Advances in Mathematics **163**, 34–73 (2001)
doi:10.1006/aima.2001.1995, available online at http://www.idealibrary.com on IDEAL®

The Additivity of Traces in Triangulated Categories

J. P. May[1]

Department of Mathematics, The University of Chicago, Chicago, Illinois 60637
E-mail: may@math.uchicago.edu

Communicated by Mark Hovey

Received October 2, 2000; accepted March 28, 2001

We explain a fundamental additivity theorem for Euler characteristics and generalized trace maps in triangulated categories. The proof depends on a refined axiomatization of symmetric monoidal categories with a compatible triangulation. The refinement consists of several new axioms relating products and distinguished triangles. The axioms hold in the examples and shed light on generalized homology and cohomology theories. © 2001 Academic Press

Contents.

Let $\mathscr{C}$ be a closed symmetric monoidal category with a compatible triangulation. We shall give a precise definition that explains what we mean by this in Section 4. We write S for the unit object of $\mathscr{C}$, $\wedge$ for the product, F for the internal hom functor, and $DX = F(X, S)$ for the dual object of X. The reader so inclined should read $\otimes$ for $\wedge$ and Hom for F. For any object X, we have an evaluation map ε: $DX \wedge X \longrightarrow X$. As recalled in [11,

[1] It is a pleasure to thank Mark Hovey for the proof of Lemmas 3.6 and other help and to thank Halvard Fausk, John Greenlees, and Gaunce Lewis for careful reading and helpful comments.

0001-8708/01 $35.00

Sect. 2], when X is dualizable we also have a coevaluation map $\eta\colon S \longrightarrow X \wedge DX$. The Euler characteristic $\chi(X)$ is then the composite

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\varepsilon} S,$$

where γ is the commutativity isomorphism. We shall prove the following theorem.

THEOREM 0.1. *Assume given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X. \tag{0.2}$$

If X, Y, and therefore Z are dualizable, then $\chi(Y) = \chi(X) + \chi(Z)$.

Some of the significance of this basic result is discussed in [11]. In fact, we shall prove a more general additivity theorem of the same nature. We discuss generalized trace maps and state the generalization in Section 1.

Philosophically, we view our additivity "theorems" as basic results that must hold in any closed symmetric monoidal category with a "compatible" triangulation. That is, our aim is less to prove the theorems than to explain the proper meaning of the word "compatible."

In Sections 2 and 3, we define triangulated categories and briefly discuss homotopy pushouts and pullbacks in such categories. We make heavy use of Verdier's axiom in our work, and we take the opportunity to show that the axiom in the definition of a triangulated category that is usually regarded as the most substantive one is in fact redundant: it is implied by Verdier's axiom and the remaining, less substantial, axioms. Strangely, since triangulated categories have been in common use for over 30 years, this observation seems to be new.

We explain our new axioms for the definition of a compatible triangulation on a symmetric monoidal category and show how they imply Theorem 0.1 in Section 4. The new axioms relate the product $\wedge$ and duality to distinguished triangles. The need for the new axioms is not so strange, since the first published formulation of compatibility conditions that I know of is only a few years old [6] and the new axioms are considerably less transparent than the others in this theory.

The axioms are folklore results in the stable homotopy category. They can also be verified in the usual derived categories in algebraic geometry and homological algebra and in the Morel–Voevodsky $\mathbb{A}^1$-stable homotopy categories. We shall explain both intuitively and model theoretically what is involved in the verifications in Sections 5–7. The model theoretical material in those sections is the technical heart of the paper. A disclaimer may be in order. In view of what is involved in the verification of the axioms, they are unlikely to be satisfied except in triangulated categories that arise as the homotopy categories of suitable model categories. Nevertheless, we shall see that, despite their complicated formulations, the axioms record information that is intuitively transparent. We show how to prove the generalization Theorem 1.9 of Theorem 0.1 in Section 8.

The axioms give information that has been used in stable homotopy theory for decades. Adams' 1971 Chicago lectures [1, III, Sect. 9] gave a systematic account of products in homology and cohomology theories that implicitly used one version of these axioms, and I first formulated some of the axioms in forms similar to those given here in unpublished notes written soon after. In Section 9, I will briefly indicate the role the axioms play in generalized homology and cohomology theories. The discussion applies to any symmetric monoidal category with a compatible triangulation.

One moral of this paper is that the types of structured categories we consider are still not well understood, despite their ubiquitous appearance in algebraic topology, homological algebra, and algebraic geometry. We will leave several problems about them unresolved.

1. GENERALIZED TRACE MAPS

We recall the following definition from [8, III.7.1]. We do not need the triangulation of $\mathscr{C}$ here, just the closed symmetric monoidal structure.

Definition 1.1. Let X be a dualizable object of $\mathscr{C}$ with a self-map $f: X \longrightarrow X$. Let C be any object of $\mathscr{C}$ and suppose given a map $\Delta = \Delta_X: X \longrightarrow X \wedge C$. Define the *trace of f with respect to Δ*, denoted $\tau(f)$, to be the composite

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\mathrm{id} \wedge f} DX \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} DX \wedge X \wedge C \xrightarrow{\varepsilon \wedge \mathrm{id}} S \wedge C \cong C.$$

Since $(f \wedge \mathrm{id}) \circ \eta = (\mathrm{id} \wedge Df) \circ \eta$ and $\varepsilon \circ (\mathrm{id} \wedge f) = \varepsilon \circ (Df \wedge \mathrm{id})$, easy diagram chases show that the same map $\tau(f)$ is obtained if we insert any of the following four composites between $\gamma \circ \eta$ and $\varepsilon \wedge \mathrm{id}$:

$$DX \wedge X \underset{Df \wedge \mathrm{id}}{\overset{\mathrm{id} \wedge f}{\rightrightarrows}} DX \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} DX \wedge X \wedge C$$

$$DX \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} DX \wedge X \wedge C \underset{Df \wedge \mathrm{id} \wedge \mathrm{id}}{\overset{\mathrm{id} \wedge f \wedge \mathrm{id}}{\rightrightarrows}} DX \wedge X \wedge C.$$

If $C = S$ and Δ is the unit isomorphism $X \cong X \wedge S$, then $\tau(f)$ is denoted $\chi(f)$ and is called the *trace* or *Lefschetz constant* of f. The trace of the identity map is the Euler characteristic of X. If $C = X$, then Δ is thought of as a diagonal map and $\tau(\mathrm{id})\colon S \to X$ is called the *transfer map* of X with respect to Δ.

The definition includes a variety of familiar maps in algebra, algebraic geometry, and algebraic topology. If $\mathscr{C}$ is the category of vector spaces over a field and X is a finite dimensional vector space, then $\chi(f)$ is just the classical trace of the linear transformation f. If X is graded, then $\chi(X)$ is just the classical Euler characteristic. The classical (reduced) Euler characteristics and Lefschetz numbers in algebraic topology are also special cases. The essential point in the verification of assertions such as these is the additivity theorem that we prove in this paper.

In the most interesting situations, C is a comonoid (or coalgebra) with coproduct $\Delta\colon C \to C \wedge C$ and counit $\xi\colon C \to S$ and $\Delta\colon X \to X \wedge C$ is a coaction of C on X, meaning that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \wedge C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge\Delta} \\ X \wedge C & \xrightarrow[\Delta\wedge\mathrm{id}]{} & X \wedge C \wedge C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & & \\ {\scriptstyle\Delta}\downarrow & \searrow\!\!\!\!= & \\ X \wedge C & \xrightarrow[\mathrm{id}\wedge\xi]{} & X. \end{array}$$

The second diagram implies the commutativity of the diagram

$$S \xrightarrow{\tau(f)} C \xrightarrow{\xi} S, \qquad \xi \circ \tau(f) = \chi(f),$$

which is familiar and important in a variety of contexts. We recall the following further formal properties of generalized trace maps from [8, III, Sect. 7]. The proofs are easy diagram chases, some of which use the alternative descriptions of $\tau(f)$ given in Definition 1.1. Assume that X and Y are dualizable.

Lemma 1.2 (Unit Property). *For any map $f\colon S \to S$, $\chi(f) = f$.*

Lemma 1.3 (Fixed Point Property). *If $h\colon C \to C$ is a map such that the following diagram commutes, then $h \circ \tau(f) = \tau(f)$:*

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \wedge C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\wedge h} \\ X & \xrightarrow[\Delta]{} & X \wedge C. \end{array}$$

For example, when $C = X$ and Δ is a diagonal of the usual sort, we have $(f \wedge f) \circ \Delta = \Delta \circ f$ and can take $h = f$. This property is closely related to the Lefschetz fixed point theorem.

LEMMA 1.4 (Invariance under Retraction). *Let $i: X \longrightarrow Y$ and $r: Y \longrightarrow X$ be a retraction, $r \circ i = \mathrm{id}$. Let $\Delta_X: X \longrightarrow X \wedge C$, $\Delta_Y: Y \longrightarrow Y \wedge D$, and $h: C \longrightarrow D$ be maps such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \wedge C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i \wedge h} \\ Y & \xrightarrow[\Delta_Y]{} & Y \wedge D. \end{array}$$

Then $h \circ \tau(f) = \tau(i \circ f \circ r)$ for any map $f: X \longrightarrow X$.

For example, we can take $C = D$ and $\Delta_Y = (i \wedge \mathrm{id}) \circ \Delta_X \circ r$. When i is an isomorphism with inverse r, this gives invariance under isomorphism.

Duality and traces are natural with respect to (lax) symmetric monoidal functors, by [8, III.1.9, III.7.7].

PROPOSITION 1.5. *Let $F: \mathscr{C} \longrightarrow \mathscr{D}$ be a symmetric monoidal functor such that the unit map $\lambda: T \longrightarrow FS$ is an isomorphism, where T is the unit object of $\mathscr{D}$. Let X be a dualizable object of $\mathscr{C}$ such that the product map*

$$\phi: FX \wedge FD(X) \longrightarrow F(X \wedge DX)$$

is an isomorphism. Then FX is dualizable in $\mathscr{D}$, the natural map $FDX \longrightarrow DFX$ is an isomorphism, and

$$\phi: FX \wedge FZ \longrightarrow F(X \wedge Z)$$

is an isomorphism for every object Z of $\mathscr{C}$. Given $\Delta_X: X \longrightarrow X \wedge C$, define $\Delta_{FX} = \phi^{-1} \circ F\Delta_X: FX \longrightarrow FX \wedge FC$. Then, regarding λ as an identification, $\tau(Ff) = F\tau(f): T \longrightarrow FC$ for any map $f: X \longrightarrow X$.

Returning to the algebraic properties of trace maps, we first record their behavior with respect to $\wedge$-products, coproducts, and suspension, and then formulate our additivity theorem.

LEMMA 1.6 (Commutation with $\wedge$-Products). *Given maps $\Delta_X: X \longrightarrow X \wedge C$ and $\Delta_Y: Y \longrightarrow Y \wedge D$, define*

$$\begin{aligned} \Delta_{X \wedge Y} = (id \wedge \gamma \wedge id) \circ (\Delta_X \wedge \Delta_Y): X \wedge Y &\longrightarrow (X \wedge C) \wedge (Y \wedge D) \\ &\longrightarrow (X \wedge Y) \wedge (C \wedge D). \end{aligned}$$

Then $\tau(f \wedge g) = \tau(f) \wedge \tau(g)\colon S \longrightarrow C \wedge D$ *for any* $f\colon X \longrightarrow X$ *and* $g\colon Y \longrightarrow Y$.

Now assume that $\mathscr{C}$ is additive with coproduct $\vee$; it follows that $\wedge$ is bilinear.

LEMMA 1.7 (Commutation with Sums). *Given maps* $\Delta_X\colon X \longrightarrow X \wedge C$ *and* $\Delta_Y\colon Y \longrightarrow Y \wedge C$, *define*

$$\Delta_{X \vee Y} = \Delta_X \vee \Delta_Y\colon X \vee Y \longrightarrow (X \wedge C) \vee (Y \wedge C) \cong (X \vee Y) \wedge C.$$

Then $\tau(h) = \tau(f) + \tau(g)\colon S \longrightarrow C$ *for any map* $h\colon X \vee Y \longrightarrow X \vee Y$, *where* $f\colon X \longrightarrow X$ *and* $g\colon Y \longrightarrow Y$ *are obtained from* h *by restriction and retraction.*

That is, as one would expect of a trace, the cross terms $X \longrightarrow Y$ and $Y \longrightarrow X$ of h make no contribution. Now assume our original hypothesis that $\mathscr{C}$ has a triangulation compatible with its symmetric monoidal structure. A diagram chase from (TC1) of Definition 4.1 gives the following generalization of [11, 4.7].

LEMMA 1.8 (Anticommutation with Suspension). *Given* $\Delta_X\colon X \longrightarrow X \wedge C$, *define* $\Delta_{\Sigma X}\colon \Sigma X \longrightarrow (\Sigma X) \wedge C$ *by suspending* Δ_X *and using the canonical isomorphism* $\Sigma(X \wedge C) \cong (\Sigma X) \wedge C$. *Then* $\tau(\Sigma f) = -\tau(f)$ *for any map* $f\colon X \longrightarrow X$.

The following result is our generalization of Theorem 0.1. For reasons that will become clear in Section 8, we now assume that $\mathscr{C}$ is the homotopy category of a closed symmetric monoidal model category $\mathscr{B}$ that satisfies the usual properties that lead to a triangulation on $\mathscr{C}$ that is compatible with its smash product. These properties are made precise at the start of Sections 5 and 6.

THEOREM 1.9 (Additivity on Distinguished Triangles). *Let* X, Y, *and therefore* Z *be dualizable in the distinguished triangle* (0.2). *Assume given maps* $\phi\colon X \longrightarrow X$ *and* $\psi\colon Y \longrightarrow Y$ *and maps* $\Delta_X\colon X \longrightarrow X \wedge C$ *and* $\Delta_Y\colon Y \longrightarrow Y \wedge C$ *such that the left squares commute in the following two diagrams:*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \Sigma\phi} \\
X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & \Sigma X
\end{array}$$

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \Delta_X} & & \downarrow{\scriptstyle \Delta_Y} & & \downarrow{\scriptstyle \Delta_Z} & & \downarrow{\scriptstyle \Sigma\Delta_X} \\
X \wedge C & \xrightarrow[f \wedge id]{} & Y \wedge C & \xrightarrow[g \wedge id]{} & Z \wedge C & \xrightarrow[h \wedge id]{} & \Sigma(X \wedge C).
\end{array}$$

Then there are maps $\omega\colon Z \longrightarrow Z$ and $\Delta_Z\colon Z \longrightarrow Z \wedge C$ such that these diagrams commute and the additivity relation $\tau(\psi) = \tau(\omega) + \tau(\phi)$ holds.

A result like this was first formulated in [8, III.7.6], in the context of equivariant stable homotopy theory. It has important calculational consequences in that subject, and it should be of comparable significance in other areas.

Remark 1.10. I do not know whether or not the conclusion holds for *every* choice of ω and Δ_Z that make the displayed diagrams commute, but I would expect not. This was claimed to hold in [8, III.7.6], but even in that special context the proof is incomplete. The question is related to Neeman's work in [12], where it is emphasized that some fill-ins in diagrams such as these are better than others. The theorem has a slight caveat in the generality of traces, as opposed to Lefschetz constants; see Remark 8.3.

2. TRIANGULATED CATEGORIES

We recall the definition of a triangulated category from [17]; see also [2, 6, 10]. Actually, one of the axioms in all of these treatments is redundant, namely the one used to construct the maps ω and Δ on Z in the additivity theorem just stated. The most fundamental axiom is called *Verdier's axiom*, or the *octahedral axiom* after one of its possible diagrammatic shapes. However, the shape that I find most convenient, a braid, does not appear in the literature of triangulated categories. It does appear in Adams [1, p. 212], who used the term "sine wave diagram" for it. We call a diagram (0.2) a "triangle" and use the notation (f, g, h) for it.

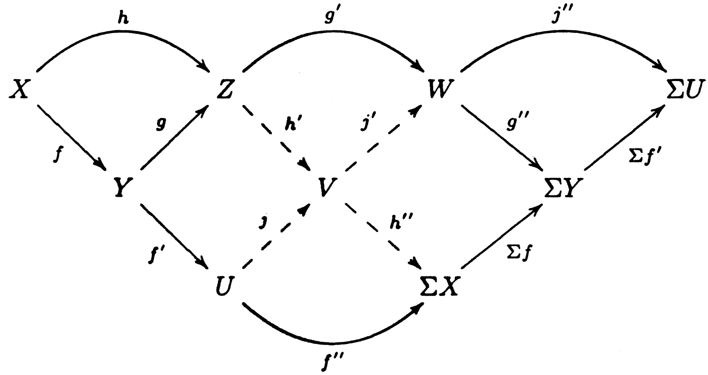
Definition 2.1. A triangulation on an additive category $\mathscr{C}$ is an additive self-equivalence $\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$ together with a collection of triangles, called the *distinguished triangles*, such that the following axioms hold.

Axiom (T1). Let X be any object and $f\colon X \longrightarrow Y$ be any map in $\mathscr{C}$.

(a) The triangle $X \xrightarrow{\text{id}} X \longrightarrow * \longrightarrow \Sigma X$ is distinguished.
(b) The map $f\colon X \longrightarrow Y$ is part of a distinguished triangle (f, g, h).
(c) Any triangle isomorphic to a distinguished triangle is distinguished.

Axiom (T2). If (f, g, h) is distinguished, then so is $(g, h, -\Sigma f)$.

Axiom (T3) (Verdier's Axiom). Consider the following diagram.



Assume that $h = g \circ f$, $j'' = \Sigma f' \circ g''$, and (f, f', f'') and (g, g', g'') are distinguished. If h' and h'' are given such that (h, h', h'') is distinguished, then there are maps j and j' such that the diagram commutes and (j, j', j'') is distinguished. We call the diagram a *braid of distinguished triangles generated by* $h = g \circ f$ or a *braid cogenerated by* $j'' = \Sigma f' \circ g''$.

We have labeled our axioms (T?), and we will compare them with Verdier's original axioms (TR?). Our (T1) is Verdier's (TR1) [17], our (T2) is a weak form of Verdier's (TR2), and our (T3) is Verdier's (TR4). We have omitted Verdier's (TR3), since it is exactly the conclusion of the following result.

LEMMA 2.2 (TR3). *If the rows are distinguished and the left square commutes in the following diagram, then there is a map k that makes the remaining squares commute.*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'
\end{array}$$

Proof. This is part of the 3×3 lemma, which we state and prove below. The point is that the construction of the commutative diagram in that proof requires only (T1), (T2), and (T3), not the conclusion of the present lemma; compare [2, 1.1.11]. ∎

Verdier's (TR2) includes the converse, (T2′) say, of (T2). That too is a consequence of our (T1), (T2), and (T3). A standard argument using only

(T1), (T2), (TR3), and the fact that Σ is an equivalence of categories shows that, for any object A, a distinguished triangle (f, g, h) induces a long exact sequence upon application of the functor $\mathscr{C}(A, -)$. Here we do not need the converse of (T2) because we are free to replace A by $\Sigma^{-1}A$. In turn, by the five lemma and the Yoneda lemma, this implies the following addendum to the previous lemma.

LEMMA 2.3. *If i and j in* (TR3) *are isomorphisms, then so is k.*

LEMMA 2.4 (T2′). *If $(g, h, -\Sigma f)$ is distinguished, then so is (f, g, h).*

Proof. Choose a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X$. By (T2), the triangles $(-\Sigma f, -\Sigma g', -\Sigma h')$ and $(-\Sigma f, -\Sigma g, -\Sigma h)$ are distinguished. By Lemmas 2.2 and 2.3, they are isomorphic. By desuspension, (f, g, h) is isomorphic to (f', g', h'). By (T1), it is distinguished. ▮

Similarly, we can derive the converse version, (T3′) say, of Verdier's axiom (T3).

LEMMA 2.5 (T3′). *In the diagram of* (T3), *if j and j' are given such that (j, j', j'') is distinguished, then there are maps h' and h'' such that the diagram commutes and (h, h', h'') is distinguished.*

Proof. Desuspend a braid of distinguished triangles generated by $j'' = \Sigma f' \circ g''$. ▮

LEMMA 2.6 (The 3×3 Lemma). *Assume that $j \circ f = f' \circ i$ and the two top rows and two left columns are distinguished in the following diagram.*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
\downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle \Sigma i'} \\
X'' & \overset{f''}{\dashrightarrow} & Y'' & \overset{g''}{\dashrightarrow} & Z'' & \overset{h''}{\dashrightarrow} & \Sigma X'' \\
\downarrow{\scriptstyle i''} & & \downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle k''} & & \downarrow{\scriptstyle -\Sigma i''} \\
\Sigma X & \xrightarrow[\Sigma f]{} & \Sigma Y & \xrightarrow[\Sigma g]{} & \Sigma Z & \xrightarrow[-\Sigma h]{} & \Sigma^2 X
\end{array}$$

Then there is an object Z'' and there are dotted arrow maps f'', g'', h'', k, k', k'' such that the diagram is commutative except for its bottom right square, which commutes up to the sign -1, and all four rows and columns are distinguished.

Proof. The bottom row is isomorphic to the triangle $(-\Sigma f, -\Sigma g, -\Sigma h)$ and is thus distinguished by (T2); similarly the right column is distinguished. Applying (T1), we construct a distinguished triangle

$$X \xrightarrow{j \circ f} Y' \xrightarrow{p} V \xrightarrow{q} \Sigma X.$$

Applying (T3), we obtain braids of distinguished triangles generated by $j \circ f$ and $f' \circ i$. These give distinguished triangles

$$Z \xrightarrow{s} V \xrightarrow{t} Y'' \xrightarrow{\Sigma g \circ j''} \Sigma Z$$
$$X'' \xrightarrow{s'} V \xrightarrow{t'} Z' \xrightarrow{\Sigma i' \circ h'} \Sigma X''$$

such that

$$p \circ j = s \circ g, \qquad t \circ p = j', \qquad q \circ s = h, \qquad j'' \circ t = \Sigma f \circ q$$
$$p \circ f' = s' \circ i', \qquad t' \circ p = g', \qquad q \circ s' = i'', \qquad h' \circ t' = \Sigma i \circ q.$$

Define $k = t' \circ s \colon Z \to Z'$. Then $k \circ g = g' \circ j$ and $h' \circ k = \Sigma i \circ h$, which already completes the promised proof of Lemma 2.2. Define $f'' = t \circ s'$ and apply (T1) to construct a distinguished triangle

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} \Sigma X''.$$

Applying (T3), we obtain a braid of distinguished triangles generated by $f'' = t \circ s'$. Here we start with the distinguished triangles $(s', t', \Sigma i' \circ h')$ and $(t, \Sigma g \circ j'', -\Sigma s)$, where the second is obtained by use of (T2). This gives a distinguished triangle

$$Z' \xrightarrow{k'} Z'' \xrightarrow{k''} V \xrightarrow{-\Sigma k} \Sigma Z'$$

such that the squares left of and above the bottom right square commute and

$$g'' \circ t = k' \circ t' \qquad \text{and} \qquad -\Sigma s \circ k'' = \Sigma s' \circ h''.$$

The commutativity (and anti-commutativity of the bottom right square) of the diagram follow immediately. It also follows immediately that (f'', g'', h'') and $(k', k'', -\Sigma k)$ are distinguished. Lemma 2.4 implies that (k, k', k'') is distinguished. ∎

Remark 2.7. Conversely, Verdier's axiom is implied by (T1), (T2), and the 3×3 lemma. To see this, apply the 3×3 lemma starting with the top left square

$$\begin{array}{ccc} X & \xrightarrow{h = g \circ f} & Z \\ {\scriptstyle f}\downarrow & & \| \\ Y & \xrightarrow[g]{} & Z. \end{array}$$

3. WEAK PUSHOUTS AND WEAK PULLBACKS

In any category, weak limits and weak colimits satisfy the existence but not necessarily the uniqueness in the defining universal properties. They need not be unique and need not exist. When constructed in particularly sensible ways, they are called homotopy limits and colimits and are often unique up to non-canonical isomorphism. As we recall here, there are such homotopy pushouts and pullbacks in triangulated categories. Homotopy colimits and limits of sequences of maps in triangulated categories are studied in [3, 13], but a complete theory of homotopy limits and colimits in triangulated categories is not yet available. The material in this section is meant to clarify ideas and will not be used in the proofs of the additivity theorems. However, it seems to me that there should be better proofs that do make use of this material, although I have not been able to find them.

DEFINITION 3.1. A *homotopy pushout* of maps $f: X \longrightarrow Y$ and $g: X \longrightarrow Z$ is a distinguished triangle

$$X \xrightarrow{(f, -g)} Y \vee Z \xrightarrow{(j, k)} W \xrightarrow{i} \Sigma X.$$

A *homotopy pullback* of maps $j: Y \longrightarrow W$ and $k: Z \longrightarrow W$ is a distinguished triangle

$$\Sigma^{-1} W \xrightarrow{-\Sigma^{-1} i} X \xrightarrow{(f, g)} Y \vee Z \xrightarrow{(j, -k)} W.$$

The sign is conventional and ensures that in the isomorphism of extended triangles

$$\begin{array}{ccccccccc}
\Sigma^{-1}W & \xrightarrow{-\Sigma^{-1}i} & X & \xrightarrow{(f,-g)} & Y\vee Z & \xrightarrow{(j,k)} & W & \xrightarrow{i} & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle (\mathrm{id},-\mathrm{id})} & & \| & & \| \\
\Sigma^{-1}W & \xrightarrow{-\Sigma^{-1}i} & X & \xrightarrow{(f,g)} & Y\vee Z & \xrightarrow{(j,-k)} & W & \xrightarrow{i} & \Sigma X,
\end{array}$$

the top row displays a homotopy pushout if and only if the bottom row displays a homotopy pullback.

At this point we introduce a generalization of the distinguished triangles.

DEFINITION 3.2. A triangle (f, g, h) is *exact* if it induces long exact sequences upon application of the functors $\mathscr{C}(-, W)$ and $\mathscr{C}(W, -)$ for every object W of $\mathscr{C}$.

The following is a standard result in the theory of triangulated categories [17].

LEMMA 3.3. *Every distinguished triangle is exact.*

If (f, g, h) is distinguished, then $(f, g, -h)$ is exact but generally not distinguished. These exact triangles $(f, g, -h)$ give a second triangulation of $\mathscr{C}$, which we call the negative of the original triangulation.

Problem 3.4. The relationship between distinguished and exact triangles has not been adequately explored in the literature. Can a triangulated category $\mathscr{C}$ admit a triangulation with a given functor Σ that differs from both the original triangulation and its negative? Consideration of automorphisms of objects shows that there usually are exact triangles in $\mathscr{C}$ that are in neither the original triangulation nor its negative. Nevertheless, it seems possible that the answer is no.

The fact that the triangles in Definition 3.1 give rise to weak pushouts and weak pullbacks depends only on the fact that they are exact, not on the assumption that they are distinguished. This motivates the following definition.

DEFINITION 3.5. For exact triangles of the form displayed in Definition 3.1, we say that the following commutative diagram, which displays both a weak pushout and a weak pullback, is a *pushpull square*.

$$\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\
Y & \xrightarrow[j]{} & W
\end{array}$$

LEMMA 3.6. *The central squares in any braid of distinguished triangles generated by $h = g \circ f$ are pushpull squares. More precisely, with the notations of* (T3), *the following triangles are exact.*

$$Y \xrightarrow{(f', g)} U \vee Z \xrightarrow{(j, h')} V \xrightarrow{-g'' \circ j'} \Sigma Y$$

$$Y \xrightarrow{j \circ f'} V \xrightarrow{(j', h'')} W \vee \Sigma X \xrightarrow{(g'', -\Sigma f)} \Sigma Y.$$

Proof. Although rather lengthy, this is an elementary diagram chase. ▮

Remark 3.7. We would like to conclude that the triangles displayed in the lemma are distinguished and not just exact. Examples in [12] imply that this is not true for all choices of j and j'. The braid in (T3) gives rise to a braid of distinguished triangles that is cogenerated by $-g'' \circ j'$ or, equivalently, generated by $\Sigma^{-1}(j'' \circ g')$. Here $\Sigma^{-1}(j'' \circ g') = 0$ since $j'' = \Sigma f' \circ g''$. This implies that the central term in the braid splits as $U \vee Z$. Application of (T3) gives a distinguished triangle

$$Y \xrightarrow{\alpha} U \vee Z \xrightarrow{\beta} V \xrightarrow{-g'' \circ j'} \Sigma Y.$$

Inspecting the relevant braid, we see that $\alpha = (\bar{f}', g)$ and $\beta = (j, \bar{h}')$. However, we cannot always replace $\bar{f}'$ and $\bar{h}'$ by f' and h' and still have a distinguished triangle.

This leaves open the possibility that the triangles displayed in Lemma 3.6 are distinguished for some choices of j and j'. It was stated without proof in [2, 1.1.13] that j and j' can be so chosen in the main examples, and we shall explain why that is true in Section 5. It was suggested in [2, 1.1.13] that this conclusion should be incorporated in Verdier's axiom if the conclusion were needed in applications. This course was taken in [10], and we believe it to be a sensible one. However, rather than try to change established terminology, we offer the following modified definition.

DEFINITION 3.8. A triangulation of $\mathscr{C}$ is *strong* if the maps j and j' asserted to exist in (T3) can be so chosen that the two exact triangles displayed in Lemma 3.6 are distinguished.

Remark 3.9. Neeman has given an alternative definition of a triangulated category that is closely related to our notion of a strong triangulated category; compare [12, 1.8; 13, Sect. 1.4]. It is based on the existence of particularly good choices of the map k in (TR3).

4. THE COMPATIBILITY AXIOMS

In the rest of the paper, we return to our standing hypothesis that $\mathscr{C}$ is a closed symmetric monoidal category with a "compatible" triangulation. In this section, we state and explain the compatibility axioms. Let $S^n = \Sigma^n S$ for any integer n, where Σ^n is the n-fold iterate of Σ if n is positive or the $(-n)$-fold interate of Σ^{-1} if n is negative. Of course, S^{-n} is isomorphic to DS^n [11, 2.9].

DEFINITION 4.1. The triangulation on $\mathscr{C}$ is *compatible* with its closed symmetric monoidal structure if axioms (TC1)–(TC5) are satisfied.

Axiom (TC1). There is a natural isomorphism $\alpha\colon X \wedge S^1 \longrightarrow \Sigma X$ such that the composite

$$S^2 = \Sigma S^1 \xrightarrow{\alpha^{-1}} S^1 \wedge S^1 \xrightarrow{\gamma} S^1 \wedge S^1 \xrightarrow{\alpha} \Sigma S^1 = S^2$$

is multiplication by -1.

Axiom (TC2). For a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and an object W, each of the following triangles is distinguished.

$$X \wedge W \xrightarrow{f \wedge \mathrm{id}} Y \wedge W \xrightarrow{g \wedge \mathrm{id}} Z \wedge W \xrightarrow{h \wedge \mathrm{id}} \Sigma(X \wedge W)$$
$$W \wedge X \xrightarrow{\mathrm{id} \wedge f} W \wedge Y \xrightarrow{\mathrm{id} \wedge g} W \wedge Z \xrightarrow{\mathrm{id} \wedge h} \Sigma(W \wedge X)$$
$$F(W, X) \xrightarrow{F(\mathrm{id}, f)} F(W, Y) \xrightarrow{F(\mathrm{id}, g)} F(W, Z) \xrightarrow{F(\mathrm{id}, h)} \Sigma F(W, X)$$
$$\Sigma^{-1} F(X, W) \xrightarrow{-F(h, \mathrm{id})} F(Z, W) \xrightarrow{F(g, \mathrm{id})} F(Y, W) \xrightarrow{F(f, \mathrm{id})} F(X, W).$$

Remark 4.2. In (TC2) and in later axioms, we implicitly use isomorphisms such as

$$(\Sigma X) \wedge Y \cong \Sigma(X \wedge Y) \cong X \wedge (\Sigma Y)$$
$$\text{and} \qquad F(\Sigma^{-1} X, Y) \cong \Sigma F(X, Y) \cong F(X, \Sigma Y)$$

that are implied by (TC1). We often write ΣDX where the canonically isomorphic object $D(\Sigma^{-1} X)$ might seem more natural. We can deduce from (TC1) that $\varepsilon\colon D(\Sigma^{-1} X) \wedge \Sigma^{-1} X \longrightarrow S$ agrees with $\varepsilon\colon DX \wedge X \longrightarrow S$ under the canonical isomorphism of sources, and similarly for η when X is dualizable.

The first triangle displayed in (TC2) is isomorphic to the second, by application of γ to all terms, so that the second one is redundant.

Remark 4.3. Our (TC1) and (TC2) are equivalent to the compatibility conditions specified by Hovey *et al.* [6, A.2]. Indeed, using associativity isomorphisms implicitly, we have the composite natural isomorphism

$$(\Sigma X)\wedge Y \xrightarrow{\alpha^{-1}\wedge \mathrm{id}} X\wedge S^1\wedge Y \xrightarrow{\mathrm{id}\wedge\gamma} X\wedge Y\wedge S^1 \xrightarrow{\alpha} \Sigma(X\wedge Y).$$

Calling this map $e_{X,Y}$, we see that the conditions prescribed in [6, A.2] are satisfied. Conversely, isomorphisms $e_{X,Y}$ as prescribed there are determined by the $e_{S,Y}$ via the diagram on [6, p. 105], and the $e_{S,Y}$ determine and are determined by the maps

$$\alpha_Y\colon Y\wedge S^1 \xrightarrow{\gamma} S^1\wedge Y \xrightarrow{e_{S,Y}} \Sigma(S\wedge Y)\cong \Sigma Y.$$

These maps give a natural isomorphism α that satisfies (TC1) and (TC2).

The need for the axioms (TC1) and (TC2) is clear, and we view them as analogues of the elementary axioms (T1) and (T2) for a triangulated category. The new axioms (TC3)–(TC5) encode information about the $\wedge$-product of distinguished triangles that holds in the examples but is not implied by (TC1) and (TC2). The reader may recoil in horror at first sight of the diagram in the following axiom but, as we shall explain shortly, it is really quite natural.

Axiom (TC3) (The Braid Axiom for Products of Triangles). Suppose given distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'.$$

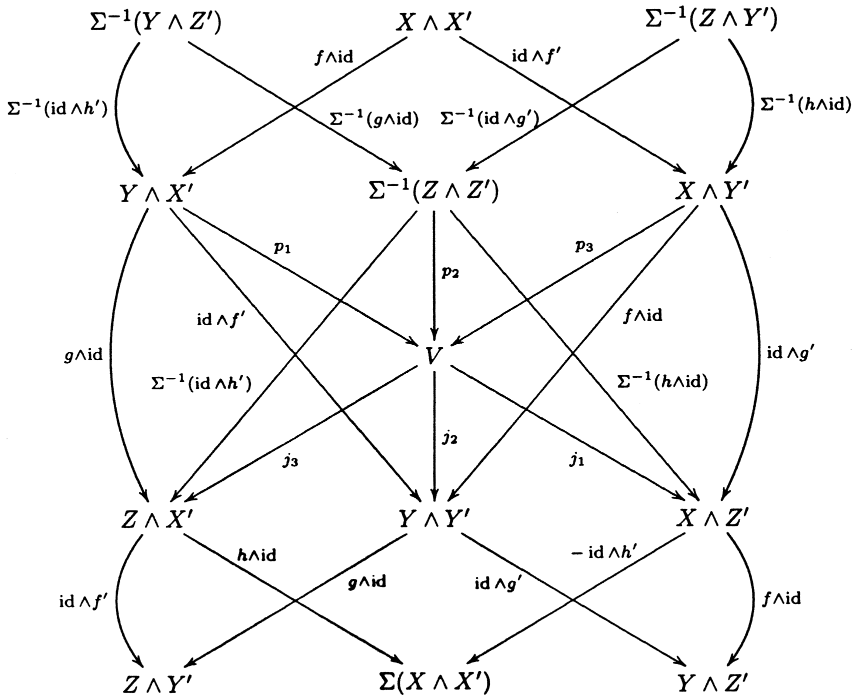
Then there are distinguished triangles

$$Y\wedge X' \xrightarrow{p_1} V \xrightarrow{j_1} X\wedge Z' \xrightarrow{f\wedge h'} \Sigma(Y\wedge X')$$

$$\Sigma^{-1}(Z\wedge Z') \xrightarrow{p_2} V \xrightarrow{j_2} Y\wedge Y' \xrightarrow{-g\wedge g'} Z\wedge Z'$$

$$X \wedge Y' \xrightarrow{p_3} V \xrightarrow{j_3} Z \wedge X' \xrightarrow{h \wedge f'} \Sigma(X \wedge Y')$$

such that the following diagram commutes.



There are several ways to understand (TC3). In concrete terms, one can pretend that X and X' are subobjects of Y and Y' with quotient objects Z and Z' and that V is the pushout

$$V = (Y \wedge X') \cup_{X \wedge X'} (X \wedge Y'). \tag{4.4}$$

Then the map $j_2 \colon V \longrightarrow Y \wedge Y'$ corresponds to the evident inclusion, while $j_1 \colon V \longrightarrow X \wedge Z'$ and $j_3 \colon V \longrightarrow Z \wedge X'$ correspond to the maps obtained by quotienting out $Y \wedge X'$ and $X \wedge Y'$. The diagram then corresponds to a folklore diagram in classical algebraic topology. Starting from this idea, we shall explain how to use standard cofiber sequences to verify the axiom in Section 6.

In axiomatic terms, it is more instructive to explain (TC3) in terms of Verdier's axiom. Consider the canonical 3×3 diagram

$$
(4.5)\qquad
\begin{array}{ccccccc}
X\wedge X' & \xrightarrow{f\wedge \mathrm{id}} & Y\wedge X' & \xrightarrow{g\wedge \mathrm{id}} & Z\wedge X' & \xrightarrow{h\wedge \mathrm{id}} & \Sigma(X\wedge X') \\
{\scriptstyle \mathrm{id}\wedge f'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge f'} & & \downarrow{\scriptstyle \mathrm{id}\wedge f'} & & \downarrow{\scriptstyle \Sigma(\mathrm{id}\wedge f')} \\
X\wedge Y' & \xrightarrow{f\wedge \mathrm{id}} & Y\wedge Y' & \xrightarrow{g\wedge \mathrm{id}} & Z\wedge Y' & \xrightarrow{h\wedge \mathrm{id}} & \Sigma(X\wedge Y') \\
{\scriptstyle \mathrm{id}\wedge g'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge g'} & & \downarrow{\scriptstyle \mathrm{id}\wedge g'} & & \downarrow{\scriptstyle \Sigma(\mathrm{id}\wedge g')} \\
X\wedge Z' & \xrightarrow{f\wedge \mathrm{id}} & Y\wedge Z' & \xrightarrow{g\wedge \mathrm{id}} & Z\wedge Z' & \xrightarrow{h\wedge \mathrm{id}} & \Sigma(X\wedge Z') \\
{\scriptstyle \mathrm{id}\wedge h'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge h'} & & \downarrow{\scriptstyle \mathrm{id}\wedge h'} & & \downarrow{\scriptstyle -\Sigma(\mathrm{id}\wedge h')} \\
\Sigma(X\wedge X') & \xrightarrow{\Sigma(f\wedge \mathrm{id})} & \Sigma(Y\wedge X') & \xrightarrow{\Sigma(g\wedge \mathrm{id})} & \Sigma(Z\wedge X') & \xrightarrow{-\Sigma(h\wedge \mathrm{id})} & \Sigma^2(X\wedge X').
\end{array}
$$

All squares except the bottom right one commute, and that square commutes up to the sign -1. To see this, observe that we have implicitly used the identification

$$(\Sigma X)\wedge Z' \cong (X\wedge S^1)\wedge Z' \cong (X\wedge Z')\wedge S^1 \cong \Sigma(X\wedge Z').$$

It is the anticommutativity of this square that forces the sign in the diagram of (TC3). By (TC2) and (T2), the rows and columns are distinguished. The signs are inserted in the bottom right square to ensure this. Each square gives two composites to which Verdier's axiom can be applied.

The diagram in (TC3) arranges in a single picture parts of braids generated by desuspending the composites

$$
\begin{aligned}
(g\wedge \mathrm{id})\circ(\mathrm{id}\wedge g') &= g\wedge g' = (\mathrm{id}\wedge g')\circ(g\wedge \mathrm{id})\\
(\mathrm{id}\wedge h')\circ(f\wedge \mathrm{id}) &= f\wedge h' = (f\wedge \mathrm{id})\circ(\mathrm{id}\wedge h')\\
(\mathrm{id}\wedge f')\circ(h\wedge \mathrm{id}) &= h\wedge f' = (h\wedge \mathrm{id})\circ(\mathrm{id}\wedge f').
\end{aligned}
$$

By expanding the relevant diagrams (T3) slightly, we see that pairs of these six composites appear in each of three distinct Verdier braids. To avoid expanding an already complicated diagram, we have omitted from the diagram in (TC3) the generating and cogenerating triangles from the three relevant braids as displayed in (T3), thus including only those subdiagrams from (T3) that involve at least one dotted arrow.

The point of (TC3) is that the cited three braids are duplicative. If we start with a given distinguished triangle $(p_2, j_2, -g\wedge g')$, then applications of Verdier's axiom to the two composite descriptions of the desuspension of $g\wedge g'$ construct distinguished triangles $(p_1, j_1, f\wedge h')$ and $(p_3, j_3, h\wedge f')$. On the other hand, application of Verdier's axiom to the desuspension of the composite $(f\wedge \mathrm{id})\circ(-\mathrm{id}\wedge h')$ constructs $(p_3, j_3, h\wedge f')$ from $(p_1, j_1, f\wedge h')$. The axiom (TC3) says that we can use the same maps in these a priori different ways of generating braids with the same objects. This discussion leads to the following addendum. Compare Definition 3.8.

LEMMA 4.6. *In the diagram of Axiom* TC3, *the six squares that have V as a vertex are pushpull squares.*

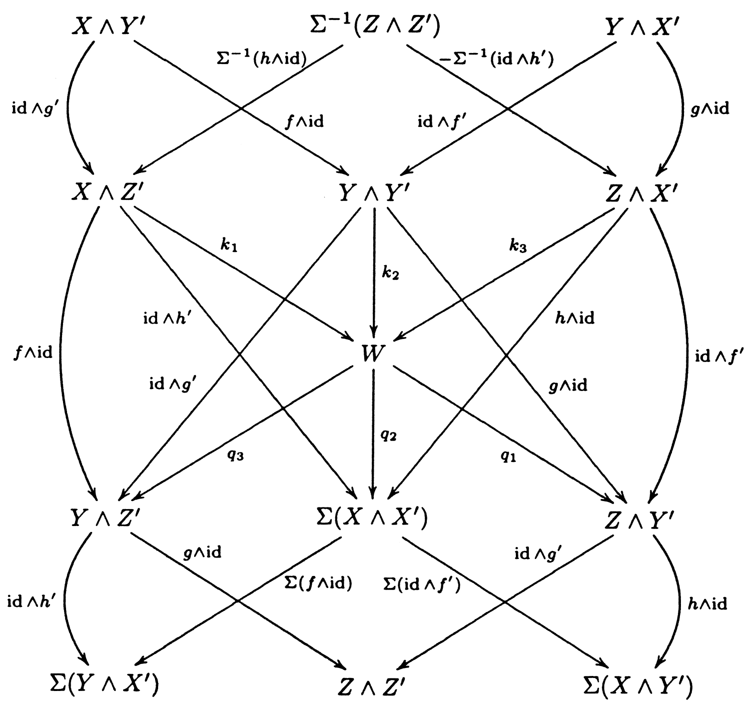
Proof. Three of the squares have side arrows (p_1, p_2), (p_1, p_3), (p_2, p_3) with target V, three have side arrows (j_1, j_2), (j_1, j_3), (j_2, j_3) with source V. These squares pair up as the central squares in the three braids cited in the paragraph above, and the conclusion is immediate from Lemma 3.6. ▮

Applying (TC3) to the distinguished triangles $(-\Sigma^{-1}h, f, g)$ and $(-\Sigma^{-1}h', f', g')$, we obtain the following equivalent form of that axiom.

LEMMA 4.7 (TC3′). *For the distinguished triangles* (f, g, h) *and* (f', g', h') *displayed in* (TC3), *there are distinguished triangles*

$$X \wedge Z' \xrightarrow{k_1} W \xrightarrow{q_1} Z \wedge Y' \xrightarrow{h \wedge g'} \Sigma(X \wedge Z')$$
$$Y \wedge Y' \xrightarrow{k_2} W \xrightarrow{q_2} \Sigma(X \wedge X') \xrightarrow{-\Sigma(f \wedge f')} \Sigma(Y \wedge Y')$$
$$Z \wedge X' \xrightarrow{k_3} W \xrightarrow{q_3} Y \wedge Z' \xrightarrow{g \wedge h'} \Sigma(Z \wedge X')$$

such that the following diagram commutes.



For intuition, pretending that we have inclusions of X in Y and X' in Y' with quotient objects Z and Z', we can think of W as the quotient object

$$W = (Y \wedge Y')/(X \wedge X'), \tag{4.8}$$

with k_2 being the quotient map and k_1 and k_3 being the inclusions of

$$X \wedge Z' \cong (X \wedge Y')/(X \wedge X') \quad \text{and} \quad Z \wedge X' \cong (Y \wedge X')/(X \wedge X').$$

LEMMA 4.9. *In the diagram of Axiom* TC3$'$, *the six squares that have* W *as a vertex are pushpull squares.*

The following obvious remark is quite useful.

Remark 4.10. We can reverse the order of our given triangles (f, g, h) and (f', g', h') and apply (TC3) and (TC3$'$). We agree to write $\bar{V}$, $\bar{W}$, and similarly for maps in such resulting diagrams. By (TR3), we can obtain equivalences $\gamma\colon V \longrightarrow \bar{V}$ and $\gamma\colon W \longrightarrow \bar{W}$ such that $\bar{p}_2 \circ \gamma = \gamma \circ p_2$, $\bar{j}_2\gamma = \gamma j_2$, $\bar{k}_2 \circ \gamma = \gamma \circ k_2$, and $\bar{q}_2\gamma = \gamma q_2$. We can then redefine the remaining maps $\bar{r}$ ($\bar{r} = \bar{p}_i$, $\bar{j}_i$, $\bar{k}_i$, $\bar{q}_i$, $i = 1$ and 3) by taking $\bar{r} = \gamma \circ r \circ \gamma^{-1}$. It follows from the axioms that the new diagrams still satisfy the properties specified in (TC3) and (TC3$'$) for the interchanged triangles. We say that the new diagrams are *involutions* of the original diagrams for (f, g, h) and (f', g', h').

The heart of our work concerns the interplay between (TC3) and (TC3$'$); we retain their notations.

Axiom (TC4) (The Additivity Axiom). The maps j_i and k_i can be so chosen that the following diagram is a pushpull square.

$$\begin{array}{ccc}
V & \xrightarrow{\quad j_2 \quad} & Y \wedge Y' \\
{\scriptstyle (j_1, j_3)}\big\downarrow & & \big\downarrow{\scriptstyle k_2} \\
(X \wedge Z') \vee (Z \wedge X') & \xrightarrow[(k_1, k_3)]{} & W
\end{array}$$

In particular, $k_2 \circ j_2 = k_1 \circ j_1 + k_3 \circ j_3$.

We will show how to use (4.4) and (4.8) to derive this axiom in Section 6. In fact, we will see that, in practice, the square comes from a distinguished triangle; compare Remark 3.7 and Definition 3.8. This suggests the following strengthened alternative to the concept of compatibility that we are in the process of defining.

DEFINITION 4.11. A strong triangulation of $\mathscr{C}$ is *strongly compatible* with its symmetric monoidal structure if the maps of (TC3) and (TC3′) can be so chosen that the pushpull squares of Lemmas 4.4 and 4.6 and of (TC4) all arise from distinguished triangles.

To see the plausibility of (TC4), observe that there is yet another Verdier braid in sight, coming from the relation

$$(j_1, j_3) \circ p_2 = (\Sigma^{-1}h \wedge \mathrm{id}, \Sigma^{-1}\, \mathrm{id} \wedge h').$$

We have a given distinguished triangle

$$\Sigma^{-1}(Z \wedge Z') \xrightarrow{p_2} V \xrightarrow{j_2} Y \wedge Y' \xrightarrow{-g \wedge g'} Z \wedge Z'.$$

If the triangulation is strong, we also have distinguished triangles

$$V \xrightarrow{(j_1, j_3)} (X \wedge Z') \vee (Z \wedge X') \xrightarrow{(\mathrm{id} \wedge h', h \wedge \mathrm{id})} \Sigma(X \wedge X') \xrightarrow{\Sigma(p_3 \circ (\mathrm{id} \wedge f'))} \Sigma V$$
$$\Sigma^{-1}(Z \wedge Z') \xrightarrow{(\Sigma^{-1}h \wedge \mathrm{id},\, \Sigma^{-1}\mathrm{id} \wedge h')} (X \wedge Z') \vee (Z \wedge X') \xrightarrow{(k_1, k_3)} W \xrightarrow{(g \wedge \mathrm{id})\, q_3} Z \wedge Z'.$$

Taking these three triangles as input in Verdier's axiom and noting that

$$\Sigma j_2 \circ \Sigma(p_3 \circ (\mathrm{id} \wedge f')) = -\Sigma(f \wedge f'),$$

(TC4) states that the maps asserted to exist by Verdier's axiom can be taken to be the maps k_2 and q_2 of the given distinguished triangle

$$Y \wedge Y' \xrightarrow{k_2} W \xrightarrow{q_2} \Sigma(X \wedge X') \xrightarrow{-\Sigma(f \wedge f')} \Sigma(Y \wedge Y').$$

Finally, we need an axiom that relates duality to $\wedge$-products of distinguished triangles. Keeping the original distinguished triangle (f, g, h), we specialize the distinguished triangle (f', g', h') to

$$DZ \xrightarrow{Dg} DY \xrightarrow{Df} DX \xrightarrow{D(\Sigma^{-1}h)} \Sigma DZ.$$

We can construct diagrams as in (TC3) and (TC3′) for both this pair of distinguished triangles and for the same pair of distinguished triangles in the reverse order. We adopt the notations of Remark 4.10 for the relevant objects and maps. Observe that we have a natural map $\rho\colon X \longrightarrow DDX$ and a natural composite

$$\xi\colon X \wedge DX \xrightarrow{\rho \wedge \mathrm{id}} DDX \wedge DX \xrightarrow{\wedge} D(DX \wedge X), \tag{4.12}$$

both of which are isomorphisms when X is dualizable. We have the following pleasant observation. The duals of the diagrams in (TC3) and (TC3′), if flipped over (or read from bottom to top), give further diagrams of the same shape. This is not an accident.

LEMMA 4.13. *Let X, Y, and therefore Z be dualizable. Then, using isomorphisms ξ, the dual of a diagram as in* (TC3) *for the triangles (f, g, h) and $(Dg, Df, D\Sigma^{-1}h)$ is a diagram as in* (TC3′) *for the same triangles in the reverse order.*

That is, taking $\bar{W} = DV$ with V as in (TC3) and taking

$$(\bar{k}_1, \bar{k}_2, \bar{k}_3) = (Dj_3, Dj_2, Dj_1) \qquad \text{and} \qquad (\bar{q}_1, \bar{q}_2, \bar{q}_3) = (Dp_3, Dp_2, Dp_1),$$

we obtain a diagram as in (TC3′) for $(Dg, Df, D\Sigma^{-1}h)$ and (f, g, h). We can now formulate our last compatibility axiom. Despite considerable effort, I have not been able to deduce it from the others. Recall Remark 4.10.

Axiom (TC5) (The Braid Duality Axiom). There is a diagram as in (TC3′) for the triangles $(Dg, Df, D\Sigma^{-1}h)$ and (f, g, h) which satisfies the following properties.

(a) There is a map $\bar{\varepsilon}: \bar{W} \longrightarrow S$ such that the following diagram commutes.

$$\begin{array}{ccccc}
(DZ \wedge Z) \vee (DX \wedge X) & \xrightarrow{(\bar{k}_1, \bar{k}_3)} & \bar{W} & \xleftarrow{\bar{k}_2} & DY \wedge Y \\
& {\scriptstyle (\varepsilon, \varepsilon)} \searrow & \downarrow {\scriptstyle \bar{\varepsilon}} & \swarrow {\scriptstyle \varepsilon} & \\
& & S & &
\end{array}$$

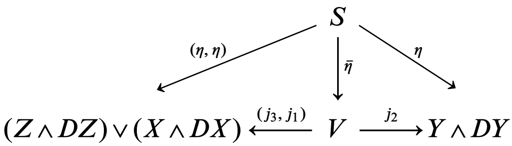
(b) If X, Y, and Z are dualizable, then the chosen diagram as in (TC3′) is isomorphic to the dual of a diagram as in (TC3) for the triangles (f, g, h) and $(Dg, Df, D\Sigma^{-1}h)$ and satisfies the additivity axiom (TC4) with respect to an involution of the latter (TC3) type diagram.

Here (b) ensures that the dual of (a) also holds. A diagram chase [8, III.1.4] shows that the coevaluation map $\eta: S \longrightarrow X \wedge DX$ of a dualizable object X is

$$D\varepsilon: S \cong DS \to D(DX \wedge X) \cong X \wedge DX.$$

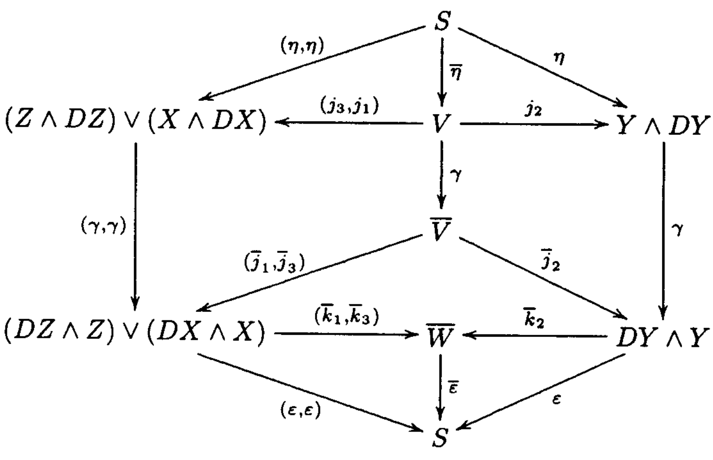
Therefore, via isomorphisms ρ and ξ, the dual of the commutative diagram in (a) gives the commutative diagram in the following result.

LEMMA 4.14 (TC5a′). *With the diagram* (TC3) *for* (f, g, h) *and* $(Dg, Df, D\Sigma^{-1}h)$ *taken as in* (TC5b), *there is a map* $\bar{\eta}\colon S \longrightarrow V$ *such that the following diagram commutes.*



This reduces the proof of Theorem 0.1 to quotation of the axioms.

Proof of Theorem 0.1. We are assuming Axioms (TC1)–(TC5), and we have the following commutative diagram. The desired formula $\chi(Y) = \chi(X) + \chi(Z)$ follows by traversing its outer edge.



Here the top and bottom pairs of triangles are given by (TC5a) and (TC5a′), the trapezoids involving maps γ are given by (TC5b) and Remark 4.10, and the remaining trapezoid is given by (TC5b) and (TC4). ∎

5. HOW TO PROVE VERDIER'S AXIOM

To prepare for the proofs of the compatibility axioms, we first recall the standard procedure for proving Verdier's axiom (T3).

We assume that our given category $\mathscr{C}$ is the "derived category" or "homotopy category" obtained from some Quillen model category $\mathscr{B}$. One can give general formal proofs of our axioms that apply to the homotopy categories associated to "simplicial", "topological", or "homological" model categories that are enriched over based simplicial sets, based spaces, or chain complexes, respectively. We shall be informal, but we shall give arguments in forms that should make it apparent that they apply equally well to any of these contexts. An essential point is to be careful about the passage from arguments in the point-set level model category $\mathscr{B}$, which is complete and cocomplete, to conclusions in its homotopy category $\mathscr{C}$, which generally does not have limits and colimits.

We assume that $\mathscr{B}$ is tensored and cotensored over the category in which it is enriched. We then have canonical cylinders, cones, and suspensions, together with their Eckmann–Hilton duals. The duals of cylinders are usually called "path objects" in the model theoretic literature (although in based contexts that term might more sensibly be reserved for the duals of cones). When we speak of homotopies, we are thinking in terms of the canonical cylinder $X \otimes I$ or path object Y^I, and we need not concern ourselves with left versus right homotopies in view of the adjunction

$$\mathscr{B}(X \otimes I, Y) \cong \mathscr{B}(X, Y^I).$$

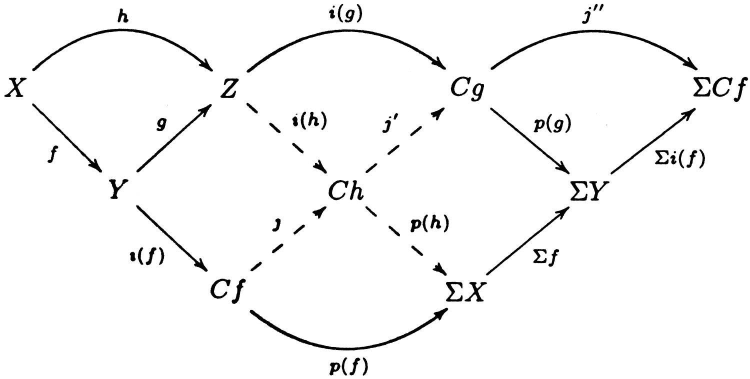
Hovey [5] gives an exposition of much of the relevant background material on simplicial model categories. Discussions of topological model categories appear in [4, 9]. Homological model categories appear implicitly in [5; 7, III, Sect. 1]. Of course, we must assume that the functor Σ on $\mathscr{B}$ induces a self-equivalence of $\mathscr{C}$. This is enough for the verification of most of the axioms but, to verify parts of (TC2) and (TC5), we assume more precisely that the adjunction between $\Sigma\colon \mathscr{B} \longrightarrow \mathscr{B}$ and its right adjoint Ω is a Quillen equivalence of model categories. (See, e.g., [5, 1.3.3] for a discussion of this notion.)

The distinguished triangles in $\mathscr{C}$ are the triangles that are isomorphic in $\mathscr{C}$ to a canonical distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{i(f)} Cf \xrightarrow{p(f)} \Sigma X \tag{5.1}$$

in $\mathscr{B}$. Here $Cf = Y \cup_f CX$, where CX is the cone on X, and $i(f)$ and $p(f)$ are the evident canonical maps. Then (T1) is clear and (T2) is a standard argument with cofiber sequences. One uses formal comparison arguments (as in [17, II.1.3.2]) to reduce the verification of (T3) in $\mathscr{C}$ to consideration of canonical cofiber sequences in $\mathscr{B}$. In $\mathscr{B}$, one writes down the following version of the braid in (T3).

(5.2)



Here $h = g \circ f$, j and j' are evident canonical induced maps, $j'' = \Sigma i(f) \circ p(g)$, and the diagram commutes in $\mathscr{B}$. One proves (T3) by writing down explicit inverse homotopy equivalences

$$\xi\colon Cg \longrightarrow Cj \qquad \text{and} \qquad \nu\colon Cj \longrightarrow Cg$$

such that $j' = \nu \circ i(j)$ and $j'' = p(j) \circ \xi$. Details of the algebraic argument are in [17, pp. 75–77], and the analogous topological argument is an illuminating exercise.

We could go on to use these Verdier braids to prove (TC3) and (TC4), but there are simpler proofs that give more information. To see this, we need a reformulation of the original triangulation.

Assuming, as can be arranged by cofibrant approximation, that f is a cofibration between cofibrant objects, the quotient Y/X is cofibrant. Let Mf be the mapping cylinder of f. Passage to pushouts from the evident commutative diagram

$$\begin{array}{ccccc} * & \longleftarrow & X & \longrightarrow & Mf \\ \| & & \| & & \downarrow \simeq \\ * & \longleftarrow & X & \underset{f}{\longrightarrow} & Y \end{array}$$

gives a quotient map $q(f)\colon Cf \to Y/X$. By [5, 5.2.6], we have the following standard result. It is central to our way of thinking about triangulated categories.

Lemma 5.3. *Let $f\colon X \to Y$ be a cofibration between cofibrant objects. Then the quotient map $q(f)\colon Cf \to Y/X$ is a weak equivalence.*

Now define

$$\delta(f)\colon Y/X \longrightarrow \Sigma X$$

to be the map in $\mathscr{C}$ represented by the formal "connecting map"

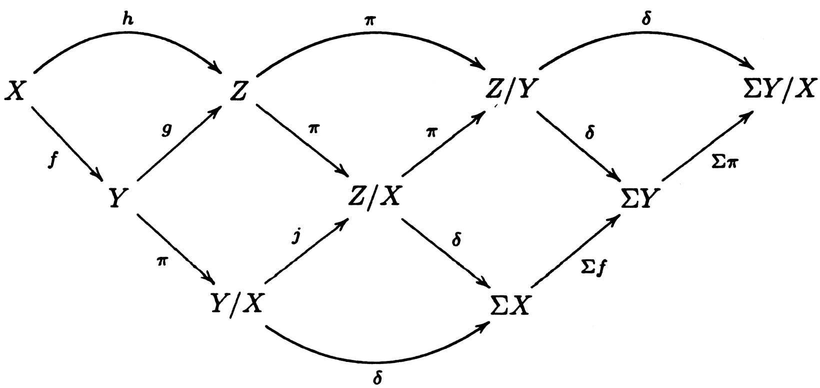
$$(5.4)\qquad Y/X \xleftarrow{q(f)} Cf \xrightarrow{p(f)} \Sigma X$$

in $\mathscr{B}$. Observe that (5.4) gives a functor from cofibrations in $\mathscr{B}$ to diagrams in $\mathscr{B}$. The composite $q(f)\circ i(f)\colon Y \longrightarrow Y/X$ is the evident quotient map, which we denote by $\pi(f)$. Therefore, when we pass to $\mathscr{C}$, our canonical distinguished triangle (5.1) is isomorphic to the triangle represented by the diagram

$$(5.5)\qquad X \xrightarrow{f} Y \xrightarrow{\pi(f)} Y/X \xrightarrow{\delta(f)} \Sigma X$$

in $\mathscr{B}$, and our triangulation consists of all triangles in $\mathscr{C}$ that are isomorphic to one of this alternative canonical form. This reformulation has distinct advantages.

Returning to Verdier's axiom, we can replace the given maps f, g, and thus $h = g\circ f$ by cofibrations between cofibrant objects, and then the quotient objects Y/X, Z/X and Z/Y are cofibrant. The point of Verdier's axiom now reduces to just the observation that Z/Y is canonically isomorphic in $\mathscr{B}$ to $(Z/X)/(Y/X)$. Using our new canonical cofibrations (5.5) starting from f, g, h, and the cofibration $j\colon Y/X \longrightarrow Z/X$, we obtain the following braid.



Expanding the arrows δ as in (5.4), we find that this braid in $\mathscr{C}$ is represented by an actual commutative diagram in $\mathscr{B}$, but of course with some wrong way arrows. With this proof of Verdier's axiom, there is no need to introduce the explicit homotopies ξ and v of our first proof. Modulo equivalences, the two central braids in (5.7) are as follows. Here and later, we generally write $C(Y, X)$ instead of Cf for a given cofibration $f: X \to Y$.

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y/X & \longrightarrow & Z/X \end{array} \qquad \begin{array}{ccc} C(Z, X) & \longrightarrow & C(Z, Y) \\ \downarrow & & \downarrow \\ \Sigma X & \longrightarrow & \Sigma Y \end{array}$$

These are both pushouts in which the horizontal arrows are cofibrations and all objects are cofibrant. By the following lemma, this implies that, in $\mathscr{C}$, these two squares give pushpull diagrams that arise from distinguished triangles. We conclude that $\mathscr{C}$ is strongly triangulated in the sense of Definition 3.8.

Lemma 5.7. *Suppose given a pushout diagram in* $\mathscr{B}$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[k]{} & W, \end{array}$$

in which f and therefore k are cofibrations and all objects are cofibrant. Then there is a distinguished triangle

$$X \xrightarrow{(-f,g)} Y \vee Z \xrightarrow{(j,k)} W \longrightarrow \Sigma X$$

in $\mathscr{C}$. Thus the original square gives rise to a pushpull square in $\mathscr{C}$.

Proof. Standard topological arguments work model theoretically to give a weak pushout (double mapping cylinder) $M(f, g)$ in $\mathscr{B}$ which fits into a canonical triangle

$$X \vee Y \xrightarrow{(j',k')} M(f,g) \xrightarrow{\pi} \Sigma X \xrightarrow{\delta} \Sigma X \vee \Sigma Y$$

as in (5.5). It is easy to check that $\delta = (f, -g)$ in $\mathscr{C}$ and that there is a weak equivalence $M(f, g) \to W$ under $Y \vee Z$ in $\mathscr{B}$. The conclusion follows. ∎

6. HOW TO PROVE THE BRAID AND ADDITIVITY AXIOMS

We now consider axioms (TC1)–(TC4). Here, in addition to the assumptions of the previous section, we assume that the closed symmetric monoidal structure on the homotopy category $\mathscr{C}$ is induced from a closed symmetric monoidal structure on $\mathscr{B}$. There are model theoretic axioms, specified in [5, 15], that codify the relationship between products $\wedge$ and cofibrations in reasonable monoidal model categories, and we assume that such standard properties hold in $\mathscr{B}$. They are known to hold in the usual examples. The most important axiom, which is called the *pushout-product axiom* in [15], asserts that, for cofibrations $f\colon X \longrightarrow Y$ and $f'\colon X' \longrightarrow Y'$, the evident induced map

$$V = (X \wedge Y') \cup_{X \wedge X'} (Y \wedge X') \longrightarrow Y \wedge Y'$$

is a cofibration which is acyclic if either f or g is acyclic. It follows that, for any object T, the dual induced map

$$F(Y \wedge Y', T) \longrightarrow F(V, T) \cong F(X \wedge Y', T) \times_{F(X \wedge X', T)} F(Y \wedge X', T)$$

is a fibration. Some equivalent conditions are given in [5, 4.2.2].

The verification of (TC1) is trivial, and the verification of (TC2) is standard; see Hovey [5, 6.41., 6.6.3]. For the cofiber sequences of (TC2) that involve $\wedge$, one uses the triangulation by cofibrations. For the cofiber sequences of (TC2) that involve the internal hom functor F, one verifies that the negative of the triangulation by cofiber sequences is the triangulation given by fiber sequences in $\mathscr{B}$, which are Eckmann–Hilton dual to cofiber sequences and whose development is word-for-word dual to that described in Section 5. For a given map f with fiber Ff and cofiber Cf, there is a map $\eta\colon Ff \longrightarrow \Omega Cf$ that is suitably related to the unit and counit of the (Σ, Ω) adjunction. This can be used in a direct verification of (TC2), as in [8, III, Sect. 2]; see also [4, II.6.4]. The argument in the homological context is easier.

We consider the new axiom (TC3). We may assume without loss of generality that the given distinguished triangles are of the form (5.5). We write them as

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'.$$

Thus all objects are cofibrant, f and f' are cofibrations, $Z = Y/X$ and $Z' = Y'/X'$, g and g' are quotient maps, and h and h' are connecting maps δ. We are thinking of these as diagrams in $\mathscr{B}$, the arrows h and h' being shorthand for pairs of arrows as displayed in (5.4).

As in (4.4) and (4.8), we set

$$V = (Y \wedge X') \cup_{X \wedge X'} (X \wedge Y') \qquad \text{and} \qquad W = (Y \wedge Y')/(X \wedge X').$$

We have many canonical isomorphisms of quotients, such as

$$X \wedge Z' \cong (X \wedge Y')/(X \wedge X'),$$
$$X \wedge C(Y', X') \cong C(X \wedge Y', X \wedge X'),$$
$$V/(X \wedge X') \cong (Z \wedge X') \vee (X \wedge Z'),$$
$$(Y \wedge Y')/V \cong Z \wedge Z'.$$

These are used heavily in verifying the claims that we are about to make.

By the cited axioms for a monoidal model category, or standard verifications in the usual examples, we have the following canonical triangles as in (5.5). Thus, in each case, the first map is a cofibration, the second map is a quotient map, and the third map is a connecting map as in (5.4). Note that j_2 is a cofibration by the pushout-product axiom. The maps p_2 and q_2 are defined in terms of displayed connecting maps, and the symbol $\simeq$ indicates an identification of the class of a given map δ in $\mathscr{C}$; each such identification can be verified by an elementary diagram chase.

$$Y \wedge X' \xrightarrow{p_1} V \xrightarrow{j_1} X \wedge Z' \xrightarrow{\delta \simeq f \wedge h'} \Sigma(Y \wedge X')$$
$$V \xrightarrow{j_2} Y \wedge Y' \xrightarrow{g \wedge g'} Z \wedge Z' \xrightarrow{\delta \equiv \Sigma p_2} \Sigma V$$
$$X \wedge Y' \xrightarrow{p_3} V \xrightarrow{j_3} Z \wedge X' \xrightarrow{\delta \simeq h \wedge f'} \Sigma(X \wedge Y')$$
$$X \wedge Z' \xrightarrow{k_1} W \xrightarrow{q_1} Z \wedge Y' \xrightarrow{\delta \simeq h \wedge g'} \Sigma(X \wedge Z')$$
$$X \wedge X' \xrightarrow{f \wedge f'} Y \wedge Y' \xrightarrow{k_2} W \xrightarrow{\delta \equiv q_2} \Sigma(X \wedge X')$$
$$Z \wedge X' \xrightarrow{k_3} W \xrightarrow{q_3} Y \wedge Z' \xrightarrow{\delta \simeq g \wedge h'} \Sigma(Z \wedge X')$$

Note that there are no signs here. The sign inserted in one of the corresponding distinguished triangles listed in each of (TC3) and (TC3') is dictated by (T2). Straightforward diagram chases, using commutative diagrams in $\mathscr{B}$, show that the diagrams displayed in (TC3) and (TC3′) commute in $\mathscr{C}$. Each of these diagrams has one arrow whose label is given with a minus sign. Without the sign, the square of which the arrow is one side would anticommute for the same reason that the bottom right square of (4.5) anticommutes.

This completes the verification of (TC3) and (TC3′). While (TC3′) also follows formally from (TC3), its present proof makes (TC4) obvious. Indeed, the following square is a pushout in $\mathscr{B}$ to which Lemma 5.7 applies.

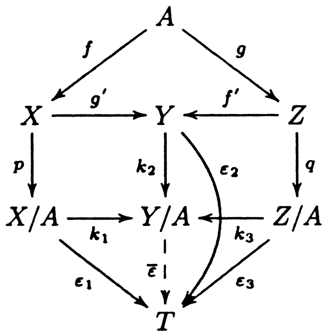
$$\begin{array}{ccc} V & \xrightarrow{j_2} & Y \wedge Y' \\ {\scriptstyle (j_1, j_3)}\downarrow & & \downarrow{\scriptstyle k_2} \\ (X \wedge Z') \vee (Z \wedge X') & \xrightarrow[(k_1, k_3)]{} & W \end{array}$$

Reinterpreting the squares of (TC3) and (TC3′) in terms of equivalent Verdier braids, we conclude that the strong form of the axioms (TC3) and (TC4) specified in Definition 4.11 holds.

7. HOW TO PROVE THE BRAID DUALITY AXIOM

We must still verify (TC5). We retain the assumptions of the previous two sections. We will use the following elementary observation.

LEMMA 7.1. *Suppose given a commutative diagram*



in $\mathscr{B}$ in which the maps f, g, f', and g' are cofibrations between cofibrant objects, the maps p, k_2, and q are quotient maps, and the maps k_1 and k_3 are induced by g' and f'. Let $\bar{\varepsilon}\colon Y/A \longrightarrow T$ be the map induced by passage to quotients from ε_2. Then $\bar{\varepsilon} \circ k_1 = \varepsilon_1$ and $\bar{\varepsilon} \circ k_3 = \varepsilon_3$.

As in Section 4, replace (f', g', h') by the distinguished triangle

(7.2) $$DZ \xrightarrow{Dg} DY \xrightarrow{Df} DX \xrightarrow{D(\Sigma^{-1}h)} \Sigma DZ.$$

A priori, this lies in $\mathscr{C}$. As we discuss in more detail shortly, we can represent it in $\mathscr{B}$ by a canonical triangle of the form displayed in (5.5). We can then define

$$V = (Y \wedge DZ) \cup_{X \wedge DZ} (X \wedge DY) \qquad \text{and} \qquad W = (Y \wedge DY)/(X \wedge DZ)$$

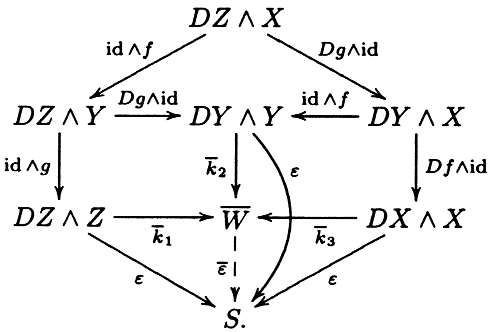
as in Section 6, with canonical maps j_i, p_i, q_i, and k_i. We can also define

$$\bar{V} = (DY \wedge X) \cup_{DZ \wedge X} (DZ \wedge Y) \qquad \text{and} \qquad \bar{W} = (DY \wedge Y)/(DZ \wedge X)$$

with canonical maps $\bar{j}_i$, $\bar{p}_i$, $\bar{q}_i$, and $\bar{k}_i$. The commutativity isomorphism γ for $\wedge$ in $\mathscr{B}$ induces isomorphisms $\gamma\colon V \longrightarrow \bar{V}$ and $\gamma\colon W \longrightarrow \bar{W}$ under which

$\gamma \circ r_i \circ \gamma^{-1} = \bar{r}_i$ for $r = j, p, q, k$, as in Remark 4.10. Thus, with these choices, the involution condition of (TC5b) is immediate. We must verify (TC5a) and the rest of (TC5b). For (TC5a), the idea is to apply Lemma 7.1 to the diagram

(7.3)



The problem is that, a priori, the solid arrow part of the diagram only commutes in $\mathscr{C}$. We must show that we can arrange representative objects and maps in $\mathscr{B}$ so that the diagram is already defined and commutative there.

We have internal hom objects $F(X, Y)$ in $\mathscr{B}$. When X is cofibrant and Y is fibrant, $F(X, Y)$ is fibrant. We need the following small observation.

Remark 7.4. In topological examples, S is fibrant, but in cases where that is not so we must use a fibrant approximation $\lambda\colon S \longrightarrow T$. Here λ is an acyclic cofibration. Since a pushout of an acyclic cofibration is an acyclic cofibration, it follows from the pushout-product axiom that we have a composite of acyclic cofibrations

$$T \cong S \wedge T \longrightarrow (T \wedge S) \cup_{S \wedge S} (S \wedge T) \longrightarrow T \wedge T.$$

Since T is fibrant, there is a retraction $r\colon T \wedge T \longrightarrow T$, and r is clearly a weak equivalence.

For a cofibrant object X, a cofibrant approximation $\phi\colon DX \longrightarrow F(X, T)$ gives a fibrant and cofibrant representative DX in $\mathscr{B}$ for the dual of X in $\mathscr{C}$. Moreover, the composite

$$DX \wedge X \xrightarrow{\phi \wedge \mathrm{id}} F(X, T) \wedge X \xrightarrow{\varepsilon} T$$

in $\mathscr{B}$ represents the evaluation map $\varepsilon\colon DX \wedge X \longrightarrow S$ in $\mathscr{C}$.

We have the cofibration $f\colon X \longrightarrow Y$ and the quotient map $g\colon Y \longrightarrow Z$. The composite

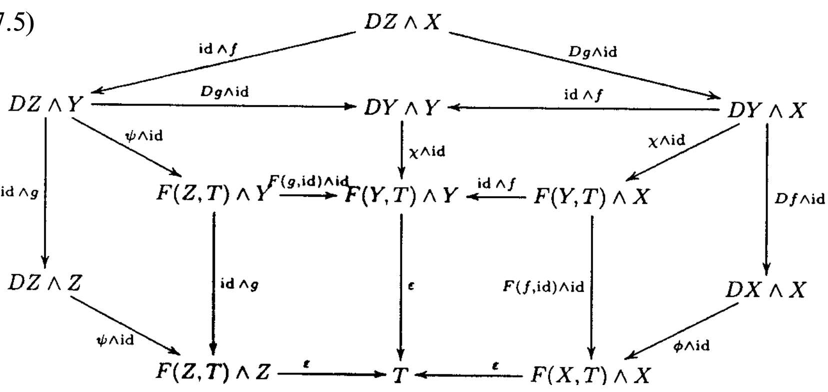
$$F(Z,T) \xrightarrow{F(g,\,\mathrm{id})} F(Y,T) \xrightarrow{F(f,\,\mathrm{id})} F(X,T)$$

is the trivial map, and $F(f, \mathrm{id})$ is a fibration. Choose a cofibrant approximation $\psi\colon DZ \longrightarrow F(Z,T)$ as above and factor $F(g,\mathrm{id}) \circ \psi$ as the composite of a cofibration $Dg\colon DZ \longrightarrow DY$ and an acyclic fibration $\chi\colon DY \longrightarrow F(Y,T)$. Define $DX = DY/DZ$ and let $Df\colon DY \longrightarrow DX$ be the quotient map. Since the composite $F(f,\mathrm{id}) \circ \chi\colon DY \longrightarrow F(X,T)$ is trivial when restricted to DZ, it factors as $\phi \circ Df$ for a map $\phi\colon DX = DY/DZ \longrightarrow F(X,T)$. Clearly χ is a cofibrant approximation, and it is implicit in the verification of (TC2) by use of fiber sequences that ϕ is a weak equivalence and thus a cofibrant approximation. Setting

$$D(\Sigma^{-1}h) = \delta\colon DX \longrightarrow \Sigma DZ,$$

we have the required canonical triangle (7.2). Moreover, we have the following commutative diagram in $\mathscr{B}$. It represents the diagram (7.3) in $\mathscr{C}$ and allows us to apply Lemma 7.1 to construct a map $\bar{\varepsilon}\colon \bar{W} \longrightarrow T$ in $\mathscr{B}$ that represents the map $\bar{\varepsilon}$ in $\mathscr{C}$ that is required to verify (TC5a).

(7.5)



To complete the verification of (TC5b), it remains to show that the diagram of (TC3′) centered around $\bar{W}$ is isomorphic in $\mathscr{C}$ to the dual of the diagram of (TC3) centered around V. We need a standard observation that can be verified by comparing cofiber and fiber sequences as in [5, 6.3; or 8, III.2.3].

Remark 7.6. Let $p\colon E \longrightarrow B$ be a fibration in $\mathscr{B}$ with fiber $i\colon F \longrightarrow E$, so that F is the pullback of p along $* \longrightarrow B$. We have a fiber sequence in canonical form

$$\Omega B \xrightarrow{\delta} F \xrightarrow{i} E \xrightarrow{p} B.$$

This is Eckmann–Hilton dual to (5.5). Shifting to the right, it gives rise to a distinguished triangle

$$F \xrightarrow{i} E \xrightarrow{p} B \xrightarrow{\Sigma\delta} \Sigma F \tag{7.7}$$

in $\mathscr{C}$. We have an explicit comparison of this triangle with the canonical distinguished triangle

$$F \xrightarrow{i} E \longrightarrow C(i) \longrightarrow \Sigma F.$$

In fact, the canonical composite $q\colon Ci \longrightarrow E/F \longrightarrow B$ in $\mathscr{B}$ is a weak equivalence that restricts to p on E and makes the following diagram commute in $\mathscr{C}$:

$$\begin{array}{ccc} C(i) & \longrightarrow & \Sigma F \\ {\scriptstyle q}\downarrow & & \| \\ B & \underset{\Sigma\delta}{\longrightarrow} & \Sigma F. \end{array}$$

We use isomorphisms $\xi\colon DX \wedge Y \longrightarrow D(X \wedge DY)$ in $\mathscr{C}$ of (4.12) to identify all entries other than DV and $\bar{W}$ in our diagrams (TC3′) and (TC3), but we must again distinguish between $\mathscr{B}$ and $\mathscr{C}$. We use the duals and cofibrant approximations in $\mathscr{B}$ discussed above. In $\mathscr{B}$, we have a map

$$\rho\colon X \longrightarrow F(DX, T),$$

namely the adjoint of the composite

$$DX \wedge X \xrightarrow{\xi \wedge \mathrm{id}} F(X, T) \wedge X \xrightarrow{\varepsilon} T.$$

Using this and the map $r\colon T \wedge T \longrightarrow T$ of Remark 7.4, we obtain a map

$$DZ \wedge X \xrightarrow{\psi \wedge \rho} F(Z, T) \wedge F(DX, T) \xrightarrow{F(\mathrm{id}, r) \circ \wedge} F(Z \wedge DX, T).$$

We write ξ for this map and for other similarly defined maps. They are representatives in $\mathscr{B}$ for maps ξ as in (4.12), hence they are weak equivalences. Observe that this depends on arguments in $\mathscr{C}$ that are based on the assumption that X, Y, and Z are dualizable.

Let us write $D'(-)$ for the functor $F(-,T)$ on $\mathscr{B}$. We define $\xi\colon \bar{W} \longrightarrow D'(V)$ as follows. The composite of the map

$$(\mathrm{id}\wedge g,\, Df\wedge \mathrm{id})\colon DY\wedge Y \longrightarrow (DY\wedge Z)\times_{DX\wedge Z}(DX\wedge Y)$$

and the cofibration $Dg\wedge f\colon DZ\wedge X \longrightarrow DY\wedge Y$ is the trivial map, hence $(\mathrm{id}\wedge g, Df\wedge \mathrm{id})$ factors through a map

$$\bar{W} \longrightarrow (DY\wedge Z)\times_{DX\wedge Z}(DX\wedge Y).$$

The maps ξ for $DY\wedge Z$, $DX\wedge Z$, and $DX\wedge Y$ are compatible, hence they induce a map

$$(DY\wedge Z)\times_{DX\wedge Z}(DX\wedge Y) \longrightarrow D'(Y\wedge DZ)\times_{D'(X\wedge DZ)} D'(X\wedge DY).$$

Since the functor D' converts pushouts to pullbacks, the target here is isomorphic to $D'(V)$. The composite of the last two maps is the desired map

$$\xi\colon \bar{W} \longrightarrow D'(V).$$

Immediate diagram chases from the definitions give that the following diagrams commute in $\mathscr{B}$.

$$\begin{array}{ccccc}
DZ\wedge Z & \xrightarrow{\bar{k}_1} & \bar{W} & \xrightarrow{\bar{q}_1} & DX\wedge Y \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
D'(Z\wedge DZ) & \xrightarrow[D'(j_3)]{} & D'(V) & \xrightarrow[D'(p_3)]{} & D'(X\wedge DY)
\end{array}$$

$$\begin{array}{ccccc}
DX\wedge X & \xrightarrow{\bar{k}_3} & \bar{W} & \xrightarrow{\bar{q}_3} & DY\wedge Z \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
D'(X\wedge DX) & \xrightarrow[D'(j_1)]{} & D'(V) & \xrightarrow[D'(p_1)]{} & D'(Y\wedge DZ)
\end{array}$$

Now consider the following diagram.

$$\begin{array}{ccccccc}
DZ\wedge X & \xrightarrow{Dg\wedge f} & DY\wedge Y & \xrightarrow{\bar{k}_2} & \bar{W} & \xrightarrow{\bar{q}_2} & \Sigma DZ\wedge X \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \Sigma\xi} \\
D'(Z\wedge DX) & \xrightarrow[D'(g\wedge Df)]{} & D'(Y\wedge DY) & \xrightarrow[D'(j_2)]{} & D'(V) & \xrightarrow[D'(p_2)]{} & \Sigma D'(Z\wedge DX)
\end{array}$$

The left square clearly commutes in $\mathscr{B}$, and another immediate diagram chase from the definitions shows that the middle square commutes in $\mathscr{B}$. We must prove that $\xi\colon \bar{W} \longrightarrow D'(V)$ is a weak equivalence and that the right hand square, which is the only one in sight that involves connecting maps, commutes in $\mathscr{C}$. This will give that our cofibrant version in $\mathscr{B}$ of the diagram of (TC3′) centering around $\bar{W}$ is essentially a cofibrant approximation of a fibrant version in $\mathscr{B}$ of the dual of the diagram of (TC3) centering around V.

In the bottom row of the last diagram, $D'(j_2)$ is a fibration, $D'(g \wedge Df)$ is its fiber, and, by inspection of duality, $D'(p_2)$ can be identified in $\mathscr{C}$ with a map of the form $\Sigma\delta$ as in (7.7). Since the left square commutes in $\mathscr{B}$ and its vertical arrows are weak equivalences, there results a weak equivalence $\xi\colon C(Dg \wedge f) \rightarrow CD'(g \wedge Df)$ that fits into a comparison of canonical distinguished triangles

$$\begin{array}{ccccccc}
DZ \wedge X & \xrightarrow{Dg \wedge f} & DY \wedge Y & \longrightarrow & C(Dg \wedge f) & \longrightarrow & \Sigma DZ \wedge X \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \Sigma\xi} \\
D'(Z \wedge DX) & \xrightarrow[D'(g \wedge Df)]{} & D'(Y \wedge DY) & \longrightarrow & CD'(g \wedge Df) & \longrightarrow & \Sigma D'(Z \wedge DX).
\end{array}$$

Moreover, the following diagram commutes in $\mathscr{B}$, where the bottom arrow q is as in Remark 7.6:

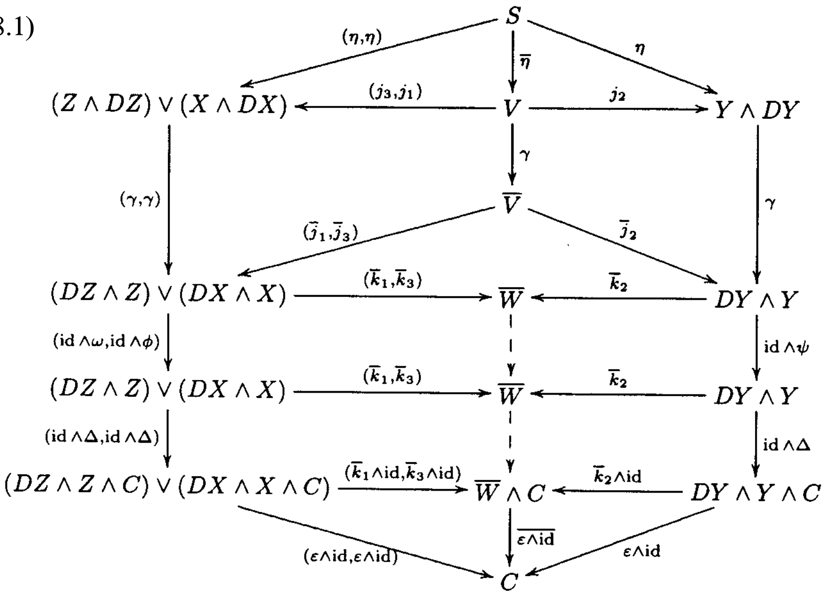
$$\begin{array}{ccc}
C(Dg \wedge f) & \xrightarrow{q} & \bar{W} \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
CD'(g \wedge Df) & \xrightarrow[q]{} & D'(V).
\end{array}$$

Since both maps q and the left map ξ are weak equivalences, so is $\xi\colon \bar{W} \longrightarrow D'(V)$. Moreover, a diagram chase from the two diagrams above and the diagram in Remark 7.6 shows that the right hand square in the third diagram above commutes in $\mathscr{C}$. This completes the proof of (TC5b).

8. THE PROOF OF THE ADDITIVITY THEOREM FOR TRACES

We adopt the methods of the previous section to prove Theorem 1.9. We retain the assumptions and notations there. The idea is to construct a commutative diagram as follows in $\mathscr{C}$.

(8.1)



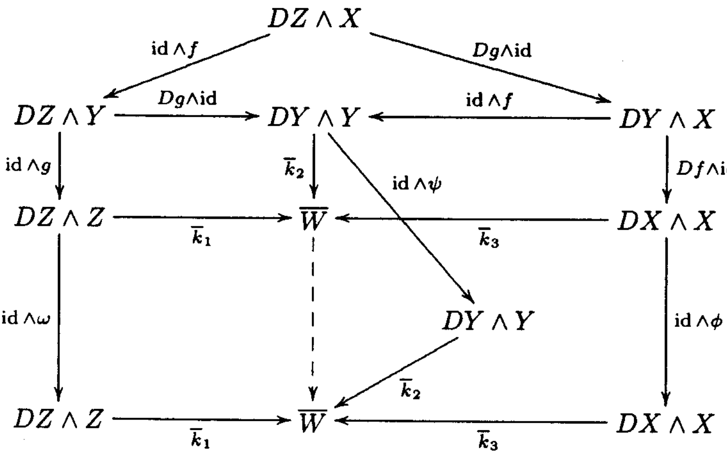
Traversing the outer edge of (8.1), we read off the additivity relation of Theorem 1.9. In view of the diagram in the proof of Theorem 0.1 at the end of Section 4, it remains only to construct dotted arrows that make (8.1) commute.

We first concentrate on the upper dotted arrow. We are given the solid arrow portion of the following diagram in $\mathscr{C}$.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \Sigma\phi} \\
X & \xrightarrow[f\wedge \mathrm{id}]{} & Y & \xrightarrow[g\wedge \mathrm{id}]{} & Z & \xrightarrow[h\wedge \mathrm{id}]{} & \Sigma X
\end{array}$$

As in the previous section, we may take this to be a diagram in $\mathscr{B}$, where f is a cofibration between cofibrant objects, $Z = Y/X$, g is the quotient map, and h is the canonical connecting map of (5.4). We may as well assume further that X and Y are fibrant, although Z need not be.

Since maps in $\mathscr{C}$ between fibrant and cofibrant objects are homotopy classes of maps, the left square is homotopy commutative. We may apply the homotopy extension property [14, p. 1.7] to a homotopy $\psi \circ f \simeq f \circ \phi$ to obtain a homotopy from ψ to a map ψ' such that $\psi' \circ f = f \circ \phi$. Replacing ψ by ψ', we may as well assume that the left square commutes. It then induces a map $\omega\colon Z \longrightarrow Z$ by passage to quotients. With this choice of ω, the middle square commutes and the right square induces a commutative diagram in $\mathscr{C}$. Now the solid arrow portion of the following diagram is easily checked to commute, and Lemma 7.1 applies to give the required dotted arrow.



In the case of Lefschetz constants of maps, where $C = S$, this completes the proof of Theorem 1.9.

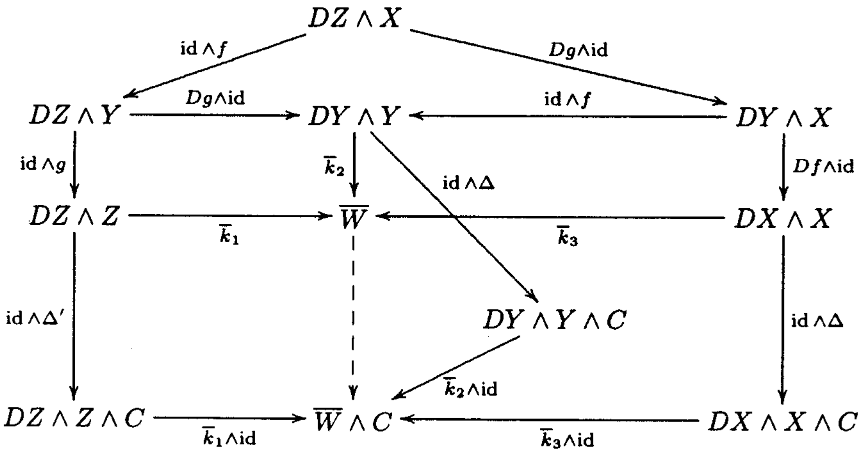
Now consider the lower dotted arrow in (8.1). We are given the solid arrow portion of the following diagram, which we take as above as a diagram in $\mathscr{B}$:

(8.2)
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \vdots{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Sigma\Delta} \\ X \wedge C & \xrightarrow[f\wedge \mathrm{id}]{} & Y \wedge C & \xrightarrow[g\wedge \mathrm{id}]{} & Z \wedge C & \xrightarrow[h\wedge \mathrm{id}]{} & \Sigma(X \wedge C). \end{array}$$

We may as well assume that C is fibrant and cofibrant. However, there is a slight catch to applying the argument just given to arrange that the left square commutes on the nose rather than just up to homotopy.

Remark 8.3. The object $Y \wedge C$ need not be fibrant, hence $p_0\colon (Y \wedge C)^I \longrightarrow Y \wedge C$ need not be a fibration and the model theoretic version of the homotopy extension property may not apply; see [14, p. 1.6, 1.7]. In topological situations, all objects are fibrant and the problem disappears. Moreover, in the applications to natural diagonal maps that I have in mind, $C = Y$ and Δ for X is the composite of $\mathrm{id} \wedge f$ and the diagonal $X \longrightarrow X \wedge X$. In such cases, the left square does commute in $\mathscr{B}$. It seems that a fairly elaborate diagram chase using functorial fibrant approximation can circumvent this problem, but I will leave the details to the interested reader.

Once we have that the left square commutes in (8.2), we can define Δ on Z by passage to quotients. Then the solid arrow portion of the following diagram is easily checked to commute, and Lemma 7.1 applies to give the required dotted arrow.



9. HOMOLOGY AND COHOMOLOGY THEORIES

When $\mathscr{C}$ is the stable homotopy category, one can give a general treatment of the products in homology and cohomology theories that is based solely on the structure of $\mathscr{C}$ as a symmetric monoidal category with a compatible triangulation. There are four basic products here, two of which are called "slant products." A systematic exposition is given by Adams [1, III, Sect. 9] and followed by Switzer [16, pp. 270–284]. We warn the reader that the treatment of slant products in the literature is chaotic. No other two sources seem to give the same signs, and some standard references actually confuse the slant product $\backslash$ with a product that differs only by a sign from the slant product $/$. We run through a version of Adams' definitions and pinpoint the role played by the new axioms. If we were starting from scratch, our preferred version of slant products would differ by signs from those below, but the logical advantage of writing variables in their most natural order is outweighed by the need for consistency in the literature. Adams and Switzer make no use of function spectra $F(X, Y)$, which were only obtainable by use of Brown's representability theorem at the time they were writing, and this obscures the formal nature of their definitions of the products.

For an object X of $\mathscr{C}$ and an integer n, define

$$\pi_n(X) = \mathscr{C}(S^n, X).$$

When $\mathscr{C}$ is the stable homotopy category, $\pi_n(X)$ is the nth homotopy group of the spectrum X. When $\mathscr{C}$ is the derived category of chain complexes over a commutative ring R, S^n is the trivial chain complex given by R in degree n and $\pi_n(X)$ is the nth homology group of the chain complex X. Applying the product $\wedge$ ($\otimes$ in algebraic settings), we obtain a natural pairing

$$\pi_m(X) \otimes \pi_n(Y) \longrightarrow \pi_{m+n}(X \wedge Y). \tag{9.1}$$

For objects X and E, algebraic topologists define

$$E_n(X) = \pi_n(E \wedge X) \qquad \text{and} \qquad E^n(X) = \pi_{-n}F(X, E).$$

Equivalently, $E^n(X) \cong \mathscr{C}(X, \Sigma^n E)$. The four products referred to above are

$$\wedge : D_p(X) \otimes E_q(Y) \longrightarrow (D \wedge E)_{p+q}\,(X \wedge Y), \tag{9.2}$$

$$\cup : D^p(X) \otimes E^q(Y) \longrightarrow (D \wedge E)^{p+q}\,(X \wedge Y), \tag{9.3}$$

$$/ : D^p(X \wedge Y) \otimes E_q(Y) \longrightarrow (D \wedge E)^{p-q}\,(X), \tag{9.4}$$

$$\backslash : D^p(X) \otimes E_q(X \wedge Y) \longrightarrow (D \wedge E)_{q-p}\,(Y). \tag{9.5}$$

The naturality of slant products is better seen by rewriting them in adjoint form

$$/ : D^p(X \wedge Y) \longrightarrow \operatorname{Hom}(E_q(X), (D \wedge E)^{p-q}\,(X)), \tag{9.6}$$

$$\backslash : E_q(X \wedge Y) \longrightarrow \operatorname{Hom}(D^p(X), (D \wedge E)_{q-p}\,(Y)). \tag{9.7}$$

The four products are obtained by passing to π_* and applying the pairing (9.1) and functoriality, starting from formally defined canonical maps

$$D \wedge X \wedge E \wedge Y \longrightarrow D \wedge E \wedge X \wedge Y, \tag{9.8}$$

$$F(X, D) \wedge F(Y, E) \longrightarrow F(X \wedge Y, D \wedge E), \tag{9.9}$$

$$F(X \wedge Y, D) \wedge E \wedge Y \longrightarrow F(X, D \wedge E), \tag{9.10}$$

$$F(X, D) \wedge E \wedge X \wedge Y \longrightarrow D \wedge E \wedge Y. \tag{9.11}$$

Here (9.10) is obtained by permuting E and Y and using the natural isomorphism

$$F(X \wedge Y, D) \cong F(Y, F(X, D)),$$

the evaluation map ε: $F(Y, F(X, D)) \wedge Y \longrightarrow F(X, D)$, and the natural map

$$\nu\colon F(X, D) \wedge E \longrightarrow F(X, D \wedge E),$$

while (9.11) is obtained by permuting E and X and using the evaluation map ε: $F(X, D) \wedge X \longrightarrow D$.

Of course, when $D = E$ is a monoid in $\mathscr{C}$ (ring spectrum in the algebraic topology setting), we can compose the given external products with maps induced by the product $E \wedge E \longrightarrow E$ to obtain internal products. Similarly, when $X = Y$ has a coproduct $X \longrightarrow X \wedge X$ or product $X \wedge X \longrightarrow X$, we can obtain internal products by composition. In topology, we are thinking of reduced cohomology and the diagonal map $A_+ \longrightarrow (A \times A)_+ \cong A_+ \wedge A_+$ on spaces A. The internalization of the product $\backslash$ is the cap product.

There are many unit, associativity, and commutativity relations relating the four products, and these are catalogued in [1, 16]. Without exception, these formulas are direct consequences of our axioms for a symmetric monoidal category with a compatible triangulation. In particular, Adams [1, pp. 235–244] and Switzer [16, pp. 276–283] catalogue many formulas and commutative diagrams that relate the four products to the connecting homomorphisms in the homology and cohomology of pairs (X, A) and (Y, B), the crucial point being the correct handling of signs. Modulo change of notation, they are considering the behavior of smash products and function spectra with respect to pairs of distinguished triangles in the stable homotopy category. Our compatibility axioms give what is needed to make the derivations of these formulas and diagrams formal consequences of the axioms.

REFERENCES

1. J. F. Adams, "Stable Homotopy and Generalized Homology," Univ. of Chicago Press, Chicago, 1974.
2. A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982), 5–171.
3. M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, *Compositio Math.* **86** (1993), 209–234.
4. A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, "Rings, Modules, and Algebras in Stable Homotopy Theory," Mathematical Surveys and Monographs, Vol. 47, Amer. Math. Soc., Providence, 1997.

5. M. Hovey, “Model Categories,” Mathematical Surveys and Monographs, Vol. 63, Amer. Math. Soc., Providence, 1999.
6. M. Hovey, J. H. Palmieri, and N. P. Strickland, Axiomatic stable homotopy theory, *Mem. Amer. Math. Soc.* **610** (1997).
7. I. Kriz and J. P. May, Operads, algebras, modules, and motives, *Astérisque* **233** (1995).
8. L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure), “Equivariant Stable Homotopy Theory,” Lecture Notes in Mathematics, Vol. 1213, Springer-Verlag, New York/Berlin, 1986.
9. M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, Model categories of diagram spectra, *Proc. London Math. Soc.* **82** (2001), 441–512.
10. H. Margolis, “Spectra and the Steenrod Algebra,” North-Holland, Amsterdam, 1983.
11. J. P. May, Picard groups, Grothendieck rings, and Burnside rings of categories, *Adv. Math.*, in press.
12. A. Neeman, Some new axioms for triangulated categories, *J. Algebra* **139** (1991), 221–255.
13. A. Neeman, “Triangulated Categories,” Annals of Mathematics Studies, Vol. 148, Princeton Univ. Press, Princeton, NJ, 2001.
14. D. G. Quillen, “Homotopical Algebra,” Lecture Notes in Mathematics, Vol. 43, Springer-Verlag, New York/Berlin, 1967.
15. S. Schwede and B. Shipley, Algebras and modules in monoidal model categories, *Proc. London Math. Soc.* **80** (2000), 491–511.
16. R. M. Switzer, “Algebraic Topology—Homotopy and Homology,” Springer-Verlag, New York/Berlin, 1975.
17. J. L. Verdier, Catégories dérivées, Lecture Notes in Mathematics, Vol. 569, pp. 262–311, Springer-Verlag, New York/Berlin, 1971.