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# Homotopy Theory of $A_\infty$ Ring Spectra and Applications to $MU$-Modules

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**Abstract.** We give a definition of a derivation of an $A_\infty$ ring spectrum and relate this notion to topological Hochschild cohomology. Strict multiplicative structure is introduced into Postnikov towers and generalized Adams towers of $A_\infty$-ring spectra. An obstruction theory for lifting multiplicative maps is constructed. The developed techniques are then applied to show that a broad class of complex-oriented spectra admit structures of $MU$-algebras where $MU$ is the complex cobordism spectrum. Various computations of topological derivations and topological Hochschild cohomology are made.

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## 1. Introduction

It has long been proved useful in topology to exploit multiplicative structures. The notion of an $A_\infty$ ring spectrum or, in more modern terminology, an $S$-algebra is an analogue in stable homotopy theory of the algebraic notion of an associative ring. The analogue of a commutative ring is the so-called $E_\infty$ ring spectrum or a commutative $S$-algebra. There has been much interest recently to these objects, especially after the appearance of the seminal work [7]. For instance, Mandell used them in [14] to give an algebraic characterization of the category of $p$-complete nilpotent spaces. Another prominent example is the Hopkins–Miller theorem (cf. [16]) asserting the existence of the action of the Morava stabilizer group on the Johnson–Wilson spectrum $\hat{E}_n$.

Until recently only a handful of spectra were known to possess an $A_\infty$ or $E_\infty$ structure. Those are $K$-theoretic spectra arising from permutative categories, Eilenberg–MacLane spectra, various bordism spectra and their Bousfield localizations. The main purpose of the present paper is to devise a general method of proving the existence of $A_\infty$-structures.

We give a definition of the topological derivation spectrum of an $S$-algebra in the sense of [7] and, more generally, of an $R$-algebra for an $S$-algebra $R$. This definition is based on the notion of a derivation of an $R$-algebra. A derivation of an $R$-algebra $A$ with coefficients in an $A$-bimodule $M$ is a map of $R$-algebras $A \to A \vee M$ commuting with the canonical projection onto $A$. An analogous theory for commutative $R$-algebras was developed by Basterra in [3]. The topological derivation spectrum and the topological Hochschild (co)homology of an $R$-algebra are related via cofibre sequences (2) and in good many cases they can be recovered from each other. We use these tools to develop an obstruction theory for lifting $R$-algebra maps and to introduce (strict) multiplicative structures into Postnikov towers of $R$-algebras and generalized Adams towers. We compute the topological derivations of the Eilenberg–MacLane spectrum $HZ/p$ as an $MU$-algebra and compare the result with the classical computation of [8]. A curious consequence is that the mod $p$ Moore spectrum does not admit a structure of an $A_\infty$ ring spectrum for any $p$. Further we prove that Morava $K$-theories (as well as a much broader class of complex-oriented spectra) admit structures of $MU$-algebras. By neglect of structure they also admit an $A_\infty$-structure. The problem of introducing multiplicative structures into $MU$-modules was considered by Elmendorf, Kriz, Mandell and May in [7] and by Strickland in [18]. However these authors only prove the existence of such structures up to homotopy whereas the present paper deals with strict products. Such strict products (for Morava $K$-theories only) were first studied in [17]. However, in the cited reference the existence of an $A_\infty$ structure on $K(n)$ was established with a weaker notion of an $A_\infty$ ring spectrum (without the unit condition). Our method involves building up a complex-oriented spectrum starting from the Eilenberg–MacLane spectrum (which reduces to the Postnikov tower in the case of Morava $K$-theories) in the spirit of algebraic deformation theory.

Finally we compute the topological derivations and topological Hochschild cohomology of $K(n)$ (modulo additive extensions). We show how to deal with these extensions in a forthcoming paper. Our interest in topological Hochschild (co)homology of general spectra was inspired by the work of McClure and Staffeldt [15] where computations were made for the connective $K$-theory spectrum.

This paper is based on two preprints of the author [11] and [12], though the content of these has been considerably revised. Further extensive revisions were made following suggestions of the referees.

## 2. Derived Module of Differentials

Let $R$ be a commutative $S$-algebra (in older terminology, an $E_\infty$ ring spectrum), $A$ a (not necessarily commutative) $R$-algebra. Without loss of generality we assume that $R$ is $q$-cofibrant as a commutative $S$-algebra and $A$ is $q$-cofibrant as an $R$-algebra in the sense of [7]. Throughout the paper the notation $\wedge$ will mean $\wedge_R$.

Denote by $\tilde{\Omega}_{A/R}$ the (functorial) homotopy fibre of the multiplication map taken in the category of $A$-bimodules

$$\tilde{\Omega}_{A/R} \to A \wedge A \to A.$$

That is, $\tilde{\Omega}_{A/R}$ is the pullback $A \wedge A \times_A F(I, A)$ where $I$ is the unit interval $R$-module. The pullbacks of $A$-bimodules are created in the category of $R$-modules and thus $\Omega_{A/R}$ is an $A$-bimodule.

DEFINITION 2.1. The (derived) module of differentials $\Omega_{A/R}$ or just $\Omega_A$ if the 'ground ring' $R$ is clear from the context is the canonical cell approximation of $\tilde{\Omega}_{A/R}$ in the category of $A$-bimodules.

So we still have the homotopy fibre sequence of $A$-bimodules

$$\Omega_{A/R} \to A \wedge A \to A. \tag{1}$$

The exact triangle (1) is split by the map

$$A \cong A \wedge R \stackrel{id\wedge 1}{\to} A \wedge A.$$

in the homotopy category of left $A$-modules. Here (and later on in the paper) $1\colon R \to A$ stands for the unit map of the $R$-algebra $A$. Therefore $\Omega_A$ is weakly equivalent as a left $A$-module to $A \wedge A/R$. Symmetrically, $\Omega_A$ is weakly equivalent to $A/R \wedge A$ as a right $A$-module.

Now let $M$ be a cell $A$-bimodule. Recall from [7], IX.1 that $M$ is then a left $A \wedge A^{\mathrm{op}}$ module where $A^{\mathrm{op}}$ is an $R$-algebra which as an $R$-module coincides with $A$ but with opposite multiplication. Define the topological derivation homology spectrum $\mathbf{Der^R}(A, M)$ as

$$\mathbf{Der^R}(A, M) = \Omega_A \wedge_{A\wedge A^{\mathrm{op}}} M$$

and the topological derivation cohomology spectrum $\mathbf{Der_R}(A, M)$ as

$$\mathbf{Der_R}(A, M) = F_{A\wedge A^{\mathrm{op}}}(\Omega_A, M).$$

Here **Der** stands for 'derivations', and the meaning of this notation will become clear shortly.

Recall that the standard definition of Hochschild homology and cohomology (cf. [7]) is respectively

$$\mathbf{THH^R}(A, M) = A \wedge_{A\wedge A^{\mathrm{op}}} M,$$

$$\mathbf{THH_R}(A, M) = F_{A\wedge A^{\mathrm{op}}}(A, M).$$

Now from (1) we see that the two definitions are related via cofibre sequences of $R$-modules

$$\begin{aligned} &\mathbf{Der^R}(A, M) \to M \to \mathbf{THH^R}(A, M),\\ &\mathbf{THH_R}(A, M) \to M \to \mathbf{Der_R}(A, M). \end{aligned} \tag{2}$$

Now define the derived module of derivations of $A$ with values in $M$ as

$$Der(A, M) = [A, A \vee M]_{R-\mathrm{alg}/A}. \tag{3}$$

The right-hand side of the last formula means homotopy classes of maps in the category of $R$-algebras over $A$, that is, $R$-algebras $B$ supplied with a map $B \to A$. This category inherits the structure of a topological model category from the category of $R$-algebras and therefore it is legitimate to consider homotopy classes of maps in $R - \mathrm{alg}/A$. Further, $A$ is an $R$-algebra over $A$ in the obvious way and $A \vee M$ is given the structure of an $R$-algebra as follows:

$$(A \vee M) \wedge (A \vee M) = A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge M \to A \vee M,$$

where the last map is induced by the multiplication on the first summand, by the $A$-bimodule structure on $M$ on the second and third summands and zero on the last summand. The structure map $A \vee M \to A$ is the usual projection.

We will see shortly that $Der(A, M)$ is an Abelian group and

$$Der(A, M) = [\Omega_A, M]_{A-\mathrm{bimod}}.$$

The right-hand side of the last equality is the homotopy classes of maps from $\Omega_A$ to $M$ in the category of $A$-bimodules (or, equivalently, in the category of left $A \wedge A^{\mathrm{op}}$-modules). This gives, therefore, another definition of topological derivation groups, namely

$$\mathbf{Der_R}^n(A, M) = Der(A, \Sigma^n M)$$

(and explains the notation $\mathbf{Der_R}(A, M)$).

Denote the category of $R$-algebras over $A$ by $\mathcal{C}_{R/A}$ and its homotopy category by $ho\mathcal{C}_{R/A}$. Let $B \in \mathcal{C}_{R/A}$ and denote the $A$-bimodule

$$A \wedge_B \Omega_B \wedge_B A \cong A \wedge A^{\mathrm{op}} \wedge_{B \wedge B^{\mathrm{op}}} \Omega_B \tag{4}$$

by $\Omega_A^B$. Since $\Omega_B$ is by definition a cell $B \wedge B^{\mathrm{op}}$-module the smash products in (4) represent the derived smash product in the category of $B$-bimodules. Therefore the functor $\Omega_A^B$ could be interpreted as a point-set functor from the category of $R$-algebras over $A$ to the category of $A$-bimodules or as a homotopy functor between the corresponding homotopy categories.

THEOREM 2.2. *There is a natural equivalence*

$$[B, A \vee M]_{R-\mathrm{alg}/A} \cong [\Omega_A^B, M]_{A-\mathrm{bimod}}.$$

*In other words the functors*

$$B \to \Omega_A^B \colon ho\mathcal{C}_{R/A} \to ho(A - \mathrm{bimod}) \quad \textit{and}$$

$$M \to A \vee M \colon ho(A - \mathrm{bimod}) - ho\mathcal{C}_{R/A}$$

*are adjoint.*

*Proof.* Replace the $R$-algebra $A \vee M$ by its fibrant cofibrant approximation $\overline{A \vee M}$ in the category of $R$-algebras over $A$. As a first step we will construct a 'universal map' of $\overline{A \vee M}$-bimudules

$$\Omega_A^{\overline{A\vee M}} \to M.$$

Since $\Omega_A^{\overline{A\vee M}} = A \wedge_{\overline{A\vee M}} \Omega_{\overline{A\vee M}} \wedge_{\overline{A\vee M}} A$ it is enough to construct a map of $\overline{A \vee M}$-bimodules $\Omega_{\overline{A\vee M}} \to M$.

Consider the module $\Omega_{A\vee M}$ defined as the cell approximation of the homotopy fibre of the multiplication map $(A \vee M) \wedge (A \vee M) \to A \vee M$ in the category of $A \vee M$-bimodules. Even though $A \vee M$ is not a $q$-cofibrant $R$-algebra the smash product $(A \vee M) \wedge (A \vee M)$ clearly represents the derived smash product and therefore $\Omega_{A\vee M}$ has the correct homotopy type.

Since the $R$-algebras $(A\vee M)\wedge(A\vee M)^{\mathrm{op}}$ and $\overline{(A \vee M)}\wedge\overline{(A \vee M)}^{op}$ are weakly equivalent the homotopy categories of $A\vee M$-bimodules and of $\overline{A \vee M}$-bimodules are also equivalent and therefore it suffices to construct a map of $A\vee M$-bimodules $\Omega_{A\vee M} \to M$.

Consider the following diagram in the homotopy category of $A\vee M$-bimodules:

$$\begin{array}{ccc}
A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge M & \to & A \vee M \\
\downarrow & & \| \\
A \vee M \vee M & \to & A \vee M
\end{array}$$

Here the left vertical arrow is determind by the $R$-algebra structure on $A$ and an $A$-bimodule structure on $M$ on the first three summands and is zero on the last summand. The lower horizontal arrow is zero on $A$, $id$ on the first summand and $-id$ on the last summand. The homotopy fibre of the upper row is equivalent to $\Omega_{A\vee M}$ by definition and the homotopy fibre of the lower row is equivalent to $M$. There results the map of $A \vee M$-bimodules $\Omega_{A\vee M} \to M$ as desired.

Now for a map $f\colon B \to \overline{A \vee M}$ we have a composite map $\Omega_A^B \to \Omega_A^{\overline{A\vee M}} \to M$ and therefore a correspondence

$$D\colon [B, \overline{A \vee M}]_{R-alg/A} \to [\Omega_A^B, M]_{A-bimod}$$

Notice that $D$ is actually functorial in $B$ on a point-set level. This more precise result will be used below. We need a special case of the Theorem 2.2 to prove the general case. □

LEMMA 2.3. *For*

$$B = TV = \{R \vee V \vee V^{\wedge 2} \vee \ldots \xrightarrow{\mu} A\} \in \mathcal{C}_{R/A},$$

*the free associative algebra on an $R$-module $V$ over $A$ this correspondence establishes an isomorphism*

$$[TV, A \vee M]_{R-\mathrm{alg}/A} \to [\Omega_A^{TV}, M]_{A-\mathrm{bimod}}. \tag{5}$$

*Proof.* First compute both sides of (5) separately. Obviously, the left-hand side is equal to

$$[V, A \vee M]_{R-\text{mod}/A} = [V, M]_{R-\text{mod}}.$$

We have the following homotopy fibre sequence of $TV$-bimodules:

$$(TV) \wedge V \wedge TV \stackrel{f}{\to} TV \wedge TV \stackrel{m}{\to} TV, \tag{6}$$

where $f = f_1 - f_2$ and

$$f_1\colon\ TV \wedge V \wedge TV \stackrel{m\wedge id}{\to} TV \wedge TV,$$

$$f_2\colon\ TV \wedge V \wedge TV \stackrel{id\wedge m}{\to} TV \wedge TV,$$

$m$ is the multiplication map. Smashing (6) with $A$ on both sides over $TV$ we get

$$A \wedge V \wedge A \to A \wedge A \to A \wedge_{TV} A.$$

Therefore $\Omega_A^{TV} = A \wedge V \wedge A$ and

$$[\Omega_A^{T(V)}, M]_{A-\text{bimod}} = [V, M]_{R-\text{mod}} = [T(V), A \vee M]_{ho\mathcal{C}_{R/A}}.$$

The second equality uses the (obvious) fact that the functor

$$M \to A \vee M\colon\ ho(R - \text{mod}) \to ho(R - \text{mod}/A)$$

is right adjoint to the forgetful functor

$$ho(R - \text{mod}/A) \to ho(R - \text{mod}).$$

So we identified each side of (5) with $[V, M]_{R-\text{mod}}$. To prove our lemma we have to show that $D$ respects this identification. In other words for an $R$-module map $f\colon\ V \to M$ the composite

$$\Omega_{TV} \to \Omega_{A\vee M} \to M,$$

should correspond under this identification to the map $f$.

Consider now the diagram

$$\begin{array}{ccccc}
TV \wedge V \wedge TV & \longrightarrow & \Omega_{A\vee M} & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
TV \wedge TV & \longrightarrow & A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge M & \longrightarrow & A \vee M \vee M \\
\downarrow & & \downarrow & & \downarrow \\
TV & \longrightarrow & A \wedge M & = & A \wedge M.
\end{array}$$

Here all three columns are homotopy cofibre sequences. Also recall that the upper left vertical map is $m \wedge id - id \wedge m$. Clearly the composite

$$V \to TV \wedge V \wedge TV \to TV \wedge TV \to$$

$$A \wedge A \vee A \wedge M \vee M \wedge A \vee M \wedge \to A \vee M \vee M \to M \vee M,$$

coincides with the composite

$$V \to V \vee V \to A \wedge M \vee M \wedge A \to M \vee M,$$

where the first map is id $\vee 0 - 0 \vee$ id, the second map is $\mu \wedge f - f \wedge \mu$ and the last map is induced by the $A$-bimodule structure on $M$. Therefore the composite map $V \to M \vee M$ is $f \vee 0 - 0 \vee f$. Taking into account that the exact triangle

$$M \to M \vee M \to M,$$

splits we see that there is a unique up to homotopy lifting

$$V \to TV \wedge V \wedge TV \to M$$

and it coincides with $f$. The lemma is proved.

Return now to the general case. Without loss of generality we assume that $B$ is a $q$-cofibrant $R$-algebra and resolve it by the monadic bar-construction $\mathbf{B}_\star$. Recall from [7], Chapter 12 that $\mathbf{B}_\star = \{B_i\}$ is a simplicial $R$-algebra with $B_i = T^i B$ and there is a weak equivalence $|\mathbf{B}_\star| \to B$ where $|\mathbf{B}_\star|$ is the geometric realization of $\mathbf{B}_\star$. According to [7], Proposition 3.3, Chapter VII this geometric realization is the same in the category of $R$-algebras as in the underlying category of $R$-modules).

The problem with $\mathbf{B}_\star$ is that its $i$th simplices need not be $q$-cofibrant $R$-algebras which means that its realization $|\mathbf{B}_\star|$ is not necessarily $q$-cofibrant. Therefore we can't use it to compute the homotopy type of the function space out of $B$. To make the cofibrant resolution of $B$ we use the argument due to Basterra, cf. [3]. Namely, replace $\mathbf{B}_\star$ by the simplicial $R$-algebra $\Gamma\mathbf{B}_\star$ that is obtained from $\mathbf{B}_\star$ by the application of the functorial cell $R$-algebra approximation functor $\Gamma$ componentwise. Then each component $\Gamma B_i$ is a cell $R$-algebra, all degeneracy operators are cell inclusions and the face maps are sequentially cellular. Therefore we conclude by [7], X, 2.7 that the realization $|\Gamma\mathbf{B}_\star|$ is a cell $R$-algebra. Moreover, the simplicial components of $\Gamma\mathbf{B}_\star$ are weakly equivalent to free $R$-algebras. Also $|\Gamma\mathbf{B}_\star|$ is weakly equivalent to $B$. That means that we have a weak equivalence of topological spaces

$$[B, A \vee M]_{R-\mathrm{alg}/A} \simeq [|\Gamma\mathbf{B}_\star|, A \vee M]_{R-\mathrm{alg}/A}.$$

Denote $F_{\mathcal{C}_{A/R}}(-,-)$ the topological space of maps between two objects in $\mathcal{C}_{A/R}$ (not homotopy classes of maps). We have

$$F_{\mathcal{C}_{A/R}}(|\Gamma\mathbf{B}_\star|, A \vee M) \cong F_{\mathcal{C}_{A/R}}(\Gamma\mathbf{B}_\star, A \vee M),$$

where $F_{\mathcal{C}_{A/R}}(\Gamma\mathbf{B}_\star, A \vee M)$ is the topological space of maps between two simplicial $R$-algebras $\Gamma\mathbf{B}_\star$ and $A \vee M$, the latter regarded as a constant simplicial algebra.

Next, $F_{\mathcal{C}_{A/R}}(\Gamma\mathbf{B}_\star, A \vee M)$ is the total space of the cosimplicial topological space

$$T^\star = \{F_{\mathcal{C}_{A/R}}(\Gamma\mathbf{B}_i, A \vee M)\}. \tag{7}$$

The monadic bar-construction $\mathbf{B}_\star$ is a proper simplicial spectrum and so is $\Gamma\mathbf{B}_\star$. That ensures that the cosimplicial space (7) is fibrant in the sense that there is a tower of fibrations

$$Tot^0(T) \leftarrow Tot^1(T) \leftarrow \dots,$$

where $Tot^i(T)$ is the $i$th cosimplicial approximation of $T^\star$ and whose inverse limit is $Tot(T^\star)$.

Likewise

$$F_{A-\mathrm{bimod}}(\Omega_A^{\Gamma\mathbf{B}_\star}, M) = F_{\Gamma\mathbf{B}_\star-\mathrm{bimod}}(\Omega_{\Gamma\mathbf{B}_\star}, M)$$

is the total space of the cosimplicial space

$$\{T^{\star\prime}\} = F_{\Gamma B_i-\mathrm{bimod}}(\Omega_{\Gamma B_i}, M)\}. \tag{8}$$

The cosimplicial space $T^{\star\prime}$ *does not* necesssarily give rise to a tower of fibrations since the simplicial spectrum $\Omega_{\Gamma B_i}$ need not be proper. (Recall that the definition of the module of differentials involves taking the homotopy fibre which could destroy properness). Replace $\{\Omega_{\Gamma B_i}\}$ functorially by a proper simplicial spectrum and denote the corresponding cosimplicial space by $\tilde{T}^\star$. Then $\tilde{T}^\star$ is fibrant and we still have a map $T^\star \to \tilde{T}^\star$. According to the previous lemma this map induces an equivalence on the $i$th cosimplices for all $i$. It follows that this map induces an isomorphism of the corresponding spectral sequences. Therefore it is a weak equivalence and Theorem 2.2 is proved. □

*Remark* 2.4. It will be useful to generalize the notion of the module of differentials a little. Let $R'$ be a (not necessarily commutative) $R$-algebra and consider the category of $R'$-algebras which means just $R$-algebras $A$ equipped with a map $R' \to A$, not necessarily central as usually required. For an $R'$-algebra $A$ we can define $\Omega_{A/R'}$ just as we did before (replacing $\wedge$ by $\wedge_{R'}$). The analogue of Theorem (2.2) reads as follows: for an $R'$-algebra $B$ over an $R'$-algebra $A$ one has a natural equivalence

$$[B, A \vee M]_{ho\mathcal{C}_{R'/A}} \cong [\Omega^B_{A/R'}, M]_{A-\mathrm{bimod}},$$

where $\Omega^B_{A/R'} := A \wedge_B \Omega_{A/R'} \wedge_B A$.

The proof is the same, one only has to notice that there still are adjunctions

$$R' - \text{algebras} \leftrightarrows R' - \text{bimodules},$$

$$A - \text{bimodules} \leftrightarrows R' - \text{bimodules}$$

where the 'free' $R'$-algebra functor is given for an $R'$-bimodule $V$ as

$$TV = R' \vee V \vee V \wedge_{R'} V \dots$$

and the 'free' $A$-bimodule on a $R'$-bimodule $V$ is $A \wedge_{R'} V \wedge_{R'} A$. The $R'$-bimodule $F_{A-\text{bimod}}(\Omega_{A/R'}, M)$ will be called $R'$-relative topological derivation spectrum of $A$ with coefficients in $M$ and denoted as $\mathbf{Der}(A; R', M)$. Notice, that even if $R'$ happens to be commutative $\mathbf{Der}(A; R', M)$ is not the same as $\mathbf{Der}_{\mathbf{R}'}(A, M)$ since the latter is relevant to the category of $R'$-relative $A$-bimodules, that is, those $A$-bimodules whose induced $R'$-bimodule structure is symmetric.

COROLLARY 2.5. *The functor*

$$B \to \Omega_A^B : \mathcal{C}_{\mathcal{R}/\mathcal{A}} \to A - \text{bimod},$$

*respects homotopy colimits.*

PROPOSITION 2.6. (Transitivity exact sequence). *Let $A \to B \to C$ be morphisms of $R$-algebras. Then one has the following homotopy cofibre sequence of $C$-bimodules:*

$$C \wedge_B \Omega_{B/A} \wedge_B C \to \Omega_{C/A} \to \Omega_{C/B}.$$

*Proof.* Without loss of generality we assume that both maps $A \to B$ and $B \to C$ are $q$-cofibration of $R$-algebras. We have the following homotopy cofibre sequence of $B$-bimodules:

$$\Omega_{B/A} \to B \wedge_A B \to B.$$

Smashing it with $C$ over $B$ on both sides we get the homotopy cofibre sequence of $C$-bimodules

$$C \wedge_B \Omega_{B/A} \wedge_B C \to C \wedge_A C \to C \wedge_B C.$$

Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} C \wedge_B \Omega_{B/A} \wedge_B C & \longrightarrow & C \wedge_A C & \longrightarrow & C \wedge_B C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & C. \end{array}$$

Here the right and the middle vertical arrows are induced by multiplication on $C$. Taking the homotopy fibres of the vertical arrows we get the desired cofibre sequence. □

COROLLARY 2.7. *For a $C$-bimodule $M$ we have the following homotopy cofibre sequence:*

$$\mathbf{Der}(C; B, M) \to \mathbf{Der}(C; A, M) \to \mathbf{Der}(B; A, M).$$

*Proof.* Apply the functor $F_{C-\text{bimod}}(?, M)$ to the transitivity exact sequence. □

*Remark* 2.8. Notice that the transitivity exact sequence is functorial with respect to the strict morphisms of triples $\{A \to B \to C\}$ (essentially because the construction of the modules of differentials is functorial).

## 3. Extensions of *R*-algebras

In this section we define and study extensions of $R$-algebras. Our results turn out to be in some ways analogous to the classical theory for associative algebras (cf., for example, [13], Chapter X). A more modern treatment suitable for operadic algebras is found in the recent work of Hopkins and Goerss [9].

DEFINITION 3.1. An extension of an $R$-algebra $B$ by an $R$-algebra $I$ without unit is the following homotopy fibre sequence of $R$-modules:

$$I \to A \to B. \tag{9}$$

where $A$ is an algebra, and both arrows are algebra maps.

Next denote $I/I^2$ the homotopy cofibre

$$I \wedge_A I \to I \to I/I^2.$$

(notice that $I$ is naturally an $A$-bimodule.) Then we have the following.

LEMMA 3.2 (Conormal exact sequence). *The $R$-module $I/I^2$ is weakly equivalent as an $R$-module to a $B$-bimodule (which we will still denote as $I/I^2$) and there is a homotopy cofibre sequence of $B$-bimodules*

$$I/I^2 \to B \wedge_A \Omega_A \wedge_A B \to \Omega_B. \tag{10}$$

*Proof.* We have the following transitivity exact sequence (corresponding to algebra maps $R \to A \to B$):

$$B \wedge_A \Omega_A \wedge_A B \to \Omega_B \to \Omega_{B/A}.$$

So it suffices to show that $\Sigma(I/I^2) = \Omega_{B/A}$. Consider the following diagram:

$$\begin{array}{ccccc} I \wedge_A I & \longrightarrow & I & \longrightarrow & I/I^2 \\ \downarrow & & \downarrow & & \downarrow \\ I & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ I \wedge_A B & \longrightarrow & B & \longrightarrow & B \wedge_A B. \end{array}$$

Here the arrows going to the right lower corner are both equal to $1 \wedge id$, the arrows going out of the left upper corner are multiplication maps on $I$ and the map from $I$ to $I \wedge_A B$ is $id \wedge 1$. This diagram is commutative in the homotopy category of $R$-modules and all rows and columns are homotopy cofibre sequences of $R$-modules. Indeed, the middle row and column are cofibre sequences by hypothesis, the first column and last row are likewise, since smash product preserves such sequences, the first row is a cofibre sequence by definition of $I/I^2$, and the last column of induced maps is also a cofibre sequence. To be precise, the map from the first column to the middle column can be rigidified in the category of $R$-modules, hence the last column can be considered as the functorial cofibre of the relevant maps and is itself therefore a cofibre sequence. Next we have the following commutative diagram in the homotopy category of $R$- modules:

$$\begin{array}{ccccc} I/I^2 & \longrightarrow & 0 & \longrightarrow & \Omega_{B/A} \\ \downarrow & & \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & B & \longrightarrow & B \wedge_A B \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B. \end{array}$$

Here the rightmost column is a cofibre sequence by definition of $\Omega_{B/A}$, the middle row is also such by the previous discussion. It follows that the upper row is a cofibre sequence. Therefore $\Omega_{B/A} \cong \Sigma I/I^2$ and the lemma is proved.

A derivation $d\colon\ B \to B \vee \Sigma I$ gives rise to an extension as follows.

Consider the diagram

$$\begin{array}{ccc} I & \cong & I \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \xrightarrow{d} & B \vee \Sigma I. \end{array} \tag{11}$$

Here the lower vertical map is the canonical inclusion of a retract; the lower square is a homotopy pullback in the category of $R$-algebras (which is also the homotopy pullback of underlying $R$-modules), therefore the left column is a homotopy cofibre sequence of $R$-modules and an extension of an $R$-algebra $B$ by $I$. We will call $I \to X \to B$ the extension associated with the derivation $d$. □

DEFINITION 3.3. An extension $I' \to X' \to B'$ is called singular if there exists an extension $I \to X \to\ B$ associated with a derivation $d\colon\ B \to B \vee \Sigma I$ and a strictly commutative diagram of algebras

$$\begin{array}{ccccc} I & \to & X & \to & B \\ \downarrow & & \downarrow & & \downarrow \\ I' & \to & X' & \to & B, \end{array}$$

where the vertical arrows are weak equivalences.

The next proposition explains the analogy with usual singular extensions of algebras (hence the name); the multiplication map on the 'ideal' $I$ (at least up to homotopy) is zero.

PROPOSITION 3.4. *Let $I \to X \to B$ be a singular extension. Then the multiplication map $I \wedge_X I \to I$ is homotopic to zero as a map of $R$-modules, so the homotopy cofibre sequence*

$$I \wedge_X I \to I \to I/I^2,$$

*splits and $I/I^2$ is weakly equivalent as an $R$-module to $I \vee \Sigma(I \wedge_X I)$.*

*Proof.* Let us consider first the 'universal' singular extension

$$I \to B \to B \vee \Sigma I.$$

Then the multiplication map $I \wedge_B I \to I$ can be factored up to homotopy as follows:

$$\begin{array}{ccc} I \wedge_B I & \to & I \\ & \searrow & \| \\ & & B \wedge_B I. \end{array}$$

Since the map $I \to B$ is homotopically zero we conclude that the map $I \wedge_B I \to I$ is also so.

Next if we have a singular extension $I \to X \to B$ associated with a derivation $B \to B \vee \Sigma I$ it is clear that the multiplication map $I \wedge_X I \to I$ factors through $I \wedge_B I \to I$ and is therefore homotopic to zero.

The next proposition describes more concretely the boundary map

$$\Omega_B \to \Sigma I/I^2$$

in the case of a singular extension. □

PROPOSITION 3.5. *Let* (9) *be a singular extension, so* $\Sigma I/I^2 \simeq \Sigma I \vee \Sigma I \wedge_A \Sigma I$. *Denote by*

$$d\colon\ B \to \Sigma I$$

*the boundary map in* (9). *Recall that there is an* $R$*-module weak equivalence* $\Omega_B \cong B \wedge B/R$. *Then for the boundary map in* (10)

$$\partial\colon\ B \wedge B/R \to \Sigma I \vee \Sigma I \wedge_A \Sigma I,$$

*we have* $\partial = \partial_1 + \partial_2$ *where*

$$\partial_1\colon\ B \wedge B/R \overset{id\wedge d}{\to} B \wedge \Sigma I \to B \wedge_B \Sigma I \simeq \Sigma I,$$

*and* $\partial_2$ *is the composite map*

$$B \wedge B/R \overset{d\wedge d}{\to} \Sigma I \wedge \Sigma I \to \Sigma I \wedge_A \Sigma I.$$

*Proof.* Consider the following homotopy commutative diagram of $R$-modules:

$$\begin{array}{ccccccc} B & = & B \wedge_A A & \to & B \wedge_A B & \overset{id\wedge d}{\to} & B \wedge_A \Sigma I \\ \| & & & & \uparrow & & \uparrow id \wedge d \\ B & = & B \wedge_A A & \to & B \wedge B & \to & B \wedge B/R. \end{array}$$

Here both rows are split homotopy cofibre sequences , the $R$-module $B \wedge_A \Sigma I$ is homotopy equivalent to $\Sigma I/I^2$ as we saw in the proof of Lemma 3.2 and the right vertical map is the map $\partial$ that we are interested in. Now the proposition follows from the following diagram, where again both rows are split homotopy cofibre sequences of $R$-modules:

$$\begin{array}{ccccc} \Sigma I & = A \wedge_A \Sigma I & \to & B \wedge_A \Sigma I & \to & \Sigma I \wedge_A \Sigma I \\ \uparrow & & & \uparrow{\scriptstyle id \wedge d} & & \uparrow{\scriptstyle id \wedge d} \\ A \wedge B/R & & \to & B \wedge B/R & \to & \Sigma I \wedge B/R. \end{array}$$

□

*Remark* 3.6. There is a slight catch in the above proposition. We considered the map

$$id \wedge d\colon\ B \wedge B/R \to B \wedge \Sigma I,$$

denoting the extension of the map $B \to \Sigma I$ to $B/R$ by the same letter $d$. This extension is not unique, but the indeterminacy vanishes upon smashing with $B$.

## 4. The Universal Derivation

In this section we will compute the homotopy type of the 'universal derivation', that is, of the composite map

$$A \to A \vee \Omega_A \to \Omega_A,$$

where the first map is the algebra map adjoint to the identity map $\Omega_A \to \Omega_A$ and the second map is the projection.

Let $A$ be an $R$-algebra. Recall that there is an equivalence of $R$-modules

$$\Omega_A \cong A \wedge A/R. \tag{12}$$

THEOREM 4.1. *Upon the identification* (12) *the universal derivation $A \to \Omega_A$ coincides with the map*

$$A \overset{id\wedge d}{\to} A \wedge A/R,$$

*where $d$ is defined by the homotopy cofibre sequence*

$$R \to A \overset{d}{\to} A/R.$$

*More symmetrically: the universal derivation is the unique (in the homotopy category of $R$-modules) lifting $A \to \Omega_A$ in the diagram*

$$\begin{array}{ccc} & & \Omega_A \\ & & \downarrow \\ A \cong A & \overset{f}{\to} & A \wedge A, \end{array}$$

*where $f = 1 \wedge id - id \wedge 1$.*

*Proof.* Consider the following extension:

$$I \to R \to A,$$

where the second map is just the unit map and $I$ is the kernel of this map. From the conormal sequence of this extension we see that there is a map

$$\Omega_A \to \Sigma I/I^2 \tag{13}$$

and, consequently, a singular extension with the kernel $I/I^2$

$$I/I^2 \to X \to I. \tag{14}$$

We have the following homotopy commutative diagram of $R$-modules:

$$\begin{array}{ccl} \Omega_A & \xrightarrow{g} & \Sigma I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2 \\ \uparrow & \nearrow & \\ A & & . \end{array}$$

Here the horizontal row is a fragment of the conormal sequence corresponding to (13), the vertical arrow is the universal derivation and the slanted arrow is the boundary map in (13).

We claim that the map $g$ is the canonical inclusion of a retract

$$f\colon \Omega_A \simeq \Sigma I/I^2 \to \Sigma I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2.$$

To see that observe that there is a map of extensions

$$\begin{array}{ccccc} I & \to & R & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ I/I^2 & \to & X & \to & A \end{array}$$

and therefore the following homotopy commutative diagram of $R$-modules:

$$\begin{array}{rccccc} \Omega_A \simeq I/I^2 & \to & A & \stackrel{1\wedge id}{\to} & A \wedge A \\ & & \downarrow & & \downarrow \\ \Omega_{A/X} \simeq I/I^2 \vee \Sigma^2 I/I^2 \wedge_X I/I^2 & \to & A & \stackrel{1\wedge id}{\to} & A \wedge_X A. \end{array}$$

Notice that since both rows split there exists only one up to homotopy map

$$\Omega_A \to \Omega_{A/X},$$

making the diagram commute and this map is obviously the inclusion of a retract. Our claim is proved. Therefore, our universal derivation is just the boundary map (14)

$$A \to \Sigma I/I^2.$$

Next from the diagram

$$\begin{array}{ccccccccc} I & \to & R & \to & A & \to & \Sigma I & = & A/R \\ \downarrow & & \downarrow & & \| & & \downarrow & & \\ I/I^2 & \to & R/I^2 & \to & A & \to & \Sigma I/I^2 & = & A \wedge A/R, \end{array}$$

we see that the boundary map $A \to \Sigma I/I^2$ is the composition

$$A \to A/R \simeq \Sigma I \to \Sigma I/I^2$$

and the diagram

$$\begin{array}{ccccccc} I & \to & R & \to & A & \to & A/R \\ \downarrow & & \downarrow & & 1 \wedge id \downarrow & & 1 \wedge id \downarrow \\ I/I^2 & \to & A & \stackrel{1\wedge id}{\to} & A \wedge A & \to & A \wedge A/R, \end{array}$$

tells us that the map $\Sigma I \to \Sigma I/I^2$ is the same as

$$A/R \stackrel{1\wedge id}{\to} A \wedge A/R$$

and the first part of our theorem is proved.

Consider now the following homotopy commutative diagram

$$\begin{array}{ccccc} \Omega_A & \to & A \wedge A & \to & A \wedge A/R \\ & & f \uparrow & \nearrow & \\ & & A & . & \end{array}$$

Here the slanted arrow is $id \wedge d$ and the composite of the two horizontal arrows is an isomorphism identifying $\Omega_A$ with $A \wedge A/R$. Clearly the map $A \to \Omega_A$ fitting in the above diagram exists and is unique up to homotopy. With this our theorem is proved.

The universal derivation allows us to define a useful forgetful map

$$l\colon \mathbf{Der_R}(A, M) \to F_R(A, M)$$

as the composite

$$\mathbf{Der_R}(A, M) = F_{A-\mathrm{bimod}}(\Omega_A, M) \to F(\Omega_A, M) \to F(A, M),$$

where the last map is induced by $d$.

We will also denote by $l$ the corresponding map on homotopy groups

$$Der_R^\bullet(A, M) \to [A, M]_R^\bullet.$$

At the level of homotopy groups the map $l$ has the following simple description: let $d\colon A \to A \vee M$ be a derivation of $A$ with coefficients in $M$. Then $l(d)$ is the homotopy class of the following composite map:

$$A \stackrel{d}{\to} A \vee M \to M,$$

where the second map is just the projection onto the wedge summand. □

LEMMA 4.2. *Let* $d \in Der^{\bullet}_R(A, A)$. *Then* $l(d) \in [A, A]^{\bullet}_R$ *is contained in the subspace of primitive cohomology operations from A into itself.*

*Proof.* Let $d$ have degree $i$, so it determines a map of algebras $A \to A \vee \Sigma^i A$. Then the following diagram is homotopy commutative:

$$\begin{array}{ccc} A \wedge A & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Sigma^i A \wedge A \vee A \wedge \Sigma^i A & \to & A. \end{array}$$

Here the upper arrow is the multiplication map $m$, the lower arrow is $m \vee m$, the right vertical arrow is $d$ and the left vertical arrow is $d \wedge 1 \vee 1 \wedge d$. This is the definition of a primitive operation and the lemma is proved. □

## 5. Pullbacks and Pushouts of Singular Extensions

Here we will describe some basic constructions with singular extensions. Let

$$E\colon\ I \to A \to B \tag{15}$$

be a singular extension and

$$f\colon\ C \to B$$

a map of algebras. Consider the strictly commutative diagram

$$\begin{array}{ccccc} I & \to & X & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ I & \to & A & \to & B, \end{array} \tag{16}$$

where the right vertical map is $f$ and the right square is a homotopy pullback of $R$-algebras. It is easy to see that the upper row in (16) is a singular extension (the associated derivation is induced by the map $C \to B$). We will denote it $f^\star(E)$.

We have the following simple interpretation of $f^\star(E)$ in terms of the topological module of differentials. Let

$$\Omega_B \to \Sigma I$$

be the derivation corresponding to the singular extension $E$. Then it is easy to see that the composite map

$$\Omega_C \to \Omega_B \to \Sigma I,$$

represents the element in $\mathbf{Der_R}^1(C, I)$ that corresponds to $f^\star(E)$.

Next consider again an extension (15) and a map of $B$-bimodules

$$g\colon I \to M.$$

The map

$$d\colon \Omega_B \to \Sigma I \to \Sigma M,$$

gives rise to an element in $\mathbf{Der_R}^1(B, M)$ and, therefore, to a singular extension of $B$ by $M$. We will call this extension the pushout of $E$ by $g$ and denote it $g_\star(E)$.

PROPOSITION 5.1. *There is a map from $E$ to $g_\star(E)$ which is the identity on $B$.*

*Proof.* Consider the diagram

$$\begin{array}{ccccc} I & & & & \\ \downarrow & & & & \\ A & \rightarrow & B & \cong & B \\ \downarrow & & \downarrow & & \downarrow \\ B & \stackrel{d}{\rightarrow} & B \vee \Sigma I & \rightarrow & B \vee \Sigma M. \end{array}$$

Here the left square is a homotopy pullback, and from the commutativity of the outer square it follows that there is a map from it to the homotopy pullback diagram

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ B & \rightarrow & B \vee \Sigma I \end{array}$$

and therefore from $E$ to $g_\star(E)$ and the proposition is proved. □

*Remark* 5.2. Consider the diagram

$$\begin{array}{ccccc} I & \rightarrow & A & \rightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ M & \rightarrow & Y & \rightarrow & B \end{array}$$

where the upper row is $E$ and the lower row is $g_\star(E)$. Since both rows are homotopy cofibre sequences of $R$-modules, $Y$ is homotopically the same as $A \vee_I M$ but it is not clear *apriori* how to introduce strict multiplication on $A \vee_I M$.

## 6. Obstruction Theory

In this section we consider the problem of lifting an algebra map. We restrict ourselves to the case of a singular extension only. However, in the next section we will see that a fairly broad class of algebra maps can be decomposed as a sequence of singular extensions, which we call a generalized Adams resolution. Together these results provide a sequence of obstructions for lifting an arbitrary map of $R$-algebras.

THEOREM 6.1. *Let $I \rightarrow B/A$ be a singular extension of $R$-algebras associated with a derivation $d: A \rightarrow A \vee \Sigma I$ and $f: X \rightarrow A$ a map of $R$-algebras where $X$ is $q$-cofibrant. Then:* (i) *$f$ lifts to an $R$-algebra map $X \rightarrow B$ iff the induced derivation $d \circ f \in Der^1(X, I)$ is homotopic to zero.* (ii) *Assuming that a lifting exists in the fibration*

$$F_{R-\mathrm{alg}}(X, B) \rightarrow F_{R-\mathrm{alg}}(X, A), \tag{17}$$

*the homotopy fibre over the point $f \in F_{R-\mathrm{alg}}(X, A)$ is weakly equivalent to $\Omega^\infty$ $\mathbf{Der}_R(X, I)$ (the* 0*th space of the spectrum $\mathbf{Der}_R(X, I)$).*

*Proof.* (i) If a lifting of $f$ exists, then $d \circ f$ factors as

$$X \overset{f}{\to} A \overset{\epsilon}{\to} A \vee \Sigma I,$$

which means that the derivation $d \circ f$ is trivial, Conversely, if $d \circ f$ is trivial then the diagram of $R$-algebras

$$\begin{array}{ccc} X & \overset{f}{\to} & A \\ f\downarrow & & \downarrow\epsilon \\ A & \overset{d}{\to} & A \vee \Sigma I, \end{array}$$

commutes up to homotopy and by the universal property of the homotopy pullback there is a map $\tilde{f}\colon X \to B$ lifting $f$.

(ii) Suppose that the lifting of $f$ exists, so the (homotopy) fibre of the map (17) is nonempty. We have the following diagram of $R$-algebras:

$$\begin{array}{ccccc} B & & \longrightarrow & & A \\ \Big\downarrow & \searrow & & \swarrow & \Big\downarrow \\ & & A & & \\ & \nearrow & & \nwarrow & \\ A & & \overset{d}{\longrightarrow} & & A \vee \Sigma I. \end{array}$$

Here the northwest arrow is the canonical projection onto the wedge summand. Changing $B$ in its homotopy class if necessary we can arrange that this diagram be strictly commutative. Notice that the outer square is a homotopy pullback of $R$-algebras. Applying the functor $F_{R-\mathrm{alg}}(X, -)$ to this diagram we get the diagram of spaces

$$\begin{array}{ccccc} F_{R-\mathrm{alg}}(X, B) & & \longrightarrow & & F_{R-\mathrm{alg}}(X, A) \\ \Big\downarrow & \searrow & & \swarrow & \Big\downarrow \\ & & F_{R-\mathrm{alg}}(X, A) & & \\ & \nearrow & & \nwarrow & \\ F_{R-\mathrm{alg}}(X, A) & & \longrightarrow & & F_{R-\mathrm{alg}}(X, A \vee \Sigma I). \end{array}$$

Again the outer square is a homotopy pullback (of spaces). Taking the homotopy fibres of the maps from the outer square to the center (over $f \in F_{R-\mathrm{alg}}(X, A)$) we get the following homotopy pullback of spaces:

$$\begin{array}{ccc} hofib F_{R-\mathrm{alg}}(X, B) \to F_{R-\mathrm{alg}}(X, A) & \longrightarrow & pt \\ \downarrow & & \downarrow \\ pt & \longrightarrow & hofib F_{R-\mathrm{alg}}(X, A \vee \Sigma I) \to F_{R-\mathrm{alg}}(X, A). \end{array} \tag{18}$$

Notice that the space in the right lower corner of (18) is canonically weakly equivalent to $\Omega^{\infty}\mathbf{Der}_R(X, \Sigma I)$. Since according to our assumption the space

$$hofib\ F_{R-\mathrm{alg}}(X, B) \to F_{R-\mathrm{alg}}(X, A)$$

is nonempty the images of the lower and right arrows in (18) coincide and therefore

$$\begin{aligned} hofib\ F_{R-\mathrm{alg}}(X, B) \to F_{R-\mathrm{alg}}(X, A) &\cong \Omega(\Omega^\infty \mathbf{Der}_R(X, \Sigma I)) \\ &\cong \Omega^\infty \mathbf{Der}_R(X, I). \end{aligned}$$

With this our theorem is proved. □

## 7. Ideals of *R*-algebras and Generalized Adams Resolutions

Let

$$I \to A \to B \tag{19}$$

be an extension of $R$-algebras (not necessarily singular.) In such a situation we will call $I$ an ideal of $A$. To stress the analogy with algebra we will use the notation $A/I$ instead of $B$ in the sequel so (19) becomes

$$I \to A \to A/I. \tag{20}$$

One essential feature of ideals is that one can define $R$-algebras of the form $A/I^n$ for $n > 1$. More precisely:

THEOREM 7.1. *Associated to the extension* (20) *is a tower of $R$-algebras (a generalized Adams resolution) of the form*

$$\ldots \to A/I^n \to A/I^{n-1} \to \ldots \to A/I \tag{21}$$

*such that* (i) *there are projections $p_n\colon A \to A/I^n$ compatible with the maps in the tower and the corresponding extensions*

$$I^n \to A \to A/I^n,$$

*where*

$$I^n := I \wedge_A \ldots \wedge_A I,$$

(ii) *each successive stage $A/I^n$ is a singular extension of the previous stage $A/I^{n-1}$ by a certain $A/I^{n-1}$-bimodule which we denote $I^n/I^{n-1}$.*

*Proof.* We know that there is a weak equivalence

$$\Omega_{(A/I)/A} \simeq \Sigma I/I^2,$$

where $I/I^2$ is determined as the homotopy cofibre of the multiplication map $I \wedge_A I \to I$. Associated to the (universal) derivation

$$d_{A/I}\colon A/I \to \Omega_{(A/I)/A} \simeq \Sigma I/I^2$$

is the following singular extension

$$I/I^2 \to X \to A/I.$$

We claim that there exists a map $A \to X$ making the diagram

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ A & \xrightarrow{f} & A/I \end{array}$$

commute. Indeed, the obstruction to the existence of such a map is an element

$$f^\star(d_{A/I}) \in Der(A, \Sigma I/I^2),$$

which is zero by virtue of the transitivity exact sequence corresponding to the morphisms of algebras

$$R \to A \to A/I.$$

Therefore we have a morphism of extensions

$$\begin{array}{ccccc} I & \to & A & \to & A/I \\ \downarrow & & \downarrow & & \| \\ I/I^2 & \to & X & \to & A/I. \end{array} \tag{22}$$

Now set $A/I^2 := X$. From (22) it is easy to see that there is an extension

$$I \wedge_A I \to A \to A/I^2$$

(which justifies the notation $A/I^2$).

Proceeding by induction assume that we have constructed the tower $\{A/I^k\}$ for $k \leqslant n$ together with algebra maps $p_n\colon\ A \to\ A/I^n$. Consider the following homotopy commutative diagram of $A/I^n$-bimodules where both rows are homotopy fibre sequences:

$$\begin{array}{ccccc} ? & \to & A/I^n \wedge_A A/I & \to & A/I \\ g\uparrow & & \uparrow & & \uparrow \\ \Omega_{(A/I^n)/A} & \to & A/I^n \wedge_A A/I^n & \to & A/I^n. \end{array}$$

Here the lower row is the definition of $\Omega_{(A/I^n)/A}$ and the upper row represents the (left) action of $A/I^n$ on $A/I$. The right and middle vertical arrows are obvious maps of $A/I^n$-bimodules. Therefore, we can choose the left vertical arrow (denoted by $g$) to be also a map of $A/I^n$-bimodules. Notice, that despite the choices made in the underlying category of $R$-modules this map still remains unique (up to homotopy).

Further notice that the $A/I^n$-bimodule ? is weakly equivalent as an $R$-module to $\Sigma I^n \wedge_A A/I$. Denoting by $I^n/I^{n+1}$ the homotopy cofibre of the multiplication map $I^n \wedge_A I \to I^n$ we see that $I^n/I^{n+1}$ is weakly equivalent to $I^n \wedge_A A/I$. So the $A/I^n$-bimodule ? is weakly equivalent as an $R$-module to $\Sigma I^n/I^{n+1}$. From now on we will (slightly abusing notations) use the symbol $\Sigma I^n/I^{n+1}$ to denote the corresponding $A/I^n$-bimodule.

Now we can construct the singular extension

$$I^n/I^{n+1} \to A/I^{n+1} \to A/I^n,$$

associated with the derivation

$$g\colon\ \Omega_{(A/I^n)/A} \to ? = \Sigma I^n/I^{n+1}.$$

The induction step is concluded and our theorem is proved. □

*Remarks.* (1) In the special case when $A = S$, the sphere spectrum, the tower (21) is just the canonical Adams resolution of $S$, cf. [1] and its homotopy inverse limit is the nilpotent completion of $S$ at the ideal $I$ (cf. [5]), which under favorable circumstances coincides with the localization of $S$ with respect to the spectrum $S/I$. This is why we called the tower (21) the generalized Adams resolution.

(2) It is not hard to show that the tower (21) is determined in the homotopy category of $R$-algebras up to a (noncanonical) isomorphism. We leave the details to the interested reader.

EXAMPLES. (1) Take $A = k(n)$, the $n$th connective Morava $K$-theory, $A/I := HZ/p$, $A \to A/I$ the canonical map inducing an isomorphism on $\pi_0$. Then the generalized Adams resolution of $k(n)$ is exactly its Postnikov tower.

(2) Truncated polynomial algebra. Denote $R[t]$ the free associative algebra on the $R$-module $\Sigma^d R$, that is,

$$R[t] \simeq R \vee \Sigma^d R \vee \Sigma^{2d} R \vee \dots .$$

The parameter $t$ has formal degree $d$. The coefficient ring of $R[t]$ is then $R_\bullet[t]$ – the polynomial algebra over $R_\bullet$ on one generator of degree $d$. Notice, that the $S$-algebra $R[t]$ need not be commutative unless $d = 0$.

Define the ideal $tR[t]$ from the following homotopy fibre sequence:

$$tR[t] \to R[t] \to R.$$

Then it is easy to see that the $n$th Adams stage

$$R^{n-1} := R[t]/(tR[t])^n,$$

splits as a spectrum as

$$R^{n-1} \simeq R \vee \Sigma^d R \vee \Sigma^{2d} R \vee \dots \vee \Sigma^{d(n-1)} R$$

and $R_\bullet^{n-1} = R_\bullet[t]/t^n$. Moreover, the homotopy fibre sequence

$$\Sigma^{d(n-1)} R \to R^{n-1} \to R^{n-2}$$

is a singular extension.

(3) Let $E$ be a complex-oriented cohomology theory obtained by killing a regular ideal $I_\bullet = (x_{i_1}, x_{i_2}, \dots)$ in $MU_\bullet = Z[x_1, x_2, \dots]$. By [7], Chapter V and [18] we

know that $E$ is an $MU$-module with homotopy associative multiplication (or $MU$-ring spectrum). We will see that $E$ possesses the structure of an $MU$-algebra (i.e. with *strict* multiplication). Therefore, we have the generalized Adams resolution

$$\to MU/I^n \to \dots \to MU/I^2 \to MU/I = E,$$

which is related to the algebraic tower $\{MU_\bullet/I_\bullet^n\}$ and whose homotopy inverse limit is the spectrum $MU$ itself.

(4) Let $E_n$ be the Johnson–Wilson theory with coefficient ring $E_n = Z_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$. It can be constructed from $MU$ by killing a collection of polynomial generators and is, therefore, an $MU$-algebra. We have a canonical $MU$-algebra map $E_n \to K(n)$ with the kernel $I_n$. Associated with this map is a generalized Adams resolution

$$K(n) = E_n/I_n \leftarrow E_n/I_n^2 \leftarrow \dots ,$$

whose homotopy inverse limit is equivalent to the $K(n)$-localization of $E_n$. This tower is closely related to the so-called $I_n$-adic tower studied in, for Example [2].

## 8. Postnikov Towers of *R*-algebras

In this section we prove, as an application of the techniques developed before, that a Postnikov tower of an $R$-algebra is a tower of $R$-algebras. The analogous statement for commutative $R$-algebras was proved in [10] and [3]. The proof presented here is somewhat sketchy since it differs little from the one found in the above mentioned references.

THEOREM 8.1. *Assume that $R$ is a connective commutative $S$-algebra and $A$ is a connective $R$-algebra. Then there exists a tower of $R$-algebras*

$$\dots \leftarrow A_i \leftarrow A_{i+1} \leftarrow \dots$$

*and maps of $R$-algebras*

$$f_i\colon\ A \to A_i$$

*compatible with the maps in the tower such that* (1) $\pi_k(A_i) = 0$ *for* $k > i$ *and* (2) $f_i$ *induces an isomorphism* $\pi_k(A) \to \pi_k(A_i)$ *for* $k \leqslant i$. *Moreover, the homotopy fibre sequence*

$$K(\pi_{i+1}(A), i+1) \to A_{i+1} \to A_i$$

*is a singular extension and the whole construction is functorial in the homotopy category of $R$-algebras.*

*Remark* 8.2. The connectivity assumptions on $R$ and $A$ are needed in order to be able to kill higher homotopy groups.

*Proof.* Take $A_0$ to be equal to $K(\pi_0(A,)0)$. This is an $R$-algebra and according to [7], Proposition IV, 3.1 there exists an $R$-algebra map $A \to A_0$ realizing the isomorphism on $\pi_0$.

Suppose by induction that the map $A \to A_i$ has already been constructed. Consider the following diagram:

$$\begin{array}{ccccc} A & \to & A_i & \overset{k}{\to} & \Sigma K(\pi_{i+1}(A), i+1) \\ & & \downarrow & \nearrow & \\ & & \Omega_{A_i/A} & & . \end{array}$$

Here the vertical map is the canonical derivation, $k$ is the $i$th $k$-invariant. The existence of the diagonal arrow can be seen as follows. The first nontrivial homotopy group of $\Omega_{A_i/A}$ is the same as the single nontrivial homotopy group of $\Sigma K(\pi_{i+1}(A), i+1)$, so we could construct a map

$$\Omega_{A_i/A} \to \Sigma K(\pi_{i+1}(A), i+1),$$

by attaching cells in the category of $A_i$-bimodules. This gives a derivation

$$A_i \to A_i \vee \Sigma K(\pi_{i+1}(A), i+1)$$

and the corresponding singular extension

$$K(\pi_{i+1}(A), i+1) \to A_{i+1} \to A_i,$$

which is the $(i+1)$th stage of the Postnikov tower and the theorem is proved. □

## 9. The Spectral Sequence for Topological Hochschild Cohomology

In this section we discuss the spectral sequence that relates topological Hochschild cohomology and ordinary Hochschild cohomology of graded algebras. Since we know of no published source which describes algebraic Hochschild cohomology the way we need it we begin with an outline of the classical Hochschild cohomology.

DEFINITION 9.1. Let $R_\bullet$ be a commutative graded ring, $A_\bullet$ is an algebra over $R_\bullet$, $M_\bullet$ is a graded module over $A_\bullet$. Then Hochschild cohomology of $A_\bullet$ with coefficients in $M_\bullet$ is the module

$$HH^\bullet_{R_\bullet}(A_\bullet, M_\bullet) = Ext_{A_\bullet \otimes^L_{R_\bullet} A_{\bullet\mathrm{op}}}(A_\bullet, M_\bullet),$$

where $\otimes^L_{R_\bullet}$ denotes derived tensor product over $R_\bullet$.

From now on we will write $\otimes$ instead of $\otimes_{R_\bullet}$. Notice that if $A_\bullet$ is flat over $R_\bullet$ then this definition coincides with the standard one (cf.[6]). There is also a generalization of the standard complex which computes Hochschild cohomology;

let $\tilde{A}_\bullet$ be a differential graded $R_\bullet$-algebra which is quasi-isomorphic to $A_\bullet$ and $R_\bullet$-flat. Then $HH^\bullet_{R_\bullet}(A_\bullet, M_\bullet)$ is the cohomology of the bicomplex

$$C^{ij}(A_\bullet, M_\bullet) = Hom^i_{R_\bullet}(\tilde{A}_\bullet^{\otimes j}, M_\bullet).$$

The differentials in $C^{\bullet\bullet}$ are the internal one induced by the differential in $\tilde{A}_\bullet$ and the standard bar-differential.

We will be interested mostly in the truncated version of Hochschild cohomology which will be denoted as $Der^\bullet_{R_\bullet}(A_\bullet, M_\bullet)$ (in keeping with the topological notations).

To define it let us introduce the module of differentials $\Omega^\bullet_{A_\bullet/R_\bullet}$ from the following short exact sequence of complexes:

$$\Omega^\bullet_{A_\bullet/R_\bullet} \to \tilde{A}_\bullet \otimes \tilde{A}_\bullet \stackrel{m}{\to} \tilde{A}_\bullet,$$

the second arrow being the multiplication map.

Now define the truncated Hochschild cohomology of $A_\bullet$ with coefficients in $M_\bullet$ as

$$Der^\bullet_{R_\bullet}(A_\bullet, M_\bullet) = Ext^\bullet_{\tilde{A}_\bullet \otimes \tilde{A}_\bullet}(\Omega^\bullet_{A_\bullet/R_\bullet}, M_\bullet).$$

As in the topological case there is a 'universal derivation'

$$d\colon \tilde{A}_\bullet \to \Omega^\bullet_{A_\bullet/R_\bullet},$$

whose composition with the inclusion

$$\Omega^\bullet_{A_\bullet/R_\bullet} \hookrightarrow \tilde{A}_\bullet \otimes \tilde{A}_\bullet$$

is the map $1 \wedge id - id \wedge 1$. It gives, therefore, the forgetful map

$$l\colon\ Der^\bullet_{R_\bullet}(A_\bullet, M_\bullet) \to Ext^\bullet_{R_\bullet}(A_\bullet, M_\bullet).$$

LEMMA 9.2. *Consider the truncated Hochschild complex*

$$\overline{C}^{ij}(A_\bullet, M_\bullet) = Hom^i_{R_\bullet}(\tilde{A}_\bullet^{\otimes j-1}, M_\bullet)$$

*with the same differentials as in* $C^{\bullet\bullet}(A_\bullet, M_\bullet)$. *Then the cohomology of* $\overline{C}^{\bullet\bullet}(A_\bullet, M_\bullet)$ *coincides with* $Der_{R_\bullet}(A_\bullet, M_\bullet)$ *and the forgetful map* $l$ *is induced by the projection*

$$\overline{C}^{\bullet\bullet}(A_\bullet, M_\bullet) \to \overline{C}^{\bullet 0}(A_\bullet, M_\bullet) = Hom^\bullet_{R_\bullet}(\tilde{A}_\bullet, M_\bullet)$$

*times* $(-1)$.

*Proof.* Considering the standard two-sided bar-resolution of $\tilde{A}_\bullet$

$$\tilde{A}_\bullet \stackrel{m}{\leftarrow} \tilde{A}_\bullet \otimes \tilde{A}_\bullet \leftarrow \tilde{A}_\bullet^{\otimes 3} \leftarrow \dots,$$

we see that the complex $\tilde{A}_\bullet^{\otimes 3} \leftarrow \tilde{A}_\bullet^{\otimes 4} \leftarrow \dots$ maps quasi-isomorphically onto the complex $\Omega^\bullet_{A_\bullet/R_\bullet} = Ker(m)$. The universal derivation

$$d\colon\ \tilde{A}_\bullet \to \Omega^\bullet_{A_\bullet/R_\bullet}$$

lifts to the map

$$\tilde{d}\colon\ \tilde{A}_\bullet \to \tilde{A}_\bullet^{\otimes 3},$$

$$\tilde{d}(a) = -1 \otimes a \otimes 1.$$

Now the statement of the lemma becomes clear after we apply the functor $Hom_{\tilde{A}_\bullet \otimes \tilde{A}_\bullet^{\mathrm{op}}}(-, M_\bullet)$ to the complex $\tilde{A}_\bullet^{\otimes 3} \leftarrow \tilde{A}_\bullet^{\otimes 4} \leftarrow \dots$.

We now return to the topological situation; as before $A$ is an $R$-algebra, $M$ is an $A$-bimodule; denote $\pi_\bullet(A)$ as $A_\bullet$, $\pi_\bullet(R)$ as $R_\bullet$ and $\pi_\bullet(M)$ as $M_\bullet$.

PROPOSITION 9.3. *Suppose that* (1) *the Kunneth spectral sequence for* $\pi_\bullet(A \wedge A^{\mathrm{op}})$ *collapses so we have*

$$\pi_\bullet(A \wedge A^{\mathrm{op}}) = Tor_\bullet{}^{R_\bullet}(A_\bullet, A_\bullet);$$

(2) *the algebras* $A_\bullet \otimes^L_{R_\bullet} A_\bullet$ *and* $Tor_\bullet^{R_\bullet}(A_\bullet, A_\bullet)$ *are quasi-isomorphic.*
*Then there are the following spectral sequences:*

$${}^1E_2^{ij} = Der^{ij}_{R_\bullet}(A_\bullet, M_\bullet) \Longrightarrow \mathbf{Der_R}^{i+j}(A, M);$$

$${}^2E_2^{ij} = Ext^{ij}_{R_\bullet}(A_\bullet, M_\bullet) \Longrightarrow \pi_{i+j} F_R(A, M)$$

*and a map*

$${}^1E_2^{ij} \to {}^2E_2^{ij},$$

*which at the level of* $E_\infty$ *terms gives the forgetful map*

$$l\colon\ gr\mathbf{Der}_{R_\bullet}(A, M) \to gr[A, M]_\bullet.$$

*Proof.* There is a spectral sequence (cf. [7], IV, 4.1)

$$E_2^{\bullet,\bullet} = Ext_{\pi_\bullet(A\wedge A^{\mathrm{op}})}(\pi_\bullet \Omega_A, M_\bullet) \Longrightarrow \pi_\bullet F_{A\wedge A^{\mathrm{op}}}(\Omega_A, M) = \mathbf{Der_R}^\bullet(A, M).$$

Our assumptions (1) and (2) ensure that

$$\pi_\bullet(A \wedge A^{\mathrm{op}}) = A_\bullet \otimes^L_{R_\bullet} A_\bullet$$

and

$$\pi_\bullet(\Omega_A) = \Omega_{A_\bullet/R_\bullet},$$

so we identify the $E_2$-term of our spectral sequence as $Der^{\bullet\bullet}_{R_\bullet}(A_\bullet, M_\bullet)$.

The second spectral sequence is just the hypercohomology spectral sequence for calculating $\pi_\bullet F_R(A, M)$. Next from the naturality of the hypercohomology spectral sequences it follows that we have the following commutative diagram of spectral sequences:

$$\begin{array}{ccccc} Ext^\bullet_{\pi_\bullet(A\wedge A^{\mathrm{op}})}(\pi_\bullet\Omega_A, M_\bullet) & \to & Ext^\bullet_{R_\bullet}(\pi_\bullet\Omega_A, M_\bullet) & \to & Ext^\bullet_{R_\bullet}(A_\bullet, M_\bullet) \\ \| & & \| & & \| \\ Der^\bullet_{R_\bullet}(A_\bullet, M_\bullet) & \to & Ext^\bullet_{R_\bullet}(\Omega_{A_\bullet/R_\bullet}, M_\bullet) & \to & Ext^\bullet_{R_\bullet}(A_\bullet, M_\bullet) \end{array} \qquad (23)$$

and since the composite map

$$A_\bullet \to \pi_\bullet\Omega_A \simeq \Omega_{A_\bullet/R_\bullet}$$

is just the universal derivation of the algebra $A_\bullet$, we conclude that the lower row in (23) gives us the forgetful map

$$l\colon\ Der^{\bullet\bullet}_{R_\bullet}(A_\bullet, M_\bullet) \to Ext^\bullet_{R_\bullet}(A_\bullet, M_\bullet).$$

Our proposition is proved. □

## 10. Calculations with $HZ/p$

In this section we calculate the topological Hochschild cohomology of the Eilenberg–MacLane spectrum $HZ/p$ which will be denoted (in this section only) by $H$ considered as an algebra over the complex cobordism spectrum $MU$. Before starting any calculations we record the following result of a general nature. It really should have belonged to [7], but unfortunately, is not found there. The argument below was communicated to the author by Mandell.

LEMMA 10.1. *Let $M$ and $N$ are $R$-ring spectra. Then the spectral sequence*

$$Tor_{R_\bullet}(M_\bullet, N_\bullet) \Longrightarrow \pi_\bullet(M \wedge_R N) \qquad (24)$$

*is one of differential graded $R_\bullet$-algebras.*

*Sketch proof.* To deal with multiplicative structures we need a slightly different construction then the one given in [EKMM], IV, Section 5. Let

$$\ldots \to F_p \xrightarrow{d_p} F_{p-1} \to \ldots \xrightarrow{\epsilon} M_\bullet$$

be an $R_\bullet$-free resolution of $M_\bullet$. Let $Q_0 = Ker\ \epsilon$ and $Q_p = ker\ d_p$. For $p \geqslant 0$ denote by $\mathbf{F_p}$ the wedge of sphere $R$-modules with $\pi_\bullet(\mathbf{F_p}) = F_p$. Now let $M'_0 = \mathbf{F_0}$ and choose a map $\phi_0\colon\ M'_0 \to M$ that represents $\epsilon$ on the level of homotopy groups. Let $\mathbf{Q_0}$ be the homotopy fibre of $\phi_0$. Then $\pi_\bullet(\mathbf{Q_0}) = Q_0$, so we can choose a map $\mathbf{F_1} \to \mathbf{Q_0}$ the given map on the homotopy groups. Take $M'_1$ to be the cofibre of the composite map

$$\mathbf{F_1} \to \mathbf{Q_0} \to \mathbf{F_0}.$$

Further the map $\phi_0$ canonically extends to a map

$$\phi_1\colon\ M'_1 \to M.$$

Denote the homotopy fibre of this map by $\mathbf{Q_1}$. Then

$$\pi_\bullet(\Sigma^{-1}\mathbf{Q_1}) = Q_1,$$

so we can find a map $\Sigma^{-1}\mathbf{F_2} \to \mathbf{Q_1}$ realizing the given map on the homotopy groups etc.

Continuing in this way we construct a direct system

$$M'_0 \to M'_1 \to \dots . \tag{25}$$

(Notice that we used the notation $M'_p$ to distinguish it from $M_p$ of [7].) The telescope $M'$ of this direct system is weakly equivalent to $M$ and we can assume that all consecutive maps are inclusions of cell subcomplexes, that is, we have a filtration on $M'$. Smashing this with $N$ we get a filtration on $M' \wedge_R N$ and the associated spectral sequence (24) converging to $\pi_\bullet(M \wedge_R N)$.

Now recall that $M$ (and therefore $M'$) is an $R$-ring spectrum. Smashing (25) with itself we get a filtration on $M' \wedge_R M'$

$$\dots \to \vee_{i+j=k} M'_i \wedge M'_j \to \vee_{i+j=k+1} M'_i \wedge M'_j \to . \tag{26}$$

Proceeding by induction we realize the multiplication map $M' \wedge_R M' \to M'$ as a map of filtered $R$-modules so that on the cofibres of (26) it agrees with the pairing of algebraic resolutions $F_\bullet \otimes_{R_\bullet} F_\bullet \to F_\bullet$.

So, we constructed a collection of maps $M'_i \wedge_R M'_j \to M'_{i+j}$. Using these maps and the multiplication on $N$ we construct the maps

$$M'_i \wedge_R N \wedge_R M'_j \wedge_R N \to M'_{i+j} \wedge_R N.$$

This induces the required pairing of spectral sequences and we are done. □

THEOREM 10.2.
(i)

$$\mathbf{THH}^\bullet_{MU}(H, H) = Z/pZ[y_2, y_4, \dots, y_{2n}, \dots],$$

$$\mathbf{Der_{MU}}(H, H)^{\bullet-1} = Z/pZ[y_2, y_4, \dots]/(Z/pZ).$$

(ii)

$$[H, H]^\bullet_{MU} = \Lambda_{Z/pZ}(z_1, z_3, \dots, z_{2n+1}, \dots)$$

*and the map*

$$\mathbf{Der}_{\mathrm{MU}}(H, H)^\bullet \to [H, H]^\bullet_{MU}$$

*sends the elements* $y_{2i} \in \mathbf{Der}_{\mathrm{MU}}(H, H)^{2i-1}$ *to* $z_{2i-1} \in [H, H]_{MU}$.
(iii) *Under the forgetful map*

$$[H, H]^{\bullet}_{MU} \to [H, H]^{\bullet}_{S} = \mathcal{A}_p \tag{27}$$

*the elements* $z_{2p^i-1}$ *get mapped to the elements* $Q_i$ *which form the set of primitive elements in the Steenrod algebra, and all other* $z$*'s get mapped to zero.*

LEMMA 10.3. *Let* $HZ$ *denote the integral Eilenberg–MacLane spectrum. Then*

$$\pi_{\bullet}(H \wedge_{MU} HZ) = \Lambda(x_3, x_5, \ldots, x_{2n+1}, \ldots).$$

*Proof.* Consider first the spectrum $HZ \wedge_{MU} HZ$. We have a strongly converging spectral sequence

$$Tor^{MU_{\bullet}}(Z, Z) = \Lambda(x_3, \ldots x_{2n+1}, \ldots) \to \pi_{\bullet}(H \wedge_{MU} H). \tag{28}$$

To prove that it collapses we compare it to the spectral sequence

$$\begin{aligned} Tor^{MU_{\bullet}}(Z, HQ) &= Tor^{MU_{Q\bullet}}(HQ, HQ) \\ &= \Lambda_Q(x_3, \ldots x_{2n+1}, \ldots) \\ &\to \pi_{\bullet}(H \wedge_{MU} H). \end{aligned} \tag{29}$$

Here $HQ$ is the rational Eilenberg–MacLane spectrum and $MU_Q$ is the rationalization of $MU$. We see that the integral spectral sequence injects into the rational one and therefore it is enough to prove the collapse of the latter. This is an essentially trivial fact and can be seen as follows. The rational spectrum $MU_Q$ is equivalent as a commutative $S$-algebra to the infinite smash product of $S$-algebras of the form $HQ[x_n]$ where

$$HQ[x_n] = HQ \vee \Sigma^{2n} HQ \vee \ldots.$$

Here $x_n$ is a formal symbol of degree $2n$ which corresponds to the polynomial generator of $MU$ in degree $2n$. Then we have $\pi_{\bullet} HQ[x_n] = Q[x_n]$. Further the spectral sequence

$$Tor_{HQ[x_n]_{\bullet}}(HQ, HQ) = \Lambda_Q(y_{n+1}) \Longrightarrow \pi_{\bullet}(H \wedge_{HQ[x_n]} H),$$

collapses for dimensional reasons and it follows that (29) splits as an infinite product of these little spectral sequence and therefore itself collapses. So we showed that (28) collapses. The spectral sequence for computing $\pi_{\bullet}(H \wedge_{MU} HZ)$ is just the reduction of (28) modulo $p$ and the statement of our lemma follows. □

*Proof of Theorem 10.2.* First compute $\pi_{\bullet}(H \wedge_{MU} H)$. We have the following spectral sequence:

$$Tor^{MU_{\bullet}}(Z/pZ, Z/pZ) = \Lambda_{Z/pZ}(x_1, x_3, \ldots x_{2n+1}, \ldots) \Longrightarrow \pi_{\bullet}(H \wedge_{MU} H). \tag{30}$$

We want to show that (30) collapses. To this end consider the following homotopy cofibre sequence of $MU$-modules:

$$H \wedge_{MU} HZ \stackrel{id\wedge p}{\to} H \wedge_{MU} HZ \to H \wedge_{MU} H.$$

From this cofibre sequence and taking into account that $\pi_\bullet(H \wedge_{MU} HZ)$ is $p$-torsion we see that

$$\pi_\bullet(H \wedge_{MU} H) = \pi_\bullet(H \wedge_{MU} HZ) \oplus \Sigma\pi_\bullet(H \wedge_{MU} HZ).$$

Here $\Sigma$ means shifting the grading by 1. It follows that (30) collapses (the permanent cycle $x_1$ in (30) is accountable for the second direct summand in the last equality). So we conclude that

$$\pi_\bullet(H \wedge_{MU} H) = \Lambda_{Z/pZ}(x_1, x_3, \dots, x_{2n+1}, \dots).$$

Next consider the spectral sequence

$$Ext^{\bullet\bullet}_{\pi_\bullet(H\wedge_{MU}H)}(Z/pZ, Z/pZ) \Longrightarrow \mathbf{THH}^\bullet_{MU}(H, H).$$

Its $E_2$-term is easily computed and equal to

$$Ext_{\Lambda_{Z/pZ}(x_1,x_3,\dots x_{2n+1},\dots)}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4, \dots, y_{2n} \dots].$$

Again, all higher differentials are zero this time because the spectral sequence is even. Now we know that our spectral sequence lies in the right half plane and converges conditionally. Since all higher differentials are zero Boardman's theorem (cf. [4], Thm 7.1) tells us that it converges strongly. All extension problems are clearly trivial and we conclude that

$$\mathbf{THH}^\bullet_{MU}(H, H) = Z/pZ[y_2, y_4, \dots, y_{2n} \dots].$$

The formula for $\mathbf{Der}^\bullet_{\mathrm{MU}}(H, H)$ follows from the homotopy cofibre sequence

$$\mathbf{THH}_{MU}(H, H) \to H \to \mathbf{Der}_{\mathrm{MU}}(H, H)$$

and part (i) of our theorem is proved.

For part (ii) notice that $H \wedge_{MU} H$ is a left $H$-module and we have an equality

$$F_H(H \wedge_{MU} H, H) = F_{MU}(H, H)$$

in the homotopy category of $H$-modules. Therefore $[H, H]^\bullet_{MU}$ is the $Z/pZ$ dual of the cooperation algebra $\pi_\bullet(H \wedge_{MU} H)$ and (ii) follows.

To compute the image of the map

$$\mathbf{Der}^\bullet_{\mathrm{MU}}(H, H) \to [H, H]^\bullet_{MU}, \tag{31}$$

we first look at the map

$$Der_{MU_\bullet}(Z/pZ, Z/pZ) \to Ext_{MU_\bullet}(Z/pZ, Z/pZ)$$

and then apply Proposition 9.3.

It is an easy exercise in homological algebra to show the elements

$$y_{2i} \in HH^{1,2i-1}_{MU_\bullet}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4 \ldots]$$

correspond to the elements

$$z_{2i+1} \in Ext^{1,2i}_{Z[x_1,x_2\ldots]}(Z/pZ, Z/pZ),$$

so Proposition 9.3 tells us that the map (31) is as stated at the level of $E_2$-terms. However, the image of (31) should be contained in the subgroup of primitive cohomology operations by Lemma 4.2, which is spanned by the elements $z_{2i+1} \in [H, H]_{MU}$ and therefore no elements of higher bar filtration might appear in this image. Part (ii) is proved.

To prove part (iii) notice that the $MU$-Steenrod algebra as well as the usual Steenrod algebra are Hopf algebras and the map (27) respects this structure. Therefore, the primitive elements $z_{2i+1}$ should map to the primitive elements in the Steenrod algebra. Hence only the elements $z_{2p^i-1}$ could have a nonzero image (namely, $Q_i$). We will show now that this is indeed the case. Consider the spectrum $k(n)$, the $n$th connective Morava $K$-theory. According to [7], Chapter V, Section 4 the spectrum $k(n)$ is an $MU$-module. Therefore, the Postnikov tower of $k(n)$ can be constructed within the category of $MU$-modules, and in particular the first nontrivial $k$-invariant of $k(n)$ (which is just $Q_n$) is given as a map of $MU$-modules $H \to H$. In other words the cohomological operation $Q_n$ can be lifted as a map of $MU$-modules and part (iii) is proved. □

*Remark* 10.4. Recall that for the ordinary topological Hochschild cohomology of $K$ we have (cf.[8],Thm 7.3)

$$\mathbf{THH}^\bullet(H, H) = Sym(e_0, e_1, \ldots e_h \ldots)/ < (e_h)^p, h \in \mathbf{N} >,$$

where $e_h \in \mathbf{THH}^{2p^h}(H, H)$. It follows that

$$\mathbf{Der}^{\bullet-1}_{\mathrm{MU}}(H, H) = \mathbf{THH}^\bullet(H, H)/(Z/pZ).$$

Then under the map

$$l\colon \mathbf{Der}^\bullet_{\mathrm{MU}}(H, H) \to [H, H]^\bullet_S = \mathcal{A}_p, \qquad (32)$$

the elements $e_h$ correspond to $Q_h \in \mathcal{A}_p$ and the rest get mapped to zero. Indeed, the Hochschild cohomology classes should give rise to primitive cohomology operations, therefore only elements $e_h$ may have a nonzero image. But we know that the elements $Q_i$ do lift to Hochschild cohomology classes (even in the category of $MU$-modules). Therefore the map (32) is as described.

COROLLARY 10.5. *The mod p Moore spectrum $M_p$ cannot be given a structure of an $A_\infty$ ring spectrum for any p.*

Indeed, for $p = 2$ $M_p$ is not even a ring spectrum up to homotopy. For an odd $p$ the second nontrivial homotopy group of $M_p$ is $\mathbf{Z}/p\mathbf{Z} = \pi_{2p-3}(M_p)$ and therefore the first $k$-invariant of $M_p$ is a cohomology operation of degree $2p - 2$ and cannot be one of $Q_i$'s which is a cohomology operation of odd degree. (In fact, the first Postnikov $k$-invariant of $M_p$ is $\mathcal{P}^\infty$). We see that already the second Postnikov stage of $M_p$ cannot be $A_\infty$.

## 11. Morava *K*-theories as *MU*-algebras

THEOREM 11.1. *For an odd prime p let $L_n$ be the MU-module obtained by killing the regular ideal generated by $(p, x_1, x_2, \ldots, \hat{x}_n, \ldots)$ in the ring $MU_\bullet = Z[x_1, x_2, \ldots]$. Then $L_n$ admits a MU-algebra structure (which extends the $MU_\bullet$-algebra structure on $L_{n\bullet} = Z/pZ[x_n]$).*

*Proof.* We will prove this theorem by induction up the Postnikov tower of $L_n$. The 1st Postnikov stage of $L_n$ is $HZ/p$, the Eilenberg–MacLane spectrum mod $p$ which is, of course, an $MU$-algebra. Suppose that we proved that $L_n^{(i)}$, the $2n(i-1)+1$st Postnikov stage of $L_n$, is an $MU$-algebra and show that $L_n^{(i+1)}$ is also such. Notice that the $MU$-module $L_n^{(i)}$ is gotten from $MU$ by killing the elements $x_n^i$ plus all the remaining polynomial generators and the prime $p$ in the ring $MU_\bullet$, so that

$$\pi_\bullet L_n^{(i)} = Z/pZ[x_n]/x_n^i. \qquad \square$$

LEMMA 11.2.

$$THH^\bullet_{MU}(L_n^{(i)}, HZ/p) = Z/pZ[y_2, y_4, \ldots],$$

*where deg $y_{2m} = 2m$ for $2m \neq 2n + 2$ and deg $y_{2n+2} = 2ni + 2$.*

*Proof.* The first step is to compute $\pi_\bullet(L_n^{(i)} \wedge_{MU} L_n^{(i)})$. Consider the spectral sequence

$$Tor_{st}^{Z[x_1,x_2,\ldots]}(Z/pZ[x_n]/x_n^i, Z/pZ[x_n]/x_n^i) \Longrightarrow \pi_{s+t}(L_n^{(i)} \wedge_{MU} L_n^{(i)}).$$

To compute the $E_2$-term of this spectral sequence we introduce a differential graded algebra

$$A_* = Z[x_1, x_2, \ldots] \otimes \Lambda(z_1, z_3, \ldots),$$

where

$$d(z_1) = p,\ d(z_3) = x_1, \ldots,\ d(z_{2n-1}) = x_{n-1},\ d(z_{2n+1}) = x_n^i,$$
$$d(z_{2n+2}) = x_{n+1} \ldots$$

Here $deg\ x_m = 2m$, $deg\ z_{2m+1} = 2m + 1$ if $m \neq n$ and $deg\ z_{2n+1} = 2ni + 1$. Then $A_*$ is quasi-isomorphic to $Z/pZ[x_n]/x_n^i$ and therefore

$$\begin{aligned} &Tor^{Z[x_1,x_2,\ldots]}(Z/pZ[x_n]/x_n^i, Z/pZ[x_n]/x_n^i \\ &\quad = A_* \otimes_{Z[x_1,x_2,\ldots]} Z/pZ[x_n]/x_n^{i+1} \\ &\quad = \Lambda_{Z/pZ}(z_1, z_3, \ldots) \otimes Z/pZ[x_n]/x_n^i. \end{aligned}$$

To show that all higher differentials are zero we introduce $MU$-modules $\tilde{L}_n^{(i)}$ with

$$\pi_\bullet(\tilde{L}_n^{(i)}) = Z[x_n]/x_n^i.$$

The $MU$-modules $\tilde{L}_n^{(i)}$ are obtained from $MU$ similar to $L_n^{(i)}$ but without killing the prime $p$. Then the rationalization arguments as in the proof of Lemma (10.3) with $\tilde{L}_n^{(i)}$ in place of the integral Eilenberg–MacLane spectrum show that our spectral sequence collapses.

Clearly there is no room for no multiplicative extensions in the spectral sequence and we have

$$\pi_\bullet({L_n}^{(i)} \wedge_{MU} {L_n}^{(i)}) = \Lambda_{Z/pZ}(z_1, z_3, \ldots) \otimes Z/pZ[x_n]/x_n^i.$$

Next we have the following spectral sequence:

$$\begin{aligned} E_2^{st} &= Ext^{st}_{\Lambda_{Z/pZ}(z_1,z_3,\ldots)\otimes Z/pZ[x_n]/x_n^i}(Z/pZ[x_n]/x_n^i, Z/pZ) \\ &\Longrightarrow THH^{s+t}_{MU}(L_n^{(i)}, HZ/p). \end{aligned}$$

We have

$$E_2^{st} = Ext^{st}_{\Lambda_{Z/pZ}(z_1,z_3,\ldots)}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4, \ldots],$$

where $deg\ y_{2m} = 2m$ for $m \neq n+1$ and $deg\ y_{2n+2} = 2ni+2$. This spectral sequence collapses since it is even and our lemma is proved.

*Remark* 11.3. We don't claim that the isomorphism of Lemma 11.2 is actually multiplicative. It is likely to be the case, however the proof of this seems to require the fact that the Yoneda product in the $Ext$-spectral sequence is associated with the composition product. This is claimed in [7], IV, Proposition 4.4 . Unfortunately the proof of this fact given in the cited reference is incorrect.

The next lemma computes the cohomology operations from $L_n$ to $HZ/p$ in the category of $MU$-modules.

LEMMA 11.4.

$$[L_n^{(i)}, HZ/p]^\bullet_{MU} = \Lambda^*_{Z/pZ}(z_1, z_3, \ldots),$$

*where $deg\ z_{2m+1} = 2m+1$ for $m \neq n$ and $deg\ z_{2n+1} = 2ni+1$ and $*$ denotes the dual vector space (over $Z/pZ$).*

*Proof.* Notice that both $HZ/p$ and $L_n^{(i)} \wedge_{MU} L_n^{(i)}$ are naturally left $L_n^{(i)}$-modules. We have the following weak equivalence of $MU$-modules:

$$F_{L_n^i}(L_n^{(i)} \wedge_{MU} L_n^{(i)}, HZ/p) = F_{MU}(L_n^{(i)}, HZ/p).$$

Now the statement follows from the computation of $L_n^{(i)} \wedge_{MU} L_n^{(i)}$ in Lemma 11.2.

The next step is to compute the module of derivations from $L_n^{(i)}$ into $HZ/p$ and its image under the forgetful map into $[L_n, HZ/p]_{MU}$. □

LEMMA 11.5.

$$\begin{aligned} Der^{\bullet}_{MU}(L^{(i)}_n, HZ/p) &= THH^{\bullet -1}_{MU}(L^{(i)}_n, HZ/p)/Z/pZ \\ &= \Sigma^{-1} Z/pZ[y_2, y_4, \ldots]/Z/pZ, \end{aligned}$$

*where as before deg* $y_{2m} = 2m$ *for* $m \neq n$ *and deg* $y_{2n+2} = 2ni + 2$*.*

*Proof.* We have the following homotopy cofibre sequence:

$$HZ/p \to THH_{MU}(L^{(i)}_n, HZ/p) \to \Sigma Der_{MU}(L^{(i)}_n, HZ/p)$$

from which the result follows. □

*Remark* 11.6. Using this result we can talk unambigously about derivations and singular extensions corresponding to Hochschild cohomology classes of $L^{(i)}_n$ with coefficients in $HZ/p$ having positive degree.

Consider now the element $\Sigma^{-1}(y_{2n+2}) \in Der^{2ni+1}_{MU}(L^{(i)}_n, HZ/p)$ and its image $l^*(\Sigma^{-1}(y_{2n+2})$ in the group $[L_n, HZ/p]_{MU}$, where $l$ is the canonical forgetful map

$$l\colon\ Der_{MU}(L^{(i)}_n, HZ/p) \to [L_n, HZ/p]_{MU}.$$

LEMMA 11.7.

$$l^*(\Sigma^{-1}(y_{2n+2})) = z^*_{2n+1} \in \Lambda^*_{Z/pZ}(z_1, z_3, \ldots) = [L^{(i)}_n, HZ/p]^{\bullet}_{MU}.$$

*Here in the coalgebra* $[L^{(i)}_n, HZ/p]^{\bullet}_{MU}$ $z^*_{2n+1}$ *denotes the primitive element dual to* $z_{2n+1}$*.*

*Proof.* This is analogous to the statement (ii) of Theorem 10.2 of and is proved similarly. First we find by comparing the spectral sequences for $Der^{\bullet}_{MU}(L^{(i)}_n, HZ/p)$ and $[L^{(i)}_n, HZ/p]_{MU}$ that the image of $\Sigma^{-1}(y_{2n})$ is as stated on the level of the $E_\infty$-terms. Next, this image is contained in the subspace of primitive cohomology operations from $L^{(i)}_n$ to $HZ/p$, that is, those operations which are derivations up to homotopy and therefore no higher filtration terms will appear. The lemma is proved.

We now begin to perform the inductive step – proving that $L^{(i+1)}_n$ is an $MU$-algebra. The cases $i = 1$ and $i > 1$ differ and will be considered separately.

(1)When $i = 1$ $L^{(i)}_n = HZ/p$. Take the element

$$y_{2n+2} \in Z/pZ[y_2, y_4, \ldots] = THH^{\bullet}_{MU}(L^{(i)}_n, HZ/p)$$

and consider the associated singular extension of $MU$-algebras

$$\Sigma^{2ni} HZ/p \to X \to L^{(i)}_n. \tag{33}$$

We want to prove that (33) is the $2n + 1$th Postnikov stage of $L_n$. For this it suffices to show that the $MU_\bullet$-module structure on $X_\bullet$ is given by the formula

$$X_\bullet = MU_\bullet/(p, x_1, x_2, \ldots, x^2_n, x_{n+1}, \ldots).$$

Notice that (33) gives rise to the following (algebraic) singular extension of $MU_\bullet$ - algebras:

$$\Sigma^{2n} Z/pZ \to X_\bullet \to Z/pZ. \tag{34}$$

If (34) is nontrivial (as an extension of $MU_\bullet$-algebras) then we are done – it is elementary to verify that there is only one (up to a scalar) nontrivial singular graded extension of $Z/pZ$ by $\Sigma^{2n} Z/pZ$, namely

$$\Sigma^{2n} Z/pZ \to Z/pZ[x_n]/x_n^2 \to Z/pZ$$

which is what we need. Therefore, we have to eliminate the possibility for (34) to be trivial, that is split. Suppose, on the contrary, that this is the case. We will deduce from it that (33) is split in the homotopy category of $MU$-modules, which would be a contradiction by Lemma (11.7).

Indeed, consider the spectral sequence

$$\begin{aligned} E_2^{st} &= Ext_{MU_\bullet}^{st}(Z/pZ, Z/pZ \oplus \Sigma^{2n} Z/pZ) \\ &= Ext_{MU_\bullet}^{st}(L_n^{(i)}, X_\bullet) \Rightarrow [L_n^{(i)}, X]_{MU}^\bullet. \end{aligned}$$

Then

$$E_2^{st} = (\Lambda_{Z/pZ}(z_1, z_3, \ldots) \oplus \Sigma^{2n} \Lambda_{Z/pZ}(z_1, z_3, \ldots))^*.$$

Comparing $E_2^{st}$ with the spectral sequence for $[X, X]_{MU}^\bullet$ we see that the former collapses. The element $1 \in E_2^{00}$ corresponds the $MU$-module map $L_n^{(i)} \to X$ splitting the cofibre sequence (33) and we are done.

(2) The case $i > 1$ is a little harder. We want to obtain the singular extension

$$\Sigma^{2ni} HZ/p \to Y \to L_n^{(i)},$$

where $Y_\bullet = MU_\bullet/(p, x_1, \ldots, x_{n-1}, x_n^{i+1}, x_{n+1}, \ldots)$.

However, the relevant subspace in $THH^\bullet(L_n^{(i)}, HZ/p)$ is two-dimensional, it is spanned by the elements

$$y_{2n+2}, y_{2ni+2} \in THH^\bullet(L_n^{(i)}, HZ/p) = Z/pZ[y_2, y_4, \ldots],$$

since $y_{2n+2}$ and $y_{2ni+2}$ both have degree $2ni + 2$. Take any nonzero element $y$ in this subspace, that is a linear combination of $y_{2n+2}$ and $y_{2ni+2}$. Associated to $y$ is a certain singular extension of $L_n^{(i)}$ by $\Sigma^{2ni} HZ/p$ which gives rise to the following algebraic singular extension of $MU$-algebras:

$$\Sigma^{2ni} Z/pZ \to Y_\bullet \to L_{n\bullet}^{(i)}. \tag{35}$$

Just as before, we show that (35) is nontrivial. But now the space of extensions like (35) is also two-dimensional. The basis in this space corresponds to two different $MU_\bullet$-algebra structures on $Y_\bullet$. One is given as

$$Y_\bullet = MU_\bullet/(p, x_1, \ldots, x_{n-1}, x_n^{i+1}, x_{n+1}, \ldots)$$

and the other as

$$Y_\bullet = MU_\bullet/(p, x_1, \dots, x_{n-1}, x_n^i, x_{n+1}, \dots, x_{2ni}^2, x_{2ni+2}, \dots).$$

Let us denote the corresponding algebraic extensions as $E_1$ and $E_2$ respectively. We need to choose the element $y$ which gives rise to the extension $E_1$. □

CLAIM. *The element* $y_{2ni+2} \in THH^{2ni+2}(L_n^{(i)}, HZ/p)$ *gives rise to the extension* $E_2$ *(or a multiple of it).* Indeed, consider the canonical map $L_n^{(i)} \to L_n^1 = HZ/p$ and the induced map on $THH$:

$$\begin{aligned} f\colon THH(L_n^{(1)}, HZ/p) &= Z/pZ[\overline{y}_2, \overline{y}_4, \dots] \to THH(L_n^{(i)}, HZ/p) \\ &= Z/pZ[y_2, y_4, \dots] \end{aligned}$$

(here we put bars over the polynomial generators of $THH^\bullet(L_n^{(1)}, HZ/p)$ to distinguish them from the corresponding elements in $THH^\bullet(L_n^{(i)}, HZ/p)$. Then $f(\overline{y}_{2ni+2}) = y_{2ni+2}$. Indeed, the map $f$ is so on the level of $E_2$-terms of the corresponding spectral sequences. Furthermore, since $f(\overline{y}_{2ni+2})$ is primitive no higher degree correction terms will appear.

The considerations as in the case $i = 1$ tell us that the element $\overline{y}_{2ni+2}$ gives rise to the algebraic extension (up to a scalar factor)

$$\Sigma^{2ni} Z/pZ \to \overline{Y}_\bullet \to L_{n\bullet}^{(1)} = Z/pZ, \tag{36}$$

where

$$\overline{Y}_\bullet = MU_\bullet/(p, x_1, \dots, x_{ni}, x_{ni+1}^2, \dots).$$

Therefore the extension corresponding to $\overline{y}_{2ni+2}$ is the extension induced from (36) by the augmentation map $Z/pZ[x_n]/x_n^i \to Z/p$ which is nothing but the extension $E_2$.

Next take an element $y \in THH^\bullet(L_n^{(i)}, HZ/p)$ which is a nonzero linear combination of $y_{2n+2}$ and $y_{2ni+2}$ and not proportional to $y_{2ni+2}$. The corresponding algebraic extension will have the form $\alpha E_1 + \beta E_2$, where $\alpha$ and $\beta$ are scalars with $\alpha \neq 0$ and '+' means the Baer sum of extensions. Altering $y$ by adding to it a multiple of $y_{2ni+2}$ and perhaps multiplying it by a scalar we obtain an element $\tilde{y}$ whose corresponding algebraic extension is $E_1$. (we leave it to the reader to check that addition of elements in $THH^\bullet(L_n^{(i)}, HZ/p)$ corresponds to the Baer sum of associated extensions.)

We proved that the element $\tilde{y}$ descends to the $2ni + 1$st $k$-invariant of $L_n$. With this our theorem is proved.

COROLLARY 11.8. *The spectrum* $k(n)$*, the* $n$*th connective Morava* $K$*-theory admits a structure of an* $MU$*-algebra.*

*Remark* 11.9. P. Goerss pointed out to the author that there are in fact uncountably many different $MU$-algebra structures on $L_n$. For example, when considering case 1) above we could have altered the element $y_{2n+2}$ by adding to it any decomposable element of degree $2n+2$ in the graded algebra $Z/pZ[y_2, y_4, \dots] = THH^{\bullet}_{MU}(L^i_n, HZ/p)$. In fact, to have a unique $MU$-algebra structure on $L_n$ we should fix an $A_\infty$-$MU_\bullet$-algebra structure on $L_{n\bullet}$ which amounts to infinitely many conditions.

*Remark* 11.10. Let $L$ be an $MU$-module obtained by killing an arbitrary subset of polynomial generators of $MU_\bullet$ and/or $p$. Then the exact same method as above and obvious induction show that $L$ admits a structure of an $MU$-algebra.

By comparison the results of [7] and [18] give only structures of $MU$-ring spectra on various $MU$-modules, which is an up to homotopy notion. It should be noted, however, that in contrast with the cited papers we never address the question of commutativity of our products, up to homotopy or otherwise.

## 12. Topological Hochschild Cohomology of Morava *K*-theories

Now we know that the spectra $k(n)$ are $MU$-algebras (hence so are the nonconnective Morava $K$-theories $K(n)$ as being Bousfield localizations of $k(n)$'s) and therefore, we can talk about their topological Hochschild cohomology. The computation of topological Hochschild (co)homology of spectra other than Eilenberg–MacLane was initiated by McClure and Staffeldt (cf. [15]). In this section we compute $THH_{MU}(K(n), K(n))$ and $THH_S(K(n), K(n))$ for odd primes modulo additive extensions. The possibly nontrivial additive extensions might arise because $K(n)$ is not a *commutative* $MU$-algebra. We hope to return to this problem in a future paper. Apart from this complication the results are similar to the calculation of topological Hochschild cohomology of the Eilenberg–MacLane spectrum $HZ/p$ in Section 9.

In the following propsotion $gr(?)$ stands for the associated graded of a filtered module.

PROPOSITION 12.1.
(i)

$$grTHH^{\bullet}_{MU}(K(n), K(n)) = K(n)_\bullet[[y_2, y_4, \dots \hat{y}_{2p^n}, \dots]];$$

$$grDer^{\bullet-1}_{MU}(K(n), K(n)) = K(n)_\bullet[[y_2, y_4, \dots \hat{y}_{2p^n}, \dots]]/K(n)_\bullet.$$

(ii)

$$[K(n), K(n)]^{\bullet}_{MU} = \Lambda_{K(n)_\bullet}(z_1, z_3, \dots, \hat{z}_{2p^n-1}, \dots).$$

*and the map*

$$Der^{\bullet}_{MU}(K(n), K(n)) \to [K(n), K(n)]^{\bullet}_{MU}$$

*sends the (representatives of) elements* $y_{2i} \in grDer^{2i-1}_{MU}(K(n), K(n))$ *to* $z_{2i-1} \in [K(n), K(n)]^{2i-1}_{MU}$.

(iii) *The forgetful map*

$$[K(n), K(n)]^\bullet_{MU} \longrightarrow [K(n), K(n)]^\bullet_S,$$

*maps the elements* $z_{2p^k-1}$ *to the operations* $Q_k \in [K(n), K(n)]^{2p^k-1}_S$ *(cf., for instance* [2]*) and all other z's go to zero.*

*Proof.* The proof is along the same lines as the proof of Theorem 10.2, and we will be somewhat sketchy therefore. First compute $\pi_\bullet(K(n) \wedge_{MU} K(n))$. We have our usual $Tor$-spectral sequence

$$\begin{aligned} Tor^{Z[x_1,x_2,\ldots]}(K(n)_\bullet, K(n)_\bullet) &= \Lambda_{K(n)_\bullet}(z_1, z_3, \ldots, \hat{z}_{2p^n-1}, \ldots) \\ &\Longrightarrow \pi_\bullet(K(n) \wedge_{MU} K(n)) \end{aligned}$$

which collapses so we have multiplicatively

$$\pi_\bullet K(n) \wedge_{MU} K(n) = \Lambda_{K(n)_\bullet}(z_1, z_3, \ldots, \hat{z}_{2p^n-1}, \ldots).$$

Next consider the spectral sequence

$$E_2^{ij} = Ext^{ij}_{\pi_\bullet K(n)\wedge_{MU} K(n)}(K(n)_\bullet, K(n)_\bullet) \Longrightarrow THH^{i+j}_{MU}(K(n), K(n)).$$

We have

$$E_2^{\bullet\bullet} = Ext^{\bullet\bullet}_{\Lambda_{K(n)_\bullet}(z_1,z_3,\ldots)}(K(n)_\bullet, K(n)_\bullet) = K(n)_\bullet[[y_2, y_4, \ldots]].$$

This spectral sequence collapses since it is even, and so part (i) of Proposition 12.1 is proved. Notice, however, that there could possibly be some nontrivial additive extensions. Indeed, $K(n)$ is not a commutative $MU$-algebra, and $THH_{MU}(K(n), K(n))$ need not be a $K(n)$-module. In particular we don't know whether $THH^\bullet_{MU}(K(n), K(n))$ is a $Z/pZ$-algebra.

Part (ii) is proved similarly to Theorem 10.2. An easy calculation shows that

$$[K(n), K(n)]_{MU} = \hat{\Lambda}_{K(n)_\bullet}(z_1, z_3, \ldots, \hat{z}_{2p^n-1}, \ldots),$$

where $\hat{\Lambda(?)}$ denotes the completed exterior algebra. Next we first show that the elements $y_{2i}$ can map to either $z_{2i-1}$ or zero and then considering the induced map on the corresponding spectral sequences we see, that the image of $y_{2i-1}$ is not zero. Part (ii) is proved.

Part (iii) is also proved similarly to the corresponding statement in Theorem 10.2. The only nontrivial thing is to prove that the element $z_{2p^i-1} \in [K(n), K(n)]^{2p^i-1}_{MU}$ considered as a $K(n)$-operation in the category of $S$-modules is not homotopic to zero. Denote $MU(i,n)$ the $MU$-module obtained by killing the prime $p$ and all polynomial generators in $MU_\bullet$ except for $x_{p^i-1}$ and $x_{p^i}$. Consider the following homotopy cofibre sequence of $MU$-modules:

$$\Sigma^{2p^i-2} MU(i,n) \xrightarrow{x_{p^i-1}} MU(i,n) \xrightarrow{p} K(n) \xrightarrow{\partial} \Sigma^{2p^i-1} MU(i,n).$$

Now the map of $MU$-modules

$$p \circ \partial\colon K(n) \to \Sigma^{2p^i-1}K(n)$$

is exactly the $K(n)$-operaton $Q_i$ and we are done. □

The next proposition describes topological Hochschild cohomology of $K(n)$ as an $S$-algebra.

PROPOSITION 12.2. *Let p be an odd prime. Then*

$$grTHH_S^\bullet(K(n), K(n)) = K(n)_\bullet[[\alpha_0, \alpha_2, \dots, \alpha_{n-1}]],$$

*where* $\alpha_k \in THH^{2p^k}(K(n), K(n))$. *Moreover under the forgetful map*

$$THH_{MU}^\bullet(K(n), K(n)) \to THH_S^\bullet(K(n), K(n)$$

*the image of* $y_{2p^i}$ *for* $i = 0, 1 \dots, n-1$ *is* $\alpha_i$ *and the rest of* $y_{2i}$ *get mapped to* 0 *(on the level of associated graded modules).*

*Proof.* We have a spectral sequence

$$E_2^{\bullet\bullet} = Ext^{\bullet\bullet}_{K(n)_\bullet K(n)}(K(n), K(n)) \Longrightarrow THH_S^\bullet(K(n), K(n))$$

and the calculation of Robinson in [17] shows that

$$E_2^{ij} = K(n)_\bullet[[\alpha_0, \alpha_2, \dots, \alpha_{n-1}]].$$

This spectral sequence converges strongly to its target and collapses at the $E_2$-term.

To see that the image of $y_{2i}$ is as stated consider the forgetful map

$$\pi_\bullet(K(n) \wedge_S K(n) \to \pi_\bullet(K(n) \wedge_{MU} K(n).$$

We know (from, e.g., [19]) that

$$\pi_\bullet(K(n) \wedge_S K(n) \cong K(n)_\bullet[t_1, t_2, \dots, ]/(v_n t_i^{p^n} - v_n^{p^i} t_i) \otimes \Lambda(a_0, a_1, \dots a_{n-1}),$$

where the degree of $a_i$ is $2p^i - 1$ and that of $t_i$ is $2(p^i - 1)$. It follows that the exterior generators $a_i$ correspond to $z_{2p^i-1} \in \pi_\bullet(K(n) \wedge_{MU} K(n))$. After that the statement about the image of $y_{2i}$ follows from comparing spectral sequences for $THH_S^\bullet(K(n), K(n))$ and $THH_{MU}^\bullet(K(n), K(n))$. □

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