CLASSIFYING SPACES OF TOPOLOGICAL MONOIDS AND CATEGORIES

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0. Introduction. In recent years, a number of curious and seemingly paradoxical properties of the bar constuction have come to light. These results have the general form that, in certain situations, the bar construction on a topological group, monoid, or category can be largely independent of the topological structure of the underlying object. Among the most prominent of these results is that of Thurston [28] which states that if $G = \mathrm{Homeo}(M)$ is the topological group of self-homeomorphisms of a compact manifold and G^δ is the same group endowed with the discrete topology, then $BG^\delta \to BG$ is a homology equivalence (cf. also MacDuff [16]). In a similar spirit there are two rather amazing results, due to Kan-Thurston [14] and MacDuff [15], that given any connected CW complex X there is a discrete group G and a homology equivalence $BG \to X$ and a discrete monoid M and a homotopy equivalence $BM \simeq X$. Most recently Friedlander and Milnor have conjectured that for any Lie group G, $BG^\delta \to BG$ is a homology equivalence with finite coefficients.

This paper analyzes in detail one particular class of these phenomena, the case when $BM^\delta \to BM$ or $B\mathcal{C}^\delta \to B\mathcal{C}$ is a weak homotopy equivalence for a topological monoid M or a topological category $\mathcal{C}$. The author's interest in this sort of phenomenon arose in trying to understand Waldenhausen's work on the algebraic K-theory of spaces (cf. [30]). If X is a connected space, Waldhausen considers the topological category $\mathcal{C}_{n,k}$ of G-equivalences of spaces having the homotopy type of $G_+ \wedge \vee^k S^n$, where G is the Kan loop group of X. He then defines the algebraic K-theory $A(X)$ in various ways, among them (1) $A(X) = \mathbf{Z} \times \varinjlim_{n,k}(B\mathcal{C}^\delta_{n,k})^+$ and (2) $A(X) = \mathbf{Z} \times \varinjlim_{n,k}(B\mathcal{C}_{n,k})^+$. Here $\mathcal{C}^\delta_{n,k}$ is the discrete category obtained from $\mathcal{C}_{n,k}$ by discarding the topology on the function spaces of $\mathcal{C}_{n,k}$. Both of these definitions play an important role in Waldhausen's theory: (1) is required to compare $A(X)$ with Hatcher's higher simple homotopy theory;

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(2) is required to relate $A(X)$ to something whose (rational) homotopy groups can be computed (cf. [30], [26], [18], [8], [10]).

This paper consists of eight sections. The first three deal directly with questions relating to the equivalence of these two definitions of $A(X)$. In the first section we derive a criterion for a category $\mathcal{C}$ and its discretization $\mathcal{C}^{\delta}$ to have equivalent classifying spaces (Theorem 1.2). In the second section we apply that criterion to the case Waldhausen considers (Theorems 2.1, 2.2). Proceeding further along these lines we obtain that the topological monoid $H(X)$ of self-equivalences of X has the same weak homotopy type as the discrete monoid $\varinjlim_n H(X \times I^n)^{\delta}$ (Theorem 2.4). In the third section we give an analogous procedure for replacing any topological monoid whatsoever by a discrete monoid with equivalent classifying space (Theorem 3.4). This gives an alternate proof of the aforementioned result of MacDuff that for any path connected space there is a weak equivalence $BM \simeq X$ with M a discrete monoid. Our method moreover gives a somewhat stronger result: M is a functor of X and the weak equivalence is natural (Theorem 3.5).

Sections 4 and 5 deal with the question of determining the homotopy type of the classifying spaces of amalgamated free products of discrete monoids. Here we establish the analog of J. H. C. Whitehead's classical theorem that the classifying space of an amalgamated free product of discrete groups is the pushout of the classifying spaces of the factors (Theorem 4.1). We also give a "flat mapping cylinder" construction for discrete monoids (Proposition 4.5) which can be used along with the result on amalgamated free products of monoids to furnish another proof of MacDuff's result, along the lines of Kan-Thurston (Theorem 4.7).

In section 6 we use the results of the preceding two sections to give a simple proof of the folklore result that the classifying space of the James construction JX on any space X has the homotopy type of ΣX (Theorem 6.10). We also introduce a Moore suspension functor Ξ left adjoint to the Moore loop functor Λ and obtain a natural homorphism of topological monoids $JX \to \Lambda\Xi X$ which is an equivalence if X is path connected and a "group completion" otherwise. This is a sort of natural monoidal version of James' classical result.

The last two sections deal with the following question: How can we characterize the classifying space construction as a functor on the category of topological monoids? This is not merely an academic question, since in addition to the bar construction and its minor variants, there is another family of classifying space functors based on the two sided bar construction of Beck and May. This latter family of constructions is the basis of May's theory of iterated loop spaces and is connected with other important topics in homotopy theory such as homology operations and stable splittings of free loop spaces. In [27], Thomason compared these two types of classifying space constructions and showed that they are naturally weakly equivalent. Unfortunately Thomason's proof is rather long and complicated. In section 7 we show how the ideas of section 6 can be used to streamline and simplify Thomason's arguments. Finally in section 8 we take another approach to this question and derive a definitive axiomatic characterization of classifying space functors. Our main result (Theorem 8.5) says that classifying space functors on topological monoids are completely determined by their restrictions to the category of discrete free monoids.

Let us proceed to fix some conventions. Throughout the paper we take the word equivalence to mean weak homotopy equivalence. All spaces will be taken to be compactly generated weak Hausdorff. Products and function spaces will be topologized appropriately to stay in this category. All based spaces will be assumed to have non-degenerate basepoints. All simplicial spaces considered will be taken to be proper: that is we assume that all the degeneracy maps are cofibrations. Whenever this is not obvious we will comment as to why our simplicial constructions are of this type. We collect here for general reference some basic properties of geometric realization.

LEMMA 0. (a) *Let* $f_*: X_* \to Y_*$ *be a map of proper simplicial spaces such that each* $f_n: X_n \to Y_n$ *is a (weak) homotopy equivalence. Then* $|f_*|: |X_*| \to |Y_*|$ *is a (weak) homotopy equivalence.*

(b) *Realization commutes with fiber products*:

$$|X_* \times_{Z_*} Y_*| \cong |X_*| \times_{|Z_*|} |Y_*|$$

(c) *If X is a topological space regarded as a constant simplicial space* (i.e. $X_n = X$ *all n and all the faces and degeneracies are identity maps*) *then* $|X| \equiv X$.

(d) *If* X_{**} *is a bisimplicial space, there are natural homeomorphisms*

$$|m \mapsto |n \mapsto X_{mn}|| \cong |m \mapsto X_{mm}| \cong |n \mapsto |m \mapsto X_{mn}||$$

(e) *If* X_* *is a based proper simplicial space such that each* X_n *is path connected, there is a natural weak equivalence* $|n \mapsto \Omega X_n| \xrightarrow{\simeq} \Omega |X_*|$

(f) *If X_* is a based proper simplicial space such that each X_n is path connected, there is natural homomorphism of topological monoids $|n \mapsto \Lambda X_n| \overset{\simeq}{\to} \Lambda|X_*|$ which is also a weak equivalence. Here Λ denotes the Moore loop functor.*

For proofs, the reader is referred to [25; Appendix A] or [20; Appendix] (cf. also [5; Appendix, Proposition 4.8]) for part (a). The reference for parts (b) and (c) is [21; 11.6, 11.8]. A convenient source for (d) is [23; page 10]. For (e) the reader should consult [2]; Theorem 12.3]. Part (e) also follows from the fact that geometric realization preserves fibrations over a path connected base (cf. [29; Lemma 5.2], [21; Theorem 12.7] or [1], [7] for more general versions). Finally (f) is a slight modification of (e).

I would like to take this opportunity to express my thanks to Peter May, Mark Steinberger and Bob Thomason for many stimulating discussions which contributed to the writing of this paper. I would also like to thank the referee for suggesting numerous improvements in the original draft.

2. A discretization criterion for topological categories. In this section we derive a general criterion for determining when a topological category $\mathcal{C}$ and its discretization $\mathcal{C}^\delta$ have equivalent classifying spaces. While our applications will deal with topological categories having a discrete space of objects, our discretization criterion is equally easy to state and prove in the more general setting.

However we do need to impose one important restriction on the type of topological categories $\mathcal{C}$ we consider. In order for $B\mathcal{C}$ to be a proper simplicial space it is necessary to require that the map which assigns to each object the identity map associated to the object be a closed cofibration from the space of objects to the space of morphisms. Note that this is a nonvacuous assumption even when $\mathcal{C}$ has a discrete space of objects. If $\mathcal{C}$ does not satisfy this condition, then we can rectify this by "growing a whisker" from each identity of $\mathcal{C}$. More precisely we consider the category $\mathcal{C} \times I$ where $\mathrm{Ob}(\mathcal{C} \times I) = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}(\mathcal{C} \times I) = (\mathrm{Mor}\,\mathcal{C}) \times I$ with the obvious source and target maps. Composition in $\mathcal{C} \times I$ is defined by $(f, s)\cdot(g, t) = (f\cdot g, st)$ whenever $f\cdot g$ is defined in $\mathcal{C}$. The identities of $\mathcal{C} \times I$ are the morphisms of the form (id, 1). We now take $\mathcal{C}'$ to be the subcategory of $\mathcal{C} \times I$ having the same objects as $\mathcal{C}$ and whose morphisms have one of the forms $(f, 0)$ or (id, t) $t \in I$. Then $\mathcal{C}'$ satisfies the cofibration condition on identities and the projection $\mathcal{C}' \to \mathcal{C}$ is an equivalence of topological categories.

We shall henceforth assume in this section and the succeeding ones that our topological categories shall have cofibered identities or that the above whiskering construction has been employed prior to taking classifying spaces. We will use the notation $B_*\mathcal{C}$ for the nerve of the category $\mathcal{C}$ and $B\mathcal{C} = |B_*\mathcal{C}|$ for the classifying space of $\mathcal{C}$. We will denote by $\mathcal{C}^\delta$ the discretization of the topological category $\mathcal{C}$ (i.e. we discard the topology on both objects and morphisms of $\mathcal{C}$).

Our discretization criterion is based on the following function space construction for categories.

Definition 1.1. Let $\mathcal{C}$ be a topological category and let X be a topological space. The function category $\mathcal{C}^X$ is the category with objects $(\mathrm{Ob}\ \mathcal{C})^X$ (i.e. continuous functions $X \to \mathrm{Ob}\ \mathcal{C}$) and morphisms $(\mathrm{Mor}\ \mathcal{C})^X$. The source, target, identity and composition of $\mathcal{C}^X$ are those induced from $\mathcal{C}$ by passage to function spaces.

The construction $\mathcal{C}^X$ is a covariant functor of $\mathcal{C}$ and a contravariant functor of X. In particular there is a natural functor

$$(*) \qquad J : \mathcal{C} \to \mathcal{C}^X$$

induced by the constant map.

In the following theorem and throughout the paper, we use the notation I to denote the unit interval, Δ^n to denote the standard n-simplex, and J to denote the functor $(*)$ above. In this notation our discretization criterion may be started as follows:

THEOREM 1.2. *Let $\mathcal{C}$ be a topological category. Suppose that for every n*

$$J^\delta : \mathcal{C}^\delta \to (\mathcal{C}^{\Delta^n})^\delta$$

induces an equivalence of classifying spaces. Then the natural map $B\mathcal{C}^\delta \to B\mathcal{C}$ is an equivalence.

Proof. Construct a discrete simplicial category $\mathcal{C}^\delta_*$ such that $\mathcal{C}^\delta_n = (\mathcal{C}^{\Delta^n})^\delta$ and whose faces and degeneracies are induced by the edges and collapses of the standard simplex.

The nerve of the simplicial category $\mathcal{C}^\delta_*$ is a bisimplicial set which in bidegree (m, n) is

$$(\underbrace{(\mathrm{Mor}\ \mathcal{C} \times_{\mathrm{Ob}\ \mathcal{C}} \mathrm{Mor}\ \mathcal{C} \times_{\mathrm{Ob}\ \mathcal{C}} \cdots \times_{\mathrm{Ob}\ \mathcal{C}} \mathrm{Mor}\ \mathcal{C})^{\Delta^n})^\delta}_{m \text{ factors}}.$$

By Lemma 0(d) its geometric realization $B\mathcal{C}_*^\delta = |B_*\mathcal{C}_*^\delta|$ is homeomorphic to $|m \mapsto TB_m\mathcal{C}|$ where $TB_m\mathcal{C}$ is the geometric realization of the total singular complex of the space of m-simplices of the nerve $B_*\mathcal{C}$. By Lemma 0(a), since the natural map $TB_m\mathcal{C} \to B_m\mathcal{C}$ is an equivalence, so is the induced map $B\mathcal{C}_*^\delta \to B\mathcal{C}$.

The inclusion of vertices $\mathcal{C}_0^\delta \to \mathcal{C}_n^\delta$ is the functor

$$J^\delta : \mathcal{C}^\delta \to (\mathcal{C}^{\Delta^n})^\delta.$$

By assumption, it induces an equivalence of classifying spaces. It follows (Lemma 0(a), (c)) that the induced map $B\mathcal{C}^\delta \to B\mathcal{C}_*^\delta$ is an equivalence. Combining all the information above we obtain that the functor $\mathcal{C}^\delta \to \mathcal{C}$ induces an equivalence

$$B\mathcal{C}^\delta \to B\mathcal{C}.$$

Remark 1.3. What happens to the above argument when $\mathcal{C}$ does not satisfy the cofibration condition on identities? Well the discrete categories $\mathcal{C}^\delta$, $(\mathcal{C}^{\Delta^n})^\delta$ are perfectly fine in this regard. On the other hand consider the category $\mathcal{C}'$ obtained by whiskering $\mathcal{C}$. The functor $\mathcal{C}' \to \mathcal{C}$ induces a homotopy equivalence $TB_m\mathcal{C}' \to TB_m\mathcal{C}$ for all m (notation as in the proof above) and hence an equivalence $B(\mathcal{C}'_*)^\delta \to B\mathcal{C}_*{}^\delta$. Under the hypothesis of Theorem 1.2 one therefore has a chain of equivalences

$$B\mathcal{C}^\delta \to B\mathcal{C}_*{}^\delta \leftarrow B(\mathcal{C}'_*)^\delta \to B\mathcal{C}'$$

COROLLARY 1.4. *Let $\mathcal{C}$ be a topological category and let $\mathcal{C}^\delta$ denote its discretization. Then the natural map $B\mathcal{C}^\delta \to B\mathcal{C}$ is an equivalence under either of the following two circumstances*:

(i) *For each n there is a functor*

$$K : (\mathcal{C}^{I^n})^\delta \to \mathcal{C}^\delta$$

such that the composite

$$(\mathcal{C}^{I^n})^\delta \xrightarrow{K} \mathcal{C}^\delta \xrightarrow{J^\delta} (\mathcal{C}^{I^n})^\delta$$

induces an equivalence on classifying spaces.

(ii) *There is a continuous functor*

$$K : \mathcal{C}^I \to \mathcal{C}$$

and a (continuous) natural transformation between

$$\mathcal{C}^I \xrightarrow{K} \mathcal{C} \xrightarrow{J} \mathcal{C}^I$$

and the identity functor.

Proof. Since Δ^n is homeomorphic to I^n, there is an isomorphism of topological categories $\mathcal{C}^{I^n} \cong \mathcal{C}^{\Delta^n}$. In view of Theorem 1.2 it suffices to show that for each n

$$J^\delta : \mathcal{C}^\delta \to (\mathcal{C}^{I^n})^\delta$$

induces an equivalence of classifying spaces. Moreover any map $* \to I^n$ induces a functor

$$S : (\mathcal{C}^{I^n})^\delta \to \mathcal{C}^\delta$$

such that SJ^δ is the identity functor on $\mathcal{C}^\delta$. Clearly then, under the hypothesis of (i), J^δ induces an equivalence.

Now suppose that (ii) holds. Observe that there is an obvious isomorphism of topological categories $(\mathcal{C}^I)^{I^n} \cong \mathcal{C}^{I^{n+1}}$. Hence by iteration (and functoriality in the function space variable) we obtain a functor

$$K' : \mathcal{C}^{I^n} \to \mathcal{C}$$

and a natural transformation between JK' and the identity of $\mathcal{C}^{I^n}$. We conclude that (i) holds and hence we get the required equivalence.

2. Categories and monoids of homotopy equivalences. In this section we apply our discretization criterion to categories and monoids of self-equivalences, and consider some questions raised by those aspects of Waldhausen's work discussed in the introduction. The reader should be forewarned that in the statements of the results below we are implicitly using the conventions adopted in section 1 to deal with cofibration problems.

THEOREM 2.1. *Let $\mathcal{C}$ be a topological category whose objects are a set of topological spaces and whose morphisms are all the homotopy equivalences between these spaces. (The morphisms in $\mathcal{C}$ are given the function space topology). Then $B\mathcal{C}^{\delta}$ and $B\mathcal{C}$ have the same weak homotopy type provided the following condition holds: If X is an object of $\mathcal{C}$, then $\mathcal{C}$ also contains an object homeomorphic to $X \times I$.*

Proof. We apply Corollary 1.4(ii): it suffices to construct a continuous functor

$$K : \mathcal{C}^I \to \mathcal{C}$$

and a natural transformation from the identity of $\mathcal{C}^I$ to the composite JK.

Since $\mathcal{C}$ has a discrete set of objects, $\mathcal{C}^I$ has the same objects as $\mathcal{C}$. We may regard a morphism from X to Y in $\mathcal{C}^I$ to be a map $F : I \times X \to Y$. The composite of $F : I \times X \to Y$ and $G : I \times Y \to Z$ is the map $H : I \times X \to Z$ where

$$H(s, x) = G(s, F(s, x))$$

The functor $K : \mathcal{C}^I \to \mathcal{C}$ is defined by $K(X) = X \times I$ on objects. On morphisms, given $F : I \times X \to Y$ we define $K(F) : X \times I \to Y \times I$ by $K(F)(x, t) = (F(t, x), t)$. We define the natural transformation

$$U : \mathrm{id}_{\mathcal{C}^I} \to JK$$

to be given by the map $U : I \times X \to X \times I = JK(X)$ where $U(s, x) = (x, s)$.

It is easy to check that the diagram

$$\begin{array}{ccc} X & \xrightarrow{U} & JK(X) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle JK(F)} \\ Y & \xrightarrow{U} & JK(Y) \end{array}$$

commutes:

$$U(s, F(s, x)) = (F(s, x), s) = JK(F)(s, (x, s)) = JK(F)(s, U(s, x)).$$

This completes the proof.

The above argument is based on ideas of Mark Steinberger and Bob Thomason. This kind of result also appears in a simplicial setting in Dwyer-Kan [11].

There are numerous variations of this theorem: e.g. based versions, equivariant versions, etc. The particular variation which is directly relevant to Waldhausen's work on the algebraic K-theory of spaces is the following.

THEOREM 2.2. *Let G be a topological group. Let $\mathcal{C}$ be a topological category whose objects are a set of based G-spaces and whose morphisms are all the based G-equivariant homotopy equivalences between these spaces. (The morphisms are given the function space topology.) Let $\mathcal{C}^\delta$ denote the discretization of $\mathcal{C}$. Then $B\mathcal{C}^\delta$ and $B\mathcal{C}$ have the same weak homotopy type provided the following condition holds: If X is an object of $\mathcal{C}$, then $\mathcal{C}$ also contains an object G-homeomorphic to $X \wedge I_+ = X \times I/_* \times I$ (with G acting trivially on I).*

The proof of Theorem 2.2, like that of many other variants, follows an identical pattern to that of Theorem 2.1.

Before proceeding on, we draw the following curious consequence of Theorem 2.1, which was pointed out to me by M. Steinberger. For a topological space X, let $H(X)$ denote the topological monoid of self homotopy equivalences of X.

COROLLARY 2.3. *Let M be a Hilbert cube manifold and let $H(M)^\delta$ denote the discretization of $H(M)$. Then $BH(M)^\delta \to BH(M)$ is a weak homotopy equivalence.*

Proof. By the stability theorem for Hilbert cube manifolds (cf. [2]) we have $M \times I \cong M$. Hence we can apply Theorem 2.1 to the topological category whose single object is M and whose morphisms are the self equivalences of M (with the function space topology).

A similar sort of argument can be applied to the self equivalences of any topological space X. For if Q denotes the Hilbert cube, then arguing as above we see that $BH(X \times Q)^\delta$ has the same weak homotopy type as $BH(X \times Q)$ and hence also as $BH(X)$.

Now what all the preceding results show is that if we have a topological category or monoid which is "stable" with respect to the functor $\_\_ \times I$, then the weak homotopy type of the classifying space of that category or monoid does not depend on the underlying function space topology on the morphisms. This suggests the following theorem.

THEOREM 2.4. *Let X be a topological space. Let the map $H(X) \hookrightarrow H(X \times I)$ be given by $f \mapsto f \times 1$ and let $M(X)$ be the colimit of the sequence*

$$(*) \qquad H(X) \hookrightarrow H(X \times I) \hookrightarrow H(X \times I^2) \hookrightarrow \cdots$$

Then there is an equivalence $BM(X)^\delta \to BH(X)$, where $M(X)^\delta$ denotes the discretization of $M(X)$.

Proof. Clearly the inclusion map $H(X) \hookrightarrow M(X)$ is an equivalence and hence induces an equivalence on classifying spaces (by Lemma 0(a), (b)).

It therefore suffices to show that $BM(X)^\delta \to BM(X)$ is a weak homotopy equivalence. We do this by applying Theorem 1.2. We regard monoids, such as $M(X)$, as categories with a single object.

By Corollary 1.4 it suffices to show that for all m there is a functor

$$(*) \qquad K: M(X)^{I^m} \to M(X)$$

such that the composite

$$(M(X)^{I^m})^\delta \xrightarrow{K^\delta} M(X)^\delta \xrightarrow{J^\delta} (M(X)^{I^m})^\delta$$

induces an equivalence on classifying spaces.

We begin by noting that since I^m is compact, $M(X)^{I^m}$ may be regarded as the colimit of the sequence

$$H(X)^{I^m} \hookrightarrow H(X \times I)^{I^m} \hookrightarrow H(X \times I^2)^{I^m} \hookrightarrow \cdots.$$

We construct functors

$$K_n: H(X \times I^n)^{I^m} \to H(X \times I^{m+n})$$

as follows. As in the preceding proof, we consider the elements of $H(X \times I^n)^{I^m}$ to be maps $F: I^m \times X \times I^n \to X \times I^n$ with composition given by

$$(G \cdot F)(s, x, t) = G(s, F(s, x, t))$$

We define

$$K_n(F)(x, t_1, t_2, \ldots, t_{m+n})$$

$$= (F_1(t_1, \ldots, t_m, x, t_{m+1}, \ldots, t_{m+n}), t_1, \ldots, t_m,$$

$$F_2(t_1, \ldots, t_m, x, t_{m+1}, \ldots, t_{m+n})).$$

It is easy to see that the K_n's are a compatible family of functors and thus define a functor $K : M(X)^{I^m} \to M(X)$.

We construct natural transformations U_n (not a compatible family) from the inclusion functor

$$H(X \times I^n)^{I^m} \xrightarrow{S_{n,m}} H(X \times I^{m+n})^{I^m}$$

to the composite

$$H(X \times I^n)^{I^m} \xrightarrow{K_n} H(X \times I^{m+n}) \xrightarrow{J} H(X \times I^{m+n})^{I^m}$$

by specifying $U_n : I^m \times X \times I^{m+n} \to X \times I^{m+n}$ by the formula

$$U_n(s_1, s_2, \ldots, s_m, x, t_1, t_2, \ldots, t_{m+n})$$

$$= (x, s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n)$$

It is easy to see that the following diagram commutes for any element $F \in H(X \times I^n)^{I^m}$

$$\begin{array}{ccc} \cdot & \xrightarrow{U_n} & \cdot \\ \downarrow {\scriptstyle S_{n,m}(F)} & & \downarrow {\scriptstyle JK_n(F)} \\ \cdot & \xrightarrow{U_n} & \cdot \end{array}$$

We now observe that the inclusion functor

$$R_n : H(X \times I^n)^{I^m} \to M(X)^{I^m}$$

factors as the composite

$$H(X \times I^n)^{I^m} \xrightarrow{S_{n,m}} H(X \times I^{m+n})^{I^m} \xrightarrow{R_{m+n}} M(X)^{I^m},$$

while the composite

$$H(X \times I^n)^{I^m} \overset{K_n}{\hookrightarrow} H(X \times I^{m+n}) \xrightarrow{J} H(X \times I^{m+n})^{I^m} \xrightarrow{R_{m+n}} M(X)^{I^m}$$

can also be written as the composite

$$H(X \times I^n)^{I^m} \xrightarrow{R_n} M(X)^{I^m} \xrightarrow{K} M(X) \xrightarrow{J} M(X)^{I^m}.$$

Therefore $R_{m+n}U_n$ is a natural transformation $R_n \to JKR_n$. This implies that the classifying map of the composite

$$(M(X)^{I^m} \xrightarrow{K^\delta} M(X)^\delta \xrightarrow{J^\delta} (M(X)^{I^m})^\delta$$

is weakly homotopic to the identity map (i.e. homotopic on compact subspaces of $BM(X)^\delta$), and hence is an equivalence as desired.

We conclude this section with an application of our discretization criterion to André homology, which was brought to my attention by Emmanuel Dror. Let X be a topological space. Let $\mathfrak{M}$ denote any set of topological spaces having the following properties

(i) If $M \in \mathfrak{M}$ then M is contractible
(ii) If $M \in \mathfrak{M}$ then $M \times I \in \mathfrak{M}$ (up to homeomorphism).

The standard example is $\mathfrak{M} = \{\Delta^n \mid n \geq 0\}$. Let $\mathcal{C}(X)^\delta$ denote the discrete category whose objects are maps $\{M \to X \mid M \in \mathfrak{M}\}$. Morphisms in $\mathcal{C}(X)^\delta$ are commutative diagrams

$$\begin{array}{ccc} M & \longrightarrow & M' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Let $T: \mathcal{C}(X)^{\delta} \to \mathrm{Iso}(\mathrm{Ab})$ be a functor with values in the category of abelian groups and isomorphisms. Then André (cf. [32]) defines a chain complex $A_*(X; T)$ by

$$A_n(X; T) = \bigoplus_{\substack{M_0 \to M_1 \to \cdots \to M_n \\ \searrow \ \downarrow \ \swarrow \\ X}} T(M_0 \to X)$$

with differentials given as the alternating sum of faces, with the i-th face given by dropping M_i (and applying the functor T if $i = 0$). The André homology $H_*{}^A(X; T)$ is defined to be the homology of this chain complex and it fits in a broad categorical framework. We also note that the functor T defines a local coefficient system $\tilde{T}$ on X as follows. We pick a fixed object $M_0 \in \mathfrak{M}$. To each point $x \in X$ we assign the abelian group $T(c_x : M_0 \to X)$ where c_x is the constant map at x. To each path $\alpha : I \to X$ we assign the isomorphism

$$T(c_{\alpha(0)} : M_0 \to X) \xrightarrow{T\left(\begin{smallmatrix} M_0 \xrightarrow{i_0} M_0 \times I \\ \searrow \ \swarrow \\ X \end{smallmatrix}\right)} T(\tilde{\alpha} : M_0 \times I \to X)$$

$$\xrightarrow{T\left(\begin{smallmatrix} M_0 \xrightarrow{i_1} M_0 \times I \\ \searrow \ \swarrow \\ X \end{smallmatrix}\right)^{-1}} T(c_{\alpha(1)} : M_0 \to X)$$

where $i_t : M_0 \to M_0 \times I$, $\tilde{\alpha} : M_0 \times I \to X$ are given by $i_t(m_0) = (m_0, t)$, $\tilde{\alpha}(m_0, t) = \alpha(t)$ respectively. It is easy to see that this assignment gives a local coefficient system $\tilde{T}$ on X which (up to isomorphism) does not depend on the choice of M_0. We now use Corollary 1.4 to give a simple proof of the following theorem of André [32, p. 48].

THEOREM 2.5. *There is a natural isomorphism*

$$H_*{}^A(X; T) \cong H_*(X; \tilde{T}).$$

Proof. Consider the topological category $\mathcal{C}(X)$ where the objects and morphisms of $\mathcal{C}(X)^{\delta}$ are endowed with the function space topology. Then there is a continuous natural transformation $K: \mathcal{C}(X)^I \to \mathcal{C}(X)$ which assigns to each parametrized family of objects $I \to \mathrm{Ob}\ \mathcal{C}(X) = \coprod_{M \in \mathfrak{M}} X^M$ the adjoint map $M \times I \to X$ regarded as an object in $\mathcal{C}(X)$ and to each parametrized family of morphisms $I \to \mathrm{Mor}\ \mathcal{C}(X) = \coprod_{M, M' \in \mathfrak{M}} M'^M \times X^{M'}$ the corresponding adjoint diagram

$$\begin{array}{ccc} M \times I & \longrightarrow & M' \times I \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

regarded as a morphism in $\mathcal{C}(X)$. It is easy to see that K satisfies the hypothesis of Corollary 1.4(ii) and hence $B\mathcal{C}(X)^\delta \to B\mathcal{C}(X)$ is an equivalence. Next let $\mathcal{C}'(X)$ denote the full subcategory of $\mathcal{C}(X)$ whose objects are all the constant maps $\{c_x : M \to X \mid M \in \mathfrak{M}\}$. Then it follows immediately from the contractibility of the objects of $\mathfrak{M}$ and Lemma 0(a) that $B\mathcal{C}'(X) \hookrightarrow B\mathcal{C}(X)$ is an equivalence. Now consider the topological category $\mathfrak{X}$ whose objects are the points of X and whose morphisms are the identities. Then the functor $\mathfrak{X} \to \mathcal{C}'(X)$ which assigns to each object x of $\mathfrak{X}$ the object $c_x : M_0 \to X$ evidently induces an equivalence $X = B\mathfrak{X} \to B\mathcal{C}'(X)$.

We thus obtain a natural chain of equivalences

$$X = B\mathfrak{X} \to B\mathcal{C}'(X) \hookrightarrow B\mathcal{C}(X) \leftarrow B\mathcal{C}(X)^\delta.$$

Now the functor $T: \mathcal{C}(X)^\delta \to \mathrm{Iso(Ab)}$ defines a local coefficient system T on $B\mathcal{C}(X)^\delta$ and it is easy to see that T corresponds under the above chain of equivalences to the system $\tilde{T}$ on X. Thus we get a natural isomorphism $H_*(X; \tilde{T}) \cong H_*(B\mathcal{C}(X)^\delta; T)$. But the André chain complex $A_*(X; T)$ is evidently the same as the cellular chain complex $C_*^{\mathrm{cell}}(B\mathcal{C}(X)^\delta; T)$, so $H_*(B\mathcal{C}(X)^\delta; T) = H_*^A(X; T)$. This completes the proof.

3. Discretizing topological monoids. In this section we construct an analog of the stabilization procedure of Theorem 2.4 thus obtaining a functor which replaces any topological monoid by a discrete monoid whose classifying space has the same weak homotopy type. Specializing to the case of Moore loops, we recover MacDuff's result that any connected CW homotopy type can be realized as the classifying space of a discrete monoid.

As usual we restrict our attention to topological monoids for which the unit element is a nondegenerate basepoint.

The following wreath product construction serves as the basis of our replacement procedure.

Definition 3.1. Let M, N be topological monoids. Suppose N acts from the left on a space X. The wreath product $N \int M^X$ is the following topological monoid

(a) $N \int M^X = N \times M^X$ as a space
(b) multiplication in $N \int M^X$ is defined by

$$(\alpha, f)(\beta, g) = (\alpha\beta, u)$$

where $u(x) = f(\beta x) \cdot g(x)$

(c) the unit of $N \int M^X$ is $(1, c)$ where $c(x) = 1$ for all $x \in X$.

In the case when N is a finite permutation group acting on a finite set X, our construction $N \int M^X$ is the usual wreath product $N \int M$.

We next make an observation on constructing maps between wreath products.

Observation 3.2. Let $\phi : N \to N'$ be a homomorphism of topological monoids. Let X be an N-space, X' and N' space and $r : X' \to X$ a map such that

$$r(\phi(n)x') = nr(x') \qquad \forall n \in N, x' \in X'$$

Then the map $\phi \int r^* : N \int M^x \to N' \int M^{X'}$ given by

$$\left(\phi \int r^*\right)(\alpha, f) = (\phi(\alpha), f \cdot r)$$

is a homomorphism of topological monoids.

Example 3.3. (1). Let X be an N space, $\{1\}$ the trivial monoid. Then $\phi : \{1\} \to N$ and $\mathrm{id}_X : X \to X$ satisfy the hypothesis, so there is an induced homomorphism

$$\phi \int \mathrm{id}_X : M^X \to N \int M^X$$

(2) Let $\pi : X \times I \to X$ be the projection, and let $\phi : H(X) \to H(X \times I)$ be given by $\phi(\alpha) = \alpha \times 1_I$. Then

$$\pi(\phi(\alpha)(x, t)) = \pi(\alpha x, t) = \alpha x = \alpha\pi(x, t)$$

Hence there is an induced homomorphism

$$\phi \int \pi * : H(X) \int M^X \hookrightarrow H(X \times I) \int M^{X \times I}$$

Using these constructions we produce the desired procedure for converting topological monoids into discrete monoids with equivalent classifying spaces. Since the argument is basically a reprise of the proof of Theorem 2.4, we shall be content to sketch a brief outline leaving details to the diligent reader.

THEOREM 3.4. *Let M be a topological monoid. Let $H(I^n)$ denote the topological monoids of self-equivalence of the n-cube. Let $\bar{M}$ denote the colimit of the sequence*

$$M \hookrightarrow H(I) \int M^I \hookrightarrow H(I^2) \int M^{I^2} \hookrightarrow \cdots$$

(where the maps are as in 3.3(2) *above). Then there is an equivalence $B\bar{M}^\delta \to BM$.*

Proof. Since the inclusion $M \hookrightarrow \bar{M}$ is a homotopy equivalence, it suffices to show $B\bar{M}^\delta$ is weakly equivalent to $B\bar{M}$.

As in the proof of Theorem 2.4, it suffices to produce two pieces of data:

(1) a compatible family of functors (i.e. homomorphisms)

$$K_n : \left(H(I^n) \int M^{I^n}\right)^{I^m} \to H(I^{m+n}) \int M^{I^{m+n}}$$

where m is a fixed nonnegative integer and $n \geq 0$.

(2) a (incompatible) family of natural transformations U_n from the inclusion functor

$$S_{n,m} : \left(H(I^n) \int M^{I^n}\right)^{I^m} \to \left(H(I^{n+m}) \int M^{i^{n+m}}\right)^{I^n}$$

to the composite

$$\left(H(I^n) \int M^{I^n}\right)^{I^m} \xrightarrow{K_n} H(I^{m+n}) \int M^{I^{m+n}} \xrightarrow{J} \left(H(I^{m+n}) \int M^{I^{m+n}}\right)^{I^m}$$

We denote elements of $(H(I^n) \int M^{I^n})^{I^m}$ as parametrized families of elements (α_s, f_s), $s \in I^m$ in $H(I^n) \int M^{I^n}$. In this notation we define K_n by the formula

$$K_n(\alpha_s, f_s) = (\bar{\alpha}, \bar{f})$$

where $\bar{\alpha} : I^m \times I^n \to I^m \times I^n$ is given by

$$\bar{\alpha}(s, t) = (s, \alpha_s(t)) \qquad s \in I^m, t \in I^n$$

and where $\bar{f} : I^m \times I^n \to M$ is given by

$$\bar{f}(s, t) = f_s(t)$$

It is easy to check that the K_n are a compatible family of functors and thus define a functor

$$K : \bar{M}^{I^m} \to \bar{M}.$$

We now define an element $(u_s, i_s) \in (H(I^{m+n}) \int M^{I^{m+n}})^{I^m}$ with $u_s : I^{m+n} \to I^{m+n}$ given by

$$u_s(t_1, t_2, \ldots, t_{m+n}) = (s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n)$$

and $i_s : I^{m+n} \to M$ given by

$$i_s(t_1, t_2, \ldots, t_{m+n}) = 1$$

It is easily checked that the following diagram commutes for any element $(\alpha_s, f_s) \in (H(I^n) \int M^{I^n})^{I^m}$

$$\begin{array}{ccc} \cdot & \xrightarrow{(u_s, i_s)} & \cdot \\ \Big\downarrow{\scriptstyle S_{n,m}(\alpha_s, f_s)} & & \Big\downarrow{\scriptstyle JK_n(\alpha_s, f_s)} \\ \cdot & \xrightarrow{(u_s, i_s)} & \cdot \end{array}$$

The rest of the argument then follows the pattern of the proof of Theorem 2.4.

An immediate consequence of Theorem 3.4 is the following functorial and natural version of MacDuff's result on the homotopy type of classifying spaces of discrete monoids.

Let $\mathfrak{M}^\delta$ denote the category of discrete monoids and $\mathfrak{J}_0$ the category of based path connected spaces.

THEOREM 3.5. *There is a functor $D : \mathfrak{J}_0 \to \mathfrak{M}^\delta$ and a chain of natural equivalences*

$$BDX \simeq X$$

Thus every path connected space has the weak homotopy type of the classifying space of a discrete monoid.

Proof. Let ΛX denote the topological monoid of Moore loops on X. According to [19; Lemma 15.4] there is a natural equivalence

$$\xi : B\Lambda X \to X$$

Let $\overline{\Lambda X}$, $\overline{\Lambda X}^\delta$ be as in Theorem 3.4. Obviously $\overline{\Lambda X}$, $\overline{\Lambda X}^\delta$ are functors of X and we have a natural chain of equivalences

$$B\overline{\Lambda X}^\delta \to B\overline{\Lambda X} \leftarrow B\Lambda X \to X$$

Hence we can define $DX = \overline{\Lambda X}^\delta$.

4. The classifying spaces of pushouts of discrete monoids. In this section we extend a result of J. H. C. Whitehead [31] on the classifying spaces of amalgamated free products of discrete groups to the case of discrete monoids and discuss its relation to Theorem 3.5. This result will also play a crucial role in our axiomatic characterization of classifying space constructions on topological monoids.

Of course, Theorem 3.5 is strongly reminiscent of the result of Kan and Thurston [14], which states that, for any path connected space X, there is a discrete group G and a homology equivalence $BG \to X$. One of the main ingredients in their proof is the following result.

THEOREM 4.0 (J. H. C. Whitehead). *Let $K = \coprod \{G \xrightarrow{\lambda_i} H_i\}_{i \in I}$ be a pushout diagram of discrete groups with each λ_i a monomorphism.* (*i.e. K is the free product of the H_i's amalgamated over G.*) *Then the natural map*

$$\coprod_{BG} BH_i \to BK$$

is a homotopy equivalence.

The corresponding result for discrete monoids, which will be proved in the next section, is

THEOREM 4.1. *Let* $P = \amalg \{ W \xrightarrow{\psi_i} M_i \}_{i \in I}$ *be a pushout diagram of discrete monoids. Assume that*

(i) *all the* ψ_i*'s are injections*
(ii) *the monoid ring* $\mathbf{Z}[M_i]$ *is flat as a left* $\mathbf{Z}[W]$ *module for all i*

Then the natural map

$$\amalg_{BW} BM_i \to BP$$

is a homotopy equivalence.

A special case of Theorem 4.1 was proved by MacDuff in [15].

Remark 4.2. The proof of Theorem 4.1 will show that if hypothesis (i) is deleted while (ii) is retained, then we obtain that the natural map of the homotopy pushout of the BM_i's over BW into BP is a homotopy equivalence.

Remark 4.3. In the case of groups, hypothesis (i) implies (ii). For monoids however flatness phenomena over the monoid ring appear to be much more complex and seem to play a very important role in determining the homotopy type of the classifying space. For instance it is well known that, for a discrete monoid M, $\pi_i BM = 0$ for $i \geq 2$ if M is either a group or a commutative monoid [17]. On the other hand, according to Theorem 3.5, $\pi_i BM$ can be anything at all in general. The following result accounts for this startling difference.

PROPOSITION 4.4. *Let M be a discrete monoid and let GM be the nonabelian Grothendieck group of M (i.e. GM is the free group on M modulo relations of the form* $[m_1 m_2] = [m_1][m_2]$*). Then the following statements are equivalent:*

(i) $\pi_i BM = 0$ *for* $i \geq 2$
(ii) $BM \to BGM$ *is an equivalence*
(iii) $\mathrm{Tor}_i^{\mathbf{Z}[M]}(\mathbf{Z}, \mathbf{Z}[GM]) = 0$ *for all* $i \geq 1$.

Proof. It follows from the standard calculation of the fundamental group of a reduced simplicial set that $\pi_1 BM = GM$. It follows that (i) $\Leftrightarrow$ (ii). Next one observes that the universal covering space of BM is the two-sided bar construction $B(*, M, GM)$ (cf. [19]). To see this, it suffices to note that $GM = \pi_1 BM$ acts freely on $B(*, M, GM)$ with orbit space $B(*, M, *) = BM$. If $C_*(\ )$ denotes cellular chains, then

$$H_i(B(*, M, GM)) \cong H_i(C_*(B(*, M, M)) \otimes_{\mathbf{Z}[M]} \mathbf{Z}[GM])$$

$$\cong \operatorname{Tor}_i^{\mathbf{Z}[M]}(\mathbf{Z}, \mathbf{Z}[GM])$$

since $C_*(B(*, M, M))$ is the standard $\mathbf{Z}[M]$—free resolution of $\mathbf{Z}$. This completes the proof. Alternatively one can appeal to [9; Chapter X, Proposition 3.1].

We recover immediately the aforementioned results about $\pi_i BM = 0$ for M a discrete group or commutative monoid. In both cases $\mathbf{Z}[GM]$ is flat over $\mathbf{Z}[M]$. In the first case $M = GM$, while in the second case $\mathbf{Z}[GM] = \mathbf{Z}[M][M]^{-1}$ is a localization and hence flat over $\mathbf{Z}[M]$.

Theorem 4.1 can be used together with the following "flat mapping cylinder construction" to give a simplified version of MacDuff's original proof of Theorem 3.5.

PROPOSITION 4.5. *Let $f: M \to N$ be a homomorphism of discrete monoids. Then there is a discrete monoid Mf together with homomorphisms $i: M \to Mf$, $\rho: Mf \to N$, and a nonunital homomorphism $j: N \to Mf$ such that $\rho \cdot j = \mathrm{id}_N$ and*

(i) *the diagram*

$$\begin{array}{ccc} & & Mf \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow{f} & N \end{array}$$

commutes

(ii) *i is an injection*

(iii) *$\mathbf{Z}[Mf]$ is free as a left $\mathbf{Z}[M]$ module*

(iv) *$B\rho: BMf \to BN$ is a homotopy equivalence*

(v) *Mf is functorial in f: a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f'} & N' \end{array}$$

induces a homomorphism $Mf \to Mf'$ and this correspondence is functorial.

Proof. Let $N \cup \tilde{1}$ denote N with another unit adjoined. Define $Mf = M \times (N \cup \tilde{1})$ with the following multiplication

$$(m, x)(m', y) = \begin{cases} (mm', y) & \text{if} \quad x = \tilde{1} \\ (m, xf(m')y) & \text{if} \quad x \in N \end{cases}$$

Define $i(m) = (m, \tilde{1})$, $\rho(m, x) = f(m)x$, $j(n) = (1, n)$. Then $\rho \cdot j = \mathrm{id}_N$, (i), (ii), (iii) and (v) follow immediately. To prove (iv) we proceed as follows. If we regard Mf as a category with one object, the commutative diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{(m,x)} & \cdot \\ \downarrow {\scriptstyle (1,1)} & & \downarrow {\scriptstyle (1,1)} \\ \cdot & \xrightarrow{(1,f(m)x)} & \cdot \end{array}$$

provides a natural transformation between id_{Mf} and the "nonunital functor" $j \cdot \rho$. Now since j is nonunital, $Bj : BN \to BMf$ preserves faces but not degeneracies. However it is well known that the homotopy type of the geometric realization of a simplicial set depends only its faces and not on its degeneracies [25, Appendix A]. Hence Bj provides a homotopy inverse for $B\rho$, which is therefore an equivalence.

Remark 4.6. In the special case of the trivial homomorphism $\{1\} \to N$ we have $Mf = N \cup \tilde{1})$. Hence 4.5 implies that $B(N \cup \tilde{1}) \to BN$ is an equivalence.

We now obtain the following version of Theorem 3.5 which follows more closely the approach taken by MacDuff and Kan-Thurston. For technical reasons, we prefer to work in the category of Δ-sets (simplicial sets with faces but not degeneracies, (cf. [24]).

THEOREM 4.7. *There is a functor which assigns to each connected Δ-set X a discrete monoid RX such that BRX is equivalent to $|X|$.*

Proof. We inductively construct a sequence of functors R_n : connected Δ-sets of dimension $\leq n \to$ discrete monoids, having the following properties

(i) There are natural nonunital homomorphisms

$$R_{n-1}X^{n-1} \to R_n X$$

(ii) There is an equivalence $|X| \xrightarrow{\simeq} BR_nX$ which is natural up to homotopy and such that the following diagram homotopy commutes.

$$\begin{array}{ccc} |X^{n-1}| & \longrightarrow & |X| \\ \downarrow \simeq & & \downarrow \simeq \\ BR_{n-1}X & \longrightarrow & BR_nX \end{array}$$

Here and throughout X^m denotes the m-skeleton of the Δ-set X.

If X is of dimension ≤ 1, we define $R_1X = \pi_1|X|$. It is a classical result of homotopy theory that there is an equivalence $|X| \to BR_1X$ which is natural up to homotopy.

Now assume that we have defined R_{n-1}. Let X be a connected Δ-set of dimension $\leq n$. We then have a pushout diagram

$$\begin{array}{ccc} \amalg\, |\partial\Delta^n| & \longrightarrow & \amalg\, |\Delta^n| \\ \downarrow & & \downarrow \\ |X^{n-1}| & \longrightarrow & |X| \end{array} \tag{1}$$

where Δ^n is the Δ-set consisting of all the nondegenerate faces of the standard n-simplex and the coproduct $\amalg$ is taken over the (possibly empty) set of n-simplices of X. By induction the map $\alpha: \amalg R_{n-1}(\partial\Delta^n) \to R_{n-1}X^{n-1}$ is defined. We now perform the flat mapping cylinder construction of Proposition 4.5 on the map α and on the trivial map $0: \amalg R_{n-1}(\partial\Delta^n) \to \{1\}$ and define R_nX via the pushout diagram of monoids

$$\begin{array}{ccc} \amalg R_{n-1}(\partial\Delta^n) & \longrightarrow & M0 \\ \downarrow & & \downarrow \\ M\alpha & \longrightarrow & R_nX \end{array} \tag{2}$$

It is clear from construction that $X \mapsto R_nX$ defines a functor on n-dimensional Δ-sets. By Proposition 4.5 we have a natural nonunital homomorphism

$$R_{n-1}X^{n-1} \xrightarrow{j} M\alpha \to R_nX$$

By Theorem 4.1 the classifying space of diagram (2) is also a pushout diagram. Thus the equivalence $|\ | \to BR_{n-1}(\ )$ on $(n-1)$-dimensional Δ-sets induces an equivalence of pushout diagrams from (1) to the classifying space of (2). Hence we obtain an equivalence $|X| \to BR_nX$ which is natural up to homotopy and compatible up to homotopy with $BR_{n-1}X^{n-1} \to BR_nX$.

Now let X be an arbitrary connected Δ-set. Then $\varinjlim R_nX^n$ is a semigroup, i.e. it has an associative multiplication but no unit. Let $RX = (\varinjlim R_nX^n) \cup \tilde{1}$. Clearly $RX = \varinjlim(R_nX^n \cup \tilde{1})$, and by Remark 4.6 $BRX = \varinjlim B(R_nX^n \cup \tilde{1}) \simeq \varinjlim BR_nX^n$. The equivalence $|X^n| \to BR_nX^n$ thus induces an equivalence $|X| \to BRX$.

We conclude our discussion of classifying spaces of discrete monoids with an amusing example whose details we leave as an exercise for the reader. Let M be the five element monoid consisting of a unit 1 and elements x_{ij}, $i, j = 1, 2$ which multiply according to the rule $x_{ij}x_{k\ell} = x_{i\ell}$. Then $BM \simeq S^2$. It seems likely that any finite simply connected complex should be equivalent to the classifying space of a finite monoid.

5. Proof of Theorem 4.1. Our proof of Theorem 4.1 follows the same line of argument as Gruenberg's homological proof of J. H. C. Whitehead's Theorem 4.0 (cf. [13]). In the following we will use the notation $I(M)$ for the augmentation ideal of the monoid ring $\mathbf{Z}[M] \xrightarrow{\epsilon} \mathbf{Z}$.

LEMMA 5.1. *For any pushout of monoids there is an isomorphism*

$$I(\amalg_W M_i) \cong \amalg_{I(W)\otimes_{\mathbf{Z}[W]}\mathbf{Z}[\amalg_W M_i]} I(M_i) \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_i]$$

of right $\mathbf{Z}[\amalg_W M_i]$ *modules.*

Proof. We construct a category $\mathfrak{Monmod}$ whose objects are pairs (M, A) where M is a monoid and A is a right $\mathbf{Z}[M]$ module. A morphism in $\mathfrak{Monmod}$ from (M, A) to (N, B) is a pair (ϕ, ψ) where $\phi : M \to N$ is homomorphism of monoids and $\psi : A \to B$ is a homomorphim of abelian groups such that

$$\psi(am) = \psi(a)\phi(m)$$

There is a functor $F : \mathfrak{Monoids} \to \mathfrak{Monmod}$ given by $F(M) = (M, I(M))$. The functor $G : \mathfrak{Monmod} \to \mathfrak{Monoids}$ given by $G(M, A) = M \ltimes A$ (split extension of M by A) is a right adjoint for F. Hence F preserves all colimits, in particular pushouts. Since the pushout in $\mathfrak{M}$onmod of $\{(W, I(W)) \to (M_i, I(M_i))\}_{i\in I}$ is

$$(\amalg_W M_i,\ \amalg_{I(W)\otimes_{\mathbf{Z}[W]}\mathbf{Z}[\amalg_W M_i]} I(M_i) \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_i]),$$

the result follows.

LEMMA 5.2. *Let* $\{W \to M_i\}_{i\in I}$ *be a pushout diagram of monoids such that each* $\mathbf{Z}[M_i]$ *is flat as a left* $\mathbf{Z}[W]$*-module. Then*

(a) $\mathbf{Z}[\amalg_W M_i]$ *is flat both as a left* $\mathbf{Z}[W]$ *module and as a left* $\mathbf{Z}[M_j]$ *module for each* $j \in I$

(b) $I(W) \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] \to I(M_j) \otimes_{\mathbf{Z}[M_j]} \mathbf{Z}[\amalg_W M_i]$ *is an injection for all* $j \in I$

(c) *The sequence*

$$0 \to \bigoplus_{I-i_0} \mathbf{Z} \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] \to \bigoplus_{I} \mathbf{Z} \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_j] \to \mathbf{Z} \to 0$$

is exact.

Proof. (a) This is immediate since $\mathbf{Z}[\amalg_W M_i]$ is the direct limit of

$$\mathbf{Z}[M_j] \otimes_{\mathbf{Z}[W]} \mathbf{Z}[M_{i_1}] \otimes_{\mathbf{Z}[W]} \mathbf{Z}[M_{i_2}] \otimes_{\mathbf{Z}[W]} \cdots \otimes_{\mathbf{Z}[W]} \mathbf{Z}[M_{i_k}]$$

$$i_1, i_2, \ldots i_k \in I$$

as a left $\mathbf{Z}[M_j]$ module and hence is flat over $\mathbf{Z}[M_j]$ and therefore also over $\mathbf{Z}[W]$

(b) We have a commutative diagram

$$\begin{array}{ccc} I(W) \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] & \longrightarrow & I(M_j) \otimes_{\mathbf{Z}[M_j]} \mathbf{Z}[\amalg_W M_i] \\ \downarrow & & \downarrow \\ \mathbf{Z}[W] \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] & \xleftrightarrow{\simeq} & \mathbf{Z}[M_j] \otimes_{\mathbf{Z}[M_j]} \mathbf{Z}[\amalg_W M_i] \end{array}$$

The left hand vertical arrow is an injection since $\mathbf{Z}[\amalg_W M_i]$ is flat over $\mathbf{Z}[W]$. Hence the top arrow must also be an injection.

(c) We have a commutative diagram of right $\mathbf{Z}[\amalg_W M_i]$ modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{I-i_0} I(W) \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] & \to & \bigoplus_I I(M_i) \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_i] & \to & I(\amalg_W M_i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{I-i_0} \mathbf{Z}[W] \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] & \to & \bigoplus_I \mathbf{Z}[M_i] \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_i] & \to & \mathbf{Z}[\amalg_W M_i] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \bigoplus_{I-i_0} \mathbf{Z} \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i] & \to & \bigoplus_I \mathbf{Z} \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_i] & \to & \mathbf{Z} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The left and middle columns are exact by part (a) while the right column is clearly exact. The top row is exact by Lemma 5.1 and part (b), while the middle row is obviously exact. It follows that the bottom row is also exact.

LEMMA 5.3. *The Grothendieck construction G : $\mathfrak{M}$onoids $\to$ $\mathcal{G}$roups preserves pushouts.*

Proof. The Grothendieck construction is left adjoint to the forgetful functor $\mathcal{G}$roups $\to$ $\mathfrak{M}$onoids.

In what follows we assume the hypothesis of Theorem 4.1: that $\{W \xrightarrow{\psi_i} M_i\}_{i \in I}$ is a pushout diagram of monoids such that each ψ_i is an injection and every $\mathbf{Z}[M_i]$ is flat as a left $\mathbf{Z}[W]$ module. We denote by

$$\pi : E(\amalg_W M_i) = B(*, \amalg_W M_i, \amalg_W M_i) \to B(\amalg_W M_i)$$

$$= B(*, \amalg_W M_i, *)$$

the standard bar construction on the monoid $\amalg_W M_i$ (cf. [19]). We also note that under the hypothesis $\amalg_{BW} BM_i$ can be considered as a subcomplex of $B(\amalg_W M_i)$.

LEMMA 5.4. *The space $\pi^{-1}(\amalg_{BW} BM_i)$ is acyclic.*

Proof. We have a Mayer-Victoris sequence

$$\cdots \to \bigoplus_{I-i_0} H_s(\pi^{-1}(BW)) \to \bigoplus_I H_s(\pi^{-1}(BM_i))$$

$$\to H_s(\pi^{-1}(\amalg_{BW} BM_i)) \to \bigoplus_{I-i_0} H_{s-1}(\pi^{-1}(BW)) \to \cdots$$

On the level of cellular chains we have

$$C_*(\pi^{-1}(BK)) \cong C_*(EK) \otimes_{\mathbf{Z}[K]} \mathbf{Z}[\amalg_W BM_i] \qquad K = W \quad \text{or} \quad M_j, j \in I.$$

Consequently

$$H_s(\pi^{-1}(BK)) \cong \mathrm{Tor}_s^{\mathbf{Z}[K]}(\mathbf{Z}, \mathbf{Z}[\amalg_W M_i])$$

$$= \begin{cases} 0 & \text{if} \quad s > 0 \quad \text{by Lemma 5.2(a)} \\ \mathbf{Z} \otimes_{\mathbf{Z}[K]} \mathbf{Z}[\amalg_W M_i] & \text{if} \quad s = 0 \end{cases}$$

It follows that $H_s(\pi^{-1}(\amalg_{BW} BM_i)) = 0$ for $s \geq 2$ and that the rest of the Mayer-Victoris sequence above collapses to

$$0 \to H_1(\pi^{-1}(\amalg_{BW} BM_i)) \to \bigoplus_{I - i_0} \mathbf{Z} \otimes_{\mathbf{Z}[W]} \mathbf{Z}[\amalg_W M_i]$$

$$\to \bigoplus_I \mathbf{Z} \otimes_{\mathbf{Z}[M_i]} \mathbf{Z}[\amalg_W M_j] \to H_0(\pi^{-1}(\coprod_{BW} BM_i)) \to 0$$

Now it follows from Lemma 5.2(c) that

$$H_1(\pi^{-1}(\amalg_{BW} BM_i)) = 0, \qquad H_0(\pi^{-1}(\amalg_{BW} BM_i)) = \mathbf{Z},$$

which completes the proof.

Proof of Theorem 4.1. It suffices to show that

$$\amalg_{BW} BM_i \hookrightarrow B(\amalg_W M_i)$$

induces isomorphisms on homotopy groups.

Isomorphism on π_1 follows from the van Kampen theorem and Lemma 5.3. Consider the universal cover

$$\tilde{\pi} : B(*, \amalg_W M_i, G(\amalg_W M_i)) \to B(\amalg_W M_i)$$

and its restriction

$$\tilde{\pi} : \tilde{\pi}^{-1}(\amalg_{BW} BM_i) \to \amalg_{BW} BM_i$$

This is a covering with covering group $G(\amalg_W M_i) = \pi_1(\amalg_{BW} BM_i)$. Hence $\tilde{\pi}^{-1}(\amalg_{BW} BM_i)$ is the universal cover of $\amalg_{BW} BM_i$. The proof now boils down to showing that

$$\tilde{\pi}^{-1}(\amalg_{BW} BM_i) \hookrightarrow B(*, \amalg_W M_i, G(\amalg_W M_i))$$

is a homology equivalence.

This is easy. On the level of cellular chains we have a commutative diagram

$$\begin{array}{ccc} C_*(\tilde{\pi}^{-1}(\amalg_{BW} BM_i)) & \cong & C_*(\pi^{-1}(\amalg_{BW} BM_i)) \otimes_{\mathbf{Z}[\amalg_W M_i]} \mathbf{Z}[G(\amalg_W M_i)] \\ \downarrow & & \downarrow \\ C_*(B(*, \amalg_W M_i, G(\amalg_W M_i))) & \cong & C_*(E(\amalg_W M_i)) \otimes_{\mathbf{Z}[\amalg_W M_i]} \mathbf{Z}[G(\amalg_W M_i)] \end{array}$$

But $C_*(\pi^{-1}(\amalg_{BW} BM_i)) \hookrightarrow C_*(E(\amalg_W M_i))$ is a chain homotopy equivalence of right $\mathbf{Z}[\amalg_W M_i]$ complexes, since, by Lemma 5.4, both are free $\mathbf{Z}[\amalg_W M_i]$ resolutions of $\mathbf{Z}$. It follows that

$$C_*(\tilde{\pi}^{-1}(\amalg_{BW} BM_i)) \to C_*(B(*, \amalg_W M_i, G(\amalg_W M_i))$$

is also a chain homotopy equivalence. Hence

$$H_*(\tilde{\pi}^{-1}(\amalg_{BW} BM_i)) \to H_*(B(*, \amalg_W M_i, G(\amalg_W M_i))$$

is an isomorphism.

6. The Moore suspension functor. One of the major advances in homotopy theory was James' construction of a tractable model JX for the free loop space $\Omega\Sigma X$ on a based path connected space X. James showed that if one takes

$$JX = \text{free topological monoid on } X$$

$$= \amalg_{n \geq 0} X^n \Big/ \begin{array}{l}\text{equivalence relation generated by} \\ \text{insertion and deletion of basepoint}\end{array}$$

then there is an equivalence $JX = \Omega\Sigma X$ provided that X is path connected.

One would like to extend the natural based map $\eta : X \to \Omega\Sigma X$ to a morphism of monoids $JX \to \Omega\Sigma X$, but $\Omega\Sigma X$ is not a monoid. The obvious course is to replace the ordinary loop functor Ω by the Moore loop functor Λ. Unfortunately there is no natural map $X \to \Lambda\Sigma X$. The most natural procedure would be to replace the suspension functor Σ by a "Moore suspension" functor Ξ. Then one has a natural based map $X \to \Lambda\Xi X$ which extends to the desired homomorphism $JX \to \Lambda\Xi X$. Unfortunately, Ξ does not appear in the literature, and we need its properties for applications to the uniqueness theorems for classifying spaces in the following sections. This section is devoted to this topic.

Since the suspension Σ is left adjoint to the loop functor Ω, the Moore suspension Ξ should be left adjoint to the Moore loop functor Λ. At first glance it would seem that no such functor exists. For if a functor has a left adjoint, it must preserve products, which Λ evidently does not. (Indeed no monoid-valued loop functor can preserve products [6; 6.1].) However upon closer inspection we see that $\Lambda(X \times Y) = \Lambda X \times_{\mathbf{R}_+} \Lambda Y$ where the fiber product is taken over the parameter length functions. This is a product but not in the usual category of topological spaces. This insight gets us underway.

Definition 6.1. Let $\mathfrak{J}_*[\mathbf{R}_+]$ denote the category whose objects are based spaces X together with a continuous map $p : X \to \mathbf{R}_+$ (the nonnegative real numbers) such that $p^{-1}(0) = *$. The morphisms from (X, p) to (Y, q) are commutative triangles

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathbf{R}_+ & \end{array}$$

We say that $(X, p) \to (Y, q)$ is an equivalence if $X \to Y$ is an equivalence.

The Moore suspension functor is the functor $\Xi : \mathfrak{J}_*[\mathbf{R}_+] \to \mathfrak{J}_*$ (the category of based spaces) given on objects by

$$\Xi(X, p) = X \times \mathbf{R}_+ / \{(x, t) \mid t = 0 \quad \text{or} \quad t \geq p(x)\}$$

and extended to morphisms in the obvious way.

LEMMA 6.2. (i) *$\Xi : \mathfrak{J}_*[\mathbf{R}_+] \to \mathfrak{J}_*$ is left adjoint to $\Lambda : \mathfrak{J}_* \to \mathfrak{J}_*[\mathbf{R}_+]$.*

(ii) *For any object* (X, p) *in* $\mathfrak{T}_*[\mathbf{R}_+]$ *there is a natural homeomorphism*

$$\Xi(X, p) \cong \Sigma X$$

Proof. (i) is obvious. As for (ii) the mutually inverse homeomorphisms are

$$\Sigma X \to \Xi(X, p) \qquad [x, t] \to [x, tp(x)]$$

$$\Xi(X, p) \to \Sigma X \qquad [x, t] \to [x, t/p(x)]$$

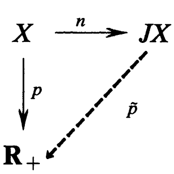
Remark 6.3. The reason we imposed the condition $p^{-1}(0) = *$ on the objects (X, p) of $\mathfrak{T}_*[\mathbf{R}_+]$ is precisely because we wanted (ii) to hold. If this condition is deleted, the above argument shows that

$$\Xi(X, p) \cong \Sigma(X/p^{-1}(0)).$$

If (X, p) is an object of $\mathfrak{T}_*[\mathbf{R}_+]$ then the unit of the adjunction (Ξ, Λ) gives us a natural map $X \to \Lambda\Xi(X, p)$ which extends uniquely to a natural map of topological monoids $JX \to \Lambda\Xi(X, p)$. If, however, we are to fully exploit all the structure inherent in this situation, we need to interpret the functor J in the category $\mathfrak{T}_*[\mathbf{R}_+]$.

Definition 6.4. A monoid in $\mathfrak{T}_*[\mathbf{R}_+]$ is a pair (N, p) such that $p: N \to \mathbf{R}_+$ is a monoid homomorphism. A morphism $f: (N, p) \to (N', p')$ of monoids in $\mathfrak{T}_*[\mathbf{R}_+]$ is a morphism in $\mathfrak{T}[\mathbf{R}_+]$ such that $f: N \to N'$ is a monoid homomorphism. We denote by $\mathfrak{M}[\mathbf{R}_+]$ the category of monoids in $\mathfrak{T}_*[\mathbf{R}_+]$.

LEMMA 6.5. (i) *The free monoid* $J(X, p)$ *on a space* (X, p) *in* $\mathfrak{T}_*[\mathbf{R}_+]$ *is the pair* $(JX, \tilde{p})$ *where* $\tilde{p}$ *is the unique continuous monoid homomorphism filling in the following diagram*



(ii) *The free monoid functor* $J : \mathfrak{J}_*[\mathbf{R}_+] \to \mathfrak{J}_*[\mathbf{R}_+]$ *is a monad in* $\mathfrak{J}_*[\mathbf{R}_+]$.

(iii) *There is a morphism of monads* $\lambda : J \to \Lambda\Xi$ in $\mathfrak{J}_*[\mathbf{R}_+]$.

Proof. (i) is obvious, the only point to check being that $\tilde{p}^{-1}(0) = *$. (ii) follows immediately. For (iii) one uses freeness to fill in the following diagram

$$\begin{array}{ccc} (X, p) & \longrightarrow & J(X, p) \\ \downarrow & \swarrow & \\ \Lambda\Xi(X, p) & & \end{array}$$

Since the domain of Ξ is the category $\mathfrak{J}_*[\mathbf{R}_+]$ while real life goes on in the category $\mathfrak{J}_*$ we need a functor $R : \mathfrak{J}_* \to \mathfrak{J}_*[\mathbf{R}_+]$ with good properties.

LEMMA 6.6. (i) *The forgetful functor* $L : \mathfrak{J}_*[\mathbf{R}_+] \to \mathfrak{J}_*$ *has a right adjoint* $R : \mathfrak{J}_* \to \mathfrak{J}_*[\mathbf{R}_+]$.

(ii) *The unit* $(X, p) \to RL(X, p)$ *and counit* $LRY \to Y$ *are equivalences*

(iii) *R commutes with geometric realizations: if* X_* *is a simplicial space, there is a natural homeomorphism* $|n \to RX_n| \cong R|X_*|$

(iv) *If M is a (well-based) topological monoid, then RM has a natural structure of a topological monoid in* $\mathfrak{J}_*[\mathbf{R}_+]$

(v) *The adjunction* (L, R) *induces an adjunction* $L : \mathfrak{M}[\mathbf{R}_+] \to \mathfrak{M}$ $\mathfrak{R} : \mathfrak{M} \to \mathfrak{M}[\mathbf{R}_+]$ *on the corresponding categories of monoids.*

Proof. We define RX as the subspace of $(X \times \mathbf{R}_+, \pi : X \times \mathbf{R}_+ \to \mathbf{R}_+)$ consisting of $(*, 0)$ and all (x, t) such that $t > 0$; (i) immediately follows. The unit $(X, p) \to RL(X, p)$ is given by $x \mapsto (x, p(x))$ and the counit $LRY \to Y$ is given by $(y, t) \mapsto y$. Clearly both are equivalences. Parts (b), (c) of Lemma 0 imply (iii). The monoid structure on RM is the one induced by the direct product structure on $M \times \mathbf{R}_+$.

LEMMA 6.7. *The adjunctions* (Ξ, Λ) *and* $(\Sigma L, R\Omega)$ *are isomorphic.*

Proof. According to Lemma 6.2(ii) there is a natural isomorphism $h : \Xi \cong \Sigma L$. There is also an obvious natural isomorphism $k : \Lambda \cong R\Omega$. One checks immediately that the following diagram commutes for any object $(X, p) \in \mathfrak{J}_*[\mathbf{R}_+]$ and any object $Y \in \mathfrak{J}_*$

$$\begin{array}{ccc} \mathfrak{T}_*(\Sigma L(X, p), Y), Y) \cong \mathfrak{T}_*(L(X, p), \Omega Y) & \cong & \mathfrak{T}_*[\mathbf{R}_+]((X, p), R\Omega Y) \\ \Big\| h^* & & \Big\| k_* \\ \mathfrak{T}_*(\Xi(X, p), Y) & \xleftarrow{\ \simeq\ } \!\!\!\!\rightarrow & \mathfrak{T}_*[\mathbf{R}_+]((X, p), \Lambda Y) \end{array}$$

To avoid notational clutter we shall usually suppress any mention of the forgetful functor L. It will be clear from context when we are regarding objects in $\mathfrak{T}_*[\mathbf{R}_+]$ as objects of $\mathfrak{T}_*$.

A minor modification of the usual proof that JX and $\Omega\Sigma X$ have the same weak homotopy type when X is path connected yields the following result.

THEOREM 6.8. *If (X, p) is an object in $\mathfrak{T}_*[\mathbf{R}_+]$ such that X is path connected, then the natural map of monoids*

$$\lambda : J(X, p) \to \Lambda\Xi(X, p)$$

is an equivalence in $\mathfrak{T}_*[\mathbf{R}_+]$.

The rest of this section is devoted to analyzing the natural map $\lambda : J(X, p) \to \Lambda\Xi(X, p)$ when X is not path connected. We will show that this map is a "group completion" in the sense of Thomason [27], that is, $B\lambda$ is an equivalence.

In what follows we denote by $\iota : \Sigma M \to BM$ the natural map whose adjoint $\hat{\iota} : M \to \Omega BM$ is an equivalence when $\pi_0 M$ is a group. Our starting point is the following elementary consequence of Theorem 4.1.

LEMMA 6.9. *If X is a discrete based space then the composite map*

$$\Sigma X \xrightarrow{\Sigma\eta} \Sigma JX \xrightarrow{\iota} BJX$$

is an equivalence.

Proof. We have $X = \vee_{X-*} S^0$ and $JX = \coprod_{X-*}\mathbf{N}$ where $\mathbf{N} = JS^0$ is the monoid of natural numbers and the coproduct is taken in the category of discrete monoids. We have a commutative diagram

$$\begin{array}{ccccc} \Sigma X & \xrightarrow{\Sigma\eta} & \Sigma JX & \xrightarrow{\iota} & BJX \\ \Big\| & & \uparrow & & \uparrow \\ \vee_{X-*}\Sigma S^0 & \xrightarrow{\Sigma\eta} & \vee_{X-*}\Sigma\mathbf{N} & \xrightarrow{\vee\iota} & \vee_{X-*}B\mathbf{N} \end{array}$$

and by Theorem 4.1 the right hand vertical map is an equivalence. This reduces us to the case $X = S^0$, for which we appeal to the commutative diagram

$$\begin{array}{ccccc} & \nearrow & \Sigma \mathbf{N} & \longrightarrow & B\mathbf{N} \\ \Sigma S^0 & & \downarrow & & \downarrow \\ & \searrow & \Sigma \mathbf{Z} & \longrightarrow & B\mathbf{Z} \end{array}$$

The composite across the bottom is classically known to be an equivalence. The right hand vertical arrow is an equivalence by Proposition 4.4 (and the succeeding remarks). This completes the proof.

We now easily deduce the corresponding result for arbitrary spaces. This is of interest in its own right (cf. [3]).

THEOREM 6.10. *For any space the composite*

$$\Sigma X \xrightarrow{\Sigma\eta} \Sigma JX \xrightarrow{\iota} BJX$$

is an equivalence.

Proof. Let $TX = |T_*X|$ be the geometric realization of the total singular complex of X. Let $\xi : TX \to X$ be the natural equivalence. Consider the commutative diagram

$$\begin{array}{ccccc} |n \mapsto \Sigma T_n X| & \xrightarrow{|\Sigma\eta_*|} & |n \mapsto \Sigma JT_n X| & \xrightarrow{|\iota *|} & |n \to BJT_n X| \\ \Vert\wr & & \Vert\wr & & \Vert\wr \\ \Sigma TX & \xrightarrow{\Sigma\eta} & \Sigma JTX & \xrightarrow{\iota} & BJTX \\ \downarrow{\scriptstyle \Sigma\xi} & & \downarrow{\scriptstyle \Sigma J\xi} & & \downarrow{\scriptstyle BJ\xi} \\ \Sigma X & \xrightarrow{\Sigma\eta} & \Sigma JX & \xrightarrow{\iota} & BJX \end{array}$$

The vertical homeomorphisms arise from the fact that the functors Σ, J (cf. [21, 12.1, 12.2]) and B (cf. Lemma 0(b), (d)) commute with geometric realization. The bottom vertical maps are equivalences since the functors Σ, J (cf. [2]; 2.6]) and B (cf. Lemma 0(a)) preserve equivalences. The composite of the horizontal maps across the top is an equivalence by Lemma 6.9 and Lemma 0(a). Hence so is the composite across the bottom.

LEMMA 6.11. *For any space X the following diagram commutes*

$$\begin{array}{ccc} \Xi\Lambda\dot{X} \cong \Sigma\Lambda X & \xrightarrow{\ \iota\ } & B\Lambda X \\ \downarrow{\scriptstyle \epsilon} & \swarrow{\scriptstyle \xi} & \\ X & & \end{array}$$

Here ϵ is the counit of the adjunction (Ξ, Λ) and ξ is the natural map $B\Lambda \to 1$ ([19; 15.4]).

Proof. The maps ϵ, ι, ξ are given by the following formulas

$$\iota[z, t] = [z, (1 - t, t)],\ \epsilon[\omega, t] = \omega(t),$$

and

$$\xi[(\omega_1, \omega_2, \ldots, \omega_m), (t_0 t_1, \ldots, t_m)] = \omega_1\omega_2 \cdots \omega_m(\Sigma_{i=1}^m \ell(\omega_i)t_i).$$

Here ℓ refers to the parameter length of a Moore loop. Commutativity can be checked by direct calculation.

Combining these results we obtain that $J(X, p) \to \Lambda\Xi(X, p)$ is a group completion.

THEOREM 6.12. *For any object (X, p) in $\mathfrak{J}_*[\mathbf{R}_+]$ the natural map*

$$BJ(X, p) \xrightarrow{B\lambda} B\Lambda\Xi(X, p)$$

is an equivalence.

Proof. By naturality of ι, Lemma 6.11, the definition of λ, and general properties of adjunctions, the following diagram commutes

$$\begin{array}{ccccc} & \nearrow^{\Xi\eta} & \Xi J(X, p) & \xrightarrow{\ \iota\ } & BJ(X, p) \\ & & \downarrow{\scriptstyle \Xi\lambda} & & \downarrow{\scriptstyle B\lambda} \\ \Xi(X, p) & \xrightarrow{\ \Xi\eta\ } & \Xi\Lambda\Xi(X, p) & \xrightarrow{\ \iota\ } & B\Lambda\Xi(X, p) \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon} & \swarrow_{\xi} & \\ & & \Xi(X, p) & & \end{array}$$

The composite across the top is an equivalence by Theorem 6.10. The map ξ is an equivalence by [19; Lemma 15.4] since $\Xi(X, p)$ is path connected. The result follows.

For subsequent use we derive the following easy consequence of Theorem 6.12.

COROLLARY 6.13. *Let X be a discrete based space and let GX denote the free group on X (based version). Then the natural map*

$$BJX \to BGX$$

is an equivalence.

Proof. Consider the space $\Lambda\Xi RX$. The discretization map (which collapses each path component to a point)

$$\rho : \Lambda\Xi RX \to \pi_0\Lambda\Xi RX = GX$$

is an equivalence since X is discrete. The result now follows from the theorem via the commutative diagram

$$\begin{array}{ccc} BJRX & \xrightarrow{B\lambda} & B\Lambda\Xi RX \\ \downarrow{\scriptstyle BJ\gamma} & & \downarrow{\scriptstyle B\rho} \\ BJX & \longrightarrow & BGX \end{array}$$

where $\gamma : RX = LRX \to X$ is the counit of the adjunction (L, R) (cf. 6.6(ii)), and Theorem 6.12.

We conclude with a related result on homological group completions. Recall that it has been shown in [17] (cf. also [3] and [20] that for any topological monoid M such that $\pi_0 M$ is in the center of $H_*(M)$ the natural map $\hat{\iota} : M \to \Omega BM$ induces a group completion in homology with coefficients in any commutative ring k. In the present context this may be formulated as saying that

$$(*) \qquad \begin{array}{ccc} H_0(M; k) = k[\pi_0 M] & \longrightarrow & k[G\pi_0 M] = H_0(\Omega BM; k) \\ \downarrow & & \downarrow \\ H_*(M; k) & \xrightarrow{\hat{\iota}_*} & H_*(\Omega BM; k) \end{array}$$

is a pushout diagram of graded associative algebras over k. Here we are using the notation of 4.4, letting $G\pi_0 M$ denote the Grothendieck group of $\pi_0 M$. It is natural to ask whether the centrality hypothesis can be weakened. The following example shows it cannot be eliminated entirely.

Example 6.14. Let X be any connected complex. By Theorem 3.5 (or Theorem 4.7) we can find a discrete topological monoid M such that $BM \simeq X$. If (*) is a pushout diagram in this case, then we would conclude that $H_*(\Omega X; k) \cong H_0(\Omega X; k)$. If we take, say $X = S^n\ n > 1$, we obtain an absurdity!

Actually [17] proves the stronger result that (*) is a pushout diagram under the hypothesis that $(H_*(M),\ \pi_0 M)$ admits a calculus of fractions. The following rather curious result, which I owe partly to Fred Cohen and Michael Barratt, proves this under a seemingly diametrically opposite hypothesis.

THEOREM 6.15. *For any topological space the natural map $\hat{\iota} : JX \to \Omega BJX \simeq \Omega\Sigma X$ induces a group completion in homology with field coefficients, i.e.*

$$\begin{array}{ccc} k[J\pi_0 X] & \longrightarrow & k[G\pi_0 X] \\ \big\downarrow & & \big\downarrow \\ H_*(JX; k) & \xrightarrow{\hat{\iota}_*} & H_*(\Omega BJX; k) \cong H_*(\Omega\Sigma X; k) \end{array}$$

is a pushout diagram in the category of graded associative algebras over k.

Proof. Since $H_*(JX; k) \cong T(\tilde{H}_*(X; k))$, the tensor algebra on the reduced homology $\tilde{H}_*(X; k)$, this amounts to proving that $H_*(\Omega\Sigma X; k)$ is the quotient of $T(\oplus_{n>0} H_n(X; k) \oplus I(G\pi_0 X))$ by the relations $g_1 \otimes g_2 = g_1 g_2,\ g_1,\ g_2 \in G\pi_0 X$. Here $I(G\pi_0 X)$ denotes the augmentation ideal of the group ring $k[G\pi_0 X]$.

By passing to the geometric realization of the total singular complex, we may assume that X is a CW complex. Denote $X = \coprod_{\alpha \in \pi_0 X} X_\alpha$ and choose a basepoint in each component X_α. Then by a simple geometric argument, which we leave as an exercise, we see that $\Sigma X \simeq \Sigma(\vee_\alpha X_\alpha) \vee \Sigma\pi_0 X$. Thus without loss of generality we may assume $X = X_0 \vee \pi_0 X$ where X_0 is connected. Now $\Sigma X = \Sigma X_0 \vee \Sigma\pi_0 X$, ΣX_0 is simply connected, and $\Sigma\pi_0 X$ is a wedge of circles. It follows that the universal cover $\widetilde{\Sigma X}$ is a "panoply of balloons:" one takes the tree which is the universal cover of

$\Sigma\pi_0 X$ and attaches a copy of ΣX_0 to each node. Hence $\widetilde{\Sigma X} \simeq \vee_{g\in G\pi_0 X}\Sigma X_0$. The covering group $G\pi_0 X$ acts by permuting coordinates by translation.

By considering the split fibration

$$\Omega\widetilde{\Sigma X} \to \Omega\Sigma X \rightleftarrows \Omega\Sigma\pi_0 X$$

we compute that

$$H_*(\Omega\Sigma X; k) \cong k[G\pi_0 X] \otimes_k T(\oplus_{g\in G\pi_0 X}\tilde{H}_*(X_0; k))$$

with multiplication given by $(g_1 \otimes u_1)(g_2 \otimes u_2) = g_1 g_2 \otimes (u_1 \cdot g_2)u_2$. It follows that $H_*(\Omega\Sigma X; k)$ has the required generators and relations.

7. Uniqueness of classifying space constructions *I*. In this section we begin to consider the question raised in the introduction: how can we characterize classifying space functors on topological monoids? As we mentioned there, historically there are two basic types of classifying space constructions. The first type is represented by the familiar bar construction which assigns to a topological monoid M the geometric realization BM of the simplicial space

$$* \leftleftarrows M \;\overset{\leftarrow}{\leftleftarrows}\; M^2 \;\overset{\leftleftarrows}{\leftleftarrows}\; M^3 \;\cdots$$

The second type of construction is the two sided bar construction $B(\Sigma, C_1, M)$ due to Beck [4] and May [21] which can be described as the geometric realization of the simplicial space

$$\Sigma M \leftleftarrows \Sigma C_1 M \;\overset{\leftarrow}{\leftleftarrows}\; \Sigma C_1 C_1 M \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots$$

where C_1 denotes a generalized James construction.

These constructions look very different and it is not at all clear that they produce equivalent results. The fact that they do was first proved by Thomason [27]. His proof is very complicated, relying partially on brute force arguments to force a direct comparison between these two very different constructions.

In this section we give a much simpler proof of the equivalence between the bar construction and the May construction. Our approach avoids the difficult problems of directly comparing these two constructions, by introducing a third classifying space construction which is intermediate between the two. The new construction, which is based on the Moore suspension of section 6, looks very much like the May construction. On the other hand, in certain very crucial ways its behavior resembles much more closely that of the bar construction. In the last section we will turn to more general considerations, and we will give a second independent proof of the equivalence of the various classifying space constructions by giving an axiomatic characterization.

Since both the May construction and the new construction we will introduce are special cases of the monadic two sided bar construction, we recall here for the convenience of the reader some basic facts about it (cf. [21] for details.)

Definition 7.0. Let D be a monad in some category of topological spaces (e.g. $\mathfrak{J}_*$ or $\mathfrak{J}_*[\mathbf{R}_+]$). Thus D is an endofunctor together with natural transformations $\mu : DD \to D$ and $\eta : 1 \to D$ satisfying associativity and unicity.

A D-algebra is an object together with a map $\theta : DX \to X$ compatible with μ and η. Similarly a D-functor is a functor F together with a natural transformation $\lambda : FD \to F$ also compatible with μ and η. Given a monad D, a D-algebra X and a D-functor F, the two sided bar construction $B(F, D, X)$ is the geometric realization $|n \mapsto FD^nX|$ where $D^n = DD \cdots D$. The faces of this simplicial space are induced by $\lambda : FD \to F$, $\mu : DD \to D$ and $\theta : DX \to X$. The degeneracies are induced by $\eta : 1 \to D$.

One basic property of the two sided bar construction is that

(1) $X \xrightarrow{\eta} B(D, D, X)$ and $B(D, D, X) \xrightarrow{\theta} X$
(2) $FX \xrightarrow{F\eta} B(F, D, DX)$ and $B(F, D, DX) \xrightarrow{\lambda} X$

are inverse homotopy equivalences.

The May classifying space functor assigns to a topological monoid M the classifying space $B(\Sigma, C_1, M)$. We now introduce our own variant of this construction.

Definition 7.1. Let Λ, Ξ denote Moore loops, Moore suspension respectively. Since Ξ is a $\Lambda\Xi$-functor and there is a map of monads $J \to \Lambda\Xi$ (Lemma 6.5), Ξ is a J-functor. Thus if M is a topological monoid, then RM is a monoid in $\mathfrak{J}_*[\mathbf{R}_+]$ and is thus a J-algebra. Hence the two sided bar construction $B(\Xi, J, RM)$ is defined. This gives a functor $B(\Xi, J, R\_\_) : \mathfrak{M} \to \mathfrak{T}_*$ together with the following natural chain of maps

$$\Sigma M \overset{\sigma}{\underset{\simeq}{\leftarrow}} \Xi RM \xrightarrow{\bar{\tau}} B(\Xi, J, RM)$$

where σ denote the counit of the (L, R) adjunction and $\bar{\tau}$ is given by inclusion of 0-simplices. By inverting σ we obtain a map $\iota : \Sigma M \to B(\Xi, J, RM)$ which is natural up to homotopy.

To the casual eye our construction appears to closely resemble May's construction. Hence one might naturally expect it to be much easier to relate this construction to the May construction than to the bar construction. Surprisingly the opposite is true.

Before we proceed with our analysis, however, we will find it convenient to establish some notation.

Definition 7.2. A pseudomap is a chain of maps from one space to another in which the wrong way maps are equivalences. We will use the notation $\rightarrowtail$ to denote pseudomaps. By abuse we will often refer to maps when we actually mean pseudomaps. A pseudomap is said to be an equivalence if every map in the chain is an equivalence.

Naturality of pseudomaps is taken in the following sense: all objects along the chain are functors and all maps along the chain are natural. Diagrams involving pseudomaps are said to homotopy commute if they commute in the category obtained by inverting all weak equivalences. (This category is equivalent to the standard homotopy category of CW complexes).

THEOREM 7.3. *For any topological monoid M there is a natural equivalence $\zeta : B(\Xi, J, RM) \rightarrowtail BM$ such that the following diagram homotopy commutes*

$$\begin{array}{ccc} & \Sigma M & \\ {\scriptstyle \iota}\swarrow & & \searrow{\scriptstyle \iota} \\ B(\Xi, J, RM) & \overset{\zeta}{\rightarrowtail} & BM \end{array}$$

Proof. One takes ζ to be the chain

$$B(\Xi, J, RM) \overset{\xi}{\underset{\simeq}{\leftarrow}} B\Lambda B(\Xi, J, RM) \overset{B\chi}{\underset{\simeq}{\leftarrow}} B(B(\Lambda\Xi, J, RM))$$

$$\xleftarrow[\simeq]{B(B(\lambda,1,1))} B(B(J, J, RM) \xrightarrow[\simeq]{B\theta} BRM \xrightarrow[\simeq]{B\gamma} BM$$

Here $\xi : B\Lambda X \to X$ is the classical equivalence [19; 15.4], χ is the equivalence of $0(f)$, $B(B(\lambda, 1, 1))$ is induced by the equivalence of 6.12 and $0(a)$, θ is the equivalence of 7.0 and γ is the counit of the (L, R) adjunction. With the help of Lemma 6.11, the verification of the homotopy commutative diagram reduces to an easy exercise.

We turn now to the comparison of $B(\Xi, J, R\_\_)$ with the May construction $B(\Sigma, C_1, \_\_)$. In order to carry out this comparison we need to work in the framework of May's theory of A_∞ spaces. We therefore summarize for the convenience of the reader the basic notions of the theory (cf. [21] for details).

7.4. *A_∞ operads, monads and spaces.* An A_∞ operad $\mathcal{C}$ is a collection $\{\mathcal{C}(n) \mid n \geq 0\}$ of contractible spaces such that $\mathcal{C}(0) = *$ together with a distinguished element $1 \in \mathcal{C}(1)$ and maps $\gamma : \mathcal{C}(n) \times \mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(k_1 + \cdots + k_n)$ which satisfy an associativity diagram and unicity with respect to $1 \in \mathcal{C}(1)$. (Note we are here using the non-Σ form of an A_∞ operad [21; 3.12]). A $\mathcal{C}$-space is a based topological space X together with maps $\theta : \mathcal{C}(n) \times X^n \to X$ which satisfy $\theta(1, x) = x$ and an associativity condition with respect to the maps γ. The $\mathcal{C}$ action on X gives X the structure of an H-space, $\mathcal{C}(n)$ parametrizing n-fold multiplications. The contractibility of the $\mathcal{C}(n)$ gives a precise formulation of the statement that X is associative and unital up to all higher homotopies. A space is said to be an A_∞ space if it is a $\mathcal{C}$ space for some A_∞ operad $\mathcal{C}$. For any based topological Y there is a free $\mathcal{C}$-space $CY = \coprod_{n \geq 0} \mathcal{C}(n) \times Y^n / \approx$ where $\approx$ is an equivalence relation induced by insertion and deletion of basepoints. The association $Y \mapsto CY$ defines a monad $C : \mathfrak{T}_* \to \mathfrak{T}_*$. We say that C is the monad associated to the operad $\mathcal{C}$.

There are two A_∞ operads in common use: the trivial operad $\mathcal{J} = \{* \mid n \geq 0\}$ and the "little intervals" operad $\mathcal{C}_1$ due to Boardman and Vogt [5]. A $\mathcal{J}$-space is exactly the same thing as a topological monoid and the monad associated to J is the James construction J. The operad $\mathcal{C}_1$ has as n^{th} space

$$\mathcal{C}_1(n) = \{(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \mid 0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq 1\}$$

each element of which is to be thought of as an ordered collection $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$ of subintervals of the unit interval. The structure maps γ of $\mathcal{C}_1$ are given by composition of intervals. The element $\{[0, 1]\} \in \mathcal{C}_1(1)$ is the unit. The operad $\mathcal{C}_1$ acts naturally on any loop space

ΩX: given $c = \{[a_1, b_1], \ldots, \ldots, [a_n, b_n]\} \in \mathcal{C}_1(n)$ and $(\omega_1, \ldots, \omega_n) \in (\Omega X)^n$, $\theta(c)(\omega_1, \ldots, \omega_n)$ is the loop which is ω_i (suitably reparametrized) on $[a_i, b_i]$ $i = 1, 2, \ldots, n$ and trivial otherwise. The monad associated to $\mathcal{C}_1$ is the May monad C_1 and the map of monads $C_1 \to \Omega\Sigma$ is that induced by the $\mathcal{C}_1$ action on Ω.

For any A_∞ operad $\mathcal{C}$ the unique map $\mathcal{C} \to \mathcal{J}$ induces a map of monads $C \to J$ with each $CX \to JX$ an equivalence.

In order to relate the two constructions $B(\Xi, J, R\_\_)$ and $B(\Sigma, C_1, \_\_)$ we have to relate the two maps of monads

$$J \to \Lambda\Xi \quad \text{in} \quad \mathfrak{J}_*[\mathbf{R}_+]$$

$$C_1 \to \Omega\Sigma \quad \text{in} \quad \mathfrak{J}_*$$

The tools for doing this are provided by Lemma 6.7 and an A_∞ operad $\mathcal{M}_1$ ("Moore little intervals") constructed below which bridges the gap between the operads $\mathcal{J}$ and $\mathcal{C}_1$.

LEMMA 7.5. *There is an* A_∞ operad $\mathcal{M}_1$ having the following properties

(i) $\mathcal{M}_1$ contains $\mathcal{J}$ and $\mathcal{C}_1$ as suboperads
(ii) $\mathcal{M}_1$ acts naturally on Moore loop spaces ΛX
(iii) The restriction of the $\mathcal{M}_1$ action on ΛX to $\mathcal{J}$ gives ΛX the usual monoid structure
(iv) The restriction of the $\mathcal{M}_1$ action on ΛX to $\mathcal{C}_1$ gives ΛX the $\mathcal{C}_1$-structure induced by the inclusion $\Lambda X \cong R\Omega X \subseteq \Omega X \times \mathbf{R}_+$.

Proof. We first construct an auxiliary A_∞ operad $\hat{\mathcal{C}}_1$ ("extended little intervals") by taking $a_i \leq b_i$ (instead of $a_i < b_i$) in the definition of $\mathcal{C}_1(n)$, thus allowing subintervals with vacuous interiors). (In itself, the operad $\hat{\mathcal{C}}_1$ appears to be totally useless. For instance the natural $\mathcal{C}_1$ action on loop spaces fails spectacularly to extend to a $\hat{\mathcal{C}}_1$ action.)

We now define the operad $\mathcal{M}_1$ by taking

$$\mathcal{M}_1(n) = \{f : \mathbf{R}_+^n \to \hat{\mathcal{C}}_1(n) \mid f \text{ is continuous and the } i\text{-th subinterval of } f(t_1, t_2, \ldots, t_n) \text{ has nonempty interior if } t_i \neq 0\}.$$

Contractibility of $\mathcal{M}_1(n)$ follows from the contractibility of $\mathbf{R}_+^n$ and $\hat{\mathcal{C}}_1(n)$.

The structure map $\gamma : \mathfrak{M}_1(n) \times \mathfrak{M}_1(k_1) \times \cdots \times \mathfrak{M}_1(k_n) \to \mathfrak{M}_1(k_1 + \cdots + k_n)$ is given in terms of the structure map of $\hat{\mathcal{C}}_1$ by

$$\gamma(f; g_1, g_2, \ldots, g_n)(t_1, t_2, \ldots, t_{k_1+\cdots+k_n}) = \gamma(f(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n); c_1, c_2, \ldots, c_n)$$

where $\bar{t}_i = \Sigma_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} t_j$ and $c_i = g_i(t_{k_1+\cdots+k_{i-1}+1}, \ldots, t_{k_1+\cdots+k_i})$.

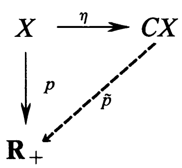
A routine check shows that $\mathfrak{M}_1$ is an A_∞ operad. The inclusion $\mathcal{J} \hookrightarrow \mathfrak{M}_1$ sends the unique element $* \in \mathcal{J}(n)$ to the function $f_n \in \mathfrak{M}_1(n)$ given by $f_n(t_1, t_2, \ldots, t_n) = \{[0, s_1], [s_1, s_2], \ldots, [s_{n-1}, 1]\}$ where $s_i = (\Sigma_{j=1}^{i} t_j)/(\Sigma_{j=1}^{n} t_j)$. The inclusion $\mathcal{C}_1 \hookrightarrow \mathfrak{M}_1$ sends an element of $\mathcal{C}_1(n)$ to the constant function at that element.

The operad $\mathfrak{M}_1$ acts on $\Lambda X \cong R\Omega X \subseteq \Omega X \times \mathbf{R}_+$ as follows: $f \in \mathfrak{M}_1(n)$ sends $(\omega_1, a_1, \omega_2, a_2, \ldots, \omega_n, a_n) \in (\Lambda X)^n$ to the element $(\psi, a_1 + a_2 + \cdots + a_n)$ where Ω is the loop which is ω_i (rescaled) on the i-th subinterval of $f(a_1, a_2, \ldots, a_n)$ $i = 1, 2, \ldots, n$ and is trivial otherwise. It is obvious that this action restricts to $\mathcal{J}$ and $\mathcal{C}_1$ as stated.

Before proceeding, we take a brief pause to describe what should be meant by actions of operads on objects in $\mathfrak{J}_*[\mathbf{R}_+]$ and how to define the associated monads in this category. We just do the obvious things (as in 6.4 and 6.5).

Definition 7.6. Let $\mathcal{C}$ be an A_∞ operad. We say that an object (X, p) of $\mathfrak{J}_*[\mathbf{R}_+]$ is a $\mathcal{C}$-object if X is a $\mathcal{C}$-space and $p : X \to \mathbf{R}_+$ is a map of $\mathcal{C}$-spaces. (We give $\mathbf{R}_+$ the $\mathcal{C}$-action induced by $\mathcal{C} \to \mathcal{J}$.)

LEMMA 7.7. (i) *The free $\mathcal{C}$-object $C(X, p)$ on an object (X, p) of $\mathfrak{J}_*[\mathbf{R}_+]$ is the pair $(CX, \tilde{p})$ where $\tilde{p}$ is the unique $\mathcal{C}$-map filling in the diagram below*



(ii) *The free $\mathcal{C}$-space functor $C : \mathfrak{J}_*[\mathbf{R}_+] \to \mathfrak{J}_*[\mathbf{R}_+]$ is a monad in* $\mathfrak{J}_*[\mathbf{R}_+]$. (Note that $LC = CL$.)

(iii) $\delta : C(X, p) = (CX, \tilde{p}) \to RL(CX, \tilde{p}) = RCL(X, p)$ *specifies a* *map of monads in* $\mathfrak{J}_*[\mathbf{R}_+]$, *where* RCL *is a monad via* $(R\eta L)\cdot\delta : 1 \to RL \to RCL$ *and* $(R\mu L)\cdot(RC\gamma CL) : RCLRCL \to RCCL \to RCL$.

(iv) *There is a map of monads* $M_1 \to \Lambda\Xi$ *in* $\mathfrak{J}_*[\mathbf{R}_+]$ *where* M_1 *is the monad associated to the* A_∞ *operad* $\mathfrak{M}_1$.

(v) *The following is a commutative diagram of monads in* $\mathfrak{J}_*[\mathbf{R}_+]$

$$\begin{array}{ccccccc} J & \longrightarrow & M_1 & \longleftarrow & C_1 & \longrightarrow & RC_1L \\ {\scriptstyle\lambda}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda\Xi & = & \Lambda\Xi & = & \Lambda\Xi & \equiv & R\Omega\Sigma L \end{array}$$

where λ *is the map of* 6.5(iii) *and the right vertical arrow is induced by* $C_1 \to \Omega\Sigma$. *The horizontal maps are equivalences.*

An immediate consequence is

THEOREM 7.8. *For any topological monoid N there is a natural equivalence* $\zeta : B(\Xi, J, RN) \to B(\Sigma, C_1, N)$ *such that the following diagram homotopy commutes*

$$\begin{array}{ccc} & \Sigma N & \\ {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle\iota} \\ B(\Xi, J, RN) & \xrightarrow{\ \zeta\ } & B(\Sigma, C_1, N) \end{array}$$

Proof. The equivalence is given by the following chain $B(\Xi, J, RN) \to B(\Xi, M_1, RN) \leftarrow B(\Xi, C_1, RN) \cong B(\Sigma L, C_1, RN) = B(\Sigma, C_1, LRN) \overset{\simeq}{\to} B(\Sigma, C_1, N)$ where the isomorphisms in the middle right arise from the isomorphism $\Xi \cong \Sigma L$ as C_1-functors (cf. 7.7(v) above) and the identity $LC_1 = C_1L$ (cf. 7.7(ii)). The homotopy commutativity of the diagram is immediate.

Combining Theorems 7.3 and 7.8 we obtain the result we were looking for

COROLLARY 7.9. *For any topological monoid N there is a natural equivalence* $\zeta : BN \to B(\Sigma, C_1, N)$ *such that the following diagram homotopy commutes*

$$\begin{array}{ccc} & \Sigma N & \\ {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle\iota} \\ BN & \xrightarrow{\ \zeta\ } & B(\Sigma, C_1, N) \end{array}$$

Remark 7.10. There is a remarkable connection between the monads C_1X and JX which is worth noting: RC_1X is homeomorphic to the quotient topological monoid of JRX obtained by dividing out by the relations $(*, s)(*, t) = (*, s + t)$. This connection is of importance in proving uniqueness of n-fold delooping machines $2 \leq n < \infty$ which is the subject of a forthcoming paper.

8. Uniqueness of classifying space constructions *II*. In this section we consider the general question of uniqueness of classifying space constructions. In a practical sense we have already solved the main problem in this area by showing that the two principal classifying space constructions now in use are equivalent to each other. Nevertheless ideally we would like something more: a useful axiomatic characterization of classifying space construction analogous to those achieved for infinite loop space constructions (cf. [12], [22]).

The arguments of the preceding section are direct and fairly straightforward. Nevertheless in some sense they are rather ad hoc, using features specific to the classifying space constructions involved. It is unclear how to go about modifying the arguments so that they would apply to any conceivable classifying space construction. A good axiomatization would not only be more generally applicable, but would also give a better understanding of what is really involved in proving that two classifying space constructions are equivalent.

In this section we give a very simple axiomatic characterization of classifying space constructions: they are completely determined by their behavior on discrete free monoids. Any functor with any reasonable pretensions to being called a classifying space construction is morally compelled to satisfy the axioms and is therefore naturally equivalent to the bar construction. In particular this method provides an alternative proof that the May classifying space is equivalent to the bar construction.

Definition 8.1. A classifying space construction on the category $\mathfrak{M}$ of topological monoids is a pair (W, ι), where $W : \mathfrak{M} \to \mathfrak{T}_*$ is a functor and $\iota : \Sigma M \rightarrow WM$ is a natural transformation subject to the following conditions.

Axiom E (*Equivalence*). If $f : M \to N$ is a monoid homomorphism which is an equivalence in $\mathfrak{T}_*$, then $Wf : WM \to WN$ is also an equivalence.

Axiom R (*Realization*). If M_* is a simplicial topological monoid, there is a natural equivalence $\chi : |n \mapsto WM_n| \rightarrow W|M_*|$ which is compatible up to homotopy with ι.

Axiom GC (*Group Completion*). For any discrete free monoid J, WJ is an Eilenberg-MacLane space $K(G, 1)$ and the composite

$$J \xrightarrow{\eta} \Omega\Sigma J \xrightarrow{\Omega\iota} \Omega WJ$$

induces group completion on the 0-th homotopy group.

Remark 8.2 (i) We are using the conventions of 7.2. In connection with Axiom GC, note that any pseudomap induces an honest map on homotopy groups, since equivalences induce isomorphisms on homotopy groups.

(ii) In all classifying space constructions we are aware of, the equivalence χ of Axiom R is actually a homeomorphism. However imposing the stronger axiom would not simplify the analysis in any way. On the other hand, in its present form Axiom R (as well as E and GC) are homotopy invariant. That is, any construction naturally equivalent to one satisfying the axioms will itself satisfy the axioms.

Before proceeding with our analysis, let us check how the specific space constructions we have considered fit into this framework.

LEMMA 8.3. *The bar construction* (B, ι) *satisfies Axiom* E, R *and* GC.

Proof. Axioms E and R follow directly from Lemma 0(a), (b), (d). Axiom GC follows from Lemma 6.9, Corollary 6.13 and the well-known fact, that for a discrete group G, BG is an Eilenberg-MacLane space $K(G, 1)$.

LEMMA 8.4. *The May construction* $B(\Sigma, C_1, \_\_)$ *satisfies Axioms* E, R *and* G.

Proof. Axioms E and R follow directly from Lemma 0 and the fact that the functors Σ and C_1 preserve equivalences and realizations [21; 12.1

and 12.2]. To prove Axiom GC, let X be a discrete space and consider the diagram

$$\begin{array}{ccccc} C_1X & \xrightarrow{\eta} & \Omega\Sigma C_1X & \xrightarrow{\Omega\iota} & \Omega B(\Sigma, C_1, C_1X) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ JX & \xrightarrow{\eta} & \Omega\Sigma JX & \xrightarrow{\Omega\iota} & \Omega B(\Sigma, C_1, JX) \end{array}$$

where the vertical arrows are equivalences induced by the equivalence of monads $C_1X \to JX$. By 7.0 we have an equivalence $\lambda : B(\Sigma, C_1, C_1X) \to \Sigma X$ so $B(\Sigma, C_1, JX)$ is an Eilenberg-MacLane space $K(G, 1)$. To show that the map induced on π_0 by the composite across the bottom is a group completion it suffices to prove the same for the composite

$$C_1X \xrightarrow{\eta} \Omega\Sigma C_1X \xrightarrow{\Omega\iota} \Omega B(\Sigma, C_1, C_1X) \xrightarrow{\Omega\lambda} \Omega\Sigma X$$

However by inspection this composite is given by the map of monads $C_1X \to \Omega\Sigma X$ which does induce a group completion on π_0 [21; 8.14].

A similar argument shows that the construction $B(\Xi, J, R\_\_)$ also satisfies Axioms E, R and GC.

Thus not only are Axioms E, R and GC compulsively reasonable, they are also fairly painless to check directly for all known classifying space constructions. Therefore the following result may be regarded as the definitive uniqueness theorem for classifying space constructions.

THEOREM 8.5. *Let (W, ι) be a classifying space construction on $\mathfrak{M}$ satisfying Axioms E, R and GC. Then for any topological monoid M there is a natural equivalence $\zeta : WM \to BM$ such that the following diagram homotopy commutes*

$$\begin{array}{ccc} & \Sigma M & \\ {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle\iota} \\ WM & \xrightarrow{\zeta} & BM \end{array}$$

Before proceeding with the proof of Theorem 8.5, we need a few preliminaries.

Remark 8.6. (i) In the sequel we will use the notation $\rho : X \to \pi_0 X$ to denote the discretization map which collapses each path component to a point. This map is defined only if X is semilocally path connected (i.e. path components are open). This is the case for instance if X has CW homotopy type.

(ii) We note here for future use that $\rho : X \to \pi_0 X$ is natural and is a monoid homomorphism if X is a topological monoid.

(iii) It will be convenient for our purposes to assume hereafter that W takes values in the category of spaces of CW homotopy type. This is no restriction since we can always replace WM functorially by $W'M = TWM$, the geometric realization of the total singular complex of WM. We then have a natural equivalence $W'M \to WM$.

(iv) Let J be a discrete free monoid. By naturality of ρ we have a commutative diagram

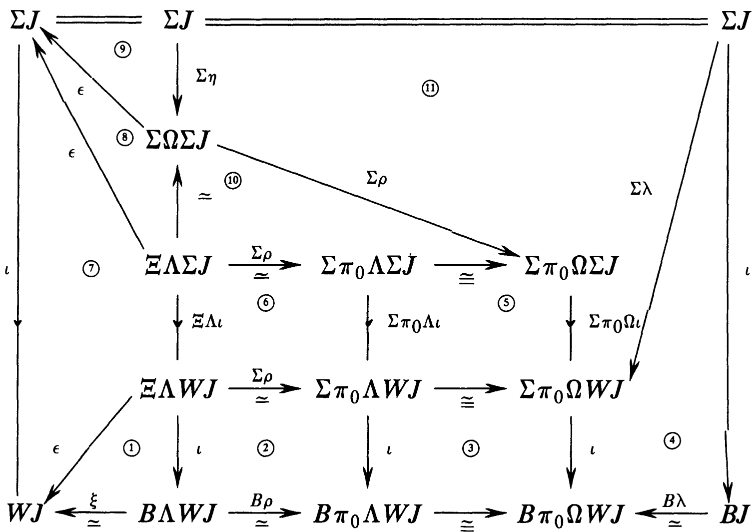
$$\begin{array}{ccccc} J & \xrightarrow{\eta} & \Omega\Sigma J & \xrightarrow{\Omega\iota} & \Omega WJ \\ \Big\| \rho & & \Big\downarrow \rho & & \Big\downarrow \rho \\ J & \xrightarrow{\pi_0\eta} & \pi_0\Omega\Sigma J & \xrightarrow{\pi_0\Omega\iota} & \pi_0\Omega WJ \end{array}$$

(Note that ρ is defined since, if X has CW homotopy type, so does ΩX). The left and middle vertical arrows are equivalences by the discreteness of J. Axiom GC implies that the right vertical arrow is an equivalence and that the composite across the bottom is a group completion. Hereafter we shall denote this composite by $\lambda : J \to \pi_0\Omega WJ$.

LEMMA 8.7. *For any discrete free monoid J there is a natural equivalence $\bar{\zeta} : WJ \to BJ$ such that the following diagram naturally commutes*

$$\begin{array}{ccc} & \Sigma J & \\ \swarrow \iota & & \searrow \iota \\ WJ & \xrightarrow{\bar{\zeta}} & BJ \end{array}$$

Proof. Consider the following diagram



Here ξ denotes the classical equivalence $B\Lambda \to 1$, ϵ is the counit of the (Ξ, Λ) adjunction or of the (Σ, Ω) adjunction and η is the unit of the (Σ, Ω) adjunction. The unmarked arrows are induced by the equivalence $\Lambda X \to \Omega X$.

By Corollary 6.13 the map $B\lambda$ is an equivalence. Thus the bottom row of the diagram defines a chain of equivalences $\bar{\zeta}: WJ \to BJ$ which is natural in J. The diagram labelled 2 through 6 commute by the naturality of ι. Diagrams 1 and 8 commute by Lemmas 6.11 and 6.7 respectively. Naturality of ϵ and ρ makes diagrams 7 and 10 commute. Commutativity of diagram 9 follows from general properties of adjunctions. Finally commutativity of diagram 11 follows from 8.6(iv).

Remark 8.8. The careful reader will note the use of the undefined term "naturally commutes" in the preceding lemma. What this means is that the diagram can be filled in with functors and natural maps as in the proof above so that the resulting diagram commutes. To be more precise the diagram commutes in the category of fractions of the functor category. Natural commutativity implies homotopy commutativity, but it is a stronger condition, essential to the proof of Theorem 8.5. However the reader can ignore all of this by regarding the "naturally commutative" diagram in the statement of Lemma 8.7 merely as a convenient shorthand for the more elaborate but conventional commutative diagram of the proof of Lemma 8.7.

LEMMA 8.9. *There is a functor F_* from topological monoids to simplicial monoids together with a natural equivalence $\kappa : |F_*| \to 1$ such that for each n, $F_n M$ is a discrete free monoid.*

Proof. Let T_*M denote the total singular complex of the topological monoid. Then T_*M is a simplicial (discrete) monoid. Define F_*M = diagonal of the bisimplicial set $B_*(J, J, T_*M)$. Thus $F_nM = J^{n+1}T_nM$ is a discrete free monoid. The equivalence κ is the composite

$$|n \mapsto F_nM| \cong B(J, J, |T_*M|) \xrightarrow{\theta} |T_*M| \xrightarrow{\xi} M$$

where θ is the equivalence of 7.0 and ξ is the natural equivalence between a space and the geometric realization of its total singular complex.

8.10. *Proof of Theorem* 8.5. Consider the diagram

$$\begin{array}{ccccccccc}
\Sigma M & \xleftarrow[\simeq]{\Sigma\kappa} & \Sigma|F_*M| & \xleftarrow{\cong} & |n \mapsto \Sigma F_nM| & \xrightarrow{\cong} & \Sigma|F_*M| & \xrightarrow[\simeq]{\Sigma\kappa} & \Sigma M \\
\downarrow \iota & ① & \downarrow \iota & ② \quad \iota \swarrow & ③ & \searrow \iota \quad ④ & \downarrow \iota & ⑤ & \downarrow \iota \\
WM & \xleftarrow[\simeq]{Wk} & W|F_*M| \overset{\chi}{\underset{\simeq}{-}} |n \mapsto WF_nM| & & \overset{|\bar\zeta_*|}{\underset{\simeq}{-}} & & |n \mapsto BF_nM| \cong B|F_*M| & \xrightarrow[\simeq]{B\kappa} & BM
\end{array}$$

where $|\bar\zeta_*|$ is induced by the natural equivalence of Lemma 8.7. Thus by Axioms E and R the bottom row defines a natural equivalence $\zeta : WM \to BM$. The diagrams labelled 1 and 2 commute by the naturality of ι. Diagrams 2 and 4 homotopy commute by Axiom R for W and B respectively. Finally diagram 3 naturally commutes and hence homotopy commutes by Lemma 8.7. (Note that if the diagram of Lemma 8.7 were known only to homotopy commute instead of naturally commute we would be unable to deduce that this commutativity passed on to the level of geometric realizations in diagram 3.) Skeptical readers who are suspicious of the virtues of natural commutativity may prefer to fill in diagram 3 with a suitably modified version of the diagram used to prove Lemma 8.7.

We conclude with a brief discussion on how our results extend to more general A_∞ input data. This gives rise to uniqueness theorem for one-fold delooping machines which subsumes Thomason's results in [27].

Remark 8.11. One-fold delooping machines accept as input the data some kind of notion of an H-space which is associative and unital up to all higher coherence homotopies. In 7.4 we discussed one version of this notion, that of an A_∞-space, due to May. Another version, due to Segal [25], is that of a special Δ-space. This is essentially a simplicial space X_* such that $X_n \simeq X_1^n$. Thomason produced a common generalization of these two notions and showed that all these various categories of input data for one-fold delooping machines are essentially equivalent.

Rather than recapitulating Thomason's results, we note that his work leads us to conclude that any reasonable category $\mathcal{A}$ of input data for a one-fold delooping machine has the following two properties

(1) $\mathcal{A}$ contains the category $\mathfrak{M}$ of topological monoids

(2) There is a functor $\Gamma : \mathcal{A} \to \mathfrak{M}$ and a natural equivalence $\zeta : \Gamma A \to A$ for objects A in $\mathcal{A}$.

Since any creditable delooping machine must preserve equivalences, any such machine on $\mathcal{A}$ is determined up to natural equivalence by its restriction to $\mathfrak{M}$. Theorem 8.5 therefore implies that such delooping machines are unique.

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