MAT9580: Spectral Sequences

Chapter 11: The Adams Spectral Sequence

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Outline

Outline

The classical Adams spectral sequence

- The classical mod p Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

  aims to study the abelian group

$$[X, Y] = \mathrm{Ho}(Sp^{\mathbb{O}})(X, Y)$$

  of stable morphisms $f\colon X \to Y$.
- It takes as input the A-modules $H^*(X)$ and $H^*(Y)$ and the derived functors of Hom_A, where A denotes the mod p Steenrod algebra and $H = H\mathbb{F}_p$.
- It was introduced by Adams in [Ada58].

Homological formulation

- There is also a homological formulation

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

  of the Adams spectral sequence.
- It is defined in terms of the dual mod p Steenrod algebra A_* and the A_*-comodules $H_*(X)$ and $H_*(Y)$.
- This is a little more generally applicable than the cohomological version.

The Adams–Novikov spectral sequence

The generalization to the study of $[X, Y]$ by means of

- the E^*E-modules $E^*(X)$ and $E^*(Y)$, or
- the E_*E-comodules $E_*(X)$ and $E_*(Y)$,

for a suitable ring spectrum E, is known as

- the Adams–Novikov spectral sequence (principally for $E = MU$ [Nov67] and $E = BP$), or as
- the E-based Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{E_*E}^{s,t}(E_*(X), E_*(Y)) \Longrightarrow_s [X, Y_E^\wedge]_{t-s} \, .$$

Outline

The degree of a map

The degree $\deg(f)$ of a map $f\colon M^n \to N^n$ of closed, connected, oriented n-manifolds with fundamental classes $[M]$ and $[N]$ is the integer satisfying

$$f_*([M]) = \deg(f)[N]$$

in $H_n(N;\mathbb{Z}) \cong \mathbb{Z}$. The d-invariant is defined to detect similar information.

The homological d-invariant

- Let the (mod p homology) d-invariant be the homomorphism

$$\begin{aligned} d\colon [X, Y]_* &\longrightarrow \mathrm{Hom}^*_{A_*}(H_*(X), H_*(Y)) \\ [f] &\longmapsto f_* \,. \end{aligned}$$

- $[X, Y]_n = [S^n \wedge X, Y]$ denotes the degree n morphisms $X \to Y$ in the stable category.
- $\mathrm{Hom}^n_{A_*}(M, N) = \mathrm{Hom}_{A_*}(\Sigma^n M, N)$ denotes the A_*-comodule homomorphisms $M \to N$ of homological degree n, for (graded) A_*-comodules M and N.
- Hence d maps the homotopy class of $f\colon S^n \wedge X \to Y$ to the induced homomorphism $f_*\colon \Sigma^n H_*(X) \cong H_*(S^n \wedge X) \to H_*(Y)$.

The cohomological d-invariant

- For spectra X and Y, let the (mod p cohomology) d-invariant be the homomorphism

$$\begin{aligned} d\colon [X, Y]_* &\longrightarrow \mathrm{Hom}_A^*(H^*(Y), H^*(X)) \\ [f] &\longmapsto f^* . \end{aligned}$$

- $\mathrm{Hom}_A^n(M, N) = \mathrm{Hom}_A(M, \Sigma^n N)$ denotes the A-module homomorphisms $M \to N$ of cohomological degree $-n$, for (graded) A-modules M and N.
- Hence d maps the homotopy class of $f\colon S^n \wedge X \to Y$ to the induced homomorphism $f^*\colon H^*(Y) \to H^*(S^n \wedge X) \cong \Sigma^n H^*(X)$.

Maps from spheres

When $X = S$, the homology d-invariant specializes to a homomorphism

$$d \colon \pi_*(Y) \longrightarrow \mathrm{Hom}^*_{A_*}(\mathbb{F}_p, H_*(Y))\,,$$

while the cohomology d-invariant specializes to

$$d \colon \pi_*(Y) \longrightarrow \mathrm{Hom}^*_{A}(H^*(Y), \mathbb{F}_p)\,.$$

Dualization

Lemma

The cohomology d-invariant is obtained by dualization from the homology d-invariant, in the sense that it equals the composition

$$[X, Y]_* \xrightarrow{d} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(Y)) \xrightarrow{D} \mathrm{Hom}^*_{A}(H^*(Y), H^*(X)) .$$

The dualization homomorphism D is an isomorphism whenever $H_*(Y)$ is bounded below and of finite type over $\mathbb{F}_p$.

H-injective spectra

The d-invariant is particularly sensitive for maps to spectra of the form

$$W = H \wedge T\,,$$

where T is an arbitrary spectrum.
These are the H-injective spectra of [Mil81], and can be expressed as sums or products of suspensions of Eilenberg–MacLane spectra.

Lemma

Let $W_* = H_*(T)$. There are isomorphisms

$$H \wedge T \xleftarrow{\cong} \bigvee_n \Sigma^n H(W_n) \xrightarrow{\cong} \prod_n \Sigma^n H(W_n)$$

in the stable category, each inducing the identity map of W_n on π_n for $n \in \mathbb{Z}$.

Proof

- Choose a basis for $W_n = H_n(T)$ as an $\mathbb{F}_p$-vector space, and represent its elements by morphisms $f_\alpha \colon S^n \to H \wedge T$.
- Use the product $\mu \colon H \wedge H \to H$ to extend these to morphisms

$$\bar{f}_\alpha = (\mu \wedge 1)(1 \wedge f_\alpha) \colon \Sigma^n H \cong H \wedge S^n \to H \wedge T\,,$$

  and form their sum

$$g_n \colon \Sigma^n H(W_n) \cong \bigvee_\alpha \Sigma^n H \longrightarrow H \wedge T\,.$$

Proof (cont.)

- The sum
$$g\colon \bigvee_n \Sigma^n H(W_n) \longrightarrow H \wedge T$$
over $n \in \mathbb{Z}$ then induces the isomorphism $g_*\colon W_* \xrightarrow{\cong} H_*(T)$ in homotopy, hence is a stable equivalence.
- The canonical map
$$\bigvee_n \Sigma^n H(W_n) \longrightarrow \prod_n \Sigma^n H(W_n)$$
induces the identity of W_* on graded homotopy groups, hence is also a stable equivalence. □

A d-isomorphism

Proposition

In the case $W \cong H \wedge T$, the homological d-invariant

$$d\colon [X, W]_* \xrightarrow{\cong} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(W))$$

is an isomorphism.

If, furthermore, W is bounded below with mod p homology of finite type, then the cohomological d-invariant

$$d\colon [X, W]_* \xrightarrow{\cong} \mathrm{Hom}^*_{A}(H^*(W), H^*(X))$$

is an isomorphism.

Proof

- By the Künneth theorem, the homology smash product

$$\wedge\colon H_*(H) \otimes H_*(T) \xrightarrow{\cong} H_*(H \wedge T)$$

  is an isomorphism.
- Here $H_*(H) \cong A_*$, and the source has the diagonal A_*-coaction.
- By the untwisting isomorphism

$$A_* \otimes H_*(T) \cong A_* \otimes UH_*(T)$$

  this is isomorphic to the extended A_*-comodule on the underlying graded $\mathbb{F}_p$-vector space of $H_*(T)$.

Proof (cont.)

- By adjunction, there is an isomorphism

$$\mathrm{Hom}^*_{A_*}(H_*(X), A_* \otimes UH_*(T)) \cong \mathrm{Hom}^*(UH_*(X), UH_*(T))\,.$$

- Omitting the forgetful functor U from the notation, the composite homomorphism

$$[X, H \wedge T]_* \xrightarrow{d} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(H \wedge T)) \cong \mathrm{Hom}^*(H_*(X), H_*(T))$$

  defines a morphism of cohomology theories for (spaces or) spectra X, since $H_*(T)$ is automatically injective as a graded $\mathbb{F}_p$-vector space.
- Moreover, this morphism is an isomorphism for $X = S$. Hence it, and d, is an isomorphism for every spectrum X. □

Outline

Towers in $Sp^{\mathbb{O}}$

By a tower $Y_\star$ of (orthogonal) spectra we mean a diagram of the form

$$\dots \longrightarrow Y_{s+1} \xrightarrow{\alpha} Y_s \longrightarrow \dots \longrightarrow Y_1 \xrightarrow{\alpha} Y_0$$

in $Sp^{\mathbb{O}}$. We write

$$Y_{s,r} = C(\alpha^r \colon Y_{s+r} \to Y_s) = Y_s \cup CY_{s+r}$$

for the mapping cone of $\alpha^r \colon Y_{s+r} \to Y_s$, so that we have a homotopy cofiber sequence

$$Y_{s+r} \xrightarrow{\alpha^r} Y_s \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r}$$

for each $s \geq 0$ and $r \geq 0$.

Chains of homotopy cofiber sequences

In particular, when $r = 1$ we have a Puppe sequence

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1} \,,$$

for each $s \geq 0$. We often display the tower, and the homotopy cofiber sequences for $r = 1$, as follows.

$$\begin{array}{cccccccccc}
\cdots \longrightarrow & Y_{s+1} & \xrightarrow{\alpha} & Y_s & \longrightarrow \cdots \longrightarrow & Y_2 & \xrightarrow{\alpha} & Y_1 & \xrightarrow{\alpha} & Y_0 \\
 & & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & & & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\
 & & & Y_{s,1} & & & & Y_{1,1} & & Y_{0,1}
\end{array}$$

Here the dashed arrows refer to maps to the suspension of the indicated target, i.e., of degree -1.

Maps of towers

By a (strict) map of towers $\phi_\star \colon Y_\star \to Z_\star$ we mean a sequence of maps $\phi_s \colon Y_s \to Z_s$ such that each square

$$\begin{array}{ccc} Y_{s+1} & \xrightarrow{\alpha} & Y_s \\ \downarrow{\scriptstyle \phi_{s+1}} & & \downarrow{\scriptstyle \phi_s} \\ Z_{s+1} & \xrightarrow{\alpha} & Z_s \end{array}$$

commutes in $Sp^{\mathbb{O}}$.

There are then well-defined maps $\phi_{s,r} \colon Y_{s,r} \to Z_{s,r}$ for all $s \geq 0$ and $r \geq 0$, making the diagrams

$$\begin{array}{ccccccc} Y_{s+r} & \xrightarrow{\alpha^r} & Y_s & \longrightarrow & Y_{s,r} & \longrightarrow & \Sigma Y_{s+r} \\ \downarrow{\scriptstyle \phi_{s+r}} & & \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_{s,r}} & & \downarrow{\scriptstyle \Sigma\phi_{s+r}} \\ Z_{s+r} & \xrightarrow{\alpha^r} & Z_s & \longrightarrow & Z_{s,r} & \longrightarrow & \Sigma Z_{s+r} \end{array}$$

commute.

Resolutions in $\mathrm{Ho}(Sp^{\mathbb{O}})$

These chains have the following images in the stable category.

By a resolution $(Y_\star, Y_{\star,1})$ in the stable category, we mean a diagram of the form

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Y_{s+1} & \xrightarrow{\alpha} & Y_s & \longrightarrow \cdots \longrightarrow & Y_2 & \xrightarrow{\alpha} & Y_1 & \xrightarrow{\alpha} & Y_0 \\
 & & & \overset{\gamma}{\nwarrow} & \downarrow{\scriptstyle\beta} & & & \overset{\gamma}{\nwarrow} & \downarrow{\scriptstyle\beta} & \overset{\gamma}{\nwarrow} & \downarrow{\scriptstyle\beta} \\
 & & & & Y_{s,1} & & & & Y_{1,1} & & Y_{0,1}
\end{array}$$

in $\mathrm{Ho}(Sp^{\mathbb{O}})$, where each triangle

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1}$$

is distinguished.

Maps of resolutions

By a (weak) map of resolutions $\phi_\star\colon (Y_\star, Y_{\star,1}) \to (Z_\star, Z_{\star,1})$ we mean sequences of morphisms

$$\phi_s\colon\ Y_s \longrightarrow Z_s$$
$$\phi_{s,1}\colon\ Y_{s,1} \longrightarrow Z_{s,1}$$

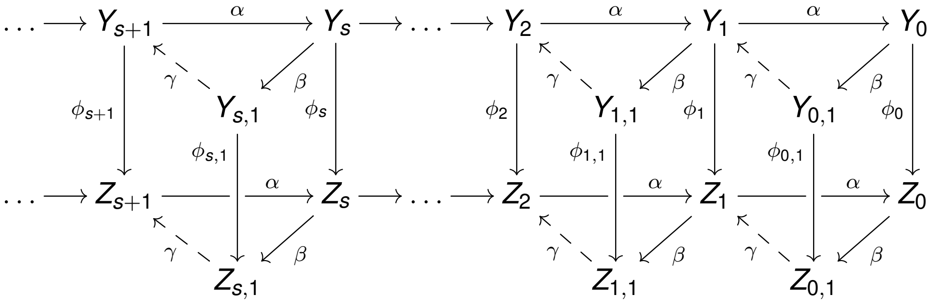
in $\mathrm{Ho}(Sp^{\mathbb{O}})$, such that the diagrams

$$\begin{array}{ccccccc}
Y_{s+1} & \xrightarrow{\alpha} & Y_s & \xrightarrow{\beta} & Y_{s,1} & \xrightarrow{\gamma} & \Sigma Y_{s+1} \\
\downarrow{\scriptstyle \phi_{s+1}} & & \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_{s,1}} & & \downarrow{\scriptstyle \Sigma\phi_{s+1}} \\
Z_{s+1} & \xrightarrow{\alpha} & Z_s & \xrightarrow{\beta} & Z_{s,1} & \xrightarrow{\gamma} & \Sigma Z_{s+1}
\end{array}$$

commute in the stable category.

Maps of resolutions (cont.)

Here is a different view of a map of resolutions.



The homotopy exact couple

The homotopy exact couple (A, E) associated to a spectrum X and a resolution $(Y_\star, Y_{\star,1})$ is the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow [X, Y_{s+2}]_* & \xrightarrow{\alpha} & [X, Y_{s+1}]_* & \xrightarrow{\alpha} & [X, Y_s]_* & \xrightarrow{\alpha} & [X, Y_{s-1}]_* & \longrightarrow \cdots \\ & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & \\ & & [X, Y_{s+1,1}]_* & & [X, Y_{s,1}]_* & & [X, Y_{s-1,1}]_* & , \end{array}$$

where

$$\cdots \to [X, Y_{s+1}]_n \xrightarrow{\alpha} [X, Y_s]_n \xrightarrow{\beta} [X, Y_{s,1}]_n \xrightarrow{\gamma} [X, Y_{s+1}]_{n-1} \to \cdots$$

is a long exact sequence for each $s \geq 0$. The bigraded abelian groups A and E are given by

$$A^{s,t} = [X, Y_s]_{t-s} = [S^{t-s} \wedge X, Y_s]$$
$$E^{s,t} = [X, Y_{s,1}]_{t-s} = [S^{t-s} \wedge X, Y_{s,1}] .$$

The homotopy spectral sequence

The homotopy spectral sequence

$$(E_r, d_r)_{r\geq 1}$$

associated to X and $(Y_\star, Y_{\star,1})$ is the spectral sequence associated to the homotopy exact couple, with

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} = [S^{t-s} \wedge X, Y_{s,1}]$$

and

$$d_1^{s,t} = \beta\gamma \colon E_1^{s,t} \longrightarrow E_1^{s+1,t}$$

for all $s \geq 0$ and $t \in \mathbb{Z}$. The d_r-differentials

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

then have (s,t)-bidegree $(r, r-1)$, for each $r \geq 1$.

Remark on grading

- We treat the total degree $t - s$ as a homological grading, so that the differentials have total degree -1, which means that the internal degree t is homological and the filtration degree s is cohomological.
- Since the filtration degree s interacts most directly with the term number r for the spectral sequence, we write E_r^s for the filtration s part of the E_r-term.
- It is then traditional to write $E_r^{s,t}$ for the internal degree t part of this graded group, even if $(E_r^s)_t$ might have been more consistent.

The target for convergence

Definition

The abutment of the homotopy exact couple of X and $Y_\star$ is the graded abelian group $[X, Y_0]_*$ with the descending, exhaustive filtration

$$\cdots \subset F^{s+1}[X, Y_0]_* \subset F^s[X, Y_0]_* \subset \cdots \subset F^0[X, Y_0]_* = [X, Y_0]_*$$

given by

$$F^s[X, Y_0]_* = \operatorname{im}([X, Y_s]_* \xrightarrow{\alpha^s} [X, Y_0]_*)$$

for $s \geq 0$.

Degreewise discrete convergence

- There are injective homomorphisms

$$\frac{F^s[X, Y_0]_n}{F^{s+1}[X, Y_0]_n} \overset{\zeta}{\rightarrowtail} E_\infty^{s,s+n}$$

  for all $s \geq 0$ and $n \in \mathbb{Z}$.
- If for each n the groups $[X, Y_s]_n$ vanish for all sufficiently large s, then the filtration $(F^s[X, Y_0]_*)_s$ is degreewise discrete, and the homotopy spectral sequence

$$E_r^{s,t} \Longrightarrow_s [X, Y_0]_{t-s}$$

  converges (strongly), so that each ζ is an isomorphism.

The case of homotopy groups

When $X = S$, the homotopy exact couple of $(Y_\star, Y_{\star,1})$ is the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \pi_*(Y_{s+2}) & \xrightarrow{\alpha} & \pi_*(Y_{s+1}) & \xrightarrow{\alpha} & \pi_*(Y_s) & \xrightarrow{\alpha} & \pi_*(Y_{s-1}) & \longrightarrow \cdots \\
& \nwarrow^{\gamma} & \downarrow{\scriptstyle\beta} & \nwarrow^{\gamma} & \downarrow{\scriptstyle\beta} & \nwarrow^{\gamma} & \downarrow{\scriptstyle\beta} & \\
& & \pi_*(Y_{s+1,1}) & & \pi_*(Y_{s,1}) & & \pi_*(Y_{s-1,1}) & ,
\end{array}$$

where

$$\cdots \to \pi_n(Y_{s+1}) \xrightarrow{\alpha} \pi_n(Y_s) \xrightarrow{\beta} \pi_n(Y_{s,1}) \xrightarrow{\gamma} \pi_{n-1}(Y_{s+1}) \to \cdots$$

is a long exact sequence for each $s \geq 0$.

The case of homotopy groups (cont.)

The bigraded abelian groups A and $E = E_1$ are given by

$$A^{s,t} = \pi_{t-s}(Y_s)$$
$$E^{s,t} = E_1^{s,t} = \pi_{t-s}(Y_{s,1})$$

and $d_1^{s,t} = \beta\gamma \colon E_1^{s,t} \to E_1^{s+1,t}$ equals the composite

$$\pi_{t-s}(Y_{s,1}) \xrightarrow{\gamma} \pi_{t-s-1}(Y_{s+1}) \xrightarrow{\beta} \pi_{t-s-1}(Y_{s+1,1})\,.$$

The case of homotopy groups (cont.)

Definition

The abutment of the homotopy exact couple of $Y_\star$ is the graded abelian group $\pi_*(Y_0)$ with the descending, exhaustive filtration given by

$$F^s\pi_*(Y_0) = \mathrm{im}(\pi_*(Y_s) \xrightarrow{\alpha^s} \pi_*(Y_0))$$

for $s \geq 0$.

The case of homotopy groups (cont.)

- There are injective homomorphisms

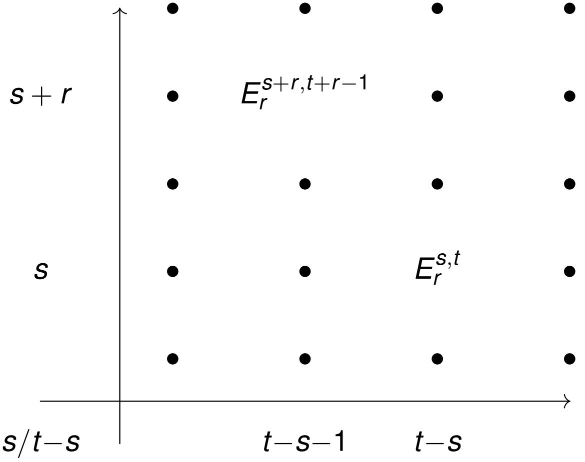
$$\frac{F^s\pi_n(Y_0)}{F^{s+1}\pi_n(Y_0)} \overset{\zeta}{\rightarrowtail} E_\infty^{s,s+n}$$

  for all $s \geq 0$ and $n \in \mathbb{Z}$.
- If the connectivity of the spectra Y_s increases to infinity with s, then the filtration $(F^s\pi_*(Y_0))_s$ is degreewise discrete and the homotopy spectral sequence

$$E_r^{s,t} \Longrightarrow_s \pi_{t-s}(Y_0)$$

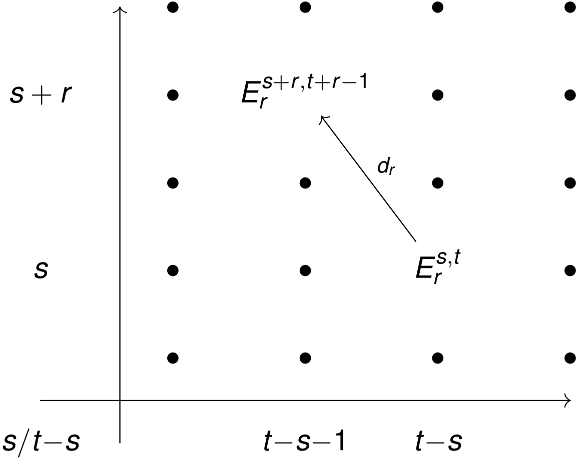
  converges (strongly), so that each ζ is an isomorphism.

Adams grading



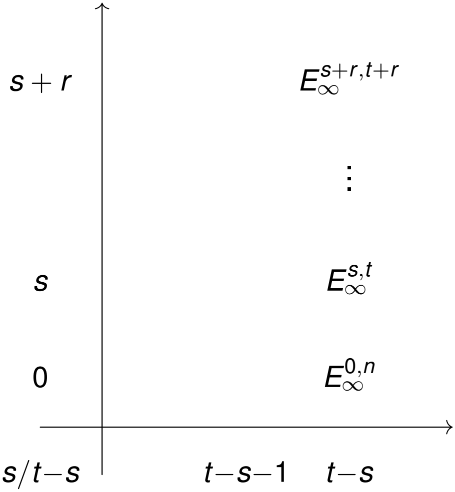
We use $(t-s, s)$-coordinates for homotopy spectral sequences, placing each group $E_r^{s,t}$ at the position with horizontal coordinate $t-s$ and vertical coordinate s.

Adams differentials



The d_r-differentials then have $(t-s, s)$-bigrading $(-1, r)$, mapping one column to the left and r rows up.

Vertical filtrations



The associated graded groups of the filtration $(F^s[X, Y_0]_n)_s$ lie in the column with $t - s = n$.

Tower of extensions

There is then a tower of short exact sequences

$$\begin{array}{ccc}
\cdots & & \\
\downarrow & & \\
F^{s+1}[X, Y_0]_n & \twoheadrightarrow & \dfrac{F^{s+1}[X, Y_0]_n}{F^{s+2}[X, Y_0]_n} \cong E_\infty^{s+1,s+1+n} \\
\downarrow & & \\
F^{s}[X, Y_0]_n & \twoheadrightarrow & \dfrac{F^{s}[X, Y_0]_n}{F^{s+1}[X, Y_0]_n} \cong E_\infty^{s,s+n} \\
\downarrow & & \\
F^{s-1}[X, Y_0]_n & \twoheadrightarrow & \dfrac{F^{s-1}[X, Y_0]_n}{F^{s}[X, Y_0]_n} \cong E_\infty^{s-1,s-1+n} \\
\downarrow & & \\
\cdots & &
\end{array}$$

mapping down and across, ending with an edge homomorphism induced by $\beta\colon Y_0 \to Y_{0,1}$.

$$[X, Y_0]_n \twoheadrightarrow \frac{[X, Y_0]_n}{F^1[X, Y_0]} \cong E_\infty^{0,n} \rightarrowtail E_1^{0,n} = [X, Y_{0,1}]_n$$

Cartan–Eilenberg systems

- We can associate an extended Cartan–Eilenberg system (π_*, η, ∂) to a spectrum X and a tower of spectra $Y_\star$.
- We set $Y_\infty = *$ and $Y_s = Y_0$ for $-\infty \leq s \leq 0$, and consider the graded groups

$$\pi_*(s, s+r) = [X, Y_{s,r}]_*$$

  for $r \geq 0$.
- The exact couple underlying this Cartan–Eilenberg system is the same as the homotopy exact couple of (the resolution in the stable category associated to) the tower of spectra.

Outline

Adams resolutions

Let Y be an (orthogonal) spectrum. A mod p Adams resolution of Y is a resolution

$$\begin{array}{ccccccccccc}
\dots \longrightarrow & Y_{s+1} & \xrightarrow{\alpha} & Y_s & \longrightarrow \dots \longrightarrow & Y_2 & \xrightarrow{\alpha} & Y_1 & \xrightarrow{\alpha} & Y_0 \\
& & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & & & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\
& & & Y_{s,1} & & & & Y_{1,1} & & Y_{0,1}
\end{array}$$

in $\mathrm{Ho}(Sp^{\mathbb{O}})$, with a stable equivalence $Y \sim Y_0$, such that

1. $Y_{s,1}$ is H-injective, and
2. $\alpha_*\colon H_*(Y_{s+1}) \to H_*(Y_s)$ is zero,

for each $s \geq 0$.

Remarks

- A spectrum W is H-injective if it has the form $H \wedge T$ for some spectrum T, which means that it is stably equivalent to a wedge sum of suspensions of Eilenberg–MacLane spectra.
- In view of the long exact sequences

$$\cdots \to H_*(Y_{s+1}) \xrightarrow{\alpha_*} H_*(Y_s) \xrightarrow{\beta_*} H_*(Y_{s,1}) \xrightarrow{\gamma_*} H_{*-1}(Y_{s+1}) \to \dots$$

$$\cdots \to H^{*-1}(Y_{s+1}) \xrightarrow{\gamma^*} H^*(Y_{s,1}) \xrightarrow{\beta^*} H^*(Y_s) \xrightarrow{\alpha^*} H^*(Y_{s+1}) \to \dots$$

  and the universal coefficient theorem, the condition that α_* is zero is equivalent to each of the following: that β_* is injective, γ_* is surjective, α^* is zero, β^* is surjective or γ^* is injective.

Adams towers

A mod p Adams tower for Y is a diagram

$$\ldots \longrightarrow Y_{s+1} \xrightarrow{\alpha} Y_s \longrightarrow \ldots \longrightarrow Y_1 \xrightarrow{\alpha} Y_0$$

in $Sp^{\mathbb{O}}$, with a stable equivalence $Y \sim Y_0$, such that the associated resolution (with $Y_{s,1} = C(\alpha\colon Y_{s+1} \to Y_s)$) is an Adams resolution.

The Adams spectral sequence

Definition

The mod p Adams spectral sequence for $[X, Y]_*$ is the homotopy spectral sequence

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

associated to a mod p Adams resolution $(Y_\star, Y_{\star,1})$ of Y.
In the case $X = S$ we write

$$E_1^{s,t}(Y) = \pi_{t-s}(Y_{s,1}) \Longrightarrow_s \pi_{t-s}(Y)$$

for this spectral sequence.

Remarks

- As stated, this depends on a choice of Adams resolution.
- We now show that Adams resolutions exist, that they are quasi-uniquely defined and natural, and that we can give algebraic descriptions of the E_1- and E_2-terms of the associated homotopy spectral sequences.
- In particular, the E_2-term will be seen to be independent of the choice of Adams resolution.

The mod p Hurewicz map and its cofiber

Definition

Let $H = H\mathbb{F}_p$, with unit map $h\colon S \to H$ and ring spectrum multiplication $\mu\colon H \wedge H \to H$, and let

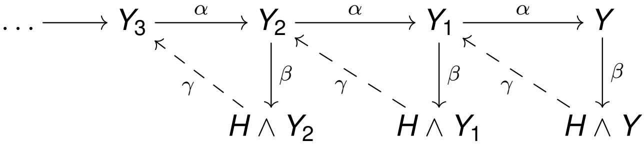
$$S \xrightarrow{h} H \xrightarrow{i} \bar{H} \xrightarrow{q} S^1$$

be the Puppe sequence generated by h, with $\bar{H} = Ch = H \cup_h CS$.

Here h induces the stable mod p Hurewicz homomorphism $\pi_*(X) \to H_*(X)$, hence the notation.

The canonical Adams resolution

The canonical Adams resolution of Y



is defined inductively by setting $Y_0 = Y$ and, for $s \geq 0$, letting

$$Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1} \xrightarrow{-\Sigma\alpha} \Sigma Y_s$$

be equal to

$$S \wedge Y_s \xrightarrow{h \wedge 1} H \wedge Y_s \xrightarrow{i \wedge 1} \bar{H} \wedge Y_s \xrightarrow{q \wedge 1} S^1 \wedge Y_s \,.$$

This implicitly defines $\alpha \colon Y_{s+1} \to Y_s$ in $\mathrm{Ho}(Sp^{\mathbb{O}})$, since Σ is an equivalence of categories.

The canonical Adams resolution (cont.)

- Equivalently,

$$\Sigma^s Y_s = \bar{H}^{\wedge s} \wedge Y$$
$$\Sigma^s Y_{s,1} = H \wedge \bar{H}^{\wedge s} \wedge Y$$

for each $s \geq 0$, with β, γ and $-\Sigma\alpha$ induced by h, i and q, respectively.

- The canonical Adams resolution of Y equals the canonical Adams resolution

$$\begin{array}{ccccccc}
\dots \longrightarrow \Sigma^{-3}\bar{H}^{\wedge 3} & \xrightarrow{\alpha} & \Sigma^{-2}\bar{H}^{\wedge 2} & \xrightarrow{\alpha} & \Sigma^{-1}\bar{H} & \xrightarrow{\alpha} & S \\
\nwarrow^{\gamma} & & \downarrow^{\beta} \quad \nwarrow^{\gamma} & & \downarrow^{\beta} \quad \nwarrow^{\gamma} & & \downarrow^{\beta} \\
 & & H \wedge \Sigma^{-2}\bar{H}^{\wedge 2} & & H \wedge \Sigma^{-1}\bar{H} & & H
\end{array}$$

of S, smashed with Y.

Existence of Adams resolutions

Lemma

- *The canonical Adams resolution* $(Y_\star, Y_{\star,1})$ *is an Adams resolution of* $Y = Y_0$*.*
- *If* Y *is bounded below with mod p homology of finite type, then each* $Y_{s,1}$ *is also bounded below with mod p homology of finite type.*

Proof

- Each spectrum $Y_{s,1} = H \wedge Y_s$ is H-injective by construction.
- Furthermore, each homomorphism

$$\beta_* \colon H_*(Y_s) \longrightarrow H_*(Y_{s,1})$$

is induced by the unit inclusion

$$H \wedge Y_s \cong H \wedge S \wedge Y_s \xrightarrow{1 \wedge h \wedge 1} H \wedge H \wedge Y_s \,,$$

which is split by the ring spectrum multiplication

$$H \wedge H \wedge Y_s \xrightarrow{\mu \wedge 1} H \wedge Y_s \,.$$

- Hence β_* is (split) injective and $\alpha_* = 0$.
- (This only uses that $\mu(1 \wedge h) = 1$ in the stable category.)

Proof (cont.)

- Note that H and $\bar{H}$ are bounded below, with $H_*(H) \cong A_*$ and $H_*(\bar{H}) \cong J(A_*)$ both being of finite type.
- It follows from the proposition on the connectivity of smash products that if Y is bounded below, then so is each $Y_{s,1}$.
- If Y furthermore has mod p homology of finite type, then the Künneth formula

$$H_*(Y_{s,1}) \cong A_* \otimes J(A_*)^{\otimes s} \otimes H_*(Y)$$

shows that each $Y_{s,1}$ also has this property. □

Homological variance

The homological image of an Adams resolution begins as follows.

$$
\begin{array}{ccccccc}
\cdots & & H_*(\Sigma^2 Y_2) & & H_*(\Sigma Y_1) & & H_*(Y) \\
 & \overset{\gamma_*}{\nwarrow} & \big\downarrow \beta_* & \overset{\gamma_*}{\nwarrow} & \big\downarrow \beta_* & \overset{\gamma_*}{\nwarrow} & \big\downarrow \beta_* \\
 & & H_*(\Sigma^2 Y_{2,1}) & & H_*(\Sigma Y_{1,1}) & & H_*(Y_{0,1})
\end{array}
$$

The Adams (E_1, d_1)-term

Proposition

Let

- *X be a spectrum and*
- *$(Y_\star, Y_{\star,1})$ be an Adams resolution of Y.*

The Adams spectral sequence

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

satisfies:

1. *The d-invariant*

$$d\colon E_1^{s,t} \xrightarrow{\cong} \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1}))$$

is an isomorphism.

The Adams (E_1, d_1)-term (cont.)

2. The diagram

$$\begin{array}{ccc} E_1^{s,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1})) \\ \downarrow d_1^{s,t} & & \downarrow \mathrm{Hom}(1, \beta_*\gamma_*) \\ E_1^{s+1,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^{s+1} Y_{s+1,1})) \end{array}$$

   commutes.

3. The A_*-comodule complex

$$\dots \leftarrow H_*(\Sigma^{s+1} Y_{s+1,1}) \xleftarrow{\beta_*\gamma_*} H_*(\Sigma^s Y_{s,1}) \xleftarrow{\beta_*\gamma_*} \dots$$
$$\dots \xleftarrow{\beta_*\gamma_*} H_*(\Sigma Y_{1,1}) \xleftarrow{\beta_*\gamma_*} H_*(Y_{0,1}) \xleftarrow{\beta_*} H_*(Y) \leftarrow 0$$

   is exact, and each $H_*(\Sigma^s Y_{s,1})$ is an extended A_*-comodule. Hence this is an injective A_*-comodule resolution of $H_*(Y)$.

Proof

Claim (1) follows from the proposition on the *d*-isomorphism, using the identification

$$\mathrm{Hom}_{A_*}^{t-s}(H_*(X), H_*(Y_{s,1})) \cong \mathrm{Hom}_{A_*}^{t}(H_*(X), H_*(\Sigma^s Y_{s,1}))\,,$$

since each $\Sigma^s Y_{s,1}$ is H-injective, i.e., has the form $H \wedge T$.

Proof (cont.)

Claim (2) follows from the commutative diagram below, since $d_1^{s,t} = \beta_* \gamma_*$.

$$\begin{array}{ccc}
E_1^{s,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}_{A_*}^t(H_*(X), H_*(\Sigma^s Y_{s,1})) \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \mathrm{Hom}(1,\gamma_*)} \\
[X, \Sigma^{s+1} Y_{s+1}]_t & \xrightarrow{d} & \mathrm{Hom}_{A_*}^t(H_*(X), H_*(\Sigma^{s+1} Y_{s+1})) \\
\downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \mathrm{Hom}(1,\beta_*)} \\
E_1^{s+1,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}_{A_*}^t(H_*(X), H_*(\Sigma^{s+1} Y_{s+1,1}))
\end{array}$$

Proof (cont.)

Claim (3) follows by splicing together the sequences

$$0 \leftarrow H_*(\Sigma^{s+1} Y_{s+1}) \xleftarrow{\gamma_*} H_*(\Sigma^s Y_{s,1}) \xleftarrow{\beta_*} H_*(\Sigma^s Y_s) \leftarrow 0$$

for all $s \geq 0$. These are all short exact, because $\alpha_* = 0$. Since each $\Sigma^s Y_{s,1}$ has the form $H \wedge T$ for some spectrum T, the Künneth formula and untwisting isomorphism show that

$$H_*(\Sigma^s Y_{s,1}) \cong H_*(H) \otimes H_*(T) \cong A_* \otimes H_*(T)$$

is an extended A_*-comodule, for each $s \geq 0$. □

The Adams E_2-term

Theorem

The Adams spectral sequence for $[X, Y]_*$ has E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y))\,,$$

which only depends on the A_*-comodules $H_*(X)$ and $H_*(Y)$. In the special case $X = S$, we write

$$E_2^{s,t}(Y) = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(Y))$$

for this E_2-term.

Proof

- Let $I_*^s = H_*(\Sigma^s Y_{s,1})$, $\delta^s = \beta_*\gamma_* \colon I_*^s \to I_*^{s+1}$ and $\eta = \beta_* \colon H_*(Y) \to I_*^0$.
- Then

$$\dots \leftarrow I_*^{s+1} \xleftarrow{\delta^s} I_*^s \longleftarrow \dots \longleftarrow I_*^1 \xleftarrow{\delta^0} I_*^0 \xleftarrow{\eta} H_*(Y) \leftarrow 0$$

  is an injective A_*-comodule resolution of $H_*(Y)$.
- Hence the cohomology groups of the cochain complex

$$\dots \leftarrow \mathrm{Hom}_{A_*}^t(H_*(X), I_*^{s+1}) \xleftarrow{\mathrm{Hom}(1,\delta^s)} \mathrm{Hom}_{A_*}^t(H_*(X), I_*^s) \xleftarrow{\mathrm{Hom}(1,\delta^{s-1})} \mathrm{Hom}_{A_*}^t(H_*(X), I_*^{s-1}) \leftarrow \dots$$

  are by definition the A_*-comodule Ext-groups $\mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y))$, for all $s \geq 0$ and t.

Proof (cont.)

- Since this cochain complex is isomorphic to

$$\ldots \leftarrow E_1^{s+1,t} \xleftarrow{d_1^{s,t}} E_1^{s,t} \xleftarrow{d_1^{s-1,t}} E_1^{s-1,t} \leftarrow \ldots ,$$

  these cohomology groups are precisely the components $E_2^{s,t}$ of the Adams spectral sequence E_2-term. □

Cohomological variance

The cohomological image of an Adams resolution begins as follows.

$$
\begin{array}{ccccccc}
\cdots & & H^*(\Sigma^2 Y_2) & & H^*(\Sigma Y_1) & & H^*(Y) \\
 & \overset{\gamma^*}{\searrow} & \uparrow \beta^* & \overset{\gamma^*}{\searrow} & \uparrow \beta^* & \overset{\gamma^*}{\searrow} & \uparrow \beta^* \\
 & & H^*(\Sigma^2 Y_{2,1}) & & H^*(\Sigma Y_{1,1}) & & H^*(Y_{0,1})
\end{array}
$$

The Adams (E_1, d_1)-term

Proposition

Let X and Y be spectra, and suppose that $(Y_\star, Y_{\star,1})$ is an Adams resolution of Y with each $Y_{s,1}$ bounded below and of finite type mod p. The Adams spectral sequence

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

satisfies

1. *The d-invariant*

   $$d\colon E_1^{s,t} \xrightarrow{\cong} \mathrm{Hom}_A^t(H^*(\Sigma^s Y_{s,1}), H^*(X))$$

   is an isomorphism.

The Adams (E_1, d_1)-term

2. The diagram

$$\begin{array}{ccc} E_1^{s,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}_A^t(H^*(\Sigma^s Y_{s,1}), H^*(X)) \\ {\scriptstyle d_1^{s,t}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\gamma^*\beta^*,1)} \\ E_1^{s+1,t} & \xrightarrow[\cong]{d} & \mathrm{Hom}_A^t(H^*(\Sigma^{s+1} Y_{s+1,1}), H^*(X)) \end{array}$$

   commutes.

3. The A-module complex

$$\cdots \to H^*(\Sigma^{s+1} Y_{s+1,1}) \xrightarrow{\gamma^*\beta^*} H^*(\Sigma^s Y_{s,1}) \xrightarrow{\gamma^*\beta^*} \dots$$
$$\dots \xrightarrow{\gamma^*\beta^*} H^*(\Sigma Y_{1,1}) \xrightarrow{\gamma^*\beta^*} H^*(Y_{0,1}) \xrightarrow{\beta^*} H^*(Y) \to 0$$

   is exact, and each $H^*(\Sigma^s Y_{s,1})$ is an extended A-module. Hence this is a projective A-module resolution of $H^*(Y)$.

The Adams E_2-term

Theorem

Let X and Y be spectra, with Y bounded below and of finite type mod p. The Adams spectral sequence for $[X, Y]_*$ has E_2-term

$$E_2^{s,t} \cong \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X)) \,,$$

which only depends on the A-modules $H^*(X)$ and $H^*(Y)$. In the special case $X = S$, we write

$$E_2^{s,t}(Y) = \mathrm{Ext}_A^{s,t}(H^*(Y), \mathbb{F}_p)$$

for this E_2-term.

Proof

- Let $P_s^* = H^*(\Sigma^s Y_{s,1})$, $\partial_s = \gamma^* \beta^* \colon P_s^* \to P_{s-1}^*$ and $\epsilon = \beta^* \colon P_0^* \to H^*(Y)$.
- Then

$$\cdots \to P_{s+1}^* \xrightarrow{\partial_{s+1}} P_s^* \xrightarrow{\partial_s} \dots \xrightarrow{\partial_2} P_1^* \xrightarrow{\partial_1} P_0^* \xrightarrow{\epsilon} H^*(Y) \to 0$$

is a projective A-module resolution of $H^*(Y)$.
- Hence the cohomology groups of the cochain complex

$$\dots \leftarrow \mathrm{Hom}_A^t(P_{s+1}^*, H^*(X)) \xleftarrow{\mathrm{Hom}(\partial_{s+1},1)} \mathrm{Hom}_A^t(P_s^*, H^*(X)) \xleftarrow{\mathrm{Hom}(\partial_s,1)} \mathrm{Hom}_A^t(P_{s-1}^*, H^*(X)) \leftarrow \dots$$

are by definition the A-module Ext-groups $\mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X))$, for all $s \geq 0$ and t.

Proof (cont.)

- Since this cochain complex is isomorphic to

$$\dots \leftarrow E_1^{s+1,t} \xleftarrow{d_1^{s,t}} E_1^{s,t} \xleftarrow{d_1^{s-1,t}} E_1^{s-1,t} \leftarrow \dots,$$

these cohomology groups are precisely the components $E_2^{s,t}$ of the Adams spectral sequence E_2-term. □

Filtration zero and the degree invariant

Lemma

The Adams spectral sequence edge homomorphism

$$[X, Y]_n \longrightarrow E_\infty^{0,n} \subset E_2^{0,n} = \mathrm{Hom}^n_{A_*}(H_*(X), H_*(Y))$$

is equal to the mod p homological d-invariant.

If Y is bounded below and of finite type mod p, then the edge homomorphism

$$[X, Y]_n \longrightarrow E_\infty^{0,n} \subset E_2^{0,n} = \mathrm{Hom}^n_{A}(H^*(Y), H^*(X))$$

is equal to the mod p cohomological d-invariant.

Proof

- The E_1-edge homomorphism $[X, Y]_* \to [X, Y_{0,1}]_* = E_1^{0,*}$ is induced by $\beta\colon Y \to Y_{0,1}$, and factors through the inclusion $E_2^{0,*} \subset E_1^{0,*}$ of the kernel of $\beta_*\gamma_*$.
- The lower row in the commutative diagram

$$\begin{array}{ccccccc}
[X, \Sigma Y_{1,1}]_* & \xleftarrow{\beta_*\gamma_*} & [X, Y_{0,1}]_* & \xleftarrow{\beta_*} & [X, Y]_* & & \\
d\downarrow\cong & & d\downarrow\cong & & \downarrow d & & \\
\mathrm{Hom}_{A_*}(H_*(X), I_*^1) & \xleftarrow{\delta_*^0} & \mathrm{Hom}_{A_*}(H_*(X), I_*^0) & \xleftarrow{\eta_*} & \mathrm{Hom}_{A_*}(H_*(X), H_*(Y)) & \leftarrow & 0
\end{array}$$

  is exact.
- Therefore the E_2-edge homomorphism corresponds under the middle isomorphism d to the right hand homomorphism d. □

The Hopf–Steenrod invariant

For $f \in [X, Y]_n$ satisfying $d(f) = 0$, then the mod p Hopf–Steenrod invariant

$$e(f) \in \mathrm{Ext}^1_{A_*}(H_*(\Sigma^{1+n}X), H_*(Y)) = \mathrm{Ext}^{1,1+n}_{A_*}(H_*(X), H_*(Y))$$

is defined to be the class of the A_*-comodule extension

$$0 \leftarrow H_*(\Sigma^{1+n}X) \xleftarrow{q_*} H_*(Cf) \xleftarrow{i_*} H_*(Y) \leftarrow 0\,.$$

If Y is bounded below and of finite type mod p, then this equals the class

$$e(f) \in \mathrm{Ext}^1_{A}(H^*(Y), H^*(\Sigma^{1+n}X)) = \mathrm{Ext}^{1,1+n}_{A}(H^*(Y), H^*(X))$$

of the A-module extension

$$0 \to H^*(\Sigma^{1+n}X) \xrightarrow{q^*} H^*(Cf) \xrightarrow{i^*} H^*(Y) \to 0\,.$$

Filtration one and the Hopf–Steenrod invariant

Proposition

The Adams spectral sequence near-edge homomorphism

$$F^1[X, Y]_n \longrightarrow E_\infty^{1,1+n} \subset E_2^{1,1+n} = \mathrm{Ext}_{A_*}^{1,1+n}(H_*(X), H_*(Y))$$

equals the mod p Hopf–Steenrod invariant, mapping f with $d(f) = 0$ to $e(f)$.

Proof

A morphism $f \in [X, Y]_n = [\Sigma^n X, Y]$ satisfies $d(f) = 0$ precisely if $\beta f = 0$, in which case there exist morphisms $f_1 \colon \Sigma^n X \to Y_1$ and $Cf \to Y_{0,1}$ making the following diagram commute.

$$
\begin{array}{ccccccccc}
\Sigma^n X & \xrightarrow{f} & Y & \xrightarrow{i} & Cf & \xrightarrow{q} & \Sigma^{1+n} X & & \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow{\scriptstyle \Sigma f_1} & & \\
Y_1 & \xrightarrow{\alpha} & Y_0 & \xrightarrow{\beta} & Y_{0,1} & \xrightarrow{\gamma} & \Sigma Y_1 & & \\
 & & & & & & & \searrow{\scriptstyle \Sigma\beta} & \\
 & & & & & & & & \Sigma Y_{1,1}
\end{array}
$$

Proof (cont.)

Passing to homology, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_*(Y) & \xrightarrow{i_*} & H_*(Cf) & \xrightarrow{q_*} & H_*(\Sigma^{1+n}X) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow \Sigma(\beta f_1)_* & & \\ 0 & \longrightarrow & H_*(Y) & \xrightarrow{\eta} & I_*^0 & \xrightarrow{\delta^0} & I_*^1 & \xrightarrow{\delta^1} & I_*^2 \end{array}$$

of A_*-comodules. Here the (well-defined) cohomology class

$$e(f) \in \mathrm{Ext}^1_{A_*}(H_*(\Sigma^{1+n}X), H_*(Y))$$

of

$$\Sigma(\beta f_1)_* \in \mathrm{Hom}_{A_*}(H_*(\Sigma^{1+n}X), I_*^1)$$

corresponds both to the A_*-comodule extension given by $H_*(Cf)$, and to the class in $E_\infty^{1,1+n} \subset E_2^{1,1+n}$ detecting f in the Adams spectral sequence. □

Outline

The Adams Spectral Sequence

- The d-invariant
- Towers of spectra
- Adams resolutions
- Comparison of resolutions
- The Adams filtration
- Ext over the Steenrod algebra
- Monoidal structure
- Composition pairings
- Products in Ext over A
- Adams differentials for S
- Homotopy of the sphere spectrum

Comparison of resolutions

Proposition

- *Let* $(Y_\star, Y_{\star,1})$ *and* $(Z_\star, Z_{\star,1})$ *be resolutions such that*
  1. $\alpha_* \colon H_*(Y_{s+1}) \to H_*(Y_s)$ *is zero and*
  2. $Z_{s,1}$ *is H-injective*

  for each $s \geq 0$*.*
- *Let* $\phi_0 \colon Y_0 \to Z_0$ *be any morphism in* $\mathrm{Ho}(Sp^{\mathbb{O}})$*.*

- *Then there exists a map of resolutions* $\phi_\star$ *that extends* ϕ_0*.*
- *Moreover, if* $\psi_\star$ *is a second map of resolutions extending* $\phi_0 = \psi_0$*, then* $\alpha\phi_s = \alpha\psi_s$ *for each* $s \geq 1$ *and* $\phi_s\alpha = \psi_s\alpha$ *for each* $s \geq 0$*.*

Proof

Suppose, by induction, that $\phi_0, \phi_{0,1}, \ldots, \phi_{s-1,1}$ and ϕ_s have been compatibly constructed. Consider the diagram below, with horizontal distinguished triangles.

$$
\begin{array}{ccccccc}
Y_{s+1} & & & & & & \\
 & \searrow^{\alpha} & & & & & \\
 & & Y_s & \xrightarrow{\beta} & Y_{s,1} & \xrightarrow{\gamma} & \Sigma Y_{s+1} \xrightarrow{-\Sigma\alpha} \Sigma Y_s \\
 & & \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_{s,1}} & & \downarrow{\scriptstyle \Sigma\phi_{s+1}} \qquad \downarrow{\scriptstyle \Sigma\phi_s} \\
 & & Z_s & \xrightarrow{\beta} & Z_{s,1} & \xrightarrow{\gamma} & \Sigma Z_{s+1} \xrightarrow{-\Sigma\alpha} \Sigma Z_s
\end{array}
$$

We claim that $\beta\phi_s\alpha\colon Y_{s+1} \to Z_{s,1}$ is zero in the stable category.

Proof (cont.)

The isomorphism

$$d\colon [Y_{s+1}, Z_{s,1}] \xrightarrow{\cong} \mathrm{Hom}_{A_*}(H_*(Y_{s+1}), H_*(Z_{s,1}))$$

maps $\beta\phi_s\alpha$ to zero because $\alpha_* = 0$. By exactness of the sequence

$$[\Sigma Y_{s+1}, Z_{s,1}] \xrightarrow{\gamma^*} [Y_{s,1}, Z_{s,1}] \xrightarrow{\beta^*} [Y_s, Z_{s,1}] \xrightarrow{\alpha^*} [Y_{s+1}, Z_{s,1}]$$

there exists an extension $\phi_{s,1}\colon Y_{s,1} \to Z_{s,1}$ of $\beta\phi_s$ over β, and by the fill-in axiom for triangulated categories there exists a morphism $\Sigma\phi_{s+1}\colon \Sigma Y_{s+1} \to \Sigma Z_{s+1}$ making all three squares commute, in $\mathrm{Ho}(Sp^{\mathbb{O}})$.

The proof of quasi-uniqueness is similar. □

Well-defined Adams E_2-spectral sequence

Theorem

- *Let X and Y be spectra.*
- *When viewed as an E_2-spectral sequence, the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow_s [X, Y]_{t-s}$$

does not depend on the choice of Adams resolution for Y.

Proof

By the previous proposition, for any morphism $\phi_0 \colon Y_0 \to Z_0$ and any two Adams resolutions $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ there is a map $\phi_\star \colon Y_\star \to Z_\star$ of resolutions that extends ϕ_0, and this induces a map

$$\begin{array}{ccccccccc}
\dots & \xleftarrow{\delta^1} & H_*(\Sigma Y_{1,1}) & \xleftarrow{\delta^0} & H_*(Y_{0,1}) & \xleftarrow{\eta} & H_*(Y_0) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{1,1*}} & & \downarrow{\scriptstyle \phi_{0,1*}} & & \downarrow{\scriptstyle \phi_{0*}} & & \\
\dots & \xleftarrow{\delta^1} & H_*(\Sigma Z_{1,1}) & \xleftarrow{\delta^0} & H_*(Z_{0,1}) & \xleftarrow{\eta} & H_*(Z_0) & \longleftarrow & 0
\end{array}$$

of injective A_*-comodule resolutions. When ϕ_0 is the composite of two stable equivalences $Y_0 \sim Y \sim Z_0$ then this chain map is a chain homotopy equivalence, well-defined up to chain homotopy, which induces a canonical isomorphism of Adams E_2-terms. □

Cohomological variant

Theorem

- *Let X and Y be spectra, with Y bounded below and of finite type mod p.*
- *When viewed as an E_2-spectral sequence, the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X)) \Longrightarrow_s [X, Y]_{t-s}$$

does not depend on the choice of Adams resolution for Y.

Proof

For any morphism $\phi_0 \colon Y_0 \to Z_0$ and any two Adams resolutions $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ there is a map $\phi_\star \colon Y_\star \to Z_\star$ of resolutions that extends ϕ_0, and this induces a map

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\partial_2} & H^*(\Sigma Y_{1,1}) & \xrightarrow{\partial_1} & H^*(Y_{0,1}) & \xrightarrow{\epsilon} & H^*(Y_0) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \phi^*_{1,1}} & & \uparrow{\scriptstyle \phi^*_{1,1}} & & \uparrow{\scriptstyle \phi^*} & & \\
\dots & \xrightarrow{\partial_2} & H^*(\Sigma Z_{1,1}) & \xrightarrow{\partial_1} & H^*(Z_{0,1}) & \xrightarrow{\epsilon} & H^*(Z_0) & \longrightarrow & 0
\end{array}$$

of projective A-module resolutions. When ϕ_0 is the composite of two stable equivalences $Y_0 \sim Y \sim Z_0$ then this chain map is a chain homotopy equivalence, well-defined up to chain homotopy, which induces a well-defined isomorphism of Adams E_2-terms. □

The homotopy limit of a tower

For any Adams resolution $(Y_\star, Y_{\star,1})$ of Y, let

$$Y_\infty = \operatorname*{holim}_s Y_s$$

be the sequential homotopy limit of the underlying tower

$$\cdots \to Y_{s+1} \xrightarrow{\alpha} Y_s \to \cdots \to Y_0 \,,$$

and write $\alpha^\infty \colon Y_\infty \to Y_0 \simeq Y$ for the evident map.

This homotopy limit, or microscope, can be defined as the homotopy equalizer of two maps

$$\prod_s Y_s \underset{\alpha}{\overset{1}{\rightrightarrows}} \prod_s Y_s \,,$$

where 1 denotes the identity map and α is the product of the maps $\alpha \colon Y_{s+1} \to Y_s$ for $s \geq 0$.

The Bousfield H-nilpotent completion

There is a natural short exact lim-Rlim sequence

$$0 \to \operatorname{Rlim}_s \pi_{n+1}(Y_s) \longrightarrow \pi_n(\operatorname{holim}_s Y_s) \longrightarrow \lim_s \pi_n(Y_s) \to 0$$

for each n. Hence $Y_\infty \sim *$ if and only if $\lim_s \pi_*(Y_s) = 0$ and $\operatorname{Rlim}_s \pi_*(Y_s) = 0$.

The Bousfield *H-nilpotent completion* $Y_H^\wedge$ of Y is defined so that there is a homotopy cofiber sequence

$$Y_\infty \xrightarrow{\alpha^\infty} Y \longrightarrow Y_H^\wedge \longrightarrow \Sigma Y_\infty\,,$$

and $Y_\infty \sim *$ if and only if $Y \to Y_H^\wedge$ is a stable equivalence.

Invariance of the homotopy limit

Proposition

The stable homotopy type of $Y_\infty = \operatorname{holim}_s Y_s$ *does not depend on the choice of Adams resolution* $(Y_\star, Y_{\star,1})$.

Proof.

- Let $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ be Adams resolutions of $Y_0 \sim Y \sim Z_0$.
- We have maps of resolutions $\phi_\star \colon Y_\star \to Z_\star$ and $\psi_\star \colon Z_\star \to Y_\star$, such that $\psi_s \phi_s \alpha = \alpha \colon Y_{s+1} \to Y_s$ and $\phi_s \psi_s \alpha = \alpha \colon Z_{s+1} \to Z_s$ in the stable category, for all $s \geq 0$.
- It follows that

$$(\pi_*(\phi_s))_s \colon (\pi_*(Y_s))_s \longrightarrow (\pi_*(Z_s))_s$$
$$(\pi_*(\psi_s))_s \colon (\pi_*(Z_s))_s \longrightarrow (\pi_*(Y_s))_s$$

are mutually inverse pro-isomorphisms of towers.

Proof (cont.)

- Hence they induce isomorphisms

$$\phi_* \colon \lim_s \pi_*(Y_s) \xrightarrow{\cong} \lim_s \pi_*(Z_s)$$
$$\phi_* \colon \operatorname{Rlim}_s \pi_*(Y_s) \xrightarrow{\cong} \operatorname{Rlim}_s \pi_*(Z_s)\,.$$

- The map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Rlim}_s \pi_{n+1}(Y_s) & \longrightarrow & \pi_n(Y_\infty) & \longrightarrow & \lim_s \pi_n(Y_s) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \\
0 & \longrightarrow & \operatorname{Rlim}_s \pi_{n+1}(Z_s) & \longrightarrow & \pi_n(Z_\infty) & \longrightarrow & \lim_s \pi_n(Z_s) & \longrightarrow & 0
\end{array}$$

of lim-Rlim short exact sequences then implies that

$$\phi_* \colon \pi_*(Y_\infty) \xrightarrow{\cong} \pi_*(Z_\infty)$$

is an isomorphism, so that Y_∞ and Z_∞ are stably equivalent. □

Conditional convergence, after Boardman

Definition

For any exact couple (A, E), let

$$A^{-\infty} = \operatorname*{colim}_s A^s$$
$$A^{\infty} = \lim_s A^s$$
$$RA^{\infty} = \operatorname*{Rlim}_s A^s \,.$$

We say that (A, E) converges conditionally to the colimit $A^{-\infty}$ if $A^{\infty} = 0$ and $RA^{\infty} = 0$ are both trivial.

If $E^s = 0$ for all $s < 0$, as is the case for each homotopy exact couple associated to an (Adams) resolution, then $A^0 \cong A^{-1} \cong \ldots \cong A^{-\infty}$.

Conditional convergence for the homotopy exact couple

Lemma

- *Let* $(Y_\star, Y_{\star,1})$ *be an Adams resolution of* Y*.*
- *The homotopy exact couple of* X *and* Y*, with* $A^{s,*} = [X, Y_s]_*$ *and* $E^{s,*} = [X, Y_{s,1}]_*$*, converges conditionally to* $[X, Y]_*$ *if and only if* $[X, Y_\infty]_* = 0$*.*
- *This holds for every* X *if (and only if)* $Y_\infty \sim *$*.*

Proof.

This follows from the short exact sequence

$$0 \to \operatorname{Rlim}_s [X, Y_s]_{n+1} \longrightarrow [X, \operatorname{holim}_s Y_s]_n \longrightarrow \lim_s [X, Y_s]_n \to 0 \,.$$

□

The RE_∞-term, after Boardman

Definition

For any spectral sequence (E_r, d_r), let

$$RE_\infty = \operatorname{Rlim}_r Z_r$$

denote the right derived E_∞-term, where

$$\cdots \subset Z_{r+1} \subset Z_r \subset \cdots \subset Z_1 = E_1 \,.$$

is the descending chain of r-th order cycles.

If $E_r^s = 0$ for $s < 0$, then $E_{r+1}^s \subset E_r^s$ for all $r > s$, and

$$\operatorname{Rlim}_r Z_r^s \xrightarrow{\cong} R\lim_r E_r^s \,,$$

which partially justifies the notation RE_∞ (rather than RZ_∞).

Vanishing criteria

- Consider a bidegree (s, t).
- If (E_r, d_r) stabilizes in that bidegree (so that $E_r^{s,t} = E_\infty^{s,t}$ for all sufficiently large r), then $RE_\infty^{s,t} = 0$.
- This is always the case of $E_r^{s,t}$ is finite for some r.
- Hence if (E_r, d_r) stabilizes in each bidegree, then $RE_\infty = 0$.
- More generally, it suffices that $(E_r^{s,t})_r$ satisfies the Mittag–Leffler condition in each bidegree.

Complete Hausdorff filtrations

Definition

A filtration

$$\cdots \subset F^{s+1}G \subset F^sG \subset \cdots \subset G$$

of (graded) abelian groups is Hausdorff if

$$\lim_s F^sG = 0$$

and it is complete if

$$\operatorname{Rlim}_s F^sG = 0\,.$$

Lemma

A filtration $(F^sG)_s$ is Hausdorff and complete if and only if the canonical map

$$G \xrightarrow{\cong} \lim_s \frac{G}{F^sG}$$

is an isomorphism.

Strong convergence

Definition

A spectral sequence (E_r, d_r) converges strongly to a filtration $(F^sG)_s$ of a (graded) abelian group G if there are isomorphisms

$$\zeta\colon \frac{F^sG}{F^{s+1}G} \xrightarrow{\cong} E^s_\infty$$

for each s, and the filtration is exhaustive, Hausdorff and complete.

If the spectral sequence arises from an exact couple, we always assume that the isomorphism ζ is the preferred homomorphism introduced earlier.

Reconstruction of the abutment

Strong convergence, together with solutions to all of the finite extension problems

$$0 \to E_\infty^s \longrightarrow \frac{F^a G}{F^{s+1} G} \longrightarrow \frac{F^a G}{F^s G} \to 0$$

is precisely sufficient to reconstruct the (graded) abelian group G by passage to algebraic colimits and limits.

Lemma

If $(F^s G)_s$ is complete Hausdorff and exhaustive, then there are isomorphisms

$$\operatorname*{colim}_a \lim_s \frac{F^a G}{F^s G} \cong G \cong \lim_s \operatorname*{colim}_a \frac{F^a G}{F^s G} .$$

A criterion for strong convergence

Theorem ([Boa99])

Let (A, E) be an exact couple with $E^s = 0$ for $s < 0$, so that $A^0 \cong A^{-\infty}$. Any two of the following conditions implies the third.

1. *The exact couple converges conditionally to the colimit A^0.*
2. $RE_\infty = 0$.
3. *The spectral sequence converges strongly to A^0, with the filtration $F^s A^0 = \mathrm{im}(\alpha^s \colon A^s \to A^0)$.*

Hence, for a conditionally convergent Adams spectral sequence, the vanishing of RE_∞ is equivalent to strong convergence.

Outline

Adams filtration

Definition

- The abutment of the Adams spectral sequence for X and Y with Adams resolution $(Y_\star, Y_{\star,1})$, is $[X, Y]_*$, with the decreasing, exhaustive filtration given by

$$F^s[X, Y]_* = \mathrm{im}(\alpha^s \colon [X, Y_s]_* \to [X, Y]_*)\,.$$

- We call this the Adams filtration of $[X, Y]_*$.
- The elements of $F^s[X, Y]_*$ have Adams filtration $\geq s$.
- The elements of $F^s[X, Y]_* \setminus F^{s+1}[X, Y]_*$ have Adams filtration exactly s.

Independence of resolution

Lemma

The Adams filtration is independent of the choice of Adams resolution.

Proof.

For any other choice of Adams resolution $(Z_\star, Z_{\star,1})$ we have a map of resolutions $\phi_*\colon Y_\star \to Z_\star$ making the diagram

$$\begin{array}{ccc} Y_s & \xrightarrow{\alpha^s} & Y \\ {\scriptstyle \phi_s}\downarrow & & \downarrow{\scriptstyle =} \\ Z_s & \xrightarrow{\alpha^s} & Y \end{array}$$

commute, so

$$\operatorname{im}(\alpha^s\colon [X, Y_s]_* \to [X, Y]_*) \subset \operatorname{im}(\alpha^s\colon [X, Z_s]_* \to [X, Y]_*)\,.$$

Reversing the roles of the two resolutions gives the opposite inclusion. Hence the two image filtrations agree. □

Maps that induce zero in mod p (co-)homology

The Adams filtration can be characterized in terms of maps that induce zero in mod p (co-)homology.

Proposition

A morphism $f \in [X, Y]_n$ has Adams filtration $\geq s$ if and only if it can be factored as a composite $f_1 \circ \cdots \circ f_s$ of s morphisms

$$\Sigma^n X = X_s \xrightarrow{f_s} X_{s-1} \xrightarrow{f_{s-1}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = Y \,,$$

each of which (for $1 \leq i \leq s$) induces the zero homomorphism $f_{i*} \colon H_*(X_i) \to H_*(X_{i-1})$ in mod p homology.

Proof

- If $f = \alpha^s g$ with $g\colon \Sigma^n X \to Y_s$, then f admits the factorization

$$\Sigma^n X = X_s \xrightarrow{\alpha g} Y_{s-1} \xrightarrow{\alpha} \dots \xrightarrow{\alpha} Y_1 \xrightarrow{\alpha} Y_0 = Y$$

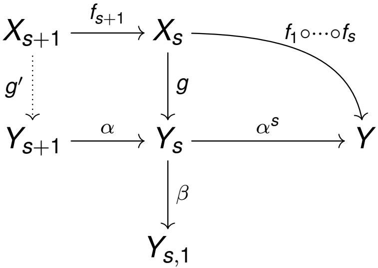
  where $(\alpha g)_* = 0$ and $\alpha_* = 0$ (in mod p homology) in each case.

- Conversely, if $f = f_1 \circ \cdots \circ f_{s+1}$ with $f_{i*} = 0$ for each i, then we may inductively assume that $f_1 \circ \cdots \circ f_s\colon X_s \to Y$ factors as

$$f_1 \circ \cdots \circ f_s = \alpha^s \circ g$$

  for some $g\colon X_s \to Y_s$.

Proof (cont.)



Then $gf_{s+1} \colon X_{s+1} \to Y_s$ followed by β induces zero in homology, and has target the H-injective spectrum $Y_{s,1}$, hence is null-homotopic. By exactness of the sequence

$$[X_{s+1}, Y_{s+1}] \xrightarrow{\alpha_*} [X_{s+1}, Y_s] \xrightarrow{\beta_*} [X_{s+1}, Y_{s,1}]$$

it follows that $gf_{s+1} = \alpha g'$ for some $g' \colon X_{s+1} \to Y_{s+1}$, which proves that f has Adams filtration $\geq s + 1$. $\square$

A tower of Moore spaces

Definition

Let $(S^1/p^v)_{v\geq 1}$ be the tower of Moore spaces given by the Puppe sequences

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow p & & \downarrow = & & \downarrow r & & \downarrow p \\
S^1 & \xrightarrow{p^3} & S^1 & \xrightarrow{i} & S^1/p^3 & \xrightarrow{q} & S^2 \\
\downarrow p & & \downarrow = & & \downarrow r & & \downarrow p \\
S^1 & \xrightarrow{p^2} & S^1 & \xrightarrow{i} & S^1/p^2 & \xrightarrow{q} & S^2 \\
\downarrow p & & \downarrow = & & \downarrow r & & \downarrow p \\
S^1 & \xrightarrow{p} & S^1 & \xrightarrow{i} & S^1/p & \xrightarrow{q} & S^2
\end{array}$$

and let $(S/p^v)_{v\geq 1}$ be its desuspension, with $S/p^v = F_1 S^1/p^v$.

Completion of spectra

- The *p*-completion of a spectrum Y is the sequential homotopy limit

$$Y_p^\wedge = \operatorname*{holim}_v Y \wedge S/p^v$$

of the tower

$$\ldots \longrightarrow Y \wedge S/p^3 \xrightarrow{1\wedge r} Y \wedge S/p^2 \xrightarrow{1\wedge r} Y \wedge S/p\,.$$

- Let $\kappa\colon Y \to Y_p^\wedge$ denote the completion map, induced by the compatible maps $i\colon S \to S/p^v$.

Higher Bockstein maps

- We use the abbreviation

$$Y/p^v = Y \wedge S/p^v$$

  for the homotopy cofiber of $p^v \colon Y \to Y$.
- There is a distinguished triangle

$$Y/p \xrightarrow{e} Y/p^{v+1} \xrightarrow{r} Y/p^v \xrightarrow{\beta_v} \Sigma Y/p$$

  for each v, where β_v is the v-th order Bockstein map.

Completion of abelian groups

- For an abelian group G, let

$$G_p^\wedge = \lim_v G/p^v$$

  denote its *p-completion*.
- In particular, let $\mathbb{Z}_p = \mathbb{Z}_p^\wedge$ denote the ring of *p-adic integers*.
- We say that G is *p-complete* if the canonical homomorphism

$$\kappa\colon\ G \longrightarrow G_p^\wedge$$

  is an isomorphism.
- If G is finite, then κ is the surjection mapping all torsion of order prime to p to zero, which maps the p-Sylow subgroup of G isomorphically to $G_p^\wedge$.

Completion of spectra of finite type

Lemma

If Y has finite type, then there are natural isomorphisms

$$\pi_*(Y_p^\wedge) \xleftrightarrow{\cong} \pi_*(Y)_p^\wedge = \lim_v \pi_*(Y)/p^v \xleftarrow{\cong} \pi_*(Y) \otimes \mathbb{Z}_p \,.$$

If, furthermore, $\pi_*(Y)$ is p-complete in each degree, then $\kappa \colon Y \to Y_p^\wedge$ is a stable equivalence.

Proof

- Let ${}_{p^v}G = \ker(p^v\colon G \to G)$.
- The tower of universal coefficient short exact sequences

$$0 \to \pi_n(Y)/p^v \longrightarrow \pi_n(Y/p^v) \longrightarrow {}_{p^v}\pi_{n-1}(Y) \to 0$$

  induces an exact sequence

$$0 \to \pi_n(Y)^\wedge_p \longrightarrow \lim_v \pi_n(Y/p^v) \longrightarrow \lim_v {}_{p^v}\pi_{n-1}(Y)\,.$$

- The right hand limit is trivial because $\pi_{n-1}(Y)$ is finitely generated.
- Hence the left hand arrow is an isomorphism.

Proof (cont.)

- In the Milnor short exact sequence

$$0 \to \operatorname{Rlim}_{v} \pi_{n+1}(Y/p^v) \longrightarrow \pi_n(Y_p^\wedge) \longrightarrow \lim_{v} \pi_n(Y/p^v) \to 0$$

  each group $\pi_{n+1}(Y/p^v)$ is finite, because $\pi_n(Y)$ and $\pi_{n+1}(Y)$ are finitely generated, so the Rlim term vanishes and the right hand arrow is an isomorphism.
- For any finitely generated abelian group G the canonical map

$$G \otimes \mathbb{Z}_p \longrightarrow \lim_{v} G \otimes \mathbb{Z}/p^v \cong \lim_{v} G/p^v$$

  is an isomorphism, since this holds for each cyclic group G.
- (The left hand side commutes with sums, the right hand side commutes with products, and finite sums and finite products agree.) □

Completion is a mod p equivalence

Proposition

There are stable equivalences

$$\kappa\colon Y/p \xrightarrow{\simeq} (Y/p)_p^\wedge$$
$$\kappa/p\colon Y/p \xrightarrow{\simeq} (Y_p^\wedge)/p$$

and an isomorphism

$$\kappa_*\colon H_*(Y) \xrightarrow{\cong} H_*(Y_p^\wedge)$$

in mod p homology (and cohomology).

Proof

- There is a homotopy (co-)fiber sequence

$$F(S[1/p], Y) \longrightarrow Y \xrightarrow{\kappa} Y_p^\wedge$$

where $S[1/p]$ is the homotopy colimit (= telescope) of the sequence

$$S \xrightarrow{p} S \xrightarrow{p} S \xrightarrow{p} S \to \dots .$$

- Since $p\colon S[1/p] \to S[1/p]$ is a stable equivalence, it follows that $F(S[1/p], Y/p) \simeq F(S[1/p], Y)/p \simeq *$, so that $\kappa\colon Y/p \to (Y/p)_p^\wedge$ and $\kappa/p\colon Y/p \to (Y_p^\wedge)/p$ are stable equivalences.
- Applying integral homology to the second of these, and noting that $H\mathbb{Z} \wedge S/p \simeq H$, we deduce that $\kappa_*\colon H_*(Y) \to H_*(Y_p^\wedge)$ is an isomorphism. □

The integral Hurewicz map and its cofiber

- Let
$$S \xrightarrow{h} H\mathbb{Z} \xrightarrow{i} \overline{H\mathbb{Z}} \xrightarrow{q} S^1$$
be the Puppe sequence generated by the unit map $h\colon S \to H\mathbb{Z}$ of the integral Eilenberg–MacLane ring spectrum.
- Note that h is 1-connected (= 2-connective).
- Hence $\overline{H\mathbb{Z}}$ is also 1-connected (= 2-connective).

The canonical $H\mathbb{Z}$-Adams resolution

For each spectrum Y let

$$\begin{array}{ccccccc}
\cdots \longrightarrow Y'_3 & \xrightarrow{\alpha} & Y'_2 & \xrightarrow{\alpha} & Y'_1 & \xrightarrow{\alpha} & Y'_0 \\
 & {}^{\gamma}\nwarrow & \downarrow \beta & {}^{\gamma}\nwarrow & \downarrow \beta & {}^{\gamma}\nwarrow & \downarrow \beta \\
 & & Y'_{2,1} & & Y'_{1,1} & & Y'_{0,1}
\end{array}$$

be the canonical $H\mathbb{Z}$-Adams resolution of Y, with $Y'_0 = Y$ and

$$Y'_s \xrightarrow{\beta} Y'_{s,1} \xrightarrow{\gamma} Y'_{s+1} \xrightarrow{-\Sigma\alpha} S^1 \wedge Y'_s$$

equal to

$$S \wedge Y'_s \xrightarrow{h \wedge 1} H\mathbb{Z} \wedge Y'_s \xrightarrow{i \wedge 1} \overline{H\mathbb{Z}} \wedge Y'_s \xrightarrow{q \wedge 1} S^1 \wedge Y'_s .$$

The canonical $H\mathbb{Z}$-Adams resolution (cont.)

- Hence

$$\Sigma^s Y'_s = \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$$
$$\Sigma^s Y'_{s,1} = H\mathbb{Z} \wedge \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$$

  for all $s \geq 0$.
- Note that $(Y'_\star, Y'_{\star,1})$ is generally not a mod p Adams resolution, since the spectra $Y'_{s,1}$ are not of the form $H \wedge T$.

Degreewise discrete convergence for Y/p

Proposition

- *Let Y be any spectrum. The canonical $H\mathbb{Z}$-Adams resolution $((Y/p)'_\star, (Y/p)'_{\star,1})$ of Y/p is a mod p Adams resolution.*
- *If Y/p is ℓ-connective, then $(Y/p)'_s$ is $(s+\ell)$-connective for each $s \geq 0$, so the homotopy exact couple*

$$\begin{array}{ccccccc}
\dots \longrightarrow \pi_*((Y/p)'_2) & \xrightarrow{\alpha} & \pi_*((Y/p)'_1) & \xrightarrow{\alpha} & \pi_*(Y/p) \\
& {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\
& & \pi_*((Y/p)'_{1,1}) & & \pi_*((Y/p)'_{0,1})
\end{array}$$

is degreewise discrete, the Adams E_1-term is concentrated in the region $t-s \geq s+\ell$, and

$$E_2^{s,t} = \operatorname{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(Y/p)) \Longrightarrow_s \pi_{t-s}(Y/p)$$

is strongly convergent.

Proof

- Each spectrum

$$\Sigma^s (Y/p)'_{s,1} = H\mathbb{Z} \wedge \overline{H\mathbb{Z}}^{\wedge s} \wedge Y/p$$

  has the form $H \wedge T$ with $T = \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$, in view of the stable equivalence $H\mathbb{Z} \wedge S/p \simeq H$.
- Each homomorphism

$$\beta_* \colon H_*((Y/p)'_s) \longrightarrow H_*((Y/p)'_{s,1})$$

  is induced by the unit inclusion

$$H \wedge (Y/p)'_s \cong H \wedge S \wedge (Y/p)'_s \xrightarrow{1 \wedge h \wedge 1} H \wedge H\mathbb{Z} \wedge (Y/p)'_s \,,$$

  which is split by the right module action

$$H \wedge H\mathbb{Z} \wedge (Y/p)'_s \xrightarrow{\rho \wedge 1} H \wedge (Y/p)'_s$$

  of $H\mathbb{Z}$ upon H.

Proof (cont.)

- Suppose that Y/p is ℓ-connective.
- Since $\overline{H\mathbb{Z}}$ is 2-connective, the smash products

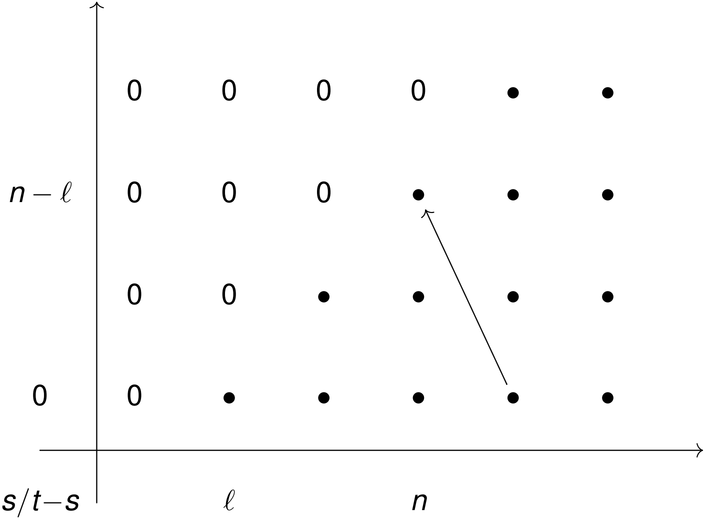
$$\Sigma^s (Y/p)'_s = (\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$$
$$\Sigma^s (Y/p)'_{s,1} = H\mathbb{Z} \wedge (\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$$

  are $(2s + \ell)$-connective.
- Hence

$$A^{s,t} = \pi_{t-s}((Y/p)'_s)$$
$$E^{s,t} = \pi_{t-s}((Y/p)'_{s,1})$$

  are trivial for $t - s < s + \ell$, which implies that the terms of the Adams spectral sequence are concentrated on and below the line $t - s = s + \ell$ in the $(t - s, s)$-plane.

The region $t - s \geq s + \ell$



Proof (cont.)

- Hence the Adams spectral sequence converges (strongly) to a degreewise discrete filtration of $\pi_*(Y/p)$.
- In particular, there are canonical isomorphisms

$$E_\infty^{s,t} \cong \frac{F^s \pi_{t-s}(Y/p)}{F^{s+1}\pi_{t-s}(Y/p)}$$

for all $s \geq 0$ and t, where

$$0 = F^{n-\ell+1}\pi_n(Y/p) \subset F^{n-\ell}\pi_n(Y/p) \subset \cdots \subset F^1\pi_n(Y/p) \subset \pi_n(Y/p)$$

for all $n \geq \ell$. □

Vanishing homotopy limit

Corollary

If Y/p is bounded below, then $(Y/p)_\infty \sim *$

Proof.

- We can calculate $(Y/p)_\infty$ using the canonical $H\mathbb{Z}$-Adams resolution of Y/p.
- If Y/p is ℓ-connective, then $\pi_n((Y/p)'_s) = 0$ for $n < s + \ell$, so $\lim_s \pi_n((Y/p)'_s) = 0$ and $\mathrm{Rlim}_s \pi_{n+1}((Y/p)'_s) = 0$.
- Together these imply that $\pi_n((Y/p)_\infty) = 0$ for all n.

□

Conditional convergence to $[X, Y_p^\wedge]_*$

Theorem

If Y/p is bounded below, then the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y_p^\wedge)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

for X and $Y_p^\wedge$ is conditionally convergent (to the achieved colimit).

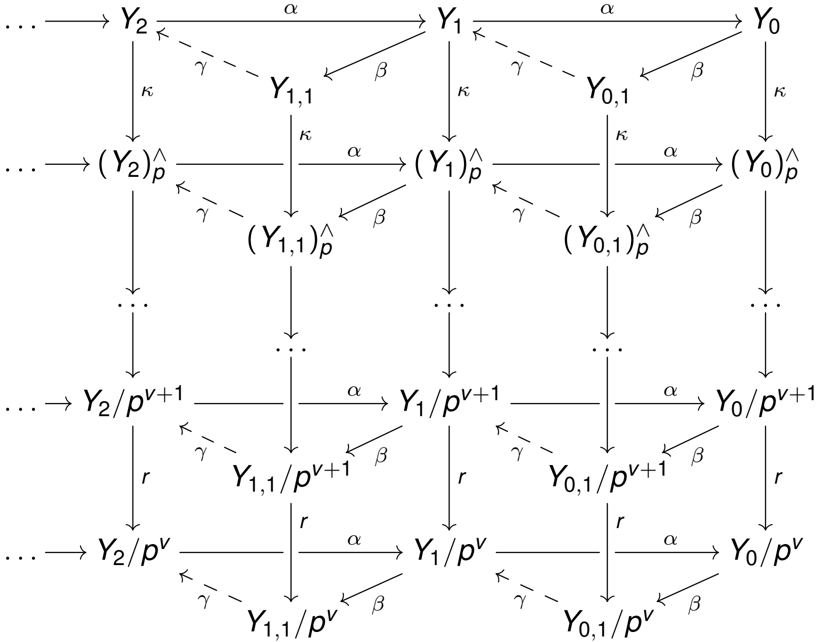
Proof.

The smash product of a fixed Adams resolution of S with the tower

$$Y \to \cdots \to Y/p^{v+1} \xrightarrow{r} Y/p^v \to \dots$$

gives a tower of Adams resolutions, as on the next page.

Tower of Adams resolutions



Proof (cont.)

- The homotopy limit over v of the lower part of the diagram gives a resolution $((Y_\star)^\wedge_p, (Y_{\star,1})^\wedge_p)$, which we claim is also an Adams resolution.
- Each H-injective $Y_{s,1}$ has the form $H \wedge T \simeq (H\mathbb{Z} \wedge T)/p$, which implies that $\kappa\colon Y_{s,1} \to (Y_{s,1})^\wedge_p$ is a stable equivalence. Hence $(Y_{s,1})^\wedge_p$ is H-injective.
- Likewise, the completion homomorphisms κ_* in the commutative square

$$\begin{array}{ccc} H_*(Y_{s+1}) & \xrightarrow{\alpha_*} & H_*(Y_s) \\ {\scriptstyle \kappa_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \kappa_*}\downarrow{\scriptstyle \cong} \\ H_*((Y_{s+1})^\wedge_p) & \xrightarrow{\alpha_*} & H_*((Y_s)^\wedge_p) \end{array}$$

  are isomorphisms, so the vanishing of the upper α_* implies the vanishing of the lower α_*. This confirms the claim.

Proof (cont.)

- We shall prove that

$$\operatorname{holim}_s (Y_s)^\wedge_p \sim * \,,$$

so that the homotopy exact couple for X and $Y^\wedge_p$ is conditionally convergent.

- First, since $(Y_\star/p, Y_{\star,1}/p)$ is an Adams resolution of Y/p, and Y/p is bounded below, we know that

$$\operatorname{holim}_s Y_s/p \sim (Y/p)_\infty \sim * \,.$$

Proof (cont.)

- Second, we have homotopy cofiber sequences

$$\operatorname{holim}_s Y_s/p \xrightarrow{e} \operatorname{holim}_s Y_s/p^{v+1} \xrightarrow{r} \operatorname{holim}_s Y_s/p^v \xrightarrow{\beta_v} \operatorname{holim}_s \Sigma Y_s/p$$

  for all $v \geq 1$, so

$$\operatorname{holim}_s Y_s/p^v \sim *$$

  in each case, by induction on v.
- This implies that

$$\operatorname{holim}_s (Y_s)^\wedge_p = \operatorname{holim}_s \operatorname{holim}_v Y_s/p^v \sim \operatorname{holim}_v \operatorname{holim}_s Y_s/p^v \sim * ,$$

  by the interchange rule for homotopy limits. □

Strong convergence to $[X, Y_p^\wedge]_*$

Theorem

Let X and Y be spectra, with Y/p bounded below. The Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y_p^\wedge)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

is strongly convergent if and only if $RE_\infty = 0$. In this case, there are isomorphisms

$$\frac{F^s[X, Y_p^\wedge]_n}{F^{s+1}[X, Y_p^\wedge]_n} \cong E_\infty^{s,s+n}$$

$$[X, Y_p^\wedge]_n \cong \lim_s \frac{[X, Y_p^\wedge]_n}{F^s[X, Y_p^\wedge]_n}$$

for all $s \geq 0$ and n.

Proof.

This is a special case of Boardman's theorem on conditional and strong convergence. □

Sufficient conditions for strong convergence

- Suppose that Y/p is bounded below.
- The condition $RE_\infty = 0$ holds if the spectral sequence terms $E_r^{s,t}$ stabilize in each bidegree, which in turn holds if $E_r^{s,t}$ is eventually finite in each bidegree.
- In particular, this holds if $E_2^{s,t}$ is finite in each bidegree, and this holds if $H_*(X)$ is bounded above and finite in each degree and $H_*(Y)$ is (bounded below and) finite in each degree.
- For example, it suffices for strong convergence that X is finite and Y/p is bounded below and of finite type.

Strong convergence to $\pi_*(Y_p^\wedge)$

The special case $X = S$ is worth emphasizing.

Theorem

Let Y/p be bounded below of finite type. The mod p Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(Y)) \\ &= \mathrm{Ext}_A^{s,t}(H^*(Y), \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(Y_p^\wedge) \end{aligned}$$

is strongly convergent, meaning that there are isomorphisms

$$\frac{F^s\pi_n(Y_p^\wedge)}{F^{s+1}\pi_n(Y_p^\wedge)} \cong E_\infty^{s,s+n} \qquad \textit{and} \qquad \pi_n(Y_p^\wedge) \cong \lim_s \frac{\pi_n(Y_p^\wedge)}{F^s\pi_n(Y_p^\wedge)}$$

for all $s \geq 0$ and n.

Outline

Ext over the Steenrod algebra

- Suppose that Y/p is bounded below and of finite type.
- To calculate the Adams E_2-term

$$E_2 = \mathrm{Ext}_A(H^*(Y), \mathbb{F}_p)$$

we consider a free, hence projective, A-module resolution

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

of $H^*(Y)$.
- The group $E_2^{s,t}$ is then given by the cohomology in degree s of the cochain complex

$$\ldots \leftarrow \mathrm{Hom}_A^t(P_2, \mathbb{F}_p) \xleftarrow{\delta^1} \mathrm{Hom}_A^t(P_1, \mathbb{F}_p) \xleftarrow{\delta^0} \mathrm{Hom}_A^t(P_0, \mathbb{F}_p) \leftarrow 0$$

with $\delta^s = \mathrm{Hom}(\partial_{s+1}, 1)$ for each $s \geq 0$.

Minimal resolutions

The passage to cohomology takes no effort if the resolution is minimal, in the following sense.

Definition

Let $I(A) \subset A$ denote the augmentation ideal. A resolution (P_*, ∂) of an A-module M is minimal if $\partial_{s+1}(P_{s+1}) \subset I(A)P_s$ for each $s \geq 0$.

Lemma

If (P_*, ∂) is minimal, then $\delta^s = 0$ for each $s \geq 0$, so that

$$\mathrm{Ext}_A^{s,t}(M, \mathbb{F}_p) = \mathrm{Hom}_A^t(P_s, \mathbb{F}_p)$$

for all $s \geq 0$ and t.

Proof.

Any A-module homomorphism $f\colon P_s \to \Sigma^t \mathbb{F}_p$ maps $I(A)P_s$ to zero, so $\delta^s(f) = \pm f\partial_{s+1}\colon P_{s+1} \to \Sigma^t \mathbb{F}_p$ will be zero when the resolution is minimal. □

Existence of minimal resolutions

Lemma

Each bounded below A-module M admits a minimal resolution (P_*, ∂). If M has finite type, then so does each P_s.

Proof.

- Choose an $\mathbb{F}_p$-linear section to the projection $M \to \mathbb{F}_p \otimes_A M$, and let

$$\epsilon\colon P_0 = A \otimes (\mathbb{F}_p \otimes_A M) \longrightarrow M$$

be left adjoint to this section, where P_0 is the free A-module induced up from $\mathbb{F}_p \otimes_A M$.

- Then $1 \otimes \epsilon\colon \mathbb{F}_p \otimes_A P_0 \to \mathbb{F}_p \otimes_A M$ is an isomorphism, and ϵ is surjective, since $\mathbb{F}_p \otimes_A \mathrm{cok}(\epsilon) = 0$ and $\mathrm{cok}(\epsilon)$ is bounded below.

Proof (cont.)

- Inductively, for $s \geq 0$ let $Z_s = \ker(\partial_s)$, which must be interpreted as $\ker(\epsilon)$ when $s = 0$.
- Choose a section to $Z_s \to \mathbb{F}_p \otimes_A Z_s$, and let

$$\tilde{\partial}_{s+1} \colon P_{s+1} = A \otimes (\mathbb{F}_p \otimes_A Z_s) \longrightarrow Z_s$$

  be left adjoint to the section.
- Then $1 \otimes \tilde{\partial}_{s+1} \colon \mathbb{F}_p \otimes_A P_{s+1} \to \mathbb{F}_p \otimes_A Z_s$ is an isomorphism, and $\tilde{\partial}_{s+1}$ is surjective.
- Let $\partial_{s+1} \colon P_{s+1} \to P_s$ be its composite with the inclusion $Z_s \subset P_s$.

Proof (cont.)

- The condition that $1 \otimes \tilde{\partial}_s$ is an isomorphism is equivalent to the condition that $\partial_{s+1}(P_{s+1}) \subset I(A)P_s$, as can be seen by chasing the following diagram with exact rows.

$$
\begin{array}{ccccccccc}
 & & P_{s+1} & & & & & & \\
 & & \downarrow{\scriptstyle \tilde{\partial}_{s+1}} & \searrow{\scriptstyle \partial_{s+1}} & & & & & \\
0 & \longrightarrow & Z_s & \longrightarrow & P_s & \xrightarrow{\tilde{\partial}_s} & Z_{s-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{F}_p \otimes_A Z_s & \longrightarrow & \mathbb{F}_p \otimes_A P_s & \xrightarrow{1 \otimes \tilde{\partial}_s} & \mathbb{F}_p \otimes_A Z_{s-1} & \longrightarrow & 0
\end{array}
$$

- If M has finite type, then P_0 is finitely generated and free over A, hence it and Z_0 are of finite type.
- Inductively, if Z_s is of finite type for $s \geq 0$, then so are P_{s+1} and Z_{s+1}. □

Robert R. Bruner's program `ext`

- For any finitely presented A-module M, at the prime $p = 2$, Bruner's program `ext` calculates a minimal resolution (P_*, ∂) of M, in a finite range of bidegrees $s \leq s_{\max}$ and $t \leq t_{\max}$.
- In essence, it calculates $Z_s = \ker(\partial_s)$ and chooses a minimal generating set for this A-module, which is then a basis for P_{s+1}.
- In cohomological (= filtration) degree $s \geq 0$, we write

$$P_s = A\{s_0^*, s_1^*, \dots, s_g^*, \dots\}$$

for the free A-module P_s, so that s_g^* denotes the g-th generator in degree s, counting from $g = 0$.

Bruner's program `ext` (cont.)

- In concrete cases we substitute numbers for s and g in this notation, leading to expressions such as 0_0^*, 1_4^* or 5_{13}^*.
- The program records the internal degree t of each generator s_g^*.
- Furthermore, it records the boundary homomorphism $\partial_{s+1}\colon P_{s+1} \to P_s$ by giving its value on each basis element in P_{s+1} as an A-linear combination

$$\sum_g \theta_g s_g^*$$

in P_s, where the $\theta_g \in A$.

Bruner's program `ext` (cont.)

- By minimality,

$$\mathrm{Ext}_A^{s,*}(M, \mathbb{F}_2) = \mathrm{Hom}_A(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{s_0, s_1, \dots, s_g, \dots\},$$

  where $s_g \colon P_s \to \mathbb{F}_2$ denotes the dual of s_g^*.
- In other words, s_g takes the value 1 on s_g^*, and 0 on the other A-module basis elements of P_s.
- In the concrete cases above, we write 0_0, 1_4 and 5_{13} for these elements in $\mathrm{Ext}_A(M, \mathbb{F}_2)$.
- The cohomological degree of s_g is s, while its internal (homological, or homotopical) degree t is equal to the internal (cohomological) of s_g^*.

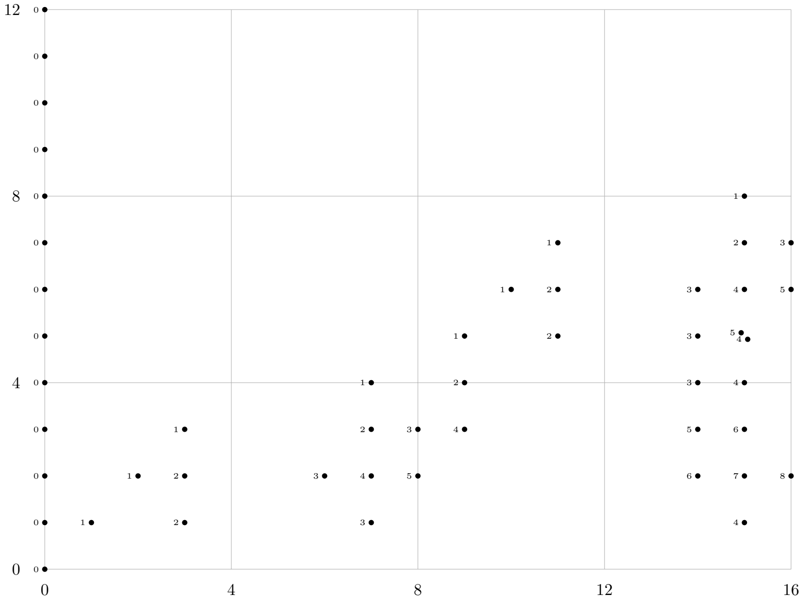
The Adams E_2-term for S

- We consider $Y = S$ at $p = 2$ with $M = \mathbb{F}_2$.
- A quick machine calculation with $s_{\max} = 12$ and $t_{\max} = 28$ suffices to compute

$$\mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{0_0\} \oplus \mathbb{F}_2\{s_g \mid s \geq 1, g \geq 0\}$$

  in the range $0 \leq s \leq 12$ and $0 \leq t \leq 28$.
- This includes the rectangular region $0 \leq s \leq 12$ and $0 \leq t - s \leq 16$ in the $(t-s, s)$-plane shown on the next page.
- A filled circle labeled "g" in bidegree $(t-s, s)$ represents the Ext-generator s_g, dual to the A-module generator s_g^* in the minimal resolution, both of which have internal degree t.

Vector space basis for $E_2^{s,t}(S) = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$



Bigraded basis

- In this range, most groups $E_2^{s,t}$ have dimension 0 or 1 as $\mathbb{F}_2$-vector spaces, but in bidegree $(t-s,s)=(15,5)$, corresponding to $(s,t)=(5,20)$, there are two generators 5_4 and 5_5, which means that

$$E_2^{5,20}(S) = \mathrm{Ext}_A^{5,20}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{5_4, 5_5\}$$

  is 2-dimensional.
- The program `ext` makes a deterministic choice of basis for this $\mathbb{F}_2$-vector space, but other methods of calculation might lead to a different choice of basis, so care is needed when comparing different approaches.

Filtration zero and one

- The minimal resolution starts

$$\cdots \to A\{2_g^* \mid g \geq 0\} \xrightarrow{\partial_2} A\{1_i^* \mid i \geq 0\} \xrightarrow{\partial_1} A\{0_0^*\} \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

  with $\epsilon(0_0^*) = 1$ and

$$\partial_1(1_i^*) = Sq^{2^i}\, 0_0^*$$

  for each $i \geq 0$.
- This way $\mathrm{im}(\partial_1) = I(A) = \ker(\epsilon)$, which is minimally generated as an A-module by the Sq^{2^i} for $i \geq 0$.

Filtration two

- Less obviously,

$$\partial_2(2_0^*) = Sq^1\, 1_0^*$$
$$\partial_2(2_1^*) = Sq^3\, 1_0^* + Sq^2\, 1_1^*$$
$$\partial_2(2_2^*) = Sq^4\, 1_0^* + Q_1\, 1_1^* + Sq^1\, 1_2^*\,,$$

  which correspond to the following Adem relations.

$$Sq^1 Sq^1 = 0$$
$$Sq^3 Sq^1 + Sq^2 Sq^2 = 0$$
$$Sq^4 Sq^1 + Q_1 Sq^2 + Sq^1 Sq^4 = 0$$

- Here $Q_1 = Sq^3 + Sq^2 Sq^1 = Sq(0, 1)$ is the Milnor primitive, dual to ξ_2 in the Milnor basis for A_*.

Comodule primitives and module indecomposables

Definition

- For an A_*-comodule M_*, with coaction $\nu \colon M_* \to A_* \otimes M_*$, let

  $$P_{A_*}(M_*) = \{x \in M_* \mid \nu(x) = 1 \otimes x\}$$

  be the subspace of A_*-comodule primitives.

- For an A-module M, let

  $$Q_{A_*}(M) = \mathbb{F}_p \otimes_A M$$

  be the quotient space of A-module indecomposables.

These should not be confused with the (coalgebra) primitives $P(C)$ of a coaugmented coalgebra and the (algebra) indecomposables $Q(A)$ of an augmented algebra.

Filtration zero and comodule primitives

Lemma

For any A_*-comodule M_*, there are natural isomorphisms

$$\mathrm{Ext}^{0,*}_{A_*}(\mathbb{F}_p, M_*) \cong \mathbb{F}_p \,\square_{A_*}\, M_* \cong P_{A_*}(M_*)$$

and

$$\mathrm{Ext}^{0,*}_{A}(M, \mathbb{F}_p) \cong \mathrm{Hom}_A(M, \mathbb{F}_p) \cong \mathrm{Hom}(Q_A(M), \mathbb{F}_p)\,.$$

In particular,

$$\mathrm{Ext}^{0,*}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}^{0,*}_{A}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{1\}\,.$$

Filtration one and coalgebra primitives

Lemma

There are natural isomorphisms

$$\mathrm{Ext}_{A_*}^{1,*}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}_{A}^{1,*}(\mathbb{F}_p, \mathbb{F}_p) \cong P(A_*) \cong \mathrm{Hom}(Q(A), \mathbb{F}_p)$$

where

$$P(A_*) = \mathbb{F}_2\{\xi_1^{2^i} \mid i \geq 0\}$$

for $p = 2$.

Definition

For $p = 2$ let

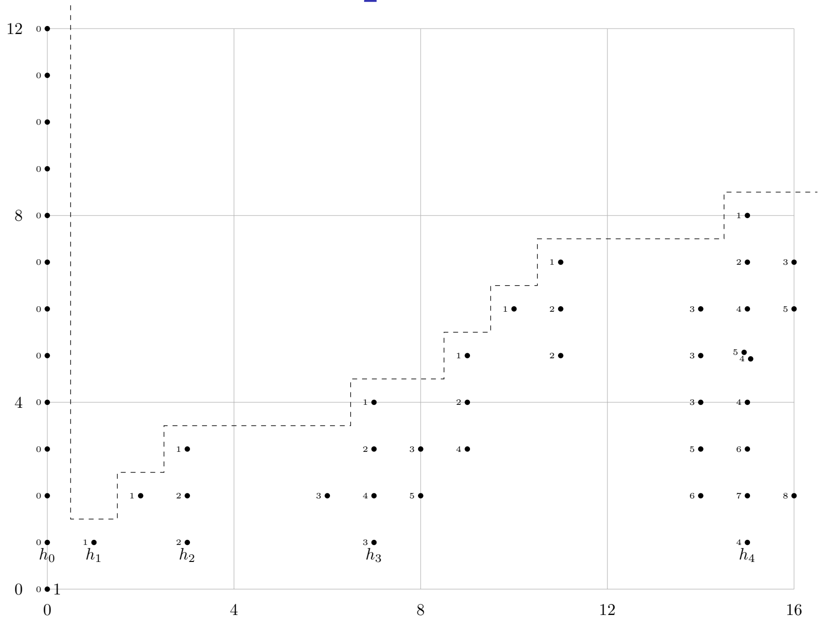
$$h_i \in \mathrm{Ext}_A^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$$

denote the class of $\xi_1^{2^i}$, dual to $Sq^{2^i} \in Q(A)$, for each $i \geq 0$.

Labels, vanishing

- In the s_g-notation of `ext`, the generator in $E_2^{0,0}(S)$ is $1 = 0_0$, while the generator in $E_2^{1,2^i}(S)$ is $h_i = 1_i$ for each $i \geq 0$.
- These classes are labeled on the next page.
- The calculation shows that $E_2^{s,t}(S)$ appears to vanish above a line of slope $1/2$ in the $(t-s, s)$-plane, except for $t - s = 0$.
- This is indeed the case, as was proved by Adams, and confirms that there are no other classes in $E_\infty^{s,t}(S)$ for $0 < t - s \leq 16$ than the ones shown.

Generators 1 and h_i in $E_2^{s,t}(S)$



Adams vanishing theorem

Theorem ([Ada66])

For $p = 2$, the groups $E_2^{s,t}(S)$ are trivial for

$$0 < t - s < \begin{cases} 2s - 1 & \textit{for } s \equiv 0 \mod 4, \\ 2s + 1 & \textit{for } s \equiv 1 \mod 4, \\ 2s + 2 & \textit{for } s \equiv 2 \mod 4, \\ 2s + 3 & \textit{for } s \equiv 3 \mod 4. \end{cases}$$

Adams' proof uses the structure of A as a union of finite sub Hopf algebras $A(n)$, and some initial calculations.

Possible differentials

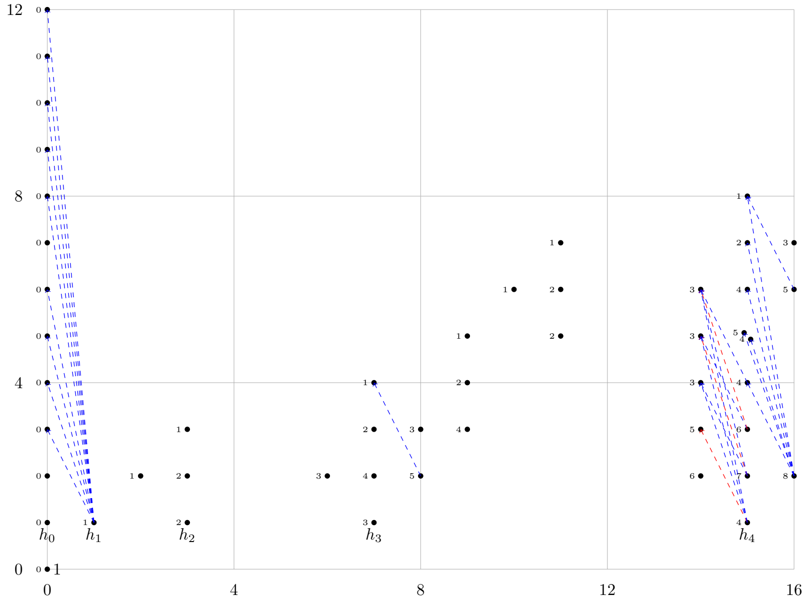
Recall that the r-th Adams differential

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

has $(t-s,s)$-bidegree $(-1,r)$. The first possibly nonzero Adams differentials for S are the following.

1. $d_{s-1}(h_1) \in \{0, s_0\}$ for $s \geq 3$;
2. $d_2(2_5) \in \{0, 4_1\}$;
3. $d_2(h_4) \in \{0, 3_5\}$.

Possible differentials in $E_r^{s,t}(S)$ (actual diff's in red)



The 0- and 1-stem

- Since this spectral sequence converges to $\pi_*(S_2^\wedge) \cong \pi_*(S)_2^\wedge$, and we know that

$$\pi_1(S) = \mathbb{Z}/2\{\eta\} \neq 0\,,$$

  it follows that $1_1 = h_1$ must survive to E_∞ and detect $\eta\colon S^1 \to S$.
- Hence each class $s_0 \in E_2^{s,s}$ also survives to E_∞.
- We shall see that it detects 2^s, so that the groups $E_\infty^{s,s}(S) \cong \mathbb{F}_2\{s_0\}$ give the associated graded of the 2-adic filtration

$$\cdots \subset 2^{s+1}\mathbb{Z}_2 \subset 2^s\mathbb{Z}_2 \subset \cdots \subset 2\mathbb{Z}_2 \subset \mathbb{Z}_2\,.$$

  on

$$\pi_0(S)_2^\wedge \cong \mathbb{Z}_2\,.$$

Stems 2 through 6

- It also follows that

$$\pi_2(S)_2^\wedge \cong \mathbb{Z}/2\,,$$

  with a generator detected by 2_1, and that $\pi_3(S)_2^\wedge$ has order $2^3 = 8$.
- However, the group structure of $\pi_3(S)_2^\wedge$ remains to be determined.
- Moreover,

$$\pi_4(S)_2^\wedge = 0 \quad \text{and} \quad \pi_5(S)_2^\wedge = 0\,,$$

  since the E_2- and E_∞-terms contain only trivial groups in these total degrees.
- Furthermore, $\pi_6(S)_2^\wedge \cong \mathbb{Z}/2$, with a generator detected by 2_3.

Stems 7 and 8

- If $d_2(2_5) = 0$, which turns out to be the case, then $\pi_7(S)^\wedge_2$ has order $2^4 = 16$ and $\pi_8(S)^\wedge_2$ has order $2^2 = 4$.
- If, on the other hand, $d_2(2_5) = 4_1$ were nonzero, then $\pi_7(S)^\wedge_2$ would have order $2^3 = 8$ and $\pi_8(S)^\wedge_2 \cong \mathbb{Z}/2$.
- To decide between these two cases we must calculate this Adams d_2-differential.

Stems 9 through 14

- Continuing, $\pi_9(S)_2^\wedge$ has order $2^3 = 8$, $\pi_{10}(S)_2^\wedge = \mathbb{Z}/2$, $\pi_{11}(S)_2^\wedge$ has order $2^3 = 8$, $\pi_{12}(S)_2^\wedge = 0$ and $\pi_{13}(S)_2^\wedge = 0$.
- We can also see that $\pi_{14}(S)_2^\wedge$ has order dividing $2^5 = 32$, but here there is room for many differentials from topological degree 15.
- To proceed, we will use that the ring spectrum structure on S makes the associated Adams spectral sequence an algebra spectral sequence.
- This severely limits the possible differential patterns that can be present in the spectral sequence.

Outline

Monoidal structure

For spectra X', X'', Y' and Y'', with smash products $X = X' \wedge X''$ and $Y = Y' \wedge Y''$ there are Adams spectral sequences

$$'E_2 = \mathrm{Ext}_{A_*}(H_*(X'), H_*(Y')) \Longrightarrow [X', Y']_*$$
$$''E_2 = \mathrm{Ext}_{A_*}(H_*(X''), H_*(Y'')) \Longrightarrow [X'', Y'']_*$$
$$E_2 = \mathrm{Ext}_{A_*}(H_*(X), H_*(Y)) \Longrightarrow [X, Y]_* \, .$$

Smash product of morphisms

- The smash product of morphisms induces a pairing

$$\wedge\colon [X', Y']_n \otimes [X'', Y'']_m \longrightarrow [X, Y]_{n+m}$$

that takes $f\colon \Sigma^n X' \to Y'$ and $g\colon \Sigma^m X'' \to Y''$ to the composite

$$\Sigma^{n+m} X = S^n \wedge S^m \wedge X' \wedge X'' \xrightarrow{1\wedge\tau\wedge 1} S^n \wedge X' \wedge S^m \wedge X'' \xrightarrow{f\wedge g} Y' \wedge Y'' = Y\,.$$

- It preserves the Adams filtrations, in the sense that $F^s[X', Y']_* \otimes F^u[X'', Y'']_*$ is mapped into $F^{s+u}[X, Y]_*$.
- If $f = f_1 \circ \cdots \circ f_s$ and $g = g_1 \circ \cdots \circ g_u$, with $H_*(f_i) = 0$ and $H_*(g_j) = 0$, then $f \wedge g$ is the composite of $s + u$ maps of the form $f_i \wedge 1$ and $1 \wedge g_j$, each of which induces zero in mod p homology.

Internal product in A_*-comodule Ext

- For Hopf algebras, the tensor product of two (co-)modules is again a (co-)module, using the diagonal (co-)action.
- Since A_* is a Hopf algebra, there is an internal product

$$\wedge\colon\ \mathrm{Ext}_{A_*}(M', N') \otimes \mathrm{Ext}_{A_*}(M'', N'') \longrightarrow \mathrm{Ext}_{A_*}(M' \otimes M'', N' \otimes N'')\,.$$

- It is given by choosing injective A_*-comodule resolutions $({}'I_*^s, \delta)_s$ and $({}''I_*^u, \delta)_u$ of N' and N'', respectively, and forming their tensor product $(I_*^\sigma, \delta)_\sigma$ with

$$I_*^\sigma = \bigoplus_{s+u=\sigma} {}'I_*^s \otimes {}''I_*^u$$

and $\delta = \delta \otimes 1 + 1 \otimes \delta$, which is an injective A_*-comodule resolution of $N' \otimes N''$.

Internal product (cont.)

- Given s- and u-cocycles

$$f\colon M' \longrightarrow {}'I^s_* \qquad \text{and} \qquad g\colon M'' \longrightarrow {}''I^u_*$$

  the internal product of the cohomology classes $[f]$ and $[g]$ is the class of the composite $(s+u)$-cocycle

$$M' \otimes M'' \xrightarrow{f \otimes g} {}'I^s_* \otimes {}''I^u_* \subset I^{s+u}_* \,.$$

- If we have given A_*-comodule homomorphisms $M \to M' \otimes M''$ and $N' \otimes N'' \to N$ then we can further internalize the product to obtain a pairing

$$\wedge\colon\ \mathrm{Ext}_{A_*}(M', N') \otimes \mathrm{Ext}_{A_*}(M'', N'') \longrightarrow \mathrm{Ext}_{A_*}(M, N)\,.$$

- If M is an A_*-comodule coalgebra and N is an A_*-comodule algebra, this makes $\mathrm{Ext}_{A_*}(M, N)$ an $\mathbb{F}_p$-algebra.

Internal product in A-module Ext

- Dually, since A is a Hopf algebra there is an internal product

$$\wedge\colon\ \mathrm{Ext}_A(M', N') \otimes \mathrm{Ext}_A(M'', N'') \longrightarrow \mathrm{Ext}_A(M' \otimes M'', N' \otimes N'')$$

- It is given by choosing projective A-module resolutions $({}'P^*_s, \partial)_s$ and $({}''P^*_u, \partial)_u$ of M' and M'', respectively, and forming their tensor product $(P^*_\sigma, \partial)_\sigma$ with

$$P^*_\sigma = \bigoplus_{s+u=\sigma} {}'P^*_s \otimes {}''P^*_u$$

and $\partial = \partial \otimes 1 + 1 \otimes \partial$, which is a projective A-module resolution of $M' \otimes M''$.

Internal product (cont.)

- Given s- and u-cocycles

$$f\colon {}'P_s^* \longrightarrow N' \qquad \text{and} \qquad g\colon {}''P_u^* \longrightarrow N''$$

the internal product of the cohomology classes $[f]$ and $[g]$ is the class of the composite $(s+u)$-cocycle

$$P_\sigma^* \to {}'P_s^* \otimes {}''P_u^* \xrightarrow{f \otimes g} N' \otimes N'' .$$

- If we have given A-module homomorphisms $M \to M' \otimes M''$ and $N' \otimes N'' \to N$ then we can further internalize the product to obtain a pairing

$$\wedge\colon \operatorname{Ext}_A(M', N') \otimes \operatorname{Ext}_A(M'', N'') \longrightarrow \operatorname{Ext}_A(M, N) .$$

- If M is an A-module coalgebra and N is an A-module algebra, this makes $\operatorname{Ext}_A(M, N)$ an $\mathbb{F}_p$-algebra. See [ML63].

Pairing of Adams spectral sequences

Theorem

(a) For spectra X', X'', Y' *and* Y'', *with* $X = X' \wedge X''$ *and* $Y = Y' \wedge Y''$, *there is a natural pairing*

$$\wedge_r \colon ({}'E_r, {}''E_r) \longrightarrow E_r$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\wedge \colon [X', Y']_* \otimes [X'', Y'']_* \longrightarrow [X, Y]_*$$

mapping $f \otimes g$ *to* $f \wedge g$.

Theorem (cont.)

(b) The pairing of E_2-terms

$$\wedge_2 \colon \operatorname{Ext}_{A_*}(H_*(X'), H_*(Y')) \otimes \operatorname{Ext}_{A_*}(H_*(X''), H_*(Y'')) \longrightarrow \operatorname{Ext}_{A_*}(H_*(X), H_*(Y))$$

is the internal product.

(c) If Y'/p and Y''/p are bounded below of finite type, then the E_2-pairing

$$\wedge_2 \colon \operatorname{Ext}_A(H^*(Y'), H^*(X')) \otimes \operatorname{Ext}_A(H^*(Y''), H^*(X'')) \longrightarrow \operatorname{Ext}_A(H^*(Y), H^*(X))$$

is the internal product (followed by the pairing $\mu \colon H^*(X') \otimes H^*(X'') \to H^*(X)$).

The case of homotopy groups

- There is a natural pairing

$$\wedge_r \colon (E_r(Y'), E_r(Y'')) \longrightarrow E_r(Y' \wedge Y'')$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\cdot \colon \pi_*(Y') \otimes \pi_*(Y'') \longrightarrow \pi_*(Y' \wedge Y'') .$$

- The pairing of E_2-terms is the internal product

$$\wedge \colon \ \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y')) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y'')) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y)) .$$

- If Y'/p and Y''/p are bounded below of finite type, then this equals the internal product

$$\wedge \colon \ \mathrm{Ext}_{A}(H^*(Y'), \mathbb{F}_p) \otimes \mathrm{Ext}_{A}(H^*(Y''), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{A}(H^*(Y), \mathbb{F}_p) .$$

Homotopy of ring spectra

- If E is a ring spectrum (up to homotopy) with multiplication $\mu\colon E \wedge E \to E$, then there is a pairing

$$\mu_r\colon (E_r(E), E_r(E)) \longrightarrow E_r(E)$$

  of Adams spectral sequences making $E_r(E)$ an algebra spectral sequence, with abutment the filtration-preserving graded ring product given by the composition

$$\pi_*(E) \otimes \pi_*(E) \stackrel{\cdot}{\longrightarrow} \pi_*(E \wedge E) \stackrel{\mu_*}{\longrightarrow} \pi_*(E)\,.$$

- The pairing of E_2-terms is the internal product

$$\mu_* \wedge\colon\ \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E))\,.$$

- If E/p is bounded below of finite type, then this equals the internal product

$$\mu_* \wedge\colon\ \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \otimes \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(H^*(E), \mathbb{F}_p)\,.$$

Homotopy of module spectra

- If M is an E-module ring spectrum (up to homotopy) with action $\lambda\colon E \wedge M \to M$, then there is a pairing

$$\lambda_r\colon (E_r(E), E_r(M)) \longrightarrow E_r(M)$$

of Adams spectral sequences making $E_r(M)$ an $E_r(E)$-module spectral sequence, with abutment the filtration-preserving module action given by the composition

$$\pi_*(E) \otimes \pi_*(M) \stackrel{\cdot}{\longrightarrow} \pi_*(E \wedge M) \stackrel{\lambda_*}{\longrightarrow} \pi_*(M)\,.$$

- The pairing of E_2-terms is the internal product

$$\lambda_*\wedge\colon\ \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(M)) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(M))\,.$$

- If E/p and M/p are bounded below of finite type, then this equals the internal product

$$\lambda_*\wedge\colon\ \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \otimes \mathrm{Ext}_A(H^*(M), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(H^*(M), \mathbb{F}_p)\,.$$

Module structure over $E_r(S)$

In particular, $E_r(S)$ is a (graded commutative) algebra spectral sequence, and each Adams spectral sequence $E_r(Y)$ is a (right) $E_r(S)$-module spectral sequence.

$$\mu_r \colon E_r(S) \otimes E_r(S) \longrightarrow E_r(S)$$
$$\rho_r \colon E_r(Y) \otimes E_r(S) \longrightarrow E_r(Y)$$

Outline

Composition product of morphisms

- For spectra X, Y and Z the composition of morphisms defines a pairing

$$\circ\colon [Y,Z]_n \otimes [X,Y]_m \longrightarrow [X,Z]_{n+m}$$

that takes $g\colon \Sigma^n Y \to Z$ and $f\colon \Sigma^m X \to Y$ to the composite

$$g \circ \Sigma^n f\colon \Sigma^{n+m} X = \Sigma^n \Sigma^m X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{g} Z\,.$$

- It preserves Adams filtrations, in the sense that $F^s[Y,Z]_* \otimes F^u[X,Y]_*$ is mapped into $F^{s+u}[X,Z]_*$.
- The combined composite of s and u maps, each of which induces zero in mod p homology, is obviously a composite of $s+u$ such maps.

Yoneda product

- For any algebra A and (left) A-modules L, M and N there is a natural Yoneda composition product

$$\circ\colon \operatorname{Ext}_A^s(M,N) \otimes \operatorname{Ext}_A^u(L,M) \longrightarrow \operatorname{Ext}_A^{s+u}(L,N)\,.$$

- To define it, let

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \to 0$$

and

$$\cdots \to Q_u \xrightarrow{\partial_u} Q_{u-1} \to \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} L \to 0$$

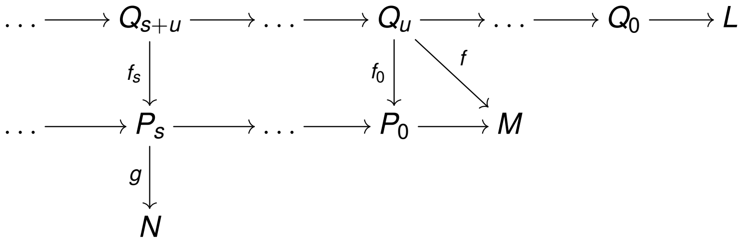
be projective A-module resolutions.

Yoneda product (cont.)

- Given cocycles

$$g\colon P_s \longrightarrow N \quad \text{and} \quad f\colon Q_u \longrightarrow M$$

choose a chain map $f_*\colon Q_{*+u} \to P_*$ of degree $-u$ lifting f.



- The composite $g \circ f_s$ is a cocycle, and its cohomology class

$$[g] \circ [f] = [g \circ f_s] \in \mathrm{Ext}_A^{s+u}(L, N)$$

defines the composition product.

Yoneda's Proposition

In the case of modules over a Hopf algebra B, the interior and composition products are related as follows.

Proposition ([Yon58])

For

$$x' \in \mathrm{Ext}_B^{s'}(M', N') \qquad y' \in \mathrm{Ext}_B^{u'}(L', M')$$
$$x'' \in \mathrm{Ext}_B^{s''}(M'', N'') \qquad y'' \in \mathrm{Ext}_B^{u''}(L'', M'')$$

the identity

$$(x' \circ y') \wedge (x'' \circ y'') = (-1)^{s''u'}(x' \wedge x'') \circ (y' \wedge y'')$$

holds in $\mathrm{Ext}_B^{s'+u'+s''+u''}(L' \otimes L'', N' \otimes N'')$.

Corollary

- Let B a Hopf algebra over k.
- For $x \in \mathrm{Ext}_B^s(k, N)$ and $y \in \mathrm{Ext}_B^u(L, k)$ the identity

$$x \wedge y = (x \wedge 1) \circ (1 \wedge y) = x \circ y$$

  holds in $\mathrm{Ext}_B^{s+u}(k \otimes L, N \otimes k) \cong \mathrm{Ext}_B^{s+u}(L, N)$.
- The identity

$$(-1)^{su} y \wedge x = (1 \wedge x) \circ (y \wedge 1) = x \circ y$$

  holds in $\mathrm{Ext}_B^{u+s}(L \otimes k, k \otimes N) \cong \mathrm{Ext}_B^{u+s}(L, N)$.
- In particular, the interior and composition products

$$\mathrm{Ext}_B^s(k, k) \otimes \mathrm{Ext}_B^u(k, k) \longrightarrow \mathrm{Ext}_B^{s+u}(k, k)$$

  agree, and make $\mathrm{Ext}_B^*(k, k)$ a graded commutative k-algebra.

Composition products

- For spectra X, Y and Z consider the Adams spectral sequences

$$
\begin{aligned}
{}'E_2 &= \mathrm{Ext}_A(H_*(Y), H_*(Z)) \Longrightarrow [Y, Z]_* \\
{}''E_2 &= \mathrm{Ext}_A(H_*(X), H_*(Y)) \Longrightarrow [X, Y]_* \\
E_2 &= \mathrm{Ext}_A(H_*(X), H_*(Z)) \Longrightarrow [X, Z]_* \,.
\end{aligned}
$$

- The interaction between the composition product in Ext and the composition in the stable category was determined by Michael Moss.

Theorem ([Mos68])

- *There is a natural pairing of Adams spectral sequences*

$$\circ_r : ({}'E_r, {}''E_r) \longrightarrow E_r$$

  with abutment the filtration-preserving pairing

$$\circ : [Y, Z]_* \otimes [X, Y]_* \longrightarrow [X, Z]_*$$

  mapping $g \otimes f$ *to* $g \circ \Sigma^{|g|} f$.
- *If* Y/p *and* Z/p *are bounded below of finite type, then the* E_2*-pairing*

$$\circ_2 : \operatorname{Ext}_A(H^*(Z), H^*(Y)) \otimes \operatorname{Ext}_A(H^*(Y), H^*(X)) \longrightarrow \operatorname{Ext}_A(H^*(Z), H^*(X))$$

  is the twisted composition product, mapping $y \otimes x$ *to* $(-1)^{|x||y|} x \circ y$*, where* $|x| = v - u$ *and* $|y| = t - s$ *for* $x \in {}''E_2^{u,v}$ *and* $y \in {}'E_2^{s,t}$.

The sphere case

Corollary

- *There is a natural pairing of Adams spectral sequences*

$$\circ_r \colon (E_r(S), E_r(S)) \longrightarrow E_r(S)$$

with abutment the filtration-preserving pairing

$$\circ \colon \pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S)$$

mapping $g \otimes f$ to $g \circ \Sigma^{|g|} f = g \wedge f$.

- *The E_2-pairing*

$$\circ_2 \colon \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) \otimes \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$$

is the twisted composition product, mapping $y \otimes x$ to $(-1)^{|x||y|} x \circ y = y \wedge x$, where $|x| = v - u$ and $|y| = t - s$ for $x \in {}'' E_2^{u,v}(S)$ and $y \in {}' E_2^{s,t}(S)$.

Outline

Products in the Adams spectral sequence for S

- In the case $X = Y = S$, the mod p Adams spectral sequence for the sphere spectrum is a graded commutative algebra spectral sequence

$$E_2(S)^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(S)_p^\wedge$$

with differentials

$$d_r^{s,t} \colon E_r^{s,t}(S) \longrightarrow E_r^{s+r,t+r-1}(S)\,.$$

- The multiplication on the E_2-term is given by the internal product

$$\wedge\colon \mathrm{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Ext}_A^{u,v}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A^{s+u,t+v}(\mathbb{F}_p, \mathbb{F}_p)\,,$$

and converges to the smash product pairing

$$\wedge\colon \pi_n(S)_p^\wedge \otimes \pi_m(S)_p^\wedge \longrightarrow \pi_{n+m}(S)_p^\wedge$$

giving the graded commutative ring structure on $\pi_*(S)_p^\wedge$.

Computation of products

- Yoneda's proposition shows that the internal product pairing is equal to the composition product in Ext, and that the smash product pairing is equal to the composition product in $\pi_*(S)^\wedge_p$.
- For $p = 2$, Bruner's program $\texttt{ext}$ can calculate the Yoneda (composition) products in Ext, by lifting cocycles to chain maps and evaluating their composites.

h_i-multiplications

- The computation of products

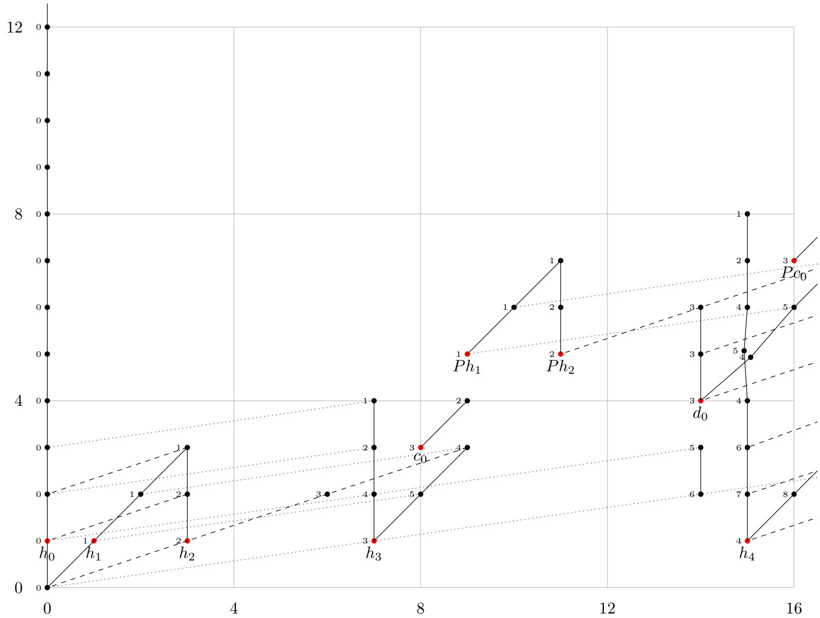
$$h_i \colon \operatorname{Ext}_A^{s,t}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_A^{s+1,t+2^i}(M, \mathbb{F}_2)$$

  with the Hopf–Steenrod classes h_i is particularly simple, and can be read off from the boundary homomorphism

$$\partial_{s+1} \colon P_{s+1} \longrightarrow P_s$$

  in a minimal resolution for M.
- In the case $M = \mathbb{F}_2$, the multiplications by h_i for $0 \leq i \leq 3$ in $\operatorname{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ are shown in the figure on the next page.

$E_2(S)$ with h_i-multiplications



Legend

- Each nonzero multiplication by $h_0 \in E_2^{1,1}(S)$ is shown by a line connecting x in bidegree $(t-s, s)$ to $h_0 x$ in bidegree $(t-s, s+1)$, i.e., by a vertical line of unit length.
- Each nonzero multiplication by $h_1 \in E_2^{1,2}(S)$ is shown by a line connecting x in bidegree $(t-s, s)$ to $h_1 x$ in bidegree $(t-s+1, s+1)$, i.e., by a line of slope $+1$.
- Each nonzero multiplication by $h_2 \in E_2^{1,4}(S)$ is shown by a dashed line connecting x in bidegree $(t-s, s)$ to $h_2 x$ in bidegree $(t-s+3, s+1)$, i.e., by a dashed line of slope $+1/3$.
- Each nonzero multiplication by $h_3 \in E_2^{1,8}(S)$ is shown by a dotted line connecting x in bidegree $(t-s, s)$ to $h_3 x$ in bidegree $(t-s+7, s+1)$, i.e., by a dotted line of slope $+1/7$.

Algebra generators for $E_2(S)$

Lemma

In the range $t - s \leq 16$, the $\mathbb{F}_2$-algebra $E_2^{*,*}(S)$ is generated by the following classes.

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_4	Pc_0
$t-s$	0	1	3	7	8	9	11	14	15	16
s	1	1	1	1	3	5	5	4	1	7

The relation $c_0^2 = h_1^2 d_0$ holds.

Proof

- The h_i-multiplications can be read off from the minimal resolution (P_*, ∂) of $\mathbb{F}_2$ calculated by `ext`.
- The classes h_i in filtration $s = 1$ must be algebra indecomposable for filtration degree reasons.
- The only other basis elements that are not h_i-multiplies are the classes denoted c_0, d_0, Ph_1, Ph_2 and Pc_0, and these must then be algebra decomposable for topological degree reasons, since these all lie in degrees $t - s \geq 8$.
- To calculate $c_0^2 = c_0 \cdot c_0$, we instead call on `ext` to lift the cocycle $f = 3_3 \colon P_3 \to \Sigma^{11}\mathbb{F}_2$ to a chain map $f_* \colon P_{*+3} \to \Sigma^{11} P_*$, and then to evaluate the composite

$$P_6 \xrightarrow{f_3} \Sigma^{11} P_3 \xrightarrow{f} \Sigma^{22}\mathbb{F}_2 \,.$$

- This turns out to map 6_5^* to 1, hence equals the cocycle 6_5, which we have already seen represents $h_1^2 d_0$. □

Nomenclature

- The prefix P refers to the periodicity operator from [Ada66].
- The notations $c_0, d_0, \ldots$ stem from computations in the range $t - s \leq 70$ made by May (unpublished) and Tangora [Tan70].
- In his work on the Hopf invariant one problem, Adams showed that there are no algebra indecomposables in filtration $s = 2$ of $E_2^{*,*}(S) = \mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, and determined the multiplicative relations satisfied by the generators h_i in filtrations $s \leq 3$.

Adams relations

Theorem ([Ada60])

The relations

$$h_i h_{i+1} = 0$$
$$h_i^2 h_{i+2} = h_{i+1}^3$$
$$h_i h_{i+2}^2 = 0$$

hold in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$*, for each* $i \geq 0$*.*

The algebra homomorphism

$$\frac{\mathbb{F}_2[h_i \mid i \geq 0]}{(h_i h_{i+1}, h_i^2 h_{i+2} + h_{i+1}^3, h_i h_{i+2}^2)} \longrightarrow \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism in filtration degrees $s \leq 2$*, and is injective in degree* $s = 3$*.*

Filtrations $0 \leq s \leq 3$

- More explicitly,

$$\mathrm{Ext}_A^{0,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{1\}$$
$$\mathrm{Ext}_A^{1,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{h_i \mid i \geq 0\}$$
$$\mathrm{Ext}_A^{2,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{h_i h_j \mid 0 \leq i \leq j-2\} \oplus \mathbb{F}_2\{h_j^2 \mid j \geq 0\}$$

- If we omit the generators $h_i h_{i+1} h_k$, $h_i h_j h_{j+1}$, $h_i h_i h_{i+2}$ and $h_i h_{i+2} h_{i+2}$ from

$$\mathbb{F}_2\{h_i h_j h_k \mid i \leq j \leq k\}$$

then the remainder maps injectively to $\mathrm{Ext}_A^{3,*}(\mathbb{F}_2, \mathbb{F}_2)$.
- The class c_0 (which is part of a family of related classes c_i for $i \geq 0$) shows that surjectivity fails for $s = 3$.

Outline

Adams d_2-differentials for S

In view of the Leibniz rule

$$d_2(xy) = d_2(x)y + xd_2(y)$$

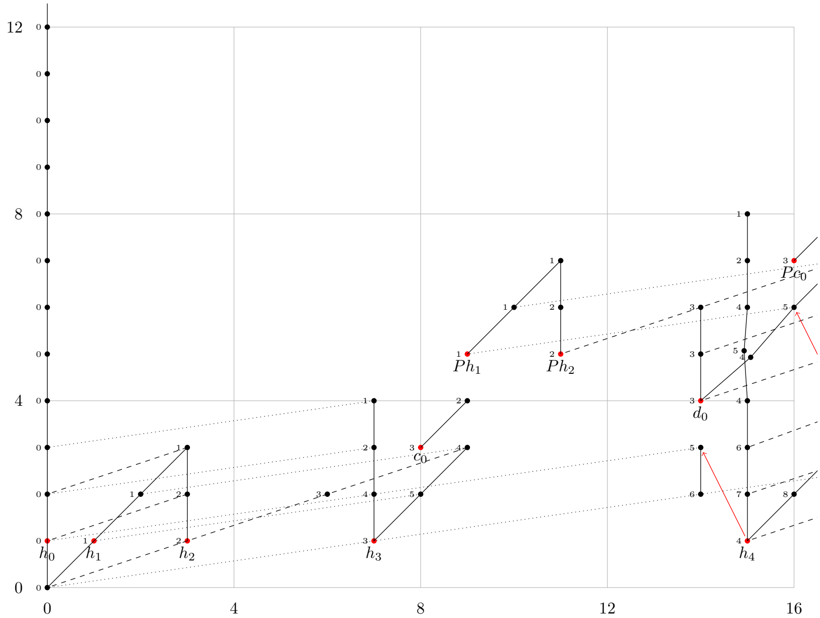
in $E_2(S)$, the d_2-differential is determined by its values on a set of algebra generators for this E_2-term.

Proposition

In the range $t - s \leq 16$, the d_2-differential on the algebra generators is given as follows.

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_4	Pc_0
$d_2(x)$	0	0	0	0	0	0	0	0	$h_0h_3^2$	0

$E_2(S)$ with d_2-differentials



Proof

- The d_2-differentials on h_0, h_2, h_3, c_0, Ph_1, Ph_2, d_0 and Pc_0 land in trivial groups, hence are zero.
- The relation $h_0 h_1 = 0$ and the Leibniz rule imply that $0 \cdot h_1 + h_0 \cdot d_2(h_1) = d_2(0) = 0$, so that $h_0 d_2(h_1) = 0$. Since $h_0 \cdot h_0^3 = h_0^4 \neq 0$, it follows that $d_2(h_1) \neq h_0^3$, and $d_2(h_1) = 0$ is the only possibility.
- The final case, of $d_2(h_4)$, deserves to be stated as a separate theorem. □

Theorem ([Ada58])

$d_2(h_4) = h_0 h_3^2$.

Proof

- The class $h_0 \in E_2^{1,1}(S)$ detects the homotopy class $2 \in \pi_0(S)_2^\wedge$.
- The class $h_3 \in E_2^{1,8}(S)$ must survive to $E_\infty(S)$ since $d_r(h_3)$ lies in a trivial group for all $r \geq 2$. Hence it detects a homotopy class $\sigma \in \pi_7(S)_2^\wedge$.
- By multiplicativity of the Adams spectral sequence for S, it follows that $2\sigma^2 = 2 \cdot \sigma \cdot \sigma$ is detected by $h_0 h_3^2 = h_0 \cdot h_3 \cdot h_3$ in $F^3\pi_*(S)_2^\wedge / F^4\pi_*(S)_2^\wedge \cong E_\infty^{3,*}$.
- However, by the graded commutativity of $\pi_*(S)_2^\wedge$, we have

  $$\sigma \cdot \sigma = -\sigma \cdot \sigma \,,$$

  since $|\sigma| = 7$ is odd. Thus $2\sigma^2 = 0$, which implies that $h_0 h_3^2 = 0$ in $E_\infty(S)$.
- This can only happen because $h_0 h_3^2 \in E_2(S)$ is the boundary of a differential, and $d_2(h_4) = h_0 h_3^2$ is the only possibility. $\square$

No map of Hopf invariant one

This recovers a result of Toda, first proved by secondary composition methods.

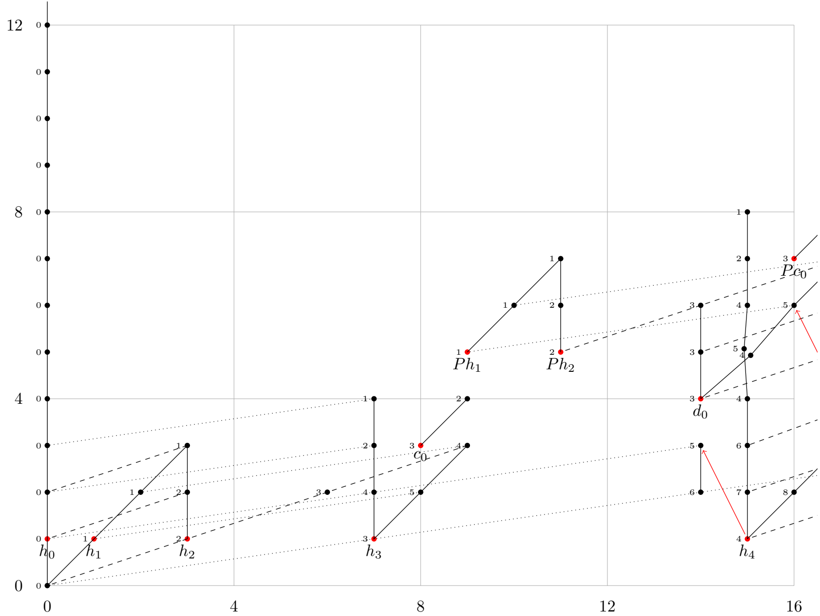
Corollary ([Tod55])

There is no stable map $S^{15} \to S$ of Hopf–Steenrod invariant one. Hence there is no map $S^{31} \to S^{16}$ of Hopf invariant one, no H-space structure on S^{15}, and no division algebra structure on $\mathbb{R}^{16}$.

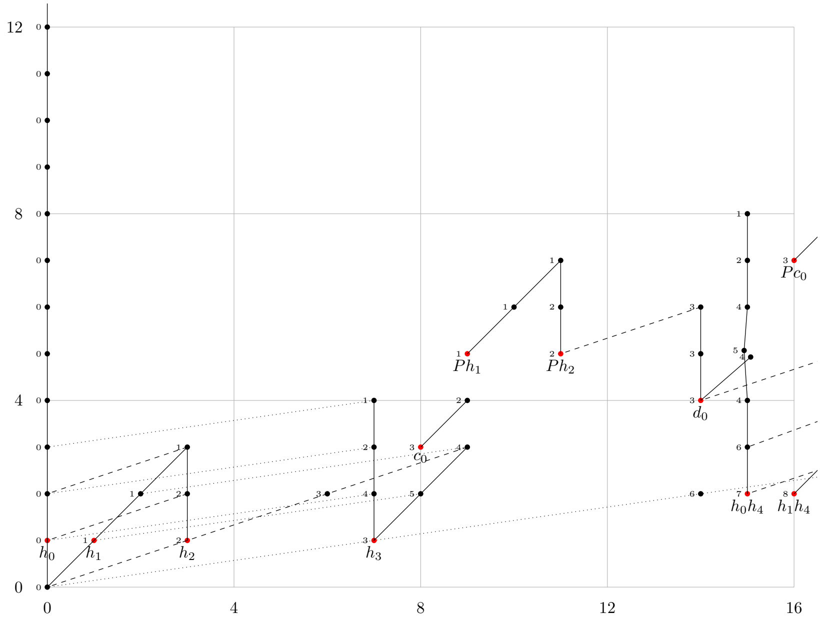
Proof.

Such a map would be detected by h_4, which would have to survive to the E_∞-term, but the nonzero differential $d_2(h_4) = h_0 h_3^2$ shows that this is not the case. □

$E_2(S)$ with d_2-differentials



$E_3(S) = H(E_2(S), d_2)$



The Adams E_3-term for S

- Passing to cohomology with respect to the d_2-differential, we can calculate $E_3(S)$ in our range, and determine its algebra indecomposables.
- Note that $h_0 h_4$ and $h_1 h_4$ were decomposable on $E_2(S)$, but are indecomposable in $E_3(S)$.

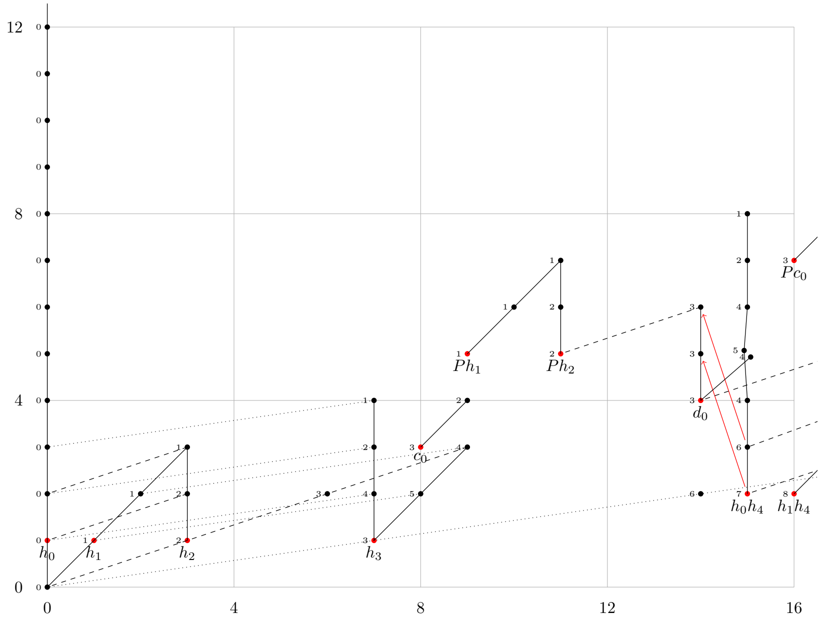
Lemma

For $t - s \leq 16$, the $\mathbb{F}_2$-algebra $E_3^{*,*}(S)$ is generated by the following classes.

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	$h_0 h_4$	$h_1 h_4$	Pc_0
$t - s$	0	1	3	7	8	9	11	14	15	16	16
s	1	1	1	1	3	5	5	4	2	2	7

The h_i-multiplications are visible in the previous figure, and the remaining products in this range are zero.

$E_3(S)$ with d_3-differentials



Adams d_3-differentials for S

Proposition

In the range $t - s \leq 16$, the d_3-differential on the algebra generators is given as follows.

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	$h_0 h_4$	$h_1 h_4$	Pc_0
$d_3(x)$	0	0	0	0	0	0	0	0	$h_0 d_0$	0	0

Proof.

- The d_3-differentials on h_0, h_2, h_3, c_0, Ph_1, Ph_2, d_0 and Pc_0 land in trivial groups, hence are zero. In particular, d_3 commutes with multiplication by any of these elements.

Proof (cont.)

- The differential on h_1 vanishes by h_0-linearity, since

$$h_0 d_3(h_1) = d_3(h_0 h_1) = d_3(0) = 0\,,$$

  while $h_0 h_0^4 \neq 0$, so $d_3(h_1) \neq h_0^4$.
- By h_0-linearity, $d_3(h_1 h_4)$ is h_0-torsion, hence lies in $\{0, h_1 d_0\}$. By calculating $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ in a larger range, we can show that $d_0 \cdot h_1 h_4 = 0$, while $d_0 \cdot h_1 d_0 = h_1 d_0^2 = 9_9 \neq 0$ in $E_2^{9,9+29}(S)$. Moreover, we claim that $h_1 d_0^2$ remains nonzero in $E_3(S)$. This follows from $d_2(k) \neq 0$, which implies $d_2(h_0 k) \neq 0$, $d_2(r) = 0$ and $d_2(h_0 r) = 0$. Hence

$$d_0 \cdot d_3(h_1 h_4) = d_3(d_0 \cdot h_1 h_4) = d_3(0) = 0$$

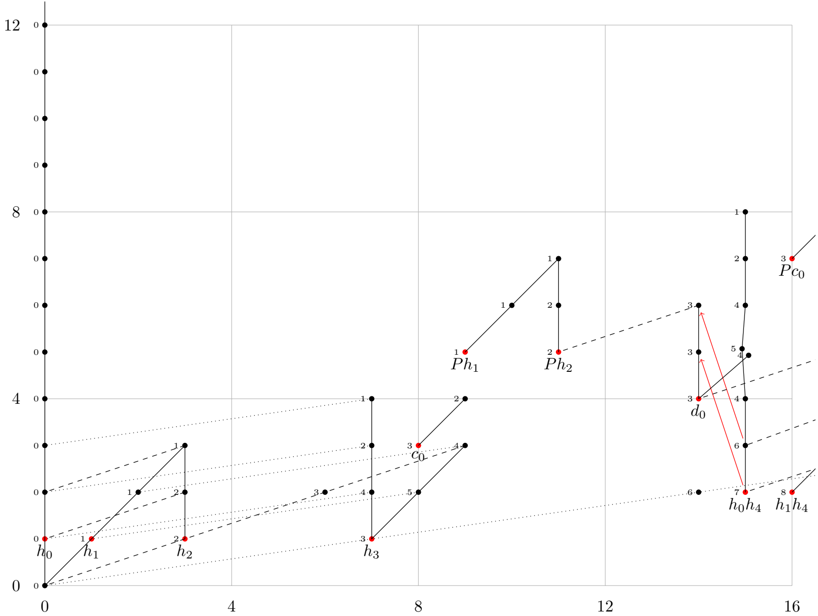
  and $d_0 \cdot h_1 d_0 \neq 0$ in $E_3(S)$ imply that $d_3(h_1 h_4) \neq h_1 d_0$. The only remaining possibility is $d_3(h_1 h_4) = 0$.
- The final case, $d_3(h_0 h_4) = h_0 d_0$, deserves a separate theorem. □

Theorem
$d_3(h_0 h_4) = h_0 d_0$.

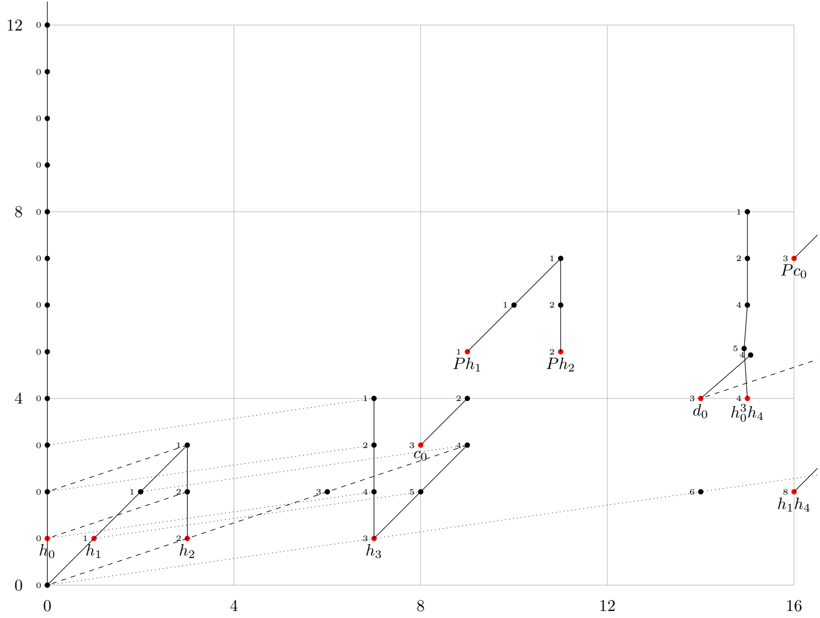
Proof.
((ETC: This can be proved by comparison with the Adams spectral sequence for $C\sigma$, or using the split surjectivity (Adams conjecture) of the Adams e-invariant
$e\colon \pi_{15}(S)^\wedge_2 \to \pi_{15}(j)^\wedge_2 \cong \mathbb{Z}/32$ based on real K-theory.)) □

The Leibniz rule for d_3 implies that $d_3(h_0^2 h_4) = h_0^2 d_0$. Passing to cohomology with respect to the d_3-differential, we can calculate $E_4(S)$ in our range, and determine its algebra indecomposables.

$E_3(S)$ with d_3-differentials



$E_4(S) = H(E_3(S), d_3)$



The Adams E_4-term for S

Lemma

For $t-s \leq 16$, the $\mathbb{F}_2$-algebra $E_4^{*,*}(S)$ is generated by the following classes.

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	$h_0^3 h_4$	$h_1 h_4$	Pc_0
$t-s$	0	1	3	7	8	9	11	14	15	16	16
s	1	1	1	1	3	5	5	4	4	2	7

The h_i-multiplications are visible in the previous figure, and the remaining products in this range are zero.

Collapse at the E_4-term

Proposition

All d_r-differentials for $r \geq 4$ are zero in the range $t - s \leq 16$. Hence $E_4(S) = E_\infty(S)$ in this range.

Proof.

- This is clear for all of the algebra generators other than h_1 and $h_1 h_4$.
- We see that $d_r(h_1) = 0$ in each case by h_0-linearity, since $h_0^{r+1} \neq 0$ in $E_r(S)$ by induction.
- Likewise, $d_r(h_1 h_4) = 0$ for $r \in \{4, 5\}$ by h_0-linearity.
- The only remaining case is $d_6(h_1 h_4) \in \{0, h_0^7 h_4\}$. ((ETC: This can be deduced by Maunder's theorem, or by the construction of a homotopy class η^* detected by $h_1 h_4$, using the quadratic construction $D_2(S^7)$.))

□

Outline

Toda's notation I

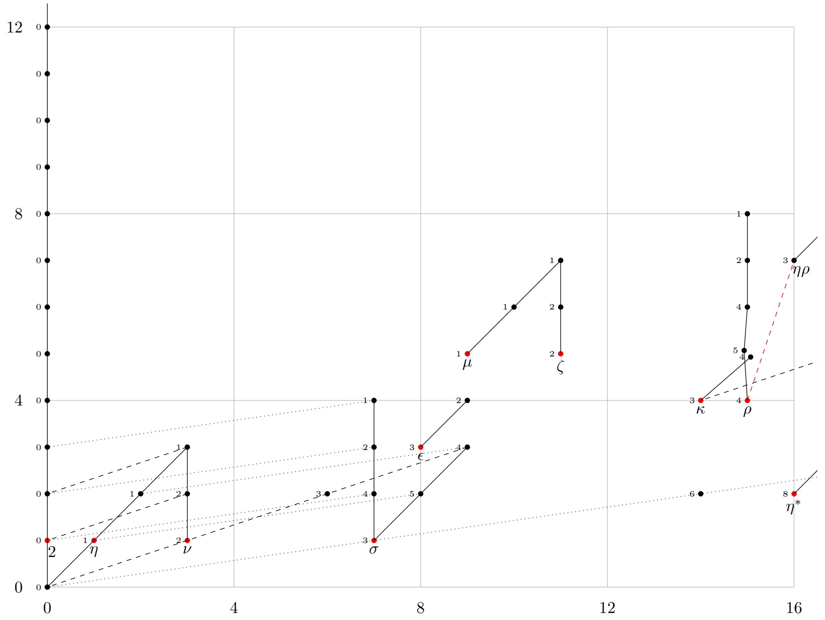
We adopt the following notations from Toda's book [Tod62].

- $\eta \in \pi_1(S)$ is the stable class of the complex Hopf fibration, detected by $h_1 \in E_\infty(S)$ in bidegree $(t-s, s) = (1, 1)$.
- $\nu \in \pi_3(S)$ is the stable class of the quaternionic Hopf fibration, detected by $h_2 \in E_\infty(S)$ in bidegree $(t-s, s) = (3, 1)$.
- $\sigma \in \pi_7(S)$ is the stable class of the octonionic Hopf fibration, detected by $h_3 \in E_\infty(S)$ in bidegree $(t-s, s) = (7, 1)$.
- $\epsilon \in \pi_8(S)_2^\wedge$ is the unique homotopy class detected by $c_0 \in E_\infty(S)$ in bidegree $(t-s, s) = (8, 3)$.
- $\mu \in \pi_9(S)_2^\wedge$ is the unique homotopy class detected by $Ph_1 \in E_\infty(S)$ in bidegree $(t-s, s) = (9, 5)$.

Toda's notation, II

- $\zeta \in \pi_{11}(S)_2^\wedge$ is detected by $Ph_2 \in E_\infty(S)$ in bidegree $(t-s, s) = (11, 5)$. This determines ζ up to an odd multiple. (A definite choice can be made using the J-homomorphism.)
- $\kappa \in \pi_{14}(S)_2^\wedge$ is the unique homotopy class detected by $d_0 \in E_\infty(S)$ in bidegree $(t-s, s) = (14, 4)$.
- $\rho \in \pi_{15}(S)_2^\wedge$ is detected by $h_0^3 h_4 \in E_\infty(S)$ in bidegree $(t-s, s) = (15, 4)$. This determines ρ up to an odd multiple, modulo $\eta\kappa$. (A definite choice can be made using the J-homomorphism.)
- $\eta^* \in \pi_{16}(S)_2^\wedge$ is detected by $h_1 h_4 \in E_\infty(S)$ in bidegree $(t-s, s) = (16, 2)$. This determines η^* modulo $\eta\rho$. (A definite choice can be made using the Adams e-invariant.)

The associated graded of $\pi_n(S)$ for $0 \leq n \leq 16$



Hidden extensions

Let Y be an S-module, so that the Adams spectral sequence $E_r(Y)$ is an $E_r(S)$-module spectral sequence converging to $\pi_*(Y)$.

Definition

Let $\alpha \in \pi_*(S)$ be detected by $a \in E_\infty(S)$, and consider nonzero classes b and $c \in E_\infty(Y)$. We say that there is an α-extension from b to c if there exists a $\beta \in \pi_*(Y)$ such that

- β is detected by b,
- $\alpha\beta$ is detected by c, and
- there is no class $\beta' \in \pi_*(Y)$ of higher Adams filtration than β for which $\alpha\beta'$ is detected by c.

This is a hidden α-extension if $ab = 0$.

Remarks

- In the definition of (hidden) α-extensions, c should be viewed as being defined modulo the classes (in the same bidegree) detecting products $\alpha\beta'$ with β' of higher Adams filtration than β.
- More generally, we can consider maps $f\colon X \to Y$ and compare the filtrations

$$\cdots \subset f_*(F^s\pi_*(X)) \subset \cdots \subset f_*(\pi_*(X))$$
$$\cdots \subset F^u\pi_*(Y) \subset \cdots \subset \pi_*(Y)$$

to form the bifiltration $\Phi^{s,u} = f_*(F^s\pi_*(X)) \cap F^u\pi_*(Y)$. The group

$$\frac{\Phi^{s,u}}{\Phi^{s+1,u} + \Phi^{s,u+1}}$$

measures filtration shifts by f_* from s to u.

A hidden η-extension

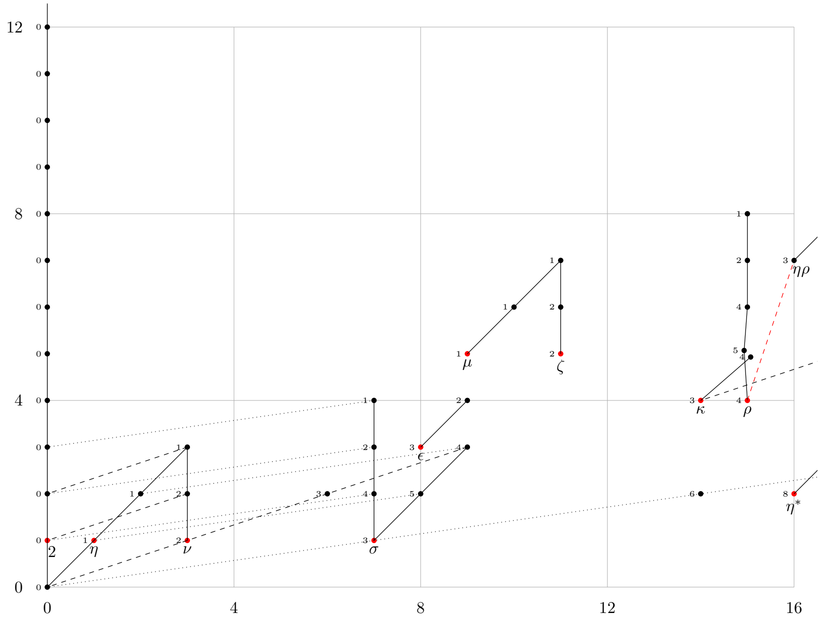
Proposition

$\eta\rho \in \pi_{16}(S)^\wedge_2$ is detected by $Pc_0 \in E_\infty(S)$ in bidegree $(t-s, s) = (16, 7)$, while $\eta^2\kappa = 0$. Hence there is a hidden η-extension from $h_0^3 h_4$ to Pc_0.

Proof.

((ETC: This can be deduced using the *e*-invariant to the image-of-*J* spectrum, or perhaps by a comparison with the Adams spectral sequence for $C\eta$.)) □

The associated graded of $\pi_n(S)$ for $0 \leq n \leq 16$



The notation $\{a\} \subset G$ for $a \in E_\infty$

Definition

- When a spectral sequence (E_r, d_r) converges to G, and $a \in E_\infty^s$ is a nonzero class, we write $\{a\} \subset G$ for the set of $\alpha \in G$ that are detected by a.
- This is the coset of $F^{s+1}G$ in F^sG that corresponds to a under the isomorphism $F^sG/F^{s+1}G \cong E_\infty^s$.
- When $F^{s+1}G = 0$ in the total degree of a, this is a single element and we write $\alpha = \{a\}$.

We next summarize these initial findings about the graded commutative ring $\pi_*(S)_2^\wedge$, in degrees $* \leq 16$. We write $\mathbb{Z}/n\{\alpha\}$ for the cyclic group of order n generated by a class α.

The graded ring $\pi_*(S)$, I

Theorem

0. $\pi_0(S)^\wedge_2 \cong \mathbb{Z}_2$;
   $2^s \in \{h_0^s\}$ *for* $s \geq 0$.

1. $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2\{\eta\}$;
   $\eta = \{h_1\}$.

2. $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2\{\eta^2\}$;
   $\eta^2 = \{h_1^2\}$.

3. $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/8\{\nu\}$;
   $\nu \in \{h_2\}$, $2\nu \in \{h_0 h_2\}$, $4\nu = \{h_0^2 h_2\}$;
   $\eta^3 = 4\nu$.

4. $\pi_4(S)^\wedge_2 = 0$.

The graded ring $\pi_*(S)$, II

Theorem

5. $\pi_5(S)^\wedge_2 = 0$.

6. $\pi_6(S)^\wedge_2 = \mathbb{Z}/2\{\nu^2\}$;
   $\nu^2 = \{h_2^2\}$.

7. $\pi_7(S)^\wedge_2 = \mathbb{Z}/16\{\sigma\}$;
   $\sigma \in \{h_3\}$, $2\sigma \in \{h_0 h_3\}$, $4\sigma \in \{h_0^2 h_3\}$, $8\sigma = \{h_0^3 h_3\}$.

8. $\pi_8(S)^\wedge_2 = \mathbb{Z}/2\{\epsilon\} \oplus \mathbb{Z}/2\{\eta\sigma\}$;
   $\eta\sigma \in \{h_1 h_3\}$, $\epsilon = \{c_0\}$.

9. $\pi_9(S)^\wedge_2 = \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta\epsilon\} \oplus \mathbb{Z}/2\{\eta^2\sigma\}$;
   $\eta^2\sigma \in \{h_1^2 h_3\}$, $\eta\epsilon \in \{h_1 c_0\}$, $\mu = \{Ph_1\}$;
   $\nu^3 = \eta\epsilon + \eta^2\sigma$.

The graded ring $\pi_*(S)$, III

Theorem

10. $\pi_{10}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\mu\}$;
    $\eta\mu = \{h_1 Ph_1\}$;
    $\eta^2\epsilon = 0$, $\nu\sigma = 0$.
11. $\pi_{11}(S)^\wedge_2 = \mathbb{Z}/8\{\zeta\}$;
    $\zeta \in \{Ph_2\}$, $2\zeta \in \{h_0 Ph_2\}$, $4\zeta = \{h_0^2 Ph_2\}$;
    $\eta^2\mu = 4\zeta$, $\nu\epsilon = 0$.
12. $\pi_{12}(S)^\wedge_2 = 0$.
13. $\pi_{13}(S)^\wedge_2 = 0$.

The graded ring $\pi_*(S)$, IV

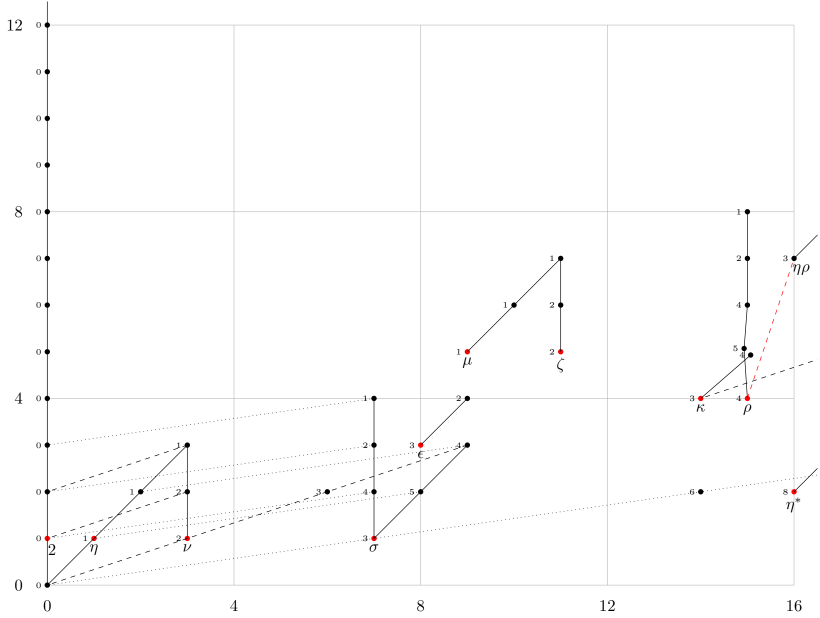
Theorem

14. $\pi_{14}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa\} \oplus \mathbb{Z}/2\{\sigma^2\}$;
$\kappa = \{d_0\}$, $\sigma^2 \in \{h_3^2\}$;
$\nu\zeta = 0$.

15. $\pi_{15}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\}$;
$\rho \in \{h_0^3 h_4\}$, $2\rho \in \{h_0^4 h_4\}$, $4\rho \in \{h_0^5 h_4\}$, $8\rho \in \{h_0^6 h_4\}$,
$16\rho = \{h_0^7 h_4\}$, $\eta\kappa \in \{h_1 d_0\}$;
$\eta\sigma^2 = 0$, $\sigma\epsilon = 0$.

16. $\pi_{16}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\rho\} \oplus \mathbb{Z}/2\{\eta^*\}$;
$\eta\rho = \{Pc_0\}$, $\eta^* \in \{h_1 h_4\}$; $\eta^2\kappa = 0$, $\sigma\mu = \eta\rho$, $\epsilon^2 = 0$.

The associated graded of $\pi_n(S)$ for $0 \leq n \leq 16$



Proof

In many cases, this is immediate from the algebra structure of the E_∞-term, keeping in mind that if α and β are detected by a and b, respectively, then $\alpha\beta$ is detected by ab if $ab \neq 0$, and has higher Adams filtration than this product if $ab = 0$. The following cases require additional argments.

(9) The spectral sequence algebra structure shows that ν^3 is detected by $h_2^2 = h_1^2 h_3$, hence equals $\eta^2\sigma$ modulo Adams filtration ≥ 4, i.e., modulo $\mathbb{F}_2\{\mu, \eta\epsilon\}$. The KO-theory d- and e-invariants, which combine to a map $e \colon S \to j$ to the image-of-J spectrum, show that we must have $\nu^3 = \eta^2\sigma + \eta\epsilon$.

(10) The map to the image-of-J detects $\eta\mu$, but not $\eta^2\epsilon$ or $\nu\sigma$, so the latter two products are zero.

Proof (cont.)

(11) The image-of-J detects ζ, 2ζ and 4ζ but not $\nu\epsilon$, so the latter product is zero.

(14) The product $\nu\zeta$ has Adams filtration $\geq 1 + 5 = 6$, hence is zero, since the E_∞-classes in total degree 14 all have lower Adams filtration.

(15) The image-of-J shows that $\eta\sigma^2$ and $\sigma\epsilon$ lie in $\mathbb{F}_2\{0, \eta\kappa\}$. ((ETC: Justify $\eta\sigma^2 = 0$ and $\sigma\epsilon = 0$.))

(16) The relations $\eta^2\kappa = 0$, $\sigma\mu = \eta\rho$ and $\epsilon^2 = 0$ are all detected in the image-of-J spectrum. Since they all lie in Adams filtrations greater than that of η^*, they also hold in the homotopy of S. $\square$

Toda's relation in $\pi_9(S)$

Remark

The relation $\nu \cdot \nu^2 = \eta^2\sigma + \eta\epsilon$ shows that the (hidden or visible) α-extensions do not completely determine the multiplicative action by α, since there may be higher filtration terms that are not seen by the α-extension. In this case there is a ν-extension from h_2^2 to $h_2^3 = h_1^2 h_3$, and $\eta\epsilon$ is the higher-filtration term.

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