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CALCULATING SIMPLICIAL LOCALIZATIONS*

W.G. DWYER
Yale University, Department of Mathematics, New Haven, CO 06520, USA

D.M. KAN
Massachusetts Institute of Technology, Cambridge, MA, USA

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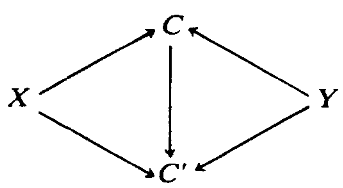
1. Introduction

1.1. Summary. This paper is essentially a continuation of [3], where we introduced a (*standard*) *simplicial localization* functor, which assigned to every category **C** and subcategory $\mathbf{W} \subset \mathbf{C}$, a *simplicial category* $L\mathbf{C}$ with in each dimension the same objects as **C** (i.e. for every two objects $X, Y \in \mathbf{C}$, the maps $X \to Y \in L\mathbf{C}$ form a simplicial set $L\mathbf{C}(X, Y)$). This simplicial localization has all kinds of nice general properties, but, except in a few extreme cases [3, Section 5], it is difficult to get a hold on the homotopy type of the simplicial sets $L\mathbf{C}(X, Y)$. In this paper we therefore consider a homotopy variation on the standard simplicial localization $L\mathbf{C}$, the *hammock localization* $L^{\mathrm{H}}\mathbf{C}$ (Section 2), which (Section 3) has some of the nice properties of the standard localization only up to homotopy, but is in other respects considerably better behaved. In particular (Sections 4 and 5) the simplicial sets $L^{\mathrm{H}}\mathbf{C}(X, Y)$ are much more accessible; *each simplicial set* $L^{\mathrm{H}}\mathbf{C}(X, Y)$ *is the direct limit of a diagram of simplicial sets which are nerves of categories* and (Section 6) *if the pair* $(\mathbf{C}, \mathbf{W})$ *admits a "homotopy calculus of fractions," then several of these nerves already have the homotopy type of* $L^{\mathrm{H}}\mathbf{C}(X, Y)$. When **W** satisfies a mild closure condition this happens, for instance, if (Section 7) the pair $(\mathbf{C}, \mathbf{W})$ *admits a calculus of left fractions* in the sense of Gabriel–Zisman [5] or if (Section 8) **W** *is closed under push outs*, in which case $L^{\mathrm{H}}\mathbf{C}(X, Y)$ has the homotopy type of the nerve of the category which has as objects the sequences $X \to C \leftarrow Y$ in **C** for which the second map is in **W** and which has as maps the commutative diagrams



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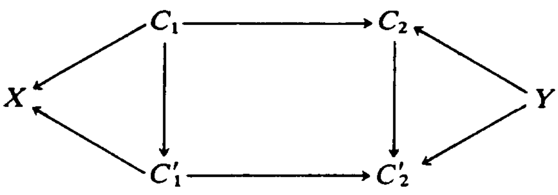
between two such sequences in which the vertical map is also in $\mathbf{W}$. It also happens (Section 8) if $\mathbf{C}$ *is a model category* in the sense of Quillen [8] and $\mathbf{W} \subset \mathbf{C}$ its subcategory of weak equivalences, in which case $L^{\mathrm{H}}\mathbf{C}(X, Y)$ has the homotopy type of the nerve of the category which has as objects the sequences $X \leftarrow C_1 \rightarrow C_2 \leftarrow Y$ in $\mathbf{C}$ for which the outside maps are in $\mathbf{W}$ and which has as maps the commutative diagrams



between two such sequences in which the vertical maps are also in $\mathbf{W}$.

In an appendix (Sections 9 and 10) we develop a two-sided version of the *Grothendieck construction* [10], which we need in order to prove that the simplicial sets $L\mathbf{C}(X, Y)$ and $L^{\mathrm{H}}\mathbf{C}(X, Y)$ have the same homotopy type and which also seems to be of interest in its own right.

1.2. Notation, terminology, etc. These will be as in [3, 1.4], with the following additions.

(i) *The category* sO-**Gr**. Let O be an arbitrary but *fixed* set. Then we denote by O-**Gr** the category of *O-graphs* [6, p. 48] and by sO-**Gr** the category of *simplicial O-graphs*, i.e. simplicial objects over O-**Gr**. If O consists of only one element, then sO-**Gr** is just the category s **Sets** of simplicial sets.

(ii) *The forgetful functor* sO-**Cat** $\rightarrow$ sO-**Gr**. By forgetting composition, every category in sO-**Cat** gives rise to a simplicial O-graph, which we usually denote by the *same* symbol.

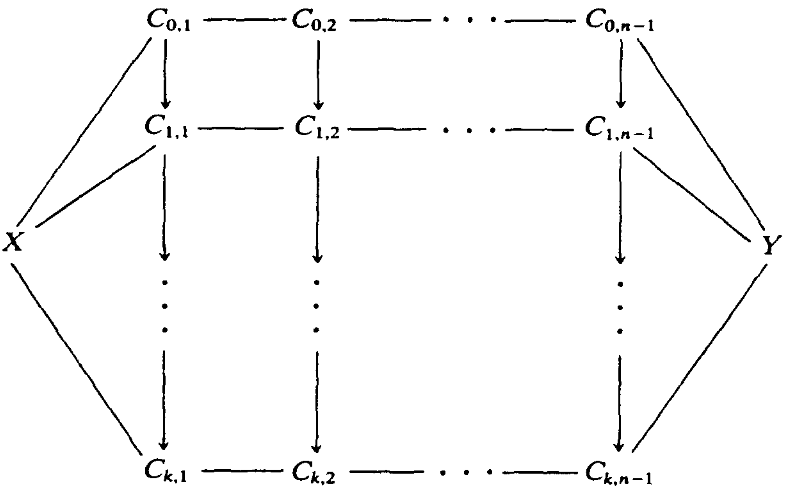
(iii) *Weak equivalences in* sO-**Gr**. These are the maps $\mathbf{A} \rightarrow \mathbf{B} \in$ sO-**Gr** which, for every two objects $X, Y \in O$, induce a weak homotopy equivalence of simplicial sets $\mathbf{A}(X, Y) \sim \mathbf{B}(X, Y)$. This terminology clearly is compatible with (ii) above and [3, 1.4(v)].

(iv) *The category* **Cat**. This is [6, p. 12] the category of all small categories.

2. The hammock localization

Given a category $\mathbf{C} \in O$-**Cat** and a subcategory $\mathbf{W} \subset \mathbf{C}$ (1.2) we construct a simplicial category $L^{\mathrm{H}}\mathbf{C}, \mathbf{W})$ (for short $L^{\mathrm{H}}\mathbf{C}) \in$ sO-**Cat**, the *hammock localization* of $\mathbf{C}$ with respect to $\mathbf{W}$, and observe that this hammock localization is a homotopy variation on the standard simplicial localization of [3].

2.1. The hammock localization. Let $\mathbf{C} \in O\text{-}\mathbf{Cat}$ be a category and $\mathbf{W} \subset \mathbf{C}$ a subcategory (1.2). The *hammock localization* of $\mathbf{C}$ with respect to $\mathbf{W}$ then is the *simplicial* category $L^{\mathrm{H}}\mathbf{C}, \mathbf{W})$ (or short $L^{\mathrm{H}}\mathbf{C}) \in sO\text{-}\mathbf{Cat}$ (1.2) defined as follows: for every two objects $X, Y \in \mathbf{C}$, the k-simplices of the simplicial set $L^{\mathrm{H}}\mathbf{C}(X, Y)$ will be the "reduced *hammocks* of width k and any length" between X and Y, i.e. the commutative diagrams in $\mathbf{C}$ of the form



(*)

in which

(i) n, the length of the hammock, is any integer $\geqslant 0$,

(ii) all vertical maps are in $\mathbf{W}$,

(iii) in each column, all maps go in the same direction; if they go to the left, then they are in $\mathbf{W}$,

(iv) the maps in adjacent columns go in different directions, and

(v) no column contains only identity maps.

Faces, degeneracies and compositions are defined in the obvious manner, i.e. the i-face is obtained by omitting the i-row and the i-degeneracy by repeating the i-row; if the resulting hammock is not reduced (i.e. does not satisfy (iv) and (v)), then it can easily be made so by repeatedly

(iv)′ composing two adjacent columns whenever their maps go in the same direction, and

(v)′ omitting any column which contains only identity maps.

This hammock localization is a homotopy variation on the standard simplicial localization L of [3]. More precisely

2.2. Proposition. *The obvious functions* [3, Section 4]

$$L^{\mathrm{H}}\mathbf{C} \leftarrow \operatorname{diag} L^{\mathrm{H}} F_* \mathbf{C} \rightarrow F_*\mathbf{C}[F_*\mathbf{W}^{-1}] = L\mathbf{C}$$

are both weak equivalences (1.2).

This follows readily from [3, 1.4(vii) and 2.6] and the following two lemmas which will be proved in Section 5.

2.3. Comparison lemma. *Let* $\mathbf{C} \in O$-**Cat** *be such that* $C = D * W$ [3, 1.4], *where* D *and* W *are free* [3, 2.1]. *Then the obvious functor* $L^{\mathrm{H}}\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ *is a weak equivalence.*

2.4. Homotopy lemma. *Let* $\mathbf{A}, \mathbf{B} \in$ sO-**Cat**, *let* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *be subcategories and let* $S: \mathbf{A} \to \mathbf{B} \in$ sO-**Cat** *be a functor which sends all of* $\mathbf{U}$ *into* $\mathbf{V}$. *If* $S: \mathbf{A} \to \mathbf{B}$ *and its restriction* $S: \mathbf{U} \to \mathbf{V}$ *are both weak equivalences, then so is the induced function* $\operatorname{diag} L^{\mathrm{H}}\mathbf{A} \to \operatorname{diag} L^{\mathrm{H}}\mathbf{B}$.

2.5. Remark. One can extend the definition of the hammock localization to sO-**Cat**, i.e., given $\mathbf{B} \in$ sO-**Cat** and a subcategory $\mathbf{V} \subset \mathbf{B}$, define the *hammock localization* of $\mathbf{B}$ with respect to $\mathbf{V}$ as $\operatorname{diag} L^{\mathrm{H}}\mathbf{B}$. The above two lemmas then imply that *the obvious functors below* [3, 6.1] *are also weak equivalences*

$$\operatorname{diag} L^{\mathrm{H}}\mathbf{B} \leftarrow \operatorname{diag} L^{\mathrm{H}}F_*\mathbf{B} \to \operatorname{diag} F_*\mathbf{B}[F_*\mathbf{V}^{-1}] = L\mathbf{B}.$$

3. Properties of the hammock localization

We now list some properties of the hammock localization L^{H}, which show that the hammock localization has disadvantages as well as advantages over the standard simplicial localization L of [3].

Definition 2.1 immediately implies

3.1. Proposition. *For every two objects* $X, Y \in \mathbf{C}$, *the components of* $L^{\mathrm{H}}\mathbf{C}(X, Y)$ *are in* 1–1 *correspondence with the maps* $X \to Y \in \mathbf{C}[\mathbf{W}^{-1}]$ [3, Section 3], *i.e.*

$$\pi_0 L^{\mathrm{H}}\mathbf{C} = \mathbf{C}[\mathbf{W}^{-1}].$$

Unlike the standard localization, the *hammock localization comes with an obvious functor* $p: \mathbf{C} \to L^{\mathrm{H}}\mathbf{C} \in$ sO-**Cat**. It has the convenient property

3.2. Proposition. *If* $\mathbf{C} = \mathbf{D} * \mathbf{W}$ [3, 1.4], *then the following natural diagram in* sO-**Cat** *is a pushout*

$$\begin{array}{ccc} \mathbf{W} & \longrightarrow & L^{\mathrm{H}}\mathbf{W} \\ {\scriptstyle \text{incl.}}\downarrow & & \downarrow{\scriptstyle \text{incl.}} \\ \mathbf{D} * \mathbf{W} = \mathbf{C} & \longrightarrow & L^{\mathrm{H}}\mathbf{C} = \mathbf{D} * L^{\mathrm{H}}\mathbf{W} \end{array}$$

Proof. This follows readily from the fact that every non-identity map of $\mathbf{C}$ admits a unique factorization into non-identity maps of $\mathbf{D}$ and $\mathbf{W}$, in which no two adjacent maps are either both in $\mathbf{D}$ or both in $\mathbf{W}$.

That the simplicial sets in the hammock localization are more accessible than those in the standard localization is, roughly speaking, due to the fact that

(i) the hammock localization is defined more directly in terms of $\mathbf{C}$ and $\mathbf{W}$, and

(ii) cancellation in any given dimension is achieved not by "imposing relations" in that same dimension, but by "imposing homotopy relations", i.e. adding maps, in the next higher dimension.

As a result one has, in contrast to [3, 4.4],

3.3. Proposition. *Let $u: X \to Y \in \mathbf{W}$. Then u induces, for every object $V \in \mathbf{C}$, weak homotopy equivalences*

$$L^{H}\mathbf{C}(V, X) \overset{u_*}{\simeq} L^{H}\mathbf{C}(V, Y) \quad \textit{and} \quad L^{H}\mathbf{C}(Y, V) \overset{u^*}{\simeq} L^{H}\mathbf{C}(X, V).$$

We end with discussing the behavior of the simplicial localizations under *functors* and *natural transformations.* First we note the existence of

3.4. Induced simplicial functors. Let $\mathbf{C} \in O$-**Cat** and $\mathbf{C}' \in O'$-**Cat** be categories and $\mathbf{W} \subset \mathbf{C}$ and $\mathbf{W}' \subset \mathbf{C}'$ subcategories. *A functor $S: \mathbf{C} \to \mathbf{C}'$ which sends all of* $\mathbf{W}$ *into* $\mathbf{W}'$ *then induces a simplicial functor $L^{H}S: L^{H}\mathbf{C} \to L^{H}\mathbf{C}'$*, i.e. S induces, for every pair of objects $X, Y \in \mathbf{C}$, a simplicial map $L^{H}\mathbf{C}(X, Y) \to L^{H}\mathbf{C}'(SX, SY)$.

Also not hard to prove is

3.5. Proposition. *Let* $\mathbf{C} \in O$-**Cat** *and* $\mathbf{C}' \in O'$-**Cat** *be categories and $W \subset C$* and *$W' \subset C'$ subcategories, let S_1, S_2*: $\mathbf{C} \to \mathbf{C}'$ *be functors which send all of* $\mathbf{W}$ *into* $\mathbf{W}'$ *and let $s: S_1 \to S_2$ be a natural transformation such that $sX \in \mathbf{W}'$ for every object $X \in \mathbf{C}$. Then, for every pair of objects $X, Y \in \mathbf{C}$, the following diagram commutes up to homotopy*

$$\begin{array}{ccccc} & & L^{H}\mathbf{C}'(S_1X, S_1Y) & & \\ & \nearrow^{L^{H}S_1} & & \searrow^{sY^*} & \\ L^{H}C(X, Y) & & & & L^{H}C'(S_1X, S_2Y) \\ & \searrow_{L^{H}S_2} & & \nearrow_{sX^*} & \\ & & L^{H}c'(S_2X, S_2Y) & & \end{array}$$

3.6. Corollary. *Let $S: C \to C'$ and $T: C' \to C$ be a pair of adjoint functors such that*

(i) *S maps all of* $\mathbf{W}$ *into* $\mathbf{W}'$ *and T maps all of* $\mathbf{W}'$ *into* $\mathbf{W}$, *and*

(ii) *for every object $X \in C$, the adjunction map $X \to TSX$ is in* $\mathbf{W}$, *and, for every object $Y' \in C'$, the adjunction map $STY' \to Y'$ is in* $\mathbf{W}'$.

Then $\pi_0 L^{\mathrm{H}} S : \pi_0 L^{\mathrm{H}}\mathbf{C} \approx \pi_0 L^{\mathrm{H}}\mathbf{C}'$ *is an equivalence of categories which has the equivalence* $\pi_0 L^{\mathrm{H}} T : \pi_0 L^{\mathrm{H}}\mathbf{C}' \approx \pi_0 L^{\mathrm{H}}\mathbf{C}$ *as an inverse, and, for every pair of objects* $X, Y \in \mathbf{C}$ *and every pair of objects* $X', Y' \in \mathbf{C}'$, *the induced maps*

$$L^{\mathrm{H}}\mathbf{C}(X, Y) \to L^{\mathrm{H}}\mathbf{C}'(SX, SY) \quad \text{and} \quad L^{\mathrm{H}}\mathbf{C}'(X', Y') \to L^{\mathrm{H}}\mathbf{C}(TX', TY')$$

are weak homotopy equivalences.

3.7. Remark. Of course 3.4 *also holds for the standard simplicial localization* L, and *so do* 3.5 *and* 3.6 (in view of 2.2). However it is rather difficult to prove the analog of 3.5 for L directly.

4. The indexing category II

The proofs of the lemmas 2.3 and 2.4 (in Section 5) will use the fact that the hammock localization is the direct limit of a diagram of simplicial graphs (1.2), indexed by a category **II**. We therefore briefly discuss here this indexing category and the behavior of limits over it.

4.1. The indexing category II. Let J denote the ordered set of the integers ≥ 1 and, for every finite subset $S \subset J$, let $|S|$ denote its number of elements. The objects of **II** then will be the ordered pairs (S, T) of disjoint finite subsets of J such that

$$S \cup T = \{1, \ldots, |S \cup T|\},$$

and the maps $(S', T') \to (S, T)$ will be the weakly order preserving functions $f : S' \cup T' \to S \cup T$ such that $f(S') \subset S$ and $f(T') \subset T$.

To get a hold on the direct limits over **II**, it is convenient to consider, for every integer $n \geq 0$, the full subcategory $\mathbf{II}_n \subset \mathbf{II}$ generated by the objects (S, T) for which $|S \cup T| \leq n$, and to denote, for a functor $\sigma : \mathbf{II} \to \mathbf{s\,Sets}$ (1.2), its restriction to the subcategories $\mathbf{II}_n$ by $\sigma_n : \mathbf{II}_n \to \mathbf{s\,Sets}$. Then one clearly has

4.2. Proposition. *For every functor* $\sigma : \mathbf{II} \to \mathbf{s\,Sets}$

$$\underrightarrow{\lim}^{\mathbf{II}}\, \sigma = \underrightarrow{\lim}^{n}\, \underrightarrow{\lim}^{\mathbf{II}_n}\, \sigma_n .$$

Next one can deal as follows with the direct limits over the $\mathbf{II}_n$. Let $I_n^0 \subset J$ (resp. $I_n^1 \subset J$) consist of the even (resp. odd) integers ≥ 1 and $\leq n$, and, for a functor $\sigma : \mathbf{II} \to \mathbf{s\,Sets}$, let

$$\mathrm{bd}\, \sigma(I_n^{\varepsilon}, I_n^{1-\varepsilon}) \subset \sigma(I_n^{\varepsilon}, I_n^{1-\varepsilon}) \quad \varepsilon = 0, 1$$

be the "boundary", i.e. the union of the images of the maps $\sigma_n f$, where f runs though all proper injections $(S, T) \to (I_n^{\varepsilon}, I_n^{1-\varepsilon}) \in \mathbf{II}_n$. Then it is not difficult to verify

4.3. Proposition. *For every functor* $\sigma : \mathbf{II} \to \mathbf{s\,Sets}$ *and every integer* $n \geq 0$, *the induced diagram*

$$\begin{array}{ccc}
\operatorname{bd} \sigma(I_n^0, I_n^1) \cup \operatorname{bd} \sigma(I_n^1, I_n^0) & \longrightarrow & \lim^{\mathbf{II}_{n-1}} \sigma_{n-1} \\
\downarrow & & \downarrow \\
\sigma(I_n^0, I_n^1) \cup \sigma(I_n^1, I_n^0) & \longrightarrow & \lim^{\mathbf{II}_n} \sigma_n
\end{array}$$

is a pushout.

To say more one has to restrict oneself to

4.4. Proper functors $\mathbf{II} \to$ s Sets. As *an injection in* $\mathbf{II}$ *is completely determined by its image*, one can, for two injections $f, g \in \mathbf{II}$ with the same range, define their *intersection* $f \cap g \in \mathbf{II}$ by the formula

$$\operatorname{im}(f \cap g) = (\operatorname{im} f) \cap (\operatorname{im} g)$$

and call a functor $\sigma: \mathbf{II} \to$ s **Sets** *proper* if it has the properties:

(i) if $f \in \mathbf{II}$ is an injection, then so is $\sigma f \in$ s **Sets**, and

(ii) if f and $g \in \mathbf{II}$ are injections with the same range, then the induced map

$$\operatorname{im} \sigma(f \cap g) \to (\operatorname{im} \sigma f) \cap (\operatorname{im} \sigma g) \in \text{s } \mathbf{Sets}$$

is an isomorphism.

Propositions 4.2 and 4.3 now imply, by a rather straightforward argument, the following homotopy invariance.

4.5. Proposition. *Let $\sigma, \sigma': \mathbf{II} \to$ s* **Sets** *be proper functors and let $t: \sigma \to \sigma'$ be a natural transformation such that $t(S, T): \sigma(S, T) \to \sigma'(S, T)$ is a weak homotopy equivalence for every object $(S, T) \in \mathbf{II}$. Then the induced map $\underline{\lim}^{\mathbf{II}} \sigma \to \underline{\lim}^{\mathbf{II}} \sigma'$ is also a weak homotopy equivalence.*

4.6. Remark. Propositions 4.2, 4.3 and 4.5 and definition 4.4 clearly also apply to the more general case of functors $\mathbf{II} \to$ sO-**Gr** (1.2).

5. The hammock graphs

In this section we show that (5.5) the hammock localization of Section 2, or rather its underlying simplicial graph (1.2), is the direct limit of a **II**-diagram (4.1) of simplicial graphs (hammock graphs) which are even more accessible than the hammock localization, because they consist of hammocks of *fixed* length and type. Using this result we then prove Lemma 2.3 and reduce Lemma 2.4 to a similar statement (5.7) concerning these hammock graphs, which in turn is an immediate consequence of Example 10.3 and Propositions 10.4 and 10.5.

We start with constructing

5.1. The hammock graphs. Let $\mathbf{C} \in O$-**Cat** be a category and $\mathbf{W} \subset \mathbf{C}$ a subcategory (1.2), let n be an integer $\geqslant 0$ and let $\mathbf{m}$ be a word of length n in $\mathbf{C}$ and $\mathbf{W}^{-1}$. Then we denote also by $\mathbf{m}$ the simplicial O-graph (1.2) such that, for every two objects $X, Y \in \mathbf{C}$, the simplicial set $\mathbf{m}(X, Y)$ has as its k-simplices the hammocks between X and Y of width k, length n and type $\mathbf{m}$, i.e. the commutative diagrams in $\mathbf{C}$ of the form 2.1(∗) in which

(i) all vertical maps are in $\mathbf{W}$, and

(ii) the maps in the ith column go to the right if the $(n+1-i)$-th letter in $\mathbf{m}$ is $\mathbf{C}$; otherwise they go to the left and are in $\mathbf{W}$.

Another way of saying this is that $\mathbf{m}(X, Y)$ *is the nerve* [3, 1.4] *of the category which has the hammocks between X and Y of length n, type* $\mathbf{m}$ *and width* 0 *and* 1 *as its objects and maps.* It therefore makes sense to denote this category by $N^{-1}\mathbf{m}(X, Y)$.

5.2. Example. The hammock graphs $\mathbf{W}^{-1}\mathbf{C}$ and $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$ are exactly the simplicial graphs mentioned in the introduction (1.1).

5.3. The functor $\lambda\mathbf{C}: \mathbf{II} \to s\,O$-Gr. This is the functor which assigns to an object $(S, T) \in \mathbf{II}$ (4.1) the hammock graph described by the word in $\mathbf{C}$ and $\mathbf{W}^{-1}$ of length $|S \cup T|$, in which the ith letter is $\mathbf{C}$ whenever $i \in S$ and is $\mathbf{W}^{-1}$ otherwise. The induced maps are obtained by adding columns of identity to the hammocks involved and/or combining adjacent columns by composing their maps.

This definition readily implies

5.4. Proposition. *The functor* $\lambda\mathbf{C}: \mathbf{II} \to sO$-**Gr** *is proper in the sense of* 4.4 *and* 4.6.

5.5. Proposition. *The reduction map* $r: \lambda\mathbf{C}(S, T) \to L^{\mathbf{H}}\mathbf{C} \in sO$-**Gr** *obtained by reducing* (2.1) *the hammocks involved, induces an isomorphism* $r: \varinjlim^{\mathbf{II}} \lambda\mathbf{C} \approx L^{\mathbf{H}}\mathbf{C}$.

Now we are ready to deal with Lemmas 2.3 and 2.4.

5.6. Proof of homotopy Lemma 2.3. Lemma 2.4 follows easily, by a diagonal argument, from 4.5, 4.6, 5.4 and 5.5 and the following homotopy lemma for hammock graphs, which in turn is an immediate consequence of Example 10.3 and Propositions 10.4 and 10.5.

5.7. Homotopy lemma. *Let* $\mathbf{A}, \mathbf{B} \in sO$-**Cat**, *let* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *be subcategories and let* $R: \mathbf{A} \to \mathbf{B} \in sO$-**Cat** *be a functor which sends all of* $\mathbf{U}$ *into* $\mathbf{V}$. *If* $R: \mathbf{A} \to \mathbf{B}$ *and its restriction* $R: \mathbf{U} \to \mathbf{V}$ are both weak equivalences, then so is, for every object (S, T) ∈ **II**, *the induced map*

$$\operatorname{diag} \lambda\mathbf{A}(S, T) \to \operatorname{diag} \lambda\mathbf{B}(S, T).$$

5.8. Proof of comparison Lemma 2.3. The proof proceeds by means of successive simplification.

(i) In view of 3.2 and [3, 3.2] it suffices to consider *the case* $\mathbf{W} = \mathbf{C}$; in addition one can, of course, assume that *$N\mathbf{W}$ is connected.*

(ii) Let $O' = O \times \pi_1 N\mathbf{W}$ and let $\tilde{\mathbf{W}} \in O'$-**Cat** be the free category such that [3, 2.8] $N^1\tilde{\mathbf{W}}$ is the universal covering of $N^1\mathbf{W}$. We have to show that, for every two objects $X, Y \in O$, every component of $L^H\mathbf{W}(X, Y)$ is contractible. But this is clearly equivalent to showing the contractibility of $L^H\tilde{\mathbf{W}}(X', Y')$ for every two objects $X', Y' \in O'$. In other words, one only has to consider *the case that* $\mathbf{W} = \mathbf{C}$ *and* $N\mathbf{W}$ *is contractible.*

(iii) Next, it is not hard to see, using 3.3, 4.2, 4.3, 5.4 and 5.5, that it suffices to show: *if* $\mathbf{W} = \mathbf{C}$ *and* $N\mathbf{W}$ *is contractible, then* $\mathbf{m}(X, Y)$ *is contractible for every hammock graph* $\mathbf{m}$ *and object* $X \in \mathbf{C}$.

To prove this last statement, one may clearly assume that $\mathbf{W}$ *has only a finite number of generators.* But in that case the proof is straightforward, by induction on the number of generators.

5.9. A slight generalization. The notion of a hammock graph can be slightly generalized by requiring that in certain columns of the hammocks in question the maps are not merely in $\mathbf{W}$ but in certain subcategories of $\mathbf{W}$. For instance, if $\mathbf{U}, \mathbf{V} \subset \mathbf{W}$ are subcategories, then $\mathbf{U}^{-1}\mathbf{C}\mathbf{V}^{-1}$ will consist of the hammocks of $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$ for which the (horizontal) maps in the first column are in $\mathbf{V}$ and those in the last column are in $\mathbf{U}$.

6. Homotopy calculi of fractions

We now give *sufficient conditions* in order that *the reduction map from* $\mathbf{W}^{-1}\mathbf{C}$, $\mathbf{C}\mathbf{W}^{-1}$ *or* $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$ *to* $L^H\mathbf{C}$ *is a weak equivalence.* Some examples will be discussed in Sections 7 and 8.

6.1. Homotopy calculi of fractions. Let $\mathbf{C} \in O$-**Cat** be a category and $\mathbf{W} \subset \mathbf{C}$ a subcategory. Then the pair $(\mathbf{C}, \mathbf{W})$ is said to admit

(i) *a homotopy calculus of* (two-sided) *fractions* if, for every pair of integers $i, j > 0$, the obvious maps

$$\mathbf{W}^{-1}\mathbf{C}^{i+j}\mathbf{W}^{-1} \to \mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}\mathbf{W}^{-1}$$

and

$$\mathbf{W}^{-1}\mathbf{W}^{i+j}\mathbf{W}^{-1} \to \mathbf{W}^{-1}\mathbf{W}^{i}\mathbf{W}^{-1}\mathbf{W}^{j}\mathbf{W}^{-1} \in sO\text{-}\mathbf{Gr}$$

are weak equivalences

(ii) *a homotopy calculus of left fractions* if, for every pair of integers $i, j \geq 0$, the obvious maps

$$\mathbf{W}^{-1}\mathbf{C}^{i+j} \to \mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j} \quad \text{and} \quad \mathbf{W}^{-1}\mathbf{W}^{i+j} \to \mathbf{W}^{-1}\mathbf{W}^{i}\mathbf{W}^{-1}\mathbf{W}^{j} \in sO\text{-}\mathbf{Gr}$$

are weak equivalences, and

(iii) *a homotopy calculus of right fractions* if, for every pair of integers $i, j \geq 0$, the obvious maps

$$\mathbf{C}^{i+j}\mathbf{W}^{-1} \to \mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}\mathbf{W}^{-1} \quad \text{and} \quad \mathbf{W}^{i+j}\mathbf{W}^{-1} \to \mathbf{W}^{i}\mathbf{W}^{-1}\mathbf{W}^{j}\mathbf{W}^{-1} \in \mathrm{s}O\text{-}\mathbf{Gr}$$

are weak equivalences.

As one might expect *a homotopy calculus of left or right fractions implies a homotopy calculus of* (two-sided) *fractions*. This is not obvious from the above definitions, but follows readily from 9.4 and 9.6.

The usefulness of homotopy calculi of fractions is due to the following proposition, which also justifies their names.

6.2. Proposition. (i) *If* $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of fractions, then the reduction maps*

$$\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1} \to L^{\mathrm{H}}\mathbf{C} \quad \textit{and} \quad \mathbf{W}^{-1}\mathbf{W}\mathbf{W}^{-1} \to L^{\mathrm{H}}\mathbf{W} \in \mathrm{s}O\text{-}\mathbf{Gr}$$

are weak equivalences.

(ii) *If* $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of left fractions, then the reduction maps*

$$\mathbf{W}^{-1}\mathbf{C} \to L^{\mathrm{H}}\mathbf{C} \quad \textit{and} \quad \mathbf{W}^{-1}\mathbf{W} \to L^{\mathrm{H}}\mathbf{W} \in \mathrm{s}O\text{-}\mathbf{Gr}$$

are weak equivalences.

(iii) *If* $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of right fractions, then the reduction maps*

$$\mathbf{C}\mathbf{W}^{-1} \to L^{\mathrm{H}}\mathbf{C} \quad \textit{and} \quad \mathbf{W}\mathbf{W}^{-1} \to L^{\mathrm{H}}\mathbf{W} \in \mathrm{s}O\text{-}\mathbf{Gr}$$

are weak equivalences.

For later reference we mention an application to

6.3. Homotopy automorphism complexes. For $\mathbf{B} \in \mathrm{s}O\text{-}\mathbf{Cat}$ and an object $Y \in \mathbf{B}$, the *homotopy automorphism complex* of Y in $\mathbf{B}$ will be the simplicial submonoid

$$\mathrm{haut}_{\mathbf{B}}\, Y \subset \mathbf{B}(Y, Y)$$

consisting of the components of $\mathbf{B}(Y, Y)$ which are invertible in $\pi_0\mathbf{B}(Y, Y)$.

If $\mathbf{C} \in O\text{-}\mathbf{Cat}$ is a category, $\mathbf{W} \subset \mathbf{C}$ a subcategory and $X \in \mathbf{C}$ an object, then the simplicial monoid $L^{\mathrm{H}}\mathbf{W}(X, X)$ is contained in $\mathrm{haut}_{L^{\mathrm{H}}\mathbf{C}}\, X$ and the simplicial group $L\mathbf{W}(X, X)$ is contained in $\mathrm{haut}_{L\mathbf{C}}\, X$, and Propositions 6.2 and 2.2 imply

6.4. Corollary. *If* $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of fractions and* $\mathbf{W}$ *is closed in* $\mathbf{C}$ [3, 3.4], *then, for every object* $X \in \mathbf{C}$, *the inclusions*

$$L^{\mathrm{H}}\mathbf{W}(X, X) \to \mathrm{haut}_{L^{\mathrm{H}}\mathbf{C}}\, X \quad \text{and} \quad L\mathbf{W}(X, X) \to \mathrm{haut}_{L\mathbf{C}}\, X$$

are weak homotopy equivalences.

6.5. Proof of Proposition 6.2. We will only prove the first half of (ii) as the proofs of the other parts are similar.

Let $A, B:\mathbf{II}\to\mathbf{II}$ be the functors given by

$$(S, T)\xrightarrow{A}(\{1,\ldots,|S|\},\emptyset)\quad\text{and}\quad(S, T)\xrightarrow{B}(S, T\cup\{|S\cup T|+1\}).$$

The desired result then follows readily from the following statements.

(i) The inclusions $i:\mathbf{W}^{-1}\mathbf{C}=(\lambda\,\mathbf{C})BA(\{1\},\emptyset)\to\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})BA$ is an isomorphism.

(ii) The map $j:\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})BA\to\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})B$, induced by the injections $A(S, T)\to(S, T)$, is a weak equivalence.

(iii) For every two objects $X, Y\in\mathbf{C}$, the composition $ji:\mathbf{W}^{-1}\mathbf{C}(X, Y)\to\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})B(X, Y)$ is homotopic to the composition

$$\mathbf{W}^{-1}\mathbf{C}(X, Y)=(\lambda\,\mathbf{C})(\{1\},\{2\})(X, Y)\xrightarrow{\text{incl.}}\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})(X, Y)\xrightarrow{k}\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})B(X, Y)$$

where the map k is induced by the inclusions $(S, T)\to B(S, T)$.

(iv) The map $k:\underrightarrow{\lim}^{\mathbf{II}}\lambda\,\mathbf{C}\to\underrightarrow{\lim}^{\mathbf{II}}(\lambda\,\mathbf{C})B$ has as a left inverse the map induced by the inclusion $\operatorname{im} B\subset\mathbf{II}$.

The verification of the statements (i), (iii) and (iv) is straightforward. To prove (ii) one notes that the functors $(\lambda\,\mathbf{C})BA$ and $(\lambda\,\mathbf{C})B$ are both proper in the sense of 4.4 and 4.6 and that it thus (4.5) suffices to show that, for every object $(S, T)\in\mathbf{II}$, the map $(\lambda\,\mathbf{C})BA(S, T)\to(\lambda\,\mathbf{C})B(S, T)$ is a weak equivalence. But this follows from an inductive argument that begins with 5.1 (ii) and continues with 9.4 and 9.6.

7. The classical calculi of fractions

In this section we show that, if $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{W}, \mathbf{W})$ admit classical calculi of left (or right) fractions [5], then

(i) $(\mathbf{C}, \mathbf{W})$ admits also a homotopy calculus of left (or right) fractions in the sense of 6.1

(ii) the simplicial localizations $L\mathbf{C}$ and $L^{\mathrm{H}}\mathbf{C}$ are weakly equivalent to the classical localization $\mathbf{C}[\mathbf{W}^{-1}]$, and

(iii) the nerve of $\mathbf{W}$ has the homotopy type of a disjoint union of $K(\pi, 1)$'s.

This last result, for categories with one object, was proved in [7].

We will actually only consider left fractions; the statements and arguments for right fractions are of course similar and will be left to the reader.

We begin with recalling from [5] the definition of a

7.1. Calculus of left fractions. Let $\mathbf{C}\in\mathcal{O}$-**Cat** be a category and $\mathbf{W}\subset\mathbf{C}$ a subcategory. Then the pair $(\mathbf{C}, \mathbf{W})$ is said to admit *a calculus of left fractions* if:

(i) For each diagram $X'\xrightarrow{u}X\xrightarrow{f}Y\in\mathbf{C}$ with $u\in\mathbf{W}$, there exists a diagram $X'\xrightarrow{f'}Y'\xrightarrow{v}Y\in\mathbf{C}$ with $v\in\mathbf{W}$ and such that $vf=f'u$.

(ii) If $f, g:X\to Y\in\mathbf{C}$ and $u:X'\to X\in\mathbf{W}$ are such that $fu=gu$, then there exists a map $v\in\mathbf{W}$ such that $vf=vg$.

Note that, *if* $\mathbf{C}, \mathbf{W}$) *admits a calculus of left fractions and has the property* (as almost always is the case):

(iii) If f and g are maps in **C** such that fg is defined and if two of f, g and fg are in **W**, then so is the third,

then (**W**, **W**) also admits a calculus of left fractions.

Now we can state our results.

7.2. Proposition. *If* (**C**, **W**) *and* (**W**, **W**) *admit a calculus of left fractions, then* (**C**, **W**) *also admits a homotopy calculus of left fractions and hence* (6.2) *the reduction maps*

$$\mathbf{W}^{-1}\mathbf{C} \to L^{\mathrm{H}}\mathbf{C} \quad \textit{and} \quad \mathbf{W}^{-1}\mathbf{W} \to L^{\mathrm{H}}\mathbf{W} \in \mathrm{s}O\text{-}\mathbf{Gr}$$

are weak equivalences.

7.3. Proposition. *If* (**C**, **W**) *and* (**W**, **W**) *admit a calculus of left fractions, then the natural map*

$$L^{\mathrm{H}}\mathbf{C} \to \pi_0 L^{\mathrm{H}}\mathbf{C} = \mathbf{C}[\mathbf{W}^{-1}] \in \mathrm{s}O\text{-}\mathbf{Cat}$$

is a weak equivalence, i.e. for every two objects $X, Y \in \mathbf{C}$, all components of $L^{\mathrm{H}}\mathbf{C}(X, Y)$ are contractible.

7.4. Proposition. *If* (**W**, **W**) *admits a calculus of left fractions, then the nerve* $N\mathbf{W}$ *has the homotopy type of a disjoint union of* $K(\pi, 1)$'s.

Proof of 7.4 (using 7.3). By [3, 4.3] $N\mathbf{W}$ and $NL\mathbf{W}$ have the same homotopy type and the desired result therefore follows from [3, 5.5], 2.2 and 7.3.

Proof of 7.3 (using 7.2). Given two objects $X, Y \in \mathbf{C}$, let $Y \downarrow \mathbf{W}$ denote the under category (which [6, p. 46] has the maps $Y \to Z \in \mathbf{W}$ as objects) and let $\bar{X}: Y \downarrow \mathbf{W} \to$ **Sets** be the functor which sends a map $Y \to Z \in \mathbf{W}$ to the set $\mathbf{C}(X, Z)$. Then it is not hard to see that [1, Ch. XII]

$$\mathbf{W}^{-1}\mathbf{C}(X, Y) \approx \underline{\mathrm{holim}}^{Y\downarrow \mathbf{W}}\, \bar{X}.$$

Moreover, the fact that (**C**, **W**) and (**W**, **W**) admit a calculus of left fractions readily implies that the category $Y \downarrow \mathbf{W}$ is right filtering [1, p. 331] and hence [1, p. 332] the natural map

$$\underline{\mathrm{holim}}^{Y\downarrow \mathbf{W}}\, \bar{X} \to \underline{\mathrm{lim}}^{Y\downarrow \mathbf{W}}\, \bar{X}$$

is a weak homotopy equivalence. The desired result now follows from 7.2 and the fact that $\underline{\mathrm{lim}}^{Y\downarrow \mathbf{W}}\, \bar{X}$ is discrete.

It thus remains to give a

Proof of 7.2. One has to show that, for every pair of integers $i, j \geqslant 0$ and every pair of objects $X, Y \in \mathbf{C}$, the obvious map

$$\mathbf{W}^{-1}\mathbf{C}^{i+j}(X, Y) \to \mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}(X, Y)$$

is a weak homotopy equivalence. This map is (5.1) the nerve of a functor, say A, and a lengthy but straightforward argument (which uses several times the fact that $(\mathbf{C}, \mathbf{W})$ and $(\mathbf{W}, \mathbf{W})$ admit a calculus of left fractions) shows that, for every object $b \in N^{-1}\mathbf{W}^{-1}\mathbf{C}^{i}\mathbf{W}^{-1}\mathbf{C}^{j}(X, Y)$, the under category $b \downarrow A$ [6, p. 46] is right filtering [1, p. 331] and hence [1, p. 332] has a contractible nerve. Quillen's theorem A [9] now immediately implies that NA is a weak homotopy equivalence.

8. Quillen model categories

We end with some further examples of homotopy calculi of (left or two-sided) fractions and indicate how (small) model categories in the sense of Quillen [8] "with functorial factorizations" give rise to such calculi.

8.1. Proposition. *Let* $\mathbf{C} \in O$-**Cat** *be a category and* $\mathbf{W} \subset \mathbf{C}$ *a subcategory saisfying* 7.1(iii) *with the following property:*

Given a diagram $X' \xleftarrow{u} X \xrightarrow{f} Y \in \mathbf{C}$ *with* $u \in \mathbf{W}$, *there is a functorial diagram* $X' \xrightarrow{g} Y' \xleftarrow{v} Y \in \mathbf{C}$ *with* $v \in \mathbf{W}$ *and* $vf = gu$. *Moreover if* f *is in* $\mathbf{W}$, *then so is* g. *Then the pair* $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of left fractions.*

This happens, for instance, if $\mathbf{W}$ *is closed under pushouts*, i.e. if every pushout of a map in $\mathbf{W}$ is again in $\mathbf{W}$.

There is also a two-sided version.

8.2. Proposition. *Let* $\mathbf{C} \in O$-**Cat** *be a category and let* $\mathbf{W} \subset \mathbf{C}$ *be a subcategory satisfying* 7.1(iii). *Let* $\mathbf{W}_1, \mathbf{W}_2 \subset \mathbf{W}$ *be subcategories with the following properties:*

(i) *Given a diagram* $X' \xleftarrow{u} X \xrightarrow{f} Y \in \mathbf{C}$ *with* $u \in \mathbf{W}_1$, *there is a functorial diagram* $X' \xrightarrow{g} Y' \xleftarrow{v} Y \in \mathbf{C}$ *with* $v \in \mathbf{W}_1$ *and* $vf = gu$. *Moreover, if* f *is in* $\mathbf{W}$, *then so is* g.

(ii) *Given a diagram* $X \xrightarrow{g} Y \xleftarrow{v} Y \in \mathbf{C}$ *with* $v \in \mathbf{W}_2$, *there is a functorial diagram* $X \xleftarrow{u} X' \xrightarrow{f} Y' \in \mathbf{C}$ *with* $u \in \mathbf{W}_2$ *and* $vf = gu$. *Moreover, if* g *is in* $\mathbf{W}$, *then so is* f.

(iii) *Every map* $w \in \mathbf{W}$ *admits a functorial factorization* $w = w_2 w_1$ *with* $w_1 \in \mathbf{W}_1$ *and* $w_2 \in \mathbf{W}_2$.

Then the pair $(\mathbf{C}, \mathbf{W})$ *admits a homotopy calculus of fractions.*

Proofs. The proofs are straightforward, combining the functorial completions of square diagrams with the fact that

(i) all the maps which have to be shown to be weak homotopy equivalences are nerves of functors, and

(ii) any natural transformation between functors induces a homotopy between their nerves.

8.3. Model categories. Let $\mathbf{M}$ be a *model category* in the sense of Quillen [8, I, Section 1] which admits *functorial factorization* [8, I, Section 1, M2], let $\mathbf{W}$ be the subcategory of the weak equivalences, let $\mathbf{M}^c$, $\mathbf{M}^f$, and $\mathbf{M}^{cf} \subset \mathbf{M}$ be the full subcategories generated by the cofibrant, fibrant, and the cofibrant-fibrant objects respectively and let $\mathbf{W}^c = \mathbf{M}^c \cap \mathbf{W}$, $\mathbf{W}^f = \mathbf{M}^f \cap \mathbf{W}$, and $\mathbf{W}^{cf} = \mathbf{M}^{cf} \cap \mathbf{W}$. Then one has:

8.4. Proposition. *The pairs* $(\mathbf{M}, \mathbf{W})$, $(\mathbf{M}^c, \mathbf{W}^c)$, $(\mathbf{M}^f, \mathbf{W}^f)$ *and* $(\mathbf{M}^{cf}, \mathbf{W}^{cf})$ *admit homotopy calculi of (two-sided) fractions. Moreover in the commutative diagram of inclusions.*

$$\begin{array}{ccc} L^{\mathrm{H}}(\mathbf{M}^{cf}, \mathbf{W}^{cf}) & \to & L^{\mathrm{H}}(\mathbf{M}^{f}, \mathbf{W}^{f}) \\ \downarrow & & \downarrow \\ L^{\mathrm{H}}(\mathbf{M}^{c}, \mathbf{W}^{c}) & \to & L^{\mathrm{H}}(\mathbf{M}, \mathbf{W}) \end{array}$$

each of the maps induces an equivalence between "the categories of components" and weak homotopy equivalences between "the simplicial hom-sets".

8.5. Corollary. *If* $\mathbf{W}$ *is closed in* $\mathbf{M}$, *then* (6.4), *for every object* $X \in \mathbf{M}$, *the classifying spaces of the simplicial monoids* $\mathrm{haut}_{L^{\mathrm{H}}\mathbf{M}}\, X$, $\mathrm{haut}_{L\mathbf{M}}\, X$ *and* $L^{\mathrm{H}}\mathbf{W}(X, X)$ *and of the simplicial group* $L\mathbf{W}(X, X)$ *have the same homotopy type.* This is, for instance, the case if $\mathbf{M}$ is a *closed model category* [8, I, Section 5].

Proposition 8.4 is not difficult to prove using 3.5, 8.1, 8.2 and the functorial factorizations.

8.6. Remark. *Every model category admits a homotopy calculus of fractions,* even if it does *not* admit functorial factorizations. However, in the general case the proof (see [4, 8.1]) becomes more complicated.

Appendix

9. The Grothendieck construction for ordinary categories

In order to prove Lemma 5.7 we have to have an inductive hold on the hammock graphs. This is provided by the (two-sided) Grothendieck constructions of this section and the next. We start here with

9.1. The (two-sided) Grothendieck construction for ordinary categories. This is the construction which assigns to any two functors (1.2)

$$\mathbf{W}^{\mathrm{op}} \xrightarrow{F} \mathbf{Cat} \quad \text{and} \quad \mathbf{W} \xrightarrow{G} \mathbf{Cat}$$

the *category* $F \otimes_{\mathbf{W}} G$ of which

(i) an *object* is a triple (A, W, B), where W is an object of $\mathbf{W}$ and A and B are objects of FW and GW respectively, and

(ii) a *map* $(A_0, W_0, B_0) \to (A_1, W_1, B_1)$ consists of a map $w: W_0 \to W_1 \in \mathbf{W}$ together with maps $A_0 \to (Fw)A_1 \in FW_0$ and $(Gw)B_0 \to B_1 \in GW_1$.

This definition is clearly *natural* in the sense that *natural transformations* $F \to F'$ *and* $G \to G'$ *induce a functor* $F \otimes_{\mathbf{W}} G \to F' \otimes_{\mathbf{W}} G'$ and that *a functor* $H: \mathbf{W}' \to \mathbf{W}$ *induces a functor* $FH \otimes_{\mathbf{W}'} GH \to F \otimes_{\mathbf{W}} G$.

9.2. Example. Let $*$ denote the trivial (covariant or contravariant) functor $\mathbf{W} \to \mathbf{Cat}$ which sends each object of $\mathbf{W}$ to the category with only one object and one (identity) map. Then $* \otimes_{\mathbf{W}} F$ is the *one-sided Grothendieck construction* of [10].

9.3. Example. Let $\mathbf{C} \in O\text{-}\mathbf{Cat}$ be a category, $\mathbf{W} \subset \mathbf{C}$ a subcategory and $\mathbf{m}$ a word in $\mathbf{C}$ and $\mathbf{W}^{-1}$. Then (5.1) the categories $N^{-1}\mathbf{m}(X, Y)$ for various $X, Y \in \mathbf{C}$ give rise to a *functor.*

$$N^{-1}\mathbf{m}(-, -): \mathbf{W} \times \mathbf{W} \to \mathbf{Cat}$$

and the variance of this functor in the first (resp. second) variable depends on whether the last (resp. first) letter of $\mathbf{m}$ is $\mathbf{C}$ or $\mathbf{W}^{-1}$. Our main observation now is that these functors $N^{-1}\mathbf{m}(-, -)$ for various $\mathbf{m}$ are related to each other by means of the Grothendieck construction as follows (the proof is straightforward):

9.4. Proposition. *Let* $\mathbf{m}$ *and* $\mathbf{m}'$ *be words in* $\mathbf{C}$ *and* $\mathbf{W}^{-1}$. *Then, for every two objects* $X, Y \in \mathbf{C}$, *the category* $N^{-1}(\mathbf{m}, \mathbf{m})(X, Y)$ *is isomorphic to*

$N^{-1}\mathbf{m}'(X, -) \otimes_{\mathbf{W}} N^{-1}\mathbf{m}(-, Y)$ *if* $\mathbf{m}'$ *starts with* $\mathbf{W}^{-1}$ *and* $\mathbf{m}$ *ends with* $\mathbf{W}^{-1}$

$N^{-1}\mathbf{m}'(-, Y) \otimes_{\mathbf{W}} N^{-1}\mathbf{m}'(X, -)$ *if* $\mathbf{m}'$ *starts with* $\mathbf{C}$ *and* $\mathbf{m}$ *ends with* $\mathbf{C}$,

$* \otimes_{\mathbf{W}} (N^{-1}\mathbf{m}'(X, -) \times N^{-1}\mathbf{m}(-, Y))$ *if* $\mathbf{m}'$ *starts with* $\mathbf{C}$ *and* $\mathbf{m}$ *ends with* $\mathbf{W}^{-1}$

$(N^{-1}\mathbf{m}'(X, -) \times N^{-1}\mathbf{m}(-, Y)) \otimes_{\mathbf{W}} *$ *if* $\mathbf{m}'$ *starts with* $\mathbf{W}^{-1}$ *and* $\mathbf{m}$ *ends with* $\mathbf{C}$.

An important property of the Grothendieck construction is its *homotopy invariance.* To formulate this properly, denote by $N(F, \mathbf{W}, G)$ the diagonal of the tri-simplicial set $N_{***}(F, \mathbf{W}, G)$ which in dimension (i, k, j) consists of the triples (f, w, g), where $w = (W_k \to \cdots \to W_0)$ is a k-simplex of $N\mathbf{W}$ and f and g are i- and j-simplices of NFW_0 and NGW_k respectively. Then one has

9.5. Proposition.. $N(F \otimes_{\mathbf{W}} G)$ *and* $N(F, \mathbf{W}, G)$ *are, in a natural manner, weakly homotopy equivalent.*

As diag $N_{*k*}(F, \mathbf{W}, G)$ is the disjoint union of the simplicial sets

$$NGW_k \times \mathbf{W}(W_k, W_{k-1}) \times \cdots \times \mathbf{W}(W_1, W_0) \times NFW_0$$

taken over all $(k+1)$-tuples $(W_k, \ldots, W_0)$ of objects of $\mathbf{W}$, Proposition 9.5 implies

9.6. Corollary. *If $F \to F'$ and $G \to G'$ are natural transformations such that, for every object $W \subset \mathbf{W}$, the induced maps $NFW \to NF'W$ and $NGW \to NG'W$ are weak homotopy equivalences, then so is the induced map $N(F \otimes_{\mathbf{W}} G) \to N(F' \otimes_{\mathbf{W}} G')$.*

To prove 9.5 we need the following two lemmas:

9.7. Lemma. *Let $P: F \otimes_{\mathbf{W}} * \to \mathbf{W}$ be the obvious functor and let $- \downarrow P: \mathbf{W}^{\mathrm{op}} \to$ **Cat** be the resulting under-category* [5, p. 46] *functor. Then there are natural weak homotopy equivalences*

$$N(F, \mathbf{W}, G) \xleftarrow{\sim} N(-\downarrow P, \mathbf{W}, G) \xrightarrow{\sim} N(*, (F \otimes_{\mathbf{W}} *), GP).$$

And dually

9.8. Lemma. *Let $Q: * \otimes_{\mathbf{W}} G \to \mathbf{W}$ be the obvious functor and let $Q \downarrow -: \mathbf{W} \to$ **Cat** be the resulting over-category functor. Then there are natural weak homotopy equivalences*

$$N(F, \mathbf{W}, G) \xleftarrow{\sim} N(F, \mathbf{W}, Q\downarrow -) \xrightarrow{\sim} N(FQ, (* \otimes_{\mathbf{W}} G), *).$$

9.9. Proof of Proposition 9.5. Note that there is an obvious isomorphism of categories

$$* \otimes_{F \otimes_{\mathbf{W}} *} GP \approx F \otimes_{\mathbf{W}} G.$$

Therefore, if $R: F \otimes_{\mathbf{W}} G \to F \otimes_{\mathbf{W}} *$ denotes the obvious functor, Lemma 9.8 implies the existence of natural weak homotopy equivalences

$$N(*, (F \otimes_{\mathbf{W}} G), GP) \xleftarrow{\sim} N(*, (F \otimes_{\mathbf{W}} *), R\downarrow -) \xrightarrow{\sim} N(*, (F \otimes_{\mathbf{W}} G), *)$$

and the desired result follows from Lemma 9.7 and the fact that $N(*, (F \otimes_{\mathbf{W}} G), *) \approx N(F \otimes_{\mathbf{W}} G)$.

It thus remains to give a

9.10. Proof of Lemma 9.7. One readily verifies that, for every object $W \in \mathbf{W}$, the obvious functor $W \downarrow P \to FW$ induces a weak homotopy equivalence $N(W \downarrow P) \sim NFW$ and the existence of the weak homotopy equivalence on the left now follows by the argument of 9.6.

To obtain the weak homotopy equivalence on the right note that $N_{i,j,*}(-\downarrow P, \mathbf{W}, G)$ is the disjoint union of the simplicial sets NGW_i, taken over all triples (a, b, w), where

$$w = (W_j \to \cdots \to W_0) \in N\mathbf{W},$$

$$a = (A_i \to \cdots \to A_0) \in N(F \otimes_{\mathbf{W}} *)$$

and b is a map $b: W_0 \to PA_0 \in \mathbf{W}$. Hence $N_{i,j,j}(- \downarrow P, \mathbf{W}, G)$ consists of the j-simplices of the disjoint union of the homotopy direct limits [1, Ch. XII]

$$\underline{\text{holim}}^{W \downarrow PA_0}[(W \to PA_0) \mapsto NGW]$$

taken over all a as above. On the other hand $N_{i,i,j}(*, (F \otimes_{\mathbf{W}} *), GP)$ consists of the j-simplices of the disjoint union of the simplicial sets $NGPA_0$, taken over all a as above. The desired result now follows from the fact that the categories $W \downarrow PA_0$ have terminal objects and that therefore they are natural weak homotopy equivalences

$$\underline{\text{holim}}^{W \downarrow PA_0}[(W \to PA_0) \mapsto NGW] \xrightarrow{\sim} GPA_0.$$

10. The Grothendieck construction for simplicial categories in sO-Cat

We saw in 9.4 that the Grothendieck construction provides an inductive hold on the hammock graphs. However, Lemma 5.7 involves *simplicial* hammock graphs and to get a proper hold on these we need a simplicial version of the Grothendieck construction which, instead of functors, involves

10.1. Covariant and contravariant transfunctors. Let $\mathbf{V} \in \mathrm{s}O\text{-}\mathbf{Cat}$. A (*covariant*) *transfunctor* $G: \mathbf{V} \to \mathbf{Cat}$ then will be an ordinary functor $* \otimes_{\Delta^{\mathrm{op}}} \mathbf{V} \to \mathbf{Cat}$ (where $\mathbf{\Delta}$ denotes the usual category of finite ordered sets and order preserving functions and $\mathbf{V}$ is considered as a functor $\mathbf{V}: \mathbf{\Delta}^{\mathrm{op}} \to O\text{-}\mathbf{Cat}$), i.e. G consists of

(i) for each integer $k \geq 0$, an ordinary functor $G_k: \mathbf{V}_k \to \mathbf{Cat}$, and

(ii) for each map $t: [k] \to [n] \in \mathbf{\Delta}^{\mathrm{op}}$, a natural transformation $G_t: G_k \to G_n \mathbf{V}_t$,

such that these transformations satisfy the obvious analogs of the simplicial identities. Similarly a *contravariant* transfunctor $F: \mathbf{V} \to \mathbf{Cat}$ will be a covariant transfunctor $F: \mathbf{V}^{\mathrm{op}} \to \mathbf{Cat}$.

If $H: \mathbf{V} \to \mathbf{Cat}$ is a transfunctor and $V \in O$ an object, then application of H to V gives rise to a *simplicial category* which we will denote by HV.

10.2. The Grothendieck construction for simplicial categories in sO-Cat. Given $V \in \mathrm{s}O\text{-}\mathbf{Cat}$ and two transfunctors

$$\mathbf{V}^{\mathrm{op}} \xrightarrow{F} \mathbf{Cat} \quad \text{and} \quad \mathbf{V} \xrightarrow{F} \mathbf{Cat},$$

we define the *Grothendieck construction* $F \otimes_{\mathbf{V}} G$ as the *simplicial category* which is $F_k \otimes_{\mathbf{V}_k} G_k$ in dimension k, while, for each map $t: [k] \to [n] \in \mathbf{\Delta}^{\mathrm{op}}$, the functor $(F \otimes_{\mathbf{V}} G)_t$ is the composition

$$F_k \otimes_{\mathbf{V}_k} G_k \to F_n \mathbf{V}_t \otimes_{\mathbf{V}_k} G \mathbf{V}_t \to F_n \otimes_{\mathbf{V}_n} G_n$$

of the functors induced by the natural transformations F_t and G_t and by the functor $\mathbf{V}_t$.

As in 9.1, this construction is of course *natural* in F, $\mathbf{V}$ and G.

10.3. Example. Let $\mathbf{B} \in sO\text{-}\mathbf{Cat}$ and let $\mathbf{V} \subset \mathbf{B}$ be a subcategory. As in 9.3 one can then, for every word in $\mathbf{B}$ and $\mathbf{V}^{-1}$ and every integer $k \geq 0$, form the functors $N^{-1}\mathbf{m}_k(-,-)$ and it is not hard to see that these functors give rise to *transfunctors*

$$N^{-1}\mathbf{m}(-,-) : \mathbf{V} \times \mathbf{V} \to \mathbf{Cat}$$

and that *these transfunctors for various* $\mathbf{m}$ *are related to each other by means of the Grothendieck construction* 10.2 *in the manner of* 9.4.

We end with showing that the Grothendieck construction 10.2 has the following *homotopy properties* which, in view of 10.3, readily imply Lemma 5.7.

10.4. Proposition. *Let* $\mathbf{V} \in sO\text{-}\mathbf{Cat}$ *and let* $F \to F'$ *and* $G \to G'$ *be natural transformations between contravariant and covariant transfunctors* $\mathbf{V} \to \mathbf{Cat}$ *respectively which, for every object* $V \in \mathbf{V}$, *induce weak homotopy equivalences* $NFV \sim NF'V$ *and* $NGV \sim NG'V$. *Then they also induce a weak homotopy equivalence* $N(F \otimes_{\mathbf{V}} G) \sim N(F' \otimes_{\mathbf{V}} G')$.

10.5. Proposition. *Let* $\mathbf{V}, \mathbf{V}' \in sO\text{-}\mathbf{Cat}$, *let F and G be a contravariant and a covariant transfunctor* $V \to Cat$ *and let* $H : V' \to V \in sO\text{-}\mathbf{Cat}$ *be a weak equivalence. Then H induces a weak homotopy equivalence* $N(FH \otimes_{\mathbf{V}'} GH) \sim N(F \otimes_{\mathbf{V}} G)$.

10.6. Proof of 10.4 and 10.5. Let F, $\mathbf{V}$ and G be as in 10.4 and let $N_{****}(F, \mathbf{V}, G)$ be the 4-simplicial set with

$$N_{n,i,k,j}(F, \mathbf{V}, G) = N_{i,k,j}(F_n, \mathbf{V}_n, G_n) \quad \text{for all } n \geq 0$$

Then (i) by 9.5 and a diagonal argument [3, 1.4] there is a natural weak homotopy equivalence between $N(F \otimes_{\mathbf{V}} G)$ and diag $N_{****}(F, \mathbf{V}, G)$, and

(ii) the partial diagonal $N_{**}(F, \mathbf{V}, G)$ obtained by taking the simplices in dimensions (i, i, k, i), consists in dimension $(*, k)$ of the disjoint union of the simplicial sets

$$\mathrm{NGV}_k \times \mathbf{V}(V_k, V_{k-1}) \times \cdots \times \mathbf{V}(V_1, V_0) \times NFV_0$$

taken over all $(k+1)$-tuples $(V_k, \ldots, V_0)$ of objects of O.

From these two facts one easily deduces propositions 10.4 and 10.5.

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