_2 = \mathscr{A}\{g_{2,0}, g_{2,1}, g_{2,2}, g_{2,3}, \dots\}$$

be generated by $g_{2,0}$ in degree $t = 2$ with $\partial_2(g_{2,0}) = \beta g_{1,0}$, by $g_{2,1}$ in degree $t = 4p - 3 = 9$ with $\partial_2(g_{2,1}) = 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}$, by $g_{2,2}$ in degree $2p^2 - 2p = 12$ with $\partial_2(g_{2,2}) = P^{p-1} g_{1,1} = P^2 g_{1,1}$, by $g_{2,3}$ in degree $2p^2 - 2p + 1 = 13$ with $\partial_2(g_{2,3}) = P^p g_{1,0} + P^2 \beta g_{1,1} - \beta g_{1,2}$, and so on. Note how $g_{2,0}$ corresponds to the relation $\beta^2 = 0$, $g_{2,1}$ corresponds to the Adem relation $P^1 \beta P^1 = \beta P^2 + P^2 \beta$ (and $P^1 P^1 = 2P^2$), and $g_{2,2}$ corresponds to the Adem relation $P^{p-1} P^1 = 0$. [[Continue with $g_{2,3}$.]] The

$$\begin{aligned}
&\beta g_{1,0}\\
&P^1\beta g_{1,0}\\
&\beta P^1\beta g_{1,0}\\
&2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}\\
&P^2\beta g_{1,0}\\
&\beta P^2 g_{1,0} - \beta P^1\beta g_{1,1}\\
&\beta P^2\beta g_{1,0}\\
&P^2 g_{1,1}\\
&P^p g_{1,0} + P^2\beta g_{1,1} - \beta g_{1,2}\\
&\beta P^2 g_{1,1}\\
&\beta P^p g_{1,0} + \beta P^2\beta g_{1,1}\\
&P^p\beta g_{1,0}\\
&\beta P^p\beta g_{1,0}
\end{aligned}$$

FIGURE 20. A basis for $\ker(\partial_1)$ at $p = 3$

boundary ∂_2 is given in Figure 21, in degrees $t \le 15$. A basis for its kernel is shown in Figure 22, in the same degrees.

10.4.3. *Filtration $s = 3$.* We continue by defining a surjection $\partial_3 \colon P_3 \to \ker(\partial_2)$. Let

$$P_3 = \mathscr{A}\{g_{3,0}, g_{3,1}, g_{3,2}, \dots\}$$

be generated by $g_{3,0}$ in degree $t = 3$ with $\partial_3(g_{3,0}) = \beta g_{2,0}$, by $g_{3,1}$ in degree $t = (?) = 13$ with $\partial_3(g_{3,1}) = P^1 g_{2,1} - \beta g_{2,2}$, by $g_{3,2}$ in degree $t = (?) = 14$ with $\partial_3(g_{3,2}) = P^p g_{2,0} + P^1\beta g_{2,1} - \beta g_{2,3}$, and so on. The boundary ∂_3 is given in Figure 23, in degrees $t \le 15$. A basis for its kernel is shown in Figure 24, in the same range.

10.4.4. *Filtrations $s \ge 4$.* From here on we get surjections $\partial_s \colon P_s \to \ker(\partial_{s-1})$ for $s \ge 4$ by letting

$$P_s = \mathscr{A}\{g_{s,0}, \dots\}$$

with $g_{s,0}$ in degree s, where $\partial_s(g_{s,0}) = \beta g_{s-1,0}$, and so on.

Theorem 10.14. *There is a minimal resolution $\epsilon \colon P_* \to \mathbb{F}_3$, with $P_0 = \mathscr{A}\{g_{0,0}\}$ and $P_s = \mathscr{A}\{g_{s,i} \mid i \ge 0\}$, where $\partial_s \colon P_s \to P_{s-1}$ is given in internal degree $t \le 15$ by*

$$\begin{aligned}
\partial_1(g_{1,0}) &= \beta g_{0,0}\\
\partial_1(g_{1,1}) &= P^1 g_{0,0}\\
\partial_1(g_{1,2}) &= P^p g_{0,0}\\
\partial_2(g_{2,0}) &= \beta g_{1,0}\\
\partial_2(g_{2,1}) &= 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}\\
\partial_2(g_{2,2}) &= P^{p-1} g_{1,1}\\
\partial_2(g_{2,3}) &= P^p g_{1,0} + P^2\beta g_{1,1} - \beta g_{1,2}\\
\partial_3(g_{3,0}) &= \beta g_{2,0}\\
\partial_3(g_{3,1}) &= P^1 g_{2,1} - \beta g_{2,2}\\
\partial_3(g_{3,2}) &= P^p g_{2,0} + P^1\beta g_{2,1} - \beta g_{2,3}\\
\partial_4(g_{4,0}) &= \beta g_{3,0}\\
&\dots\\
\partial_{15}(g_{15,0}) &= \beta g_{14,0}\,.
\end{aligned}$$

$$
\begin{aligned}
g_{2,0} &\overset{\partial_2}{\longmapsto} \beta g_{1,0}\\
\beta g_{2,0} &\longmapsto 0\\
P^1 g_{2,0} &\longmapsto P^1\beta g_{1,0}\\
\beta P^1 g_{2,0} &\longmapsto \beta P^1\beta g_{1,0}\\
P^1\beta g_{2,0} &\longmapsto 0\\
\beta P^1\beta g_{2,0} &\longmapsto 0\\
g_{2,1} &\longmapsto 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta)g_{1,1}\\
P^2 g_{2,0} &\longmapsto P^2\beta g_{1,0}\\
\beta g_{2,1} &\longmapsto 2\beta P^2 g_{1,0} - 2\beta P^1\beta g_{1,1}\\
\beta P^2 g_{2,0} &\longmapsto \beta P^2\beta g_{1,0}\\
P^2\beta g_{2,0} &\longmapsto 0\\
\beta P^2\beta g_{2,0} &\longmapsto 0\\
g_{2,2} &\longmapsto P^2 g_{1,1}\\
P^1 g_{2,1} &\longmapsto (\beta P^2 - 3P^2\beta)g_{1,1} = \beta P^2 g_{1,1}\\
\beta g_{2,2} &\longmapsto \beta P^2 g_{1,1}\\
g_{2,3} &\longmapsto P^p g_{1,0} + P^2\beta g_{1,1} - \beta g_{1,2}\\
P^p g_{2,0} &\longmapsto P^p\beta g_{1,0}\\
\beta P^1 g_{2,1} &\longmapsto 0\\
P^1\beta g_{2,1} &\longmapsto (\beta P^p - P^p\beta)g_{1,0} - 2\beta P^p\beta g_{1,1}\\
\beta g_{2,3} &\longmapsto \beta P^p g_{1,0} + \beta P^2\beta g_{1,1}\\
\beta P^p g_{2,0} &\longmapsto \beta P^p\beta g_{1,0}\\
P^p\beta g_{2,0} &\longmapsto 0\\
\beta P^1\beta g_{2,1} &\longmapsto -\beta P^p\beta g_{1,0}
\end{aligned}
$$

FIGURE 21. $\partial_2\colon P_2 \to P_1$ for $p = 3$

$$
\begin{aligned}
&\beta g_{2,0}\\
&P^1\beta g_{2,0}\\
&\beta P^1\beta g_{2,0}\\
&P^2\beta g_{2,0}\\
&\beta P^2\beta g_{2,0}\\
&P^1 g_{2,1} - \beta g_{2,2}\\
&P^p g_{2,0} + P^1\beta g_{2,1} - \beta g_{2,3}\\
&\beta P^1 g_{2,1}\\
&\beta P^p g_{2,0} + \beta P^1\beta g_{2,1}\\
&P^p\beta g_{2,0}
\end{aligned}
$$

FIGURE 22. A basis for $\ker(\partial_2)$ at $p = 3$

$$
\begin{aligned}
g_{3,0} &\xmapsto{\partial_3} \beta g_{2,0} \\
\beta g_{3,0} &\longmapsto 0 \\
P^1 g_{3,0} &\longmapsto P^1 \beta g_{2,0} \\
\beta P^1 g_{3,0} &\longmapsto \beta P^1 \beta g_{2,0} \\
P^1 \beta g_{3,0} &\longmapsto 0 \\
\beta P^1 \beta g_{3,0} &\longmapsto 0 \\
P^2 g_{3,0} &\longmapsto P^2 \beta g_{2,0} \\
\beta P^2 g_{3,0} &\longmapsto \beta P^2 \beta g_{2,0} \\
P^2 \beta g_{3,0} &\longmapsto 0 \\
\beta P^2 \beta g_{3,0} &\longmapsto 0 \\
g_{3,1} &\longmapsto P^1 g_{2,1} - \beta g_{2,2} \\
\beta g_{3,1} &\longmapsto \beta P^1 g_{2,1} \\
g_{3,2} &\longmapsto P^p g_{2,0} + P^1 \beta g_{2,1} - \beta g_{2,3} \\
P^p g_{3,0} &\longmapsto P^p \beta g_{2,0} \\
\beta g_{3,2} &\longmapsto \beta P^p g_{2,0} + \beta P^1 \beta g_{2,1}
\end{aligned}
$$

FIGURE 23. $\partial_3 \colon P_3 \to P_2$ for $p = 3$

$$
\begin{aligned}
&\beta g_{3,0} \\
&P^1 \beta g_{3,0} \\
&\beta P^1 \beta g_{3,0} \\
&P^2 \beta g_{3,0} \\
&\beta P^2 \beta g_{3,0}
\end{aligned}
$$

FIGURE 24. A basis for $\ker(\partial_3)$ at $p = 3$

Theorem 10.15. $\mathrm{Ext}^{s,*}_{\mathscr{A}}(\mathbb{F}_3, \mathbb{F}_3) \cong \mathbb{F}_3\{\gamma_{s,i}\}_i$, *where* $\gamma_{s,i} \colon P_s \to \mathbb{F}_3$ *is the* $\mathscr{A}$*-module homomorphism dual to* $g_{s,i}$. *The generators in internal degree* $t \leq 15$ *are displayed in the following figure.*

6	$\gamma_{6,0}$										$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\gamma_{5,0}$											$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
4	$\gamma_{4,0}$												$\cdot$	$\cdot$	$\cdot$	?
	$\gamma_{3,0}$										$\gamma_{3,1}$	$\gamma_{3,2}$		?	?	?
2	$\gamma_{2,0}$							$\gamma_{2,1}$			$\gamma_{2,2}$	$\gamma_{2,3}$			?	?
	$\gamma_{1,0}$			$\gamma_{1,1}$								$\gamma_{1,2}$				?
0	$\gamma_{0,0}$															
	0		2		4		6		8		10		12		14	

We have not yet computed the groups labeled $\cdot$ or ?, but by the May vanishing theorem, see Ravenel (1986, Theorem 3.4.5(b)), the groups labeled $\cdot$ are 0. In fact, many of the groups labeled ? are also zero.

The first possible differential is $d_2^{1,12}$ on $\gamma_{1,2}$), which indeed equals $\gamma_{3,1}$. Once we have proved convergence and the visible vanishing line, it follows that $\pi_*(S)^\wedge_3$ begins as follows.

n	$\pi_n(S)^\wedge_3$	gen.	rep.
0	$\mathbb{Z}_3$	ι	$\gamma_{0,0}$
1	0		
2	0		
3	$\mathbb{Z}/3$	α_1	$\gamma_{1,1}$
4	0		
5	0		
6	0		
7	$\mathbb{Z}/3$	α_2	$\gamma_{2,1}$
8	0		
9	0		
10	$\mathbb{Z}/3$	β_1	$\gamma_{2,2}$
11	$\mathbb{Z}/9$	$\alpha_{3/1}$	$\gamma_{2,3}$
12	0		

The cyclic groups in degrees $2i(p-1)-1 = 4i-1$, generated by the α-classes, equal the image $\mathrm{im}(J)_*$ of the J-homomorphism $J_* \colon \pi_{4i-1}(O) \to \pi_{4i-1}(S)$. As becomes visible in degree $2p(p-1)-1 = 11$, the order of this cyclic group $\mathrm{im}(J)_{2i(p-1)-1}$ varies with i. It is $p^{j+1} = 3^{j+1}$ where $j = v_p(i)$ is the p-valuation of i, or equivalently, the p-component of pi. The element of order p in this image is denoted α_i, for $i \geq 1$, and $\alpha_i = p^j \alpha_{i/j}$, where $\alpha_{i/j}$ is a generator of this cyclic group. This pattern persists for all odd primes p, but the case $p = 2$ is more complicated.

The first element of $\pi_*(S)^\wedge_p$ that is not in the *image of* J, hence is in the *cokernel of* J, is β_1 in $\pi_{2p(p-1)-2}(S)^\wedge_p$, represented in Adams filtration 2 by $\gamma_{2,2}$.

11. Bruner's `ext`-program

11.1. **Overview.** Robert R. Bruner (1993) has developed a package of C-programs and shell scripts, usually called `ext`, which can calculate $\mathrm{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2)$ over the mod 2 Steenrod algebra $\mathscr{A}$ for many modules M, in a finite range of filtration degrees s and internal degrees t.

The strategy is to compute a minimal free resolution $\epsilon \colon P_* \to M$ of the $\mathscr{A}$-module M, one internal degree t at a time, starting from filtration degree $s = 0$ and moving upwards. The $\mathscr{A}$-module basis $\{g_{s,i}\}_i$ for P_s then also gives an $\mathbb{F}_2$-vector space basis for $\mathbb{F}_2 \otimes_{\mathscr{A}} P_s = \mathrm{Tor}^{\mathscr{A}}_s(\mathbb{F}_2, M)$. The dual basis $\{\gamma_{s,i}\}_i$, with $\gamma_{s,i}(g_{s,j}) = \delta_{i,j}$, is then an $\mathbb{F}_2$-vector space basis for $\mathrm{Ext}^s_{\mathscr{A}}(M, \mathbb{F}_2)$.

The program can compute induced homomorphisms, Yoneda products and some Massey products, and produces output in text, TEX, Postscript and PDF formats. It can also make similar calculations over the subalgebra $A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$ of $\mathscr{A}$, but is not prepared to calculate at odd primes p.

See subsection 11.2 for a guide to how to install the current version of `ext`. Thereafter, see subsection 11.3 to see how to encode an $\mathscr{A}$-module M in a format that the `ext` program can use, and subsection 11.4 for where to store and process such module definition files. To resolve a module, first see subsection 11.5 for how to create the subdirectory where that calculation takes place, and then see subsection 11.6 for how to run the script that calculates the minimal resolution. [[ETC]]

11.2. **Installation.** At the time of writing, the most recent version of `ext` is `ext.1.8.7` from April 14th 2014. It can be downloaded via Bruner's home page at `http://www.math.wayne.edu/~rrb/papers/`, or directly from `http://www.math.wayne.edu/~rrb/papers/ext.1.8.7.tar.gz`, using a web browser. Save the file `ext.1.8.7.tar.gz` in a directory. In this guide we will assume that this directory is called `ext`. You may be offered to create such a directory when saving the file, or you can create one using `mkdir ext`.

Open a terminal window and move to the `ext` directory, using a command like `cd ext`. The file is a compressed (gzip'ed) tape archive (tar-file). First uncompress it using `gunzip ext.1.8.7.tar.gz`. This enlarges the file from about 2 MB to about 5 MB, and gives it the new name `ext.1.8.7.tar`. Then unpack the archive using `tar -xvf ext.1.8.7.tar`. To list the resulting files use `ls`, giving output like `A A2 copyright doc ext.1.8.7.tar NEW README START_HERE TODO`. The files `START_HERE` (up to date for version 1.8) and `README` (dating from versions 1.6, 1.65 and 1.66) explain the basic usage of the `ext` program. There is further documentation in the `doc` subdirectory, and an account of the changes made since version 1.66 is given in `NEW`. The subdirectory `A` will contain the code and data for making calculations over the mod 2 Steenrod algebra $\mathscr{A}$. The subdirectory `A2` will contain the corresponding code and date for calculations over the subalgebra $A(2)$.

To complete the installation, follow the instructions in section I of `START_HERE`, namely do `cd A` followed by `./Install`. This runs the shell script `Install` in `ext/A`. The script compiles several programs, and assumes that the GNU C-compiler `gcc` is already installed on the system. If not, you will need to install `gcc` first. There will be some warning messages regarding `storage.c` and `splitname.c`. Apparently it is difficult to avoid these on all different systems. It may be possible to write `Install` in place of `./Install`, but this depends on the settings of your system, i.e., whether the current directory (`.`) is in the search path variable `$PATH`. We will not assume that it is, and therefore use the explicit `./`-commands. Finally, move up to `ext` and down to the `A2` directory using `cd ../A2`, and then do `./Install` in that directory to compile the remaining programs. Again there will be some warning messages. Do `cd ..` to return to the main directory (the one we are assuming is called `ext`). This completes the installation.

11.3. **The module definition format.** In order to calculate $\mathrm{Ext}_{\mathscr{A}}^{*,*}(M, \mathbb{F}_2)$, we must first specify the $\mathscr{A}$-module M. Before version 1.5, the user was expected to provide a program (called `module.c`) that contained functions keeping track of a $\mathbb{F}_2$-vector space basis for M, and the action of elements in the Steenrod algebra on those basis elements. This is documented in `ext/doc/readme.1.0` and `ext/doc/module.doc`, but is now largely irrelevant, due to the new interface for module definitions introduced in version 1.5, partly written by Jeff Igo. It is documented in `ext/doc/modfmt.ascii` and `ext/doc/modfmt.html`, in addition to the following explanation.

The $\mathscr{A}$-module M, which may eventually have a completely different name, must be presented to the `ext` program as a finite dimensional $\mathbb{F}_2$-vector space with a chosen ordered basis $(v_0, v_1, \dots, v_{n-1})$. If there are n basis vectors, they will be numbered from 0 to $n-1$, inclusive. The $\mathscr{A}$-module action must be specified by listing the value $Sq^r(v_i)$ of each Steenrod squaring operation Sq^r on each basis vector v_i, for $r \geq 1$ and $0 \leq i < n$, except that operations that take the value 0 can be omitted. This ensured that only finitely many values need to be specified.

If one is really interested in an infinite-dimensional module M, such as $H^*\mathbb{R}P^\infty = \mathbb{F}_2[x]$, one must choose to truncate this module at some finite internal degree b, discarding all generators in internal degrees $t > b$. This will not affect $\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2)$ for $t \leq b$, so a partial calculation in a finite range of internal degrees is possible, if M is bounded below and of finite type. If M is not bounded below, or has infinitely many generators in a single degree, then the `ext`-program will not be able to calculate with it.

The module definition file will be a text file with two parts. The first part specifies the internal grading of the vector space basis. The second part specifies the action by the Steenrod operations.

The first part has the format

```
n
t0 t1 ... t(n-1)
```

where n is the $\mathbb{F}_2$-vector space dimension of M, i.e., the number of basis vectors $v_0, v_1, \dots, v_{n-1}$, and $t_0 t_1 \dots t_{n-1}$ are the internal degrees of those basis vectors. Beware that these basis vectors are assumed to be ordered so that the sequence of internal degrees is non-decreasing. In other words, a basis vector cannot be followed by a basis element in strictly lower internal degree. (If your module definition file does not satisfy this condition, the program `ext/A/samples/sortDef` can reorder the basis as needed.)

For example, if $M = H^*(S) = \mathbb{F}_2$ has a single generator in internal degree 0, the module definition file would begin:

```
1
0
```

If $M = \tilde{H}^*(\mathbb{R}P^4)$ has four generators x, x^2, x^3 and x^4 in degrees 1, 2, 3 and 4, the module definition file would begin:

```
4
1 2 3 4
```

Note that the names of the generators are irrelevant for the program; it simply considers the basis as an ordered list of n elements, and keeps track of the individual basis elements by their index in that list, which is a number between 0 and $n-1$. (This index is typically different from the internal degree of that generator.) However, the ordering of the basis elements (within a given internal degree) will be of importance when the Steenrod operations are to be specified.

The second part consists of a list of lines, one for each nonzero operation $Sq^r(v_i)$ with $r \geq 1$. If $Sq^r(v_i) = v_{j_1} + v_{j_2} + \cdots + v_{j_k}$ is a sum of k different terms, then that line will appear as follows:

```
i r k j1 j2 ... jk
```

The first entry, i, tells us which basis vector, v_i, is being acted upon. The second entry, r, tells us which Steenrod operation, Sq^r, is acting nontrivially on that basis vector. The value of $Sq^r(v_i)$ is a homogeneous element in M, hence is a sum of one or more of the basis vectors in that internal degree. The third entry (k) tells us how many different terms there are in that sum. The remainder of the line contains k entries, and these are the indices $j_1, j_2, \dots, j_k$ of the basis vectors that occur in the sum $Sq^r(v_i) = v_{j_1} + v_{j_2} + \cdots + v_{j_k}$. [[Usually $j_1 < j_2 < \cdots < j_k$. Is this necessary? Duplications are not allowed, I believe.]]

For example, if $M = H^*(S) = \mathbb{F}_2$, there are no nonzero operations Sq^r, so the second part is empty; it consists of zero lines.

If $M = \tilde{H}^*(\mathbb{R}P^4)$, the Steenrod operations satisfy $Sq^r(x^i) = \binom{i}{r} x^{r+i}$. The nonzero operations are $Sq^1(x) = x^2$, $Sq^1(x^3) = x^4$ and $Sq^2(x^2) = x^4$. The operation $Sq^1(x) = x^2$ is specified by the line

```
0 1 1 1
```

where the first 1 means that we are acting on the generator numbered 0, i.e., x, the second 1 means that we are specifying the value of Sq^1 on that generator, the third 1 means that $Sq^1(x) = x^2$ is a sum of one term only, and the last 1 means that that one term is the generator numbered 1, i.e., x^2. The operation $Sq^1(x^3) = x^4$ is specified by the line

```
2 1 1 3
```

where the first 2 means that we are acting on the generator numbered 2, i.e., x^3, the second 1 means that we are specifying the value of Sq^1 on that generator, the third 1 means that $Sq^1(x^3) = x^4$ is a sum of one term only, and the last 3 means that that one term is the generator numbered 3, i.e., x^4. The operation $Sq^2(x^2) = x^4$ is specified by the line

```
1 2 1 3
```

where the first 1 means that we are acting on the generator numbered 1, i.e., x^2, the second 2 means that we are specifying the value of Sq^2 on that generator, the third 1 means that $Sq^2(x^2) = x^4$ is a sum of one term only, and the last 3 means that that one term is the generator numbered 3, i.e., x^4. The combined second part of the module definition file for this M is therefore:

```
0 1 1 1
2 1 1 3
1 2 1 3
```

The ordering of the lines does not matter. If preferred, we could also have used the following specification

```
0 1 1 1
1 2 1 3
2 1 1 3
```

in order of the basis elements v_i, followed by the order of the squaring operations $Sq^r(v_i)$ on those basis elements. With this ordering, the whole module definition file for $\tilde{H}^*(\mathbb{R}P^4)$ would appear as follows.

```
4
1 2 3 4

0 1 1 1
1 2 1 3
2 1 1 3
```

This file can be created in a text editor.

11.4. **The `samples` directory.** Module definitions for $\mathscr{A}$-modules can conveniently be stored in the directory `ext/A/samples`. The file name can be freely chosen, but it is convenient to let it specify the $\mathscr{A}$-module, or perhaps a spectrum whose cohomology realizes that $\mathscr{A}$-module. The module definition for $\mathbb{F}_2$ can thus be saved under one of the names `F2`, `F2.def`, `S` or `S.def` in `ext/A/samples`. That directory also contains some tools for working with module definitions. See the file `ext/A/samples/README` for some documentation. The programs `tensorDef`, `dualizeDef`, `collapse` and `truncate` let you build new module definition files from old ones.

For example, if `M.def` and `N.def` contain the definitions of two $\mathscr{A}$-modules M and N, then the command `./tensorDef M.def N.def MN.def` will produce a new module definition file `MN.def`, presenting the tensor product $M \otimes N$ (with the diagonal $\mathscr{A}$-action, more on that later). If $M = H^*X$ and $N = H^*Y$, then $M \otimes N = H^*(X \wedge Y)$. If the ordered basis for M is $(v_i)_i$ and the ordered basis for N is $(w_j)_j$, the basis chosen for $M \otimes N$ will consist of the set of tensors $\{v_i \otimes w_j\}_{i,j}$, but the ordering of these basis vectors may not be obvious. The program `tensorDef` therefore outputs a list of the pairs (i, j), in the order that is chosen for $M \otimes N$. A copy of this output may be saved, since it can become useful later.

For another example, if `M.def` contains the definition of an $\mathscr{A}$-module M, then `./dualizeDef M.def DM.def` will produce a new module definition file `DM.def`, presenting the dual $\mathscr{A}$-module $M^* = \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ (with the conjugated $\mathscr{A}$-action, more on that later). If $M = H^*X$, then $M^* = H^*(DX) = H_{-*}(X)$, where $DX = F(X, S)$ is the functional dual of X. For finite CW spectra X, this is the same as the Spanier–Whitehead dual of X. [[Is the basis $\{v_i^*\}_i$ for M^* ordered by reversing the order of the basis $(v_i)_i$ for M?]]

Calling these commands without an argument, as in `./collapse` or `./truncate`, gives short messages explaining their usage.

The `consistency` command, in its improved version called `newconsistency`, checks whether the Steenrod operations listed in a module definition file actually define an $\mathscr{A}$-module, i.e., if the operations satisfy the Adem relations. If all Adem relations are satisfied, it exits quietly. If they are not, it lists the Adem relations that are not satisfied, and the generator on which this failure takes place.

[[Can use `newconsistency` to complete a partial definition of an $\mathscr{A}$-module, where only the action of the algebra indecomposables Sq^{2^i} are given, to one where the action of all Sq^r are given. To do this, start by adding operations to correct the lowest degree error message from `newconsistency`, and continue.]]

11.5. **Creating a new module.** To make $\mathrm{Ext}_{\mathscr{A}}$-calculations with an $\mathscr{A}$-module M, defined by a module definition file `M.def` in `ext/A/samples`, use `cd ..` or something similar to go to `ext/A`. Then use the command `./newmodule M samples/M.def` to create a subdirectory `ext/A/M` that contains the data and code relevant for the calculations for M. In general, replace `M` with a more memorable name for the module in question. `newmodule` calls on `newconsistency` to check that the module definition file `M.def` actually defines an $\mathscr{A}$-module. If it does not, go back and correct it before calling `newmodule again`.

A copy of the module definition file will be stored as `Def` in `ext/A/M`.

The $\mathrm{Ext}_{\mathscr{A}}$-calculations for M will be carried out by finding a finite part of a minimal resolution $P_* \to M$, in a range of filtration degrees $0 \le s \le s_{max}$, where s_{max} is the number stored in the file `ext/A/M/MAXFILT`.

$$P_{s_{max}} \longrightarrow \ldots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \longrightarrow \ldots \longrightarrow P_0 \xrightarrow{\epsilon} M \to 0\,.$$

Usually a common s_{max}-value for all $\mathscr{A}$-modules is set in the file `ext/A/MAXFILT`, and `newmodule` will copy this value into `ext/A/M/MAXFILT` when creating the directory for M. [[It should not be changed after `newmodule` has completed creating the module.]]

The data specifying the minimal resolution will be stored in the files `Diff.0, Diff.1, ...`. Here `Diff.s` will specify the internal degrees of the $\mathscr{A}$-module generators $g_{s,i}$ for P_s, and the values $\partial_s(g_{s,i})$ in P_{s-1} of the boundary homomorphism ∂_s on these generators. These values will be expressed as sums of elements in the free $\mathscr{A}$-module on the generators $g_{s-1,j}$ of P_{s-1}. [[What happens for $s = 0$?]]

The computation will be done one internal degree t at a time, assuming that the calculations for lower internal degrees have already been done. The first line of each `Diff.s` contains two numbers. The second is the internal degree t up to which the calculation of P_s and ∂_s has been completed, so far. The first is the number of generators that have been added to P_s, in internal degrees less than or equal to t. Both of these numbers are set to 0 at the outset, when the module is created with `newmodule`. [[Can this confuse the program if M starts in negative degrees, and `dims` is started at $t = 0$?]]

[[Explain format of `Diff`-files.]]

11.6. **Resolving a module.** To resolve a module M, created from a module definition file `M.def` in `ext/A/samples` using `./newmodule M samples/M.def` in `ext/A`, move into `ext/A/M` using `cd M`. (In general, replace `M` by the directory name chosen for the module.) Suppose that the module M is concentrated in internal degrees $t \ge 0$, and that we want to make the calculation up to internal degree $t = 60$. Then we use the script `dims`, which automatically starts a series of scripts `nextt`, each handling one t at a time. To calculate in the range just mentioned, use `./dims 0 60`.

In general, the command `./dims a b` in the directory `ext/A/M` will calculate the resolution P_* for $0 \le s \le s_{max}$ in the range of internal degrees $a \le t \le b$, under the assumption that the calculation is already finished for $t < a$, starting with $t = a$ and working its way up.

For each t, the calculation proceeds on s at a time, calculating the kernel of $\partial_{s-1}\colon P_{s-1} \to P_{s-2}$ in degree t, identifying the image of $\partial_s\colon P_s \to P_{s-1}$ when restricted to the generators of P_s in internal degrees less than t, and choosing an $\mathbb{F}_2$-basis for a complementary subspace. For each basis vector v_i, an $\mathscr{A}$-module generator $g_{s,i}$ is added to P_s in internal degree t, and $\partial_s(g_{s,i})$ is set equal to v_i. [[Is this a fair representation of how the program actually works?]]

The subdirectory `ext/A/M/logs` will contain log files, recording the progress made. Use `ls logs` in `ext/A/M` to get a quick look at the progress, or try `ls -lrt logs` for more detailed timing information.

After `dims` is finished, the calculation can be continued with another call to the same script, for instance by `./dims 61 100`.

[[Explain `report` and `display`.]]

[[ETC]]

12. Convergence of the Adams spectral sequence

12.1. The Hopf–Steenrod invariant. For $p = 2$, the standard notation for the class $\gamma_{1,i}$, dual to the indecomposable Sq^{2^i}, is h_i. See Adams (1958). The h is for Hopf, since these classes detect the stable maps of spheres with Hopf invariant one.

Lemma 12.1. $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A}) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$ *and* $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2) \cong \mathbb{F}_2\{h_i \mid i \geq 0\}$ *where h_i has bidegree $(s,t) = (1, 2^i)$ and is dual to Sq^{2^i}, for each $i \geq 0$.*

Proof. There exists a free resolution $\cdots \to P_1 \to P_0 \to \mathbb{F}_2 \to 0$ where $P_0 = \mathscr{A}$ and $P_1 = \mathscr{A}\{g_{1,i}\}_i$ with $\partial_1 \colon g_{1,i} \mapsto Sq^{2^i}$ for all $i \geq 0$. The resolution is exact at P_0 since the Sq^{2^i} generate the left ideal $I(\mathscr{A}) \subset \mathscr{A}$, and it is minimal there since $\partial_1(P_1) \subset I(\mathscr{A})P_0$. It is also minimal at P_1, since the surjection $P_1 \to I(\mathscr{A})$ induces an isomorphism $\mathbb{F}_2\{g_{1,i}\}_i = \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 = P_1/I(\mathscr{A})P_1 \to I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A})$, so that $\partial_2(P_2) = \ker(\partial_1) \subset I(\mathscr{A})P_1$. Hence $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 \cong Q(\mathscr{A})$ and $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \cong \mathbb{F}_2\{h_i\}_i$, as claimed. ((Proof using bar complex?)) $\square$

Lemma 12.2. *For p odd,* $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{\beta, P^{p^i} \mid i \geq 0\}$ *and* $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{a_0, h_i \mid i \geq 0\}$, *where a_0 has bidegree $(s,t) = (1,1)$ and is dual to β, and h_i has bidegree $(s,t) = (1, 2p^i(p-1))$ and is dual to P^{p^i}, for each $i \geq 0$.*

Proof. The proof is similar to the case $p = 2$, using a free resolution $\epsilon \colon P_* \to \mathbb{F}_p$, with $P_0 = \mathscr{A}$ and $P_1 = \mathscr{A}\{g_{1,0}, g_{1,i+1} \mid i \geq 0\}$, where $\partial_1(g_{1,0}) = \beta$ and $\partial_1(g_{1,i+1}) = P^{p^i}$ for each $i \geq 0$. $\square$

We shall soon prove that the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)^\wedge_2$$

converges to the 2-adic completion of the stable homotopy groups of spheres. The chart in Theorem 10.12 above displays the E_2-term in the range $t \leq 11$. [[EDIT FROM HERE TO TAKE INTO ACCOUNT THE ADAMS VANISHING LINE.]] We will see later that the pattern above the diagonal line, where $s > t-s$, continues. There is an isomorphism $\mathrm{Ext}^{s,s}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \geq 0$, while $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $t - s < 0$ and for $0 < t - s < s$. Thus the groups labeled $\cdot$ in the chart are 0. Granting this, the only possible d_r-differentials starting in total degree $t - s \leq 6$, for $r \geq 2$, are the ones starting on $\gamma_{1,1} = h_1$ and landing in the group generated by $\gamma_{r+1,0}$.

However, these differentials are all 0, as can be seen either by proving that $\gamma_{s,0}$ detected $2^s \in \pi_0(S)$, or that $\gamma_{1,1}$ detects $\eta \in \pi_1(S)$, or by appealing to multiplicative structure in the spectral sequence. Granting this, we can conclude that $E_2 = E_\infty$ in this range of degrees, so that the groups $\mathbb{F}_2\{\gamma_{s,i}\}$ in one topological degree $n = t - s$, for $s \geq 0$ and $n \leq 5$ are the filtration quotients of a complete Hausdorff filtration $\{F^s\}_s$ that exhausts $\pi_n(S)^\wedge_2$.

For $n = 0$, we already know that $\pi_0(S) = \mathbb{Z}$ so $\pi_0(S)^\wedge_2 = \mathbb{Z}_2$. The only possible filtration is the 2-adic one, with $F^s = 2^s\mathbb{Z}_2 \subset \mathbb{Z}_2$ and $F^s/F^{s+1} \cong 2^s\mathbb{Z}_2/2^{s+1}\mathbb{Z}_2 \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \geq 0$. For $n = 1$ we deduce that $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{1,1}\} = \mathbb{Z}/2\{h_1\}$. In fact $\pi_1(S) = \mathbb{Z}/2\{\eta\}$ is generated by the complex Hopf map $\eta \colon S^1 \to S$. For $n = 2$ we deduce that $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{2,1}\}$. We shall see later that $\pi_2(S) = \mathbb{Z}/2\{\eta^2\}$ is generated by the composite $\eta^2 = \eta \circ \Sigma\eta \colon S^2 \to S$. For $n = 3$ we deduce that $\pi_3(S)^\wedge_2$ is an abelian group of order 8. We shall see later that $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/(8)$ is the 2-Sylow subgroup of $\pi_3(S) \cong \mathbb{Z}/24$, generated by the quaternionic Hopf map $\nu \colon S^3 \to S$. Finally, for now, we conclude that $\pi_4(S)^\wedge_2 = 0$ and $\pi_5(S)^\wedge_2 = 0$, and in fact $\pi_4(S) = \pi_5(S) = 0$. [[EDIT TO HERE.]]

Lemma 12.3. *(Hopf, Steenrod) For $p = 2$, let $f \colon S^n \to S$ be a map with $0 = f^* \colon H^*(S) \to H^*(S^n)$, and let $C_f = \mathrm{hocofib}(f) = S \cup_f e^{n+1}$ be its mapping cone. Suppose that $Sq^{n+1} \colon H^0(C_f) \to H^{n+1}(C_f)$ is nonzero. Then $n + 1 = 2^i$ for some $i \geq 0$ and $[f] \in \pi_n(S)$ is detected in the Adams spectral sequence by $h_i \in E_2^{1,2^i}$.*

Proof. Consider the canonical Adams tower for $Y = S$, with $Y^0 = S$, $K^0 = H$, $Y^1 = \Sigma^{-1}\bar{H}$ and $K^1 = H \wedge \Sigma^{-1}\bar{H}$. The composite $j \circ f$ is null-homotopic, since $d(f) = f^* = 0$, so we have a map of cofiber sequences:

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "e"] & S \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "d"] & S^{n+1} \arrow[d, "\Sigma e"] \\
\Sigma^{-1}\bar{H} \arrow[r, "i"] \arrow[d, "j"] & S \arrow[r, "j"] & H \arrow[r, "\partial"] & \bar{H} \\
H \wedge \Sigma^{-1}\bar{H} & & &
\end{tikzcd}$$

Here $d\colon C_f \to H$ and $e\colon S^n \to \Sigma^{-1}\bar{H}$ are determined by a null-homotopy of f. Applying cohomology to the right hand part of the diagram, we get a map of $\mathscr{A}$-module extensions:

$$\begin{tikzcd}
\mathbb{F}_2 & H^*(C_f) \arrow[l, two heads] & \Sigma^{n+1}\mathbb{F}_2 \arrow[l, tail] \\
\mathbb{F}_2 \arrow[u, equal] & \mathscr{A} \arrow[l, two heads, "j^*"'] \arrow[u, "d^*"'] & I(\mathscr{A}) \arrow[l, tail, "\partial^*"'] \arrow[u, "\Sigma e^*"']
\end{tikzcd}$$

Here $d^*(1) = 1$, so by assumption $d^*(Sq^{n+1}) \neq 0$. Hence $\Sigma e^*(Sq^{n+1}) \neq 0$. This is impossible if Sq^{n+1} is decomposable, so we must have $n + 1 = 2^i$ for some $i \geq 0$. Then $e^* \neq 0$, which implies that $j \circ e\colon S^n \to H \wedge \Sigma^{-1}\bar{H}$ is essential (= not null-homotopic).

This proves that $[f] \in \pi_n(S)$ lifts to $\pi_n(Y^1)$ but not to $\pi_n(Y^2)$, hence corresponds under the isomorphism $F^1/F^2 \cong E_\infty^{1,*}$ to a nonzero class in $E_\infty^{1,2^i} \subset E_2^{1,2^i} = \mathbb{F}_2\{h_i\}$. The only possibility is that $[f]$ is detected by h_i. $\square$

Lemma 12.4. *(Hopf, Steenrod) For p odd, let $f\colon S^n \to S$ be a map with $0 = f^*\colon H^*(S) \to H^*(S^n)$, and let $C_f = \mathrm{hocofib}(f) = S \cup_f e^{n+1}$ be its mapping cone. Suppose that $P^k\colon H^0(C_f) \to H^{n+1}(C_f)$ is nonzero, with $n + 1 = 2k(p-1)$. Then $k = p^i$ for some $i \geq 0$ and $[f] \in \pi_n(S)$ is detected in the Adams spectral sequence by $h_i \in E_2^{1,2p^i(p-1)}$. Alternatively, suppose that $\beta\colon H^0(C_f) \to H^1(C_f)$ is nonzero. Then $n = 0$ and $[f] \in \pi_0(S)$ is detected by $a_0 \in E_2^{1,1}$.*

Proof. The proof is similar to the 2-primary case. $\square$

The class of $\Sigma e^* \circ \partial_1\colon P_1 \to \Sigma^{n+1}\mathbb{F}_p$ in $\mathrm{Ext}_{\mathscr{A}}^{1,n+1}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{h_i\}$ is called the Hopf–Steenrod invariant, or the cohomology e-invariant, of $[f]$. It is only defined for the $[f]$ with vanishing d-invariant. More generally, we have a diagram

$$\begin{tikzcd}
F^2 \arrow[r, tail] & F^1 \arrow[r, tail] \arrow[d, "e"] & F^0 = [X, Y]_n \arrow[d, "d"] \\
& \mathrm{Ext}_{\mathscr{A}}^{1,n+1}(H^*(X), H^*(Y)) & \mathrm{Hom}_{\mathscr{A}}^n(H^*(X), H^*(Y))
\end{tikzcd}$$

for each pair of spectra X and Y.

Theorem 12.5. *The Hopf maps $2\colon S \to S$, $\eta\colon S^1 \to S$, $\nu\colon S^3 \to S$ and $\sigma\colon S^7 \to S$ are detected in the Adams spectral sequence by the classes h_0, h_1, h_2 and h_3, respectively. These are infinite cycles in the spectral sequence.*

Proof. In each case, $f\colon S^n \to S$ is the stable form of a fibration $\Sigma^{n+1}f\colon S^{2n+1} \to S^{n+1}$, with mapping cone a projective plane P^2. Here $H^*(P^2) = P(x)/(x^3) = \mathbb{F}_2\{1, x, x^2\}$, where $|x| = n+1$, by Poincaré duality Hence $Sq^{n+1}(x) = x^2 \neq 0$, and the previous lemma applies. Quite explicitly, $\Sigma C_2 = \mathbb{R}P^2$ has a nonzero Sq^1, $\Sigma^2 C_\eta = \mathbb{C}P^2$ has a nonzero Sq^2, $\Sigma^4 C_\nu = \mathbb{H}P^2$ has a nonzero Sq^4 and $\Sigma^8 C_\sigma = \mathbb{O}P^2$ has a nonzero Sq^8. $\square$

The names η, ν and σ for the Hopf maps detected by h_1, h_2 and h_3 are supposedly unrelated to the correspondence between the initial phonemes in the Greek letters "eta", "nu" and "sigma" and in the first three Japanese numerals "ichi", "ni" and "san". We shall see later that none of the classes h_i for $i \geq 4$ survive to the E_∞-term, so there are no maps $S^n \to S$ with nonzero Hopf–Steenrod invariant for $n \geq 8$.

Theorem 12.6. *Let p be odd. There are maps $p\colon S \to S$ and $\alpha_1\colon S^{2p-3} \to S$ that are detected in the Adams spectral sequence by the classes a_0 and h_0, respectively. These are infinite cycles in the spectral sequence.*

Proof. The Bockstein homomorphism β acts nontrivially in the cohomology $H^*(C_p)$ of the mapping cone $C_p = S \cup_p e^1$ of the degree p map $p\colon S \to S$, so $[p] \in \pi_0(S)$ is detected in the Adams spectral sequence by a_0.

The map $\alpha_1 \in \pi_{2p-3}(S)$ is the stable image of the generator of $\pi_{2p}(S^3)^\wedge_p \cong \mathbb{Z}/p$ that we discussed in Theorem 5.3. It can be constructed as the stable attaching map of the $2p$-cell to the 2-cell in $\mathbb{C}P^p$, after p-completion, but this requires proving that the attaching map $\phi\colon S^{2p-1} \to \mathbb{C}P^{p-1}$ compresses into $i\colon S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^{p-1}$. For each $2 \le k \le p-1$ the obstruction to compressing a map $S^{2p-1} \to \mathbb{C}P^k$ into $\mathbb{C}P^{k-1}$ lies in $\pi_{2p-1}(S^{2k}) \cong \pi_{2(p-k)-1}(S)$, so if we assume that we know that this group is trivial, after p-completion, then ϕ compresses as $i \circ \alpha$ for a map $\alpha\colon S^{2p-1} \to S^2$. [[Another proof of this fact can be given using the action of roots of unity in $\mathbb{Z}_p$ on $(\mathbb{C}P^p)^\wedge_p$.]] Then i induces a map $j\colon C_\alpha = S^2 \cup_\alpha e^{2p} \to \mathbb{C}P^p$, and $j^*\colon H^*(\mathbb{C}P^p) = \mathbb{F}_p[y]/(y^{p+1}) \to H^*(C_\alpha)$ maps 1 and y^p to generators of $H^*(C_\alpha)$. Since $P^1(y) = y^p$ in $H^*(\mathbb{C}P^p)$, it follows that P^1 acts nontrivially in $H^*(C_\alpha)$, so the stable class α_1 of α is detected by h_0, as claimed. $\square$

12.2. **Naturality.** The essential uniqueness of free resolutions lifts to the level of spectral realizations. Consider diagrams

$$\cdots \to Y^{s+1} \xrightarrow{i} Y^s \to \cdots \to Y^0 = Y$$

and

$$\cdots \to Z^{s+1} \xrightarrow{i} Z^s \to \cdots \to Z^0 = Z$$

with cofibers $K^s = \operatorname{hocofib}(Y^{s+1} \to Y^s)$ and $L^s = \operatorname{hocofib}(Z^{s+1} \to Z^s)$ for all $s \ge 0$. There are associated chain complexes

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0$$

of $\mathscr{A}$-modules, where $P_s = H^*(\Sigma^s K^s)$, $Q_s = H^*(\Sigma^s L^s)$, $\partial_s = \partial^* j^*$ and $\epsilon = j^*$.

Theorem 12.7. *Suppose that (a) each cofiber L^s is a wedge sum of Eilenberg–Mac Lane spectra that is bounded below and of finite type, and (b) each map $i\colon Y^{s+1} \to Y^s$ induces the zero map on cohomology. (For instance, the diagrams $\{Y^s\}_s$ and $\{Z^s\}_s$ might be Adams resolutions.) Let $f\colon Y \to Z$ be any map.*

(1) *Each Q_s is a free $\mathscr{A}$-module, and the augmented chain complex $\epsilon\colon P_* \to H^*(Y) \to 0$ is exact.*

(2) *There exists a chain map $g_*\colon Q_* \to P_*$ lifting f^*, in the sense that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \arrow[r] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2"] \arrow[u, "g_2"] & Q_1 \arrow[r, "\partial_1"] \arrow[u, "g_1"] & Q_0 \arrow[r, "\epsilon"] \arrow[u, "g_0"] & H^*(Z) \arrow[r] \arrow[u, "f^*"] & 0
\end{tikzcd}$$

commutes. Furthermore, there is a map of diagrams $\{f^s\colon Y^s \to Z^s\}_s$ lifting f and realizing g_*, in the sense that there is a homotopy commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "f^2"] & Y^1 \arrow[r, "i"] \arrow[d, "f^1"] & Y \arrow[d, "f"] \\
\cdots \arrow[r] & Z^2 \arrow[r, "i"] & Z^1 \arrow[r, "i"] & Z ,
\end{tikzcd}$$

and given any choice of commuting homotopies, the induced map of homotopy cofibers $g^s\colon K^s \to L^s$ induces $g_s = (\Sigma^s g^s)^*\colon Q_s \to P_s$, for each $s \ge 0$.

(3) *If $\bar g_*\colon Q_* \to P_*$ is a second chain map lifting f^*, and $\{\bar f^s\}_s$ is a map of diagrams lifting f and realizing $\bar g_*$, then g_* and $\bar g_*$ are chain homotopic, and $\{f^s\}_s$ and $\{\bar f^s\}_s$ are homotopic in the weak sense that the composites $f^s \circ i$ and $\bar f^s \circ i\colon Y^{s+1} \to Z^s$ are homotopic for all $s \ge 0$.*

Proof. Freeness of each Q_s is clear from the wedge sum decomposition of L^s. Exactness of $\epsilon\colon P_* \to H^*(Y) \to 0$ is clear from the vanishing of i^*. The existence of a chain map g_* lifting f^* is then standard homological algebra. We need to construct the maps f^s and g^s in a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & Y^2 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd, "f^2"] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd, "f^1"] & & Y \arrow[dl, "j"] \arrow[dd, "f"] \\
\cdots & & K^1 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^1"] & & K^0 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^0"] & \\
\cdots \arrow[r] & Z^2 \arrow[rr, "i"] \arrow[dl, "j"] & & Z^1 \arrow[rr, "i"] \arrow[dl, "j"] & & Z \arrow[dl, "j"] \\
\cdots & & L^1 \arrow[ul, dashed, "\partial"] & & L^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

of spectra, inducing a commutative diagram

$$\begin{tikzcd}
& H^*(\Sigma^2 Y^2) \arrow[dr, hook, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, hook, "\partial^*"] & & H^*(Y) \\
\cdots \arrow[ur, two heads, "j^*"] \arrow[rr] & & H^*(\Sigma K^1) \arrow[ur, two heads, "j^*"] \arrow[rr] & & H^*(K^0) \arrow[ur, two heads, "j^*"] & \\
& H^*(\Sigma^2 Z^2) \arrow[uu, "(\Sigma^2 f^2)^*"] \arrow[dr, "\partial^*"] & & H^*(\Sigma Z^1) \arrow[uu, "(\Sigma f^1)^*"] \arrow[dr, "\partial^*"] & & H^*(Z) \arrow[uu, "f^*"] \\
\cdots \arrow[ur, "j^*"] \arrow[rr] & & H^*(\Sigma L^1) \arrow[uu, "(\Sigma g^1)^*"] \arrow[ur, "j^*"] \arrow[rr] & & H^*(L^0) \arrow[uu, "(g^0)^*"] \arrow[ur, "j^*"] &
\end{tikzcd}$$

of $\mathscr{A}$-modules, with $g_s = (\Sigma^s g^s)^*$.

Inductively, suppose the maps $f = f^0, \dots, f^s$ and $g^0, \dots, g^{s-1}$ are given, for some $s \geq 0$, making the diagram to the right of f^s commute up to homotopy. Then $j^* \circ g_s = (\Sigma^s f^s)^* \circ j^*$, by the assumption that g_0 lifts f^* for $s = 0$, and by the assumption that $\partial^* j^* \circ g_s = g_{s-1} \circ \partial^* j^* = \partial^* (\Sigma^s f^s)^* \circ j^*$ and the injectivity of ∂^* for $s \geq 1$.

We have an isomorphism $[K^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(K^s))$, so there is a unique homotopy class of maps $g^s\colon K^s \to L^s$ with $(\Sigma^s g^s)^* = g_s$. Note that $g^s \circ j\colon Y^s \to L^s$ is homotopic to $j \circ f^s\colon Y^s \to L^s$, because of the isomorphism $[Y^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(Y^s))$ and the fact that $(g^s \circ j)^* = (j \circ f^s)^*$. (Both isomorphisms follow from hypothesis (a)).

Choosing a commuting homotopy and passing to mapping cones, or appealing to the triangulated structure on the stable category of spectra, we can find a map of homotopy fibers $f^{s+1}\colon Y^{s+1} \to Z^{s+1}$ making the diagram

$$\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] \arrow[d, "f^{s+1}"'] & Y^s \arrow[r, "j"] \arrow[d, "f^s"'] & K^s \arrow[r, "\partial"] \arrow[d, "g^s"'] & \Sigma Y^{s+1} \arrow[d, "\Sigma f^{s+1}"'] \\
Z^{s+1} \arrow[r, "i"] & Z^s \arrow[r, "j"] & L^s \arrow[r, "\partial"] & \Sigma Z^{s+1}
\end{tikzcd}$$

commute up to homotopy. This completes the inductive step.

The uniqueness of g_* up to chain homotopy, meaning that any other lift $\bar{g}_*$ is chain homotopic to g_*, is standard homological algebra. We prove that $f^s \circ i$ is homotopic to $\bar{f}^s \circ i$ by induction on s. This is clear for $s = 0$, since $f_0 = \bar{f}_0 = f$. Suppose that $i \circ f^s \simeq f^{s-1} \circ i$ is homotopic to $i \circ \bar{f}^s \simeq \bar{f}^{s-1} \circ i\colon Y^s \to Z^{s-1}$, for some $s \geq 1$.

$$\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] & Y^s \arrow[rr, "i"] \arrow[d, shift right, "f^s"'] \arrow[d, shift left, "\bar{f}^s"] & & Y^{s-1} \arrow[d, shift right, "f^{s-1}"'] \arrow[d, shift left, "\bar{f}^{s-1}"] \\
& Z^s \arrow[rr, "i"] & & Z^{s-1} \\
& & \Sigma^{-1} L^{s-1} \arrow[ul, "\partial"] &
\end{tikzcd}$$

Then $i \circ (\bar{f}^s - f^s)$ is null-homotopic, so that $\bar{f}^s - f^s$ factors through a map $h\colon Y^s \to \Sigma^{-1}L^{s-1}$. Then $\bar{f}^s \circ i - f^s \circ i = (\bar{f}^s - f^s) \circ i$ factors through $h \circ i\colon Y^{s+1} \to \Sigma^{-1}L^{s-1}$. This map induces $i^* \circ h^* = 0$ in cohomology, hence is null-homotopic because of the isomorphism $[Y^{s+1}, \Sigma^{-1}L^{s-1}] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^{-1}L^{s-1}), H^*(Y^{s+1}))$. In other words, $f^s \circ i \simeq \bar{f}^s \circ i$. $\square$

Corollary 12.8. *Let $f\colon Y \to Z$ be a map of bounded below spectra with $H_*(Y)$ and $H_*(Z)$ of finite type. Then there is a well-defined map*

$$f_*\colon \{E_r(Y), d_r\}_r \longrightarrow \{E_r(Z), d_r\}_r$$

of Adams spectral sequences for $r \geq 2$, given at the E_2-level by the homomorphism

$$(f^*)^*\colon \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y), \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}$-module homomorphism $f^*\colon H^*(Z) \to H^*(Y)$, with expected abutment the homomorphism

$$f_*\colon \pi_*(Y) \to \pi_*(Z)\,.$$

(Similarly for the Adams spectral sequences converging to $[X, Y]_*$ and $[X, Z]_*$, for any spectrum X.)

[[The well-defined map of E_2-terms uniquely determines the maps of the following E_r-terms.]]

Lemma 12.9. *Let $\{Y^s\}_s$ and $\{Z^s\}_s$ be Adams resolutions of a bounded below spectrum Y with $H_*(Y)$ of finite type. Then there is a homotopy equivalence* $\mathrm{holim}_s\, Y^s \simeq \mathrm{holim}_s\, Z^s$.

Proof. There are maps $\{f^s\colon Y^s \to Z^s\}_s$ and $\{\tilde{f}^s\colon Z^s \to Y^s\}_s$ of resolutions covering the identity map of $Y = Y^0 = Z^0$, and homotopies $\tilde{f}^s \circ f^s \circ i \simeq i\colon Y^{s+1} \to Y^s$ and $f^s \circ \tilde{f}^s \circ i \simeq i\colon Z^{s+1} \to Z^s$, for all $s \geq 0$. Hence $\mathrm{holim}_s\, f^s$ and $\mathrm{holim}_s\, \tilde{f}^s$ are homotopy inverses. $\square$

Theorem 12.10. *Let $\{Y^s\}_s$ be an Adams resolution of Y, and let X be any spectrum. (The case $X = S$ is of particular interest.) A class $[f] \in [X, Y]_n$ has Adams filtration $\geq s$, i.e., is in the image F^s of $i^s\colon [X, Y^s]_n \to [X, Y]_n$, if and only if the representing map $f\colon \Sigma^n X \to Y$ can be factored as the composite of s maps*

$$\Sigma^n X = X_s \xrightarrow{z_s} X_{s-1} \xrightarrow{z_{s-1}} \dots \xrightarrow{z_2} X_1 \xrightarrow{z_1} X_0 = Y$$

where $0 = z_u^*\colon H^*(X_{u-1}) \to H^*(X_u)$ for each $1 \leq u \leq s$. In particular, $F^s \subset [X, Y]_*$ is independent of the choice of Adams resolution.

Proof. If $[f]$ has Adams filtration $\geq s$, let $g\colon \Sigma^n X \to Y^s$ be a lift, with $i^s \circ g \simeq f$. Let $X_u = Y^u$ and $z_u = i$ for $0 \leq u \leq s-1$, and let $z_s = ig$:

$$\Sigma^n X \xrightarrow{ig} Y^{s-1} \xrightarrow{i} \dots \xrightarrow{i} Y^1 \xrightarrow{i} Y$$

Conversely, given a factorization $f = z_1 \circ \dots \circ z_s$ as above, let $f^0\colon Y \to Y$ be the identity map. We can inductively find lifts $f^u\colon X_u \to Y^u$ making the diagram

$$\begin{CD}
X_s @>{z_s}>> X_{s-1} @>{z_{s-1}}>> \dots @>{z_2}>> X_1 @>{z_1}>> Y \\
@V{f^s}VV @V{f^{s-1}}VV @. @V{f^1}VV @VVV \\
Y^s @>{i}>> Y^{s-1} @>{i}>> \dots @>{i}>> Y^1 @>{i}>> Y
\end{CD}$$

commute, since the obstruction to lifting $f^{u-1} \circ z_u\colon X_u \to Y^{u-1}$ over $i\colon Y^u \to Y^{u-1}$ is the homotopy class of the composite $j \circ f^{u-1} \circ z_u\colon X_u \to K^{u-1}$, which is zero because $z_u^* = 0$. Let $g = f^s\colon \Sigma^n X \to Y^s$. Then $i^s \circ g \simeq f$, and $[f]$ has Adams filtration $\geq s$. $\square$

12.3. **Convergence.**

Definition 12.11. For each natural number m let the mod m Moore spectrum $S/m = S \cup_m e^1$ be defined by the homotopy cofiber sequence

$$S \xrightarrow{m} S \longrightarrow S/m \longrightarrow S^1$$

where the map m induces multiplication by m in integral (co-)homology. Note that $H_*(S/m; \mathbb{Z}) \cong \mathbb{Z}/m$ is concentrated in degree 0. For any spectrum Y let $Y/m = Y \wedge S/m$, so that there is a cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow Y/m \longrightarrow \Sigma Y\,.$$

Applying $F(-, Y)$ to the cofiber sequence

$$S^{-1} \longrightarrow S^{-1}/m \longrightarrow S \xrightarrow{m} S$$

leads to the cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow F(S^{-1}/m, Y) \longrightarrow \Sigma Y$$

and an equivalence $Y/m \simeq F(S^{-1}/m, Y)$.

Definition 12.12. For each prime p there is a horizontal tower of vertical cofiber sequences

$$\begin{tikzcd}
\dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^e"'] & \dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^2"] & S \arrow[d, "p"] \\
\dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & S \arrow[d] \\
\dots \arrow[r] & S/p^e \arrow[r] \arrow[d] & \dots \arrow[r] & S/p^2 \arrow[r] \arrow[d] & S/p \arrow[d] \\
\dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & \dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & S^1
\end{tikzcd}$$

We define the *p-completion* of Y as the homotopy limit $Y_p^\wedge = \mathrm{holim}_e\, Y/p^e$ of the tower

$$\dots \to Y \wedge S/p^e \to \dots \to Y \wedge S/p^2 \to Y \wedge S/p\,.$$

The maps $S \to S/p^e$ induce the p-completion map $Y \to Y_p^\wedge$.

Dually there is a horizontal sequence of vertical cofiber sequence

$$\begin{tikzcd}
S^{-1} \arrow[r, "p"] \arrow[d] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \arrow[r, "p"] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \\
S^{-1}/p \arrow[r] \arrow[d] & S^{-1}/p^2 \arrow[r] \arrow[d] & \dots \arrow[r] & S^{-1}/p^e \arrow[r] \arrow[d] & \dots \\
S \arrow[r, "="] \arrow[d, "p"'] & S \arrow[r, "="] \arrow[d, "p^2"'] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d, "p^e"'] & \dots \\
S \arrow[r, "p"] & S \arrow[r, "p"] & \dots \arrow[r, "p"] & S \arrow[r, "p"] & \dots
\end{tikzcd}$$

Let $S^{-1}/p^\infty = \mathrm{hocolim}_e\, S^{-1}/p^e$. Note that $H_*(S^{-1}/p^\infty;\mathbb{Z}) \cong \mathbb{Z}/p^\infty \cong \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Applying $F(-, Y)$ we get the tower defining the p-completion, so

$$Y_p^\wedge \simeq F(S^{-1}/p^\infty, Y)\,.$$

The map $S^{-1}/p^\infty \to S$ induces the p-completion map $Y \to Y_p^\wedge$.

((See Bousfield.))

Lemma 12.13. *The p-completion map induces an equivalence $Y/p^e \to (Y_p^\wedge)/p^e$ for each e. Hence it induces an isomorphism $H_*(Y) \cong H_*(Y_p^\wedge)$ in mod p homology (and cohomology). The p-completion map $Y/p^e \to (Y/p^e)_p^\wedge$ for Y/p^e is also an equivalence.*

Proof. The map $S^{-1}/p^\infty \to S$ induces an equivalence $S^{-1}/p^e \wedge S^{-1}/p^\infty \to S^{-1}/p^e \wedge S = S^{-1}/p^e$, for each e, since $p^{-1}\pi_*(S/p^e) = 0$. Apply $F(-, Y)$ to get the first conclusion. Apply integral homology to the equivalence $Y/p \to (Y_p^\wedge)/p$ to get the second conclusion. Applying $F(-, Y)$ to the interchanged equivalence $S^{-1}/p^\infty \wedge S^{-1}/p^e \to S \wedge S^{-1}/p^e$ leads to the third conclusion. $\square$

Lemma 12.14. *The p-completion of the p-completion map for Y, and the p-completion map for $Y_p^\wedge$, are equivalences $Y_p^\wedge \to (Y_p^\wedge)_p^\wedge$. In either sense, p-completion is idempotent up to equivalence.*

Proof. Use that the map $S^{-1}/p^\infty \to S$ induces equivalences $S^{-1}/p^\infty \wedge S^{-1}/p^\infty \to S^{-1}/p^\infty \wedge S$ and $S^{-1}/p^\infty \wedge S^{-1}/p^\infty \to S \wedge S^{-1}/p^\infty$, and apply $F(-, Y)$, or pass to homotopy limits over e from the previous lemma. $\square$

Lemma 12.15. *Let $\pi_n(Y)_p^\wedge = \lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e$ be the algebraic p-completion of $\pi_n(Y)$. There is a short exact sequence*

$$0 \to \pi_n(Y)_p^\wedge \to \lim_e \pi_n(Y/p^e) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(Y)) \to 0$$

and an isomorphism $\mathrm{Rlim}_e\,\pi_{n+1}(Y/p^e) \cong \mathrm{Rlim}_e\,\mathrm{Hom}(\mathbb{Z}/p^e, \pi_n(Y))$*. If* $\pi_*(Y)$ *is of finite type, i.e., if* $\pi_n(Y)$ *is finitely generated for each* n*, then* $\pi_n(Y) \otimes \mathbb{Z}_p \cong \pi_n(Y)^\wedge_p \cong \pi_n(Y^\wedge_p)$ *for all* n*.*

Proof. We have a tower of short exact sequences

$$0 \to \pi_n(Y) \otimes \mathbb{Z}/p^e \longrightarrow \pi_n(Y/p^e) \longrightarrow \mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \to 0$$

for $e \geq 1$. Each homomorphism $\pi_n(Y)\otimes\mathbb{Z}/p^{e+1} \to \pi_n(Y)\otimes\mathbb{Z}/p^e$ is surjective, so $\mathrm{Rlim}_e\,\pi_n(Y)\otimes\mathbb{Z}/p^e = 0$. Hence the associated lim-Rlim exact sequence breaks up into a short exact sequence

$$0 \to \lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e \longrightarrow \lim_e \pi_n(Y/p^e) \longrightarrow \lim_e \mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \to 0$$

and an isomorphism

$$\operatorname*{Rlim}_e \pi_n(Y/p^e) \cong \operatorname*{Rlim}_e \mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y))\,.$$

Here $\lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e = \pi_n(Y)^\wedge_p$ and $\lim_e \mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(Y))$.

If $\pi_n(Y)$ is finitely generated, then clearly $\pi_n(Y)\otimes\mathbb{Z}_p \cong \pi_n(Y)^\wedge_p$. Furthermore, each $\mathrm{Hom}(\mathbb{Z}/p^e, \pi_n(Y))$ is finite, so $\mathrm{Rlim}_e\,\pi_{n+1}(Y/p^e) = \mathrm{Rlim}_e\,\mathrm{Hom}(\mathbb{Z}/p^e, \pi_n(Y)) = 0$. If also $\pi_{n-1}(Y)$ is finitely generated, then its p-torsion subgroup is annihilated by p^N for some fixed N. Hence $\mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \subset \pi_{n-1}(Y)$ equals that p-torsion subgroup for all $e \geq N$, and the homomorphisms in the limit system induce multiplication by p, hence are nilpotent. Thus $\lim_e \mathrm{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) = 0$. Thus the lim-Rlim exact sequence

$$0 \to \operatorname*{Rlim}_e \pi_{n+1}(Y/p^e) \longrightarrow \pi_n(Y^\wedge_p) \longrightarrow \lim_e \pi_n(Y/p^e) \to 0$$

for $Y^\wedge_p = \mathrm{holim}_e\, Y/p^e$ simplifies to an isomorphism $\pi_n(Y^\wedge_p) \cong \lim_e \pi_n(Y/p^e)$, and the short exact sequence above simplifies to another isomorphism $\pi_n(Y)^\wedge_p \cong \lim_e \pi_n(Y/p^e)$. □

Example 12.16. (1) $H \simeq H^\wedge_p$ and $(H\mathbb{Z})^\wedge_p \simeq (H\mathbb{Z}_{(p)})^\wedge_p \simeq H\mathbb{Z}_p$.

(2) For $Y = H\mathbb{Z}[1/p]$ or $H\mathbb{Q}$ we have $Y/p^e \simeq *$ for all e, so $(H\mathbb{Z}[1/p])^\wedge_p \simeq (H\mathbb{Q})^\wedge_p \simeq *$.

(3) For $Y = H(\mathbb{Z}[1/p]/\mathbb{Z}) = H\mathbb{Z}/p^\infty$ or $H(\mathbb{Q}/\mathbb{Z})$ we have $Y/p^e \simeq \Sigma H\mathbb{Z}/p^e$ for all e, so $H(\mathbb{Z}[1/p]/\mathbb{Z})^\wedge_p = H(\mathbb{Z}/p^\infty)^\wedge_p \simeq H(\mathbb{Q}/\mathbb{Z})^\wedge_p \simeq \Sigma H\mathbb{Z}_p$.

Lemma 12.17. *Let* $0 \to \bigoplus_\alpha \mathbb{Z} \to \bigoplus_\beta \mathbb{Z} \to \mathbb{Z}_p \to 0$ *be a short free resolution of* $\mathbb{Z}_p$*. There is a corresponding cofiber sequence* $\bigvee_\alpha S \to \bigvee_\beta S \to S\mathbb{Z}_p$*, where* $H_*(S\mathbb{Z}_p;\mathbb{Z}) \cong \mathbb{Z}_p$ *is concentrated in degree* 0*. Then* $\pi_n(Y \wedge S\mathbb{Z}_p) \simeq \pi_n(Y) \otimes \mathbb{Z}_p$ *for all* n*. In particular,* $S^\wedge_p \simeq (S\mathbb{Z}_p)^\wedge_p \simeq S\mathbb{Z}_p$*. If* $\pi_*(Y)$ *is of finite type then the natural map* $Y \wedge S\mathbb{Z}_p \to Y^\wedge_p$ *is an equivalence, and* $H_*(Y) \to H_*(Y^\wedge_p)$ *is an isomorphism.*

Proof. ((Straightforward. TBW.)) □

Let $H\mathbb{Z}$ be the integral Eilenberg–Mac Lane spectrum, with $\pi_0(H\mathbb{Z}) = \mathbb{Z}$ and $\pi_i(H\mathbb{Z}) = 0$ for $i \neq 0$. It is a ring spectrum, with multiplication $\phi\colon H\mathbb{Z}\wedge H\mathbb{Z} \to H\mathbb{Z}$ and unit $\eta\colon S \to H\mathbb{Z}$. (Not to be confused with the Hopf map $\eta\colon S^1 \to S$.) Let $\overline{H\mathbb{Z}} = H\mathbb{Z}/S$ be the cofiber.

Lemma 12.18. $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}\{Sq^1\}$ *for* $p = 2$*, and* $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}\{\beta\}$ *for* p *odd.*

Proof. Since the unit map $S \to H\mathbb{Z}$ induces an isomorphism on π_0 and a surjection on π_1, we find that $\overline{H\mathbb{Z}}$ is 1-connected. Hence $H^1(H\mathbb{Z}) \cong H^1(\overline{H\mathbb{Z}}) = 0$.

There is a short exact sequence of $\mathscr{A}$-modules

$$0 \longleftarrow \mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow \mathscr{A} \longleftarrow \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow 0$$

where the right hand arrow takes $\Sigma 1$ to Sq^1. It is clear that $\Sigma Sq^I \mapsto Sq^I \circ Sq^1$ maps to 0, for admissible I, if and only if $I = (i_1, \dots, i_\ell)$ with $i_\ell = 1$. These Sq^I generate precisely the left ideal $\mathscr{A}\{Sq^1\}$.

There is also a cofiber sequence $H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \longrightarrow H \longrightarrow \Sigma H\mathbb{Z}$, where $2^* = 0$, so that there is an associated short exact sequence

$$0 \longleftarrow H^*(H\mathbb{Z}) \longleftarrow H^*(H) \longleftarrow \Sigma H^*(H\mathbb{Z}) \longleftarrow 0\,.$$

in cohomology. Let $\mathscr{A} \to H^*(H)$ be the isomorphism taking Sq^I to its value on the generator $1 \in H^0(H)$. The composite $\Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \to \mathscr{A} \to H^*(H) \to H^*(H\mathbb{Z})$ is zero, since the source is generated by $\Sigma 1$ in degree 1, and $H^1(H\mathbb{Z}) = 0$. Hence there is a map from the first short exact sequence of $\mathscr{A}$-modules to the second one. By induction, we may assume that the left hand homomorphism $f\colon \mathscr{A}/\mathscr{A}\{Sq^1\} \to H^*(H\mathbb{Z})$ is an isomorphism in degrees $* < t$. Then the right hand homomorphism $\Sigma f\colon \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \to \Sigma H^*(H\mathbb{Z})$ is an isomorphism in degrees $* \leq t$. Since the middle map is an isomorphism, it follows that the left hand homomorphism is an isomorphism, also in degree t.

The proof for odd p is similar, comparing the short exact sequence

$$0 \longleftarrow \mathscr{A}/\mathscr{A}\{\beta\} \longleftarrow \mathscr{A} \longleftarrow \Sigma\mathscr{A}/\mathscr{A}\{\beta\} \longleftarrow 0$$

to the short exact sequence

$$0 \longleftarrow H^*(H\mathbb{Z}) \longleftarrow H^*(H) \longleftarrow H^*(\Sigma H\mathbb{Z}) \longleftarrow 0\,.$$

□

Recall Boardman's notion of conditional convergence, meaning that $\lim_s A^s = 0$ and $\operatorname{Rlim}_s A^s = 0$, and the result that strong convergence follows from conditional convergence and the vanishing of the derived E_∞-term RE_∞. For the spectral sequence associated to an Adams resolution $\{Y^s\}_s$, conditional convergence is equivalent to the contractibility of the homotopy limit $Y^\infty = \operatorname{holim}_s Y^s$, in view of Milnor's short exact sequence

$$0 \to \operatorname*{Rlim}_s \pi_{n+1}(Y^s) \to \pi_n(\operatorname*{holim}_s Y^s) \to \lim_s \pi_n(Y^s) \to 0\,.$$

As we have seen before, the condition $\operatorname{holim}_s Y^s \simeq *$ is independent of the choice of Adams resolution.

Lemma 12.19. *Let Y be bounded below with $H_*(Y)$ of finite type. Then there is an Adams resolution $\{Z^s\}_s$ of $Z = Y/p$ with* $\operatorname{holim}_s Z^s \simeq *$.

((Enough that Y/p is bounded below with $H_*(Y/p)$ of finite type?))

Proof. The "canonical $H\mathbb{Z}$-based resolution"

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \arrow[r, "i"] \arrow[d, "j"] & S \arrow[d, "j"] \\
& H\mathbb{Z} \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} & H\mathbb{Z} \wedge \Sigma^{-1}\overline{H\mathbb{Z}} & H\mathbb{Z}
\end{tikzcd}$$

is not an Adams resolution, since $H\mathbb{Z}$ is not a wedge sum of mod p Eilenberg–Mac Lane spectra, but the ring spectrum structure ensures that $j = \eta \wedge 1 \colon X \to H\mathbb{Z} \wedge X$ induces a split injection $1 \wedge j \colon H \wedge X \to H \wedge H\mathbb{Z} \wedge X$, so that $j^* \colon H^*(H\mathbb{Z} \wedge X) \to H^*(X)$ is surjective, for each spectrum X.

Smashing this diagram with $Z = Y/p$, we get a diagram

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \wedge Y/p \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y/p \arrow[r, "i"] \arrow[d, "j"] & Y/p \arrow[d, "j"] \\
& H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \wedge Y & H \wedge \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y & H \wedge Y
\end{tikzcd}$$

where we have identified $H\mathbb{Z} \wedge X \wedge Y/p$ with $H \wedge X \wedge Y$, for suitable X. This is the desired Adams resolution, with $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$ and cofibers $L^s = H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$. The maps j are split injective, so each j^* is surjective, as before. Since $(\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$ is bounded below and $H_*((\overline{H\mathbb{Z}})^{\wedge s} \wedge Y) \cong H_*(\overline{H\mathbb{Z}})^{\otimes s} \otimes H_*(Y)$ is of finite type, it follows that each L^s is a wedge sum of suspended mod p Eilenberg–Mac Lane spectra, satisfying the finiteness condition required for an Adams resolution.

It remains to show that $\operatorname{holim}_s Z^s \simeq *$. This is true in the strong sense that in each topological degree n, $\pi_n(Z^s) = 0$ for all sufficiently large s. By assumption there is an integer N such that $\pi_n(Y) = 0$ for all $n < N$. We have seen that $\overline{H\mathbb{Z}}$ is 1-connected, so that $(\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s}$ is $(s-1)$-connected. Then $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$ is $(N+s-1)$-connected. Hence $\pi_n(Z^s) = 0$ for all $n \le N+s-1$, or equivalently, for all $s > n - N$. □

Theorem 12.20. *Let Y be bounded below with $H_*(Y)$ of finite type. Then the Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_p) \Longrightarrow \pi_{t-s}(Y_p^\wedge)$$

is strongly convergent. In particular, there is a strongly convergent Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s}(S)_p^\wedge\,.$$

More generally, the Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) \Longrightarrow [X, Y_p^\wedge]_{t-s}$$

is conditionally convergent. It is strongly convergent when $RE_\infty = 0$, which happens, for instance, if $H^*(X)$ is of finite type and bounded above, or if the spectral sequence collapses at a finite stage.

Proof. Let $\{Y^s\}_s$ be an Adams resolution of $Y^0 = Y$, with cofiber sequences

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1} .$$

Smashing with S/p^e for each $e \geq 1$, we get a tower of Adams resolutions $\{Y^s/p^e\}_s$ of $Y^0/p^e = Y/p^e$, with cofiber sequences

$$Y^{s+1}/p^e \xrightarrow{i} Y^s/p^e \xrightarrow{j} K^s/p^e \xrightarrow{\partial} \Sigma Y^{s+1}/p^e .$$

(We check that these diagrams satisfy the conditions to be Adams resolutions: Each homomorphism $j^*\colon H^*(K^s/p^e) \to H^*(Y^s/p^e)$ can be rewritten as $j^* \otimes 1\colon H^*(K^s) \otimes H^*(S/p^e) \to H^*(Y^s) \otimes H^*(S/p^e)$, hence remains surjective. Each cofiber K^s/p^e sits in a cofiber sequence

$$K^s \xrightarrow{p^e} K^s \longrightarrow K^s/p^e \longrightarrow \Sigma K^s$$

where p^e is null-homotopic, so that $K^s/p^e \simeq K^s \vee \Sigma K^s$ is still a suitably finite wedge sum of mod p Eilenberg–Mac Lane spectra.) Now pass to the homotopy limit over e of these Adams resolutions. The result is a diagram $\{(Y^s)^\wedge_p\}_s$ of spectra, with cofiber sequences

$$(Y^{s+1})^\wedge_p \xrightarrow{i} (Y^s)^\wedge_p \xrightarrow{j} (K^s)^\wedge_p \xrightarrow{\partial} \Sigma (Y^{s+1})^\wedge_p .$$

(Cofiber sequences are fiber sequences, up to a sign, hence are preserved by passage to homotopy limits, such as completions.) It is again an Adams resolution, since the completion map $K^s \to (K^s)^\wedge_p$ is an equivalence ($K^s \simeq \bigvee_u \Sigma^{n_u} H \simeq \prod_u \Sigma^{n_u} H$ and $H \to H^\wedge_p$ is easily seen to be an equivalence, see Lemma 12.13) and $j\colon (Y^s)^\wedge_p \to (K^s)^\wedge_p$ induces the "same" map as $j\colon Y^s \to K^s$ in mod p cohomology. We get the following vertical maps of Adams resolutions:

$$\begin{tikzcd}
\operatorname{holim}_s Y^s \arrow[rr] \arrow[dd] & & Y^2 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y \arrow[dl, "j"] \arrow[dd] \\
& K^2 \arrow[dd, "\simeq"'] & & K^1 \arrow[dd, "\simeq"'] & & K^0 \arrow[dd, "\simeq"'] & \\
\operatorname{holim}_s (Y^s)^\wedge_p \arrow[rr] \arrow[dd] & & (Y^2)^\wedge_p \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & (Y^1)^\wedge_p \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y^\wedge_p \arrow[dl, "j"] \arrow[dd] \\
& (K^2)^\wedge_p \arrow[dd] & & (K^1)^\wedge_p \arrow[dd] & & (K^0)^\wedge_p \arrow[dd] & \\
\operatorname{holim}_s Y^s/p^e \arrow[rr] & & Y^2/p^e \arrow[rr, "i"] \arrow[dl, "j"] & & Y^1/p^e \arrow[rr, "i"] \arrow[dl, "j"] & & Y/p^e \arrow[dl, "j"] \\
& K^2/p^e & & K^1/p^e & & K^0/p^e &
\end{tikzcd}$$

(We omit the maps $\partial\colon K^s \to \Sigma Y^{s+1}$, etc.) By the previous lemma, there exists an Adams resolution $\{Z^s\}_s$ for Y/p with $\operatorname{holim}_s Z^s \simeq *$. Since this homotopy limit is independent of the choice of resolution, we must also have $\operatorname{holim}_s Y^s/p \simeq *$.

There are cofiber sequences $S/p \to S/p^{e+1} \to S^e \to \Sigma S/p$, inducing cofiber sequences $Y^s/p \to Y^s/p^{e+1} \to Y^s/p^e \to \Sigma Y^s/p$ for all s, hence also

$$\operatorname*{holim}_s Y^s/p \longrightarrow \operatorname*{holim}_s Y^s/p^{e+1} \longrightarrow \operatorname*{holim}_s Y^s/p^e \longrightarrow \Sigma \operatorname*{holim}_s Y^s/p .$$

We deduce that $\operatorname{holim}_s Y^s/p^e \simeq *$ for all $e \geq 1$, by induction on e. Thus

$$\operatorname*{holim}_s (Y^s)^\wedge_p = \operatorname*{holim}_s \operatorname*{holim}_e Y^s/p^e \simeq \operatorname*{holim}_e \operatorname*{holim}_s Y^s/p^e \simeq *$$

by the standard exchange of homotopy limits equivalence.

Applying homotopy, we get a map of unrolled exact couples from the one for Y to the one for $Y^\wedge_p$:

$$\begin{tikzcd}
\cdots \arrow[rr] & & \pi_*(Y^2) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y^1) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y) \arrow[dl, "j"] \arrow[dd] \\
& \pi_*(K^2) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & & \pi_*(K^1) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & & \pi_*(K^0) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & \\
\cdots \arrow[rr] & & \pi_*((Y^2)^\wedge_p) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*((Y^1)^\wedge_p) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*(Y^\wedge_p) \arrow[dl, "j"] \\
& \pi_*((K^2)^\wedge_p) \arrow[ul, dashed, "\partial"] & & \pi_*((K^1)^\wedge_p) \arrow[ul, dashed, "\partial"] & & \pi_*((K^0)^\wedge_p) \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

This induces a map of spectral sequences, from the Adams spectral sequence for Y to the one associated to the lower exact couple. The equivalences $K^s \to (K^s)^\wedge_p$ induce isomorphisms

$$E_1^{s,t} = \pi_{t-s}(K^s) \xrightarrow{\cong} \pi_{t-s}((K^s)^\wedge_p)$$

of E_1-terms between these spectral sequences. By induction on r, it follows that it also induces an isomorphism of E_r-terms, for all $r \ge 1$. Hence we have two different exact couples generating the same spectral sequence. The upper one is the Adams spectral sequence for Y. The lower one is conditionally convergent to $\pi_*(Y^\wedge_p)$, since $\operatorname{holim}_s (Y^s)^\wedge_p \simeq *$. Hence the Adams spectral sequence for Y, with $E_2^{*,*} = \operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)$, is conditionally convergent to $\pi_*(Y^\wedge_p)$, as asserted. Replacing $\pi_*(-)$ by $[X, -]_*$ we get the same conclusion for the Adams spectral sequence for maps $X \to Y$.

To get strong convergence to $\pi_*(Y^\wedge_p)$ or $[X, Y^\wedge_p]_*$, we need to verify Boardman's criterion $RE_\infty = 0$. In the first case, this follows since $E_2^{s,t}(Y)$ is of finite type, i.e., is finite(-dimensional) in each bidegree (s,t). In fact, this holds already at the E_1-term if we use the canonical Adams resolution for Y, with $\Sigma^s K^s = H \wedge (\bar{H})^{\wedge s} \wedge Y$, since then

$$E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s) \cong H_t((\bar{H})^{\wedge s} \wedge Y) \cong [H_*(\bar{H})^{\otimes s} \otimes H_*(Y)]_t \,.$$

In the case of a general spectrum X, we have

$$\begin{aligned}
E_1^{s,t} &= [X, K^s]_{t-s} \cong [X, \Sigma^s K^s]_t \cong \operatorname{Hom}^t_{\mathscr{A}}(H^*(\Sigma^s K^s), H^*(X)) \\
&\cong \operatorname{Hom}^t_{\mathscr{A}}(\mathscr{A} \otimes I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \cong \operatorname{Hom}^t(I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \,.
\end{aligned}$$

This group is finite if $H^*(X)$ is of finite type and bounded above, in the sense that there exists an integer N with $H^n(X) = 0$ for $n > N$. For instance, this is the case of X is a finite CW spectrum. $\square$

Proposition 12.21. *Let Y be bounded below with $H_*(Y)$ of finite type. There is a cofiber sequence*

$$\operatorname*{holim}_s Y^s \longrightarrow Y \longrightarrow Y^\wedge_p$$

where $\{Y^s\}_s$ is any Adams resolution of Y.

Proof. We use the notation of the proof above. In view of the equivalences $K^s \simeq (K^s)^\wedge_p$, we get a chain of equivalences

$$\operatorname*{holim}_s \operatorname{hofib}(Y^s \to (Y^s)^\wedge_p) \simeq \operatorname{hofib}(Y^s \to (Y^s)^\wedge_p) \simeq \cdots \simeq \operatorname{hofib}(Y \to Y^\wedge_p)$$

for all s. Passing to homotopy limits, we find that

$$\operatorname*{holim}_s Y^s \simeq \operatorname{hofib}(\operatorname*{holim}_s Y^s \to \operatorname*{holim}_s (Y^s)^\wedge_p) \simeq \operatorname*{holim}_s \operatorname{hofib}(Y^s \to (Y^s)^\wedge_p) \simeq \operatorname{hofib}(Y \to Y^\wedge_p) \,.$$

In other words, the p-completion $Y \to Y^\wedge_p$ precisely annihilates the obstruction $\operatorname{holim}_s Y^s$ to conditional convergence for the unrolled exact couple associated to the Adams resolution of Y. $\square$

((Mention Bousfield's E-nilpotent completion $Y^\wedge_E = Y / \operatorname{holim}_s Y^s_E$ where $Y^s_E = (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y$?))

13.1. Composition and the Yoneda product. Let X, Y and Z be spectra. We have a composition pairing

$$\circ\colon [Y,Z]_* \otimes [X,Y]_* \longrightarrow [X,Z]_*$$

that takes $g\colon \Sigma^v Y \to Z$ and $f\colon \Sigma^t X \to Y$ to the composite $g \circ \Sigma^v f\colon \Sigma^{t+v} X \to Z$. More explicitly, $g\colon Y \wedge S^v \to Z$ and $f\colon X \wedge S^t \to Y$, so $\Sigma^v f = f \wedge 1\colon X \wedge S^t \wedge S^v \to Y \wedge S^v$ and $g \circ \Sigma^v f\colon X \wedge S^t \wedge S^v \to Z$. To simplify the notation we refer to f and g as maps $f\colon X \to Y$ and $g\colon Y \to Z$ of degree t and v, respectively, and write $gf = g \circ f\colon X \to Y$ for the composite of degree $t+v$.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$. If f and g have Adams filtrations $\geq s$ and $\geq u$, meaning that they factor as $f = i^s \tilde{f}$ and $g = i^u \tilde{g}$ with $\tilde{f}\colon X \to Y^s$ and $\tilde{g}\colon Y \to Z^u$ of degree t and v, respectively, then we can lift $\tilde{g}$ to a map $\{g^s\}_s$ of Adams resolutions

$$\begin{tikzcd}
X \arrow[d, "\tilde{f}"'] & & \\
Y^s \arrow[r, "i"] \arrow[d, "g^s"'] & \dots \arrow[r, "i"] & Y \arrow[d, "\tilde{g}"] \\
Z^{s+u} \arrow[r, "i"] & \dots \arrow[r, "i"] & Z^u \,.
\end{tikzcd}$$

Hence $gf = i^u \tilde{g} i^s \tilde{f} = i^{s+u} g^s f$ factors through $i^{s+u}\colon Z^{s+u} \to Z$, and has Adams filtration $\geq (s+u)$. We thus get a restricted pairing

$$F^u[Y,Z]_* \otimes F^s[X,Y]_* \longrightarrow F^{s+u}[X,Z]_*$$

that induces a pairing

$$F^u/F^{u+1} \otimes F^s/F^{s+1} \longrightarrow F^{s+u}/F^{s+u+1}$$

of filtration subquotients. When the respective spectral sequences converge, we can rewrite this as a pairing

$$E_\infty^{u,*} \otimes E_\infty^{s,*} \longrightarrow E_\infty^{s+u,*}$$

of E_∞-terms. Conversely, this pairing of E_∞-terms will determine the restricted pairings $F^u \otimes F^s \to F^{s+u}$ modulo F^{s+u+1}, i.e., modulo higher Adams filtrations. In this way the pairing of E_∞-terms determines the composition pairing $[Y,Z]_* \otimes [X,Y]_* \to [X,Z]_*$ modulo the Adams filtration.

Example 13.1. ((Example of this phenomenon: $h_2^3 = h_1^2 h_3$ so $\nu^3 \equiv \eta^2\sigma$ modulo Adams filtration ≥ 4. In fact, $\nu^3 = \eta^2\sigma + \eta\epsilon$.))

Let $P_s = H^*(\Sigma^s K^s)$ and $Q_u = H^*(\Sigma^u L^u)$, so that there are free resolutions

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_u \xrightarrow{\partial_u} Q_{u-1} \to \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0\,.$$

By definition,

$$\begin{aligned}
\mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(Y)) &= H^u(\mathrm{Hom}_{\mathscr{A}}^v(Q_*, H^*(Y))) \\
\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) &= H^s(\mathrm{Hom}_{\mathscr{A}}^t(P_*, H^*(X))) \\
\mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(H^*(Z), H^*(X)) &= H^{u+s}(\mathrm{Hom}_{\mathscr{A}}^{v+t}(Q_*, H^*(X)))\,.
\end{aligned}$$

The (opposite) Yoneda product is a pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))\,,$$

and we shall see that the Adams spectral sequence relates the Yoneda product in $E_2 = \mathrm{Ext}_{\mathscr{A}}(-,-)$ to the composition product in homotopy. (This is the opposite of the usual Yoneda pairing, meaning that the two factors in the source have been interchanged. This comes about due to the contravariance of cohomology. Working at odd primes the interchange introduces a sign.)

Let $f\colon P_s \to \Sigma^t H^*(X)$ and $g\colon Q_u \to \Sigma^v H^*(Y)$ be $\mathscr{A}$-module homomorphisms. To simplify the notation, we will refer to these as homomorphisms $f\colon P_s \to H^*(X)$ and $g\colon Q_u \to H^*(Y)$ of degree t and v, respectively. We also suppose that f and g are cocycles, meaning that $0 = f\partial_{s+1}\colon P_{s+1} \to H^*(X)$ and $0 =$

$g\partial_{u+1}\colon Q_{u+1} \to H^*(Y)$. The cohomology classes $[f]$ and $[g]$ are then elements in $\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y), H^*(X))$ and $\mathrm{Ext}^{u,v}_{\mathscr{A}}(H^*(Z), H^*(Y))$, respectively. Then g lifts to a chain map $g_* = \{g_n \colon Q_{u+n} \to P_n\}_n$, where each g_n has degree v, making the diagram

$$\begin{tikzcd}
 & H^*(X) & & & & \\
\dots \arrow[r] & P_s \arrow[u, "f"] \arrow[r, "\partial_s"] & \dots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \\
\dots \arrow[r] & Q_{u+s} \arrow[u, "g_s"] \arrow[r, "\partial_{u+s}"] & \dots \arrow[r] & Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"] & Q_u \arrow[u, "g_0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute. The composite $fg_s \colon Q_{u+s} \to H^*(X)$ is then an $\mathscr{A}$-module homomorphism of degree $(v+t)$, and satisfies $fg_s\partial_{u+s+1} = 0$. It is therefore a cocycle in $\mathrm{Hom}^{v+t}_{\mathscr{A}}(H^*(Z), H^*(X))$, and its cohomology class $[fg_s]$ in $\mathrm{Ext}^{u+s,v+t}_{\mathscr{A}}(H^*(Z), H^*(X))$ is by definition the Yoneda product of $[g]$ and $[f]$. It is not hard to check that a different choice of chain map lifting g only changes the cocycle fg_s by a coboundary, i.e., a homomorphism that factors through $\partial_{u+s}\colon Q_{u+s} \to Q_{u+s-1}$, so that its cohomology class is unchanged. Likewise, changing f or g by a coboundary only changes fg_s by a coboundary, so that the Yoneda product is well defined. [[TODO: Rewrite this as a clear definition.]]

Example 13.2. Let $X = Y = Z = S$ and let $P_* = Q_*$ be the minimal resolution of $\mathbb{F}_2$ computed earlier. We can compute the Yoneda product

$$\mathrm{Ext}^{u,v}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{u+s,v+t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

that makes $\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ into a bigraded algebra, by choosing cocycle representatives $f\colon P_s \to \mathbb{F}_2$ and $g\colon P_u \to \mathbb{F}_2$, lifting g to a chain map $g_*\colon P_{u+*} \to P_*$, and computing the composite fg_s.

Let $f = \gamma_{1,0} = h_0 \colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,0} \in P_1$ and let $g = \gamma_{1,2} = h_2\colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,2} \in P_1$. A lift $g_0\colon P_1 \to P_0$ of g is given by $g_{1,2} \mapsto g_{0,0}$ and $g_{1,i} \mapsto 0$ for $i \neq 2$.

$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "f=h_0"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & \mathbb{F}_2 \\
P_2 \arrow[u, "g_1"] \arrow[r, "\partial_2"] & P_1 \arrow[u, "g_0"] \arrow[ur, "g=h_2"'] &
\end{tikzcd}$$

The composite $g_0\partial_2\colon P_2 \to P_0$ is then given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto Sq^1 g_{0,0}$, $g_{2,3} \mapsto Sq^4 g_{0,0}$ etc. A lift $g_1\colon P_2 \to P_1$ is given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto g_{1,0}$, $g_{2,3} \mapsto g_{1,2}$ etc. Hence $fg_1\colon P_2 \to \mathbb{F}_2$ is given by $g_{2,2} \mapsto 1$ and $g_{2,i} \mapsto 0$ for $i \neq 2$ (for degree reasons), so that $[fg_1] = \gamma_{2,2}$. Thus $h_0h_2 = \gamma_{2,2}$ in bidegree $(s,t) = (2,4)$ of $\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$. In hindsight, this it the only possible nonzero value of the product, and it is realized because of the summand $Sq^1 g_{1,2}$ in $\partial_2(g_{2,2})$ and the summand $Sq^4 g_{0,0}$ in $\partial_1(g_{1,2})$, with Sq^1 detecting h_0 and Sq^4 detecting h_2.

Proposition 13.3. *Let $P_* \to \mathbb{F}_2$ and $Q_* \to H^*(Z)$ be minimal resolutions, with $P_0 = \mathscr{A}\{[]\}$, $P_1 = \mathscr{A}\{[Sq^{2^i}]\} \mid i \geq 0\}$, $Q_u = \mathscr{A}\{g_{u,j}\}_j$ and $Q_{u+1} = \mathscr{A}\{g_{u+1,k}\}_k$. Here $\partial_1([Sq^{2^i}]) = Sq^{2^i}[]$, and we can write*

$$\partial_{u+1}(g_{u+1,k}) = \sum_{i,j} \theta^k_{i,j} Sq^{2^i} g_{u,j}$$

for suitable coefficients $\theta^k_{i,j} \in \mathscr{A}$. Let $h_i \in \mathrm{Ext}^{1,2^i}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$, $\gamma_{u,j} \in \mathrm{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$ and $\gamma_{u+1,k} \in \mathrm{Ext}^{u+1,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$ be dual to $[Sq^{2^i}]$, $g_{u,j}$ and $g_{u+1,k}$, respectively. Then

$$h_i \cdot \gamma_{u,j} = \sum_k \epsilon(\theta^k_{i,j})\gamma_{u+1,k} ,$$

where $\epsilon\colon \mathscr{A} \to \mathbb{F}_2$ is the augmentation. Hence $h_i \cdot \gamma_{u,j}$ contains the term $\gamma_{u+1,k}$ if and only if $\partial_{u+1}(g_{u+1,k})$ contains the term $Sq^{2^i} g_{u,j}$.

Proof. The coefficient of $g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$ can be written as a sum $\sum_i \theta^k_{i,j} Sq^{2^i}$, since the Sq^{2^i} generate $I(\mathscr{A})$ as a left $\mathscr{A}$-module, and the coefficient lies in this augmentation ideal, by the assumption that the resolution is minimal. Consider the following diagram, for fixed choices of $i \geq 0$ and j.

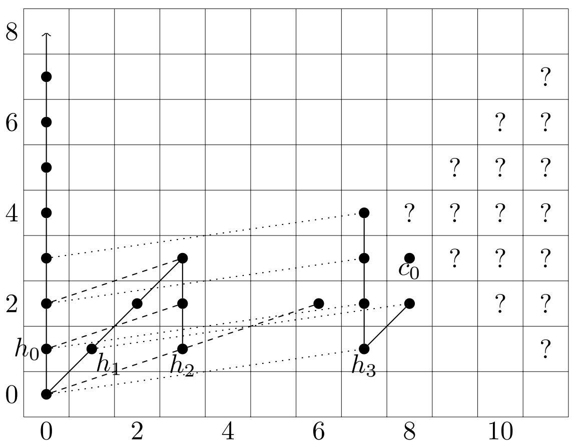
$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "h_i"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & \mathbb{F}_2 \\
Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"'] & Q_u \arrow[u, "g_0"] \arrow[ur, "\gamma_{u,j}"'] &
\end{tikzcd}$$

Here $f = h_i$ maps $[Sq^{2^i}]$ to 1 and the remaining generators of P_1 to 0. Likewise $g = \gamma_{u,j}$ maps $g_{u,j}$ to 1 and the remaining generators of Q_u to 0. We lift g to a chain map $g_* \colon Q_{u+*} \to P_*$, by first letting g_0 map $g_{u,j}$ to $1[]$ and sending the other generators of Q_u to 0. Then

$$g_0 \partial_{u+1}(g_{u+1,k}) = g_0\Big(\sum_{i,j} \theta^k_{i,j} Sq^{2^i} g_{u,j}\Big) = \sum_i \theta^k_{i,j} Sq^{2^i}[]\,,$$

so we can set $g_1(g_{u+1,k}) = \sum_i \theta^k_{i,j}[Sq^{2^i}]$. Hence the Yoneda product $f \circ g_1 \colon Q_{u+1} \to \mathbb{F}_2$ maps $g_{u+1,k}$ to $\epsilon(\theta^k_{i,j})$, and therefore contains $\gamma_{u+1,k}$ with that coefficient. $\square$

Example 13.4. From the minimal resolution in Theorem 10.11, we can read off the following nontrivial products: $h_0\gamma_{0,0} = \gamma_{1,0}$, $h_1\gamma_{0,0} = \gamma_{1,1}$, $h_2\gamma_{0,0} = \gamma_{1,2}$, $h_3\gamma_{0,0} = \gamma_{1,3}$, $h_0\gamma_{1,0} = \gamma_{2,0}$, $h_1\gamma_{1,1} = \gamma_{2,1}$, $h_2\gamma_{1,0} = \gamma_{2,2}$, $h_0\gamma_{1,2} = \gamma_{2,2}$, $h_2\gamma_{1,2} = \gamma_{2,3}$, $h_3\gamma_{1,0} = \gamma_{2,4}$, $h_0\gamma_{1,3} = \gamma_{2,4}$, $h_3\gamma_{1,1} = \gamma_{2,5}$, $h_1\gamma_{1,3} = \gamma_{2,5}$, $h_0\gamma_{2,0} = \gamma_{3,0}$, $h_2\gamma_{2,0} = \gamma_{3,1}$, $h_1\gamma_{2,1} = \gamma_{3,1}$, $h_0\gamma_{2,2} = \gamma_{3,1}$, $h_3\gamma_{2,0} = \gamma_{3,2}$, $h_0\gamma_{2,4} = \gamma_{3,2}$, $h_0\gamma_{3,0} = \gamma_{4,0}$, $h_3\gamma_{3,0} = \gamma_{4,1}$, $\dots$, $h_0\gamma_{10,0} = \gamma_{11,0}$. This gives the following multiplicative structure.



Proposition 13.5. *For p odd, let $Q_* \to H^*(Z)$ be a minimal resolution, with $Q_u = \mathscr{A}\{g_{u,j}\}_j$ and $Q_{u+1} = \mathscr{A}\{g_{u+1,k}\}_k$. Let $\gamma_{u,j} \in \mathrm{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_p)$ and $\gamma_{u+1,k} \in \mathrm{Ext}^{u+1,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_p)$ be dual to $g_{u,j}$ and $g_{u+1,k}$, respectively. Then the coefficient (in $\mathbb{F}_p$) of $\gamma_{u+1,k}$ in the Yoneda product $a_0 \cdot \gamma_{u,j}$ equals the coefficient of $\beta g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$, and the coefficient of $\gamma_{u+1,k}$ in $h_i \cdot \gamma_{u,j}$ equals the coefficients of $P^{p^i} g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$.*

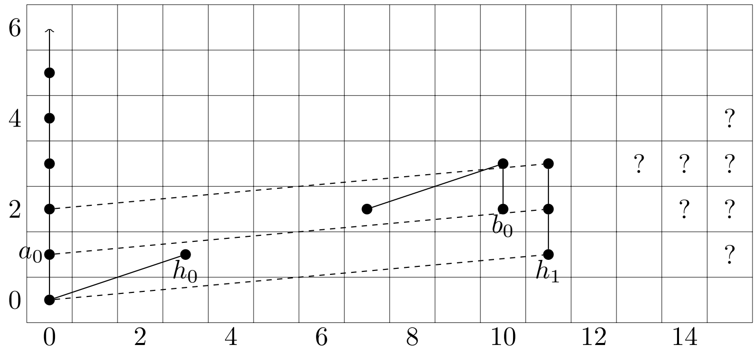
Proof. The proof is similar to the case $p = 2$. We write $\partial_{u+1}(g_{u+1,k})$ as

$$\sum_j \Big(\theta^k_j \beta + \sum_i \theta^k_{i,j} P^{p^i}\Big) g_{u,j}$$

with θ^k_j and $\theta^k_{i,j}$ in $\mathscr{A}$. This is possible, since the resolution is assumed to be minimal. Then $a_0 \cdot \gamma_{u,j} = \epsilon(\theta^k_j)\gamma_{u+1,k}$ and $h_i \cdot \gamma_{u,j} = \epsilon(\theta^k_{i,j})\gamma_{u+1,k}$. $\square$

Example 13.6. From the minimal resolution in Theorem 10.14, we can read off the following nontrivial products: $a_0\gamma_{0,0} = \gamma_{1,0}$, $h_0\gamma_{0,0} = \gamma_{1,1}$, $h_1\gamma_{0,0} = \gamma_{1,2}$, $a_0\gamma_{1,0} = \gamma_{2,0}$, $h_1\gamma_{1,0} = \gamma_{2,3}$, $a_0\gamma_{1,2} = -\gamma_{2,3}$,

$a_0\gamma_{2,0} = \gamma_{3,0}$, $h_0\gamma_{2,1} = \gamma_{3,1}$, $a_0\gamma_{2,2} = -\gamma_{3,1}$, $h_1\gamma_{2,0} = \gamma_{3,2}$, $a_0\gamma_{2,3} = -\gamma_{3,2}$, $a_0\gamma_{3,0} = \gamma_{4,0}$, $\dots$, $a_0\gamma_{14,0} = \gamma_{15,0}$. This gives the following multiplicative structure.



Definition 13.7. Consider any two complexes P_* and Q_* of $\mathscr{A}$-modules. Let

$$\mathrm{HOM}^{u,v}_{\mathscr{A}}(Q_*, P_*) = \prod_s \mathrm{Hom}^v_{\mathscr{A}}(Q_{u+s}, P_s)$$

be the abelian group of sequences $\{g_s \colon Q_{u+s} \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms, each of degree v. Thus $\mathrm{HOM}^u_{\mathscr{A}}(Q_*, P_*)$ is a graded abelian group. Let

$$\delta_u \colon \mathrm{HOM}^u_{\mathscr{A}}(Q_*, P_*) \to \mathrm{HOM}^{u+1}_{\mathscr{A}}(Q_*, P_*)$$

map $\{g_s\}_s$ to $\{\partial_{s+1} g_{s+1} + g_s \partial_{u+s+1}\}_s$. ((We are working mod 2, so there is no sign.)) Then $\delta_{u+1}\delta_u = 0$, so $\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)$ is a cocomplex of graded abelian groups.

Lemma 13.8. *The kernel*

$$\ker(\delta_0) \subset \mathrm{HOM}^0_{\mathscr{A}}(Q_*, P_*)$$

consists of the chain maps $g_* \colon Q_* \to P_*$, meaning the sequences $\{g_s \colon Q_s \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms such that $\partial_{s+1} g_{s+1} = g_s \partial_{s+1}$ for all s. The image

$$\mathrm{im}(\delta_{-1}) \subset \ker(\delta_0)$$

consists of the chain maps that are chain homotopic to 0, i.e., those of the form $\{\partial_{s+1} h_{s+1} + h_s \partial_s\}_s$ for some collection of $\mathscr{A}$-module homomorphisms $h_{s+1} \colon Q_s \to P_{s+1}$ for all s. Hence the 0-th cohomology

$$H^0(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \cong \{g_* \colon Q_* \to P_*\}/(\simeq) = [Q_*, P_*]$$

is the (graded abelian) group of chain homotopy classes of chain maps $Q_* \to P_*$. More generally, $H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*))$ is the group $[Q_{u+*}, P_*]$ of chain homotopy classes of chain maps $Q_{u+*} \to P_*$.

In the special case when $P_* = H^*(Y)$ is concentrated in filtration $s = 0$, so that $P_0 = H^*(Y)$ and $P_s = 0$ for $s \neq 0$, then $\mathrm{HOM}^{u,v}_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}^v_{\mathscr{A}}(Q_u, H^*(Y))$ and $\delta_u = (\partial_{u+1})^*$, so that $H^u(\mathrm{HOM}_{\mathscr{A}}(Q_*, H^*(Y))) \cong H^u(\mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y)))$. When Q_* is a free resolution of $H^*(Z)$, this is $\mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$.

Proposition 13.9. *Let $\epsilon \colon P_* \to H^*(Y)$ and $\epsilon \colon Q_* \to H^*(Z)$ be free $\mathscr{A}$-module resolutions. Then*

$$\epsilon_* \colon \mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \xrightarrow{\simeq} \mathrm{HOM}^*_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y))$$

is a quasi-isomorphism, in the sense that it induces an isomorphism

$$\epsilon_* \colon H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \xrightarrow{\cong} \mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$$

in cohomology, in each filtration u.

This is standard homological algebra. The first assertion only requires that Q_* is free and $P_* \to H^*(Y)$ is exact, but the final identification with Ext requires that $Q_* \to H^*(Z)$ is exact.

The composition pairing and the quasi-isomorphism

$$\begin{tikzcd} \mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(X)) \\ \mathrm{Hom}^*_{\mathscr{A}}(Q_*, H^*(Y)) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) & \end{tikzcd}$$

thus induce a pairing and an isomorphism

$$\begin{tikzcd} H^u(\mathrm{Hom}^*_{\mathscr{A}}(Q_*, P_*)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{u+s}_{\mathscr{A}}(H^*(Z), H^*(X)) \\ \mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) \arrow[ur, dashed] & \end{tikzcd}$$

in cohomology, and the Yoneda product is given by the dashed arrow. From this description it is easy to see that the Yoneda product is associative and unital. [[No evident commutativity in this generality.]]

13.2. **Pairings of spectral sequences.**

Definition 13.10. Let $\{'E_r\}_r$, $\{''E_r\}_r$ and $\{E_r\}_r$ be three spectral sequence. A pairing of these spectral sequences is a sequence of homomorphisms

$$\phi_r \colon {'E}_r^{*,*} \otimes {''E}_r^{*,*} \longrightarrow E_r^{*,*}$$

((for $r \geq 1$)) such that the Leibniz rule

$$d_r(\phi_r(x \otimes y)) = \phi_r(d_r(x) \otimes y) + (-1)^n \phi_r(x \otimes d_r(y))$$

holds, where $n = |x|$ is the total degree of x, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)]$$

where $[x] \in {'E}_{r+1}^{*,*}$ is the homology class of a d_r-cycle $x \in {'E}_r^{*,*}$, and similarly for $[y]$ and the right hand side. In other words, the diagrams

$$\begin{tikzcd} {'E}_r^{*,*} \otimes {''E}_r^{*,*} \arrow[r, "\phi_r"] \arrow[d, "d_r \otimes 1 \pm 1 \otimes d_r"'] & E_r^{*,*} \arrow[d, "d_r"] \\ {'E}_r^{*,*} \otimes {''E}_r^{*,*} \arrow[r, "\phi_r"] & E_r^{*,*} \end{tikzcd}$$

and

$$\begin{tikzcd} H^{*,*}('E_r) \otimes H^{*,*}(''E_r) \arrow[r] \arrow[d, "\cong"'] & H^{*,*}('E_r \otimes {''E_r}) \arrow[r, "(\phi_r)_*"] & H^{*,*}(E_r) \arrow[d, "\cong"] \\ {'E}_{r+1}^{*,*} \otimes {''E}_{r+1}^{*,*} \arrow[rr, "\phi_{r+1}"] & & E_{r+1}^{*,*} \end{tikzcd}$$

commute.

A spectral sequence pairing $\{\phi_r\}_r$ induces a pairing

$$\phi_\infty \colon {'E}_\infty^{*,*} \otimes {''E}_\infty^{*,*} \longrightarrow E_\infty^{*,*}$$

of E_∞-terms. ((Clear if each spectral sequence vanishes in negative filtrations, so that in each bidegree (s,t) the E_r-terms eventually form a descending sequence, with intersection equal to the E_∞-term.))

When the Künneth homomorphism $H^{*,*}('E_r) \otimes H^{*,*}(''E_r) \to H^{*,*}('E_r \otimes {''E_r})$ is an isomorphism, for each r, one can readily define a tensor product spectral sequence $\{'E_r \otimes {''E_r}\}_r$, and the pairing of spectral sequences is the same as a morphism $\{'E_r \otimes {''E_r}\}_r \to \{E_r\}_r$ of spectral sequences.

Definition 13.11. Suppose that the spectral sequences above converge to the graded abelian groups G', G'' and G, respectively, in the sense that there are filtrations $\{'F^s\}_s$, $\{''F^s\}_s$ and $\{F^s\}_s$ of these groups, and isomorphisms ${'F^s}/{'F^{s+1}} \cong {'E}^s_\infty$, ${''F^s}/{''F^{s+1}} \cong {''E}^s_\infty$ and $F^s/F^{s+1} \cong E^s_\infty$, for all s.

We say that a pairing $\{\phi_r\}_r$ of spectral sequences, as above, *converges* to a pairing $\phi\colon G' \otimes G'' \to G$ if the latter pairing restricts to homomorphisms $\phi\colon {'F^u} \otimes {''F^s} \to F^{u+s}$ for all u and s, and if the induced homomorphisms $\phi\colon {'F^u}/{'F^{u+1}} \otimes {''F^s}/{''F^{s+1}} \to F^{u+s}/F^{u+s+1}$ agree with the limit $\phi_\infty \colon {'E}^u_\infty \otimes {''E}^s_\infty \to E^{u+s}_\infty$ of the pairings ϕ_r.

In other words, the diagram

$$\begin{tikzcd} {'E}^u_\infty \otimes {''E}^s_\infty \arrow[d, "\phi_\infty"'] & {'F^u}/{'F^{u+1}} \otimes {''F^s}/{''F^{s+1}} \arrow[l, "\cong"'] \arrow[d, "\phi"'] & {'F^u} \otimes {''F^s} \arrow[l, two heads] \arrow[r] \arrow[d, "\phi"'] & G' \otimes G'' \arrow[d, "\phi"'] \\ E^{u+s}_\infty & F^{u+s}/F^{u+s+1} \arrow[l, "\cong"'] & F^{u+s} \arrow[l, two heads] \arrow[r, tail] & G \end{tikzcd}$$

commutes. ((Consequences?))

Definition 13.12. An algebra spectral sequence is a spectral sequence $\{E_r\}_r$ with a spectral sequence pairing $\{\phi_r\colon E_r \otimes E_r \to E_r\}_r$ that is associative and unital. It is commutative if the pairing satisfies $\phi_r(y \otimes x) = (-1)^{mn}\phi_r(x \otimes y)$ for all x, y and r, where $n = |x|$ and $m = |y|$ are the total degrees. ((Elaborate?))

Adams (1958) defined a join pairing in his spectral sequence for S, which is stably equivalent to a smash product pairing in that spectral sequence. We shall return to those pairings later, but first look at the case of composition pairings, since these are most closely related to the Yoneda product. ((We may also need to look at this for Moss' later theorem on Toda brackets and Massey products.))

Theorem 13.13 (Moss (1968))**.** *Let X, Y and Z be spectra, with Y and Z bounded below and $H_*(Y)$ and $H_*(Z)$ of finite type. There is a pairing of spectral sequences*

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \longrightarrow E_r^{*,*}(X,Z)$$

which agrees for $r = 2$ with the (opposite) Yoneda pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \longrightarrow [X, Z_2^\wedge]_* \,.$$

The pairing is associative and unital.

[[We omit this proof, and will instead deduce the theorem (for X and Y finite CW spectra) from a similar theorem about the smash product pairing.]]

13.3. **Modules over cocommutative Hopf algebras.** The Künneth isomorphism $H^*(Y \wedge Z) \cong H^*(Y) \otimes H^*(Z)$ and the universal coefficient theorem $H^*(F(X,S)) \cong \mathrm{Hom}^*(H^*(X), \mathbb{F}_p)$ (for finite CW spectra X) can be refined from being statements about graded $\mathbb{F}_p$-vector spaces to statements about left $\mathscr{A}$-modules. This requires making sense of the tensor product $M \otimes N = M \otimes_{\mathbb{F}_p} N$ and the homomorphism group $\mathrm{Hom}(M,N) = \mathrm{Hom}_{\mathbb{F}_p}(M,N)$ as left $\mathscr{A}$-modules, for given left $\mathscr{A}$-modules M and N.

By the Cartan formula

$$Sq^k(y \wedge z) = \sum_{i+j=k} Sq^i(y) \wedge Sq^j(z)$$

in $H^*(Y \wedge Z)$, for $y \in H^*(Y)$ and $z \in H^*(Z)$, it is clear that Sq^k must act on $y \otimes z$ in $H^*(Y) \otimes H^*(Z)$ as the sum over $i + j = k$ of the action by $Sq^i \otimes Sq^j$. Milnor (1958) proved that for $p = 2$ the rule

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

extends in a unique manner to an algebra homomorphism

$$\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A} \,.$$

Here $\mathscr{A} \otimes \mathscr{A}$ is given the algebra structure given by the composition

$$\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \overset{1\otimes\gamma\otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \overset{\phi\otimes\phi}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \,,$$

where $\gamma\colon \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$ is the (graded) twist isomorphism, so that $(\theta_1\otimes\theta_2)\cdot(\theta_3\otimes\theta_4) = (-1)^{|\theta_2||\theta_3|}\theta_1\theta_3\otimes\theta_2\theta_4$. In general, $\gamma\colon M \otimes N \to N \otimes M$ is given by

$$\gamma(m \otimes n) = (-1)^{|m||n|} n \otimes m \,.$$

For $p = 2$ the sign can be ignored. For p odd the rules

$$\beta \longmapsto \beta \otimes 1 + 1 \otimes \beta$$

and

$$P^k \longmapsto \sum_{i+j=k} P^i \otimes P^j$$

likewise extend uniquely to an algebra homomorphism $\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$. [[Give Milnor's proof?]]

It follows that the Künneth isomorphism is an isomorphism of $\mathscr{A}$-modules, if we define the tensor product $M \otimes N$ of two A-modules M and N as follows.

Definition 13.14. Let M and N be left $\mathscr{A}$-modules, with module action maps $\lambda\colon \mathscr{A} \otimes M \to M$ and $\lambda\colon \mathscr{A} \otimes N \to N$. We give $M \otimes N$ the left $\mathscr{A}$-module structure given by the composition

$$\mathscr{A} \otimes M \otimes N \overset{\psi\otimes 1\otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \otimes M \otimes N \overset{1\otimes\gamma\otimes 1}{\longrightarrow} \mathscr{A} \otimes M \otimes \mathscr{A} \otimes N \overset{\lambda\otimes\lambda}{\longrightarrow} M \otimes N \,.$$

If we write $\psi(\theta) = \sum_i \theta'_i \otimes \theta''_i$ for $\theta \in \mathscr{A}$, which we usually abbreviate to $\sum \theta' \otimes \theta''$, then

$$\theta \cdot (m \otimes n) = \sum (-1)^{|\theta''||m|} \theta' \cdot m \otimes \theta'' \cdot n$$

for $m \in M$ and $n \in N$. The sign enters from the interchange of θ'' and m, and can be ignored for $p = 2$.

The *coproduct* ψ is counital and coassociative, in the sense that the diagrams

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ \mathbb{F}_p \otimes \mathscr{A} & \mathscr{A} \otimes \mathscr{A} \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] & \mathscr{A} \otimes \mathbb{F}_p \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\psi"] \arrow[d, "\psi"'] & \mathscr{A} \otimes \mathscr{A} \arrow[d, "\psi\otimes 1"] \\ \mathscr{A} \otimes \mathscr{A} \arrow[r, "1\otimes\psi"'] & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \end{tikzcd}$$

commute. Hence $\sum \epsilon(\theta')\theta'' = \theta = \sum \theta'\epsilon(\theta'')$ and $\sum\sum (\theta')' \otimes (\theta')'' \otimes \theta'' = \sum\sum \theta' \otimes (\theta'')' \otimes (\theta'')''$. Furthermore, it is cocommutative, in the sense that the diagram

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ \mathscr{A} \otimes \mathscr{A} \arrow[rr, "\gamma"'] & & \mathscr{A} \otimes \mathscr{A} \end{tikzcd}$$

commutes, so that $\sum \theta' \otimes \theta'' = \sum (-1)^{|\theta'||\theta''|} \theta'' \otimes \theta'$. All of these properties are easily verified for the algebra generators (Sq^k for $p = 2$, β and P^k for p odd) of $\mathscr{A}$.

The counital and coassociative augmentation $\epsilon \colon \mathscr{A} \to \mathbb{F}_p$ and coproduct $\psi \colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ give $\mathscr{A}$ the structure of a *coalgebra*. By cocommutativity of ψ, it is in fact a *cocommutative coalgebra*. Both the augmentation and the coproduct are algebra morphisms. This means that $\mathscr{A}$ is a *bialgebra*, or more precisely, a *cocommutative bialgebra*.

The cocommutativity of $\mathscr{A}$ ensures that the twist isomorphism $\gamma \colon M \otimes N \to N \otimes M$ is $\mathscr{A}$-linear, since $\gamma\psi = \psi$ implies that the left hand square in the following diagram commutes. The remainder of the diagram also commutes.

$$\begin{tikzcd} \mathscr{A} \otimes M \otimes N \arrow[r, "\psi\otimes 1\otimes 1"] \arrow[d, "1\otimes\gamma"'] & \mathscr{A} \otimes \mathscr{A} \otimes M \otimes N \arrow[r, "1\otimes\gamma\otimes 1"] \arrow[d, "\gamma\otimes\gamma"'] & \mathscr{A} \otimes M \otimes \mathscr{A} \otimes N \arrow[r, "\lambda\otimes\lambda"] \arrow[d, "\gamma"'] & M \otimes N \arrow[d, "\gamma"'] \\ \mathscr{A} \otimes N \otimes M \arrow[r, "\psi\otimes 1\otimes 1"] & \mathscr{A} \otimes \mathscr{A} \otimes N \otimes M \arrow[r, "1\otimes\gamma\otimes 1"] & \mathscr{A} \otimes N \otimes \mathscr{A} \otimes M \arrow[r, "\lambda\otimes\lambda"] & N \otimes M \end{tikzcd}$$

[[If we arrange that $\otimes$ is strictly unital and associative, as we implicitly arrange when we treat the unitality and associativity isomorphisms as identities, then $\mathscr{A}$-Mod is a permutative category.]]

Furthermore, $\mathscr{A}$ admits a *conjugation* $\chi \colon \mathscr{A} \to \mathscr{A}$, a linear homomorphism satisfying the relations

$$\phi(1 \otimes \chi)\psi = \eta\epsilon = \phi(\chi \otimes 1)\psi\,.$$

Equivalently χ makes the diagram

$$\begin{tikzcd} & \mathscr{A} \otimes \mathscr{A} \arrow[r, "1\otimes\chi"] & \mathscr{A} \otimes \mathscr{A} \arrow[dr, "\phi"] & \\ \mathscr{A} \arrow[ur, "\psi"] \arrow[r, "\epsilon"] \arrow[dr, "\psi"'] & \mathbb{F}_p \arrow[rr, "\eta"] & & \mathscr{A} \\ & \mathscr{A} \otimes \mathscr{A} \arrow[r, "\chi\otimes 1"'] & \mathscr{A} \otimes \mathscr{A} \arrow[ur, "\phi"'] & \end{tikzcd}$$

commute. It follows that χ is an anti-homomorphism, i.e., satisfies $\chi(\theta_1\theta_2) = \chi(\theta_2)\chi(\theta_1)$ for all $\theta_1, \theta_2 \in \mathscr{A}$, and it is an *involution*, i.e., χ^2 equals the identity. [[Give a proof? Milnor–Moore?]]

For $p = 2$, $\chi(1) = 1$ and $\sum_{i+j=k} Sq^i \chi(Sq^j) = 0$ for all $k \geq 1$, so that

$$\chi(Sq^k) = Sq^k + \sum_{i=1}^{k-1} Sq^i \chi(Sq^{k-i}) \, .$$

For example, $\chi(Sq^1) = Sq^1$, $\chi(Sq^2) = Sq^2$, $\chi(Sq^3) = Sq^2 Sq^1$ and $\chi(Sq^2 Sq^1) = Sq^3$. For p odd we get $\chi(\beta) = -\beta$ and

$$\chi(P^k) = -P^k - \sum_{i=1}^{k-1} P^i \chi(P^{k-i}) \, .$$

A bialgebra with a conjugation is called a *Hopf algebra*. The Steenrod algebra $\mathscr{A}$ is thus an example of a *cocommutative Hopf algebra*.

Let M be a left $\mathscr{A}$-module. The functor $L \mapsto L \otimes M$ is left adjoint to the functor $N \mapsto \mathrm{Hom}^*(M, N)$, in the sense that there is a natural bijection

$$\mathrm{Hom}^*(L \otimes M, N) \cong \mathrm{Hom}^*(L, \mathrm{Hom}^*(M, N))$$

taking $f \colon L \otimes M \to N$ to $g \colon L \to \mathrm{Hom}^*(M, N)$ given by $g(\ell)(m) = f(\ell \otimes m)$. The identity map of $L \otimes M$ on the left corresponds to the adjunction unit $in \colon L \to \mathrm{Hom}^*(M, L \otimes M)$ on the right, with $in(\ell)(m) = \ell \otimes m$. The identity map of $\mathrm{Hom}^*(M, N)$ on the right corresponds to the adjunction coinit $ev \colon \mathrm{Hom}^*(M, N) \otimes N \to M$ on the left, with $ev(f \otimes n) = f(n)$. These adjunctions do not involve the symmetric structure, and do not require the introduction of signs.

Definition 13.15. Let M and N be left $\mathscr{A}$-modules, with action maps $\lambda \colon \mathscr{A} \otimes M \to M$ and $\lambda \colon \mathscr{A} \otimes N \to N$. We give $\mathrm{Hom}(M, N)$ the $\mathscr{A}$-module structure given by the homomorphism $\mathscr{A} \otimes \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N)$ with left adjoint $\mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes M \longrightarrow N$ given by the composite

$$\begin{aligned} \mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes M &\overset{\psi \otimes 1 \otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes M \overset{1 \otimes \chi \otimes 1 \otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes M \\ &\overset{1 \otimes \gamma \otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes \mathscr{A} \otimes M \overset{1 \otimes 1 \otimes \lambda}{\longrightarrow} \mathscr{A} \otimes \mathrm{Hom}(M, N) \otimes M \overset{1 \otimes ev}{\longrightarrow} \mathscr{A} \otimes N \overset{\lambda}{\longrightarrow} N \, . \end{aligned}$$

For $\theta \in \mathscr{A}$, $f \colon M \to N$ in $\mathrm{Hom}(M, N)$ and $m \in M$, this composite is

$$\sum (-1)^{|\theta''||f|} \theta' \cdot f(\chi(\theta'') \cdot m) \, .$$

Proposition 13.16. *The category $\mathscr{A}$-Mod of left $\mathscr{A}$-modules, with respect to the tensor product $- \otimes -$, the unit object $\mathbb{F}_p$, the twist isomorphism γ and the mapping object $\mathrm{Hom}(-, -)$, is closed symmetric monoidal.*

Example 13.17. For another example of this situation, consider a discrete group G and a field k. The group algebra $k[G]$ has unit and product given by the neutral element e and the multiplication in G. It admits a cocommutative coproduct $\psi \colon k[G] \to k[G] \otimes k[G]$, given by $\psi(g) = g \otimes g$ for each $g \in G$. The augmentation $\epsilon \colon k[G] \to k$ satisfies $\epsilon(e) = 1$ and $\epsilon(g) = 0$ for all group elements $g \neq e$. The conjugation $\chi \colon k[G] \to k[G]$ is the anti-homomorphism given by $\chi(g) = g^{-1}$. These maps make $k[G]$ a cocommutative Hopf algebra. The tensor product of two $k[G]$-modules M and N is again a $k[G]$-module $M \otimes N = M \otimes_k N$, with the diagonal action $g \cdot (m \otimes n) = gm \otimes gn$. The twist isomorphism $\gamma \colon M \otimes N \to N \otimes M$ is $k[G]$-linear. The homomorphism module $\mathrm{Hom}(M, N) = \mathrm{Hom}_k(M, N)$ has the conjugate $k[G]$-action, given by $(g \cdot f)(m) = gf(g^{-1}m))$. Each $k[G]$-linear homomorphism $M \otimes N \to P$ corresponds bijectively to a $k[G]$-linear homomorphism $M \to \mathrm{Hom}(N, P)$.

The following result should be compared with Lemma 9.20.

Proposition 13.18. *Let M be any $\mathscr{A}$-module, with underlying graded $\mathbb{F}_p$-module $|M|$. There is an untwisting isomorphism of $\mathscr{A}$-modules,*

$$\mathscr{A} \otimes |M| \overset{\cong}{\longrightarrow} \mathscr{A} \otimes M$$

from the induced $\mathscr{A}$-module on the left hand side (with $\mathscr{A}$ acting only on the first tensor factor), to the tensor product of $\mathscr{A}$-modules on the right hand side (with the diagonal $\mathscr{A}$-action). In particular, the diagonal tensor product $\mathscr{A} \otimes M$ is a free $\mathscr{A}$-module.

Proof. The isomorphism from left to right is the composite

$$\mathscr{A} \otimes |M| \overset{\psi \otimes 1}{\longrightarrow} \mathscr{A} \otimes \mathscr{A} \otimes |M| \overset{1 \otimes \lambda}{\longrightarrow} \mathscr{A} \otimes M \, .$$

It sends $\theta \otimes m$ to $\sum \theta' \otimes \theta'' m$, where $\psi(\theta) = \sum \theta' \otimes \theta''$. It is $\mathscr{A}$-linear, because the induced $\mathscr{A}$-module action on the left corresponds to the diagonal $\mathscr{A}$-module action on the tensor product of $\mathscr{A}$ and $\mathscr{A} \otimes |M|$ in the middle, and the left action map $\lambda \colon \mathscr{A} \otimes |M| \to M$ is $\mathscr{A}$-linear.

The inverse isomorphism, from right to left, is the composite

$$\mathscr{A} \otimes M \xrightarrow{\psi\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes M \xrightarrow{1\otimes\chi\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes M \xrightarrow{1\otimes\lambda} \mathscr{A} \otimes |M| \,.$$

It sends $\theta \otimes m$ to $\sum \theta' \otimes \chi(\theta'') m$.

One composite is visible along the upper and right hand edges of the following commutative diagram.

$$\begin{tikzcd}
\mathscr{A} \otimes |M| \arrow[r, "\psi\otimes 1"] \arrow[d, "\psi\otimes 1"'] & \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\lambda"] \arrow[d, "\psi\otimes 1\otimes 1"] & \mathscr{A} \otimes M \arrow[d, "\psi\otimes 1"] \\
\mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\psi\otimes 1"] \arrow[dd, "1\otimes\epsilon\otimes 1"'] & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[d, "1\otimes\chi\otimes 1\otimes 1"] & \mathscr{A} \otimes \mathscr{A} \otimes M \arrow[d, "1\otimes\chi\otimes 1"] \\
 & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes 1\otimes\lambda"] \arrow[d, "1\otimes\phi\otimes 1"] & \mathscr{A} \otimes \mathscr{A} \otimes M \arrow[d, "1\otimes\lambda"] \\
\mathscr{A} \otimes \mathbb{F}_p \otimes |M| \arrow[r, "1\otimes\eta\otimes 1"] & \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\lambda"] & \mathscr{A} \otimes |M|
\end{tikzcd}$$

The upper left hand rectangle commutes because ψ is coassociative, the lower left hand rectangle commutes because χ is a conjugation, the upper right hand rectangles commutes by naturality of the tensor product, and the lower right hand rectangle commutes by associativity for λ. The left hand and lower edges are the (mutually inverse) canonical isomorphisms, by counitality of ψ and unitality of λ.

[[The other composite is similar.]] □

13.4. **Smash product and tensor product.** Let T, V, Y and Z be spectra. We have a smash product pairing

$$\wedge \colon [T, Y]_* \otimes [V, Z]_* \longrightarrow [T \wedge V, Y \wedge Z]_*$$

taking $f \colon T \to Y$ and $g \colon V \to Z$ to $f \wedge g \colon T \wedge V \to Y \wedge Z$, and similarly for graded maps. In particular, for $T = V = S$ we have a pairing

$$\wedge \colon \pi_*(Y) \otimes \pi_*(Z) \longrightarrow \pi_*(Y \wedge Z) \,.$$

If Y is a ring spectrum, with unit $\eta \colon S \to Y$ and multiplication $\phi \colon Y \wedge Y \to Y$, we have a unit homomorphism

$$\eta_* \colon \pi_*(S) \longrightarrow \pi_*(Y)$$

and a product

$$\pi_*(Y) \otimes \pi_*(Y) \xrightarrow{\wedge} \pi_*(Y \wedge Y) \xrightarrow{\phi_*} \pi_*(Y)$$

that make $\pi_*(Y)$ an algebra over $\pi_*(S)$. If Y is homotopy commutative, then $\pi_*(Y)$ is a (graded) commutative $\pi_*(S)$-algebra.

When $Y = S$, the smash product $\wedge \colon \pi_*(S) \otimes \pi_*(S) \to \pi_*(S)$ agrees up to sign with the composition product $\circ \colon \pi_*(S) \otimes \pi_*(S) \to \pi_*(S)$. The smash product of $f \colon S^t \to S$ and $g \colon S^v \to S$ is $f \wedge g \colon S^{t+v} \cong S^t \wedge S^v \to S \wedge S = S$, while the composition product is $f \circ \Sigma^t g \colon S^{v+t} = \Sigma^t S^v \to \Sigma^t S = S^t \to S$. These agree up to the twist equivalence $\gamma \colon S^t \wedge S^v \cong S^v \wedge S^t$, which is a a map of degree $(-1)^{tv}$.

Now suppose that Y and Z are bounded below with $H_*(Y)$ and $H_*(Z)$ of finite type, and let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions. If $f \colon T \to Y$ and $g \colon V \to Z$ have Adams filtrations $\geq s$ and $\geq u$, respectively, then they factor as the composites of s maps

$$T = T_s \to \cdots \to T_0 = Y$$

and u maps

$$V = V_u \to \cdots \to V_0 = Z \,,$$

all inducing zero on cohomology. By the Künneth theorem, the smash product $f \wedge g$ then factors as the composite of $(s+u)$ cohomologically trivial maps

$$T \wedge V = T_s \wedge V_u \to \cdots \to T_0 \wedge V_u \to \cdots \to T_0 \wedge V_0 = Y \wedge Z \,.$$

Hence we get a restricted pairing

$$F^s[T, Y]_* \otimes F^u[V, Z]_* \longrightarrow F^{s+u}[T \wedge V, Y \wedge Z]_*$$

that descends to a pairing

$$F^s/F^{s+1} \otimes F^u/F^{u+1} \longrightarrow F^{s+u}/F^{s+u+1}$$

of filtration quotients. When the respective spectral sequences converge, we can write this as a pairing

$$E_\infty^{s,*} \otimes E_\infty^{u,*} \longrightarrow E_\infty^{s+u,*}$$

of E_∞-terms. We will relate this to an algebraically defined pairing

$$\mathrm{Ext}_{\mathscr{A}}^{s,*}(H^*(Y), H^*(T)) \otimes \mathrm{Ext}_{\mathscr{A}}^{u,*}(H^*(Z), H^*(V)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+u,*}(H^*(Y \wedge Z), H^*(T \wedge V))$$

of the Adams spectral sequence E_2-terms.

Let M, N, T and V be $\mathscr{A}$-modules.

Lemma 13.19. *Let $\epsilon\colon P_* \to M$ and $\epsilon\colon Q_* \to N$ be two resolutions. Then $\epsilon \otimes \epsilon\colon P_* \otimes Q_* \to M \otimes N$ is a resolution. If P_* or Q_* is free, then so is $P_* \otimes Q_*$.*

Proof. If $\epsilon_*\colon H_*(P_*) \to M$ and $\epsilon_*\colon H_*(Q_*) \to N$ are isomorphisms, then $(\epsilon\otimes\epsilon)_*\colon H_*(P_*\otimes Q_*) \to M\otimes N$ must also be an isomorphism, due to the Künneth isomorphism $H_*(P_*) \otimes H_*(Q_*) \cong H_*(P_* \otimes Q_*)$.

If P_* is free in each degree, then $P_s \otimes Q_u$ is a sum of copies of $\mathscr{A} \otimes Q_u$, for each s and u, hence is free by Proposition 13.18. Hence $P_* \otimes Q_*$ is free in each degree. The same argument applies if Q_* is free in each degree. $\square$

Definition 13.20. The *tensor product pairing*

$$\otimes\colon \mathrm{Ext}_{\mathscr{A}}^{s,t}(M, T) \otimes \mathrm{Ext}_{\mathscr{A}}^{u,v}(N, V) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+u,t+v}(M \otimes N, T \otimes V)$$

is given by choosing free $\mathscr{A}$-module resolutions $P_* \to M$ and $Q_* \to N$. The tensor product $P_* \otimes Q_* \to M \otimes N$ is then a free $\mathscr{A}$-module resolution of $M \otimes N$, and $T \otimes V$ is a left $\mathscr{A}$-module, in both cases using the coproduct $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ to restrict the external $\mathscr{A} \otimes \mathscr{A}$-module structure to an internal $\mathscr{A}$-module structure. The tensor product of $\mathscr{A}$-module homomorphisms induces a pairing

$$\mathrm{Hom}_{\mathscr{A}}^*(P_*, T) \otimes \mathrm{Hom}_{\mathscr{A}}^*(Q_*, V) \longrightarrow \mathrm{Hom}_{\mathscr{A}}^*(P_* \otimes Q_*, T \otimes V)$$

of complexes, and the tensor product pairing is the induced pairing in homology.

More explicitly, the pairing takes cocycles $f\colon P_s \to \Sigma^t T$ and $g\colon Q_u \to \Sigma^v V$, with $f\partial_{s+1} = 0$ and $g\partial_{u+1} = 0$, to the tensor product

$$f \otimes g\colon P_s \otimes Q_u \longrightarrow \Sigma^t T \otimes \Sigma^v V \cong \Sigma^{t+v} T \otimes V\,.$$

This is extended by zero on the remaining summands of $(P_*\otimes Q_*)_{s+u}$. Equivalently, f and g can be viewed as chain maps $P_* \to \Sigma^t T[s]$ and $Q_* \to \Sigma^v V[u]$, respectively, where $\Sigma^t T[s]$ is the chain complex with $\Sigma^t T$ concentrated in degree s, and similarly for $\Sigma^v V[u]$. Then $(f\otimes g)\partial_{s+u+1} = f\partial_{s+1}\otimes g + (-1)^{|f|} f\otimes g\partial_{u+1} = 0$, so the tensor product is a cocycle.

In particular, for $s = 0$ and $u = 0$, the tensor product pairing on Ext agrees with the Hom-pairing

$$\otimes\colon \mathrm{Hom}_{\mathscr{A}}^t(M, T) \otimes \mathrm{Hom}_{\mathscr{A}}^v(M, V) \longrightarrow \mathrm{Hom}_{\mathscr{A}}^{t+v}(M \otimes N, T \otimes V)$$

that maps $f\colon M \to \Sigma^t T$ and $g\colon N \to \Sigma^v V$ to $f \otimes g\colon M \otimes N \to \Sigma^t T \otimes \Sigma^v V \cong \Sigma^{t+v} T \otimes V$.

Alternatively, if we have given another free $\mathscr{A}$-module resolution $R_* \to M \otimes N$, then we can cover the identity of $M \otimes N$ by a chain map $\Delta\colon R_* \to P_* \otimes Q_*$, unique up to chain homotopy. Then the composite

$$R_* \xrightarrow{\Delta} P_* \otimes Q_* \xrightarrow{f\otimes g} \Sigma^t T \otimes \Sigma^v V \cong \Sigma^{t+v} T \otimes V$$

is a cocycle that represents the tensor product $[f] \otimes [g]$ in $\mathrm{Ext}_{\mathscr{A}}^{s+u,t+v}(M \otimes N, T \otimes V)$.

13.5. **The smash product pairing of Adams spectral sequences.** Let Y and Z be spectra. We have a smash product pairing

$$\wedge\colon \pi_*(Y) \otimes \pi_*(Z) \longrightarrow \pi_*(Y \wedge Z)$$

that takes $f\colon S^t \to Y$ and $g\colon S^v \to Z$ to the smash product $f \wedge g\colon S^{t+v} \cong S^t \wedge S^v \to Y \wedge Z$.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$. Let $P_s = H^*(\Sigma^s K^s)$ and $Q_u = H^*(\Sigma^u L^u)$ be the $\mathscr{A}$-modules that appear in the usual free resolutions $\epsilon\colon P_* \to H^*(Y)$ and $\epsilon\colon Q_* \to H^*(Z)$.

Let $W = Y \wedge Z$ be the smash product. Then W is bounded below and $H_*(W) \cong H_*(Y) \otimes H_*(Z)$ is of finite type. We shall construct an Adams resolution $\{W^n\}_n$ of W by geometrically mixing the Adams resolutions for Y and Z.

Traditionally, this is done by first replacing Y, Z and their Adams resolutions by homotopy equivalent spectra, so that each Y^s and Z^u is a CW spectrum, and each map $i\colon Y^{s+1} \to Y^s$ and $i\colon Z^{u+1} \to Z^u$ is the inclusion of a CW subspectrum. Then $Y^s \wedge Z^u$ is a CW subspectrum of $Y \wedge Z$, and one can form the union of these subspectra for all $s+u=n$. Hence one defines

$$W^n = \bigcup_{s+u=n} Y^s \wedge Z^u \,.$$

Then W^{n+1} is a CW subspectrum of W^n, and

$$W^n/W^{n+1} \cong \bigvee_{s+u=n} K^s \wedge L^u \,.$$

Lemma 13.21. *The diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & W^2 \arrow[r, "i"] \arrow[d, "j"] & W^1 \arrow[r, "i"] \arrow[d, "j"] & W \arrow[d, "j"] \\
& W^2/W^3 \arrow[ul, dashed, "\partial"] & W^1/W^2 \arrow[ul, dashed, "\partial"] & W/W^1 \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

is an Adams resolution of $W = Y \wedge Z$. The associated free resolution $R_* \to H^*(W)$ is the tensor product of the free resolutions $P_* \to H^*(Y)$ and $Q_* \to H^*(Z)$.

Proof. Since each K^s is a wedge sum of suspended copies of H, of finite type, and each L^u is of finite type, we know that W^n/W^{n+1} is a wedge sum of suspended copies of H, of finite type. Let

$$R_n = H^*(\Sigma^n(W^n/W^{n+1})) \cong \bigoplus_{s+u=n} P_s \otimes Q_u \,.$$

This is a free $\mathscr{A}$-module of finite type, by its geometric origin as the cohomology of W^n/W^{n+1}. The composite $W^{n-1}/W^n \to \Sigma W^n \to \Sigma(W^n/W^{n+1})$ splits as the direct sum of the maps $j\partial\wedge 1\colon K^{s-1}\wedge L^u \to \Sigma K^s \wedge L^u \cong \Sigma(K^s \wedge L^u)$ and $1 \wedge j\partial\colon K^s \wedge L^{u-1} \to K^s \wedge \Sigma L^u \cong \Sigma(K^s \wedge L^u)$. Hence the boundary map $\partial_n\colon R_n \to R_{n-1}$ is given by the usual formula

$$\partial_n(x \otimes y) = \partial_n(x) \otimes y + x \otimes \partial_n(y)$$

(we work at $p=2$, hence there is no sign), so that $R_* = P_* \otimes Q_*$ is the tensor product of the two resolutions. By the Künneth theorem, the homology of R_* is the tensor product of the homologies of P_* and Q_*, so $\epsilon\colon R_* \to H^*(Y) \otimes H^*(Z) \cong H^*(Y \wedge Z)$ is a free resolution.

In particular, $j\colon W^0 = Y \wedge Z \to K^0 \wedge L^0$ induces a surjection j^* in cohomology. It follows that $\partial\colon W/W^1 \to \Sigma W^1$ induces an injection ∂^* in cohomology, with image in $R_0 = H^*(W/W^1)$ equal to the kernel of $j^* = \epsilon$. This equals the image of $\partial_1 = \partial^* j^* : R_1 \to R_0$, by exactness at R_0 of the free resolution, which implies that j^*, induced by $j\colon W^1 \to W^1/W^2$, is surjective. Suppose inductively that $j\colon W^{n-1} \to W^{n-1}/W^n$ induces a surjection j^* in cohomology, for some $n \geq 2$. Then $\partial\colon W^{n-1}/W^n \to \Sigma W^n$ induces an injection ∂^* in cohomology. The image of ∂^* equals the kernel of j^*, hence lies in the kernel of $\partial_{n-1} = \partial^* j^*\colon R_{n-1} \to R_{n-2}$. This equals the image of $\partial_n = \partial^* j^*\colon R_n \to R_{n-1}$, by exactness at R_{n-1}, which implies that j^*, induced by $j\colon W^n \to W^n/W^{n+1}$, is surjective. $\square$

Granting a little more technology, the substitution by CW spectra can be replaced by the passage to a homotopy colimit. For a fixed $n \geq 0$, one considers the diagram of all spectra $Y^s \wedge Z^u$ for $s+u \geq n$, and forms the homotopy colimit

$$W^n = \operatorname*{hocolim}_{s+u \geq n} Y^s \wedge Z^u \,.$$

There is a natural diagram

$$\cdots \to W^2 \xrightarrow{i} W^1 \xrightarrow{i} W^0 \simeq Y \wedge Z$$

and an identification

$$W^n/W^{n+1} \cong \bigvee_{s+u=n} \mathrm{hocofib}(Y^{s+1} \to Y^s) \wedge \mathrm{hocofib}(Z^{u+1} \to Z^u)$$

where $\mathrm{hocofib}(Y^{s+1} \to Y^s) \simeq K^s$ denotes the mapping cone of the given map, etc. The proof of the lemma goes through in the same way with these conventions.

The following theorem is similar to that proved in §4 of Adams (1958).

Theorem 13.22. *There is a natural pairing*

$$E_r^{s,t}(Y) \otimes E_r^{u,v}(Z) \longrightarrow E_r^{s+u,t+v}(Y \wedge Z)$$

of Adams spectral sequences, given at the E_2-term by the tensor product pairing

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_p) \otimes \mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Y \wedge Z), \mathbb{F}_p)$$

and converging to the smash product pairing

$$\pi_{t-s}(Y_p^\wedge) \otimes \pi_{v-u}(Z_p^\wedge) \longrightarrow \pi_{t-s+v-u}((Y \wedge Z)_p^\wedge)\,.$$

More generally, there is a natural pairing

$$E_r^{s,t}(T, Y) \otimes E_r^{u,v}(V, Z) \longrightarrow E_r^{s+u,t+v}(T \wedge V, Y \wedge Z)$$

of spectral sequences, given at the E_2-term by the tensor product pairing

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(T)) \otimes \mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(V)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Y \wedge Z), H^*(T \wedge V))$$

and converging to the smash product pairing

$$[T, Y_p^\wedge]_{t-s} \otimes [V, Z_p^\wedge]_{v-u} \longrightarrow [T \wedge V, (Y \wedge Z)_p^\wedge]_{t-s+v-u}\,.$$

((Discuss the role of completion in the pairing?))

Proof. Recall that $E_r^s = Z_r^s / B_r^s$, where

$$Z_r^s = \partial^{-1} \operatorname{im}(i_*^{r-1} \colon \pi_*(Y^{s+r}) \to \pi_*(Y^{s+1}))$$

and

$$B_r^s = j \ker(i_*^{r-1} \colon \pi_*(Y^s) \to \pi_*(Y^{s+r-1}))$$

are subgroups of $E_s^1 = \pi_*(K^s)$. For the purpose of this proof, it is convenient to rewrite these groups as

$$Z_r^s = \operatorname{im}(\pi_*(Y^s/Y^{s+r}) \to \pi_*(K^s))$$

and

$$B_r^s = \operatorname{im}(\pi_*(\Sigma^{-1}(Y^{s-r+1}/Y^s)) \to \pi_*(K^s))\,.$$

These formulas can be obtained by chases in the diagrams

$$\begin{tikzcd}
Y^{s+r} \arrow[r, "i^r"] \arrow[d] & Y^s \arrow[r] \arrow[d, "j"] & Y^s/Y^{s+r} \arrow[d] \\
* \arrow[r] \arrow[d] & K^s \arrow[r, "="] \arrow[d, "\partial"] & K^s \arrow[d] \\
\Sigma Y^{s+r} \arrow[r, "\Sigma i^{r-1}"] & \Sigma Y^{s+1} \arrow[r] & \Sigma(Y^{s+1}/Y^{s+r})
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[r, "="] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[d] \\
Y^{s+1} \arrow[r, "i"] \arrow[d, "="'] & Y^s \arrow[r, "j"] \arrow[d, "i^{r-1}"] & K^s \arrow[d] \\
Y^{s+1} \arrow[r, "i^r"] & Y^{s-r+1} \arrow[r] & Y^{s-r+1}/Y^{s+1}
\end{tikzcd}$$

of horizontal and vertical cofiber sequences.

The differential $d_r^s \colon E_r^s \to E_r^{s+r}$ is determined by the homomorphism $\delta \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^{s+r}$ induced by $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and the surjection $\pi \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^s$ induced by $Y^s/Y^{s+r} \to K^s$:

$$\begin{tikzcd}
E_1^s & Z_r^s \arrow[l, tail] \arrow[d, two heads] & \pi_*(Y^s/Y^{s+r}) \arrow[l, two heads, "\pi"'] \arrow[r, "\delta"] & Z_r^{s+t} \arrow[r, tail] \arrow[d, two heads] & E_1^{s+r} \\
 & E_r^s \arrow[rr, "d_r^s"] & & E_r^{s+r} &
\end{tikzcd}$$

To compare this with the exact couple definition of d_r^s, consider the commutative diagram

$$\begin{tikzcd}
K^s \arrow[d, "k"'] & Y^s/Y^{s+r} \arrow[l] \arrow[r] \arrow[d] & \Sigma K^{s+r} \arrow[d, equal] \\
\Sigma Y^{s+1} & \Sigma Y^{s+r} \arrow[l, "i^{r-1}"'] \arrow[r, "j"] & \Sigma K^{s+r}
\end{tikzcd}$$

where the left hand square is homotopy (co-)cartesian. (It follows that $B_{r+1}^{s+r}/B_r^{s+r} \subset E_r^{s+r}$ equals the image of d_r^s.)

So far we have discussed the Adams spectral sequence for a single spectrum Y. We now relate the Adams spectral sequences for Y, Z and $W = Y \wedge Z$, where W has the Adams resolution obtained from given Adams resolutions of Y and Z.

There is a preferred inclusion $Y^s \wedge Z^u \to W^n$ for all $s, u \geq 0$ and $n = s + u$. It restricts to inclusions $Y^{s+r} \wedge Z^u \to W^{n+r}$ and $Y^s \wedge Z^{u+r} \to W^{n+r}$, that agree on $Y^{s+r} \wedge Z^{u+r}$. Hence we have a main commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r} \arrow[r] \arrow[d] \arrow[rr, bend left, "a_r"] & Y^{s+1} \wedge Z^u \cup Y^s \wedge Z^{u+1} \arrow[r, "a_1"] \arrow[d] & Y^s \wedge Z^u \arrow[r] \arrow[d] & Y \wedge Z \arrow[d, "\simeq"] \\
W^{n+r+1} \arrow[r, "i"] & W^{n+r} \arrow[r, "i^{r-1}"] \arrow[rr, bend right, "i^r"'] & W^{n+1} \arrow[r, "i"] & W^n \arrow[r] & W
\end{tikzcd}$$

where $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ denotes the pushout of $Y^{s+r} \wedge Z^u$ and $Y^s \wedge Z^{u+r}$ along $Y^{s+r} \wedge Z^{u+r}$, and U is brief notation for a similar union of $Y^{s+r+1} \wedge Z^u$, $Y^{s+r} \wedge Z^{u+1}$, $Y^{s+1} \wedge Z^u$ and $Y^s \wedge Z^{u+1}$.

Passing to horizontal cofibers for the middle part of the diagram, we get a commutative diagram

(5)
$$\begin{tikzcd}
Y^s \wedge Z^u \arrow[r] \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
W^n \arrow[r] & W^n/W^{n+r} \arrow[r] & W^n/W^{n+1}
\end{tikzcd}$$

where the maps in the upper row are smash products of the standard maps $Y^s \to Y^s/Y^{s+r}$, $Y^s/Y^{s+r} \to K^s$, etc. The vertical map $K^s \wedge L^u \to W^n/W^{n+1}$ agrees with the inclusion of a summand in $W^n/W^{n+1} \cong \bigvee_{s+u=n} K^s \wedge L^u$. Hence it induces a pairing

$$\phi_1 \colon E_1^s(Y) \otimes E_1^u(Z) \longrightarrow E_1^n(W)$$

that corresponds to the previously discussed pairing

$$\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_p) \otimes \mathrm{Hom}_{\mathscr{A}}(Q_*, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(P_* \otimes Q_*, \mathbb{F}_p)$$

under the d-invariant isomorphisms $\pi_{t-s}(K^s) \cong \mathrm{Hom}_{\mathscr{A}}^t(P_s, \mathbb{F}_p)$, etc.

Passing to horizontal cofibers further to the left in the main diagram, we get a commutative diagram

(6)
$$\begin{tikzcd}
Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & \Sigma(Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}) \arrow[r] \arrow[d] & \Sigma(K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}) \arrow[d] \\
W^n/W^{n+r} \arrow[r] & \Sigma W^{n+r} \arrow[r] & \Sigma(W^{n+r}/W^{n+r+1})
\end{tikzcd}$$

where the composite map in the upper row is the wedge sum of the smash product of the standard maps $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and $Z^u/Z^{u+r} \to L^u$, and the smash product of the standard maps $Y^s/Y^{s+r} \to K^s$ and $Z^u/Z^{u+r} \to \Sigma L^{u+r}$. The right hand vertical map is the suspension of the wedge sum of the pairings $K^{s+r} \wedge L^u \to W^{n+r}/W^{n+r+1}$ and $K^s \wedge L^{u+r} \to W^{n+r}/W^{n+r+1}$.

We now claim that (a) $\phi_1 = \tilde{\phi}_1$ restricts to a pairing

$$\tilde{\phi}_r \colon Z_r^s(Y) \otimes Z_r^u(Z) \longrightarrow Z_r^n(W)\,,$$

(b) $\tilde{\phi}_r$ descends to a pairing

$$\phi_r \colon E_r^s(Y) \otimes E_r^u(Z) \longrightarrow E_r^n(W)$$

and (c) ϕ_r satisfies the Leibniz rule

$$d_r(\phi_r(y \otimes z)) = \phi_r(d_r(y) \otimes z) + (-1)^{|y|}\phi_r(y \otimes d_r(z)) \,.$$

Here $r \geq 1$ and $n = s + u$.

Assuming these claims, which are similar to the conditions of Lemma 2.2 of Moss (1968), we can easily finish the proof of the theorem. The pairings $(\phi_r)_*$ and ϕ_{r+1} agree, under the identification $H^s(E_r^*, d_r) \cong E_{r+1}^s$, since they are both induced by a passage to quotients from $\tilde{\phi}_{r+1}$. Hence the sequence $\{\phi_r\}_r$ qualifies as a pairing of spectral sequences. In particular, $\phi_2 = (\phi_1)_*$ is the tensor product pairing of Ext-groups. This spectral sequence pairing converges to the smash product pairing in homotopy, since the pairing of E_∞-terms is induced by the pairing

$$\pi_*(Y^s) \otimes \pi_*(Z^u) \longrightarrow \pi_*(Y^s \wedge Z^u) \longrightarrow \pi_*(W^n)$$

via the surjections $\pi_*(Y^s) \to E_\infty^s$, etc., and the pairing of filtration quotients is induced by the same pairing via the surjections $\pi_*(Y^s) \to F^s \to F^s/F^{s+1}$, etc. These surjections have the same kernel, so the induced pairings of quotients are compatible under the identifications $F^s/F^{s+1} \cong E_\infty^s$.

It remains to prove the three parts of the claim.

(a) Applying $\pi_*(-)$ to the right hand square in diagram (5), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, two heads] \arrow[d] & Z_r^s(Y) \otimes Z_r^u(Z) \arrow[r, tail] \arrow[d, dashed, "\tilde{\phi}_r"'] & E_1^s(Y) \otimes E_1^u(Z) \arrow[d, "\phi_1"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, two heads, "\pi"] & Z_r^n(W) \arrow[r, tail] & E_1^n(W)
\end{tikzcd}$$

In view of the description of $Z_r^n(W)$ as the image of $\pi_*(W^n/W^{n+r}) \to \pi_*(W^n/W^{n+1}) = E_1^n(W)$, and similarly for Y and Z, it follows that there is a unique pairing $\tilde{\phi}_r$ that makes the whole diagram commute.

(b) To check that $\tilde{\phi}_r$ descends to a pairing $\phi_r \colon E_r^s(Y) \otimes E_r^u(Z) \to E_r^n(W)$, we use the diagram

$$\begin{tikzcd}
E_r^s(Y) \otimes E_r^u(Z) \arrow[d, dashed, "\phi_r"'] & Z_r^s(Y) \otimes Z_r^u(Z) \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\tilde{\phi}_r"'] & Z_{r-1}^s(Y) \otimes Z_{r-1}^u(Z) \arrow[r, two heads] \arrow[d, "\tilde{\phi}_{r-1}"'] & E_{r-1}^s(Y) \otimes E_{r-1}^u(Z) \arrow[d, "\phi_{r-1}"'] \\
E_r^n(W) & Z_r^n(W) \arrow[l, two heads] \arrow[r, tail] & Z_{r-1}^n(W) \arrow[r, two heads] & E_{r-1}^n(W) \,.
\end{tikzcd}$$

There is only something to prove for $r \geq 2$. We assume, by induction on r, that the Leibniz rule in (c) holds for d_{r-1} and ϕ_{r-1}.

Given $y \in B_r^s(Y) \subset Z_r^s(Y)$ and $z \in Z_r^u(Z)$ we must show that $\tilde{\phi}_r(y \otimes z) \in B_r^n(W) \subset Z_r^n(W)$. The image of y in $E_{r-1}^s(Y)$ has the form $[y] = d_{r-1}(x)$ for some $x \in E_{r-1}^{s-r+1}(Y)$, and the image of z in $E_{r-1}^u(Z)$ satisfies $d_{r-1}([z]) = 0$. Then $d_{r-1}(\phi_{r-1}(x \otimes [z])) = \phi_{r-1}(d_{r-1}(x) \otimes [z]) \pm \phi_{r-1}(x \otimes d_{r-1}([z])) = \phi_{r-1}([y] \otimes [z]) \pm 0 = [\tilde{\phi}_r(y \otimes z)]$. Hence $\tilde{\phi}_r(y \otimes z)$ is congruent modulo $B_{r-1}^n(W)$ to a class in $B_r^n(W)$, as we asserted. The same argument shows that $\tilde{\phi}_r$ maps $Z_r^s(Y) \otimes B_r^u(Z)$ into $B_r^n(W)$. Hence $\tilde{\phi}_r$ descends to ϕ_r, and this uniquely determines ϕ_r.

(c) [[TODO: Account for signs.]] Applying $\pi_*(-)$ to the outer rectangle in diagram (6), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, "{\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]}"] \arrow[d] & \begin{array}{c} Z_r^{s+r}(Y) \otimes Z_r^u(Z) \\ \oplus \\ Z_r^s(Y) \otimes Z_r^{u+r}(Z) \end{array} \arrow[r, tail] \arrow[d, "{[\tilde{\phi}_r\ \tilde{\phi}_r]}"'] & \begin{array}{c} E_1^{s+r}(Y) \otimes E_1^u(Z) \\ \oplus \\ E_1^s(Y) \otimes E_1^{u+r}(Z) \end{array} \arrow[d, "{[\phi_1\ \phi_1]}"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, "\delta"] & Z_r^{n+r}(W) \arrow[r, tail] & E_1^{n+r}(W)
\end{tikzcd}$$

Since the pairings $\tilde{\phi}_r$ have been defined to make the right hand square commute, the whole diagram commutes.

Combining parts of four of these diagrams, we have the commutative sprawl:

$$\begin{tikzcd}
E_r^s(Y) \otimes E_r^u(Z) \arrow[d, "\left[\begin{smallmatrix} d_r^s \otimes 1 \\ 1 \otimes d_r^u \end{smallmatrix}\right]"'] \arrow[rrr, bend left, "\phi_r"] & Z_r^s(Y) \otimes Z_r^u(Z) \arrow[l, two heads] \arrow[r, "\tilde{\phi}_r"] & Z_r^n(W) \arrow[r, two heads] & E_r^n(W) \arrow[dd, "d_r^n"] \\
& \pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[u, two heads, "\pi\otimes\pi"] \arrow[r] \arrow[d, "\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]"'] & \pi_*(W^n/W^{n+r}) \arrow[u, two heads, "\pi"'] \arrow[d, "\delta"'] & \\
\begin{matrix} E_r^{s+r}(Y) \otimes E_r^u(Z) \\ \oplus \\ E_r^s(Y) \otimes E_r^{u+r}(Z) \end{matrix} \arrow[rrr, bend right, "{[\phi_r\ \phi_r]}"'] & \begin{matrix} Z_r^{s+r}(Y) \otimes Z_r^u(Z) \\ \oplus \\ Z_r^s(Y) \otimes Z_r^{u+r}(Z) \end{matrix} \arrow[l, two heads] \arrow[r, "{[\tilde{\phi}_r\ \tilde{\phi}_r]}"] & Z_r^{n+r}(W) \arrow[r, two heads] & E_r^{n+r}(W)
\end{tikzcd}$$

Going around the outer boundary of the diagram we see that $d_r^n(\phi_r(y \otimes z)) = \phi_r(d_r^s(y) \otimes z) + \phi_r(y \otimes d_r^u(z))$, proving the Leibniz rule. □

Remark 13.23. If $y \in \pi_*(K^s)$ and $z \in \pi_*(L^u)$ lift to $\tilde{y} \in \pi_*(Y^s/Y^{s+r})$ and $\tilde{z} \in \pi_*(Z^u/Z^{u+r})$, respectively, with images $\delta y \in \pi_*(\Sigma K^{s+r})$ and $\delta z \in \pi_*(\Sigma L^{u+r})$, then $y \wedge z \in \pi_*(K^s \wedge L^u)$ lifts to $\tilde{y} \wedge \tilde{z} \in \pi_*(Y^s/Y^{s+r} \wedge Z^u/Z^{u+r})$.

$$\begin{tikzcd}
\Sigma K^{s+r} & Y^s/Y^{s+r} \arrow[l] \arrow[r] & K^s
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u & & K^s \wedge L^u & L^u \\
& Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[ul] \arrow[ur] \arrow[dr] & & Z^u/Z^{u+r} \arrow[u] \arrow[d] \\
& & \Sigma K^s \wedge L^{u+r} & \Sigma L^{u+r}
\end{tikzcd}$$

The maps $Y^s \wedge Z^u \to W^{s+u} = W^n$ induce a commutative diagram

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r} \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[l] \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
\Sigma(W^{n+r}/W^{n+r+1}) & W^n/W^{n+r} \arrow[l] \arrow[r] & W^n/W^{n+1}
\end{tikzcd}$$

and $\tilde{y} \wedge \tilde{z}$ maps to a lift $\tilde{y} \cdot \tilde{z}$ in $\pi_*(W^n/W^{n+r})$ of the image $y \cdot z$ of $y \wedge z$ in W^n/W^{n+1}. Hence $\delta(y \cdot z)$ is the image $\delta y \cdot z + y \cdot \delta z$ of $\delta y \wedge z + y \wedge \delta z$ in $\pi_*(\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r})$. The key point is that, even if $Y^s/Y^{s+r} \wedge Z^u/Z^{u+r}$ is attached to all of $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ in $Y^s \wedge Z^u$, the composite map to $W^{n+r} \to W^{n+r}/W^{n+r+1}$ factors through the quotient $K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}$, making the left hand square above commute. The bookkeeping shows that δy represents $d_r([y])$, and so on, so that $\delta(y \cdot z) = \delta y \cdot z + y \cdot \delta z$ implies the Leibniz rule for d_r.

Corollary 13.24. *Suppose that Y is a ring spectrum, with multiplication $\phi\colon Y \wedge Y \to Y$ and unit $\eta\colon S \to Y$. Then there is a natural pairing*

$$E_r^{*,*}(Y) \otimes E_r^{*,*}(Y) \longrightarrow E_r^{*,*}(Y)\,,$$

given at the E_2-term by the composite

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y \wedge Y), \mathbb{F}_p) \xrightarrow{\phi_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)\,,$$

and a unit map

$$E_r^{*,*}(S) \xrightarrow{\eta_*} E_r^{*,*}(Y)\,,$$

given at the E_2-term by

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \xrightarrow{\eta_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)\,,$$

that make the Adams spectral sequence $E_r^{*,*}(Y)$ an algebra spectral sequence over $E_r^{*,*}(S)$. If Y is homotopy commutative, then it is a commutative algebra spectral sequence.

13.6. **The bar resolution.** Let k be a commutative ring, and consider any k-algebra A. (Our principal example will be the case $k = \mathbb{F}_p$ and $A = \mathscr{A}$, the mod p Steenrod algebra.) We write $\otimes$ for $\otimes_k$ and Hom for Hom_k. Let M and N be left and right A-modules, respectively. The two-sided bar construction $\beta_\bullet(N, A, M)$ is the simplicial k-module, given in degree $q \ge 0$ by

$$\beta_q(N, A, M) = N \otimes A^{\otimes q} \otimes M\,.$$

Following Eilenberg–Mac Lane (see Mac Lane (1963, Sect. X.2)) we use the notation $n[a_1|\dots|a_q]m$ for the tensor $n \otimes a_1 \otimes \dots \otimes a_q \otimes m$ in $\beta_q(N, A, M)$, and the use of vertical bars in this notation gives the construction its name. The face operators $d_i\colon \beta_q(N, A, M) \to \beta_{q-1}(N, A, M)$ for $0 \le i \le q$ are given by

$$d_i(n[a_1|\dots|a_q]m) = \begin{cases} na_1[a_2|\dots|a_q]m & \text{for } i = 0, \\ n[a_1|\dots|a_{i-1}|a_ia_{i+1}|a_{i+2}|\dots|a_q]m & \text{for } 0 < i < q, \text{ and} \\ n[a_1|\dots|a_{q-1}]a_qm & \text{for } i = q. \end{cases}$$

The degeneracy operators $s_j\colon \beta_q(N, A, M) \to \beta_{q+1}(N, A, M)$ for $0 \le j \le q$ are given by

$$s_j(n[a_1|\dots|a_q]m) = n[a_1|\dots|a_j|1|a_{j+1}|\dots|a_q]m\,,$$

where $1 = \eta(1) \in A$ denotes the algebra unit. There is an augmentation

$$\epsilon\colon \beta_\bullet(N, A, M) \longrightarrow N \otimes_A M$$

to the balanced tensor product $N \otimes_A M$, considered as a constant simplicial object, given in degree $q = 0$ by $\epsilon(n[]m) = n \otimes_A m$. In the special case $N = A$, with the right A-module structure given by the k-algebra multiplication, there is an extra degeneracy operator $s_{-1}\colon \beta_q(A, A, M) \to \beta_{q+1}(A, A, M)$ given by

$$s_{-1}(a_0[a_1|\dots|a_q]m) = 1[a_0|a_1|\dots|a_q]m$$

and the augmentation specializes to $\epsilon\colon \beta_\bullet(A, A, M) \to M$ given by $\epsilon(a[]m) = am$. [[Can discuss how s_{-1} specifies a simplicial contraction of $\beta_\bullet(A, A, M)$ to M.]] In this case the left A-module structure on $N = A$ induces an A-module structure on each $\beta_q(A, A, M)$, making $\beta_\bullet(A, A, M)$ a simplicial A-module. (The extra degeneracy s_{-1} is not A-linear. We use the induced, not the diagonal, A-module structure on each $\beta_q(A, A, M) = A \otimes A^{\otimes q} \otimes M$, even when the latter exists.)

The associated normalized chain complex is the normalized bar construction. It is the chain complex of k-modules given in degree $q \ge 0$ by

$$B_q(N, A, M) = N \otimes J(A)^{\otimes q} \otimes M$$

where $J(A) = \mathrm{cok}(\eta\colon k \to A)$ is the unit coideal. [[Often denoted $\bar{A}$.]] The boundary operator $\partial_q\colon B_q(N, A, M) \to B_{q-1}(N, A, M)$ is the alternating sum

$$\partial_q = \sum_{i=0}^{q} (-1)^i d_i$$

of the face operators, which descends over the surjection $\beta_q(N, A, M) \to B_q(N, A, M)$. Hence

$$\partial_q(n[a_1|\dots|a_q]m) = na_1[a_2|\dots|a_q]m + \sum_{0<i<q} (-1)^i n[a_1|\dots|a_ia_{i+1}|\dots|a_q]m + (-1)^q n[a_1|\dots|a_{q-1}]a_qm$$

for $n \in N$, $a_i \in J(A)$ and $m \in M$. [[A sign may be introduced, depending on the degrees of the terms a_i.]] There is still an augmentation

$$\epsilon\colon B_0(N, A, M) \longrightarrow N \otimes_A M$$

given by the canonical surjection $N \otimes M \to N \otimes_A M$, and we get an augmented chain complex

$$\dots \to B_2(N, A, M) \xrightarrow{\partial_2} B_1(N, A, M) \xrightarrow{\partial_1} B_0(N, A, M) \xrightarrow{\epsilon} N \otimes_A M \to 0$$

of k-modules. In the special case when $N = A$, the augmentation can be rewritten as $\epsilon\colon B_0(N, A, M) \to M$, sending $a[]m$ to am. In this case the extra degeneracy gives rise to a chain contraction S of $B_*(A, A, M)$ to M. This is a chain homotopy

$$S_q\colon B_q(A, A, M) \longrightarrow B_{q+1}(A, A, M)$$

given by

$$S_q(a_0[a_1|\dots|a_q]m) = 1[a_0|a_1|\dots|a_q]m$$

for all $q \geq 0$. It satisfies

$$\partial S + S\partial = 1 - \eta\epsilon \,.$$

Here 1 denotes the identity, and $\eta\colon M \to B_0(A, A, M)$ sends m to $1[]m$, so that $\eta\epsilon(a[]m) = 1[]am$. [[Prove the chain homotopy relation. Clarify that $\partial = 0$ on $B_0(N, A, M)$. Conversely, the boundaries ∂ are inductively determined by this relation and A-linearity.]] Hence ϵ and η are chain homotopy inverse equivalences between $B_*(A, A, M)$ and M, where M is viewed as a trivial chain complex concentrated in degree 0.

The left A-module structure on $N = A$ makes $B_*(A, A, M)$ a chain complex of left A-modules. In other words,

$$\epsilon\colon B_*(A, A, M) \longrightarrow M$$

is an A-module resolution of M, called the *bar resolution*. (Note that we use the induced A-module structure on each $B_q(A, A, M) = A \otimes J(A)^{\otimes q} \otimes M$, not the diagonal structure, in case the latter exists.) If $J(A)$ and M are flat, resp. projective, as k-modules, then so is $J(A)^{\otimes q} \otimes M$. This will imply that $B_q(A, A, M)$ is flat, resp. projective, as a left A-module. Hence, under these conditions, the bar resolution is a flat, resp. projective, resolution. If k is a field, as it will be in the case when A is the mod p Steenrod algebra, then these conditions are automatically satisfied.

In particular, we can in principle use the bar resolution to calculate $\mathrm{Tor}^A_*(N, M)$ and $\mathrm{Ext}^*_A(M, L)$. If $J(A)$ and M are flat k-modules, then

$$\mathrm{Tor}^A_s(N, M) = H_s(N \otimes_A B_*(A, A, M), 1 \otimes \partial) \cong H_s(B_*(N, A, M), \partial)$$

for each $s \geq 0$, as graded k-modules. This uses the evident isomorphism $N \otimes_A B_q(A, A, M) \cong B_q(N, A, M)$ for each $q \geq 0$. If $J(A)$ and M are projective k-modules, and L is any left A-module, then

$$\mathrm{Ext}^s_A(M, L) = H^s(\mathrm{Hom}_A(B_*(A, A, M), L), \mathrm{Hom}(\partial, 1))$$

for each $s \geq 0$, again as graded k-modules. [[Can rewrite $\mathrm{Hom}_A(B_q(A, A, M), L) = \mathrm{Hom}_A(A \otimes J(A)^{\otimes q} \otimes M, L) \cong \mathrm{Hom}_k(J(A)^{\otimes q} \otimes M, L)$ in terms of $L \otimes (J(A)^*)^{\otimes q} \otimes M^*$, leading to the cobar complex $C^q(L, A^*, M^*)$ for the dual coalgebra A^*. Return to this later.]]

13.7. **Comparison of pairings.** The bar resolution grows too fast in size to be useful for efficient machine calculation, but its explicit form makes it useful for theoretical analysis. [[Calculate Yoneda composition and tensor product in terms of bar resolutions.]]

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps of spectra, and let $\tilde{g}\colon S \to F(Y, Z)$ be right adjoint to g. The smash product $\tilde{g} \wedge f\colon X \cong S \wedge X \to F(Y, Z) \wedge Y$ followed by the evaluation map $ev\colon F(Y, Z) \wedge Y \to Z$ defines a map

$$ev \circ (\tilde{g} \wedge f)\colon X \longrightarrow Z$$

that equals the composite $gf = g \circ f\colon X \to Z$. Hence the composition pairing

$$\circ\colon [Y, Z]_* \otimes [X, Y]_* \longrightarrow [X, Z]_*$$

can be rewritten in terms of the smash product pairing as the composite

$$\pi_* F(Y, Z) \otimes [X, Y]_* \xrightarrow{\wedge} [S \wedge X, F(Y, Z) \wedge Y]_* \xrightarrow{ev_*} [X, Z]_* \,.$$

In particular, for $Y = S$, the composition pairing

$$\circ\colon [S, Z]_* \otimes [X, S]_* \longrightarrow [X, Z]_*$$

equals the smash product pairing. Specializing to $X = S$, the composition and smash products give the same module action

$$\pi_*(Z) \otimes \pi_*(S) \longrightarrow \pi_*(Z) \,.$$

Specializing further to $Z = S$ the two ring structures

$$\pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S)$$

on $\pi_*(S)$ agree. The smash product pairing is graded commutative, since $\mu\colon S \wedge S \cong S$ and $\mu\gamma$ are homotopic (or equal, in some models). It follows that also the composition product is graded commutative, which is not so evident from its definition.

Conversely, given maps $f\colon T \to Y$ and $g\colon V \to Z$ of spectra, the smash product $f \wedge g\colon T \wedge V \to Y \wedge Z$ can be factored in two ways, as

$$(f \wedge 1) \circ (1 \wedge g) = f \wedge g = (1 \wedge g) \circ (f \wedge 1) \,.$$

Hence the smash product pairing

$$\wedge\colon [T,Y]_* \otimes [V,Z]_* \longrightarrow [T \wedge V, Y \wedge Z]_*$$

can be rewritten in terms of the composition pairing as the composite

$$\pi_* F(T,Y) \otimes \pi_* F(V,Z) \xrightarrow{\sigma_* \otimes \tau_*} \pi_* F(T \wedge Z, Y \wedge Z) \otimes \pi_* F(T \wedge V, T \wedge Z) \xrightarrow{\circ} \pi_* F(T \wedge V, Y \wedge Z)\,.$$

[[Explain the stabilization maps $\sigma\colon F(T,Y) \to F(T \wedge Z, Y \wedge Z)$ and $\tau\colon F(V,Z) \to F(T \wedge V, T \wedge Z)$, perhaps in terms of the adjoints $ev \wedge 1\colon F(T,Y) \wedge T \wedge Z \to Y \wedge Z$ and $\gamma(ev \wedge 1)(1 \wedge \gamma)\colon F(V,Z) \wedge T \wedge V \to T \wedge Z$.]] In particular, for $T = Z = S$, the smash product pairing

$$\wedge\colon [S,Y]_* \otimes [V,S]_* \longrightarrow [V,Y]_*$$

equals the composition pairing.

Let L, M and N be left A-modules, for a k-algebra A that is projective as a k-module. Let $\epsilon\colon B_*(A,A,M) \to M$ and $\epsilon\colon B_*(A,A,N) \to N$ be the normalized bar resolutions. The Yoneda composition

$$\circ\colon \operatorname{Ext}_A^{s,t}(M,L) \otimes \operatorname{Ext}_A^{u,v}(N,M) \longrightarrow \operatorname{Ext}_A^{s+u,t+v}(N,L)$$

takes $[f] \otimes [g]$ to $[\Sigma^v f \circ g_s]$, where $[f]$ and $[g]$ are the cohomology classes of cocycles $f\colon B_s(A,A,M) \to \Sigma^t L$ and $g\colon B_u(A,A,N) \to \Sigma^v M$, or equivalently, of chain maps $f\colon B_*(A,A,M) \to \Sigma^t L[s]$ and $g\colon B_*(A,A,N) \to \Sigma^v M[u]$, where $\Sigma^t L[s]$ denotes the chain complex with $\Sigma^t L$ in cohomological degree s and 0 in all other degrees, and similarly for $\Sigma^v M[u]$. Furthermore, $g_*\colon B_*(A,A,N) \to \Sigma^v B_*(A,A,M)[u]$ is a chain map lifting g, so that $\epsilon g_* = g$. It consists of A-module maps $g_q\colon B_{q+u}(A,A,N) \to \Sigma^v B_q(A,A,M)$ for all $q \ge 0$, commuting [[up to a sign]] with the boundary maps, and $\epsilon g_0 = g$. [[Explain the cohomological shift by u, denoted $[u]$, of a chain complex, including sign convention?]]

$$\begin{tikzcd}
B_*(A,A,N) \arrow[r, "g_*"] \arrow[d, "\epsilon", "\simeq"'] \arrow[dr, "g"] & B_*(A,A,M) \arrow[d, "\epsilon", "\simeq"'] \arrow[dr, "f"] & \\
N & M & L
\end{tikzcd}$$

The opposite Yoneda composition

$$\circ^{op}\colon \operatorname{Ext}_A^{u,v}(N,M) \otimes \operatorname{Ext}_A^{s,t}(M,L) \longrightarrow \operatorname{Ext}_A^{s+u,t+v}(N,L)$$

is given by the twist map $\gamma\colon [g] \otimes [f] \mapsto (-1)^{(t-s)(v-u)}[f] \otimes [g]$ followed by the Yoneda composition, hence maps $[g] \otimes [f]$ to $(-1)^{(t-s)(v-u)}[\Sigma^v f \circ g_s]$ in the notation above. [[Is this the correct sign?]]

It is possible to write down an explicit chain map g_* lifting g. Compare Adams (1960, p. 33).

Lemma 13.25. *Given a cocycle $g\colon B_*(A,A,N) \to \Sigma^v M[u]$, a chain map*

$$g_*\colon B_*(A,A,N) \to \Sigma^v B_*(A,A,M)[u]$$

that lifts g is given [[up to sign]] by the formula

$$g_q(a_0[a_1|\dots|a_{q+u}]n) = a_0[a_1|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n)$$

for each $q \ge 0$. Hence the Yoneda product $[f] \circ [g]$ is represented by the cocycle $\Sigma^v f \circ g_s\colon B_{s+u}(A,A,N) \to \Sigma^{t+v} L$ given by

$$a_0[a_1|\dots|a_{s+u}]n \longmapsto f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n))\,.$$

Proof. Let $g_0\colon B_u(A,A,N) \to B_0(A,A,M)$ of internal degree v be given by

$$g_0(a_0[a_1|\dots|a_u]n) = a_0[]g(1[a_1|\dots|a_u]n)\,.$$

Then g_0 is A-linear, and $\epsilon g_0 = g$. Next, define $g_1\colon B_{u+1}(A,A,N) \to B_1(A,A,M)$ to be the A-linear homomorphism of internal degree v that agrees with $S_0 g_0 \partial_{u+1}$ when restricted along $\eta \otimes 1\colon k \otimes A^{\otimes(u+1)} \otimes N \to B_{u+1}(A,A,N)$. It is given by

$$g_1(a_0[a_1|\dots|a_{u+1}]n) = a_0[a_1]g(1[a_2|\dots|a_{u+1}]n)\,,$$

since the remaining summands from ∂_{u+1} are mapped to terms of the form $a_0[1]m = 0$ in $B_1(A,A,M)$. Then $\partial_1 g_1 = \partial_1 S_0 g_0 \partial_{u+1}$ on $k \otimes A^{\otimes(u+1)} \otimes N$, which by the relation $\partial_1 S_0 + \eta\epsilon = 1$ differs from $g_0 \partial_{u+1}$ by $\eta\epsilon g_0 \partial_{u+1} = \eta g \partial_{u+1} = 0$, where we use that g_0 lifts g and g is a cocycle. Hence $\partial_1 g_1 = g_0 \partial_{u+1}$ on $k \otimes A^{\otimes(u+1)} \otimes N$. Since both sides are A-linear, it follows that $\partial_1 g_1 = g_0 \partial_{u+1}$ on all of $B_{u+1}(A,A,N)$.

For $q \ge 2$, suppose inductively we have defined g_* as a chain map, of internal degree v, up to and including $g_{q-1}\colon B_{u+q-1}(A,A,N) \to B_{q-1}(A,A,M)$. In particular, we are assuming that $\partial_{q-1} g_{q-1} = g_{q-2} \partial_{u+q-1}$. Define $g_q\colon B_{q+u}(A,A,N) \to B_q(A,A,M)$ to be the A-linear homomorphism that agrees with $S_{q-1}g_{q-1}\partial_{q+u}$ when restricted over $\eta\otimes 1\colon k\otimes A^{\otimes(q+u)}\otimes N \to B_{q+u}(A,A,N)$. Here $S_{q-1}\colon B_{q-1}(A,A,N) \to B_q(A,A,N)$ is part of the chain contraction of $B_*(A,A,N)$, and is not A-linear. By induction it follows that

$$g_q(a_0[a_1|\dots|a_{q+u}]n) = a_0[a_1|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n)\,,$$

since

$$\begin{aligned}
g_q(1[a_1|\dots|a_{q+u}]n) &= S_{q-1}g_{q-1}\partial_{q+u}(1[a_1|\dots|a_{q+u}]n)\\
&= S_{q-1}g_{q-1}(a_1[a_2|\dots|a_{q+u}]n - 1[a_1a_2|\dots|a_{q+u}]n + \dots)\\
&= S_{q-1}(a_1[a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) - 1[a_1a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) + \dots)\\
&= 1[a_1|a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) - 1[1|a_1a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) + \dots
\end{aligned}$$

where the second and the remaining terms are zero in the normalized bar resolution. To check that g_* is a chain map, we must prove that $\partial_q g_q = g_{q-1}\partial_{q+u}\colon B_{q+u}(A,A,N) \to B_{q-1}(A,A,M)$. [[This should involve a sign $(-1)^u$.]] Both sides are A-linear, so it suffices to prove this after restriction to $k\otimes A^{\otimes(q+u)}\otimes N$. Here $\partial_q g_q = \partial_q S_{q-1}g_{q-1}\partial_{q+u}$ differs from $g_{q-1}\partial_{q+u}$ by $S_{q-2}\partial q - 1 g_{q-1}\partial_{q+u}$, in view of the relation $\partial_q S_{q-1} + S_{q-2}\partial_{q-1} = 1$. This difference equals $S_{q-2}g_{q+u-1}\partial_{q+u-1}\partial_{q+u} = 0$, by the inductive hypothesis and the fact that $B_*(A,A,N)$ is a chain complex. □

[[When A is a Hopf algebra, and $M = k$, so that

$$f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n)) = f(a_0[a_1|\dots|a_s]1)\cdot g(1[a_{s+1}|\dots|a_{s+u}]n)\,,$$

the Yoneda product is induced by the diagonal approximation $B_*(A,A,N) \to B_*(A,A,k)\otimes B_*(A,A,N)$ mapping $a_0[a_1|\dots|a_q]n$ to the sum over $s+u=q$ of $a_0[a_1|\dots|a_s]1\otimes 1[a_{s+1}|\dots|a_{s+q}]n$. Dualize to a concatenation pairing of cobar complexes. What about the case when $M \ne k$?]]

We can also write down explicit diagonal approximations to calculate the tensor product pairings. Compare Adams (1960, p. 35).

Let M, N, T and V be left A-modules, still for a k-algebra A that is projective as a k-module. Let $\epsilon\colon B_*(A,A,M) \to M$ and $\epsilon\colon B_*(A,A,N) \to N$ be the normalized bar resolutions. These are projective A-module resolutions. The tensor product $\epsilon\otimes\epsilon\colon B_*(A,A,M)\otimes B_*(A,A,N) \to M\otimes N$ is then a projective $A\otimes A$-module resolution, hence is chain homotopy equivalent to the normalized bar resolution $\epsilon\colon B_*(A\otimes A, A\otimes A, M\otimes N) \to M\otimes N$.

An explicit chain equivalence

$$AW\colon B_*(A\otimes A, A\otimes A, M\otimes N) \longrightarrow B_*(A,A,M)\otimes B_*(A,A,N)$$

is given by the Alexander–Whitney map

$$AW(a_0\otimes b_0[a_1\otimes b_1|\dots|a_q\otimes b_q]m\otimes n) = \sum_{i=0}^{q} a_0[a_1|\dots|a_i]a_{i+1}\dots a_q m\otimes b_0b_1\dots b_i[b_{i+1}|\dots|b_q]n\,.$$

See Mac Lane (1964, Cor. X.7.2). Now suppose that A is a Hopf algebra, with coproduct $\psi\colon A \to A\otimes A$. Viewing $M\otimes N$ as an A-module by restricting the $A\otimes A$-module stucture along the algebra homomorphism ψ, we get a chain equivalence

$$\Psi = B(\psi,\psi,1)\colon B(A,A,M\otimes N) \longrightarrow B(A\otimes A, A\otimes A, M\otimes N)$$

of A-module resolutions of $M\otimes N$. Note that both of these are projective A-module resolutions, by our assuptions on A and the untwisting isomorphism from Proposition 13.18. The composite $\Delta = AW\circ\Psi$ is a chain equivalence

$$\Delta\colon B_*(A,A,M\otimes N) \longrightarrow B_*(A,A,M)\otimes B_*(A,A,N)$$

of A-modules, with the diagonal action on the right hand side, given by

$$\Delta(a_0[a_1|\dots|a_q]m\otimes n) = \sum_{i=0}^{q} a_0'[a_1'|\dots|a_i']a_{i+1}'\dots a_q' m\otimes a_0''a_1''\dots a_i''[a_{i+1}''|\dots|a_q'']n\,,$$

where $\psi(a_i) = \sum a_i'\otimes a_i''$ for all $0\le i\le q$.

[[TODO: State the result above as a lemma.]]

As a special case, if $M = k$, and we arrange that $a_i' \in I(A)$ for all summands $a_i'\otimes a_i''$ of $\psi(a_i)$, except for a term $1\otimes a_i$, then

$$\Delta(a_0[a_1|\dots|a_q]n) = \sum_{i=0}^{q} a_0'[a_1'|\dots|a_i']1\otimes a_0''a_1''\dots a_i''[a_{i+1}|\dots|a_q]n\,.$$

This recovers Adams' formula. [[Explain how to construct this directly?]]

The tensor product pairing

$$\otimes\colon \operatorname{Ext}_A^{s,t}(M,T)\otimes \operatorname{Ext}_A^{u,v}(N,V) \longrightarrow \operatorname{Ext}_A^{s+u,t+v}(M\otimes N, T\otimes V)$$

takes $[f]\otimes[g]$ to $[(f\otimes g)\Delta]$, where $f\colon B_s(A,A,M)\to\Sigma^t T$ and $g\colon B_u(A,A,N)\to\Sigma^v V$ are cocycles, so that $f\partial_{s+1}=0$ and $g\partial_{u+1}=0$. When viewed as chain maps $f\colon B_*(A,A,M)\to\Sigma^t T[s]$ and $g\colon B_*(A,A,N)\to \Sigma^u V[u]$, mapping to 0 in degrees other than s and u, respectively, their tensor product is a chain map

$$f\otimes g\colon B_*(A,A,M)\otimes B_*(A,A,N)\longrightarrow \Sigma^t T[s]\otimes\Sigma^u V[u]\cong \Sigma^{t+v}T\otimes V[s+u]\,.$$

The composite $(f\otimes g)\Delta$ is then the chain map determined by the cocycle

$$B_{s+u}(A,A,M\otimes N)\longrightarrow \Sigma^{t+v}T\otimes V$$

given by

$$a_0[a_1|\dots|a_{s+u}]m\otimes n\longmapsto \sum f(a_0'[a_1'|\dots|a_s']a_{s+1}'\dots a_{s+u}'m)\otimes g(a_0''a_1''\dots a_s''[a_{s+1}''|\dots|a_{s+u}'']n)\,.$$

Proposition 13.26. *Let A be a Hopf algebra, projective as a k-module, and let L and N be A-modules. The Yoneda composition pairing*

$$\circ\colon \operatorname{Ext}_A^{s,t}(k,L)\otimes \operatorname{Ext}_A^{u,v}(N,k)\longrightarrow \operatorname{Ext}_A^{s+u,t+v}(N,L)$$

agrees with the tensor product pairing

$$\otimes\colon \operatorname{Ext}_A^{s,t}(k,L)\otimes \operatorname{Ext}_A^{u,v}(N,k)\longrightarrow \operatorname{Ext}_A^{s+u,t+v}(k\otimes N,L\otimes k)\,.$$

Proof. Let $f\colon B_s(A,A,k)\to\Sigma^t L$ and $g\colon B_u(A,A,N)\to\Sigma^v k$ be A-linear cocycles. The Yoneda composite $[f]\circ[g]$ is represented by the cocycle $\Sigma^v f\circ g_s\colon B_{s+u}(A,A,N)\to\Sigma^{t+v}L$ given by

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n))\,.$$

The tensor product $[f]\otimes[g]$ is represented by the cocycle

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto \sum f(a_0'[a_1'|\dots|a_s']\epsilon(a_{s+1}'\dots a_{s+u}'))\cdot g(a_0''a_1''\dots a_s''[a_{s+1}''|\dots|a_{s+u}'']n)$$

where $\psi(a_i)=\sum a_i'\otimes a_i''$. The assumption $M=k$ implies that $g(a_0''a_1''\dots a_s''[\dots]n)=0$ if some $a_i''\in I(A)$, and $\epsilon(a_{s+1}'\dots a_{s+u}')=0$ if some $a_i'\in I(A)$, so only the summands $a_i\otimes 1$ of $\psi(a_i)$ contribute for $0\le i\le s$, and only the summands $1\otimes a_i$ contribute for $s+1\le i\le s+u$. Hence the sum simplifies to the single term

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto f(a_0[a_1|\dots|a_s]1)\cdot g(1[a_{s+1}|\dots|a_{s+u}]n)\,.$$

Since f is k-linear, this equals the cocycle $\Sigma^v f\circ g_s$. □

Theorem 13.27. *There is a natural pairing*

$$E_r^{s,t}(S,Z)\otimes E_r^{u,v}(X,S)\longrightarrow E_r^{s+u,t+v}(X,Z)$$

of Adams spectral sequences, given at the E_2-term by the opposite Yoneda product

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Z),\mathbb{F}_p)\otimes \operatorname{Ext}_{\mathscr{A}}^{u,v}(\mathbb{F}_p,H^*(X))\longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Z),H^*(X))$$

and converging to the composition pairing

$$\pi_{t-s}(Z_p^\wedge)\otimes[X,S_p^\wedge]_{v-u}\longrightarrow [X,Z_p^\wedge]_{t-s+v-u}\,.$$

Proof. This is the same as the smash and tensor product pairing of the Adams spectral sequences, since the tensor product of Ext-groups agrees with the opposite Yoneda product, and the smash product of homotopy classes agrees with the composition product. □

13.8. **The composition pairing, revisited.** Here is a geometric proof of Moss' theorem on the composition pairing, close to the one for the smash product pairing.

Proof. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$, respectively. Let $P_s = H_*(\Sigma^s K^s)$ and $Q_u = H_*(\Sigma^u L^u)$, as usual.

Consider the homotopy limit of mapping spectra

$$M^u = \operatorname*{holim}_{n \le u+s} F(Y^s, Z^n)\,.$$

Restriction from $n \le u+s+1$ to $n \le u+s$ gives a map $i\colon M^{u+1} \to M^u$. Its homotopy fiber is the product over s of the iterated homotopy fiber in the square

$$\begin{tikzcd}
F(Y^s, Z^{u+s+1}) \arrow[r] \arrow[d] & F(Y^s, Z^{u+s}) \arrow[d] \\
F(Y^{s+1}, Z^{u+s+1}) \arrow[r] & F(Y^{s+1}, Z^{u+s})\,,
\end{tikzcd}$$

which is equivalent to $F(K^s, L^{u+s})$. Hence we get a tower

$$\begin{tikzcd}
\dots \arrow[r] & M^{u+1} \arrow[r] & M^u \arrow[r] \arrow[d] & \dots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[d] \\
& & \prod_s F(K^s, L^{u+s}) \arrow[ul, dashed] & & & \prod_s F(K^s, L^s)\,. \arrow[ul, dashed]
\end{tikzcd}$$

Restriction to $(s,n) = (0,u)$ defines a map to the tower

$$\begin{tikzcd}
\dots \arrow[r] & F(Y, Z^{u+1}) \arrow[r] & F(Y, Z^u) \arrow[r] \arrow[d] & \dots \arrow[r] & F(Y, Z^1) \arrow[r] & F(Y, Z) \arrow[d] \\
& & F(Y, L^u) \arrow[ul, dashed] & & & F(Y, L^0)\,. \arrow[ul, dashed]
\end{tikzcd}$$

Applying homotopy we get a map of unrolled exact couples, from

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(M^{u+1}) \arrow[r] & \pi_*(M^u) \arrow[r] \arrow[d] & \dots \arrow[r] & \pi_*(M^1) \arrow[r] & \pi_*(M^0) \arrow[d] \\
& & \prod_s [K^s, L^{u+s}]_* \arrow[ul, dashed] & & & \prod_s [K^s, L^s]_* \arrow[ul, dashed]
\end{tikzcd}$$

to the one generating the Adams spectral sequence $\{E_r^{*,*}(Y,Z)\}_r$. Let $\{'E_r^{u,*}\}_r$ be the spectral sequence generated by the unrolled exact couple just displayed. The map $'E_1^{u,*} \to E_1^{u,*}(Y,Z)$ of E_1-terms can be identified, using the d-invariant isomorphisms

$$\prod_s [K^s, L^{u+s}]_* \cong \prod_s \operatorname{Hom}^*_{\mathscr{A}}(Q_{u+s}, P_s) = \operatorname{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*)$$
$$[Y, L^u]_* \cong \operatorname{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y))\,,$$

with the quasi-isomorphism

$$\epsilon_*\colon\ \operatorname{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \longrightarrow \operatorname{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y))$$

induced by $\epsilon\colon P_* \to H^*(Y)$. Hence the map of E_2-terms is an isomorphism, identifying $'E_2^{u,*}$ with the Adams E_2-term

$$E_2^{u,*}(Y,Z) = \operatorname{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), H^*(X))\,.$$

We shall define a pairing of spectral sequences

$$\phi_r\colon {}'E_r^{u,*} \otimes E_r^{s,*}(X,Y) \longrightarrow E_r^{u+s,*}(X,Z)$$

for $r \ge 1$, which agrees with the composition pairing

$$\operatorname{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \otimes \operatorname{Hom}_{\mathscr{A}}(P_s, H^*(X)) \to \operatorname{Hom}_{\mathscr{A}}(Q_{u+s}, H^*(X))$$

for $r = 1$. For $r \ge 2$ the source is isomorphic to

$$E_r^{u,*}(Y,Z) \otimes E_r^{s,*}(X,Y)$$

via $\epsilon_* \otimes 1$, which yields Moss' pairing and the compatibility with the Yoneda product for $r = 2$.

The pairing $\phi_1 \colon {}'E_1^{u,*} \otimes E_1^{s,*}(X,Y) \to E_1^{u+s,*}(X,Z)$ is the composition pairing

$$\prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \longrightarrow [X, L^{u+s}]_*$$

that takes $(g^s)_s \otimes f$ to $g^s f$. We show that it restricts to a pairing $\tilde{\phi}_r \colon {}'Z_r^{u,*} \otimes Z_r^{s,*}(X,Y) \to Z_r^{u+s,*}(X,Z)$ of r-th cycles, that descends to a pairing $\phi_r \colon {}'E_r^{u,*} \otimes E_r^{s,*}(X,Y) \to E_r^{u+s,*}(X,Z)$ satisfying the Leibniz rule, for each $r \ge 1$.

((EDIT FROM HERE))

We shall use the identifications

$$\begin{aligned} {}'Z_r^{u,*} &= \operatorname{im}(\pi_*(M^u/M^{u+r}) \to \pi_*(M^u/M^{u+1})) \\ Z_r^{s,*}(X,Y) &= \operatorname{im}([X, Y^s/Y^{s+r}]_* \to [X, K^s]_*) \\ Z_r^{s,*}(X,Z) &= \operatorname{im}([X, Z^{u+s}/Z^{u+s+r}]_* \to [X, L^{u+s}]_*) \end{aligned}$$

where $M^u/M^{u+1} = \prod_s F(K^s, L^{u+s})$.

Consider the commutative square

$$\begin{tikzcd} F(Y^s, Z^{u+s+r}) \arrow[r] \arrow[d] & F(Y^s, Z^{u+s}) \arrow[d] \\ F(Y^{s+r}, Z^{u+s+r}) \arrow[r] & F(Y^{s+r}, Z^{u+s}) . \end{tikzcd}$$

There are restriction maps from M^{u+r} to the upper left hand corner, and from M^u to the homotopy pullback of the rest of the square. Hence there is a map of homotopy fibers from $\Sigma^{-1}(M^u/M^{u+1})$ to $F(Y^s/Y^{s+r}, \Sigma^{-1}(Z^{u+s}/Z^{u+s+r}))$, giving a map

$$M^u/M^{u+r} \longrightarrow F(Y^s/Y^{s+r}, Z^{u+s}/Z^{u+s+r})$$

and an adjoint pairing

$$M^u/M^{u+r} \wedge Y^s/Y^{s+r} \longrightarrow Z^{u+s}/Z^{u+s+r}$$

compatible with the pairing $M^u/M^{u+1} \wedge K^s \to L^{u+s}$ for $r = 1$. This leads to the commutative diagram

$$\begin{tikzcd} \pi_*(M^u/M^{u+r}) \otimes [X, Y^s/Y^{s+r}]_* \arrow[r] \arrow[d] & {[X, Z^{u+s}/Z^{u+s+r}]_*} \arrow[d] \\ \prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \arrow[r, "\phi_1"] & {[X, L^{u+s}]_*} . \end{tikzcd}$$

The induced pairing of vertical images is ϕ_r.

((EDIT TO HERE)) □

14. The stable stems

14.1. The Adams E_2-term. We analyze the 2-primary Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)^\wedge_2 .$$

The E_2-term for $t - s \le 48$, as calculated using Bruner's `ext` package [Bru93], is displayed in Figure 25. The E_2-term for $48 \le t - s \le 72$ is displayed in Figure 26.

The algebra generators for $t - s \le 60$, with respect to the Yoneda composition in $\operatorname{Ext}_{\mathscr{A}}$, are listed in Figures 27 and 28. In many cases, the algebra generator is the unique nonzero class in its bidegree. Using the Steenrod operations in $\operatorname{Ext}_{\mathscr{A}}$, with the indexing convention that $Sq^i \colon \operatorname{Ext}_{\mathscr{A}}^{s,t} \to \operatorname{Ext}_{\mathscr{A}}^{s+i,2t}$, we follow [BMMS86, Def. VI.1.8] and let $a_{i+1} = Sq^0(a_i)$ for $a \in \{c, d, e, f, g, r, m, t, x, y\}$. Hence, once a_0 has been specified (or g_1 in the case $a = g$), the remaining a_i are also uniquely determined.

The remaining ambiguities in this range are:

1. f_0 in bidegree $(t-s, s) = (18, 4)$ is only determined modulo $h_1^3 h_4 = h_0^2 h_2 h_4$. A specific choice can be made by setting $f_0 = Sq^2(c_0)$, as in [BMMS86, VI.1, p. 178]. ((Is f_0 represented by 4_6 or $4_6 + 4_7$ in the resolution calculated by `ext`?))
2. e_1 in bidegree $(t-s, s) = (38, 4)$ is determined modulo $h_0^2 h_3 h_5$. ((Is e_1 represented by 4_{16}, satisfying $h_0 e_1 = 0$, like the choice in [Tan70], or is it $4_{16} + 4_{17}$?))
3. f_1 in bidegree $(t-s, s) = (40, 4)$ is determined modulo $h_2^3 h_5 = h_1^2 h_3 h_5$. ((Is it represented by 4_{19} or by $4_{19} + 4_{20}$?))
4. n in bidegree $(t-s, s) = (31, 5)$ is determined by the conditions that $h_0 n = 0$ and $h_2 n \neq 0$.

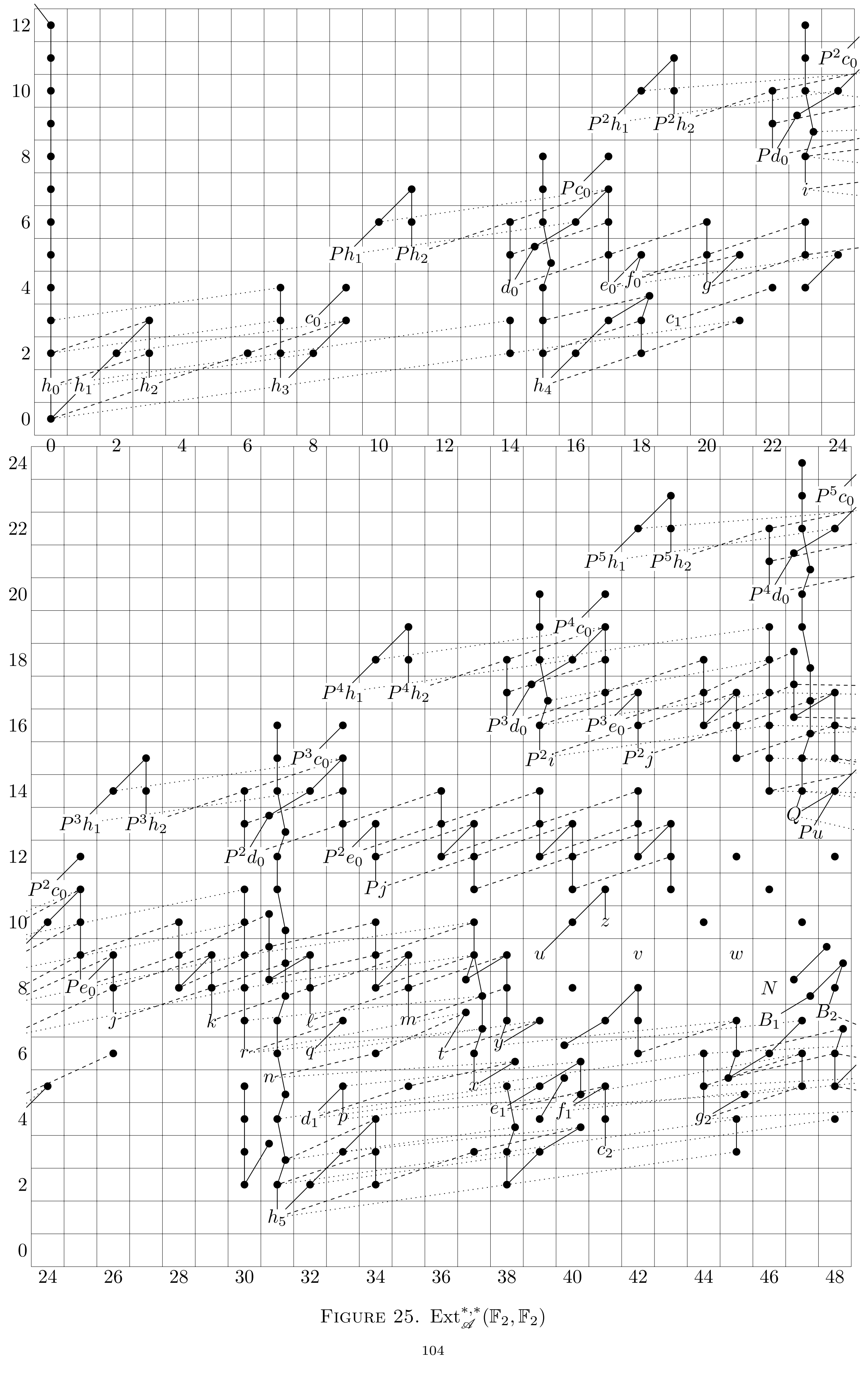


FIGURE 25. $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

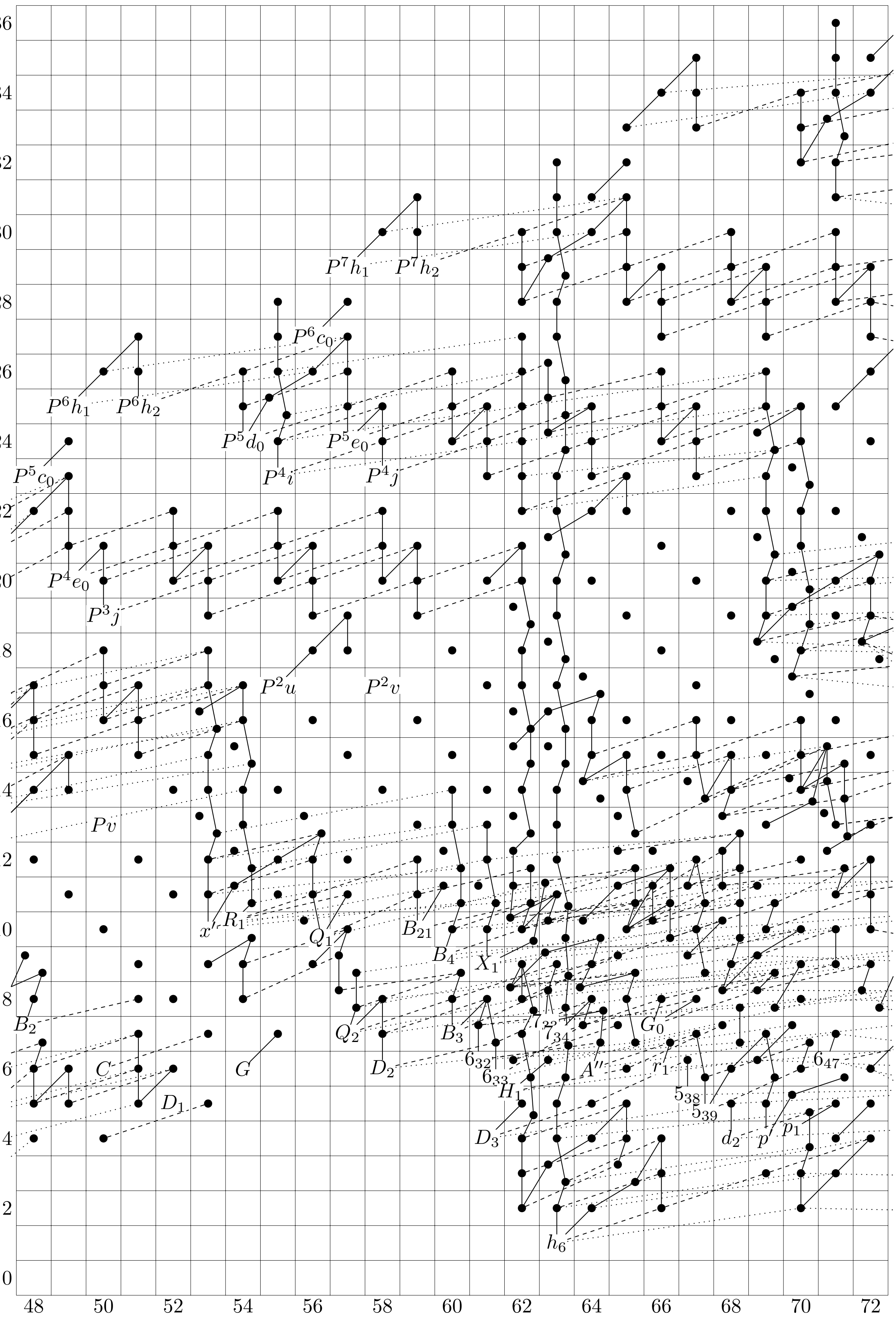


FIGURE 26. $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

$t-s$	s	t	[Tan70]	`ext`	d_2
0	1	1	h_0	1_0	0
1	1	2	h_1	1_1	0
3	1	4	h_2	1_2	0
7	1	8	h_3	1_3	0
15	1	16	h_4	1_4	$h_0h_3^2$
31	1	32	h_5	1_5	$h_0h_4^2$
8	3	11	c_0	3_3	0
19	3	22	c_1	3_9	0
41	3	44	c_2	3_{19}	h_0f_1
14	4	18	d_0	4_3	0
17	4	21	e_0	4_5	$h_1^2d_0$
18	4	22	f_0	$4_6 \mod 4_7$	$h_0^2e_0$
20	4	24	$g = g_1$	4_8	0
32	4	36	d_1	4_{13}	0
33	4	37	$p = p_0$	4_{14}	0
38	4	42	e_1	$4_{16} \mod 4_{17}$	0
40	4	44	f_1	$4_{19} \mod 4_{20}$	0
44	4	48	g_2	4_{22}	0
9	5	14	Ph_1	5_1	0
11	5	16	Ph_2	5_2	0
31	5	36	$n = n_0$	$5_{13} \mod 5_{14}$	0
37	5	42	$x = x_0$	5_{17}	0
52	5	57	D_1	5_{30}	
30	6	36	r	6_{10}	0
32	6	38	q	6_{12}	0
36	6	42	t	6_{14}	0
38	6	44	y	$6_{16} \mod 6_{17}$	h_0^3x
50	6	56	C	6_{27}	0
54	6	60	G	6_{30}	0
58	6	64	D_2	6_{31}	
16	7	23	Pc_0	7_3	0
23	7	30	i	7_5	h_0Pd_0
26	7	33	j	7_6	h_0Pe_0
29	7	36	k	7_7	$h_0d_0^2$
32	7	39	ℓ	7_{10}	$h_0d_0e_0$
35	7	42	m	7_{12}	h_0d_0g
46	7	53	B_1	7_{20}	(0)
48	7	55	B_2	$7_{22} \mod 7_{23}$	0
57	7	64	Q_2	7_{27}	
60	7	67	B_3	7_{29}	0
22	8	30	Pd_0	8_3	0
25	8	33	Pe_0	8_5	$h_1^2Pd_0$
46	8	54	N	8_{20}	0
17	9	26	P^2h_1	9_1	0
19	9	28	P^2h_2	9_2	0
39	9	48	u	9_{18}	0
42	9	51	v	9_{19}	(h_0z)
45	9	54	w	9_{20}	0
60	9	69	B_4	$9_{29} \mod 9_{30}$	
61	9	70	X_1	9_{31}	$h_0^2B_4 \mod h_1B_{21}$

FIGURE 27. Algebra generators for $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

$t-s$	s	t	[Tan70]	ext	d_2
41	10	51	z	10_{14}	0
53	10	53	x'	10_{18}	0
54	10	64	R_1	10_{19} mod 10_{20}	$h_0^2 x'$
56	10	66	Q_1	10_{22} mod 10_{21}	$h_1^2 x'$
59	10	69	B_{21}	10_{24}	0
24	11	35	$P^2 c_0$	11_3	0
34	11	45	Pj	11_7	$h_0 P^2 e_0$
30	12	42	$P^2 d_0$	12_3	0
33	12	45	$P^2 e_0$	12_5	$h_1^2 P^2 d_0$
25	13	38	$P^3 h_1$	13_1	0
27	13	40	$P^3 h_2$	13_2	0
47	13	60	Q	13_{14}	$(h_0 i^2)$
47	13	60	Pu	13_{15}	0
50	13	63	Pv	13_{16}	
32	15	47	$P^3 c_0$	15_3	0
39	15	54	$P^2 i$	15_5	$h_0 P^3 d_0$
42	15	57	$P^2 j$	15_6	$h_0 P^3 e_0$
38	16	54	$P^3 d_0$	16_3	0
41	16	57	$P^3 e_0$	16_5	$h_1^2 P^3 d_0$
33	17	50	$P^4 h_1$	17_1	0
35	17	52	$P^4 h_2$	17_2	0
55	17	72	$P^2 u$	17_{18}	0
58	17	75	$P^2 v$	17_{19}	
40	19	59	$P^4 c_0$	19_3	0
50	19	69	$P^3 j$	19_7	$h_0 P^4 e_0$
46	20	66	$P^4 d_0$	20_3	0
49	20	69	$P^4 e_0$	20_5	$h_1^2 P^4 d_0$
41	21	62	$P^5 h_1$	21_1	0
43	21	64	$P^5 h_2$	21_2	0
48	23	71	$P^5 c_0$	23_3	0
55	23	78	$P^4 i$	23_5	$h_0 P^5 d_0$
58	23	81	$P^4 j$	23_6	$h_0 P^5 e_0$
54	24	78	$P^5 d_0$	24_3	0
57	24	71	$P^5 e_0$	24_5	$h_1^2 P^5 d_0$
49	25	74	$P^6 h_1$	25_1	0
51	25	76	$P^6 h_2$	25_2	0
56	27	83	$P^6 c_0$	27_3	0
57	29	86	$P^7 h_1$	29_1	0
59	29	88	$P^7 h_2$	29_2	0

FIGURE 28. Algebra generators for $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

(5) $y = y_0$ in bidegree $(t-s, s) = (38, 6)$ is only determined modulo $h_1 x$. A specific choice can be made by setting $y = Sq^2(f_0)$, as in [BMMS86, VI.1, p. 178]. ((Is y represented by 6_{16} or by $6_{16} + 6_{17}$?))
(6) B_2 in bidegree $(t-s, s) = (48, 7)$ is only determined modulo $h_0^2 h_5 e_0$. ((Is B_2 represented by 7_{22} or by $7_{22} + 7_{23}$? Can we make a specific choice using Steenrod operations?))
(7) B_4 in bidegree $(t-s, s) = (60, 9)$ is determined modulo $h_0^2 B_3$. ((Is B_4 represented by 9_{29} or by $9_{29} + 9_{30}$?))
(8) R_1 in bidegree $(t-s, s) = (54, 10)$ is determined modulo $h_0^2 h_5 i$. ((Is R_1 represented by 10_{19} or by $10_{19} + 10_{20}$?))
(9) Q_1 in bidegree $(t-s, s) = (56, 10)$ is determined modulo tg. ((Is Q_1 represented by 10_{22} or by $10_{21} + 10_{22}$? Ravenel (1986/2004) refers to a generator R' in this bidegree.))

(10) Q in bidegree $(t-s,s)=(47,13)$ is characterized by $h_0Q\neq 0$ and $h_1Q\neq 0$. It is represented by 13_{14}. Pu in the same bidegree, $(t-s,s)=(47,13)$, is characterized by $h_0Pu=0$ and $h_1Pu\neq 0$. It is represented by 13_{15}. The sum $Q'=Q+Pu$ is characterized by $h_0Q'\neq 0$ and $h_1Q'=0$. It is represented by $13_{14}+13_{15}$. ((Check that this Pu is the Adams periodicity operator P applied to u.))

14.2. **The first 14 stems.**

Proposition 14.1. *$d_r^{s,t}=0$ for all $r\geq 2$ and $t-s\leq 14$.*

Proof. This is clear from the multiplicative structure. Each $d_r(h_0)$ lands in a trivial group. If h_1 survives to E_r then $h_0d_r(h_1)=d_r(h_0h_1)=0$ since $h_0h_1=0$, but there is no h_0-torsion with $t-s=0$, so $d_r(h_1)=0$. Hence each $d_r(h_1)=0$, by induction on $r\geq 2$. For each $r\geq 2$, the differentials $d_r(h_2)$, $d_r(h_3)$, $d_r(c_0)$, $d_r(Ph_1)$, $d_r(Ph_2)$, $d_r(h_3^2)$ and $d_3(d_0)$ land in trivial groups. In other words, all classes with $t-s\leq 14$ are permanent cycles. (This does not exclude the possibility that some classes with $t-s$ are boundaries, hence represent 0 at E_∞.) □

Theorem 14.2. (0) *$\pi_0(S)^\wedge_2=\mathbb{Z}_2\{\iota\}$, with $2^s\iota$ represented by h_0^s, for each $s\geq 0$.*
(1) *$\pi_1(S)^\wedge_2=\mathbb{Z}/2\{\eta\}$, with η represented by h_1.*
(2) *$\pi_2(S)^\wedge_2=\mathbb{Z}/2\{\eta^2\}$, with η^2 represented by h_1^2.*
(3) *$\pi_3(S)^\wedge_2=\mathbb{Z}/8\{\nu\}$, with ν represented by h_2. Here 2ν is represented by h_0h_2, and $4\nu=\eta^3$ is represented by $h_0^2h_2=h_1^3$.*
(4) *$\pi_4(S)^\wedge_2=0$.*
(5) *$\pi_5(S)^\wedge_2=0$.*
(6) *$\pi_6(S)^\wedge_2=\mathbb{Z}/2\{\nu^2\}$, with ν^2 represented by h_2^2.*
(7) *$\pi_7(S)^\wedge_2=\mathbb{Z}/16\{\sigma\}$, with σ represented by h_3. Here 2σ is represented by h_0h_2, 4σ is represented by $h_0^2h_2$ and 8σ is represented by $h_0^3h_2$.*
(8) *$\pi_8(S)^\wedge_2=\mathbb{Z}/2\{\epsilon,\eta\sigma\}$, with ϵ represented by c_0 and $\eta\sigma$ represented by h_1h_3.*
(9) *$\pi_9(S)^\wedge_2=\mathbb{Z}/2\{\mu,\eta\epsilon,\eta^2\sigma\}$, with μ represented by Ph_1, $\eta\epsilon$ represented by h_1c_0 and $\eta^2\sigma$ represented by $h_1^2h_3$. ((Claim: $\nu^3=\eta\epsilon+\eta^2\sigma$.))*
(10) *$\pi_{10}(S)^\wedge_2=\mathbb{Z}/2\{\eta\mu\}$, with $\eta\mu$ represented by h_1Ph_1. ((Claim: $\eta^2\epsilon=0$ and $\nu\sigma=0$.))*
(11) *$\pi_{11}(S)^\wedge_2=\mathbb{Z}/8\{\zeta\}$, with ζ represented by Ph_2. Here 2ζ is represented by h_0Ph_2, and $4\zeta=\eta^2\mu$ is represented by $h_0^2Ph_2=h_1^2Ph_1$. ((Claim: $\nu\epsilon=0$.))*
(12) *$\pi_{12}(S)^\wedge_2=0$.*
(13) *$\pi_{13}(S)^\wedge_2=0$.*

Proof. In degree 8, the Adams filtration gives the short exact sequence

$$0\to\mathbb{F}_2\{c_0\}\to\pi_8(S)^\wedge_2\to\mathbb{F}_2\{h_1h_3\}\to 0\,.$$

The class ϵ is represented by c_0, and the product $\eta\sigma$ is represented by h_1h_3. The extension is split, because $2\cdot\eta\sigma=0$ since $2\eta=0$.

In degree 9, there is a unique class μ represented by Ph_1. The product $\eta\epsilon$ is represented by h_1c_0 and the product $\eta^2\sigma$ is represented by $h_1^2h_3$. Hence $\pi_9(S)^\wedge_2$ is generated by μ, $\eta\epsilon$ and $\eta^2\sigma$. Here $2\cdot\mu$ is represented by $h_0Ph_1=0$ and lies in Adams filtration greater than 6, hence is 0. Furthermore $2\cdot\eta\epsilon=0$ and $2\cdot\eta^2\sigma=0$ since $2\eta=0$. □

14.3. **Higher homotopy commutativity.** The first differential can be explained using the homotopy commutativity of the pairing $\phi\colon S\wedge S\cong S$.

Proposition 14.3. *$d_2(h_4)=h_0h_3^2$.*

Proof. We know that 2ι is represented by h_0 and σ is represented by h_2, so $2\sigma^2$ is represented by $h_0h_3^2$ at E_∞. Since $2\sigma^2=0$ by graded commutativity, it follows that $h_0h_3^2=0$ at E_∞, i.e., $h_0h_3^2$ is a boundary. The only possibility is $d_2(h_4)=h_0h_3^2$. □

The next d_2-differentials can be explained using the higher order homotopy commutativity of S. This structure is derived from the fact that S is an E_∞ ring spectrum. The *quadratic construction* on a based space X is

$$D_2(X)=S^\infty_+\wedge_{C_2}(X\wedge X)=X^{\wedge 2}_{hC_2}$$

where C_2 acts freely on S^∞ by the antipodal action, $s\mapsto -s$, and acts on $X\wedge X$ by the twist action, $x\wedge y\mapsto y\wedge x$. It is filtered by the subspaces

$$D_2^k(X)=S^k_+\wedge_{C_2}(X\wedge X)$$

for $0 \leq k \leq \infty$, where C_2 also has the antipodal action on $S^k \subset S^\infty$. There is an extension of this construction to the category of spectra, see [LMSM86, (?)], denoted

$$D_2^k(Y) = S^k \ltimes_{C_2} (Y \wedge Y)$$

for $0 \leq k \leq \infty$, where we write $D_2(Y)$ for $D_2^\infty(Y)$. This twisted half-smash product, or extended power, is compatible with the space-level construction, in the sense that $D_2^k(\Sigma^\infty X) = \Sigma^\infty D_2^k(X)$ for all k. It extends the smash square, in the sense that $D_2^0(Y) \cong Y \wedge Y$, and there is a filtration

$$Y \wedge Y \cong D_2^0(Y) \subset D_2^1(Y) \subset \cdots \subset D_2^k(Y) \subset \cdots \subset D_2(Y)\,.$$

Part of the data specifying an E_∞ *ring structure* on a spectrum Y is a map

$$\xi_2 \colon D_2(Y) \longrightarrow Y$$

that restricts over $Y \wedge Y \cong D_2^0(Y) \subset D_2(Y)$ to a ring spectrum structure $\phi \colon Y \wedge Y \to Y$. The sphere spectrum S is an example of an E_∞ ring spectrum, with structure map $\xi_2 \colon D_2(S) \to S$ extending the pairing $S \wedge S \cong S$ referred to above. ((Relate to strictly commutive ring spectra.))

The relation between E_∞ ring structures and the Adams spectral sequence was studied in increasing generality by Daniel S. Kahn, Jim Milgram, Jukka Mäkinen and Robert R. Bruner. Let

$$\cdots \to Y^{s+1} \xrightarrow{i} Y^s \to \cdots \to Y^0 = Y$$

be an Adams resolution of Y, with each map i the inclusion of a CW subspectrum. Let

$$\cdots \to (Y \wedge Y)^{s+1} \xrightarrow{i} (Y \wedge Y)^s \to \cdots \to (Y \wedge Y)^0 = Y \wedge Y$$

be the product resolution of $Y \wedge Y$, with

$$(Y \wedge Y)^s = \bigcup_{i+j=s} Y^i \wedge Y^j\,.$$

Note that the twisting C_2-action on $Y \wedge Y$ restricts to an action on each $(Y \wedge Y)^s$. We get a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & (Y \wedge Y)^{s+1} \arrow[r, "i"] \arrow[d] & (Y \wedge Y)^s \arrow[r] \arrow[d] & \dots \arrow[r] & Y \wedge Y \arrow[d] \\
& \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\dots \arrow[r] & S^k \ltimes_{C_2} (Y \wedge Y)^{s+1} \arrow[r, "i"] \arrow[d] & S^k \ltimes_{C_2} (Y \wedge Y)^s \arrow[r] \arrow[d] & \dots \arrow[r] & D_2^k(Y) \arrow[d] \\
& \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\dots \arrow[r] & S^\infty \ltimes_{C_2} (Y \wedge Y)^{s+1} \arrow[r, "i"] & S^\infty \ltimes_{C_2} (Y \wedge Y)^s \arrow[r] & \dots \arrow[r] & D_2(Y)
\end{tikzcd}$$

The composite map $\phi \colon Y \wedge Y \to D_2(Y) \to Y$ can be covered by a map

$$\{\xi_{0,s} \colon (Y \wedge Y)^s \to Y^s\}_s$$

of Adams resolutions, with $\phi = \xi_{0,0}$. This map can be extended to a map from the k-skeleta of the extended powers, at the expense of a loss of k Adams filtrations, for each finite $k \geq 0$. In other words, there are maps

$$\xi_{k,s} \colon S^k \ltimes_{C_2} (Y \wedge Y)^s \longrightarrow Y^{s-k}$$

for $k \geq 0$ and $s \geq 0$, making the diagrams

$$\begin{tikzcd}
 & S^k \ltimes_{C_2} (Y \wedge Y)^{s+1} \arrow[r, "i"] \arrow[ddl, "\xi_{k,s+1}"'] & S^k \ltimes_{C_2} (Y \wedge Y)^{s} \arrow[d] \arrow[ddl, "\xi_{k,s}"'] \\
 & & S^{k+1} \ltimes_{C_2} (Y \wedge Y)^{s} \arrow[d, "\xi_{k+1,s}"] \\
Y^{s-k+1} \arrow[r, "i"] & Y^{s-k} \arrow[r, "i"] & Y^{s-k-1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
S^k \ltimes_{C_2} (Y \wedge Y)^{s} \arrow[r] \arrow[d, "\xi_{k,s}"'] & D_2(Y) \arrow[d, "\xi_2"] \\
Y^{s-k} \arrow[r, "i^{s-k}"] & Y
\end{tikzcd}$$

commute. See Bruner's [BMMS86, Theorem IV.5.2]. Concentrating on Adams filtration $2s$, for $k \geq 0$, and composing with the inclusion $Y^s \wedge Y^s \to (Y \wedge Y)^{2s}$, we get a map from the filtered quadratic construction of Y^s to the Adams resolution of Y. If $\tilde{f}\colon S^n \to Y^s$ represents a class $[f] \in \pi_n(Y)$ of Adams filtration s, with $f = i^s \tilde{f}$, we can form the following commutative diagram:

$$\begin{tikzcd}
S^n \wedge S^n \arrow[r] \arrow[d, "\tilde{f} \wedge \tilde{f}"'] & \dots \arrow[r] & D_2^k(S^n) \arrow[r] \arrow[d, "D_2^k(\tilde{f})"'] & D_2^{k+1}(S^n) \arrow[r] \arrow[d, "D_2^{k+1}(\tilde{f})"'] & \dots \\
Y^s \wedge Y^s \arrow[r] \arrow[d] & \dots \arrow[r] & D_2^k(Y^s) \arrow[r] \arrow[d] & D_2^{k+1}(Y^s) \arrow[r] \arrow[d] & \dots \\
(Y \wedge Y)^{2s} \arrow[r] \arrow[d, "\xi_{0,2s}"'] & \dots \arrow[r] & S^k \ltimes_{C_2} (Y \wedge Y)^{2s} \arrow[r] \arrow[d, "\xi_{k,2s}"'] & S^{k+1} \ltimes_{C_2} (Y \wedge Y)^{2s} \arrow[r] \arrow[d, "\xi_{k+1,2s}"'] & \dots \\
Y^{2s} \arrow[r, "i"] & \dots \arrow[r, "i"] & Y^{2s-k} \arrow[r, "i"] & Y^{2s-k-1} \arrow[r, "i"] & \dots
\end{tikzcd}$$

The quadratic construction on spheres can be rewritten in terms of stunted projective spaces. For $a \leq b \leq \infty$, let $\mathbb{R}P^b_a = \mathbb{R}P^b/\mathbb{R}P^{a-1}$ be the subquotient of $\mathbb{R}P^\infty_+$ with a single d-dimensional cell for each $a \leq d \leq b$. Then

$$D_2^k(S^n) = S^k_+ \times_{C_2} (S^n \wedge S^n) \cong \frac{S^k \times_{C_2} D^n \times D^n}{S^k \times_{C_2} \partial(D^n \times D^n)}$$

is the Thom complex of the real $2n$-bundle over $S^k/C_2 = \mathbb{R}P^k$ associated to the C_2-representation given by the twist action on $\mathbb{R}^n \times \mathbb{R}^n$. This is isomorphic to the sum of the diagonal $+1$-eigenspace and the anti-diagonal -1-eigenspace, so the vector bundle is isomorphic to $n\epsilon^1 \oplus n\gamma^1$, where ϵ^1 and γ^1 denote the trivial and the tautological line bundles over $\mathbb{R}P^k$, respectively. Hence $D_2^k(S^n) = Th(n\epsilon^1 \oplus n\gamma^1) = \Sigma^n Th(n\gamma^1)$. By a calculation of Atiyah ((is that the original reference?)), $Th(n\gamma^1) = \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1} = \mathbb{R}P^{n+k}_n$, so

$$D_2^k(S^n) \cong \Sigma^n \mathbb{R}P^{n+k}_n$$

has one d-cell for each $2n \leq d \leq 2n+k$.

We get maps of cofiber sequences

$$\begin{tikzcd}
D_2^{k-1}(S^n) \arrow[r] \arrow[d] & D_2^k(S^n) \arrow[r] \arrow[d, "\xi_{k,2s} \circ D_2^k(\tilde{f})"'] & S^{2n+k} \arrow[d] \\
Y^{2s-k+1} \arrow[r, "i"] & Y^{2s-k} \arrow[r, "j"] & K^{2s-k}
\end{tikzcd}$$

for each $k \geq 0$. By [BMMS86, Corollary IV.5.4 and Theorem IV.7.6], the right hand vertical map $S^{2n+k} \to K^{2s-k}$ represents a cocycle in $\pi_{2n+k}(K^{2s-k}) = E_1^{2s-k,2n+2s}$, whose cohomology class in $E_2^{2s-k,2n+2s}$ is given by the Steenrod operation $Sq^{s-k}(x)$, where $x \in E_2^{s,n+s}$ represents $[f]$. These Steenrod operations

$$Sq^i\colon \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s+i,2t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

can be defined in the cohomology of any cocommutative Hopf algebra, see [BMMS86, Section IV.2]. ((Note that use cohomological, rather than topological, indexing of these Steenrod operations, writing Sq^i for the operation denoted Sq_i in Bruner's chapter.)) For $x \in \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ they satisfy $Sq^i(x) = 0$ for $i < 0$ and for $i > s$, while $Sq^s(x) = x^2$. They also satisfy the Cartan formula $Sq^k(xy) = \sum_{i+j=k} Sq^i(x)Sq^j(y)$.

Proposition 14.4. *$Sq^1(h_i) = h_i^2$ and $Sq^0(h_i) = h_{i+1}$, for each $i \geq 0$.*
$Sq^3(c_0) = c_0^2$, $Sq^2(c_0) = h_0e_0$, $Sq^1(c_0) = f_0$ and $Sq^0(c_0) = c_1$.

((Proof?))

Proposition 14.5. *$\eta\sigma^2 = 0$.*

Proof. The quadratic construction on $\sigma\colon S^7 \to S$ restricts to a map

$$\Sigma^7\mathbb{R}P_7^9 \cong D_2^2(S^7) \to S\,.$$

We have $\mathbb{R}P_7^9 = S^7 \cup_2 e^8 \cup_\eta e^9$, since $Sq^1(x^7) = x^8$ and $Sq^2(x^7) = x^9$ in $H^*(\mathbb{R}P^\infty) = P(x)$, hence also in $H^*(\mathbb{R}P_7^9)$. The map

$$S^{14} \cup_2 e^{15} \cup_\eta e^{16} = \Sigma^7\mathbb{R}P_7^9 \to S$$

restricts to σ^2 on the 14-cell. Hence the map from the 15-cell gives a nullhomotopy of $2\sigma^2$, and the map from the 16-cell gives a nullhomotopy of $\eta\sigma^2$. □

Proposition 14.6. *h_1h_4 is a permanent cycle.*

Proof. The quadratic construction on $2\sigma\colon S^7 \to S$ restricts to another map

$$\Sigma^7\mathbb{R}P_7^9 \cong D_2^2(S^7) \to S\,.$$

This time, the map

$$S^{14} \cup_2 e^{15} \cup_\eta e^{16} = \Sigma^7\mathbb{R}P_7^9 \to S$$

represents $Sq^2(h_0h_3) = h_0^2h_3^2 = 0$ on the 14-cell, $Sq^1(h_0h_3) = h_0^2h_4 + h_1h_3^2 = h_0^2h_4$ on the 15-cell and $Sq^0(h_0h_3) = h_1h_4$ on the 16-cell. In more detail, this means that for an Adams resolution

$$\cdots \to Y^{s+1} \xrightarrow{i} Y^s \to \cdots \to Y^0 = S$$

of S, with homotopy cofiber sequences

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}\,,$$

the map $\sigma\colon S^7 \to S$ factors as if with $f\colon S^7 \to Y^1$, the map $2\sigma\colon S^7 \to S$ factors as i^2g with $g\colon S^7 \to Y^2$, and $f \circ 2 \simeq i \circ g$:

$$\begin{tikzcd}
S^7 \arrow[r, "2"] \arrow[d, "g"'] & S^7 \arrow[d, "f"'] \arrow[dr, "\sigma"] & \\
Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & S \arrow[d, "j"] \\
K^2 & K^1 & K^0\,.
\end{tikzcd}$$

The extensions $D_2^k(S^7) \to Y^{4-k}$ give rise to the commutative diagram

$$\begin{tikzcd}
& D_2^0(S^7) \arrow[rr] \arrow[dl] \arrow[dd, "g^2"'] & & D_2^1(S^7) \arrow[rr] \arrow[dl] \arrow[dd] & & D_2^2(S^7) \arrow[dl] \arrow[dd] \\
S^{14} \arrow[dd] & & S^{15} \arrow[dd] & & S^{16} \arrow[dd] & \\
& Y^4 \arrow[rr, "i"] \arrow[dl, "j"'] & & Y^3 \arrow[rr, "i"] \arrow[dl, "j"'] & & Y^2 \arrow[dl, "j"'] \\
K^4 & & K^3 & & K^2 &
\end{tikzcd}$$

where the vertical arrows $S^{14+k} \to K^{4-k}$ represent the various cocycles $Sq^{2-k}(h_0h_3)$, namely 0, $h_0^2h_4$ and h_1h_4. Restricting to the subcomplex $S^{14} \cup_\eta e^{16} \subset D_2^2(S^7)$, we get the following map of horizontal cofiber sequences:

$$\begin{tikzcd}
S^{14} \arrow[r] \arrow[d, "i\circ g^2"'] & S^{14} \cup_\eta e^{16} \arrow[r] \arrow[d] & S^{16} \arrow[r, "\eta"] \arrow[d, "h_1h_4"'] & S^{15} \arrow[d] \\
Y^3 \arrow[r, "i"] & Y^2 \arrow[r, "j"] & K^2 \arrow[r, "\partial"] & \Sigma Y^3 .
\end{tikzcd}$$

The obstruction to lifting $h_1h_4 \colon S^{16} \to K^2$ to Y^2 is the composite $\partial \circ h_1h_4 = \Sigma(i \circ g^2) \circ \eta$. Here $i \circ g^2 \colon S^7 \wedge S^7 \to Y^3$ factors as $(i \circ g) \cdot g \colon S^7 \wedge S^7 \to Y^1 \wedge Y^2 \to Y^3$, hence is homotopic to $(f \circ 2) \cdot g = (f \cdot g) \circ 2 \colon S^7 \wedge S^7 \to S^7 \wedge S^7 \to Y^1 \wedge Y^2 \to Y^3$. Since $2 \circ \eta = 0$, the obstruction is zero. This lift to Y^2 of the map representing h_1h_4 shows that h_1h_4 is a permanent cycle. □

Proposition 14.7. *h_2h_4 is a permanent cycle.*

Proof. ((Incomplete.)) The quadratic construction on $\eta\sigma \colon S^8 \to S$ restricts to a map

$$\Sigma^8 \mathbb{R}P_8^{10} \cong D_2^2(S^8) \to S\,.$$

We have $\mathbb{R}P_8^{10} = S^8 \vee (S^9 \cup_2 e^{10})$. The map

$$S^{16} \vee S^{15} \cup_2 e^{16} = \Sigma^8 \mathbb{R}P_8^{10} \to S$$

represents $Sq^2(h_1h_3) = h_1^2h_3^2 = 0$ on the 16-cell, $Sq^1(h_1h_3) = h_1^2h_4 + h_2h_3^2 = h_1^2h_4$ on the 17-cell and $Sq^0(h_1h_3) = h_2h_4$ on the 18-cell. Hence the 18-cell is attached by $2 \cdot P_1(\eta\sigma) = 0$ ((Explain!)), and therefore represents a permanent cycle. □

Proposition 14.8. *$d_2(f_0) = h_0^2e_0$ and c_1 is a permanent cycle. ((Claim: $d_2(c_i) = h_0f_{i-1}$ for $i \geq 2$.))*

Proof. The quadratic construction on $\epsilon \colon S^8 \to S$ restricts to a map

$$\Sigma^8 \mathbb{R}P_8^{11} \cong D_2^3(S^8) \to S\,.$$

We have $\mathbb{R}P_8^{11} = S^8 \vee (S^9 \cup_2 e^{10}) \vee S^{11}$. The map

$$S^{16} \vee (S^{17} \cup_2 e^{18}) \vee S^{19} = \Sigma^8 \mathbb{R}P_8^{11} \to S$$

represents $Sq^3(c_0) = c_0^2$ on the 16-cell, $Sq^2(c_0) = h_0e_0$ on the 17-cell, $Sq^1(c_0) = f_0$ on the 18-cell and $Sq^0(c_0) = c_1$ on the 19-cell. In more detail, this means that for an Adams resolution as above, the map $\epsilon \colon S^8 \to S$ factors through $i^3 \colon Y^3 \to S$, so $\epsilon^2 \colon S^8 \wedge S^8 = D_2^0(S^8) \to S$ factors through $i^6 \colon Y^6 \to S$. There extensions $D_2^k(S^8) \to Y^{6-k}$ give rise to the commutative diagram

$$\begin{tikzcd}
 & D_2^0(S^8) \arrow[rr] \arrow[dl] \arrow[dd, "\epsilon^2"] & & D_2^1(S^8) \arrow[rr] \arrow[dl] \arrow[dd] & & D_2^2(S^8) \arrow[rr] \arrow[dl] \arrow[dd] & & D_2^3(S^8) \arrow[dl] \arrow[dd] \\
S^{16} \arrow[dd] & & S^{17} \arrow[dd] & & S^{18} \arrow[dd] & & S^{19} \arrow[dd] & \\
 & Y^6 \arrow[rr, "i"] \arrow[dl, "j"'] & & Y^5 \arrow[rr, "i"] \arrow[dl, "j"'] & & Y^4 \arrow[rr, "i"] \arrow[dl, "j"'] & & Y^3 \arrow[dl, "j"'] \\
K^6 & & K^5 & & K^4 & & K^3 &
\end{tikzcd}$$

where the vertical arrows $S^{16+k} \to K^{6-k}$ represent the various cocycles $Sq^{3-k}(c_0)$, namely c_0^2, h_0e_0, f_0 and c_1.

Restricting to the stable summand $S^{17} \cup_2 e^{18} \subset D_2^2(S^8)$, we get a commutative diagram

$$\begin{tikzcd}
S^{17} \arrow[r] \arrow[d] & S^{17} \cup_2 e^{18} \arrow[r] \arrow[d] & S^{18} \arrow[r, "2"] \arrow[d] & S^{18} \arrow[d] \\
Y^5 \arrow[r, "i"] \arrow[d] & Y^4 \arrow[r, "j"] \arrow[d] & K^4 \arrow[r, "\partial"] \arrow[d, equal] & \Sigma Y^5 \arrow[d] \\
Y^5/Y^7 \arrow[r, "\bar\imath"] \arrow[d] & Y^4/Y^7 \arrow[r, "\bar\jmath"] \arrow[d] & K^4 \arrow[r, "\bar\partial"] \arrow[d, equal] & \Sigma Y^5/Y^7 \arrow[d] \\
K^5 \arrow[r, "\bar\imath"] & Y^4/Y^6 \arrow[r, "\bar\jmath"] & K^4 \arrow[r, "d_1"] & \Sigma K^5
\end{tikzcd}$$

where the vertical composite $S^{17} \to K^5$ represents $h_0 e_0$ and the vertical composite $S^{18} \to K^4$ represents f_0. Since f_0 is a cocycle, $d_1(f_0) = 0$, so the composite $S^{18} \to K^4 \to \Sigma Y^5/Y^7$ lifts through ΣK^6, and this lift $S^{18} \to \Sigma K^6$ represents $d_2(f_0)$, by the definition of the differential d_2.

$$\begin{tikzcd}
Y^7 \arrow[r, "i"] & Y^6 \arrow[r, "i"] \arrow[d, "j"] & Y^5 \arrow[r, "i"] \arrow[d, "j"] & Y^4 \arrow[d, "j"] \\
& K^6 \arrow[ul, dashed, "\partial"] & K^5 \arrow[ul, dashed, "\partial"] & K^4 \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

The composite $S^{18} \to K^4 \to \Sigma Y^5/Y^7$ is homotopic to the smash product of $2\colon S \to S$ and $S^{18} \to \Sigma Y^5 \to \Sigma Y^5/Y^7$. The latter two maps are represented by $h_0 \in \pi_0(K^1)$ and $h_0 e_0 \in \pi_{18}(\Sigma K^5)$, so their smash product is represented by $h_0 \cdot h_0 e_0 = h_0^2 e_0 \in \pi_{18}(\Sigma K^6)$. In other words, the lift representing $d_2(f_0)$ equals a map representing $h_0^2 e_0$. Hence $d_2(f_0) = h_0^2 e_0$.

Restricting instead to the stable summand $S^{19} \subset D_2^3(S^8)$, we get a chain of maps

$$S^{19} \to Y^3 \xrightarrow{j} K^3$$

with composite representing c_1. Hence $c_1 \in \pi_{19}(K^3)$ lifts to $\pi_{19}(Y^3)$, and is a permanent cycle. □

Corollary 14.9. *$d_2(h_0 f_0) = h_0^3 e_0$ and $d_2(e_0) = h_1^2 d_0$.*

Proof. The first claim follows from $d_2(f_0) = h_0^2 e_0$ by multiplication by h_0. We have $h_1 \cdot e_0 = h_0 f_0$ and $h_1 \cdot h_1^2 d_0 = h_0^3 e_0$, so the second claim follows from $h_1 \cdot d_2(e_0) = d_2(h_1 e_0) = d_2(h_0 f_0) = h_0^3 e_0 = h_1 \cdot h_1^2 d_0$, together with the fact the multiplication by h_1 acts injectively on $E_2^{6,22}$. □

So far we have discussed consequences of higher homotopy commutativity when applied to maps $f\colon S^n \to S$ and the permanent cycles representing them. More subtle arguments, involving a "modified Adams spectral sequence", lead to consequences also for Steenrod operations on non-permanent cycles x in the Adams spectral sequence. In the case of the sphere spectrum, these results are due to Mäkinen [Mäk73]. They were extended to H_∞ ring spectra and other cohomology theories than ordinary mod p cohomology by Bruner [BMMS86, Chapter VI]. Here is a special case:

Theorem 14.10. *Suppose that $x \in E_2^{s,t}$ survives to E_r in topological degree $n = t - s$, for some $r \geq 2$. If $r = 2$ and n is even then*

$$d_3(x^2) = Sq^{s+1}(d_2(x)) + h_0 x d_2(x)\,.$$

Otherwise,

$$d_{r+1}(x^2) = \begin{cases} Sq^{s+r-1}(d_r(x)) & \text{if } n \text{ is odd, and} \\ h_0 x d_r(x) & \text{if } r \geq 3 \text{ and } n \text{ is even.} \end{cases}$$

For the general case, Bruner uses the following notation. For elements B_1 and B_2 in $E_r^{*,*}$, let $B_1 \dotplus B_2$ denote B_1, $B_1 + B_2$ or B_2 if B_1 has lower, equal or greater Adams filtration than B_2, respectively. A formula

$$d_*(A) = B_1 \dotplus B_2$$

means that A survives to the $E_{r'}$-term, where r' is the difference betwee the Adams filtration of A and the Adams filtration of $B_1 \dotplus B_2$, and that $d_{r'}(A)$ is equal to B_1, $B_1 + B_2$ or B_2, according to the definition of $B_1 \dotplus B_2$ just given.

Let the vector fields function $v(n) \geq 1$ be maximal such that the attaching map $S^{n-1} \to \mathbb{R}P^{n-1}$ of the n-cell in $\mathbb{R}P^n$ factors through $\mathbb{R}P^{n-v(n)}$. By Adams' theorem on vector fields on spheres, $v(n) = 8a + 2^b$ if the 2-adic valuation of $n+1$ is $4a+b$ with $0 \leq b \leq 3$. Let $a(n) \in \pi_{v(n)-1}(S)$ be the top component

$$S^{n-1} \to \mathbb{R}P^{n-v(n)} \to S^{n-v(n)}$$

of such a compression. For example, if n is even, $v(n) = 1$ and $a(n) = 2\iota \in \pi_0(S)$, represented by $h_0 \in E_2^{1,1}$. If $n \equiv 1 \mod 4$ then $v(n) = 2$ and $a(n) = \eta \in \pi_1(S)$, represented by $h_1 \in E_2^{1,2}$. If $n \equiv 3 \mod 8$ then $v(n) = 4$ and $a(n) = \nu \in \pi_3(S)$ ((up to an odd multiple)), represented by $h_2 \in E_2^{1,4}$. If $n \equiv 7 \mod 16$ then $v(n) = 8$ and $a(n) = \sigma \in \pi_7(S)$ ((up to an odd multiple)), represented by $h_3 \in E_2^{1,8}$. ((ETC)) ((Claim: The class $a(n)$ lies in the image of the J-homomorphism.))

Theorem 14.11 ([BMMS86, Theorems VI.1.1 and VI.1.2])**.** *Suppose that $x \in E_2^{s,t}$ survives to E_r, for some $r \geq 2$. For $0 \leq j \leq s$, let $v = v(t-j)$ and $a = a(t-j)$, and let $\bar{a} \in E_2^{f,f+v-1}$ be the permanent cycle that detects $a \in \pi_{v-1}(S)$. Then*

$$d_*(Sq^j(x)) = Sq^{j+r-1}(d_r(x)) \dotplus \begin{cases} 0 & \text{if } v > s-j+1 \\ \bar{a}x d_r(x) & \text{if } v = s-j+1 \\ \bar{a}Sq^{j+v}(x) & \text{if } v \leq \min(s-j, 10). \end{cases}$$

Theorem 14.12 (Adams, [BMMS86, VI.1.5])**.** $d_2(h_{i+1}) = h_0 h_i^2$ *for all* $i \geq 1$.

Proof. We apply Bruner's theorem to $x = h_i$ and $Sq^0(x) = h_{i+1}$, with $i \geq 1$. In this case $r = 2$, $s = 1$, $t = 2^i$, $j = 0$, $v = 1$, $\bar{a} = h_0$ and

$$d_*(h_{i+1}) = Sq^1(d_2(x)) \dotplus h_0 Sq^1(x) = h_0 Sq^1(x)$$

gives $d_2(h_{i+1}) = h_0 h_i^2$, since $Sq^1(d_2(x))$ has Adams filtration 4 and $h_0 Sq^1(x) = h_0 h_i^2$ has Adams filtration 3. □

Theorem 14.13 ([BMMS86, VI.1.16(i)])**.** $d_3(f_1) = 0$ *and* $d_2(c_2) = h_0 f_1$.

Proof. The calculations $Sq^i(c_0) = c_0^2$, $h_0 e_0$, f_0 and c_1 for $i = 3$, 2, 1 and 0, respectively, imply that $Sq^i(c_1) = c_1^2$, $h_1 e_1$, f_1 and c_2 for $i = 3$, 2, 1 and 0, respectively.

We apply Bruner's theorem to $x = c_1$ and $Sq^1(x) = f_1$. In this case $r = 2$, $s = 3$, $t = 22$, $j = 1$, $v = 2$, $\bar{a} = h_1$ and

$$d_*(f_1) = d_*(Sq^1(c_1)) = Sq^2(d_2(c_1)) \dotplus h_1 Sq^3(c_1) = 0 + h_1 c_1^2$$

gives $d_3(f_1) = h_1 c_1^2$, since $Sq^2(d_2(c_1))$ and $h_1 Sq^3(c_1) = h_1 c_1^2$ both have Adams filtration 7. Here $h_1 c_1 = 0$, so $h_1 c_1^2 = 0$.

Next we apply Bruner's theorem to $x = c_1$ and $Sq^0(x) = c_2$. In this case $r = 2$, $s = 3$, $t = 22$, $j = 0$, $v = 1$, $\bar{a} = h_0$ and

$$d_*(c_2) = d_*(Sq^0(c_1)) = Sq^1(d_2(c_1)) \dotplus h_0 Sq^1(c_1) = h_0 Sq^1(c_1)$$

gives $d_2(c_2) = h_0 f_1$, since $Sq^1(d_2(c_1))$ has Adams filtration 6 and $h_0 Sq^1(c_1) = h_0 f_1$ has Adams filtration 5. □

14.4. **Sparsity and multiplicative structure.**

Proposition 14.14. *$P^i h_1$, $P^i h_2$, $P^i c_0$ and $P^i d_0$ are permanent cycles for all $i \geq 0$.*

Proof. We have already proved this for $i = 0$, and it is clear from the displayed E_2-term, for $1 \leq i \leq 8$, because the target groups of all these differentials are trivial. (For larger i, the result will follow from Adams periodicity.) □

Proposition 14.15. $d_2(P^i d_0 f_0) = h_0^2 P^i d_0 e_0$*, for each* $i \geq 0$.

Proof. This follows from $d_2(f_0) = h_0^2 e_0$ by multiplication with the permanent cycle $P^i d_0$. □

Corollary 14.16. $d_2(i) = h_0 P d_0$*,* $d_2(Pe_0) = h_1^2 P d_0$*,* $d_2(j) = h_0 P e_0$*,* $d_2(k) = h_0 d_0^2$*,* $d_2(\ell) = h_0 d_0 e_0$*,* $d_2(m) = h_0 d_0 g$*,* $d_2(t) = 0$*,* $d_2(y) = h_0^3 x$ *and* $d_2(r) = 0$.

Proof. Starting with $d_2(d_0f_0) = h_0^2d_0e_0$ we get the result for k by division by h_2, then for j by multiplication by h_0 and division by h_2, then for Pe_0 by multiplication by h_0^2 and division by h_1, or for i by multiplication by h_0 and division by h_2.

Heading in the opposite direction, we get the result for ℓ by division by h_0, for m by multiplication with h_2 followed by division by h_0, and for h_0y by multiplication with h_2 followed by division by h_0. For now, division by h_0 only tells us that $d_2(y)$ is either e_0g or h_0^3x.

We get $d_2(t) = 0$, since the only alternative is $d_2(t) = h_0m$, which contradicts $d_2(h_0m) = h_0^2d_0 \neq 0$. Hence $h_1d_2(y) = d_2(h_1y) = d_2(h_2t) = h_2d_(t) = 0$. Since $h_1e_0g \neq 0$, we can conclude that $d_2(y) = h_0^3x$.

Finally, $d_2(r) = 0$, since the alternative is $d_2(r) = h_0k$, but $d_2(h_0k) = h_0^2d_0^2 \neq 0$. □

Corollary 14.17. *$d_2(P^2e_0) = h_1^2P^2d_0$, $d_2(Pj) = h_0P^2e_0$ and $d_2(z) = 0$.*

Proof. Starting with $d_2(Pd_0f_0) = h_0^2Pd_0e_0$, we divide by h_2, multiply by h_0 and divide by h_2, to deduce that $d_2(Pj) = h_0P^2e_0$. Thereafter we multiply by h_0^2 and divide by h_1, to deduce that $d_2(P^2e_0) = h_1^2P^2d_0$.

We get $d_2(z) = 0$, since the only alternative is $d_2(z) = Pd_0f_0$, which contradicts $d_2(Pd_0f_0) = h_0^2Pd_0e_0 \neq 0$. □

Corollary 14.18. *$d_2(P^2i) = h_0P^3d_0$, $d_2(P^3e_0) = h_1^2P^3d_0$, $d_2(P^2j) = h_0P^3e_0$, $d_2(R_1) = h_0^2x'$, $d_2(Q_1) = h_1^2x'$, $d_2(X_1) = h_0^2B_4 \mod h_1B_{21}$. ((Are there more consequences?))*

Proof. Starting from $d_2(P^2d_0f_0) = h_0^2P^2d_0e_0$ we can divide by h_2, multiply by h_0 and divide by h_2, to get $d_2(P^2j) = h_0P^3e_0$. Multiplying by h_0^2 and dividing by h_1 gives $d_2(P^3e_0) = h_1^2P^3d_0$, while instead multiplying by h_0 and dividing by h_2 gives $d_2(P^2i) = h_0P^3d_0$.

In the opposite direction, dividing by h_0, multiplying by h_2, dividing by h_0, multiplying by h_2 and dividing by h_0^6 gives $d_2(R_1) = h_0^2x'$. Multiplying by h_2 and dividing by h_1 gives $d_2(Q_1) = h_1^2x'$. Multiplying instead by h_3 and dividing by h_0^2 gives $d_2(X_1) = h_0^2B_4 \mod h_1B_{21}$. □

Corollary 14.19. *$d_2(P^4e_0) = h_1^2P^4d_0$ and $d_2(P^3j) = h_0P^4e_0$.*

Corollary 14.20. *$d_2(P^4i) = h_0P^5d_0$, $d_2(P^5e_0) = h_1^2P^5d_0$ and $d_2(P^4j) = h_0P^5e_0$. ((There are more consequences around $t - s = 70$.))*

((Here one can keep going.))

Proposition 14.21. *d_2 is 0 on g, f_1, g_2, x, C, G, B_2, B_3, N, u, w, x', B_{21}, Pu and P^2u because the target groups are trivial, on d_1, e_1, n and q because h_0 acts trivially on the source and injectively on the target, on p because h_1 acts trivially on the source and injectively on the target, and on Pu because h_3 acts trivially on the source and injectively on the target.*

Proposition 14.22. *$d_2(v) = h_1^2u$, $d_2(B_1) = 0$ and $d_2(Q) = h_0i^2$.*

Proof. See [MT67, Theorem 1.1.4(v) and 8.9]. ((Reference for $d_2(Q)$?)) □

14.5. **The Adams E_3-term.** The d_2-differentials affecting $E_2^{s,t}$ with $t-s \leq 48$ are displayed in Figure 29. The resulting E_3-term is displayed in Figure 30, and the algebra generators in this range are listed in Figure 31. ((Extend the table for $s \geq 12$.))

14.6. **The mapping cone of σ.** To proceed, we use naturality of the Adams spectral sequence with respect to the map $i\colon S \to C_\sigma = S \cup_\sigma e^8$. The E_2-term of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(C_\sigma), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(C_\sigma)_2^\wedge$$

is displayed in Figure 35.

Proposition 14.23. *$d_r^{s,t} = 0$ for all $r \geq 2$ and $t - s \leq 14$, in the Adams spectral sequence for C_σ.*

Proof. This is clear because of the module structure of the spectral sequence for C_σ over the spectral sequence for S. For example $d_5^{1,12} = 0$, because h_1 acts trivially on the source and injectively on the target of that differential. □

Proposition 14.24. *$\nu\sigma = 0$.*

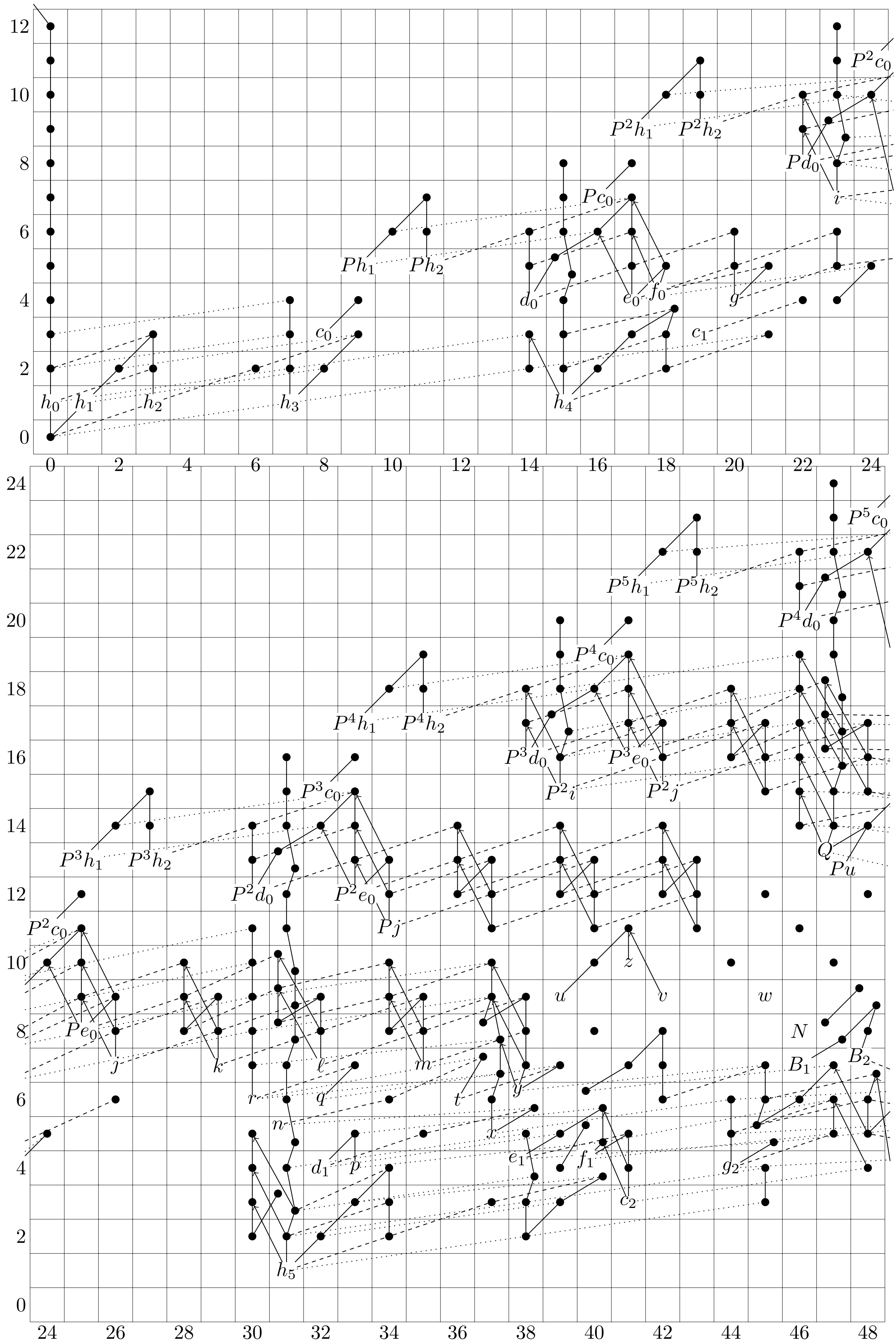


FIGURE 29. (E_2, d_2) for S

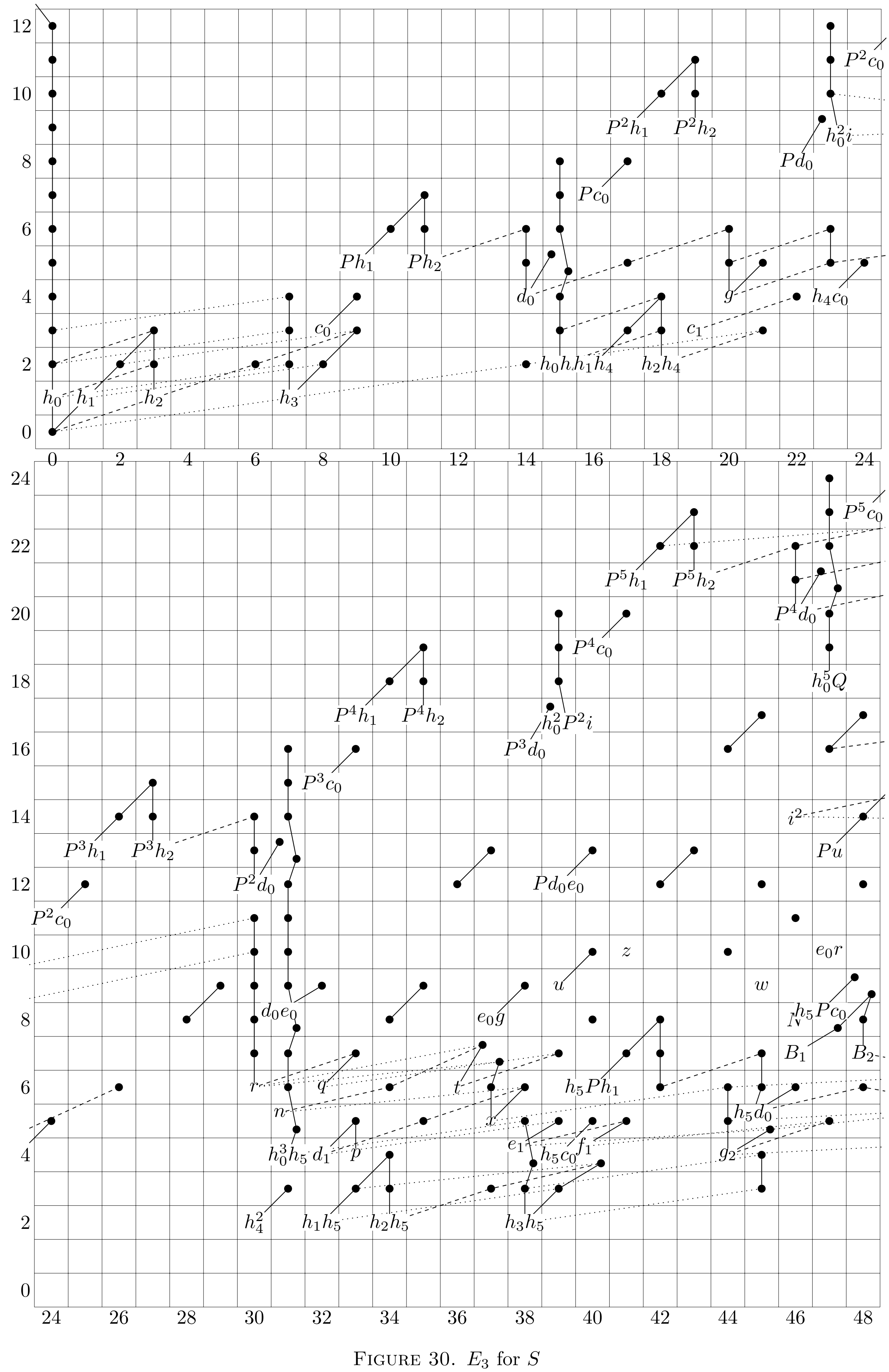


FIGURE 30. E_3 for S

$t-s$	s	t	[Tan70]	ext	d_3
0	1	1	h_0	1_0	0
1	1	2	h_1	1_1	0
3	1	4	h_2	1_2	0
7	1	8	h_3	1_3	0
15	2	17	h_0h_4	2_7	h_0d_0
16	2	18	h_1h_4	2_8	0
18	2	20	h_2h_4	2_9	0
30	2	32	h_4^2	2_{10}	0
32	2	34	h_1h_5	2_{12}	0
34	2	36	h_2h_5	2_{13}	
38	2	40	h_3h_5	2_{14}	
8	3	11	c_0	3_3	0
19	3	22	c_1	3_9	0
14	4	18	d_0	4_3	0
20	4	24	$g = g_1$	4_8	0
23	4	27	h_4c_0	4_{10}	0
31	4	35	$h_0^3h_5$	4_{12}	
32	4	36	d_1	4_{13}	0
33	4	37	$p = p_0$	4_{14}	0
38	4	42	e_1	4_{16} mod 4_{17}	
39	4	43	h_5c_0	4_{18}	0
40	4	44	f_1	4_{19} mod 4_{20}	
44	4	48	g_2	4_{22}	
9	5	14	Ph_1	5_1	0
11	5	16	Ph_2	5_2	0
31	5	36	$n = n_0$	5_{13} mod 5_{14}	0
37	5	42	$x = x_0$	5_{17}	0
45	5	50	h_5d_0	5_{24}	0
30	6	36	r	6_{10}	$(h_1d_0^2)$
32	6	38	q	6_{12}	0
36	6	42	t	6_{14}	
40	6	46	h_5Ph_1	6_{18}	
16	7	23	Pc_0	7_3	0
46	7	53	B_1	7_{20}	0
48	7	55	B_2	7_{22} mod 7_{23}	
22	8	30	Pd_0	8_3	0
31	8	39	d_0e_0	8_{10}	
37	8	45	e_0g	8_{15}	0
46	8	54	N	8_{20}	0
47	8	55	h_5Pc_0	8_{21}	
17	9	26	P^2h_1	9_1	0
19	9	28	P^2h_2	9_2	0
23	9	32	h_0^2i	9_5	0
39	9	48	u	9_{18}	0
45	9	54	w	9_{20}	0
41	10	51	z	10_{14}	
47	10	57	e_0r	10_{16}	0
24	11	35	P^2c_0	11_3	0
46	11	57	jg	11_{12}	0

FIGURE 31. Algebra generators for E_3

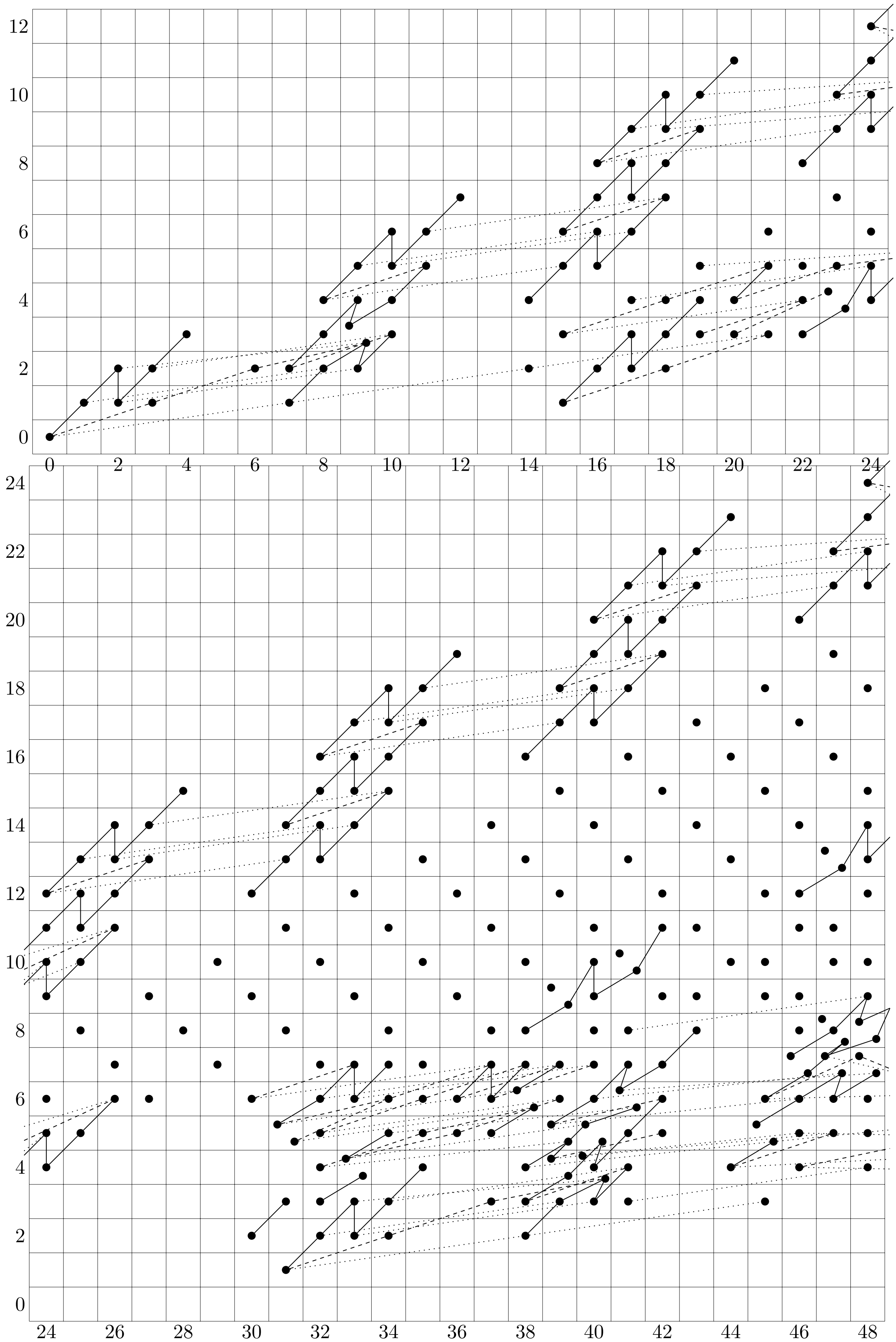


FIGURE 32. Adams E_2 for $C_2 = S \cup_2 e^1$

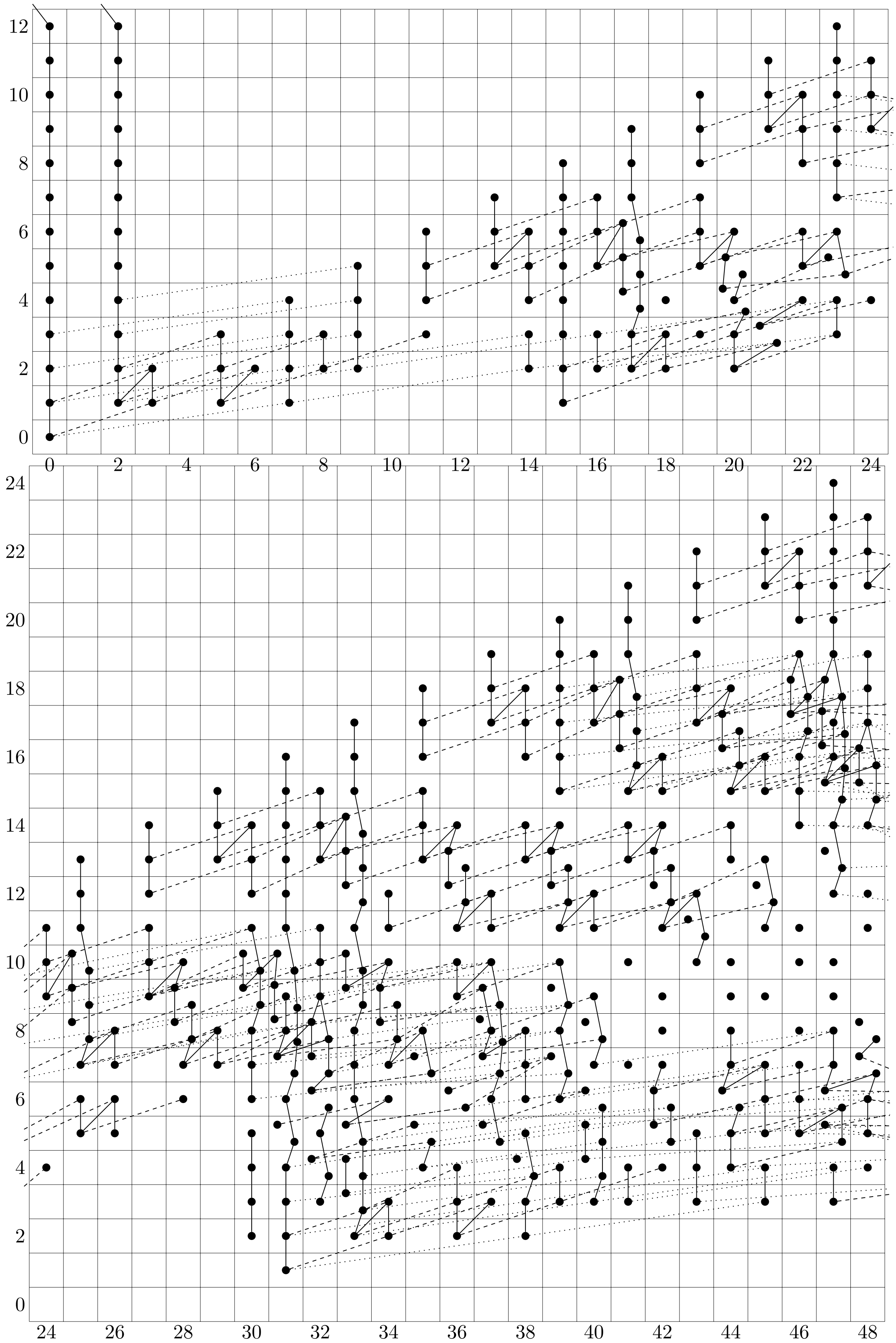


FIGURE 33. Adams E_2 for $C_\eta = S \cup_\eta e^2$

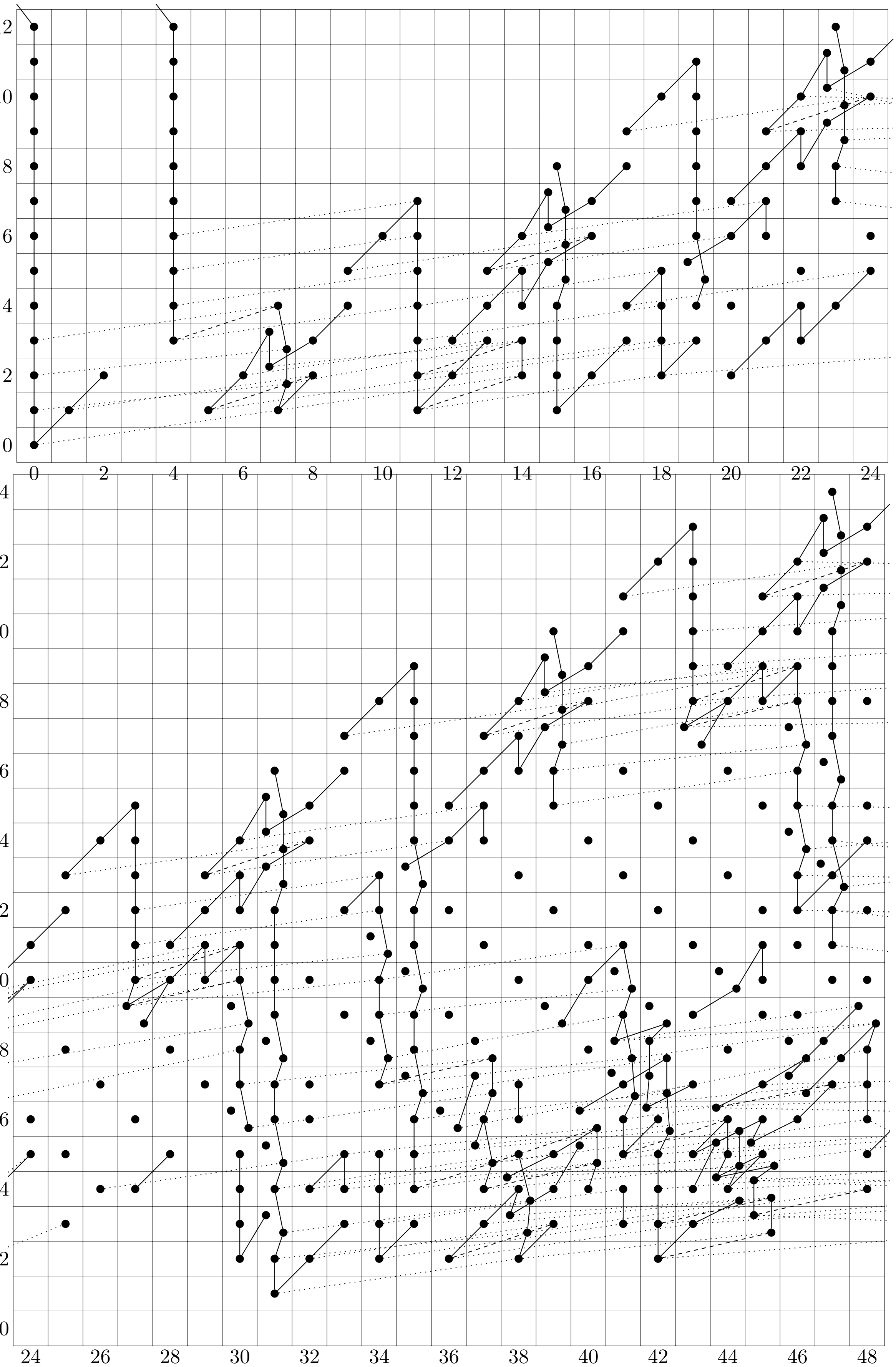


FIGURE 34. Adams E_2 for $C_\nu = S \cup_\nu e^4$

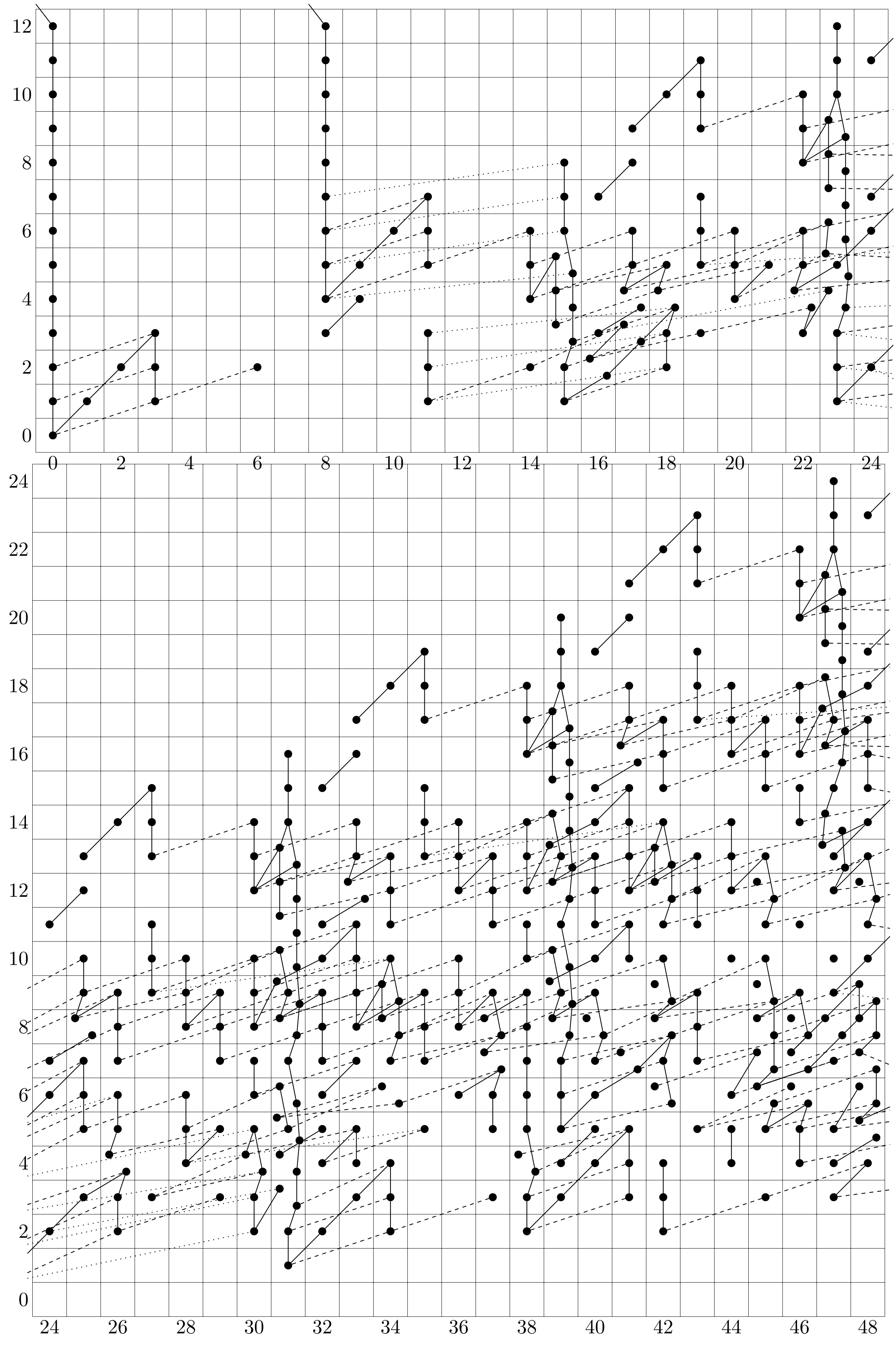


FIGURE 35. Adams E_2 for $C_\sigma = S \cup_\sigma e^8$

Proof. Consider the long exact sequence

$$\cdots \to \pi_3(S)^\wedge_2 \xrightarrow{\sigma} \pi_{10}(S)^\wedge_2 \xrightarrow{i_*} \pi_{10}(C_\sigma)^\wedge_2 \xrightarrow{j_*} \pi_2(S)^\wedge_2 \xrightarrow{\sigma} \pi_9(S)^\wedge_2 .$$

Here $\pi_{10}(C_\sigma)^\wedge_2 \cong E_\infty^{6,16}$ has order 2, and multiplication by σ acts injectively on $\pi_2(S)^\wedge = \mathbb{Z}/2\{\eta^2\}$, since $\eta^2\sigma \neq 0$, so multiplication by σ from $\pi_3(S)^\wedge_2 = \mathbb{Z}/8\{\nu\}$ to $\pi_{10}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\mu\}$ has cokernel of order 2, hence is trivial. □

The short exact sequence

$$0 \to H^*(S^8) \to H^*(C_\sigma) \xrightarrow{i^*} H^*(S) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i_*} \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(C_\sigma), \mathbb{F}_2) \to \mathrm{Ext}^{s,t-8}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s+1,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \dots$$

which we can use to determine i_* in most bidegrees.

Proposition 14.25. *In the Adams spectral sequence for C_σ, there is a unique class $\beta \in E_2^{3,18}$ with $h_2\beta = i_*(f_0)$. The differentials satisfy $d_r^{s,t} = 0$ for all $r \geq 2$ and $t - s \leq 14$, $d_2(\beta) = h_0 i_*(d_0)$, $d_2(h_0\beta) = h_0^2 i_*(d_0)$ and $d_2(i_*(f_0)) = h_0^2 i_*(e_0)$.*

((We choose the letter β, since this class maps to β in the Adams spectral sequence for *tmf*.)) ((Also $d_2^{1,24} \neq 0$, $d_2^{2,25} \neq 0$, $d_2^{2,26} \neq 0$ and $d_3^{3,26} \neq 0$.))

Proof. The existence and uniqueness of β is clear, since multiplication by h_2 is bijective from $E_2^{3,18}$. The differentials $d_r^{s,t}$ are zero for $t - s \leq 14$, either because they land in trivial groups, or as consequences of this and h_0-, h_1- and h_2-linearity. For example, $d_5^{1,12} = 0$ because h_1 is trivial on the source but nontrivial on the target of this differential.

The differential $d_2(i_*(f_0)) = h_0^2 i_*(e_0)$ follows from $d_2(f_0) = h_0^2 e_0$ by naturality with respect to i. This implies that $h_2 \cdot d_2(\beta) = h_2 \cdot h_0 i_*(d_0)$, which in turn implies that $d_2(\beta) = h_0 i_*(d_0)$, since multiplication by h_2 acts injectively on $E_2^{5,19}$. The differential $d_2(h_0\beta) = h_0^2 i_*(d_0)$ then follows by multiplication with h_0. □

Corollary 14.26. *$\pi_{14}(C_\sigma)^\wedge_2$ has order dividing 4.*

Proof. Of the generators with $t-s = 14$, only those with $s = 2$ or $s = 4$ can survive to the E_∞-term. □

Proposition 14.27. *$d_3(h_0h_4) = h_0 d_0$.*

Proof. The classes h_3^2 and d_0 in the Adams spectral sequence for S cannot be boundaries, since $d_2(h_4) = h_0 h_3^2$ implies that $d_2(h_0h_4) = h_0^2 h_3^2 = 0$ and h_4 does not survive to the E_3-term. Hence σ^2 is detected by h_3^2, which is nonzero at the E_∞-term, so σ^2 is nonzero in $\pi_{14}(S)^\wedge_2$, but $2\sigma^2 = 0$ by graded commutativity. Furthermore, there will be a class $\kappa \in \pi_{14}(S)^\wedge_2$ that is detected by d_0, so $\pi_{14}(S)^\wedge_2$ has order a multiple of 4.

Consider the long exact sequence

$$\cdots \to \pi_7(S)^\wedge_2 \xrightarrow{\sigma} \pi_{14}(S)^\wedge_2 \xrightarrow{i_*} \pi_{14}(C_\sigma)^\wedge_2 \xrightarrow{j_*} \pi_6(S)^\wedge_2 \to \pi_{13}(S)^\wedge_2 \to \dots .$$

Here $\pi_7(S)^\wedge_2 = \mathbb{Z}/16\{\sigma\}$ is generated by σ, so $\mathrm{im}(\sigma) = \ker(i_*) = \mathbb{Z}/2\{\sigma^2\}$. Furthermore, $\pi_6(S)^\wedge_2 = \mathbb{Z}/2\{\nu^2\}$ has order 2 and $\pi_{13}(S)^\wedge_2$ is trivial, so $\mathrm{cok}(i_*) = \mathrm{im}(j_*)$ has order 2. It follows that $\mathrm{im}(i_*)$ has order dividing 2, and $\pi_{14}(S)^\wedge_2$ has order dividing 4.

Combining these two bounds, we find that $\pi_{14}(S)^\wedge_2$ has order exactly 4. Hence the classes h_0d_0 and $h_0^2 d_0$ in the E_2-term with $t - s = 14$ cannot be nonzero at E_∞, and must therefore be boundaries. The only possible differential with target h_0d_0 is $d_3(h_0h_4) = h_0d_0$. □

Corollary 14.28. *$d_3(h_0^2 h_4) = h_0^2 d_0$.*

Proof. This follows by multiplication with h_0. □

Theorem 14.29. *$\pi_{14}(S)^\wedge_2 \cong \mathbb{Z}/2\{\kappa, \sigma^2\}$, with κ represented by d_0 and σ^2 represented by h_3^2.*

Proof. The Adams filtration gives the short exact sequence

$$0 \to \mathbb{F}_2\{d_0\} \to \pi_{14}(S)^\wedge_2 \to \mathbb{F}_2\{h_3^2\} \to 0 .$$

The class κ is represented by d_0, and the product σ^2 is represented by h_3^2. The extension is split, because $2 \cdot \sigma^2 = 0$ by graded commutativity. □

Proposition 14.30. *$i_*(h_4)$ is a permanent cycle, in the Adams spectral sequence for C_σ.*

Proof. The image of multiplication by σ in $\pi_{14}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa, \sigma^2\}$ is $\mathbb{Z}/2\{\sigma^2\}$, so its cokernel, which is isomorphic to $\mathrm{im}(i_*)$, is $\mathbb{Z}/2\{\kappa\}$ of order 2. Hence $\pi_{14}(C_\sigma)^\wedge_2$ has order exactly 4, and the E_∞-term of the Adams spectral sequence for C_σ must contain exactly two generators in topological degrees $t-s=14$. The generators $h_0 i_*(d_0)$ and $h_0^2 i_*(d_0)$ in filtration degrees $s=5$ and $s=6$ are d_2-boundaries. Hence the remaining two generators, in filtrations $s=2$ and $s=4$, cannot be boundaries. In particular, $d_3(i_*(h_4))$ is 0, not $i_*(d_0)$. It follows that $d_r(i_*(h_4)) = 0$ for all $r \geq 2$. □

Corollary 14.31. *$h_2 i_*(h_4)$ is a permanent cycle, in the Adams spectral sequence for C_σ.*

Proof. This follows by multiplication by h_2. □

Proposition 14.32. *h_2h_4 survives to the E_6-term.*

Proof. We know that $d_2(h_2h_4) = 0$, either by multiplication with h_2 from $d_2(h_4) = h_0h_3^2$, or because the only possible nonzero target, e_0, supports a nonzero differential $d_2(e_0) \neq 0$.

We also know that $h_2 i_*(h_4)$ and $h_0h_2 i_*(h_4)$ are permanent cycles in the Adams spectral sequence for C_σ. Hence $i_*(h_0e_0)$ in bidegree $(t-s,s) = (17,5)$ cannot be a boundary in that spectral sequence. Thus i_* induces an isomorphism of E_3-terms in that bidegree. Since $i_*(d_3(h_2h_4)) = d_3(i_*(h_2h_4)) = 0$, it follows that $d_3(h_2h_4) = 0$.

The target groups of $d_r(h_2h_4)$ are trivial for $r=4$ and $r=5$, hence h_2h_4 survives to the E_6-term. □

Proposition 14.33. *$d_3(r) = h_1 d_0^2$.*

((Give proof using quadratic construction on $\kappa\colon S^{14} \to S$, represented by d_0 with $Sq^j(d_0) = d_0^2$, 0, r, 0 and d_1 for $j = 4$, 3, 2, 1 and 0.))

14.7. **The Adams E_4-term.** The d_3-differentials affecting $E_3^{s,t}$ with $t-s \leq 24$ are displayed in Figure 36. The resulting E_4-term is displayed in Figure 37.

Proposition 14.34. *h_4c_0 is a permanent cycle.*

Proof. We know that c_0 and h_1h_4 are permanent cycles, so $h_1 d_4(h_4c_0) = d_4(c_0 \cdot h_1h_4) = 0$. Since $h_1Pd_0 \neq 0$ we cannot have $d_4(h_4c_0) = Pd_0$, and the only remaining possibility is that $d_r(h_4c_0) = 0$ for all $r \geq 2$. □

We have now shown that all the algebra generators of the E_4-term in topological degrees $t-s \leq 30$ are permanent cycles, except for h_2h_4 and g, which could support nonzero d_6- and d_7-differentials, respectively. To proceed we shall make a comparison with the image-of-J spectrum, to be introduced in the following section.

15. Topological K-theory

15.1. **Real and complex K-theory.** The set of isomorphism classes of real vector bundles over a finite CW complex X forms a commutative monoid with respect to direct (Whitney) sum of vector bundles. The additive group completion of this commutative monoid is denoted $KO(X)$, and consists of formal differences between pairs of real vector bundles over X. The corresponding construction for complex vector bundles leads to the group $KU(X)$ of formal differences of pairs of complex vector bundles. By Bott periodicity, the external tensor product of vector bundles induces natural isomorphisms $KO(X) \otimes KO(S^8) \cong KO(X \times S^8)$ and $KU(X) \otimes KU(S^2) \cong KU(X \times S^2)$. In terms of the reduced K-groups $\widetilde{KO}(X) = \ker(KO(X) \to KO(*))$ and $\widetilde{KU}(X) = \ker(KU(X) \to KU(*))$, for based finite CW-complexes X, this can be expressed as isomorphisms $\widetilde{KO}(X) \cong \widetilde{KO}(\Sigma^8 X)$ and $\widetilde{KU}(X) \cong \widetilde{KU}(\Sigma^2 X)$. Hence there are generalized (reduced) cohomology theories KO^* and KU^* defined by $KO^n(X) = \widetilde{KO}(\Sigma^m X)$, where $n + m \equiv 0 \mod 8$, and $KU^n(X) = \widetilde{KU}(\Sigma^m X)$, where $n+m \equiv 0 \mod 2$. For definiteness, we may assume $0 \leq m < 8$ in the real case, and $0 \leq m < 2$ in the complex case. The internal tensor product of vector bundles induces products in these cohomology theories. Complexification, i.e, tensoring a real vector bundle with $\mathbb{C}$ over $\mathbb{R}$ to obtain a complex vector bundle, induces a multiplicative homomorphism $c\colon KO^*(X) \to KU^*(X)$. Realification, i.e., only remembering the underlying real vector bundle of a complex vector bundle, induces a homomorphism $r\colon KU^*(X) \to KO^*(X)$, which is not multiplicative, but is linear as a map of modules over the target.

The reduced K-functors $\widetilde{KO}$ and $\widetilde{KU}$ are represented by the infinite loop spaces $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$, respectively, where $\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO)$ and $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$ by Bott periodicity. The

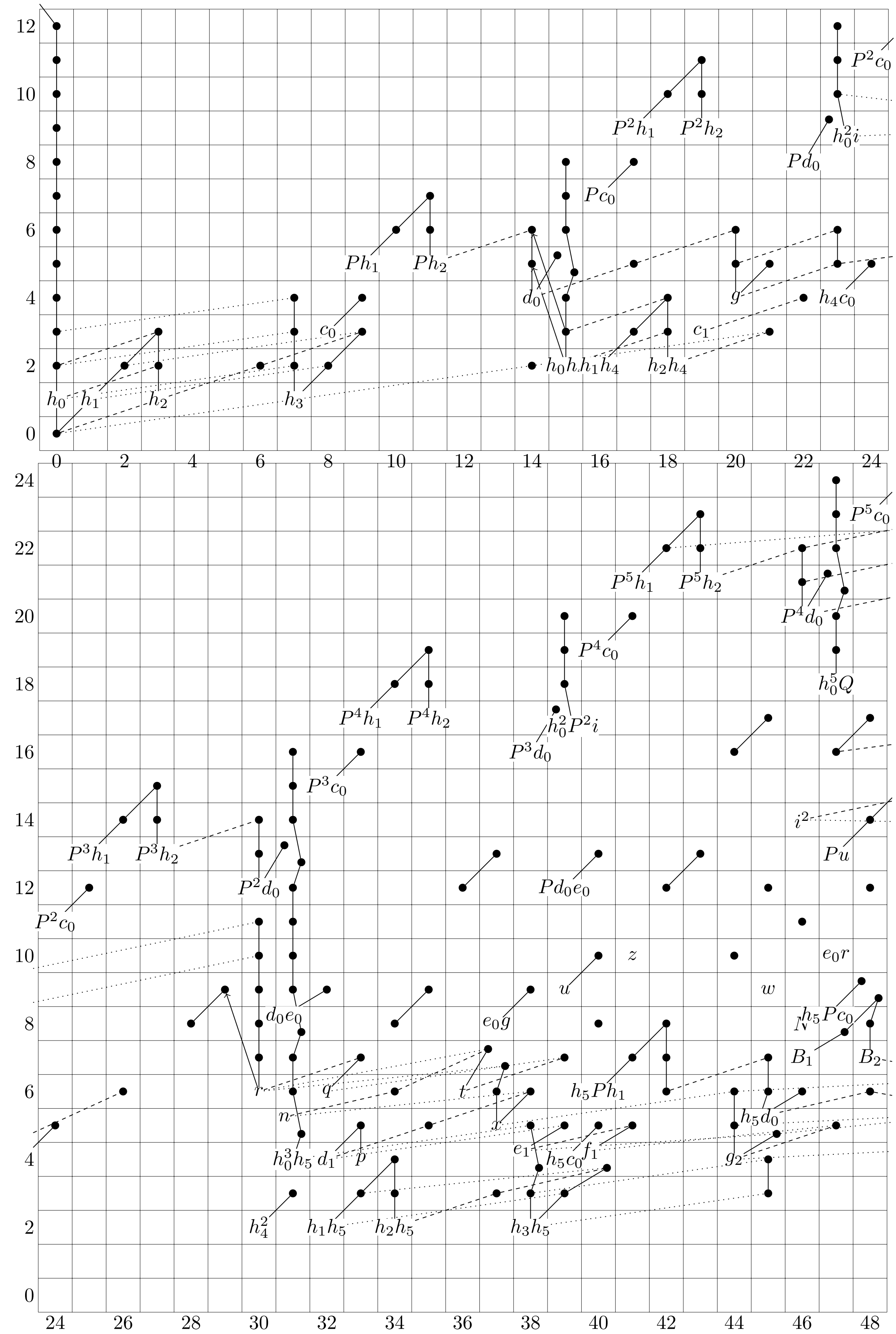


FIGURE 36. (E_3, d_3) for S

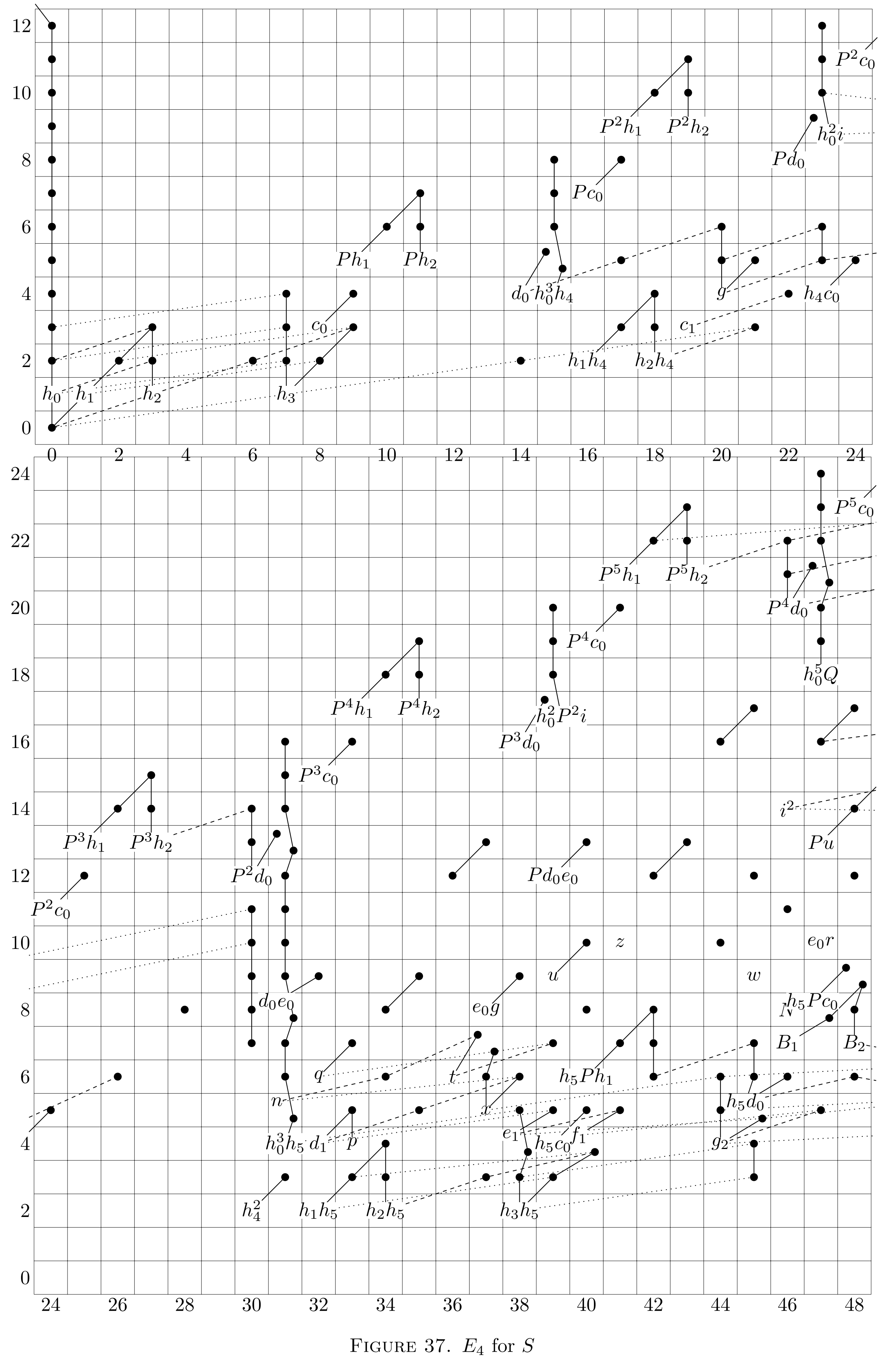


Figure 37. E_4 for S

cohomology theories KO^* and KU^* are thus represented by Ω-spectra KO and KU, respectively, with n-th spaces $\underline{KO}_n = \Omega^m(\mathbb{Z} \times BO)$ and $\underline{KU}_n = \Omega^m(\mathbb{Z} \times BU)$, where m is chosen so that $n+m \equiv 0 \mod 8$ and $0 \le m < 8$ in the real case, and $n + m \equiv 0 \mod 2$ and $0 \le m < 2$ in the complex case. The tensor product pairing is represented by pairings of spectra, that make KO and KU into E_∞ ring spectra. The unit $S \to KO$ is generated by a map $S^0 \to \mathbb{Z} \times BO$ that takes the non-base point to a point in $\{1\} \times BO$, and similarly in the complex case. Complexification is represented by a ring spectrum map $c\colon KO \to KU$, and realification is represented by a KO-module map $r\colon KU \to KO$. The homotopy groups of these ring spectra are known, by Bott periodicity, to be

$$\pi_i(KO) = \begin{cases} \mathbb{Z}\{\beta^k\} & \text{for } i = 8k, \\ \mathbb{Z}/2\{\eta\beta^k\} & \text{for } i = 8k+1, \\ \mathbb{Z}/2\{\eta^2\beta^k\} & \text{for } i = 8k+2, \\ \mathbb{Z}\{\alpha\beta^k\} & \text{for } i = 8k+4, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_i(KU) = \begin{cases} \mathbb{Z}\{u^k\} & \text{for } i = 2k \text{ even}, \\ 0 & \text{for } i \text{ odd}. \end{cases}$$

As graded rings, these are

$$\pi_*(KO) = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with η, α and β in degree 1, 4 and 8, respectively, and

$$\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$$

with u in degree 2. Complexification is given by $\eta \mapsto 0$, $\alpha \mapsto 2u^2$ and $\beta \mapsto u^4$. Realification is given by $u^{4k} \mapsto 2\beta^k$, $u^{4k+1} \mapsto \eta^2\beta^k$, $u^{4k+2} \mapsto \alpha\beta^k$ and $u^{4k+3} \mapsto 0$.

There are connective, i.e. (-1)-connected, covers of these ring spectra, denotes ko and ku, respectively, with ring spectrum maps $ko \to KO$ and $ku \to KU$ that induce isomorphisms of homotopy groups in non-negative degrees. Hence $\pi_i(ko) \cong \pi_i(KO)$ for $i \ge 0$ and $\pi_i(ko) = 0$ for $i < 0$, and similarly in the complex case. As graded rings,

$$\pi_*(ko) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

and

$$\pi_*(ku) = \mathbb{Z}[u]\,.$$

The n-th space $\underline{ko}_n$ of the spectrum ko is an $(n-1)$-connected cover of the n-the space $\underline{KO}_n$, and similarly in the complex case. For example, $\underline{ku}_0 \simeq \mathbb{Z} \times BU$, $\underline{ku}_1 \simeq U$, $\underline{ku}_2 \simeq BU$, $\underline{ku}_3 \simeq SU$ and $\underline{ku}_4 \simeq BSU$.

15.2. **Cohomology and homotopy of K-theory spectra.** Recall that $H^*(H) \cong \mathscr{A}$ and $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}Sq^1 = \mathscr{A} \otimes_{A(0)} \mathbb{F}_2 = \mathscr{A}//A(0)$, where $A(0) = E(Sq^1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1.

Let bu denote the 1-connected cover of ku, so that there is a cofiber sequence

$$bu \to ku \xrightarrow{p_0} H\mathbb{Z} \to \Sigma bu$$

and a Bott equivalence $u\colon \Sigma^2 ku \simeq bu$.

Proposition 15.1. *$H^*(ku) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} = \mathscr{A} \otimes_{E(1)} \mathbb{F}_2 = \mathscr{A}//E(1)$, where $Q_1 = [Sq^1, Sq^2] = Sq^3 + Sq^2Sq^1$ and $E(1) = E(Sq^1, Q_1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1 and Q_1. Hence there is a short exact sequence*

$$0 \to \Sigma^3\mathscr{A}//E(1) \longrightarrow \mathscr{A}//A(0) \xrightarrow{p_0^*} \mathscr{A}//E(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^3\mathbb{F}_2 \longrightarrow E(1)//A(0) \longrightarrow \mathbb{F}_2$ of $E(1)$-modules.

Proof. It is known, from calculations in $H^*(SU)$, that the bottom Postnikov k-invariant of ku, i.e., the composite $H\mathbb{Z} \to \Sigma bu \simeq \Sigma^3 ku \to \Sigma^3 H\mathbb{Z}$ viewed as a class in $H^3(H\mathbb{Z}; \mathbb{Z})$, is nonzero. This implies that $H\mathbb{Z} \to \Sigma bu$ induces an isomorphism on H^3, so that $bu \to ku$ and $u\colon S^2 \to ku$ induce zero homomorphisms on H^2. It follows that the Bott equivalence $\phi \circ (1 \wedge u)\colon bu \simeq ku \wedge S^2 \to ku \wedge ku \to ku$ induces 0 in cohomology. Hence we have a map of short exact sequences.

$$\begin{CD} 0 @>>> \Sigma^3 \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} @>>> \mathscr{A}/\mathscr{A}Sq^1 @>>> \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} @>>> 0 \\ @. @V{\Sigma^3 f}VV @V{\cong}VV @V{f}VV \\ 0 @>>> H^*(\Sigma bu) @>>> H^*(H\mathbb{Z}) @>{p_0^*}>> H^*(ku) @>>> 0 \end{CD}$$

It follows by induction that f is an isomorphism in all degrees. □

Let bo, bso, $bspin$ and $bo\langle 8\rangle$ be the 0-, 1-, 3- and 7-connected covers of ko, respectively, so that there are cofiber sequences

$$bo \to ko \xrightarrow{p_0} H\mathbb{Z} \to \Sigma bo$$
$$bso \to bo \xrightarrow{p_1} \Sigma H \to \Sigma bso$$
$$bspin \to bso \xrightarrow{p_2} \Sigma^2 H \to \Sigma bspin$$
$$bo\langle 8\rangle \to bspin \xrightarrow{p_4} \Sigma^4 H\mathbb{Z} \to \Sigma bo\langle 8\rangle$$

and a Bott equivalence $\beta\colon \Sigma^8 ko \simeq bo\langle 8\rangle$.

There is a cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \longrightarrow \Sigma^2 ko\,,$$

where c denotes the complexification map and η denotes multiplication with the Hopf map $\eta\colon S^1 \to S$. The connecting map $ku \to \Sigma^2 ko$ lifts the composite map $\Sigma^2 r \circ u^{-1}\colon KU \to \Sigma^2 KU \to \Sigma^2 KO$. The spectra ko and ku are (E_∞) ring spectra, and c is a ring spectrum map.

Proposition 15.2. *$H^*(ko) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} = \mathscr{A} \otimes_{A(1)} \mathbb{F}_2 = \mathscr{A}//A(1)$, where $A(1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1 and Sq^2. Hence there is a short exact sequence*

$$0 \to \Sigma^2 \mathscr{A}//A(1) \longrightarrow \mathscr{A}//E(1) \xrightarrow{c^*} \mathscr{A}//A(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^2\mathbb{F}_2 \longrightarrow A(1)//E(1) \longrightarrow \mathbb{F}_2$ of $A(1)$-modules.

$H^*(bo) \cong \Sigma\mathscr{A}/\mathscr{A}Sq^2 = \Sigma\mathscr{A} \otimes_{A(1)} A(1)/A(1)Sq^2$, and there is a short exact sequence

$$0 \to \Sigma^2 \mathscr{A}/\mathscr{A}Sq^2 \longrightarrow \mathscr{A}//A(0) \xrightarrow{p_0^*} \mathscr{A}//A(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^2 A(1)/A(1)Sq^2 \longrightarrow A(1)//A(0) \longrightarrow \mathbb{F}_2$ of $A(1)$-modules.

$H^*(bso) \cong \Sigma^2\mathscr{A}/\mathscr{A}Sq^3 = \Sigma^2\mathscr{A} \otimes_{A(1)} A(1)/A(1)Sq^3$, and there is a short exact sequence

$$0 \to \Sigma^3 \mathscr{A}/\mathscr{A}Sq^3 \longrightarrow \Sigma\mathscr{A} \xrightarrow{p_1^*} \Sigma\mathscr{A}/\mathscr{A}Sq^2 \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^3 A(1)/A(1)Sq^3 \longrightarrow \Sigma A(1) \longrightarrow \Sigma A(1)/A(1)Sq^2$ of $A(1)$-modules.

$H^*(bspin) \cong \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} = \Sigma^4\mathscr{A} \otimes_{A(1)} A(1)/A(1)\{Sq^1, Sq^2Sq^3\}$, and there is a short exact sequence

$$0 \to \Sigma^5 \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \longrightarrow \Sigma^2\mathscr{A} \xrightarrow{p_2^*} \Sigma^2\mathscr{A}/\mathscr{A}Sq^3 \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^5 A(1)/A(1)\{Sq^1, Sq^2Sq^3\} \longrightarrow \Sigma^2 A(1) \longrightarrow \Sigma^2 A(1)/A(1)Sq^3$ of $A(1)$-modules.

$H^*(bo\langle 8\rangle) \cong \Sigma^8\mathscr{A}//A(1)$, and there is a short exact sequence

$$0 \to \Sigma^9 \mathscr{A}//A(1) \longrightarrow \Sigma^4\mathscr{A}//A(0) \xrightarrow{p_4^*} \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^9\mathbb{F}_2 \longrightarrow \Sigma^4 A(1)//A(0) \longrightarrow \Sigma^4 A(1)/A(1)\{Sq^1, Sq^2Sq^3\}$ of $A(1)$-modules.

Proof. The map $\eta\colon S^1 \to S$ induces the zero homomorphism in cohomology, hence so does $\eta\colon \Sigma ko \to ko$, and there is a vertical map of short exact sequences:

$$\begin{CD} 0 @>>> \Sigma^2 \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} @>>> \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} @>>> \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} @>>> 0 \\ @. @V{\Sigma^2 f}VV @V{\cong}VV @V{f}VV \\ 0 @>>> H^*(\Sigma^2 ko) @>>> H^*(ku) @>{c^*}>> H^*(ko) @>>> 0 \end{CD}$$

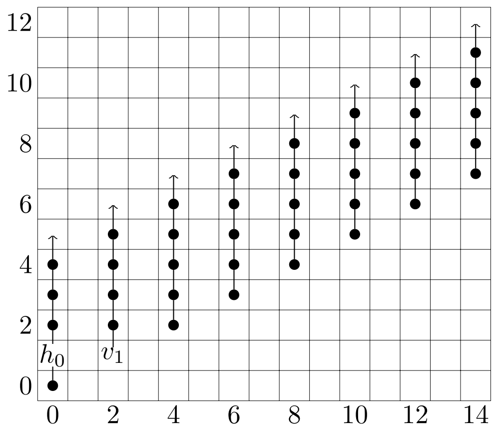


FIGURE 38. The Adams spectral sequence for ku

It follows by induction that f is an isomorphism in all degrees.

The map $p_0 \colon ko \to H\mathbb{Z}$ is 0-connected, hence $p_0^* \colon \mathscr{A}/\mathscr{A}Sq^1 \to \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$ is an isomorphism in degree 0 and surjective in all degrees. Hence p_0 is induced up from the surjection $\epsilon \colon A(1)//A(0) \to \mathbb{F}_2$ of $A(1)$-modules, with kernel $\ker(\epsilon) = \mathbb{F}_2\{Sq^2, Sq^3, Sq^2Sq^3\} \cong \Sigma^2 A(1)/A(1)Sq^2$. Hence $\Sigma H^*(bo) \cong \ker(p_0^*) \cong \mathscr{A} \otimes_{A(1)} \Sigma^2 A(1)/A(1)Sq^2 \cong \Sigma^2\mathscr{A}/\mathscr{A}Sq^2$.

((ETC)) □

Theorem 15.3 (Change of rings)**.** *Let A be any algebra, let $B \subset A$ be a subalgebra such that A is flat as a right B-module, let M be a left B-module and let N be a left A-module. Then there is a natural isomorphism*

$$\mathrm{Ext}_A^{*,*}(A \otimes_B M, N) \cong \mathrm{Ext}_B^{*,*}(M, N) .$$

Proof. Let $P_* \to M$ be a B-free resolution. Then $A \otimes_B P_* \to A \otimes_B M$ is an A-free resolution. The isomorphism $\mathrm{Hom}_A(A \otimes_B P_*, N) \cong \mathrm{Hom}_B(P_*, N)$ then induces the asserted isomorphism on passage to cohomology. □

Corollary 15.4. *There are Adams spectral sequences*

$$E_2^{s,t} = \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(ku)_2^\wedge$$

and

$$E_2^{s,t} = \mathrm{Ext}_{A(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(ko)_2^\wedge .$$

Proof. The E_2-term of the Adams spectral sequence for ku is

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ku), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E(1), \mathbb{F}_2) \cong \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

and the E_2-term of the Adams spectral sequence for ko is

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ko), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//A(1), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) ,$$

in both cases by the change-of-rings isomorphism. □

Corollary 15.5. *There is an exact sequence of $A(1)$-modules*

$$0 \to \Sigma^{12}\mathbb{F}_2 \xrightarrow{\eta} \Sigma^7 A(1)//A(0) \xrightarrow{\partial_3} \Sigma^4 A(1) \xrightarrow{\partial_2} \Sigma^2 A(1) \xrightarrow{\partial_1} A(1)//A(0) \xrightarrow{\epsilon} \mathbb{F}_2 \to 0 .$$

Proposition 15.6. $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0, v_1)$ *where h_0 in bidegree $(s,t) = (1,1)$ is dual to Sq^1 and v_1 in bidegree $(s,t) = (1,3)$ is dual to Q_1.*

The E_2-term of the Adams spectral sequence for ku is displayed in Figure 38. There is no room for differentials, and the permanent cycles h_0 and v_1 detect 2 and u, respectively, in $\pi_*(ku)_2^\wedge = \mathbb{Z}_2[u]$.

Proposition 15.7. $\mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0, h_1, v, w_1)/(h_0h_1, h_1^3, h_1v, v^2 - h_0^2w_1)$ *where h_0 in bidegree $(s,t) = (1,1)$ is dual to Sq^1, where h_1 in bidegree $(s,t) = (1,2)$ is dual to Sq^2, v is in bidegree $(s,t) = (3,7)$ and w_1 is in bidegree $(s,t) = (4,12)$.*

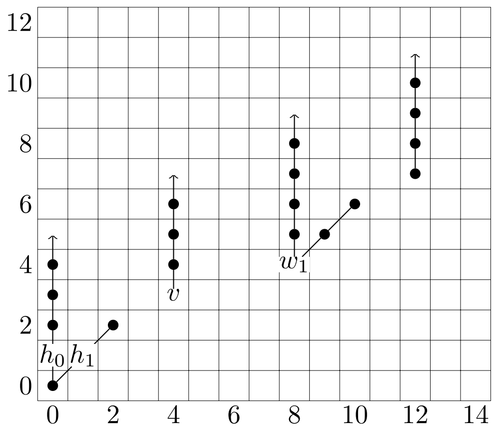


FIGURE 39. The Adams spectral sequence for ko

Proof. The central extension

$$E(Q_1) \to A(1) \to E(Sq^1, Sq^2)$$

of augmented algebras leads to a Cartan–Eilenberg spectral sequence

$$E_2^{p,q,*} = \mathrm{Ext}^{p,*}_{E(Sq^1,Sq^2)}(\mathbb{F}_2, \mathrm{Ext}^{q,*}_{E(Q_1)}(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \mathrm{Ext}^{p+q,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$$

where the $E(Sq^1, Sq^2)$-module structure on $\mathrm{Ext}^*_{E(Q_1)}(\mathbb{F}_2, \mathbb{F}_2) = P(h_{01})$ is trivial. Hence the E_2-term can be written as

$$E_2^{*,*,*} = P(h_0, h_1) \otimes P(h_{01})$$

with h_0 in bidegree $(p, q, t) = (1, 0, 1)$ dual to Sq^1, h_1 in bidegree $(p, q, t) = (1, 0, 2)$ dual to Sq^2 and h_{01} in bidegree $(p, q, t) = (0, 1, 3)$ dual to Q_1.

There are differentials $d_2(h_{01}) = h_0h_1$, so that

$$E_3^{*,*,*} = P(h_0, h_1)/(h_0h_1) \otimes P(h_{01}^2)$$

and $d_3(h_{01}^2) = h_1^3$, so that

$$E_4^{*,*,*} = P(h_0, h_1, v, w_1)/(h_0h_1, h_1^3, h_1v, v^2 - h_0^2w_1)$$

with $v = h_0h_{01}^2$ and $w_1 = h_{01}^4$. ((Justify the differentials with cobar calculations?)) Then $E_4 = E_\infty$ for degree reasons, and there is no room for multiplicative extensions between the E_∞-term and $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$. □

The E_2-term of the Adams spectral sequence for ko is displayed in Figure 39. There is no room for differentials, and the permanent cycles h_0, h_1, v and w_1 detect 2, η, α and β, respectively, in $\pi_*(ko)^\wedge_2 = \mathbb{Z}_2[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$.

The unit map $d\colon S \to ko$ induces a ring homomorphism $d_*\colon \pi_*(S)^\wedge_2 \to \pi_*(ko)^\wedge_2$ that takes $\eta \in \pi_1(S)^\wedge_2$ (detected by h_1, dual to the indecomposable Sq^2 in $\mathscr{A}$) to $\eta \in \pi_1(ko)^\wedge_2$ (detected by h_1, dual to the indecomposable Sq^2 in $A(1)$), hence also maps $\eta^2 \in \pi_2(S)^\wedge_2$ to $\eta^2 \in \pi_2(ko)^\wedge_2$. This is the KO-theory d-invariant. The classes α and β are of infinite (additive) order, hence cannot be in the image of the finite groups $\pi_4(S)^\wedge_2$ and $\pi_8(S)^\wedge_2$. However, a calculation of maps of $\mathscr{A}$-module resolutions shows that the homomorphism $d_*\colon \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ of Adams E_2-terms for S and ko is an isomorphism in the bidegrees $(t - s, s) = (8k + 1, 4k + 1)$ and $(t - s, s) = (8k + 2, 4k + 2)$ with $k \geq 0$. Hence the permanent cycles P^kh_1 and $h_1P^kh_1$ in the Adams spectral sequence for S map to the survivors $h_1w_1^k$ and $h_1^2w_1^k$ in the Adams spectral sequence for ko. It follows that there are nonzero classes μ_{8k+1} and μ_{8k+2} in $\pi_*(S)^\wedge_2$ that map to $\eta\beta^k$ and $\eta^2\beta^k$, respectively, in $\pi_*(ko)^\wedge_2$. For instance, $\mu_1 = \eta$, $\mu_2 = \eta^2$, $\mu_9 = \mu$ and $\mu_{10} = \eta\mu$, in the notation previously introduced in $\pi_*(S)^\wedge_2$. In general, $\eta\mu_{8k+1} = \mu_{8k+2}$.

((Discuss map $c\colon ko \to ku$ mapping $h_0 \mapsto h_0$, $h_1 \mapsto 0$, $v \mapsto h_0h_1^2$ and $w_1 \mapsto v_1^4$. Hence $v \mapsto 2u^2$ and $w_1 \mapsto u^4$ in homotopy.))

((After discussing the dual Steenrod algebra, and the calculation of $H_*(ku)$ and $H_*(ku)$, give alternative proof with $A(1)_*$-comodule algebra resolution $\mathbb{F}_2 \to E(\xi_1^2, \bar{\xi}_2) \otimes P(x_2, x_3)$, with $d(\xi_1^2) = x_2$ and $d(\bar{\xi}_3) = x_3$.))

15.3. **Adams vanishing.** The subalgebra $A(1)$ inherits the structure of a cocommutative Hopf algebra from $\mathscr{A}$, with the restricted coproduct and conjugation, so that the category of $A(1)$-modules has a symmetric monoidal tensor product given by the diagonal $A(1)$-action.

We start with an easy but not optimal vanishing estimate.

Lemma 15.8. *Let M be connective $A(1)$-module that is free as an $A(0)$-module. Then $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ for $t - s < s$.*

Proof. The claim is clear for $s = 0$, since M is concentrated in degrees $* \geq 0$. We prove the claim for $s \geq 1$ by induction.

Note that $A(1)//A(0) = \mathbb{F}_2\{1, Sq^2, Sq^3, Sq^2Sq^3\}$ is concentrated in degrees 0, 2, 3 and 5. The $A(1)$-module action on M induces a short exact sequence

$$0 \to \Sigma^2 K \longrightarrow A(1) \otimes_{A(0)} M \longrightarrow M \to 0$$

of $A(1)$-modules, where also K is connective. Here $A(1) \otimes_{A(0)} M \cong A(1)//A(0) \otimes M$ as $A(1)$-modules, by the untwisting isomorphism [[in the relative case for $A(0) \subset A(1)$]]. Furthermore, $A(1)//A(0) \otimes M$ is a direct sum of suspensions of $A(1)//A(0) \otimes A(0) \cong A(0) \otimes A(1)//A(0)$, as an $A(0)$-module, and the latter $A(0)$-module is free. Hence $A(1) \otimes_{A(0)} M$ is free as an $A(0)$-module, so that $\Sigma^2 K$ is stably free (and projective) as an $A(0)$-module. It follows that K is free as an $A(0)$-module.

Consider the long exact sequence

$$\dots \to \mathrm{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{A(1)}(A(1) \otimes_{A(0)} M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \to \dots .$$

Here $\mathrm{Ext}^{s,t}_{A(1)}(A(1) \otimes_{A(0)} M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2)$. Since M is free as an $A(0)$-module, $\mathrm{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2) = 0$ for $s \geq 1$, so that the connecting homomorphism δ in the long exact sequence above is surjective. Furthermore, $\mathrm{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \cong \mathrm{Ext}^{s-1,t-2}_{A(1)}(K, \mathbb{F}_2)$ is 0 for $(t-2) - (s-1) < s - 1$ by the inductive hypothesis, i.e., for $t - s < s$. Hence $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ for $t - s < s$, as asserted. $\square$

((Can we get vanishing also for $t - s = s$ when $s = 3$? If so, we may use $\epsilon'(s) = 2$ for $s \equiv 3 \mod 4$, $\epsilon''(s) = 1$ and 2 for $s \equiv 0$ and $3 \mod 4$, and $\epsilon(s) = 3$ and 2 for $s \equiv 0$ and $1 \mod 4$, in the following results.))

Proposition 15.9. *Let $\epsilon'(s) = 0$, 1, 2 and 3 for $s \equiv 0, 1, 2$ and $3 \mod 4$, respectively, and let M be a connective $A(1)$-module that is free as an $A(0)$-module. Then $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ for $t - s < 2s - \epsilon'(s)$.*

Proof. As remarked above, we may assume that this has been proved for $0 \leq s \leq 3$. We prove the claim for $s \geq 4$ by induction.

We tensor the exact sequence from Corollary 15.5 with M, to obtain an exact sequence

$$0 \to \Sigma^{12} M \xrightarrow{1\otimes\eta} \Sigma^7 A(1)//A(0) \otimes M \xrightarrow{1\otimes\partial_3} \Sigma^4 A(1) \otimes M \xrightarrow{1\otimes\partial_2} \Sigma^2 A(1) \otimes M \xrightarrow{1\otimes\partial_1} A(1)//A(0) \otimes M \xrightarrow{1\otimes\epsilon} M \to 0$$

of $A(1)$-modules. It splits into four short exact sequences

$$\begin{aligned}
&0 \to \mathrm{im}(1 \otimes \partial_1) \longrightarrow A(1)//A(0) \otimes M \longrightarrow M \to 0 \\
&0 \to \mathrm{im}(1 \otimes \partial_2) \longrightarrow \Sigma^2 A(1) \otimes M \longrightarrow \mathrm{im}(1 \otimes \partial_1) \to 0 \\
&0 \to \mathrm{im}(1 \otimes \partial_3) \longrightarrow \Sigma^4 A(1) \otimes M \longrightarrow \mathrm{im}(1 \otimes \partial_2) \to 0 \\
&0 \to \Sigma^{12} M \longrightarrow \Sigma^7 A(1)//A(0) \otimes M \longrightarrow \mathrm{im}(1 \otimes \partial_3) \to 0
\end{aligned}$$

of $A(1)$-modules, which induce long exact sequences for $\mathrm{Ext}^{*,*}_{A(1)}(-, \mathbb{F}_2)$. By the untwisting isomorphism, $A(1)//A(0) \otimes M \cong A(1) \otimes_{A(0)} M$, and since M is free as an $A(0)$-module, $\mathrm{Ext}^{s,t}_{A(1)}(A(1)//A(0) \otimes M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2)$ is 0 for all $s \geq 1$. Likewise, $A(1) \otimes M$ is free as an $A(1)$-module, so $\mathrm{Ext}^{s,t}_{A(1)}(A(1) \otimes M, \mathbb{F}_2)$ is 0 for all $s \geq 1$. Hence there is a chain of surjections

$$\begin{aligned}
\mathrm{Ext}^{s-4,t-12}_{A(1)}(M, \mathbb{F}_2) = \mathrm{Ext}^{s-4,t}_{A(1)}(\Sigma^{12} M, \mathbb{F}_2) &\xrightarrow{\delta} \mathrm{Ext}^{s-3,t}_{A(1)}(\mathrm{im}(1 \otimes \partial_3), \mathbb{F}_2) \\
&\xrightarrow{\delta} \mathrm{Ext}^{s-2,t}_{A(1)}(\mathrm{im}(1 \otimes \partial_2), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s-1,t}_{A(1)}(\mathrm{im}(1 \otimes \partial_1), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2)
\end{aligned}$$

for all $s \geq 4$.

By induction, we know that $\mathrm{Ext}^{s-4,t-12}_{A(1)}(M, \mathbb{F}_2) = 0$ for $(t-12) - (s-4) < 2(s-4) - \epsilon'(s-4)$, or equivalently, for $t - s < 2s - \epsilon'(s)$. This completes the inductive step. $\square$

Theorem 15.10. *Let $\epsilon''(s) = 2,\ 1,\ 2$ and 3 for $s \equiv 0,\ 1,\ 2$ and $3 \mod 4$, respectively, and let M be a connective $\mathscr{A}$-module that is free as an $A(0)$-module. Then $\operatorname{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) = 0$ for $t - s < 2s - \epsilon''(s)$.*

Proof. Since M is connective, it is clear that $\operatorname{Ext}_{\mathscr{A}}^{0,t}(M, \mathbb{F}_2) = 0$ for $t < 0$, which is stronger than the claim for $s = 0$. We prove the claim for $s \geq 1$ by induction on s. The function ϵ'' is chosen so that $\epsilon'(s) \leq \epsilon''(s)$ and $\epsilon''(s-1) - 1 \leq \epsilon''(s)$ for all $s \geq 1$.

Note that $\mathscr{A}//A(1) = \mathbb{F}_2\{1, Sq^4, \dots\}$ with the remaining generators in degrees $* \geq 4$. The $\mathscr{A}$-module action on M induces a short exact sequence

$$0 \to \Sigma^4 L \longrightarrow \mathscr{A} \otimes_{A(1)} M \longrightarrow M \to 0$$

of $\mathscr{A}$-modules, where L is connective. Hence there is a long exact sequence

$$\cdots \to \operatorname{Ext}_{\mathscr{A}}^{s-1,t}(\Sigma^4 L, \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{A(1)} M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t}(\Sigma^4 L, \mathbb{F}_2) \to \dots .$$

Here $\operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{A(1)} M, \mathbb{F}_2) \cong \operatorname{Ext}_{A(1)}^{s,t}(M, \mathbb{F}_2)$ is 0 for $t - s < 2s - \epsilon'(s)$, by the previous proposition. By induction, $\operatorname{Ext}_{\mathscr{A}}^{s-1,t}(\Sigma^4 L, \mathbb{F}_2) = \operatorname{Ext}_{\mathscr{A}}^{s-1,t-4}(L, \mathbb{F}_2)$ is 0 for $(t-4) - (s-1) < 2(s-1) - \epsilon''(s-1)$, or equivalently, for $t - s < 2s + 1 - \epsilon''(s-1)$. If $t - s < 2s - \epsilon''(s)$ then both inequalities are satisfied, which implies that $\operatorname{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) = 0$. This completes the inductive step. □

Theorem 15.11 (Adams vanishing (weak form))**.** *Let $\epsilon(s) = 4,\ 3,\ 2$ and 3 for $s \equiv 0,\ 1,\ 2$ and $3 \mod 4$, respectively. Then $\operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t - s < 2s - \epsilon(s)$.*

Proof. Define an $\mathscr{A}$-module M by the short exact sequence

$$0 \to \Sigma^2 M \to \mathscr{A}//A(0) \to \mathbb{F}_2 \to 0 .$$

Recall the basis for $\mathscr{A} = \mathbb{F}_2\{Sq^I\}$ given by the admissible monomials Sq^I, where $I = (i_1, \dots, i_\ell)$ with $i_u \geq 2i_{u+1}$ for each $1 \leq u < \ell$, and $i_\ell \geq 1$. The admissible monomials with $i_\ell \geq 2$, including the empty monomial $I = ()$, give a basis for $\mathscr{A}$ as a free right $A(0)$-module, hence also for $\mathscr{A}//A(0)$ as $\mathbb{F}_2$-vector space. The nonempty admissible monomials with $i_\ell \geq 2$ then give a basis for $\Sigma^2 M$. In particular, M is connective. Note now that M is free as a left $A(0)$-module. A basis is given by the Sq^I with I admissible, $i_1 = 2k$ even and $i_\ell \geq 2$, in view of the Adem relation $Sq^1 Sq^{2k} = Sq^{2k+1}$.

Consider the long exact sequence

$$\cdots \to \operatorname{Ext}_{\mathscr{A}}^{s-1,t}(\Sigma^2 M, \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{A(0)} \mathbb{F}_2, \mathbb{F}_2) \to \dots .$$

Here $\operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A} \otimes_{A(0)} \mathbb{F}_2, \mathbb{F}_2) \cong \operatorname{Ext}_{A(0)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is 0 for $t - s \neq 0$. Furthermore, $\operatorname{Ext}_{\mathscr{A}}^{s-1,t}(\Sigma^2 M, \mathbb{F}_2) = \operatorname{Ext}_{\mathscr{A}}^{s-1,t-2}(M, \mathbb{F}_2)$ is 0 for $(t-2) - (s-1) < 2(s-1) - \epsilon''(s-1)$, or equivalently, for $t - s < 2s - 1 - \epsilon''(s-1)$. We have defined $\epsilon(s) = \epsilon''(s-1) + 1$, hence $\operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t - s < 2s - \epsilon(s)$, as asserted. □

Remark 15.12. With more work, Adams (1966?) proved that one may deduce the same conclusion with $\epsilon(s) = 1,\ 1,\ 2$ and 3 for $s \equiv 0,\ 1,\ 2$ and $3 \mod 4$, respectively, which is the optimal result for $s \geq 1$.

((Can the optimal result be deduced from periodicity and the low-dimensional calculations?))

15.4. **Adams operations.** For each natural number r, Adams (1962) defined natural operations $\psi^r \colon KO(X) \to KO(X)$ and $\psi^r \colon KU(X) \to KU(X)$. For a sum of line bundles, $E = L_1 \oplus \dots \oplus L_k$, the Adams operation is given by the sum of tensor powers $\psi^r(E) = L_1^{\otimes r} \oplus \dots \oplus L_k^{\otimes r}$. This determines its behavior on general vector bundles by naturality and the splitting principle. A recursive construction can be given in terms of exterior powers $\Lambda^i(E)$ of vector bundles, using Newton's identities, by the formula

$$-\psi^r(E) = \sum_{i=1}^{r-1} (-1)^i \Lambda^i(E) \otimes \psi^{r-i}(E) + (-1)^r r \Lambda^r(E) .$$

The resulting operation is additive and multiplicative, hence extends over the group completion, to ring operations as indicated above. The real and complex Adams operations are compatible under complexification.

The Adams operations do not commute with the Bott periodicity isomorphisms. In the complex case, the Bott isomorphism $\widetilde{KU}(X) \cong \widetilde{KU}(\Sigma^2 X)$ is induced by the product with the generator $u = 1 - H$ of $\widetilde{KU}(S^2)$, where $KU(S^2) = \mathbb{Z}\{1, H\}$ is generated by the isomorphism classes 1 and H of the trivial and the canonical (Hopf) complex line bundles over $S^2 = \mathbb{C}P^1$, respectively. Here $H + H = 1 + H^2$, so $u^2 = (1 - H)^2 = 0$. The complex Adams operation ψ^r maps the generator u to

$$\psi^r(u) = \psi^r(1 - H) = 1 - H^r = 1 - (1 - u)^r = 1 - (1 - ru) = ru ,$$

i.e., acts by multiplication by r on $\widetilde{KU}(S^2)$. To extend the Adams operation to the graded groups $KU^n(X) = \widetilde{KU}(\Sigma^m X)$, where $n+m = 2k$, we must localize by inverting r, and define ψ^r on $KU^n(X)[1/r]$ as $(1/r^k)\psi^r$ on $\widetilde{KU}(\Sigma^m X)[1/r]$. The result is a map of ring spectra $\psi^r \colon KU[1/r] \to KU[1/r]$, which restricts to a map of connective ring spectra $\psi^r \colon ku[1/r] \to ku[1/r]$. At the level of homotopy groups, $\psi^r(u^k) = r^k u^k$ in degree $2k$, for all integers k. Similarly, the real Adams operation induces ring spectrum maps $\psi^r \colon KO[1/r] \to KO[1/r]$ and $\psi^r \colon ko[1/r] \to ko[1/r]$. If we complete at a fixed prime p, then $\psi^r \colon ko^\wedge_p \to ko^\wedge_p$ and $\psi^r \colon ku^\wedge_p \to ku^\wedge_p$ are defined for all r that are prime to p. For instance, when $p = 2$, ψ^r is defined for all odd r.

The natural numbers prime to p are dense in the topological group $\mathbb{Z}_p^\times$ of p-adic units, and it is possible to define p-complete Adams operations $\psi^r \colon KU^\wedge_p \to KU^\wedge_p$ for all p-adic units $r \in \mathbb{Z}_p^\times$. This defines actions through E_∞ ring spectrum maps of $\mathbb{Z}_p^\times$ on $KU^\wedge_p$ and $ku^\wedge_p$, with $r \in \mathbb{Z}_p^\times$ acting by $\psi^r(u) = ru$ in homotopy. In particular, ψ^{-1} acts as complex conjugation on KU and ku, taking a complex vector bundle to the same real vector bundle but with the opposite complex structure. There are compatible actions on $KO^\wedge_p$ and $ko^\wedge_p$, with $\psi^r(\alpha) = r^2\alpha$ and $\psi^r(\beta) = r^4\beta$. In this case ψ^{-1} acts as the identity.

15.5. **The image-of-J spectrum.** Let all spectra be implicitly completed at 2. The Adams operation $\psi^3 \colon ko \to ko$ is compatible with the unit map $d \colon S \to ko$, hence the latter lifts to a unit map

$$S \longrightarrow ko^{h\psi^3} = \mathrm{hoeq}(\psi^3, 1 \colon ko \to ko)$$

to the homotopy fixed points of ψ^3 acting on ko. Here $ko^{h\psi^3}$ is an E_∞ ring spectrum, and additively there is a homotopy (co-)fiber sequence

$$\Sigma^{-1} ko \longrightarrow ko^{h\psi^3} \longrightarrow ko \xrightarrow{\psi^3 - 1} ko\,.$$

The unit map $d \colon S \to ko$ is 3-connected, in the sense that $\pi_i(S) \to \pi_i(ko)$ is an isomorphism for $i \geq 2$, and is surjective for $i = 3$. Hence $\psi^3 - 1$ induces the zero homomorphism in degrees $i \leq 3$, so the unit map $S \to ko^{h\psi^3}$ is not an equivalence in low degrees. We correct for this in the following definition. Let j be the E_∞ ring spectrum defined by the right hand pullback square in the following commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "e"] \arrow[d] & j \arrow[r] \arrow[d] & ko^{h\psi^3} \arrow[d] \\
P^2 S \arrow[r, equal] & P^2 S \arrow[r] & P^2(ko^{h\psi^3})
\end{tikzcd}$$

Here $P^2 X$ denotes the second Postnikov section of X, obtained by attaching cells (in the category of E_∞ ring spectra) to kill $\pi_i(X)$ for $i \geq 3$. There is then a homotopy (co-)fiber sequence

$$\Sigma^{-1} bspin \xrightarrow{\partial} j \to ko \xrightarrow{\psi^3 - 1} bspin\,.$$

Here $\psi^3 - 1$ maps $\alpha\beta^k$ to $3^{2+4k} - 1$ times $\alpha\beta^k$, which is 8 times an odd number, for all $k \geq 0$. Likewise it maps β^k to $3^{4k} - 1$ times β^k, which has 2-valuation $4 + v_2(k)$ for all $k \geq 1$. In other words, $\psi^3 - 1$ multiplies by $16k$ in degree $8k$, up to multiplication and division by odd factors.

We can use this to calculate the homotopy groups of the connective E_∞ ring spectrum $j = j^\wedge_2$:

$$\pi_i(j) = \begin{cases} \mathbb{Z}_2\{\iota\} & \text{for } i = 0, \\ \mathbb{Z}/2\{\eta\} & \text{for } i = 1, \\ \mathbb{Z}/2\{\eta^2\} & \text{for } i = 2, \\ \mathbb{Z}/8\{\nu\} & \text{for } i = 3, \\ 0 & \text{for } i \equiv 4, 5, 6 \mod 8, \\ \mathbb{Z}_2/16k\{\rho_{8k-1}\} & \text{for } i = 8k-1, \\ \mathbb{Z}/2\{\eta\rho_{8k-1}\} & \text{for } i = 8k, \\ \mathbb{Z}/2\{\mu_{8k+1}, \eta^2\rho_{8k-1}\} & \text{for } i = 8k+1, \\ \mathbb{Z}/2\{\eta\mu_{8k+1}\} & \text{for } i = 8k+2, \\ \mathbb{Z}/8\{\zeta_{8k+3}\} & \text{for } i = 8k+3. \end{cases}$$

for $k \geq 1$, where $\rho_{8k-1} = \partial(\beta^k)$ and $\zeta_{8k+3} = \partial(\alpha\beta^k)$. (The case $i = 3$ coincides with the case $i = 8k+3$ for $k = 0$.)

The map $e\colon S \to j$ induces a homomorphism $e_*\colon \pi_*(S) \to \pi_*(j)$, called the KO-theory e-invariant. As a consequence of the Adams conjecture (proved by Quillen, by Sullivan, and by Becker–Gottlieb), this homomorphism is split surjective in each degree.

Recall that $H^*(ko) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$ and $H^*(bspin) \cong \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\}$.

Proposition 15.13 ([Dav75], [MM76], [AR05], Bruner)**.** *The lift $\psi^3 - 1\colon ko \to bspin$ induces the homomorphism $Sq^4\colon \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \to \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$, mapping $\Sigma^4\theta$ to θSq^4. It has kernel $\Sigma^8 K$ where*

$$K = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}\,,$$

and cokernel $C = \mathscr{A}//A(2) = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2, Sq^4\}$. Hence there is an $\mathscr{A}$-module extension

$$0 \to \mathscr{A}//A(2) \longrightarrow H^*(j) \longrightarrow \Sigma^7 K \to 0\,.$$

There are precisely two such extensions, and $H^*(j)$ is the nonsplit one. A presentation is

$$H^*(j) = \mathscr{A}\{\iota_0, \iota_7\}/\mathscr{A}\{Sq^1\iota_0, Sq^2\iota_0, Sq^4\iota_0, Sq^8\iota_0 + Sq^1\iota_7, Sq^7\iota_7, (Sq^4Sq^6 + Sq^6Sq^4)\iota_7\}\,.$$

The E_2-term of the Adams spectral sequence for j is shown in Figure 40. In this range, only one pattern of differentials is compatible with the known abutment $\pi_*(j)$, leaving the E_∞-term in Figure 41.

The map $e\colon S \to j$ induces a map

$$e_*\colon \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(H^*(j), \mathbb{F}_2)$$

of Adams spectral sequences, mapping the unit $1 \in E_2^{0,0}$ for S to the generator $1 \in E_2^{0,0}$ for j. Hence the map of E_2-terms is determined by the S-module structure of j and the induced $\operatorname{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$-module structure on the Adams E_2-term for j. In this range, this can be directly calculated, and shows that the map of E_∞-terms is surjective for $0 \le t-s \le 24$, except for $t-s = 15$, when the map of E_∞-terms is trivial.

Proposition 15.14. *The permanent cycles h_0^k for $k \ge 0$, h_1, h_1^2, $h_0^k h_2$ for $0 \le k \le 2$, $h_0^k h_3$ for $0 \le k \le 3$, c_0, h_1c_0, Ph_1, h_1Ph_1, $h_0^kPh_2$ for $0 \le k \le 2$, Pc_0, h_1Pc_0, P^2h_1, $h_1P^2h_1$, $h_0^kP^2h_2$ for $0 \le k \le 2$, (h_1Pd_0,) $h_0^{k+2}i$ for $0 \le k \le 3$ and P^2c_0 in the Adams spectral sequence for S map to (nonzero) survivors in the Adams spectral sequence for j, hence are themselves (nonzero) survivors.*

Corollary 15.15. *h_2h_4 and g are permanent cycles.*

Proof. These classes could only support differentials hitting h_1Pc_0, P^2h_1 or $h_0^kP^2h_2$ for $0 \le k \le 2$, which we have now shown are not the targets of differentials. □

Remark 15.16. In degree $n = 15$ (and more generally, in all degrees $n \equiv 15 \mod 32$) the homomorphism $e_*\colon \pi_n(S) \to \pi_n(j)$ induces a zero homomorphism of E_∞-terms. Nonetheless e_* is split surjective. This is a case of a shift in Adams filtration. There is a class $\rho \in \pi_{15}(S)$ that is represented by $h_0^3h_4$ in Adams filtration $s = 4$, and which maps to a generator of $\pi_{15}(j)$, which is represented in Adams filtration $s = 5$. Once we prove that $\eta\rho$ is represented by Pc_0, so that there is a hidden η-multiplication in the Adams spectral sequence for S, then since $e_*(\eta\rho)$ generates $\pi_{16}(j)$, it is clear that $e_*(\rho)$ must generate $\pi_{15}(j)$.

15.6. **The next fifteen stems.**

Theorem 15.17. (14) *$\pi_{14}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa, \sigma^2\}$, with κ represented by d_0 and σ^2 represented by h_3.*

(15) *$\pi_{15}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\}$, with $\eta\kappa$ represented by h_1d_0 and $\rho = \rho_{15}$ represented by $h_0^3h_4$.*

(16) *$\pi_{16}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\rho, \eta^*\}$, with $\eta\rho$ represented by Pc_0 and $\eta^* = \eta_4$ represented by h_1h_4. ((Check that $\eta\rho \ne 0$.)) ((Is $\sigma\mu = \eta\rho$?))*

(17) *$\pi_{17}(S)^\wedge_2 = \mathbb{Z}/2\{\bar\mu, \eta^2\rho, \nu\kappa, \eta\eta^*\}$, with $\bar\mu = \mu_{17}$ represented by P^2h_1, $\eta^2\rho$ represented by h_1Pc_0, $\nu\kappa$ represented by h_2d_0 and $\eta\eta^*$ represented by $h_1^2h_4$. ((Check that $2\nu\kappa = 0$.))*

(18) *$\pi_{18}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\bar\mu\} \oplus \mathbb{Z}/8\{\nu^*\}$, with $\eta\bar\mu$ represented by $h_1P^2h_1$ and ν^* represented by h_2h_4.*

(19) *$\pi_{19}(S)^\wedge_2 = \mathbb{Z}/8\{\bar\zeta\} \oplus \mathbb{Z}/2\{\bar\sigma\}$, with $\bar\zeta = \zeta_{19}$ represented by P^2h_2 and $\bar\sigma$ represented by c_1.*

(20) *$\pi_{20}(S)^\wedge_2 = \mathbb{Z}/8\{\bar\kappa\}$, with $\bar\kappa$ represented by $g = g_1$.*

(21) *$\pi_{21}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\bar\kappa, \nu\nu^*\}$, with $\eta\bar\kappa$ represented by h_1g and $\nu\nu^*$ represented by $h_2^2h_4$. ((Check that $2\nu\nu^* = 0$, which follows from $\eta^2\bar\kappa \ne 0$.))*

(22) *$\pi_{22}(S)^\wedge_2 = \mathbb{Z}/2\{\eta^2\bar\kappa, \nu\bar\sigma\}$, with $\eta^2\bar\kappa$ represented by Pd_0 and $\nu\nu^*$ represented by h_2c_1. ((Check that $\eta^2\bar\kappa \ne 0$ and that $2\nu\bar\sigma = 0$. The latter follows from $\eta^2\bar\kappa \ne 0$, since then $\eta^3\bar\kappa \ne 0$.))*

(23) *$\pi_{23}(S)^\wedge_2 = \mathbb{Z}/16\{\bar\rho\} \oplus \mathbb{Z}/8\{\nu\bar\kappa\} \oplus \mathbb{Z}/2\{\sigma\eta^*\}$, with $\bar\rho = \rho_{23}$ represented by h_0^2i, $\nu\bar\kappa$ represented by h_2g, $2\nu\bar\kappa$ represented by h_0h_2g, $4\nu\bar\kappa = \eta^3\bar\kappa$ represented by h_1Pd_0, and $\sigma\eta^*$ represented by h_4c_0. ((Check that $\sigma\eta^*$ is represented by h_4c_0.))*

(24) $\pi_{24}(S)^\wedge_2 = \mathbb{Z}/2\{\sigma\bar{\mu}\} \oplus \mathbb{Z}/2\{\eta\sigma\eta^*\}$, *with* $\sigma\bar{\mu}$ *represented by* P^2c_0 *and* $\eta\sigma\eta^*$ *represented by* $h_1h_4c_0$. *((Check that* $\eta\bar{\rho} \neq 0$.*)) ((Is* $\sigma\bar{\mu} = \mu\rho = \eta\bar{\rho}$*?))*
(25) $\pi_{25}(S)^\wedge_2 = \mathbb{Z}/2\{\mu_{25}, \eta^2\bar{\rho}\}$, *with* μ_{25} *represented by* P^3h_1 *and* $\eta^2\bar{\rho}$ *represented by* $h_1P^2c_0$.
(26) $\pi_{26}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\mu_{25}, \nu^2\bar{\kappa}\}$, *with* $\eta\mu_{25}$ *represented by* $h_1P^3h_1$ *and* $\nu^2\bar{\kappa}$ *represented by* h_2^2g.
(27) $\pi_{27}(S)^\wedge_2 = \mathbb{Z}/8\{\zeta_{27}\}$, *with* ζ_{27} *represented by* P^3h_2, $2\zeta_{27}$ *represented by* $h_0P^3h_2$ *and* $4\zeta_{27} = \eta^2\mu_{25}$ *represented by* $h_0^2P^3h_2$.
(28) $\pi_{28}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa^2\}$, *with* κ^2 *represented by* d_0^2.
(29) $\pi_{29}(S)^\wedge_2 = 0$. *((This assumes that the differential* $d_3(r) = h_1d_0^2$ *is known.))*
(30) $\pi_{30}(S)^\wedge_2 = \mathbb{Z}/2\{\theta_4\}$, *with* θ_4 *represented by* h_4^2. *((This assumes that the differentials from* $t-s = 31$ *are known.))*

Alternatively, we might just list $\ker(e_*) \subset \pi_*(S)^\wedge_2$, also known as the cokernel of J. These are the homotopy groups of the homotopy fiber $c = \mathrm{hofib}(e)$. Note that e_* maps both ϵ and $\eta\sigma$ to the generator of $\pi_8(j)$, so $\bar{\nu} = \epsilon + \eta\sigma$ generates $\pi_8(c)$. Here $\eta\bar{\nu} = \nu^3$. ((Is $\nu\nu^* = \sigma^3$?))

((ETC))

16. Topological modular forms

((Calculations involving $A(2)$. Adams periodicity.))

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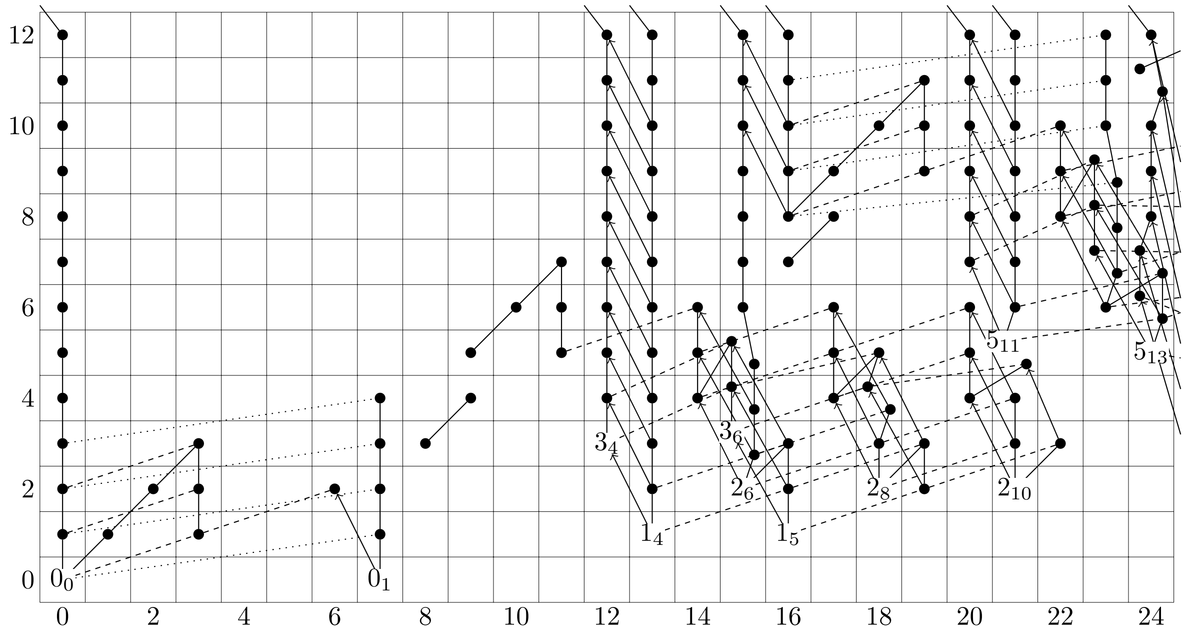

FIGURE 40. Adams (E_2, d_2)-term for j

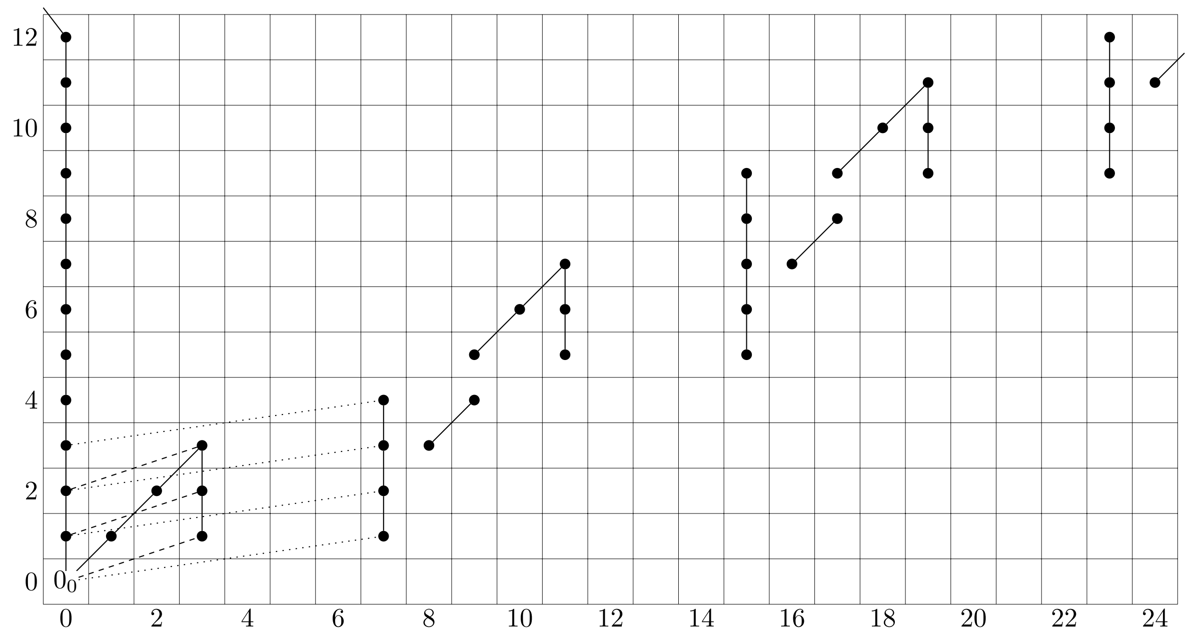

FIGURE 41. Adams E_∞-term for j

n	$\pi_n(c)$	gen.	rep.
6	$\mathbb{Z}/2$	ν^2	h_2^2
8	$\mathbb{Z}/2$	$\bar{\nu}$	h_1h_3
9	$\mathbb{Z}/2$	$\eta\bar{\nu}$	$h_1^2h_3$
14	$(\mathbb{Z}/2)^2$	κ	d_0
		σ^2	h_3^2
15	$\mathbb{Z}/2$	$\eta\kappa$	h_1d_0
16	$\mathbb{Z}/2$	η^*	h_1h_4
17	$(\mathbb{Z}/2)^2$	$\nu\kappa$	h_2d_0
		$\eta\eta^*$	$h_1^2h_4$
18	$\mathbb{Z}/8$	ν^*	h_2h_4
19	$\mathbb{Z}/2$	$\bar{\sigma}$	c_1
20	$\mathbb{Z}/8$	$\bar{\kappa}$	g
21	$(\mathbb{Z}/2)^2$	$\eta\bar{\kappa}$	h_1g
		$\nu\nu^*$	$h_2^2h_4$
22	$(\mathbb{Z}/2)^2$	$\eta^2\bar{\kappa}$	Pd_0
		$\nu\bar{\sigma}$	h_2c_1
23	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\nu\bar{\kappa}$	h_2g
		$\sigma\eta^*$	h_4c_0
24	$\mathbb{Z}/2$	$\eta\sigma\eta^*$	$h_1h_4c_0$
26	$\mathbb{Z}/2$	$\nu^2\bar{\kappa}$	h_2^2g
28	$\mathbb{Z}/2$	κ^2	d_0^2
30	$\mathbb{Z}/2$	θ_4	h_4^2