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On the Computation of Stable Stems

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ABSTRACT. This paper corrects the computation of the stable homotopy groups of spheres in degrees 54 through 64 which were made in [3] using the Atiyah–Hirzebruch spectral sequence. Many relations and Toda brackets as well as the differentials in the Adams spectral sequence in this range of dimensions are also determined.

1 Introduction

All spectra in this paper are localized at the prime two. In [3] the first author computed the first 64 stable stems π_n^S, $0 \leq n \leq 64$, by analyzing the Atiyah–Hirzebruch spectral sequence (AHSS) for the homotopy of the Brown–Peterson spectrum BP:

$$E^2_{q,n} = H_q BP \otimes \pi_n^S \Longrightarrow \pi_* BP. \tag{1.1}$$

Since H_*BP and π_*BP are known [1], the structure of π_n^S was determined by induction on n as explained in detail in [3, Section 1]. From this computation, the differentials in the Adams spectral sequence (ASS)

$$E_2^{p,n} = Ext^p_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)_n \Longrightarrow \pi_*^S \tag{1.2}$$

were deduced in this range of dimensions. The origin of this paper is the observation by the second author, using ASS computations, that $\eta A[54, 2] = 0$ in π_{55}^S, contradicting the computation in [3]. Since the method of computing stable stems in [3] is recursive, this error in the 55 stem results in other errors in the analysis of the 56 to 64 stems on pages 226 to 253 of [3]. In addition, the material in Section 7.6 and the Appendices which refer to the 56 to 64 stems are based upon those computations and consequently also contain errors. In Sections 2 and 3, we use a simultaneous analysis of the ASS and AHSS to correct those computations. These sections replace [3, pp.226–253]. In particular, we use the methods of [3] to analyze the AHSS using the known structure of $\mathbb{E}_2$ of the ASS [5] to solve some of the technical problems which arise. We also deduce the differentials in the ASS. This method of computation is easier and more accurate than computing solely in either the AHSS or the ASS. We include new Toda

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bracket constructions for the homotopy classes in this range. In Section 4, we give the table of leaders in this range which summarizes these computations. In Section 5, we give tables which summarize the values of these stems as well as the multiplicative relations given by multiplication by η, ν and σ. In Section 6, we depict the structure of the ASS in this range which follows from the values of the stable stems which were computed in Sections 2 and 3. Interestingly, we show that $\pi_{61}^S = 0$ and consequently the Kervaire invariant element θ_5 exists by a straightforward argument. (See [4].)

We use the notation of [3] to describe elements in π_*^S. In particular, $A[n,k]$ denotes the k^{th} element of π_n^S of order two. We also use $\alpha_n = \mu_n, \ \beta_n, \ \gamma_n$ to denote a generator of the image of J in degree $8n+1, \ 8n+3, \ 8n+7$, respectively. We assume that the reader is familiar with the methods of [3] and with the computation there of the first 53 stable stems.

2 Computation of π_N^S, $54 \leq N \leq 59$

We begin by establishing a new nonzero differential $d_5(Ph_5e_0) = Pe_0r$ in degree 56 in the ASS. We then determine the immediate implications of this differential on the values of the 54 to 56 stems. We show how an error was made in [3] where $2A[54,1]$ was determined to be zero instead of its correct value $A[54,2]$. We conclude this section by correcting the computations in [3] of the 57 to 59 stems by the AHSS. The following technical result will be used to analyze a representative of Pe_0r.

LEMMA 2.1 *Let $A[32,2]$ be a representative of q such that $\alpha_1 A[32,2] = 0$. Then $\langle 8\sigma, 2, A[32,2]\rangle$ contains 0.*

PROOF. Observe that $A[8]A[32,2] = \langle \nu,\eta,\nu\rangle A[32,2] \subset \langle \nu,\eta,\nu A[32,2]\rangle = \langle \nu,\eta,\eta A[14]C[20]\rangle \supset \langle \nu,\eta,\eta A[14]\rangle C[20] = 2C[20]^2$. Thus, $A[8]A[32,2] - 2C[20]^2 \in Indet\ \langle \nu,\eta,\nu A[32,2]\rangle \subset \nu\cdot\pi_{37}^S$. Since $A[8]A[32,2] - 2C[20]^2$ lies in Adams filtration nine and all the nonzero elements of $\nu\cdot\pi_{37}^S$ project to nonzero elements of Adams filtration degrees four and five, it follows that

$$A[8]A[32,2] = 2C[20]^2.$$

Since all elements of $\langle 8\sigma, 2, A[32,2]\rangle$ have Adams filtration degree at least ten, this Toda bracket contains $k\eta\gamma_4$. Thus, $\eta\langle 8\sigma,2,A[32,2]\rangle = \langle \eta,8\sigma,2\rangle A[32,2] = \alpha_1 A[32,2] = 0$, and $k=0$.

We use this result to deduce the following new differential in the 56 stem of the ASS.

THEOREM 2.2 *In the ASS, we have the following nonzero differential:*

$$d_5(Ph_5e_0) = Pe_0r.$$

PROOF. We show that Pe_0r is represented by an element of π_{55}^S which is zero. By the computations in [3], the differential of this theorem is the only possible one that can kill Pe_0r. Since $Ph_4 = h_2g$ in $Ext_A(\mathbb{Z}/2, \mathbb{Z}/2)$,

$$\nu C[20] \in \langle \sigma^2, 2, 8\sigma, 2\rangle.$$

By [3, 6.3, 7.20 and Lemma 7.4.7(c)], $\eta A[54,2] = \nu C[20]A[32,2] = \eta A[14]C[20]^2$. In the notation of the ASS, this element is denoted by $\nu\overline{\kappa}q$ and projects to Pe_0r in $E_\infty^{14,55}$. Therefore,

$$\nu C[20]A[32,2] \in \langle \sigma^2, 2, 8\sigma, 2\rangle A[32,2] \subset \langle \sigma^2, 2, \langle 8\sigma, 2, A[32,2]\rangle\rangle \supset \langle \sigma^2, 2, 0\rangle.$$

Thus, Pe_0r is the projection of an element of $Indet\ \langle \sigma^2, 2, \langle 8\sigma, 2, A[32,2]\rangle\rangle = \sigma^2 \cdot \pi_{41}^S + \pi_{15}^S \cdot \left(\pi_8^S \cdot A[32,2] + 8\sigma \cdot \pi_{33}^S\right) = \sigma^2 \cdot \eta\pi_{40}^S + \pi_{15}^S \cdot 2C[20]^2 = 0$.

Combining this new differential with the other differentials in the ASS determined in [3], we see that there is no possibility for a nonzero element in $CokJ_{55}$.

Corollary 2.3 $\pi_{55}^S = \mathbb{Z}/16\ \gamma_6$

This result contradicts the computation in [3] of $CokJ_{55}$ as $\mathbb{Z}/2\ \eta A[54,2]$. The error there is the circular reasoning used on [3, pp. 226, 237] to determine whether $2A[54,1]$ equals $A[54,2]$. Those arguments are in fact inconclusive. By Theorem 2.2, $\eta A[54,2] = 0$, and $A[54,2]M_1$ must bound in the AHSS. As noted on [3, p.228], $2\beta_2 M_1^{19}$ survives to E^{36} and $d^{36}\left(2\beta_2 M_1^{19}\right) = 2A[54,1]M_1$. This is the only possible way that $A[54,2]M_1$ can bound. Thus, $2A[54,1] = A[54,2]$, and we rename $A[54,1]$ as $B[54]$. By [3, 7.15]

$$2B[54] \in \langle \nu, \eta, \eta^2 A[45,2], \eta\rangle. \tag{2.3}$$

By [3, Lemma 1.2.10], $B[54]$ is indecomposable. The resulting modifications in the proofs of [3, Theorems 7.3.4, 7.3.5, 7.4.2 and Lemma 7.4.7(b)] prove the following theorem.

Theorem 2.4

$$\begin{aligned} \pi_{54}^S &= \mathbb{Z}/4\ B[54]\ \oplus\ \mathbb{Z}/2\ \eta A[8]D[45] \\ \pi_{56}^S &= \mathbb{Z}/2\ \nu^2 A[50,2]\ \oplus\ \mathbb{Z}/2\ \eta\gamma_6 \end{aligned}$$

where $B[54]$ *is indecomposable and* $2B[54] = A[14]C[20]^2$.

We have the following leaders remaining in degrees 58, 59 and 60 of the AHSS:
degree 58: $4C[44]M_1^7$, $A[52,1]M_1^3$, $A[52,2]\overline{M}_2$;
degree 59: $\eta^2\gamma_1 M_1^{15}\overline{M}_2^2$, $\sigma A[32,1]M_1^4M_2^2$, $2D[45]M_1^4M_2^2$, $\sigma C[44]M_1^4$, $\nu A[50,1]M_1^3$;
degree 60: $\eta A[23]M_1^{15}\overline{M}_2$, $A[30]\langle M_4\rangle$, $A[36]M_1^6\overline{M}_2^2$, $A[40,1]\overline{M}_2\langle M_3\rangle$, $\eta\sigma C[44]M_1\overline{M}_2$, $2B[54]M_1^3$.

As shown in [3, pp.232, 235],

$$\begin{aligned} d^{10}\left(A[30]\langle M_4\rangle\right) &= \sigma A[32,1]M_1^4M_2^2, \\ d^{18}\left(A[40,1]\overline{M}_2\langle M_3\rangle\right) &= \sigma C[44]M_1^4, \\ d^{22}\left(\eta A[23]M_1^{15}\overline{M}_2\right) &= 2D[45]M_1^4M_2, \\ d^{19}\left(A[36]M_1^6\overline{M}_2^2\right) &= \nu A[50,1]M_1^3. \end{aligned}$$

From the ASS, we see that $\nu B[54] \neq 0$. Thus, there must be a hidden differential on $\beta_2 M_1^{17} M_2$ which lands below the 54 row making $B[54]M_1^2$ a new leader of degree 58.

If $A[52,2]\overline{M}_2$ does not bound then $X = d^6\left(A[52,2]\overline{M}_2\right) \neq 0$, there is no possibility for XM_1 to bound and $\eta X \neq 0$. Moreover, $d^6\left(\eta A[52,2]M_1\overline{M}_2\right) = \eta X M_1$. However, the proof of [3, Lemma 7.4.7] shows that $2C[20]^3 = \eta A[59,2]$ where $A[59,2] = d^6\left(2B[54]M_1^3\right)$. Since $C[20]^2 = d^8\left(\eta A[32,2]M_1\overline{M}_2\right)$, the only possibility for $C[20]^3$ is $B[60] = d^8\left(\eta A[52,2]M_1\overline{M}_2\right)$. Thus, $A[52,2]\overline{M}_2$ must bound. As shown in [3, p.232], $A[52,1]M_1^3$ must bound. If $d^{34}\left(\beta_2 M_1^{17} M_2\right) = A[52,2]\overline{M}_2$ then there would have to be a hidden differential on $\beta_2 M_1^{14}\langle M_3\rangle$, an impossibility. Thus,

$$\begin{aligned} d^{34}\left(\beta_2 M_1^{17} M_2\right) &= A[52,1]M_1^3, \\ d^{36}\left(\eta^2\gamma_1 M_1^{15}\overline{M}_2^2\right) &= A[52,2]\overline{M}_2. \end{aligned}$$

Now the leaders $4C[44]M_1^7$, $B[54]M_1^2$ do not bound, and $A[57] = d^{14}\left(4C[44]M_1^7\right)$, $\nu B[54]$ are nonzero. As on [3, p.233],

$$A[57] \in \langle \sigma, 4C[44], \nu, \eta\rangle. \tag{2.4}$$

The argument on [3, p.234] shows that $2A[57] = 0$. As on [3, p.301], $A[57]$ is indecomposable. In addition, the leader $A[57]M_1$ can not bound, and $\eta A[57] \neq 0$. Since $A[57]M_1^2$ and $A[57]\overline{M}_2$ are d^{14}–boundaries,

$$\nu A[57] = 0. \tag{2.5}$$

Since $\nu^2 B[54]$ will be seen to be nonzero,

$$\sigma A[50,2] = 0. \tag{2.6}$$

We have thus proved the following theorem.

THEOREM 2.5

$$\begin{aligned} \pi_{57}^S &= \mathbb{Z}/2\ A[57] \ \oplus\ \mathbb{Z}/2\ \nu B[54] \ \oplus \mathbb{Z}/2\ \alpha_7 \ \oplus\ \mathbb{Z}/2\ \eta^2\gamma_6 \\ \pi_{58}^S &= \mathbb{Z}/2\ \eta A[57] \ \oplus\ \mathbb{Z}/2\ \eta\alpha_7 \end{aligned}$$

where $A[57]$ *is indecomposable.*

We have the following remaining leaders in degrees 60 and 61 of the AHSS:
degree 60: $\eta\sigma C[44]M_1\overline{M}_2$, $2B[54]M_1^3$, $\eta A[57]M_1$.
degree 61: $A[39,1]M_1M_2M_3$, $2D[45]M_1\langle M_3\rangle$, $\sigma C[44]M_1^2\overline{M}_2$, $\eta A[52,2]M_1\overline{M}_2$.

By the arguments on [3, pp.236,238],

$$d^{14}\left(A[32,1]M_1^8\overline{M}_3\right) = 2D[45]M_1\langle M_3\rangle.$$

Define $A[59,1] = d^8\left(\eta\sigma C[44]M_1\overline{M}_2\right)$. Then

$$A[59,1] \in \langle \eta, \nu, \eta\sigma C[44], \eta\rangle, \tag{2.7}$$
$$A[59,2] \in \langle \eta, \nu, 2B[54]\rangle. \tag{2.8}$$

Thus, $2A[59,2] \in 2\langle \eta, \nu, 2B[54]\rangle = \langle 2, \eta, \nu\rangle 2B[54] = 0$. By the argument on [3, p.236], $2A[59,1] = 0$. Thus, $\eta^2 A[57]$ can not be divisible by two, and $\eta A[57]M_1$ must bound. To continue our analysis of $A[57]$ we require the following analogue of [3, Theorem 2.4.6].

PROPOSITION 2.6 *Let* $d^{14}\left(XM_1^7\right) = Y$ *in the AHSS, and assume that* $\langle \sigma, \nu, X, \eta\rangle$ *is defined in* π_*^S. *Then* $Y \in \langle \sigma, \nu, X, \eta\rangle$.

PROOF. In the notation of [3, Appendix 7], XM_1^7 is represented by

$$\phi = \mu_4 \wedge \mu_2 \wedge X \wedge \mu_1 \ \cup \ \mu_4 \wedge \mu_2 \wedge B_{X\eta} \cup \ \mu_4 \wedge B_{\nu X} \wedge \mu_1 \ \cup \ B_{\langle \sigma, X, \nu\rangle} \wedge \mu_1 \ \cup \mu_4 \wedge B_{\langle \nu, X, \eta\rangle} \ \cup \ \langle \mu_{02}\rangle \wedge X \wedge \mu_1 \ \cup \ \langle \mu_{02}\rangle \wedge B_{X\eta}.$$

Then Y is represented by

$$\partial\phi = \sigma \wedge B_{\langle \nu, X, \eta\rangle} \ \cup \ B_{\langle \sigma, \nu, X\rangle} \wedge \eta \ \cup \ B_{\sigma\nu} \wedge B_{X\eta} \in \langle \sigma, \nu, X, \eta\rangle.$$

Since $CokJ_{55} = 0$ and $CokJ_{49} = 0$, $\langle \sigma, \nu, 4C[44], \eta\rangle$ is defined. By Proposition 2.6,

$$\begin{aligned}
A[57] &\in \langle \sigma, \nu, 4C[44], \eta\rangle,\\
A[57] &\in \langle \sigma, 2\nu C[44], 2, \eta\rangle = \langle \sigma, 0, 2, \eta\rangle,\\
A[57] &\in \langle \sigma, Y, \eta\rangle,\\
A[57] &\in \langle \sigma, \eta Y_0, \eta\rangle,\\
A[57] &\in \langle \sigma, Y_0, \eta^2\rangle \textit{ and}\\
\eta A[57] &\in \langle \sigma, Y_0, \eta^3\rangle = \langle \sigma, Y_0, 4\nu\rangle
\end{aligned}$$

where $Y \in CokJ_{48}$ which is contained in the ideal (η, ν). Moreover, $\langle \sigma, \nu Y_1, \eta\rangle$ contains $\langle \sigma, \nu, \eta\rangle Y_1 = 0$. Thus, $\langle \sigma, \nu Y_1, \eta\rangle$ is contained in the indeterminacy of $\langle \sigma, \eta Y_0, \eta\rangle$. Since $\sigma CokJ_{47} = 0$, $\eta A[57] \in \langle \sigma, Y_0, \nu\rangle 4 = 0$ modulo the ideal spanned by σ. Thus, $\eta A[57]$ is divisible by σ. The only possibility is

$$\sigma^2 C[44] = \eta A[57] \tag{2.9}$$

and therefore

$$d^8\left(\sigma C[44]M_1^2\overline{M}_2\right) = \eta A[57]M_1.$$

Observe that $2B[54]M_1^3$ can only bound from below the 19 row, and thus does not bound. Thus $A[59,2] \neq 0$. Assume that $d^r\left(A[39,1]M_1M_2M_3\right) = X$ where X is either $\eta\sigma C[44]M_1\overline{M}_2$ or a nonzero element of π_{60}^S. Since $A[39,1] = \sigma A[32,3]$, the boundary of a representative of $A[32,3]\left(M_1^5M_2M_3 + M_1^9M_2^2\right)$ shows that $\nu A[32,3]\left(M_1^6M_3 + M_1^4M_2^3 + M_1^7M_2^2 + M_1^{13}\right)$ also has boundary X. Thus, $d^{r+4}\left(\nu A[32,3]\left(M_1^5M_2M_3 + M_1^6M_2^3 + M_1^{15}\right)\right) = XM_1^2$. However,

$$\nu A[32,3]\left(M_1^5M_2M_3 + M_1^6M_2^3 + M_1^{15}\right) = d^4\left(A[32,3]\left(M_1^7M_2M_3 + M_1^{10}M_3 + M_1^{11}M_2^2\right)\right).$$

Therefore, XM_1^2 must bound from the 37 row and there is no such leader of degree 65. Thus, $A[39,1]M_1M_2M_3$ must be a boundary. The only possibility is

$$d^{18}\left(\nu A[19]M_1^7M_2^2\langle M_3\rangle\right) = A[39,1]M_1M_2M_3.$$

Thus $\eta\sigma C[44]M_1\overline{M}_2$ can not bound, and $A[59,1] \neq 0$.

LEMMA 2.7 *$A[59,1]$ is indecomposable.*

PROOF. By [3, Lemma 1.3.10 and 7.31] and the ASS, the only possibilities for $A[59,1]$ to be decomposable are (1) $\alpha_1 A[50,1]$, (2) $A[14]A[45,1]$, (3) $A[14]D[45]$, (4) $\gamma_1 C[44]$, (5) $\alpha_2 C[42]$, (6) $A[19]A[40,1]$, (7) $A[19]A[40,2]$ and (8) $C[20]A[39,3]$. We eliminate all of these possibilities.
(1) Note that $\alpha_1 A[50,1] \in \langle \sigma, 16, \eta \rangle A[50,1] = \sigma \langle 16, \eta, A[50,1] \rangle = 0$.
(2) $A[14]A[45,1] \in A[45,1]\langle 2, A[8], \nu, \eta \rangle \subset \langle \langle A[45,1], 2, A[8] \rangle, \nu, \eta \rangle$. Thus, $A[14]A[45,1]$ is a d^6–boundary and can not equal $A[59,1]$.
(3) Since $d^8\left(4D[45]M_1^3\overline{M}_2\right) = A[52,1]M_1^2$ and $d^{12}\left(4\nu M_1^3\overline{M}_2\right) = A[14]$, we can choose choose a representative ξ for $4D[45]M_1^3\overline{M}_2$ whose boundary is $A[52,1]\mu_2 \cup B_{A[52,1]\nu}$. Then $\nu\xi \cup B_{\nu A[52,1]} \wedge \mu_2$ has boundary $A[14]D[45]$. Therefore, $A[14]D[45] \in \langle \nu, A[52,1], \nu \rangle$. It follows that $A[14]D[45] = d^8\left(A[52,1]M_1^4\right)$ which is zero in E^8. Thus, $A[14]D[45]$ can not equal $A[59,1]$.
(4) By [3, 7.8] we have $\gamma_1 C[44] \in \gamma_1 \langle \sigma, [A[31], \nu], \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ \eta A[30] \end{bmatrix} \rangle \subset$

$$\langle \langle \gamma_1, \sigma, [A[31], \nu] \rangle, \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ \eta A[30] \end{bmatrix} \rangle \equiv \langle \langle \gamma_1, \sigma, A[31] \rangle, \eta, \nu \rangle$$

modulo (η, ν). Therefore, $\gamma_1 C[44]$ is boundary in E^6 and can not equal $A[59,1]$.
(5) $\alpha_2 C[42] \in \alpha_2 \langle 2, \eta, \eta\sigma A[32,1] \rangle = \langle \alpha_2, 2, \eta \rangle \eta\sigma A[32,1] = 0$.
(6), (7) For $k = 1, 2$, we have $A[19]A[40,k] = \langle \nu, \eta, \sigma^2 \rangle A[40,k] = \nu \langle \eta, \sigma^2, A[40,k] \rangle = 0$.
(8) $C[20]A[39,3]$ is a d^6–boundary and therefore can not equal $A[59,1]$.

In Lemma 3.1, we will show that $A[59,2] = A[14]A[45,2]$. Thus, we have the following theorem.

THEOREM 2.8 $\pi_{59}^S = \mathbb{Z}/2\ A[59,1] \oplus \mathbb{Z}/2\ A[59,2] \oplus \mathbb{Z}/8\ \beta_7$ *where $A[59,1]$ is indecomposable.*

3 Computation of π_N^S, $60 \le N \le 64$

We continue the analysis of the AHSS of the preceding section to compute the stable stems in degrees 60 through 64. We have the following leaders remaining in degrees 61, 62 and 63:
degree 61: $\eta A[52,2]M_1\overline{M}_2$, $A[59,1]M_1$, $A[59,2]M_1$;
degree 62: $B[38]M_1^2\overline{M}_2\overline{M}_3$, $\nu^2 A[50,2]M_1^3$;
degree 63: $\eta^2\sigma M_1^{21}M_2^2$, $4\beta_2 M_1^{19}\overline{M}_2$, $\gamma_2 M_1^{20}$, $\sigma A[32,1]M_1^6 M_2^2$, $4D[45]M_1^6 M_2$, $\eta A[50,2]M_1^3\overline{M}_2$, $A[8]D[45]M_1^2\overline{M}_2$, $\nu A[50,1]M_1^2 M_2$, $\nu B[54]M_1^3$.

Since $\eta A[52,2]M_1\overline{M}_2$, $A[59,1]M_1$ and $A[59,2]M_1$ can only bound from below the 54 row, $\nu^2 A[50,2]M_1^3$ transgresses.

Assume that $\eta A[59,1] \neq 0$. By (2.7),

$$A[59,1] \in \langle \eta, \nu, \eta C[44], \eta\sigma \rangle.$$

However, $A[59,1] \notin \langle \eta, \nu, \eta^2 C[44], \sigma \rangle$ because that would imply that $A[59,1]M_1 = d^{14}\left(\eta^2 C[44]M_1^5\overline{M}_2\right)$ and $\eta A[59,1] = 0$. Thus, $A[59,1] \in Indet\langle \eta, \nu, \eta^2 C[44], \sigma \rangle$.

The only possibility is $A[59,1] \in \langle \eta, A[50,1], \sigma \rangle$. Then

$$\eta A[59,1]M_1 = d^{10}\left(\eta A[50,1]M_1^3\overline{M}_2\right).$$

Since $\eta A[50,1] = 0$, the element $\eta A[59,1]M_1$ must bound from above the 51 row. The only possibilities are $d^8\left(\nu A[50,1]M_1^2 M_2\right) = \eta A[59,1]M_1$, $d^4\left(\nu B[54,1]M_1^3\right) = \eta A[59,1]M_1$ or $d^8\left(A[8]D[45]M_1^2\overline{M}_2\right) = \eta A[59,1]M_1$. The first possibility can not occur because $\nu A[50,1]M_1^2M_2 = r_{\Delta_1}\left(\nu A[50,1]M_1^3M_2\right)$ while $\eta A[59,1]M_1 \notin$ *Image* r_{Δ_1}. The second possibility implies the $\nu^2 B[54] = \eta A[59,1]$. However,

$$\nu^2 B[54] \in \langle \eta, \nu, \eta \rangle B[54] = \eta \langle \nu, \eta, B[54] \rangle.$$

Then $\langle \nu, \eta, B[54] \rangle$ would contain $A[59,1]$, and $A[59,1]$ would be a d^6-boundary, a contradiction. The third possibility implies that $A[14]D[45] = A[59,1]$ because $d^8\left(A[8]M_1^2\overline{M}_2\right) = \eta A[14]M_1$. This contradicts Lemma 2.7. Therefore, $\eta A[59,1] = 0$. Thus $A[59,1]M_1$ must bound. The only possibility is

$$d^{22}\left(B[38]M_1^2\overline{M}_2\overline{M}_3\right) = A[59,1]M_1.$$

Let $B[60] = d^8\left(\eta A[52,2]M_1\overline{M}_2\right)$. We now state the corrected version of [3, Lemma 7.4.7]. The proof is analogous to that of the original lemma.

LEMMA 3.1 ***(a)*** $\eta^2 A[45,2] = \eta A[14]A[32,2]$ *and*
$\eta A[45,2] \equiv A[14]A[32,2]$ *modulo* $\left(\eta^2 C[44], \eta A[45,1]\right)$.

(b) $2B[54] = A[14]C[20]^2$.

(c) $\nu A[52,2] = 0$.

(d) $A[59,2] = A[14]A[45,2]$.

(e) $B[60] = C[20]^3$ *and* $2B[60] = \eta A[59,2]$.

(f) $\eta B[47] = A[8]C[20]^2$.

Thus, $\pi_{60}^S = \mathbb{Z}/4\; B[60]$, and the remaining leaders of degree 62 are $\nu^2 A[50,2]M_1^3$, $B[60]M_1$ and $2B[60]M_1$. Observe that $2B[60]M_1 = d^8\left(\eta A[52,1]M_1^2\overline{M}_2\right)$ which is zero in E^8. Thus, $2B[60]M_1$ must bound from above the 53 row. The only possibility is

$$d^4\left(\nu B[54]M_1^3\right) = 2B[60]M_1.$$

It follows that

$$\nu^2 B[54] = \eta A[59,2], \tag{3.10}$$
$$A[59,2] \in \langle \nu, \eta, B[54] \rangle, \tag{3.11}$$

and $B[54]M_2$ is homologous to $2B[54]M_1^3$ in E^{36}. In the notation of [3, Appendix 7], $\eta\overline{M}_2$ is represented by

$$\rho = \eta\overline{\mu}_{01} \cup B_{\eta\nu} \wedge \mu_1 \; \textit{and}$$

$$\partial(\rho) \;=\; \eta \wedge B_{\nu\eta} \;\cup\; B_{\eta\nu} \wedge \eta \;\in\; \langle \eta, \nu, \eta \rangle \;=\; \nu^2.$$

It follows that

$$d^6 \left(\eta A[50,2] M_1^3 \overline{M}_2\right) = \nu^2 A[50,2] M_1^3.$$

Now the only remaining leader of degree 62 is $B[60]M_1$ and $d^2(B[60]M_1) = \eta B[60]$. Since $B[60] = C[20]^3$, we have that

$$B[60] \in \langle \nu, \eta, 2, A[14] \rangle C[20]^2 \subset \langle \nu, \eta, 2, A[14]C[20]^2 \rangle = \langle \nu, \eta, 2, 2B[54] \rangle.$$

By an analogue of [3, Theorem 2.4.5(b)], $d^{42}\left(4\beta_2 M_1^{19} \overline{M}_2\right)$ equals an element of $\langle \nu, \eta, 4, B[54] \rangle M_1$. Therefore,

$$d^{42}\left(4\beta_2 M_1^{19} \overline{M}_2\right) = B[60] M_1$$

and $\eta B[60] = 0$.

The following argument shows directly that $\eta B[60] = 0$. All of the triple products below have zero indeterminacy because $\pi_5^S = 0$, $\sigma A[14] = 0$ and $\nu \cdot \pi_{58}^S = 0$.

$$\begin{aligned} \eta B[60] &= \eta C[20]^3 \in \langle \nu^2, 2, A[14] \rangle C[20]^2 = \langle \nu^2, 2, A[14]C[20]^2 \rangle \\ &= \langle \nu^2, 2, 2B[54] \rangle = \langle \langle \eta, \nu, \eta \rangle, 2, 2B[54] \rangle = \eta \langle \nu, \eta, 2, 2B[54] \rangle \\ &= \langle \nu, \eta, \langle 2, 2B[54], \eta \rangle \rangle = k \langle \nu, \eta, \nu^2 A[50,2] \rangle \\ &= k \langle \nu, \eta, \nu \rangle \nu A[50,2] = k A[8] \nu A[50,2] = 0. \end{aligned}$$

Therefore, $\pi_{61}^S = 0$ and

$$\sigma B[54] = 0. \tag{3.12}$$

By [3, Lemma 7.4.7(g)], $\nu A[59,2] = \eta^2 B[60]$ and therefore

$$\nu A[59,2] = 0. \tag{3.13}$$

Thus, we have proved the following theorem.

THEOREM 3.2

$$\begin{aligned} \pi_{60}^S &= \mathbb{Z}/4\; B[60] \textit{ and } 2B[60] = \eta A[59,2] = \nu^2 B[54] \\ \pi_{61}^S &= 0 \end{aligned}$$

where $B[60] = C[20]^3$.

We have the following leaders remaining in degrees 63, 64, 65 and 66:
degree 63: $\eta^2 \sigma M_1^{21} M_1^2$, $\gamma_2 M_1^{20}$, $\sigma A[32,1] M_1^6 M_2^2$, $4D[45] M_1^6 M_2$, $A[8]D[45] M_1^2 \overline{M}_2$, $\nu A[50,1] M_1^2 M_2$;
degree 64: $4C[18] M_1^7 \overline{M}_2 M_2^2 \overline{M}_3$, $2B[34] M_1^5 \overline{M}_2 \overline{M}_3$, $\eta \sigma A[32,1] M_1^5 \langle M_3 \rangle$, $A[52,1] M_1^3 M_2$, $\eta A[57] M_1^3$;
degree 65: $\beta_1 M_1^{20} \overline{M}_3$, $\beta_2 M_1^{14} \overline{M}_2^3$, $2D[45] \langle M_1^4 \rangle \langle M_2^2 \rangle$, $\nu C[44] M_1^3 \langle M_2^2 \rangle$, $\sigma C[44] M_1^4 \overline{M}_2$, $A[59,1] M_2$, $A[59,1] \overline{M}_2$;
degree 66: $A[32,1] M_1^2 \langle M_4 \rangle$, $\eta A[39,3] M_1^3 \overline{M}_2 \langle M_3 \rangle$, $2C[44] M_1^2 M_2^3$, $\eta^2 C[44] M_1^7 \overline{M}_2$, $A[52,1] M_1^7$, $A[52,2] M_1^4 \overline{M}_2$, $\eta A[8] D[45] M_1^3 \overline{M}_2$, $B[60] M_1^3$, $B[60] \overline{M}_2$.

By the arguments on [3, pages 243, 245],

$$\begin{aligned} d^{12}\left(2B[34]M_1^5\overline{M}_2\overline{M}_3\right) &= 4D[45]\left(M_1^6M_2+M_1^2M_3\right), \\ d^{22}\left(4C[18]M_1^7\overline{M}_2M_2^2\overline{M}_3\right) &= \sigma A[32,1]M_1^6M_2^2. \end{aligned}$$

Thus $A[62,1] = d^{54}\left(\eta^2\sigma M_1^{21}M_2^2\right)$, $A[62,2] = d^{10}\left(\nu A[50,1]M_1^2M_2\right)$, $B[62] = d^{40}\left(\gamma_2 M_1^{20}\right)$ and $A[62,3] = d^{10}\left(A[8]D[45]M_1^2\overline{M}_2\right)$ are nonzero. As on [3, p.248],

$$d^{40}\left(\gamma_2 M_1^{18}\overline{M}_2\right) = A[62,3]M_1$$

and thus $A[62,3] = 2B[62]$. By [3, Theorem 2.4.5],

$$A[62,2] \in \langle \nu, \eta, \nu A[50,1], \nu\rangle, \tag{3.14}$$

$$2B[62] \in \langle \eta, \nu, A[8]D[45], \nu\rangle. \tag{3.15}$$

Clearly both $A[62,2]$ and $A[62,3]$ have order two. As on [3, pp.246, 248], $A[62,1]$ represents θ_5, $2A[62,1] = 0$ and $d^{54}\left(\beta_1 M_1^{20}\overline{M}_3\right) = B[64]$ with $2B[64] = \eta^2 A[62,1] \neq 0$. As on [3, p.251],

$$d^{14}\left(A[32,1]M_1^2\langle M_4\rangle\right) = 2D[45]\langle M_1^4\rangle\langle M_2^2\rangle.$$

By the argument on [3, p.246],

$$\nu A[59,1] = 0. \tag{3.16}$$

LEMMA 3.3 *$A[62,1]$, $A[62,2]$ and $B[62]$ are indecomposable.*

PROOF. By [3, Lemma 1.3.10] and the ASS we see that the only possibilities for a decomposable element of of π_{62}^S are (1) $C[18]C[44]$, (2) $A[30]A[32,1]$, (3) $A[30]A[32,2]$ and (4) $A[30]A[32,3]$. We eliminate all of these possibilities.
(1) By [3, 7.8],

$$\begin{aligned} C[18]C[44] &\in C[18]\langle \sigma, [A[31],\nu], \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ \eta A[30] \end{bmatrix}\rangle \\ &\subset \langle\langle C[18],\sigma,[A[31],\nu]\rangle, \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ \eta A[30] \end{bmatrix}\rangle \\ &= \langle\langle C[18],\sigma,A[31]\rangle,\eta,\nu\rangle \end{aligned}$$

Thus, $C[18]C[44]$ is a boundary in E^6 and must therefore be zero.
(2) $A[30]A[32,1] \in \langle \eta, 2, A[30]\rangle A[30] = \eta\langle 2, A[30], A[30]\rangle \subset \eta\cdot\pi_{61}^S = 0$.
(3) Since $A[32,2] = d^{22}\left(2\beta_1 M_1^8\overline{M}_2\right)$ and $d^{16}\left(\beta_1 M_1^8\overline{M}_2\right) = \nu^2 C[20]\overline{M}_2$,

$$A[32,2] \in \langle 2, \nu^2 C[20], \nu, \eta\rangle. \tag{3.17}$$

Therefore,

$$A[30]A[32,2] \in A[30]\langle 2,\nu^2C[20],\nu,\eta\rangle \subset \langle\langle A[30],2,\nu^2C[20]\rangle,\nu,\eta\rangle$$

which is a boundary in E^6 and must therefore be zero.
(4) By [3, 7.5], $A[30]A[32,3] \in A[30]\langle A[19],\sigma,\nu,\eta\rangle \subset \langle\langle A[30],A[19],\sigma\rangle,\nu,\eta\rangle$ which is also a boundary in E^6 and must therefore be zero.

In the notation of [3, Appendix 7], $\nu\langle M_2^2\rangle$ is represented by

$$\begin{aligned}\rho &= \nu\langle\mu_{02}\rangle \ \cup \ B_{\nu\sigma}\wedge\mu_2 \ and \\ \partial(\rho) &= B_{\nu\sigma}\wedge\nu \ \cup \ \nu\wedge B_{\sigma\nu} \ \in \ \langle\nu,\sigma,\nu\rangle \ = \ \sigma^2.\end{aligned}$$

Since $\eta A[57] = \sigma^2 C[44]$, it follows that

$$d^{12}\left(\nu C[44]M_1^3\langle M_2^2\rangle\right) = \eta A[57]M_1^3.$$

The ASS shows that the order of $CokJ_{63}$ is at least eight. Thus, $\eta A[62,1]$, $A[63,1] = d^{24}\left(\eta\sigma A[32,1]M_1^5\langle M_3\rangle\right)$ and $A[63,2] = d^{12}\left(A[52,1]M_1^3M_2\right)$ are nonzero. Since in the ASS $\eta A[62,1]$, $A[63,1]$, $A[63,2]$ has filtration degree 3, 6, 7, respectively, it follows that $2A[63,1] = 2A[63,2] = 0$. By [3, Lemma 1.3.10] and the ASS, the only possibility for $A[63,1]$ or $A[63,2]$ to be decomposable is as $C[18]D[45]$. However, $C[18]D[45] \in \langle\nu,\sigma,2\sigma\rangle D[45] = \nu\langle\sigma,2\sigma,D[45]\rangle \subset \nu\cdot\pi_{61}^S = 0$. Thus $A[63,1]$ and $A[63,2]$ are indecomposable. Since $B[60] = C[20]^3$,

$$\nu B[60] = 0. \tag{3.18}$$

All differentials in our picture of the ASS which originate in degree 65 follow easily from differentials in lower degrees. Consequently, the order of π_{64}^S is 128. Thus, two of the remaining leaders of degree 65 must bound. Since $\beta_2 M_1^{14}\overline{M}_2^3$, $A[59,1]M_2$ can only bound from below the 0 row, 38 row, respectively, neither of these elements can bound. Let $A[64,3] = d^6\left(A[59,1]M_2\right)$ and $A[64,1] = d^{46}\left(\beta_2 M_1^{14}\overline{M}_2^3\right)$. As noted above $\beta_1 M_1^{20}\overline{M}_3$ and $\eta A[62,1]M_1$ can not bound. In the ASS, $d_5(A') = h_1 B_{21}$, $d_4(X_2) = h_2 B_{21}$ and B_{21} represents $A[59,1]$. It follows that the nonbounding infinite cycle h_2A' represents elements of $\langle A[59,1],\eta,\nu\rangle$ and $\langle\eta,A[59,1],\nu\rangle$ while the nonbounding infinite cycle h_1X_2 represents an element of $\langle A[59,1],\nu,\eta\rangle$ which projects to $A[64,2] = d_6\left(A[59,1]\overline{M}_2\right)$ in the AHSS. Thus, $A[59,1]\overline{M}_2$ can not bound. In the ASS, h_3Q_2 is a nonbounding infinite cycle, and thus $\sigma A[57]$ is nonzero. By (2.4),

$$\sigma A[57] \in \sigma\langle 2\sigma, 2C[44],\nu,\eta\rangle \subset \langle\langle\sigma,2\sigma,2C[44]\rangle,\nu,\eta\rangle.$$

Note that $\eta\langle\sigma,2\sigma,2C[44]\rangle = \sigma\langle 2\sigma,2C[44],\eta\rangle = 0$. Thus, $\langle\sigma,2\sigma,2C[44]\rangle$ is either zero or $A[59,1]$. If it equals $A[59,1]$ then

$$A[59,1]M_1 = d^{20}\left(2\sigma M_1^5\nu A[30]M_2^2\right) = 0$$

and $A[59,1]M_1$ would bound from above the 40 row, a contradiction. Thus, $\langle\sigma,2\sigma,2C[44]\rangle = 0$ and $\sigma A[57] \in \langle 0,\nu,\eta\rangle = (\eta)$. Therefore,

$$\sigma A[57] = \eta A[63,1] \tag{3.19}$$

because $A[63,1]M_1$ can only bound from below the 40 row. Now $A[63,2]M_1$ and $\sigma C[44]M_1^4\overline{M}_2$ must bound. Since $A[63,2]M_1$, $\sigma C[44]M_1^4\overline{M}_2$ can only bound from below the 52 row, 44 row, respectively,

$$\begin{aligned}d^{12}\left(\eta\left(A[39,3]+\sigma A[32,1]\right)M_1^3\overline{M}_2\langle M_3\rangle\right) &= \sigma C[44]M_1^4\overline{M}_2, \\ d^{18}\left(\eta^2 C[44]M_1^7\overline{M}_2\right) &= A[63,2]M_1.\end{aligned}$$

Thus, $A[64,1]$, A[64,2], $A[64,3]$, $B[64]$, $\eta A[63,1]$ and $\eta^2 A[62,1]$ are nonzero. By [3, Theorems 2.4.4 and 2.4.6(c)],

$$A[64,2] \in \langle \eta, \nu, A[59,1] \rangle, \tag{3.20}$$
$$A[64,3] \in \langle A[59,1], \eta, \nu \rangle. \tag{3.21}$$

Thus, $2A[64,3] \in 2\langle A[59,1], \eta, \nu \rangle = \langle 2, A[59,1], \eta \rangle \nu = 0$. By (3.20),

$$2A[64,2] \in 2\langle \eta, \nu, A[59,1] \rangle = \langle 2, \eta, \nu \rangle A[59,1] = 0.$$

Since $A[64,1]$ is represented by q_1 of filtration degree 10 in the ASS, $2A[64,1] = 0$.

LEMMA 3.4 *$A[64,1]$, $A[64,3]$ and $B[64]$ are indecomposable.*

PROOF. By [3, Lemma 1.3.10], $A[64,1]$ and $B[64]$ are indecomposable. From the ASS, the only possibilities for $A[64,3]$ to decompose are as (1) $A[32,1]^2$, (2) $A[32,1]A[32,3]$, (3) $A[30]B[34]$ or (4) $A[19]D[45]$. We eliminate all of these possibilities.
(1), (2) For $k = 1,3$, $A[32,1]A[32,k] \in \langle \eta, 2, A[30] \rangle A[32,k] = \eta \langle 2, A[30], A[32,k] \rangle$ which can not equal $A[64,3]$.
(3) $A[30]B[34] \in A[30]\langle A[32,1], 2, \eta \rangle = \langle A[30], A[32,1], 2 \rangle \eta$ which can not equal $A[64,3]$.
(4) $A[19]D[45] \in D[45]\langle \sigma^2, \eta, \nu \rangle = \langle D[45], \sigma^2, \eta \rangle \nu = 0$.

Since $A[64,2]$ projects to $h_0 h_2 (A + A') = d_1^2$, it follows that $A[64,2] - A[32,3]^2$ has Adams filtration greater than 8 and hence equals a multiple of $A[64,1]$. We have thus proved the following theorem.

THEOREM 3.5

$$\begin{aligned}
\pi_{62}^S &= \mathbb{Z}/4\, B[62] \oplus \mathbb{Z}/2\, A[62,1] \oplus \mathbb{Z}/2\, A[62,2] \\
\pi_{63}^S &= \mathbb{Z}/2\, A[63,1] \oplus \mathbb{Z}/2\, A[63,2] \oplus \mathbb{Z}/2\, \eta A[62,1] \oplus \mathbb{Z}/128\, \gamma_7 \\
\pi_{64}^S &= \mathbb{Z}/4\, B[64] \oplus \mathbb{Z}/2\, A[64,1] \oplus \mathbb{Z}/2\, A[64,2] \oplus \mathbb{Z}/2\, A[64,3] \\
&\quad \oplus \mathbb{Z}/2\, \eta A[63,1] \oplus \mathbb{Z}/2\, \eta\gamma_7
\end{aligned}$$

where $2B[64] = \eta^2 A[62,1]$ and $A[62,1]$, $A[62,2]$, $B[62]$, $A[63,1]$, $A[63,2]$, $A[64,1]$, $A[64,3]$, $B[64]$ are indecomposable.

4 Leaders

The following tables of leaders summarize the structure of the AHSS in degrees 54 through 64 as determined in Sections 2 and 3.

54		55		56		57	
$\eta A[47]M_1^3$	$\longleftarrow$	$\eta^2\gamma_1 M_1^{19}$		$A[50,1]M_2$	$\longleftarrow$	$4\beta_1 M_1^7\overline{M}_2^3\overline{M}_3$	
$\eta\sigma C[44]M_1$	$\longleftarrow$	$\nu C[44]M_1M_2$		$\eta A[8]D[45]M_1$	$\longleftarrow$	$\nu^2 D[45]M_1^3$	
$B[54]$	$\longleftarrow$	$\beta_2 M_1^{18}$		$2B[54]M_1$	$\longleftarrow$	$2\beta_2 M_1^{16}\overline{M}_2$	
$2B[54]$	$\longleftarrow$	$\eta^2 A[45,2]M_1\overline{M}_2$		$A[52,1]M_1^2$	$\longleftarrow$	$4D[45]M_1^3M_2$	
$\eta A[8]D[45]$	$\longleftarrow$	$A[8]D[45]M_1$				$A[47]M_1^2\overline{M}_2$	$\longleftarrow$
$\longleftarrow$ $A[36]M_1^2\overline{M}_3$				$\nu^2 A[50,2]$	$\longleftarrow$	$\nu A[50,2]M_1^2$	
$\longleftarrow$ $\nu A[45,1]M_1^3$		$2\sigma C[44]M_1^2$	$\longleftarrow$	$2C[44]M_1^6$		$A[57]$	$\longleftarrow$
$\longleftarrow$ $2\nu D[45]M_1^3$		$\eta A[52,2]M_1$	$\longleftarrow$	$\eta B[47]M_1\overline{M}_2$			
$\longleftarrow$ $A[52,2]M_1$						$\nu B[54]$	$\longleftarrow$
$\longleftarrow$ $A[50,1]M_1^2$							
$\longleftarrow$ $A[50,2]M_1^2$							

58		59		60		61	
$A[52,2]\overline{M}_2$	$\longleftarrow$	$\eta^2\gamma_1 M_1^{15}\overline{M}_2^2$		$\eta A[57]M_1$	$\longleftarrow$	$\sigma C[44]M_1^2\overline{M}_2$	
		$\sigma A[32,1]M_1^4M_2^2$	$\longleftarrow$	$A[30]\langle M_4\rangle$		$A[39,1]M_1M_2M_3$	$\longleftarrow$
$\eta A[57]$	$\longleftarrow$	$A[57]M_1$		$\eta A[59,2]$	$\longleftarrow$	$A[59,2]M_1$	
$\longleftarrow$ $A[40,2]M_1^6\overline{M}_2$		$2D[45]M_1^4M_2$	$\longleftarrow$	$\eta A[23]M_1^{15}\overline{M}_2$		$2D[45]M_1\langle M_3\rangle$	$\longleftarrow$
		$\sigma C[44]M_1^4$	$\longleftarrow$	$A[40,1]\overline{M}_2\langle M_3\rangle$		$A[59,1]M_1$	$\longleftarrow$
		$\nu A[50,1]M_1^3$	$\longleftarrow$	$A[36]M_1^6\overline{M}_2^2$			
$\longleftarrow$ $4C[44]M_1^7$		$A[59,1]$	$\longleftarrow$	$\eta\sigma C[44]M_1\overline{M}_2$			
$A[52,1]M_1^3$	$\longleftarrow$	$\beta_2 M_1^{17}M_2$		$B[60]$	$\longleftarrow$	$\eta A[52,2]M_1\overline{M}_2$	
$\longleftarrow$ $B[54]M_1^2$		$A[59,2]$	$\longleftarrow$	$2B[54]M_1^3$			

<u>62</u>	<u>63</u>	<u>64</u>	<u>65</u>
$B[60]M_1$ ←	$4\beta_2 M_1^{19}\overline{M}_2$	$B[64]$ ←	$\beta_1 M_1^{20}\overline{M}_3$
← $\nu A[19]M_1^7M_2^2\langle M_3\rangle$		$\eta A[57]M_1^3$ ←	$\nu C[44]M_1^3\langle M_2^2\rangle$
$A[62,1]$ ←	$\eta^2\sigma M_1^{21}M_2^2$	$2B[62]M_1$ ←	$2\gamma_2 M_1^{18}\overline{M}_2$
← $A[32,1]M_1^8\overline{M}_3$	$\sigma A[32,1]M_1^6M_2^2$ ←	$4C[18]M_1^7\overline{M}_2M_2^2\overline{M}_3$	$2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$ ←
← $B[38]M_1^2\overline{M}_2\overline{M}_3$	$\eta A[62,1]$ ←	$A[62,1]M_1$	$\sigma C[44]M_1^4\overline{M}_2$ ←
	$4D[45]M_1^6M_2$ ←	$2B[34]M_1^5\overline{M}_2\overline{M}_3$	$A[63,2]M_1$ ←
$B[62]$ ←	$\gamma_2 M_1^{20}$	$A[64,3]$ ←	$A[59,1]M_2$
$2B[62]$ ←	$A[8]D[45]M_1^2\overline{M}_2$	$\eta A[63,1]$ ←	$A[63,1]M_1$
$A[62,3]$ ←	$\nu A[50,1]M_1^2M_2$	$\eta^2 A[62,1]$ ←	$\eta A[62,1]M_1$
$\eta A[59,2]M_1$ ←	$\nu B[54]M_1^3$	$A[64,2]$ ←	$A[59,1]\overline{M}_2$
$\nu^2 A[50,2]M_1^3$ ←	$\eta A[50,2]M_1^3\overline{M}_2$	$A[64,1]$ ←	$\beta_2 M_1^{14}\overline{M}_2^3$
	$A[63,1]$ ←	$\eta\sigma A[32,1]M_1^5\langle M_3\rangle$	
	$A[63,2]$ ←	$A[52,1]M_1^3M_2$	

<u>66</u>

$A[52,2]M_1^4\overline{M}_2$

$B[60]M_2,\ B[60]M_1^3$

$A[52,1]M_1^7$

← $A[32,1]M_1^2\langle M_4\rangle$

← $\eta\,(A[39,3]+\sigma A[32,1])\,M_1^3\overline{M}_2\langle M_3\rangle$

← $\eta^2 C[44]M_1^7\overline{M}_2$

$2C[44]M_1^2M_2^3$

$\eta A[8]D[45]M_1^3\overline{M}_2$

$\eta A[57]M_1\overline{M}_2$

$A[62,1]M_1^2,\ A[62,2]M_1^2$

$B[64]M_1,\ A[64,1]M_1,\ A[64,3]M_1,\ \eta A[63,1]M_1$

5 Group Structure and Multiplicative Relations

The first table gives the abelian group structure of the stable stems in degrees 54 through 64. The second table gives the structure of $CokJ_*$ as a module over

(η, ν, σ) in these degrees. It includes a column labeled "DEC." in which we enter D if the element is decomposable or I if the element is indecomposable. All the information in these tables was determined in Sections 2 and 3.

DEGREE	*STABLE STEM*
54	$\mathbb{Z}/4\, B[54] \oplus \mathbb{Z}/2\, \eta A[8]D[45]$
55	$\mathbb{Z}/16\, \gamma_6$
56	$\mathbb{Z}/2\, \nu^2 A[50,2] \oplus \mathbb{Z}/2\, \eta\gamma_6$
57	$\mathbb{Z}/2\, A[57] \oplus \mathbb{Z}/2\, \nu B[54] \oplus \mathbb{Z}/2\, \alpha_7 \oplus \mathbb{Z}/2\, \eta^2\gamma_6$
58	$\mathbb{Z}/2\, \eta A[57] \oplus \mathbb{Z}/2\, \eta\alpha_7$
59	$\mathbb{Z}/2\, A[59,1] \oplus \mathbb{Z}/2\, A[59,2] \oplus \mathbb{Z}/8\, \beta_7$
60	$\mathbb{Z}/4\, B[60]$
61	0
62	$\mathbb{Z}/4\, B[62] \oplus \mathbb{Z}/2\, A[62,1] \oplus \mathbb{Z}/2\, A[62,2]$
63	$\mathbb{Z}/2\, A[63,1] \oplus \mathbb{Z}/2\, A[63,2] \oplus \mathbb{Z}/2\, \eta A[62,1] \oplus \mathbb{Z}/128\, \gamma_7$
64	$\mathbb{Z}/4\, B[64] \oplus \mathbb{Z}/2\, A[64,1] \oplus \mathbb{Z}/2\, A[64,2] \oplus \mathbb{Z}/2\, A[64,3]$ $\oplus \mathbb{Z}/2\, \eta A[63,1] \oplus \mathbb{Z}/2\, \eta\gamma_7$

DEG.	*X*	*ADAMS*	ηX	νX	σX	*DEC.*
54	$B[54]$	h_0h_5i	0	$\nu B[54]$	0	I
	$\eta A[8]D[45]$	h_1x'	0	0	0	D
56	$\nu^2 A[50,2]$	gt	0	0	0	D
57	$A[57]$	Q_2	$\eta A[57]$	0	$\eta A[63,1]$	I
	$\nu B[54]$	$h_1Ph_5e_0$	0	$\eta A[59,2]$	0	D
58	$\eta A[57]$	h_1Q_2	0	0	$*$	D
59	$A[59,1]$	B_{21}	0	0	$*$	I
	$A[59,2]$	d_0w	$\eta A[59,2]$	0	$*$	D
60	$B[60]$	g^3	0	0	$*$	D
62	$A[62,1]$	h_5^2	$\eta A[62,1]$	$*$	$*$	I
	$A[62,2]$	h_5n	0	$*$	$*$	I
	$B[62]$	$C_0 + h_0^6h_5^2$	0	$*$	$*$	I
63	$A[63,1]$	h_1H_1	$\eta A[63,1]$	$*$	$*$	I
	$A[63,2]$	C'	0	$*$	$*$	I
	$\eta A[62,1]$	$h_1h_5^2$	$2B[64]$	$*$	$*$	D

Note that $B[62]$ must be represented by $x \in E_\infty^{8,62}$ such that $h_0x = h_1x = 0$. The only possibility is $x = C_0 + h_0^6h_5^2$. Also, $A[63,2]$ must be represented by $y \in E_\infty^{7,63}$ such that $h_1y = 0$. The only possibility is $y = C'$.

The product of elements in degrees less than 54 times η, ν and σ which have degree greater than or equal to 54 are the same as those given in [3, Appendix 2] with three exceptions: $\sigma A[50,2] = 0$, $\sigma^2 C[44] = \eta A[57]$ *and* $\nu A[52,2] = 0$.

6 The Adams Spectral Sequence

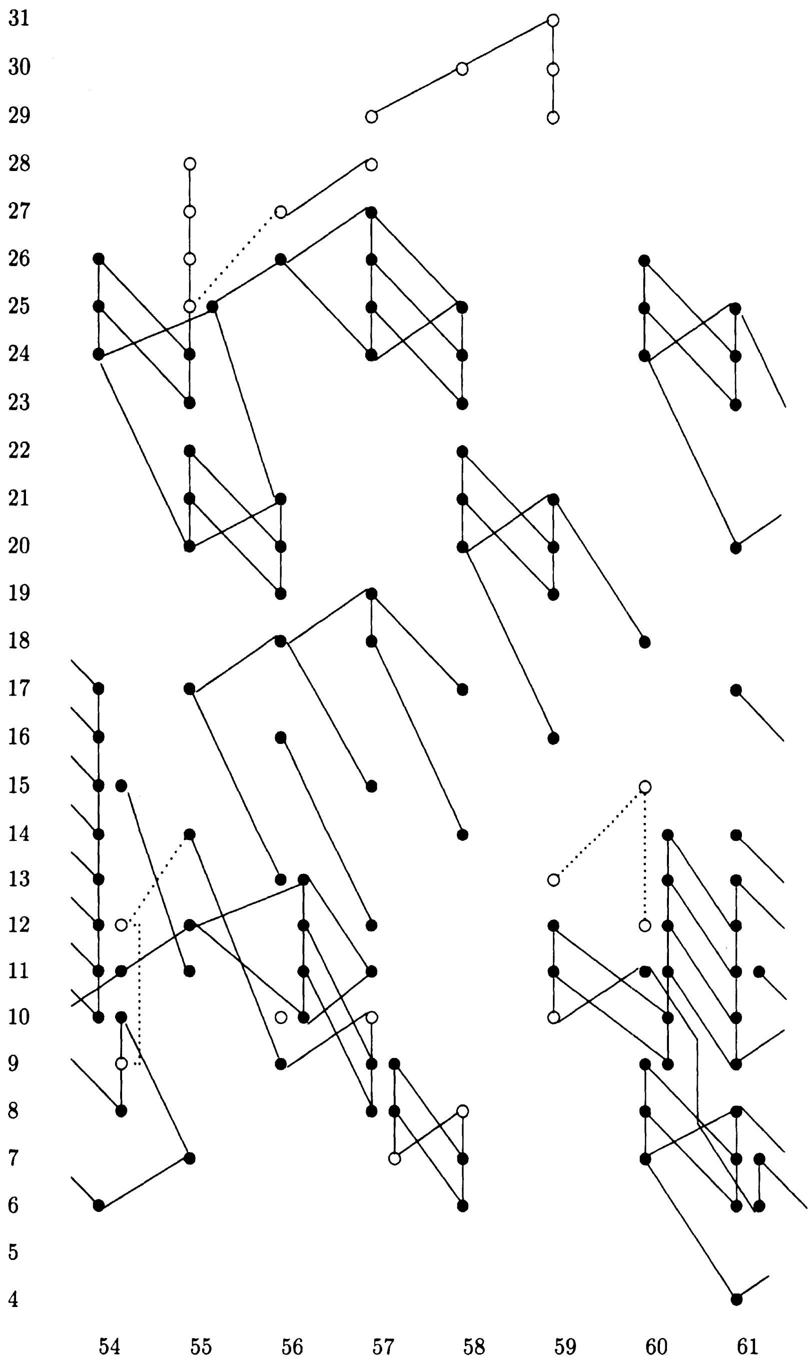
The tables of this section depict the structure of the ASS in degrees 54 through 66. (The entries in degree 66 are complete only through filtration degree four.) The group structure of E_2 was computed by Tangora [5] using the May spectral sequence. The product of h_0 and h_1 with most of these elements was computed by

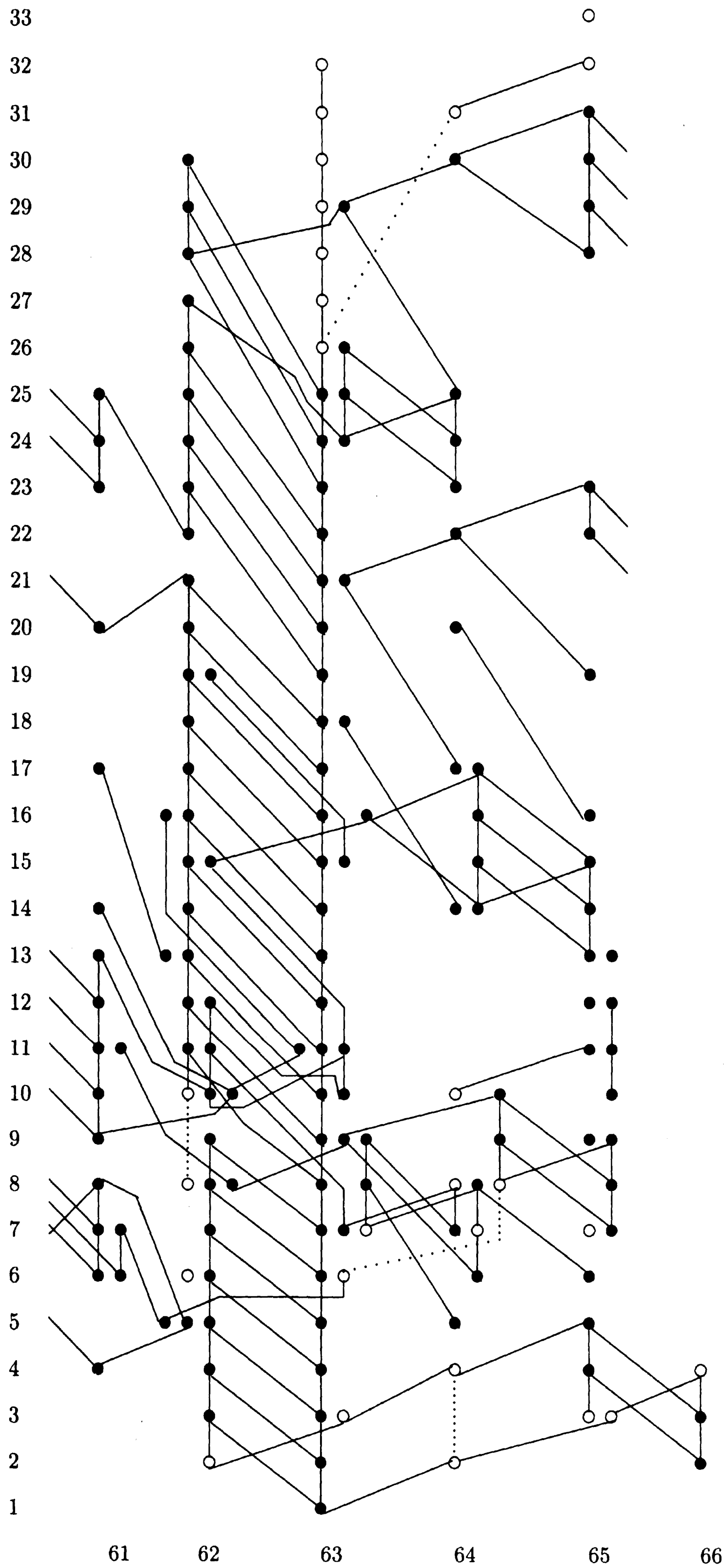
Tangora [5]. The remaining products , which correspond to nontrivial extensions in the May spectral sequence in degrees 60 through 64, were computed by Bruner [2]. We begin with a table of notation which labels the elements in the following two tables. The symbol $X \leftarrow \xi$ indicates that the element $\xi \in \pi_*^S$ defined from the AHSS projects to the infinite cycle X in the ASS. In each bidegree, elements are labeled from left to right. In the last two tables we use vertical lines to indicate multiplication by h_0, lines of positive slope to indicate multiplication by h_1 and lines of negative slope to indicate differentials. Dotted vertical lines indicate nontrivial extensions given by multiplication by two while dotted lines of positive slope indicate nontrivial extensions given by multiplication by η. Infinite cycles are indicated by circles.

Notation:

(6,54)	G	(8,54)	$h_5 i$	(9,54)	$h_0 h_5 i \leftarrow B[54]$
(10,54)	R_1	(12,54)	$e_0^2 g \leftarrow 2B[54]$	(15,54)	Pgj
(24,54)	$P^5 d_0$	(11,55)	gm	(14,55)	$Pe_0 r$
(17,55)	$P^2 u$	(20,55)	$P^3 e_0 d_0$	(23,55)	$P^4 i$
(25,55)	$h_0^2 P^4 i \leftarrow \gamma_6$	(9,56)	$Ph_5 e_0$	(10,56)	$gt \leftarrow \nu^2 A[50,2]$
(10,56)	R'	(13,56)	$d_0 v$	(16,56)	$P^2 g_2$
(19,56)	$P^3 l$	(27,56)	$P^6 c_0 \leftarrow \eta\gamma_6$	(7,57)	$Q_2 \leftarrow A[57]$
(8,57)	$h_5 j$	(10,57)	$h_1 Ph_5 e_0 \leftarrow \nu B[54]$	(12,57)	$e_0 g^2$
(15,57)	Pgk	(18,57)	$P^2 z$	(24,57)	$P^5 e_0$
(29,57)	$P^7 h_1 \leftarrow \alpha_7$	(6,58)	D_2	(14,58)	Pgr
(17,58)	$P^2 v$	(20,58)	$P^3 e_0^2$	(23,58)	$P^4 j$
(10,59)	$B_{21} \leftarrow A[59,1]$	(13,59)	$d_0 w \leftarrow A[59,2]$	(16,59)	Pe_0^3
(19,59)	$P^3 m$	(29,59)	$P^7 h_2 \leftarrow \beta_7$	(7,60)	B_3
(9,60)	B_4	(12,60)	$g^3 \leftarrow B[60]$	(15,60)	$Pgl \leftarrow \eta A[59,2]$
(18,60)	$P^2 d_0 r$	(24,60)	$P^5 g$	(4,61)	D_3
(6,61)	A	(6,61)	A'	(9,61)	X_1
(11,61)	rn	(14,61)	gz	(17,61)	$P^2 w$
(20,61)	$P^3 e_0 g$	(23,61)	$P^4 k$	(2,62)	$h_5^2 \leftarrow A[62,1]$
(5,62)	H_1	(6,62)	$h_5 n \leftarrow A[62,2]$	(8,62)	$C_0 + h_0^6 h_5^2 \leftarrow B[62]$
(8,62)	$C_0 + E_1$	(10,62)	$R \leftarrow 2B[62]$	(10,62)	B_{22}
(10,62)	PG	(13,62)	gv	(15,62)	$P^2 B_1$
(16,62)	$Pe_0^2 g$	(19,62)	$P^2 gj$	(22,62)	$P^4 r$
(28,62)	$P^6 d_0$	(1,63)	h_6	(6,63)	$h_1 H_1 \leftarrow A[63,1]$
(7,63)	X_2	(7,63)	$C' \leftarrow A[63,2]$	(8,63)	$h_2 B_3$
(10,63)	$h_2 B_4$	(15,63)	Pgm	(18,63)	$P^2 e_0 r$
(21,63)	$P^3 u$	(24,63)	$P^4 d_0 e_0$	(26,63)	$h_0^{25} h_6 \leftarrow \gamma_7$
(2,64)	$h_1 h_6 \leftarrow B[64]$	(5,64)	$h_2 D_3$	(6,64)	A''
(7,64)	$h_2 A$	(7,64)	$h_0 A'' \leftarrow A[64,3]$	(8,64)	$h_0 h_2 A \leftarrow A[64,2]$
(8,64)	$h_3 Q_2 \leftarrow \eta A[63,1]$	(10,64)	$q_1 \leftarrow A[64,1]$	(14,64)	$d_0 gr$
(14,64)	PQ_1	(15,64)	$P^2 B_2$	(17,64)	$Pd_0 v$
(20,64)	$P^3 g_2$	(23,64)	$P^4 l$	(31,64)	$P^7 c_0 \leftarrow \eta\gamma_7$
(3,65)	$h_2 h_5^2$	(6,65)	$h_2 H_1$	(7,65)	$h_2 h_5 n$
(7,65)	$h_3 D_2$	(9,65)	$h_2 C_0$	(10,65)	B_{23}
(12,65)	$Ph_5 j$	(13,65)	R_2	(13,65)	gw

(16, 65)	Pe_0g^2	(19, 65)	P^2gk	(22, 65)	P^3z
(28, 65)	P^6e_0	(33, 65)	$P^8h_1 \leftarrow \alpha_8$	(2, 66)	h_2h_6

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