MAT9580: Spectral Sequences

Chapters 1, 2 and 3: Spectral Sequences, Exact Couples and Filtrations

John Rognes

University of Oslo, Norway

May 4, 2021

Outline

Outline

Overview, I

- General algebraic theory of spectral sequences

$$E^r_{s,t} \Longrightarrow_s G_{s+t}$$

- (E^r, d^r)-terms
- E^∞-term
- Filtered abutment
- Convergence

Jean Leray

Overview, II

- The Serre spectral sequence

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)$$

- Applications relating homotopy and homology
- Cohomological version
- Cup product structure
- Steenrod operations

Henri Cartan, Jean-Pierre Serre

Overview, III

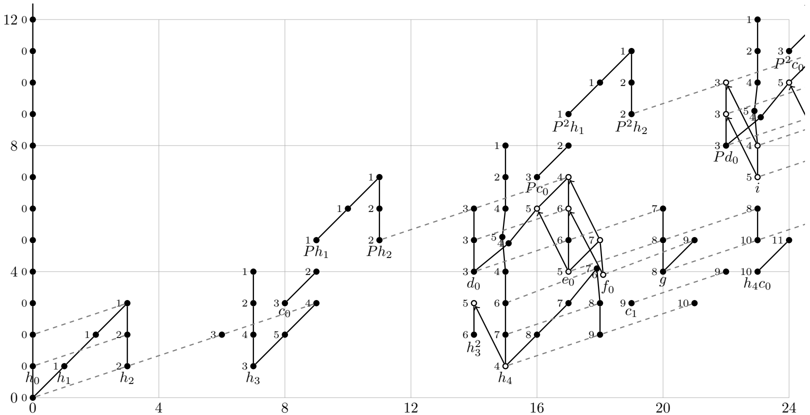
- The Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y;\mathbb{F}_p),\mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(Y_p^\wedge)$$

- Orthogonal spectra
- Steenrod algebra
- Ext-calculations
- Product structure
- Toda brackets
- Power operations

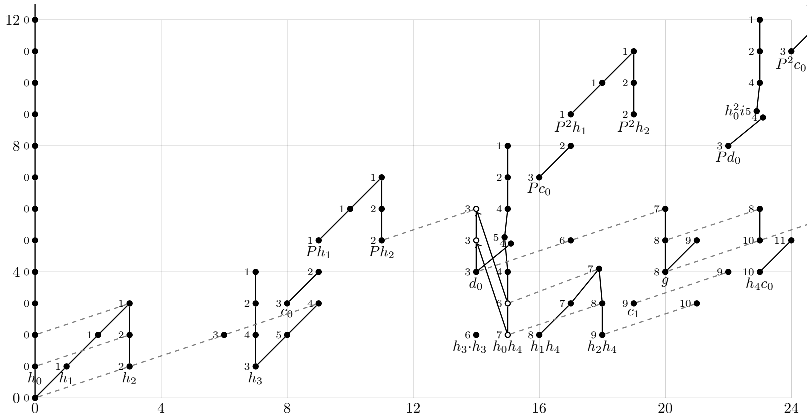
Frank Adams

Adams spectral sequence for the sphere, I



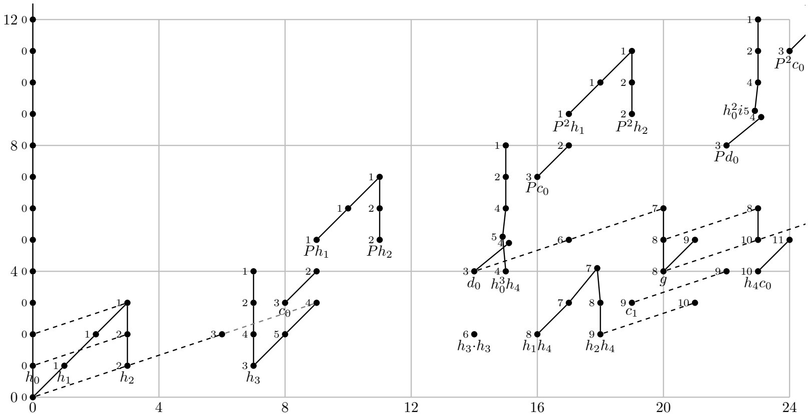
(E_2, d_2)-term for $\pi_*(S)$

Adams spectral sequence for the sphere, II



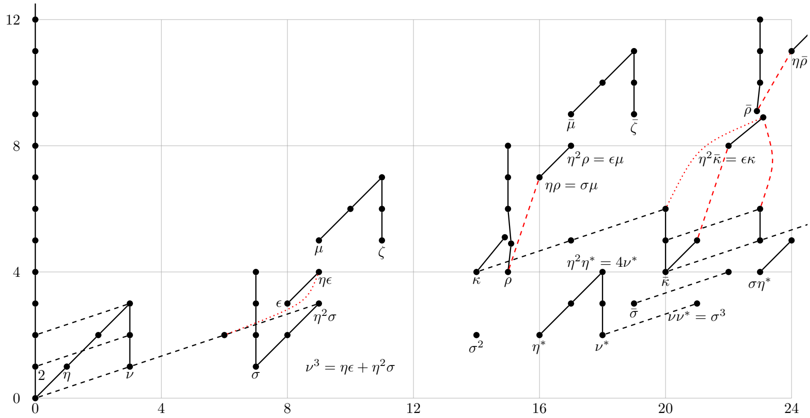
(E_3, d_3)-term for $\pi_*(S)$

Adams spectral sequence for the sphere, III



E_∞-term for $\pi_*(S)$

Adams spectral sequence for the sphere, IV



Hidden extensions for $\pi_*(S)$

Outline

Spectral Sequences

- start with the abstract definition of a spectral sequence;
- same concepts as the definition of a chain complex and its homology, but involves multiple indices;
- next discuss in what sense a spectral sequence can calculate a given abutment;
- some relatively simple examples, to get accustomed to the roles of the indices, and the meaning of convergence.

Outline

Bigraded abelian groups

Definition

- A bigraded abelian group $A = A_{*,*}$ is a doubly-indexed sequence
$$A_{*,*} = (A_{s,t})_{s,t}$$
of abelian groups, where $s, t \in \mathbb{Z}$.
- A morphism $f\colon A \to B$ of bigraded abelian groups is a sequence of group homomorphisms
$$f_{s,t}\colon A_{s,t} \longrightarrow B_{s,t}\,.$$

Bigraded morphisms

Definition

- A morphism $f\colon A \to B$ of bidegree (u, v) is a sequence of group homomorphisms

$$f_{s,t}\colon A_{s,t} \longrightarrow B_{s+u,t+v}$$

  for all $s, t \in \mathbb{Z}$.
- The composite of f followed by a morphism $g\colon B \to C$ of bidegree (u', v') is a morphism $gf\colon A \to C$ of bidegree $(u + u', v + v')$.

Differentials

Definition

- $E = E_{*,*}$ a bigraded abelian group; r an integer.
- A differential $d\colon E \to E$ of bidegree (u, v) is a morphism of bidegree (u, v) such that $dd = 0$.
- For each pair $s, t \in \mathbb{Z}$ we have a homomorphism

$$d_{s,t}\colon E_{s,t} \longrightarrow E_{s+u,t+v}$$

and the composite

$$E_{s-u,t-v} \xrightarrow{d_{s-u,t-v}} E_{s,t} \xrightarrow{d_{s,t}} E_{s+u,t+v}$$

is the zero homomorphism.

Bigraded kernel and image

Definition

Let the kernel $\ker(d) = \ker(d)_{*,*}$ be the bigraded abelian group

$$\ker(d)_{s,t} = \ker(d_{s,t})$$

and let the image $\mathrm{im}(d) = \mathrm{im}(d)_{s,t}$ be the bigraded abelian group

$$\mathrm{im}(d)_{s,t} = \mathrm{im}(d_{s-u,t-v})\,.$$

Then

$$\mathrm{im}(d)_{s,t} \subset \ker(d)_{s,t} \subset E_{s,t}$$

for all $s, t \in \mathbb{Z}$.

Cycles, boundaries, homology

Definition

We call $\ker(d)$ and $\operatorname{im}(d)$ the d-cycles and d-boundaries in E, respectively. The homology of (E, d) is the bigraded abelian group

$$H(E, d) = \frac{\ker(d)}{\operatorname{im}(d)}$$

given in bidegree (s, t) by the subquotient

$$H(E, d)_{s,t} = H_{s,t}(E, d) = \frac{\ker(d)_{s,t}}{\operatorname{im}(d)_{s,t}} = \frac{\ker(d_{s,t})}{\operatorname{im}(d_{s-u,t-v})}$$

of $E_{s,t}$. We write $[x] \in H(E, d)$ for the homology class of a d-cycle $x \in \ker(d)$.

Homological spectral sequence

Definition

A homological spectral sequence $(E^r, d^r)_{r \geq 1}$ is a sequence of bigraded abelian groups $E^r = E^r_{*,*}$ and differentials

$$d^r \colon E^r \longrightarrow E^r$$

of bidegree $(-r, r-1)$, together with isomorphisms

$$H(E^r, d^r) \cong E^{r+1}$$

for all integers $r \geq 1$.

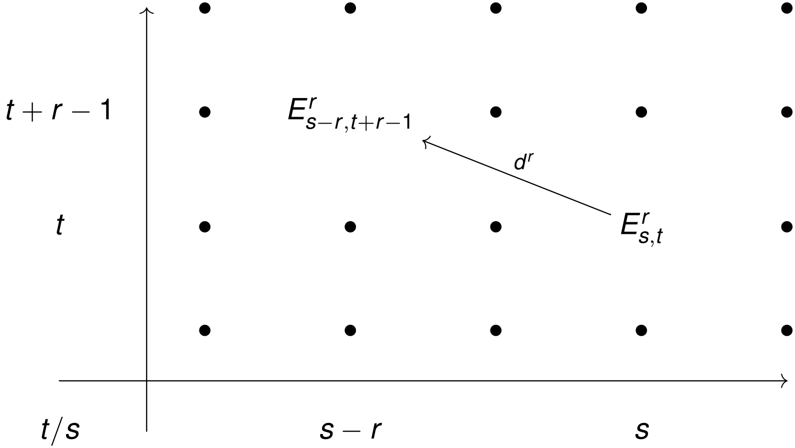
Remarks

- We call E^r and d^r the E^r-term and d^r-differential of the spectral sequence, respectively.
- In each bidegree (s, t) we refer to
  - s as the filtration degree,
  - t as the complementary degree, and
  - $s + t$ as the total degree.

  Each d^r-differential reduces the total degree by 1.
- The isomorphisms $H(E^r, d^r) \cong E^{r+1}$ are part of the structure of the spectral sequence.
- An E^p-spectral sequence $(E^r, d^r)_{r \geq p}$ is a sequence of bigraded abelian groups and differentials, as above, but indexed on the integers $r \geq p$.

Visualization

- Spread $E_{*,*}$ out in the (s,t)-plane, with $E_{s,t}$ at horizontal coordinate s and vertical coordinate t.
- View each component $d^r_{s,t}\colon E_{s,t} \to E_{s+u,t+v}$ of a d^r-differential as an arrow from position (s,t) to position $(s-r, t+r-1)$.



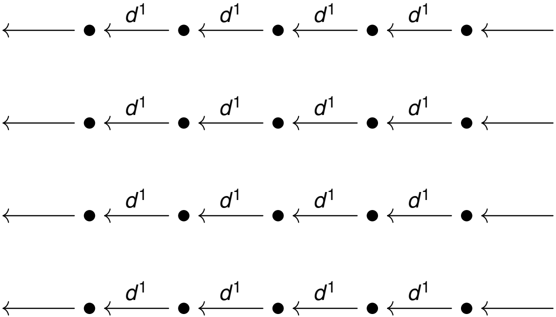
Surviving classes

- If $d^r_{s,t}(x) = y$ we say that x supports a d^r-differential, and that y is hit (or "killed") by a d^r-differential.
- The classes that support a nonzero d^r-differential are not present at the E^{r+1}-term, and the classes that are hit by a d^r-differential are set equal to zero at the E^{r+1}-term.
- Informally, the classes that support differentials, or are hit by differentials, do not "survive" to the next term.

Pages

Some authors refer to the E^r-term as the E^r-page. The transition from E^r to its subquotient E^{r+1} can be viewed as turning one page over to reveal the next.

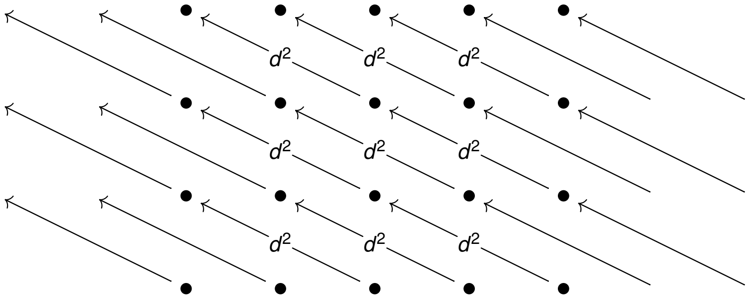
E^1 :



Pages

Some authors refer to the E^r-term as the E^r-page. The transition from E^r to its subquotient E^{r+1} can be viewed as turning one page over to reveal the next.

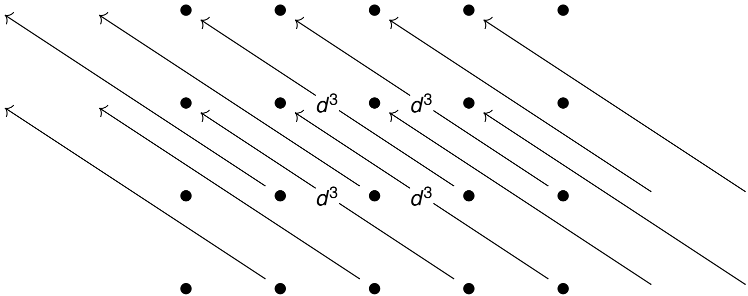
E^2 :



Pages

Some authors refer to the E^r-term as the E^r-page. The transition from E^r to its subquotient E^{r+1} can be viewed as turning one page over to reveal the next.

E^3:



Other gradings

- Most spectral sequences are bigraded, as in the definition above.
- Often one grading comes from a filtration and the other comes from a degree shift present in a long exact sequence.
- There are also cases where the complementary degree t is not present, or appears with the opposite sign, or is itself a multigrading.
- The key feature of a homological spectral sequence is that the d^r-differential reduces the filtration degree from s to $s - r$.

Morphisms of differential bigraded groups

Definition

- (E, d) and $('E, 'd)$ bigraded abelian groups with differentials of bidegree (u, v).
- A morphism $\phi\colon (E, d) \to ('E, 'd)$ is a morphism $\phi\colon E \to 'E$ that commutes with the differentials:

$$\begin{array}{ccc} E_{s,t} & \xrightarrow{\phi_{s,t}} & 'E_{s,t} \\ {\scriptstyle d_{s,t}}\downarrow & & \downarrow{\scriptstyle 'd_{s,t}} \\ E_{s+u,t+v} & \xrightarrow{\phi_{s+u,t+v}} & 'E_{s+u,t+v} \end{array}$$

- There is then an induced morphism

$$\phi_*\colon H(E, d) \longrightarrow H('E, 'd)$$

given by $\phi_*[x] = [\phi(x)]$ for each d-cycle x in E.

Morphisms of spectral sequences

Definition

- $E = (E^r, d^r)_{r\geq 1}$ and $'E = ('E^r, 'd^r)_{r\geq 1}$ spectral sequences.
- A morphism $\phi\colon E \to {}'E$ of spectral sequences is a sequence of morphisms

$$\phi^r : (E^r, d^r) \longrightarrow ('E^r, 'd^r)$$

of differential bigraded abelian groups, such that the diagram

$$\begin{array}{ccc} H(E^r, d^r) & \xrightarrow{\phi^r_*} & H('E^r, 'd^r) \\ \cong\downarrow & & \downarrow\cong \\ E^{r+1} & \xrightarrow{\phi^{r+1}} & 'E^{r+1} \end{array}$$

commutes for each $r \geq 1$.

Historical remarks, I

- Sheaves, sheaf cohomology and spectral sequences were invented by Jean Leray around 1943.
- First published references [Ler46a] and [Ler46b].
- For a map $f\colon X \to Y$ of spaces, Leray constructed a sheaf of graded abelian groups over Y, and obtained a spectral sequence with initial term given by the cohomology of Y with coefficients in this sheaf, converging to the cohomology of X.
- The current algebraic formalism, where the E^{r+1}-term is expressed as the homology of a d^r-differential acting on the E^r-term, is due to Jean-Louis Koszul [Kos47].

Historical remarks, II

- Similar structures were implicitly present in the 1946 PhD thesis of Roger C. Lyndon [Lyn48].
- The name “suite spectrale” is due to Jean–Pierre Serre [Ser51], merging the names “anneau spectral” of [Ler50] and “suite de Leray–Koszul”.
- See the articles by John McCleary [McC99] and Haynes Miller [Mil00] for more on the history of spectral sequences.

Outline

The E^∞-term

- To each spectral sequence (E^r, d^r) we will associate a limiting bigraded abelian group $E^\infty = E^\infty_{*,*}$, called the E^∞-term.
- The general definition requires some details that we will discuss later.
- Now describe some special cases for which the E^∞-term can be read off from the E^r-terms for finite r.

Collapse at E^q

Definition

- A spectral sequence (E^r, d^r) collapses at the E^q-term if $d^r = 0$ for all $r \geq q$.
- It stabilizes in each bidegree if for each bidegree (s, t) there is a $q(s, t)$ such that

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$
$$d^r_{s+r,t-r+1} \colon E^r_{s+r,t-r+1} \longrightarrow E^r_{s,t}$$

  are both zero for all $r \geq q(s, t)$.

The latter condition is strictly weaker.

E^r-terms stabilize

Lemma

If (E^r, d^r) collapses at the E^q-term, then $E^r \cong H(E^r, d^r) \cong E^{r+1}$ for all $r \geq q$, so that there are isomorphisms

$$E^q \cong E^{q+1} \cong \cdots . E^r \cong \ldots$$

for all $r \geq q$.

Proof.

If $d^r = 0$ then $\ker(d^r) = E^r$ and $\mathrm{im}(d^r) = 0$, so $H(E^r, d^r) = E^r/0 \cong E^r$. By the assumption that (E^r, d^r) is a spectral sequence, this is isomorphic to E^{r+1}. □

$E^r_{s,t}$-terms stabilize

Lemma

If (E^r, d^r) stabilizes in each bidegree, then for each bidegree (s,t) there are isomorphisms

$$E^q_{s,t} \cong E^{q+1}_{s,t} \cong \cdots \cdot E^r_{s,t} \cong \dots$$

for all $r \geq q = q(s,t)$.

Proof.

For each (s,t) and $r \geq q(s,t)$ we have $\ker(d^r)_{s,t} = E^r_{s,t}$ and $\operatorname{im}(d^r)_{s,t} = 0$, so $H(E^r, d^r)_{s,t} = E^r_{s,t}/0 \cong E^r_{s,t}$, and this is isomorphic to $E^{r+1}_{s,t}$. □

Preliminary definition of E^∞

Let (E^r, d^r) be a spectral sequence.

Lemma

If (E^r, d^r) collapses at the E^q-term, then $E^\infty \cong E^q$ is isomorphic to the common value of E^r for $r \geq q$.

Let $\phi \colon E \to {}'E$ be a morphism of spectral sequences.

Lemma

If (E^r, d^r) and $({}'E^r, {}'d^r)$ both collapse at the E^q-term, then $\phi^\infty \colon E^\infty \to {}'E^\infty$ corresponds to $\phi^r \colon E^r \to {}'E^r$ for each $r \geq q$.

Preliminary definition of E^∞

Let (E^r, d^r) be a spectral sequence.

Lemma

If (E^r, d^r) stabilizes in each bidegree, then for each bidegree (s, t) there are isomorphisms $E^\infty_{s,t} \cong E^r_{s,t}$ for all sufficiently large r.

Let $\phi\colon E \to {}'E$ be a morphism of spectral sequences.

Lemma

If (E^r, d^r) and $({}'E^r, {}'d^r)$ stabilize in each bidegree, then $\phi^\infty_{s,t}\colon E^\infty_{s,t} \to {}'E^\infty_{s,t}$ corresponds, for each bidegree (s, t), to $\phi^r_{s,t}\colon E^r_{s,t} \to {}'E^r_{s,t}$ for all sufficiently large r.

Filtrations

Definition

- An increasing filtration $(F_s G)_s$ of an abelian group G is a sequence of subgroups

$$\cdots \subset F_{s-1}G \subset F_s G \subset \cdots \subset G\,.$$

- For each filtration degree s there is a short exact sequence

$$0 \to F_{s-1}G \longrightarrow F_s G \longrightarrow \frac{F_s G}{F_{s-1}G} \to 0 \qquad (1)$$

  that expresses $F_s G$ as an extension.

- The graded abelian group

$$(F_s G / F_{s-1}G)_s$$

  is called the associated graded of the filtration $(F_s G)_s$.

Bounded filtrations

Definition

The filtration is bounded if there are integers a and b such that $F_{a-1}G = 0$ and $F_bG = G$.

In this case the sequence is determined by the finitely many terms

$$0 = F_{a-1}G \subset F_aG \subset \cdots \subset F_{b-1}G \subset F_bG = G\,,$$

extended by identities on both sides.

Extension problems

- If we have inductively determined $F_{s-1}G$, and know the filtration quotient $F_sG/F_{s-1}G$, then the next term F_sG is partially determined by the short exact sequence (1).
- There can be several non-isomorphic abelian group extensions with the same subgroup and quotient group, and the task of determining which of these is realized by F_sG is known as the extension problem in filtration s.
- If the filtration is bounded, then this inductive argument involves finitely many extension problems, starting with $s = a$ and ending with $s = b$.

Graded filtrations

Definition

- An increasing filtration of a graded abelian group $G_* = (G_n)_n$, where $n \in \mathbb{Z}$, is a sequence of graded subgroups

$$\cdots \subset F_{s-1}G_* \subset F_sG_* \subset \cdots \subset G_* \,.$$

- We call s the filtration degree and n the total degree.
- For each s there is a short exact sequence

$$0 \to F_{s-1}G_* \longrightarrow F_sG_* \longrightarrow \frac{F_sG_*}{F_{s-1}G_*} \to 0\,.$$

- This consists of an extension

$$0 \to F_{s-1}G_n \longrightarrow F_sG_n \longrightarrow \frac{F_sG_n}{F_{s-1}G_n} \to 0$$

in each total degree n.

Degreewise bounded filtrations

Definition

- The associated graded $F_sG_n/F_{s-1}G_n$ of the filtration is bigraded, either by (s, n) or by $(s, t) = (s, n - s)$.
- The filtration of G_* is bounded if there are integers a and b such that $F_{a-1}G_* = 0$ and $F_bG_* = G_*$.
- It is degreewise bounded if for each total degree n there are integers $a = a(n)$ and $b = b(n)$ such that $F_{a-1}G_n = 0$ and $F_bG_n = G_n$.
- In these cases the filtration in total degree n is determined by finitely many terms, extended by identities in both directions.

Convergence

- $(E^r_{*,*}, d^r)$ a spectral sequence.
- $(F_s G_*)_s$ a filtration of a graded abelian group G_*.
- Suppose that the spectral sequence stabilizes in each bidegree, and that the filtration is degreewise bounded.

Definition

We say that the spectral sequence converges to G_*, written

$$E^r_{*,*} \Longrightarrow G_* \,,$$

if there are isomorphisms

$$E^\infty_{s,t} \cong \frac{F_s G_{s+t}}{F_{s-1} G_{s+t}}$$

in all bidegrees (s, t).

Abutment

- The choice of filtration of G_*, and of the isomorphisms displayed above, are implicitly part of the convergence assertion.
- We call G_* the abutment of the spectral sequence.
- To emphasize the filtration degree s, and the relation between the complementary degree and the total degree, we may write
$$E^r_{s,t} \Longrightarrow_s G_{s+t} \,.$$

Strategy, I

When $E^r_{*,*} \Longrightarrow G_*$, the strategy for using the spectral sequence $(E^r_{*,*}, d^r)_{r \geq p}$ to calculate G_* is the following:

- We assume that the initial term $E^p_{*,*}$ can somehow be calculated.
- Furthermore, for each $r \geq p$ we assume that the differentials d^r can be calculated, so that we can inductively obtain $E^{r+1}_{*,*}$ as $H(E^r, d^r)_{*,*}$, for each $r \geq p$.
- Under the hypothesis that the spectral sequence stabilizes in each bidegree, we can let $E^\infty_{s,t} = E^r_{s,t}$ for $r \geq q(s,t)$ sufficiently large.
- By convergence, these are also the groups $F_s G_n / F_{s-1} G_n$ for $n = s + t$.

Strategy, II

- Consider one total degree n.
- Assuming that the filtration is degreewise bounded, we know that $F_s G_n = 0$ for $s < a(n)$ sufficiently small.
- For each $s \geq a(n)$ we must inductively solve an extension problem to determine $F_s G_n$ from $F_{s-1} G_n$ and $E^{\infty}_{s,n-s}$.
- Once $s = b(n)$ is sufficiently large, this recovers $F_s G_n = G_n$, which is the total degree n component of the abutment of the spectral sequence.

Filtration-preserving morphisms

Definition

- Let G and $'G$ be abelian groups, filtered by $(F_s G)_s$ and $(F_s{}'G)_s$, respectively.
- A homomorphism $\psi\colon G \to {}'G$ is filtration-preserving if $\psi(F_s G) \subset F_s{}'G$ for each s.
- If G_* and $'G_*$ are filtered graded abelian groups, and $\psi\colon G_* \to {}'G_*$ is a degree-preserving morphism, then the same definitions apply.

Induced maps of extensions

Definition

- Let $\psi_s \colon F_sG \to F_s{}'G$ be the restriction of ψ, and let $\bar{\psi}_s \colon F_sG/F_{s-1}G \to F_s{}'G/F_{s-1}{}'G$ be the induced homomorphism between the filtration quotients.
- We obtain a vertical map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{s-1}G & \rightarrowtail & F_sG & \twoheadrightarrow & \dfrac{F_sG}{F_{s-1}G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_{s-1}} & & \downarrow{\scriptstyle \psi_s} & & \downarrow{\scriptstyle \bar{\psi}_s} & & \\ 0 & \longrightarrow & F_{s-1}{}'G & \rightarrowtail & F_s{}'G & \twoheadrightarrow & \dfrac{F_s{}'G}{F_{s-1}G} & \longrightarrow & 0 \end{array} \qquad (2)$$

for each s.

Convergence to a morphism

Definition

- Let $(E^r_{*,*}, d^r)$ and $({}'E^r_{*,*}, {}'d^r)$ be spectral sequences converging to G_* and ${}'G_*$.
- Let $\phi\colon E \to {}'E$ be a morphism of bigraded spectral sequences, and let $\psi\colon G_* \to {}'G_*$ be a morphism of filtered graded abelian groups.
- We say that the spectral sequence morphism ϕ converges to the filtration-preserving morphism ψ if the diagram

$$\begin{array}{ccc} E^\infty_{s,*} & \xrightarrow{\cong} & \dfrac{F_s G_*}{F_{s-1} G_*} \\ \Big\downarrow{\scriptstyle \phi^\infty_{s,*}} & & \Big\downarrow{\scriptstyle \bar\psi_{s,*}} \\ {}'E^\infty_{s,*} & \xrightarrow{\cong} & \dfrac{F_s {}'G_*}{F_{s-1} {}'G_*} \end{array} \tag{3}$$

commutes for each s.

Strategy for morphisms

- Suppose we have resolved the extension problems for spectral sequences (E^r, d^r) and $({}'E^r, {}'d^r)$ converging to $G = G_*$ and ${}'G = {}'G_*$.
- Suppose also that there is a morphism $\phi\colon E \to {}'E$ converging to $\psi\colon G \to {}'G$.
- Then we can inductively attempt to determine ψ from ϕ^∞.
- Assuming that we have determined ψ_{s-1}, we obtain $\bar{\psi}_s$ from ϕ^∞_s via the commutative diagram (3).
- It then remains to identify ψ_s in diagram (2).
- In general there can be several different homomorphisms $F_s G \to F_s{}'G$ that make the diagram commute.

Outline

LES as spectral sequence, I

Pair of spaces (X, A), with associated long exact sequence

$$\to H_{n+1}(X, A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{j} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \to$$

To analyze $H_*(X)$ in terms of $H_*(A)$ and $H_*(X, A)$:

- Determine the connecting homomorphisms ∂_n
- Calculate their kernels and cokernels
- Recover result from the extension

$$0 \to \mathrm{cok}(\partial_{n+1}) \longrightarrow H_n(X) \longrightarrow \ker(\partial_n) \to 0\,.$$

LES as spectral sequence, II

- Spectral sequences provide a similar framework when the pair $A \subset X$ is generalized to a longer sequence

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

  of subspaces of X.
- Now spell out how the study of $H_*(X)$ in terms of the long exact sequence above can be expressed in terms of the spectral sequence formalism.

(E^1, d^1)-term

Let (X, A) be a pair of spaces. We will specify an associated spectral sequence $(E^r, d^r)_{r\geq 1}$. First, let $E^1 = E^1_{*,*}$ be given by

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{if } s = 0, \\ H_{1+t}(X, A) & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, let $d^1 \colon E^1_{s,t} \to E^1_{s-1,t}$ be given by

$$d^1_{1,t} = \partial_{1+t} \colon H_{1+t}(X, A) \longrightarrow H_t(A)$$

for $s = 1$, and $d^1_{s,t} = 0$ otherwise.

(s, t)-planar chart of (E^1, d^1)

Depict the (E^1, d^1)-term in the (s, t)-plane, with horizontal coordinate s and vertical coordinate t. Concrete case on the left, abstract notation on the right:

$$
\begin{array}{c|ccc}
 & \vdots & & \vdots \\
t & H_t(A) & \xleftarrow{\partial_{1+t}} & H_{1+t}(X,A) \\
 & \vdots & & \vdots \\
1 & H_1(A) & \xleftarrow{\partial_2} & H_2(X,A) \\
0 & H_0(A) & \xleftarrow{\partial_1} & H_1(X,A) \\
\hline
t/s & 0 & & 1
\end{array}
\qquad\qquad
\begin{array}{c|ccc}
 & \vdots & & \vdots \\
t & E^1_{0,t} & \xleftarrow{d^1_{1,t}} & E^1_{1,t} \\
\vdots & \vdots & & \vdots \\
1 & E^1_{0,1} & \xleftarrow{d^1_{1,1}} & E^1_{1,1} \\
0 & E^1_{0,0} & \xleftarrow{d^1_{1,0}} & E^1_{1,0} \\
\hline
t/s & 0 & & 1
\end{array}
$$

Columns, rows, quadrants

- The columns with $s < 0$ or $s > 1$ consist of trivial groups, so we have a two-column spectral sequence.
- To simplify the diagrams let us assume that $H_0(X, A) = 0$, so that the rows with $t < 0$ also consist of trivial groups.
- Then the E^1-term is concentrated in the first quadrant in the (s, t)-plane, and we speak of a first quadrant homological spectral sequence.

d^1 is a differential

- Clearly $d^1 d^1 = 0$, since

  $$d^1_{s,t} d^1_{s+1,t} \colon E^1_{s+1,t} \to E^1_{s-1,t}$$

  maps from a trivial group, or to a trivial group, or both, for each pair (s, t).
- Hence (E^1, d^1) is a bigraded abelian group with differential of bidegree $(-1, 0)$.
- Same as a chain complex of graded abelian groups.

d^1-cycles, d^1-boundaries and E^2-term

The E^2-term of this spectral sequence must be given by the homology groups $E^2_{s,t} = H(E^1, d^1)_{s,t}$. The d^1-cycles are

$$\ker(d^1)_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the d^1-boundaries are

$$\operatorname{im}(d^1)_{s,t} = \begin{cases} \operatorname{im}(\partial_{1+t}) & \text{for } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$E^2_{s,t} \cong \begin{cases} \operatorname{cok}(\partial_{1+t}) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(s,t)-planar chart of E^2

Depict E^2-term in the (s,t)-plane, with $E^2_{s,t}$ in the position where we had $E^1_{s,t}$ earlier. Concrete case on the left, generic notation on the right:

t/s	0	1
	$\vdots$	$\vdots$
t	$\mathrm{cok}(\partial_{1+t})$	$\ker(\partial_{1+t})$
	$\vdots$	$\vdots$
1	$\mathrm{cok}(\partial_2)$	$\ker(\partial_2)$
0	$\mathrm{cok}(\partial_1)$	$\ker(\partial_1)$

t/s	0	1
	$\vdots$	$\vdots$
t	$E^2_{0,t}$	$E^2_{1,t}$
$\vdots$	$\vdots$	$\vdots$
1	$E^2_{0,1}$	$E^2_{1,1}$
0	$E^2_{0,0}$	$E^2_{1,0}$

d^2 is trivial

- Since the E^2-term consists of subquotients of the E^1-term, it remains concentrated in the first quadrant, under our assumption that $H_0(X, A)$ vanishes.
- All components

$$d^2_{s,t} \colon E^2_{s,t} \to E^2_{s-2,t+1}$$

  of the d^2-differential must be zero, because the source can only be nonzero for $0 \leq s \leq 1$, in which case $s - 2 < 0$ and the target is trivial.
- Hence we must have $d^2 = 0$, and then $d^2 d^2 = 0$ is obvious.
- Hence $H(E^2, d^2) \cong E^2$, since $\ker(d^2) = E^2$ and $\mathrm{im}(d^2) = 0$, so that $E^3 \cong E^2$.

Collapse at E^2

- Likewise $d^r = 0$ for all $r \geq 2$, and $E^r \cong E^2$ for all $r \geq 2$.
- The spectral sequence collapses at the E^2-term.
- The limiting term is thus $E^\infty \cong E^2$, with components

$$E^\infty_{s,t} \cong \begin{cases} \mathrm{cok}(\partial_{1+t}) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- The picture of the E^∞-term in the (s, t)-plane equals that of the E^2-term, except that the group labeled $E^2_{s,t}$ is now labeled $E^\infty_{s,t}$.

Filtration of the abutment

We specify a filtration of $G_* = H_*(X)$ by setting

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < 0, \\ \operatorname{im}(i\colon H_n(A) \to H_n(X)) & \text{for } s = 0, \\ H_n(X) & \text{for } s \geq 1. \end{cases}$$

Then

$$0 = F_{-1}H_*(X) \subset F_0 H_*(X) \subset F_1 H_*(X) = H_*(X)$$

is a bounded filtration of the graded abelian group $H_*(X)$.

Convergence, I

- The convergence claim $E^r_{s,t} \Longrightarrow H_{s+t}(X)$ is the assertion that there are isomorphisms

$$E^\infty_{s,t} \cong \frac{F_s H_{s+t}(X)}{F_{s-1} H_{s+t}(X)}$$

for all s and t. This is obvious if $s < 0$ or $s > 1$.

- When $s = 0$, the assertion is that

$$\operatorname{cok}(\partial_{1+t}) \cong \frac{\operatorname{im}(i\colon H_t(A) \to H_t(X))}{0}$$

for each t.

- When $s = 1$, the assertion is that

$$\ker(\partial_{1+t}) \cong \frac{H_{1+t}(X)}{\operatorname{im}(i\colon H_{1+t}(A) \to H_{1+t}(X))}$$

for each t.

Convergence, II

Both of these follow from the part

$$H_{1+t}(A) \xrightarrow{i_{1+t}} H_{1+t}(X) \xrightarrow{j_{1+t}} H_{1+t}(X, A) \xrightarrow{\partial_{1+t}} H_t(A) \xrightarrow{i_t} H_t(X)$$

of the long exact sequence in homology for the pair (X, A), in view of the isomorphisms

$$\mathrm{cok}(\partial_{1+t}) = \frac{H_t(A)}{\mathrm{im}(\partial_{1+t})} = \frac{H_t(A)}{\ker(i_t)} \cong \mathrm{im}(i_t)$$

and

$$\ker(\partial_{1+t}) = \mathrm{im}(j_{1+t}) \cong \frac{H_{1+t}(X)}{\ker(j_{1+t})} = \frac{H_{1+t}(X)}{\mathrm{im}(i_{1+t})} \,.$$

Extension problems

- It remains to find $F_0H_*(X)$ and $F_1H_*(X) = H_*(X)$.
- Convergence in bidegree $(s, t) = (0, n)$ gives

$$F_0H_n(X) = \operatorname{im}(i\colon H_n(A) \to H_n(X)) \cong E^{\infty}_{0,n} = \operatorname{cok}(\partial_{1+n}) .$$

- Convergence in bidegree $(s, t) = (1, n-1)$ gives

$$\frac{H_n(X)}{F_0H_n(X)} \cong \ker(\partial_n) .$$

- Hence the extension

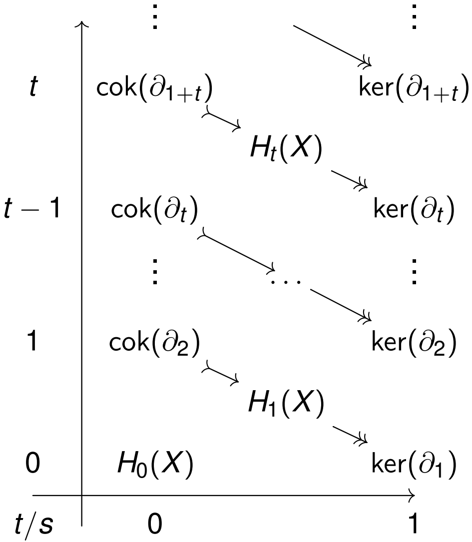
$$0 \to F_0H_n(X) \longrightarrow H_n(X) \longrightarrow \frac{H_n(X)}{F_0H_n(X)} \to 0$$

is nothing but the short exact sequence

$$0 \to \operatorname{cok}(\partial_{1+n}) \longrightarrow H_n(X) \longrightarrow \ker(\partial_n) \to 0 .$$

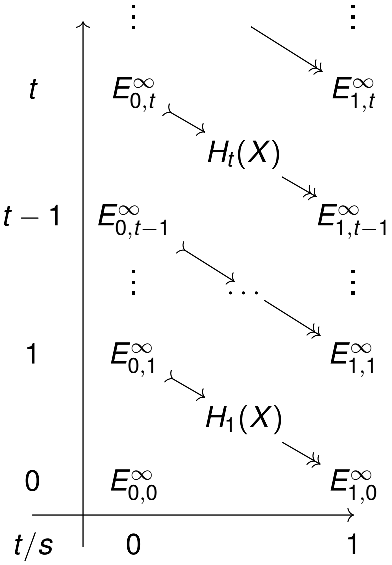
Visualization of extension problems,I

We place the degree n extension on the line of total degree n.
In the (s,t)-plane, this amounts to lines of slope -1.



Visualization, II

In the generic notation:



Visualization, III

- Draw the filtration and the filtration quotients as follows

$$\begin{array}{ccccc} 0 & \rightarrowtail & F_0 H_n(X) & \rightarrowtail & H_n(X) \\ & & \| & & \downarrow\!\!\downarrow \\ & & F_0 H_n(X) & & H_n(X)/F_0 H_n(X) \,. \end{array}$$

- Imagine the upper row being placed along the the line $s + t = n$, with $F_s H_n(X)$ in bidegree $(s, t) = (s, n - s)$, and with the quotients in the lower row appearing as the E^∞-term in the same bidegree.
- In a homological spectral sequence, the differentials map to the left, while the inclusions in the filtration map to the right.

Summary

We have spelled out what we have in mind when we say that there is a convergent spectral sequence

$$E^r_{s,t} \Longrightarrow H_{s+t}(X)$$

with

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ H_{1+t}(X, A) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we might add detail, such as saying that the d^1-differential is given by $d^1_{1,t} = \partial_{1+t} \colon H_{1+t}(X, A) \to H_t(A)$, or that the convergence is with respect to the filtration with $F_0 H_n(X) = \operatorname{im}(i \colon H_n(A) \to H_n(X))$ and $F_1 H_n(X) = H_n(X)$.

Outline

A triple of spaces

We now consider the case of a triple (X, K, A) of spaces, with $A \subset K \subset X$. This leads to the following diagram of (spaces and) pairs of spaces

$$
\begin{array}{ccccc}
A & \longrightarrow & K & \longrightarrow & X \\
 & & \downarrow & & \downarrow \\
 & & (K, A) & \longrightarrow & (X, A) \\
 & & & & \downarrow \\
 & & & & (X, K)
\end{array}
$$

Associated long exact sequences

$$\cdots \to H_n(A) \xrightarrow{i_{K,A}} H_n(K) \xrightarrow{j_{K,A}} H_n(K,A) \xrightarrow{\partial_{K,A}} H_{n-1}(A) \to \dots$$

$$\cdots \to H_n(A) \xrightarrow{i_{X,A}} H_n(X) \xrightarrow{j_{X,A}} H_n(X,A) \xrightarrow{\partial_{X,A}} H_{n-1}(A) \to \dots$$

$$\cdots \to H_n(K) \xrightarrow{i_{X,K}} H_n(X) \xrightarrow{j_{X,K}} H_n(X,K) \xrightarrow{\partial_{X,K}} H_{n-1}(K) \to \dots$$

and

$$\cdots \to H_n(K,A) \xrightarrow{i_{X,K,A}} H_n(X,A) \xrightarrow{j_{X,K,A}} H_n(X,K) \xrightarrow{\partial_{X,K,A}} H_{n-1}(K,A) \to \dots .$$

The last connecting homomorphism can be factored as the composite

$$\partial_{X,K,A} = j_{K,A}\partial_{X,K} \colon H_n(X,K) \xrightarrow{\partial_{X,K}} H_{n-1}(K) \xrightarrow{j_{K,A}} H_{n-1}(K,A) .$$

Goal

- We would like to calculate $H_*(X)$, supposing that we know the homologies

  $$H_*(A)\,,\ H_*(K,A)\,,\ H_*(X,K)$$

  of the "minimal" pairs along the diagonal in the diagram

  $$\begin{array}{ccccc} A & \longrightarrow & K & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & (K,A) & \longrightarrow & (X,A) \\ & & & & \downarrow \\ & & & & (X,K) \end{array}$$

- These involve pairs that are closer together than

  $$H_*(K)\,,\ H_*(X,A)\,,\ H_*(X)\,,$$

  and may therefore be easier to determine.

Long exact sequence approach

- Using only exact sequences, the calculation might be done in two steps, in two different ways.
- On one hand, we might first calculate $H_*(K)$ from $H_*(A)$ and $H_*(K, A)$, and then calculate $H_*(X)$ from $H_*(K)$ and $H_*(X, K)$.
- On the other hand, we might first calculate $H_*(X, A)$ from $H_*(K, A)$ and $H_*(X, K)$, and then calculate $H_*(X)$ from $H_*(A)$ and $H_*(X, A)$.

Spectral sequence approach

- Either approach involves passing to subquotients, resolving extensions, passing to subquotients again, and resolving extensions again.
- Instead, we will express the calculation in terms of a single spectral sequence, where all of the passages to subquotients is performed first, in a symmetric manner, and only thereafter are the extension problems resolved.

Homology spectral sequence of a triple

Proposition

Let (X, K, A) be a triple of spaces. There is a convergent spectral sequence

$$E^r_{s,t} \Longrightarrow_s H_{s+t}(X)$$

with

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ H_{1+t}(K, A) & \text{for } s = 1, \\ H_{2+t}(X, K) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (cont.)

The d^1-differentials are given by the connecting homomorphisms

$$d^1_{s,t} = \begin{cases} \partial_{K,A} \colon H_{1+t}(K,A) \to H_t(A) & \text{for } s = 1, \\ \partial_{X,K,A} \colon H_{2+t}(X,K) \to H_{1+t}(K,A) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The abutment is filtered by

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < 0, \\ \operatorname{im}(i_{X,A} \colon H_n(A) \to H_n(X)) & \text{for } s = 0, \\ \operatorname{im}(i_{X,K} \colon H_n(K) \to H_n(X)) & \text{for } s = 1, \\ H_n(X) & \text{for } s \geq 2. \end{cases}$$

Plan

We show that

- (E^1, d^1) as given is part of a spectral sequence (E^r, d^r)
- that collapses at $E^3 = E^\infty$,
- which is isomorphic to the associated graded of the given filtration of $H_*(X)$.

Note that the description of the E^1-term and the d^1-differential only depend on two of the long exact sequences listed above, namely the ones associated to the pairs (K, A) and (X, K).

Two exact triangles

We can wrap each of these up into an exact triangle, and the two exact triangles are then linked together at a common vertex, given by $H_*(K)$.

$$\begin{array}{ccccc} H_*(A) & \xrightarrow{i_{K,A}} & H_*(K) & \xrightarrow{i_{X,K}} & H_*(X) \\ & \nwarrow{\scriptstyle \partial_{K,A}} & \downarrow{\scriptstyle j_{K,A}} & \nwarrow{\scriptstyle \partial_{X,K}} & \downarrow{\scriptstyle j_{X,K}} \\ & & H_*(K,A) & & H_*(X,K) \end{array} \qquad (4)$$

- The dashed arrows denote homomorphisms of degree -1, sending $H_n(K,A)$ to $H_{n-1}(A)$ and $H_n(X,K)$ to $H_{n-1}(K)$.
- The E^1-term is then given by $H_*(A)$ and the groups in the lower row.
- The d^1-differentials are given by $\partial_{K,A}$ and the composite $j_{K,A}\partial_{X,K}$, all of which are visible in this diagram.

The filtration

$$
\begin{array}{ccccc}
H_*(A) & \xrightarrow{i_{K,A}} & H_*(K) & \xrightarrow{i_{X,K}} & H_*(X) \\
 & \nwarrow^{\partial_{K,A}} & \downarrow j_{K,A} & \nwarrow^{\partial_{X,K}} & \downarrow j_{X,K} \\
 & & H_*(K,A) & & H_*(X,K)
\end{array}
$$

The filtration on the abutment is also visible in this diagram, being given by

- the image of the composite $i_{X,K} i_{K,A}$ for $s = 0$,
- the image of $i_{X,K}$ for $s = 1$, and
- by $H_*(X)$ itself for $s = 2$.

(E^1, d^1)-term

We depict the (E^1, d^1)-term in the (s, t)-plane.
The columns with $s < 0$ or $s > 2$ consist of trivial groups.

$$
\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \vdots \\
t+1 & H_{t+1}(A) & \xleftarrow{\partial_{K,A}} & H_{t+2}(K,A) & \xleftarrow{\partial_{X,K,A}} & H_{t+3}(X,K) \\
t & H_t(A) & \xleftarrow{\partial_{K,A}} & H_{t+1}(K,A) & \xleftarrow{\partial_{X,K,A}} & H_{t+2}(X,K) \\
 & \vdots & & \vdots & & \vdots \\
1 & H_1(A) & \xleftarrow{\partial_{K,A}} & H_2(K,A) & \xleftarrow{\partial_{X,K,A}} & H_3(X,K) \\
0 & H_0(A) & \xleftarrow{\partial_{K,A}} & H_1(K,A) & \xleftarrow{\partial_{X,K,A}} & H_2(X,K) \\
\hline
t/s & 0 & & 1 & & 2
\end{array}
$$

Three-column spectral sequence

In abstract notation, this appears as below.

$$
\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \vdots \\
t+1 & E^1_{0,t+1} & \xleftarrow{d^1_{1,t+1}} & E^1_{1,t+1} & \xleftarrow{d^1_{2,t+1}} & E^1_{2,t+1} \\
t & E^1_{0,t} & \xleftarrow{d^1_{1,t}} & E^1_{1,t} & \xleftarrow{d^1_{2,t}} & E^1_{2,t} \\
 & \vdots & & \vdots & & \vdots \\
1 & E^1_{0,1} & \xleftarrow{d^1_{1,1}} & E^1_{1,1} & \xleftarrow{d^1_{2,1}} & E^1_{2,1} \\
0 & E^1_{0,0} & \xleftarrow{d^1_{1,0}} & E^1_{1,0} & \xleftarrow{d^1_{2,0}} & E^1_{2,0} \\
\hline
t/s & 0 & & 1 & & 2
\end{array}
$$

d^1 is a differential

The condition that $d^1_{s,t} d^1_{s+1,t} = 0$ needs only be verified for $s = 1$, when it asserts that the composite

$$\partial_{K,A}\partial_{X,K,A} \colon H_{n+1}(X,K) \xrightarrow{\partial_{X,K,A}} H_n(K,A) \xrightarrow{\partial_{K,A}} H_{n-1}(A)$$

is zero.
This follows from the factorization $\partial_{X,K,A} = j_{K,A}\partial_{X,K}$ and the fact that $\partial_{K,A} j_{K,A} = 0$, both of which are visible in the diagram (4) with two linked exact triangles.

d^1-cycles and d^1-boundaries

By the defining property of a spectral sequence, the E^2-term must be

$$E^2 \cong H(E^1, d^1) = \ker(d^1)/\operatorname{im}(d^1)\,.$$

The d^1-cycles are

$$\ker(d^1)_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ \ker(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A)) & \text{for } s = 1, \\ \ker(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A)) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The d^1-boundaries are

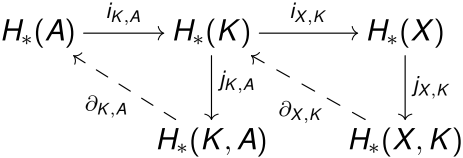
$$\operatorname{im}(d^1)_{s,t} = \begin{cases} \operatorname{im}(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A)) & \text{for } s = 0, \\ \operatorname{im}(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A)) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

E^2-term

Hence the E^2-term satisfies

$$E^2_{s,t} \cong \begin{cases} \operatorname{cok}(\partial_{K,A} \colon H_{1+t}(K,A) \to H_t(A)) & \text{for } s = 0, \\ \dfrac{\ker(\partial_{K,A} \colon H_{1+t}(K,A) \to H_t(A))}{\operatorname{im}(\partial_{X,K,A} \colon H_{2+t}(X,K) \to H_{1+t}(K,A))} & \text{for } s = 1, \\ \ker(\partial_{X,K,A} \colon H_{2+t}(X,K) \to H_{1+t}(K,A)) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

These can be visualized in the two exact triangles:



The d^2-differential

- We must now specify the d^2-differentials in the spectral sequence.
- They can only be nonzero when mapping from bidegree (s, t) with $s = 2$, since for other values of s the source or target (or both) is a trivial group.
- The interesting case is therefore

$$d^2_{2,t} \colon E^2_{2,t} = \ker(\partial_{X,K,A})_{t+2} \longrightarrow \operatorname{cok}(\partial_{K,A})_{t+1} = E^2_{0,t+1}$$

  of bidegree $(-2, 1)$.
- Here $\ker(\partial_{X,K,A})_{t+2} \subset H_{t+2}(X, K)$ while $\operatorname{cok}(\partial_{K,A})_{t+1}$ is a quotient of $H_{t+1}(A)$.

Construction of $d^2_{2,t}$

$$\begin{array}{ccc} \operatorname{cok}(\partial_{K,A})_{t+1} & \xrightarrow[\cong]{\bar{i}_{K,A}} & \operatorname{im}(i_{K,A})_{t+1} \\ & \nwarrow^{d^2_{2,t}} \qquad \nwarrow^{\tilde{\partial}_{X,K}} & \\ & & \ker(\partial_{X,K,A})_{t+2} \end{array} \tag{5}$$

Since $\partial_{X,K,A} = j_{K,A}\partial_{X,K}$, the restriction of $\partial_{X,K}$ defines a homomorphism $\tilde{\partial}_{X,K}$ where $\operatorname{im}(i_{K,A})_{t+1} \subset H_{t+1}(K)$.

$$\tilde{\partial}_{X,K} \colon \ker(\partial_{X,K,A})_{t+2} \longrightarrow \ker(j_{K,A})_{t+1} = \operatorname{im}(i_{K,A})_{t+1}$$

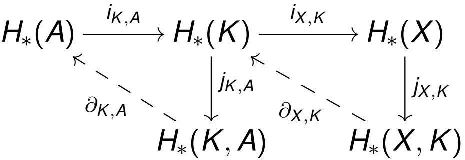
Furthermore, $i_{K,A}$ induces an isomorphism

$$\bar{i}_{K,A} \colon \operatorname{cok}(\partial_{K,A})_{t+1} = \frac{H_{t+1}(A)}{\ker(i_{K,A})_{t+1}} \xrightarrow{\cong} \operatorname{im}(i_{K,A})_{t+1} .$$

We then define $d^2_{2,t}$ to be $\tilde{\partial}_{X,K}$ followed by the inverse of $\bar{i}_{K,A}$.

$$d^2_{2,t} = \bar{i}^{-1}_{K,A}\tilde{\partial}_{X,K}$$

Element-wise definition of d^2



We calculate $d^2_{2,t}(x)$ for a class

$$x \in E^2_{2,t} = \ker(\partial_{X,K,A})_{t+2} \subset H_{t+2}(X,K)$$

by applying $\partial_{X,K}$ to get an element

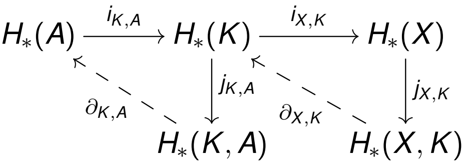
$$\partial_{X,K}(x) \in \ker(j_{K,A})_{t+1} = \operatorname{im}(i_{K,A})_{t+1} \subset H_{t+1}(K)\,,$$

writing this in the form

$$\partial_{X,K}(x) = i_{K,A}(y)$$

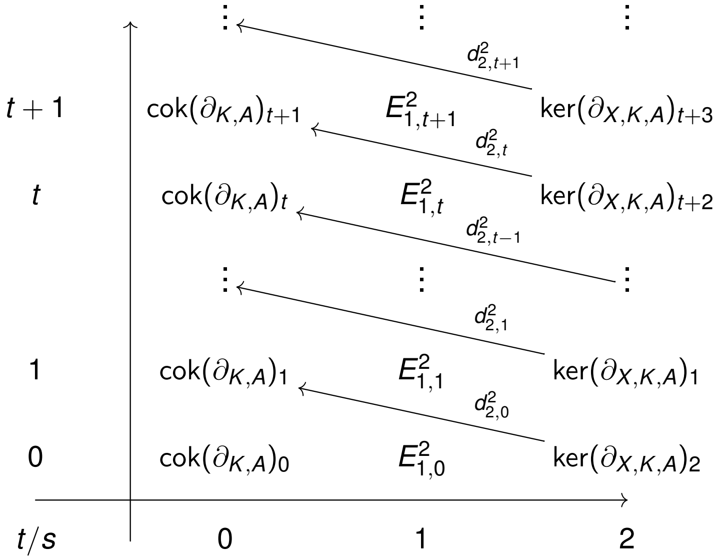
for an element $y \in H_{t+1}(A)$, and setting $d^2_{2,t}(x) = [y]$ to be the homology class of y in the quotient $E^2_{0,t+1} = \operatorname{cok}(\partial_{K,A})_{t+1}$ of $H_{t+1}(A)$.

Independence of choice



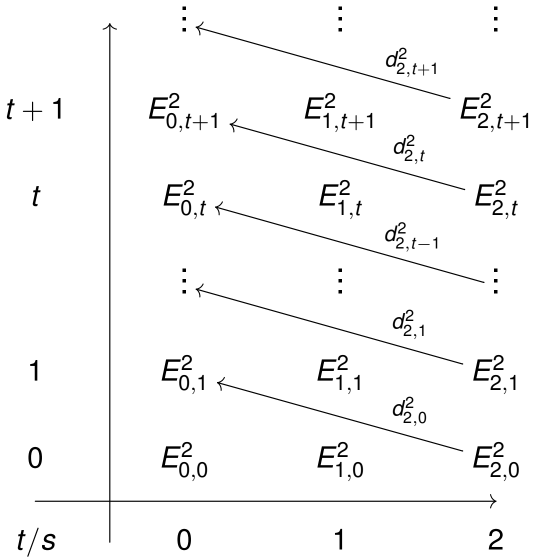
Any two choices y and y' with the same image under $i_{K,A}$ differ by an element in $\ker(i_{K,A}) = \mathrm{im}(\partial_{K,A})$, hence define the same class $[y] = [y']$ in $\mathrm{cok}(\partial_{K,A})$.

(E^2, d^2)-term



$t+1$
t
1
0
t/s
0
1
2
$\mathrm{cok}(\partial_{K,A})_{t+1}$
$E^2_{1,t+1}$
$\ker(\partial_{X,K,A})_{t+3}$
$\mathrm{cok}(\partial_{K,A})_t$
$E^2_{1,t}$
$\ker(\partial_{X,K,A})_{t+2}$
$\mathrm{cok}(\partial_{K,A})_1$
$E^2_{1,1}$
$\ker(\partial_{X,K,A})_1$
$\mathrm{cok}(\partial_{K,A})_0$
$E^2_{1,0}$
$\ker(\partial_{X,K,A})_2$
$d^2_{2,t+1}$
$d^2_{2,t}$
$d^2_{2,t-1}$
$d^2_{2,1}$
$d^2_{2,0}$

Generic three-column (E^2, d^2)-term



Collapse at E^3-term

It is clear that $d^2 d^2 = 0$, and that $d^r = 0$ for $r \geq 3$, since for each of these homomorphisms the source or target, or both, must be a trivial group.

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

Hence the spectral sequence collapses at the E^3-term, which equals the E^r-term for each $3 \leq r \leq \infty$.

E^∞-term

t/s	0	1	2
	$\vdots$	$\vdots$	$\vdots$
$t+1$	$\operatorname{cok}(d^2_{2,t})$	$\ker(\partial_{K,A})_{t+2}/\operatorname{im}(\partial_{X,K,A})_{t+2}$	$\ker(d^2_{2,t+1})$
t	$\operatorname{cok}(d^2_{2,t-1})$	$\ker(\partial_{K,A})_{t+1}/\operatorname{im}(\partial_{X,K,A})_{t+1}$	$\ker(d^2_{2,t})$
	$\vdots$	$\vdots$	$\vdots$
1	$\operatorname{cok}(d^2_{2,0})$	$\ker(\partial_{K,A})_2/\operatorname{im}(\partial_{X,K,A})_2$	$\ker(d^2_{2,1})$
0	$\operatorname{cok}(\partial_{K,A})_0$	$\ker(\partial_{K,A})_1/\operatorname{im}(\partial_{X,K,A})_1$	$\ker(d^2_{2,0})$

Generic three-column E^∞-term

	$\vdots$	$\vdots$	$\vdots$
$t+1$	$E^\infty_{0,t+1}$	$E^\infty_{1,t+1}$	$E^\infty_{2,t+1}$
t	$E^\infty_{0,t}$	$E^\infty_{1,t}$	$E^\infty_{2,t}$
	$\vdots$	$\vdots$	$\vdots$
1	$E^\infty_{0,1}$	$E^\infty_{1,1}$	$E^\infty_{2,1}$
0	$E^\infty_{0,0}$	$E^\infty_{1,0}$	$E^\infty_{2,0}$
t/s	0	1	2

Bounded convergence

Recall that

$$\begin{aligned} F_0 H_n(X) &= \operatorname{im}(i_{X,A})_n \\ F_1 H_n(X) &= \operatorname{im}(i_{X,K})_n \\ F_2 H_n(X) &= H_n(X)\,, \end{aligned}$$

so that

$$0 \subset F_0 H_*(X) \subset F_1 H_*(X) \subset F_2 H_*(X) = H_*(X)$$

is a bounded filtration of $H_*(X)$. The following three lemmas will therefore complete the proof of the proposition.

Three $= \{0, 1, 2\}$ lemmas

Lemma (0)

There is a preferred isomorphism

$$E^{\infty}_{0,n} \cong F_0 H_n(X)\,.$$

Lemma (1)

There is a preferred isomorphism

$$E^{\infty}_{1,n-1} \cong \frac{F_1 H_n(X)}{F_0 H_n(X)}\,.$$

Lemma (2)

There is a preferred isomorphism

$$E^{\infty}_{2,n-2} \cong \frac{H_n(X)}{F_1 H_n(X)}\,.$$

Proof of Lemma (0).

$$\begin{array}{ccc}
\mathrm{cok}(\partial_{K,A})_{t+1} & \xrightarrow[\cong]{\bar{i}_{K,A}} & \mathrm{im}(i_{K,A})_{t+1} \\
 & \nwarrow^{d^2_{2,t}} \qquad\qquad \nwarrow^{\tilde{\partial}_{X,K}} & \\
 & & \ker(\partial_{X,K,A})_{t+2}
\end{array}$$

The cokernel

$$E^\infty_{0,n} = E^3_{0,n} = \mathrm{cok}(d^2_{2,n-1})$$

maps isomorphically by $\bar{i}_{K,A}$ to the cokernel

$$\frac{\mathrm{im}(i_{K,A})_n}{\mathrm{im}(\tilde{\partial}_{X,K})_n} = \frac{\mathrm{im}(i_{K,A})_n}{\mathrm{im}(i_{K,A})_n \cap \mathrm{im}(\partial_{X,K})_n} = \frac{\mathrm{im}(i_{K,A})_n}{\mathrm{im}(i_{K,A})_n \cap \ker(i_{X,K})_n},$$

which maps isomorphically by $i_{X,K}$ to

$$i_{X,K}(\mathrm{im}(i_{K,A}))_n = \mathrm{im}(i_{X,A})_n = F_0 H_n(X).$$

□

Proof of Lemma (1).

$$
\begin{array}{ccccc}
H_*(A) & \xrightarrow{i_{K,A}} & H_*(K) & \xrightarrow{i_{X,K}} & H_*(X) \\
& \nwarrow_{\partial_{K,A}} & \big\downarrow j_{K,A} & \nwarrow_{\partial_{X,K}} & \big\downarrow j_{X,K} \\
& & H_*(K,A) & & H_*(X,K)
\end{array}
$$

The quotient group

$$E^{\infty}_{1,n-1} = E^{2}_{1,n-1} = \frac{\ker(\partial_{K,A})_n}{\operatorname{im}(\partial_{X,K,A})_n} = \frac{\operatorname{im}(j_{K,A})_n}{\operatorname{im}(j_{K,A}\partial_{X,K})_n} = \frac{\operatorname{im}(j_{K,A})_n}{j_{K,A}(\ker(i_{X,K}))_n} .$$

receives an isomorphism induced by $j_{K,A}$ from

$$\frac{H_n(K)}{\ker(j_{K,A})_n + \ker(i_{X,K})_n} = \frac{H_n(K)}{\operatorname{im}(i_{K,A})_n + \ker(i_{X,K})_n} ,$$

and this group maps isomorphically under $i_{X,K}$ to

$$\frac{i_{X,K}(H_n(K))}{i_{X,K}(\operatorname{im}(i_{K,A}))_n} = \frac{F_1 H_n(X)}{F_0 H_n(X)} . \quad \square$$

Proof of Lemma (2).

$$\operatorname{cok}(\partial_{K,A})_{t+1} \xrightarrow[\cong]{\bar{i}_{K,A}} \operatorname{im}(i_{K,A})_{t+1}$$

$$d^2_{2,t} : \ker(\partial_{X,K,A})_{t+2} \to \operatorname{cok}(\partial_{K,A})_{t+1}, \qquad \tilde{\partial}_{X,K} : \ker(\partial_{X,K,A})_{t+2} \to \operatorname{im}(i_{K,A})_{t+1}$$

The subgroup

$$E^{\infty}_{2,t} = E^{3}_{2,t} = \ker(d^2_{2,t}) = \ker(\tilde{\partial}_{X,K})_{t+2} = \ker(\partial_{X,K})_{t+2} = \operatorname{im}(j_{X,K})_{t+2}$$

of $H_n(X, K)$ receives an isomorphism induced by $j_{X,K}$ from

$$\frac{H_n(X)}{\ker(j_{X,K})_n} = \frac{H_n(X)}{\operatorname{im}(i_{X,K})_n} = \frac{H_n(X)}{F_1 H_n(X)}.$$

□

Imperfect precision

- The d^2-differentials in this three-column spectral sequence were not fully determined by the statement of the proposition.
- For instance, we could have reversed the sign of some of the d^2-differentials and obtained a slightly different spectral sequence, with the same (E^1, d^1)-term and filtered abutment.
- In order to be clear about which spectral sequence one has in mind one must therefore be more specific about how the spectral sequence arises, beyond just giving the initial term.
- In many cases this complete precision is not necessary, but one should be aware of the issue.

Staircase visualization

Another way to depict the two exact triangles in (4) is the following pair of long exact sequences, each shown as a "staircase" shape.

$$
\begin{array}{cccccccc}
\ldots \to H_{n+1}(K,A) \to & H_n(A) & & & & & & \\
& \downarrow & & & & & & \\
\ldots \to H_{n+1}(X,K) \to & H_n(K) & \to & H_n(K,A) & \to & H_{n-1}(A) & & \\
& \downarrow & & & & \downarrow & & \\
& H_n(X) & \to & H_n(X,K) & \to & H_{n-1}(K) & \to & H_{n-1}(K,A) \to \ldots \\
& & & & & \downarrow & & \\
& & & & & H_{n-1}(X) & \to & H_{n-1}(X,K) \to \ldots
\end{array}
$$

Outline

Exact Couples

- Almost every spectral sequences arises from a generalization of the diagram

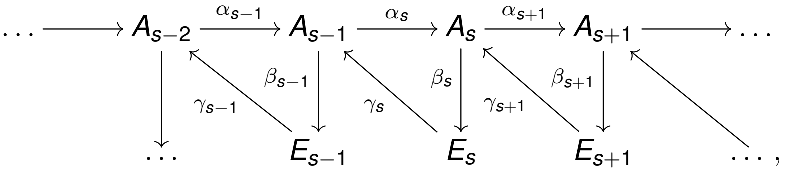
$$\begin{array}{ccccc}
H_*(A) & \xrightarrow{i_{K,A}} & H_*(K) & \xrightarrow{i_{X,K}} & H_*(X) \\
 & \nwarrow{\scriptstyle \partial_{K,A}} & \downarrow{\scriptstyle j_{K,A}} & \nwarrow{\scriptstyle \partial_{X,K}} & \downarrow{\scriptstyle j_{X,K}} \\
 & & H_*(K,A) & & H_*(X,K)
\end{array}$$

  to the case where there are infinitely many long exact sequences that are chained together at common vertices.
- This algebraic structure is called an exact couple, and was introduced by William Massey [Mas52], [Mas53].
- We prefer to display exact couples in an unrolled form, as in Michael Boardman's paper [Boa99, (0.1)].

Outline

Definition of an exact couple

An unrolled exact couple $(A, E) = (A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ is a diagram of the form



in which each triangle forms a long exact sequence

$$\cdots \to A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\beta_s} E_s \xrightarrow{\gamma_s} A_{s-1} \to \dots .$$

Here each A_s and E_s is a graded abelian group, and α_s, β_s and γ_s are graded morphisms of graded abelian groups.

Remarks

- In the long circulated preprint form of Boardman's paper, this structure was called an unraveled exact couple.
- Frequently, α_s and β_s preserve the total degree, and γ_s reduces the total degree by 1, so that we have a long exact sequence of abelian groups

$$\cdots \to (A_{s-1})_n \xrightarrow{\alpha_s} (A_s)_n \xrightarrow{\beta_s} (E_s)_n \xrightarrow{\gamma_s} (A_{s-1})_{n-1} \to \cdots$$

  for each s.
- If we set $A_{s,t} = (A_s)_{s+t}$ and $E_{s,t} = (E_s)_{s+t}$, with t a complementary degree, this appears as follows

$$\cdots \to A_{s-1,t+1} \xrightarrow{\alpha_s} A_{s,t} \xrightarrow{\beta_s} E_{s,t} \xrightarrow{\gamma_s} A_{s-1,t} \to \cdots,$$

  so that each α_s has (s,t)-bidegree $(1,-1)$, each β_s has bidegree $(0,0)$, and each γ_s has bidegree $(-1,0)$.

Morphisms

A morphism of exact couples

$$\phi\colon (A, E) \to ('A, 'E)$$

consists of degree-preserving homomorphisms

$$\phi_s\colon A_s \longrightarrow 'A_s$$
$$\phi_s\colon E_s \longrightarrow 'E_s \,,$$

for $s \in \mathbb{Z}$, making each diagram

$$\begin{array}{ccccccc} A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\beta_s} & E_s & \xrightarrow{\gamma_s} & A_{s-1} \\ \downarrow{\scriptstyle \phi_{s-1}} & & \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_{s-1}} \\ 'A_{s-1} & \xrightarrow{'\alpha_s} & 'A_s & \xrightarrow{'\beta_s} & 'E_s & \xrightarrow{'\gamma_s} & 'A_{s-1} \end{array}$$

commute.

Example: A filtered space

- A filtration of a space X is a sequence of subspaces

$$\dots \subset X_{s-1} \subset X_s \subset \dots$$

  where $s \in \mathbb{Z}$.
- The (unrolled) exact couple in homology associated to such a filtration $(X_s)_s$ is the following chain of exact triangles.

$$\begin{array}{ccccccccc}
\dots \to H_*(X_{s-2}) & \xrightarrow{\alpha_{s-1}} & H_*(X_{s-1}) & \xrightarrow{\alpha_s} & H_*(X_s) & \xrightarrow{\alpha_{s+1}} & H_*(X_{s+1}) & \longrightarrow \dots \\
 & \nwarrow{\scriptstyle \gamma_{s-1}} & \downarrow{\scriptstyle \beta_{s-1}} & \nwarrow{\scriptstyle \gamma_s} & \downarrow{\scriptstyle \beta_s} & \nwarrow{\scriptstyle \gamma_{s+1}} & \downarrow{\scriptstyle \beta_{s+1}} & \\
 & & H_*(X_{s-1}, X_{s-2}) & & H_*(X_s, X_{s-1}) & & H_*(X_{s+1}, X_s)\,. &
\end{array}$$

Notation and grading

- Here

$$A_s = H_*(X_s)$$
$$E_s = H_*(X_s, X_{s-1})$$

  and $\alpha_s = i_{X_s,X_{s-1}}$, $\beta_s = j_{X_s,X_{s-1}}$, $\gamma_s = \partial_{X_s,X_{s-1}}$.
- Hence

$$\dots \to H_*(X_{s-1}) \xrightarrow{\alpha_s} H_*(X_s) \xrightarrow{\beta_s} H_*(X_s, X_{s-1}) \xrightarrow{\gamma_s} H_{*-1}(X_{s-1}) \to \dots$$

  is the long exact sequence in homology of the pair (X_s, X_{s-1}).
- The solid arrows α_s and β_s preserve the total grading, while the dashed arrows γ_s have total degree -1.

Example: A filtered map

- Let $(X_s)_s$ and $(Y_s)_s$ be filtrations of the spaces X and Y.
- A map $\phi\colon X \to Y$ is filtration-preserving if $\phi(X_s) \subset Y_s$ for each s.
- Such a map induces a morphism ϕ of exact couples, given by the homomorphisms

$$\phi_s\colon H_*(X_s) \longrightarrow H_*(Y_s)$$
$$\phi_s\colon H_*(X_s, X_{s-1}) \longrightarrow H_*(Y_s, Y_{s-1})$$

induced by the evident restrictions of ϕ.

Massey's notation

- In Massey's paper, the exact triangles are rolled up further, by setting

$$A = \bigoplus_s A_s \qquad \text{and} \qquad E = \bigoplus_s E_s \,.$$

- An exact couple is then a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\ & & E \end{array}$$

  that is exact at each point, meaning that $\mathrm{im}(\alpha) = \ker(\beta)$, $\mathrm{im}(\beta) = \ker(\gamma)$ and $\mathrm{im}(\gamma) = \ker(\alpha)$.

- Boardman's unrolled presentation has the advantage that it visually emphasizes the filtration degree s.

Whole-plane staircase presentation

$$
\begin{array}{ccccccccc}
 & & \cdots & & & & \cdots & & \\
 & & \downarrow & & & & \downarrow & & \\
E_{s-1,t+2} & \xrightarrow{\gamma_{s-1}} & A_{s-2,t+2} & \xrightarrow{\beta_{s-2}} & E_{s-2,t+2} & \xrightarrow{\gamma_{s-2}} & A_{s-3,t+2} & \xrightarrow{\beta_{s-3}} & E_{s-3,t+2} \\
 & & \downarrow{\scriptstyle \alpha_{s-1}} & & & & \downarrow{\scriptstyle \alpha_{s-2}} & & \\
E_{s,t+1} & \xrightarrow{\gamma_{s}} & A_{s-1,t+1} & \xrightarrow{\beta_{s-1}} & E_{s-1,t+1} & \xrightarrow{\gamma_{s-1}} & A_{s-2,t+1} & \xrightarrow{\beta_{s-2}} & E_{s-2,t+1} \\
 & & \downarrow{\scriptstyle \alpha_{s}} & & & & \downarrow{\scriptstyle \alpha_{s-1}} & & \\
E_{s+1,t} & \xrightarrow{\gamma_{s+1}} & A_{s,t} & \xrightarrow{\beta_{s}} & E_{s,t} & \xrightarrow{\gamma_{s}} & A_{s-1,t} & \xrightarrow{\beta_{s-1}} & E_{s-1,t} \\
 & & \downarrow{\scriptstyle \alpha_{s+1}} & & & & \downarrow{\scriptstyle \alpha_{s}} & & \\
E_{s+2,t-1} & \xrightarrow{\gamma_{s+2}} & A_{s+1,t-1} & \xrightarrow{\beta_{s+1}} & E_{s+1,t-1} & \xrightarrow{\gamma_{s+1}} & A_{s,t-1} & \xrightarrow{\beta_{s}} & E_{s,t-1} \\
 & & \downarrow{\scriptstyle \alpha^{s+2}} & & & & \downarrow{\scriptstyle \alpha^{s+1}} & & \\
E_{s+3,t-2} & \xrightarrow{\gamma_{s+3}} & A_{s+2,t-2} & \xrightarrow{\beta_{s+2}} & E_{s+2,t-2} & \xrightarrow{\gamma_{s+2}} & A_{s+1,t-2} & \xrightarrow{\beta_{s+1}} & E_{s+1,t-2} \\
 & & \downarrow & & & & \downarrow & & \\
 & & \cdots & & & & \cdots & &
\end{array}
$$

Outline

Spectral sequence associated to exact couple

Theorem

- *Let* $(A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ *be an exact couple. Then there is a spectral sequence* $(E^r, d^r)_{r\geq 1}$ *with*
$$E^1_s = E_s$$
and
$$d^1_s = \beta_{s-1}\gamma_s \colon E^1_s \longrightarrow E^1_{s-1}$$
for all $s \in \mathbb{Z}$*.*
- *If* α_s *and* β_s *have total degree* 0 *and* γ_s *has total degree* -1*, then*
$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$
has bidegree $(-r, r+1)$*, where* $E^r_{s,t} = (E^r_s)_{s+t}$ *is a subquotient of* $E^1_{s,t} = (E_s)_{s+t}$*.*

Visualization

The E^1-term of the spectral sequence is visible in the lower row of the unrolled exact couple

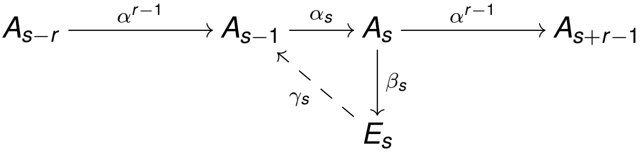
$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\alpha_{s-1}} & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\alpha_{s+1}} & A_{s+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \beta_{s-1}} & & \downarrow{\scriptstyle \beta_s} & & \downarrow{\scriptstyle \beta_{s+1}} & & \\
 & {\scriptstyle \gamma_{s-1}} \nwarrow & E_{s-1} & {\scriptstyle \gamma_s} \nwarrow & E_s & {\scriptstyle \gamma_{s+1}} \nwarrow & E_{s+1} & & ,
\end{array}
$$

with each d^1-differential being given by the composite of two homomorphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longleftarrow & E^1_{s-1} & \xleftarrow{d^1_s} & E^1_s & \xleftarrow{d^1_{s+1}} & E^1_{s+1} & \longleftarrow & \cdots \\
 & & \| & & \| & & \| & & \\
\cdots & \longleftarrow & E_{s-1} & \xleftarrow{\beta_{s-1}\gamma_s} & E_s & \xleftarrow{\beta_s\gamma_{s+1}} & E_{s+1} & \longleftarrow & \cdots
\end{array}
$$

Cycles and boundaries

To construct the E^r-term of the spectral sequence, we consider the following part of the unrolled exact couple.



Definition

For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$Z_s^r = \gamma_s^{-1} \operatorname{im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1})$$

be the r-th cycle group, and let

$$B_s^r = \beta_s \ker(\alpha^{r-1} \colon A_s \to A_{s+r-1})$$

be the r-th boundary group, both in filtration s.

Bigraded cycles and boundaries

$$Z_s^r = \gamma_s^{-1}\,\mathrm{im}(\alpha^{r-1}\colon A_{s-r} \to A_{s-1})$$
$$B_s^r = \beta_s \ker(\alpha^{r-1}\colon A_s \to A_{s+r-1})$$

- Z_s^r is the preimage under $\gamma_s\colon E_s \to A_{s-1}$ of the image of $\alpha^{r-1}\colon A_{s-r} \to A_{s-1}$.
- B_s^r is the image under $\beta_s\colon A_s \to E_s$ of the kernel of $\alpha^{r-1}\colon A_s \to A_{s+r-1}$.
- These are both graded subgroups of E_s, with components $Z_{s,t}^r$ and $B_{s,t}^r$ contained in $E_{s,t} = (E_s)_{s+t}$.

B^r- and Z^r-chains for exact couples, I

Lemma

There are inclusions

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset B_s^{r+1} \subset \cdots \subset \mathrm{im}(\beta_s)$$
$$= \ker(\gamma_s) \subset Z_s^{r+1} \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s \,.$$

Proof.

The inclusions of $\ker(\gamma_s)$ and the cycle groups follow from the inclusions

$$0 \subset \mathrm{im}(\alpha^r \colon A_{s-r-1} \to A_{s-1}) \subset \mathrm{im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1}) \,.$$

The preimage Z_s^1 of $\mathrm{im}(\alpha^0) = A_{s-1}$ is the whole of E_s.

B^r- and Z^r-chains for exact couples, II

Lemma

There are inclusions

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset B_s^{r+1} \subset \cdots \subset \mathrm{im}(\beta_s)$$
$$= \ker(\gamma_s) \subset Z_s^{r+1} \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s \,.$$

Proof (cont.)

The inclusions of boundary groups and $\mathrm{im}(\beta_s)$ follow from the inclusions

$$\ker(\alpha^{r-1} \colon A_s \to A_{s+r-1}) \subset \ker(\alpha^r \colon A_s \to A_{s+r}) \subset A_s \,.$$

The image B_s^1 of $\ker(\alpha^0) = 0$ is trivial.

B^r- and Z^r-chains for exact couples, III

Lemma

There are inclusions

$$\begin{aligned} 0 = B^1_s \subset \cdots \subset B^r_s \subset B^{r+1}_s \subset \cdots \subset \mathrm{im}(\beta_s) \\ = \ker(\gamma_s) \subset Z^{r+1}_s \subset Z^r_s \subset \cdots \subset Z^1_s = E_s \, . \end{aligned}$$

Proof (cont.)

For each finite $r \geq 1$ we have

$$B^r_s \subset \mathrm{im}(\beta_s) = \ker(\gamma_s) \subset Z^r_s$$

by exactness at E_s. □

The E^r-term

Definition

For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$E^r_s = Z^r_s / B^r_s$$

and

$$E^r_{s,t} = Z^r_{s,t} / B^r_{s,t}$$

so that $E^r = E^r_{*,*}$ is the E^r-term of the spectral sequence.
In particular, $E^1_s = E_s/0 \cong E_s$.

Decreasing upper bounds

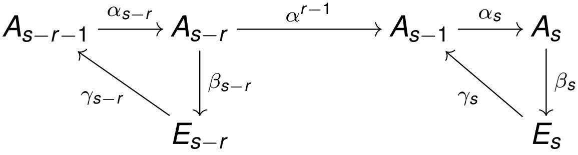
- As r increases, each E^r-term is a successively smaller subquotient of the E^1-term, since the cycle groups Z_s^r decrease and the boundary groups B_s^r increase in size.
- Each term E^q thus gives an upper bound for the subsequent terms E^r with $r \geq q$.

- If $E^q_{s,t} = 0$ for (s, t) in some region of the (s, t)-plane, then $E^r_{s,t} = 0$ for all $r \geq q$ and (s, t) in this region.
- If a term of a spectral sequence is concentrated in some region, such as the first quadrant, then so is the remainder of the spectral sequence.

- In order to have a spectral sequence, we must identify E^{r+1} as the homology of E^r with respect to a d^r-differential.

The d^r-differential

We use the following part of the unrolled exact couple.



Definition

For each $x \in Z_s^r \subset E_s$ in the r-th cycle group we write $[x] \in E_s^r$ for its equivalence class modulo the r-th boundary group. Let the d^r-differential

$$d_s^r \colon E_s^r \longrightarrow E_{s-r}^r$$

be defined by

$$d_s^r \colon [x] \longmapsto [\beta_{s-r}(y)]$$

where $y \in A_{s-r}$ is chosen to satisfy $\gamma_s(x) = \alpha^{r-1}(y)$. In particular, $d_s^1 = \beta_{s-1}\gamma_s$.

Lemma

d_s^r is well defined.

Proof.

Since $x \in Z_s^r$ we have $\gamma_s(x) \in \operatorname{im}(\alpha^{r-1}\colon A_{s-r} \to A_{s-1})$, so there exists a $y \in A_{s-r}$ with $\alpha^{r-1}(y) = \gamma_s(x)$. The image $\beta_{s-r}(y)$ then lies in $\operatorname{im}(\beta_{s-r}) \subset Z_{s-r}^r$, hence defines a class $[\beta_{s-r}(y)]$ in E_{s-r}^r. Another choice of $y' \in A_{s-r}$ with $\alpha^{r-1}(y') = \gamma_s(x)$ differs from y by a class $y' - y \in \ker(\alpha^{r-1}\colon A_{s-r} \to A_{s-1})$, hence $\beta_{s-r}(y')$ differs from $\beta_{s-r}(y)$ by a class

$$\beta_{s-r}(y' - y) \in \beta_{s-r} \ker(\alpha^{r-1}\colon A_{s-r} \to A_{s-1}) = B_{s-r}^r\,.$$

This means that $[\beta_{s-r}(y)] = [\beta_{s-r}(y')]$ as elements of E_{s-r}^r. Any other choice of $x' \in Z_s^r$ representing the same class $[x'] = [x]$ in E_s^r differs from x by an element $x' - x \in B_s^r$. Since $B_s^r \subset \ker(\gamma_s)$, it follows that $\gamma_s(x') = \gamma_s(x)$, so x and x' lead to the same choices for y and the same value of $[\beta_{s-r}(y)]$. □

Lemma

$$\ker(d^r)_s = \ker(d^r_s) = Z^{r+1}_s / B^r_s \,.$$

Proof.

First, let $x \in Z^r_s$, choose $y \in A_{s-r}$ with $\alpha^{r-1}(y) = \gamma_s(x)$, and suppose that $[x] \in \ker(d^r_s)$. This means that $\beta_{s-r}(y) \in B^r_{s-r}$, so there exists a $y' \in \ker(\alpha^{r-1}) \subset A_{s-r}$ with $\beta_{s-r}(y) = \beta_{s-r}(y')$. Then $y - y' \in \ker(\beta_{s-r}) = \mathrm{im}(\alpha_{s-r})$ equals $\alpha_{s-r}(z)$ for some $z \in A_{s-r-1}$, and

$$\alpha^r(z) = \alpha^{r-1}(y - y') = \alpha^{r-1}(y) - \alpha^{r-1}(y') = \gamma_s(x) - 0 = \gamma_s(x)\,,$$

which proves that $x \in Z^{r+1}_s$. Hence $\ker(d^r_s) \subset Z^{r+1}_s / B^r_s$. Conversely, if $x \in Z^{r+1}_s$ then we can write $\gamma_s(x) = \alpha^r(z) = \alpha^{r-1}(y)$ for some $z \in A_{s-r-1}$ and $y = \alpha_{s-r}(z) \in \mathrm{im}(\alpha_{s-r}) = \ker(\beta_{s-r})$. Then $\beta_{s-r}(y) = 0$, so d^r_s maps $[x]$ to $[0]$, and $[x] \in \ker(d^r_s)$. Hence $Z^{r+1}_s / B^r_s \subset \ker(d^r_s)$. $\square$

Higher filtrations

For d^r-boundaries in filtration s we use the following part of the unrolled exact couple.

$$
\begin{array}{ccccccc}
A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\alpha^{r-1}} & A_{s+r-1} & \xrightarrow{\alpha_{s+r}} & A_{s+r} \\
& {\scriptstyle \gamma_s} \nwarrow & \downarrow {\scriptstyle \beta_s} & & & {\scriptstyle \gamma_{s+r}} \nwarrow & \downarrow {\scriptstyle \beta_{s+r}} \\
& & E_s & & & & E_{s+r}
\end{array}
$$

Lemma

$$\mathrm{im}(d^r)_s = \mathrm{im}(d^r_{s+r}) = B^{r+1}_s / B^r_s \,.$$

Proof.

Let $x \in Z^r_{s+r}$, choose $y \in A_s$ with $\alpha^{r-1}(y) = \gamma_{s+r}(x)$, and consider $[\beta_s(y)] \in \mathrm{im}(d^r_{s+r})$. Then

$$\alpha^r(y) = \alpha_{s+r}\alpha^{r-1}(y) = \alpha_{s+r}\gamma_{s+r}(x) = 0 \,,$$

so $y \in \ker(\alpha^r \colon A_s \to A_{s+r})$ and $\beta_s(y) \in B^{r+1}_s$. Hence $\mathrm{im}(d^r_{s+r}) \subset B^{r+1}_s / B^r_s$.
Conversely, if $\beta_s(y) \in B^{r+1}_s$ with $y \in \ker(\alpha^r)$, then $\alpha^{r-1}(y) \in \ker(\alpha_{s+r}) = \mathrm{im}(\gamma_{s+r})$, so we can write $\alpha^{r-1}(y) = \gamma_{s+r}(x)$. Then $x \in Z^r_{s+r}$ and d^r_{s+r} maps $[x]$ to $[\beta_s(y)]$.
Hence $B^{r+1}_s / B^r_s \subset \mathrm{im}(d^r_{s+r})$. □

The spectral sequence condition

Lemma

- $d^r d^r = 0$
- $E_s^{r+1} \cong H(E^r, d^r)_s$

Proof.

It follows from $B_s^{r+1} \subset Z_s^{r+1}$ that $\mathrm{im}(d^r)_s \subset \ker(d^r)_s$, so $d_s^r d_{s+r}^r = 0$ and $d^r \colon E^r \to E^r$ is a differential of filtration degree $-r$. The isomorphism

$$Z_s^{r+1}/B_s^{r+1} \xrightarrow{\cong} \frac{Z_s^{r+1}/B_s^r}{B_s^{r+1}/B_s^r}$$

shows that $E_s^{r+1} \cong H(E^r, d^r)_s$, as claimed. □

Proof of Theorem

- We have
  - specified the E^r-terms,
  - specified the d^r-differentials, and
  - checked the spectral sequence condition.
- The explicit form of the E^1-differential and d^1-differential follows easily by inspection of the definitions.
- If α_s and β_s have total degree 0 while γ_s has total degree -1, then $d_s^r \colon E_s^r \to E_{s-r}^r$ has total degree -1 and reduces the filtration degree s by r. Hence it must increase the complementary degree t by $(r-1)$.

Functoriality

Lemma

Each morphism $\phi\colon (A, E) \to ({}'A, {}'E)$ of exact couples induces a morphism $\phi\colon (E^r, d^r) \to ({}'E^r, {}'d^r)$ of spectral sequences. Hence the associated spectral sequence defines a functor

$$\textit{Exact Couples} \longrightarrow \textit{Spectral Sequences}\,.$$

Proof.

It is straightforward to check that $\phi_s\colon E_s \to {}'E_s$ restricts to homomorphisms $\phi_s\colon Z^r_s \to {}'Z^r_s$, $\phi_s\colon B^r_s \to {}'B^r_s$ and $\phi_s\colon E^r_s \to {}'E^r_s$ for all $r \geq 1$ and s, and that these commute with the differentials d^r and ${}'d^r$, as well as the isomorphisms $H(E^r, d^r) \cong E^{r+1}$ and $H({}'E^r, d^r) \cong {}'E^{r+1}$. □

Remarks on indexing

- We are following the notation of [Boa99, §0], but translated into homological indexing.
- Beware that the d^r-cycles $\ker(d^r)$ are the quotient Z^{r+1}/B^r of the $(r+1)$-th cycle group, and the d^r-boundaries $\mathrm{im}(d^r)$ are the quotient B^{r+1}/B^r of the $(r+1)$-th boundary group, so that there is an offset by one from r to $(r+1)$ in the indexing of these bigraded groups.

Outline

B^r- and Z^r-chains for general spectral sequences

Lemma

Let $(E^r, d^r)_{r \geq p}$ be an E^p-spectral sequence. There are inclusions

$$0 = B^p_s \subset \cdots \subset B^r_s \subset B^{r+1}_s \subset \cdots \subset Z^{r+1}_s \subset Z^r_s \subset \cdots \subset Z^p_s = E^p_s$$

and isomorphisms $Z^r_s / Z^{r+1}_s \cong B^{r+1}_{s-r} / B^r_{s-r}$ such that

$$E^r_s \cong Z^r_s / B^r_s \, .$$

Furthermore, $d^r_s \colon E^r_s \to E^r_{s-r}$ corresponds to the composite

$$Z^r_s / B^r_s \xrightarrow{\pi} Z^r_s / Z^{r+1}_s \cong B^{r+1}_{s-r} / B^r_{s-r} \xrightarrow{\iota} Z^r_{s-r} / B^r_{s-r}$$

for all $r \geq p$ and $s \in \mathbb{Z}$.

Proof by induction on $r \geq p$

- Suppose that $E_s^r \cong Z_s^r / B_s^r$ for some r and all s. Then the subgroup $\ker(d^r)_s \subset E_s^r$ corresponds to a subgroup of Z_s^r / B_s^r, which must have the form Z_s^{r+1} / B_s^r for some $Z_s^{r+1} \subset Z_s^r$.
- Similarly, the subgroup $\operatorname{im}(d^r)_s \subset \ker(d^r)_s$ corresponds to a subgroup of Z_s^{r+1} / B_s^r, which must be of the form B_s^{r+1} / B_s^r for some $B_s^{r+1} \subset Z_s^{r+1}$.
- We have the following inclusions

$$B_s^r \subset B_s^{r+1} \subset Z_s^{r+1} \subset Z_s^r$$

and isomorphisms

$$E_s^{r+1} \cong H(E^r, d^r)_s = \frac{\ker(d^r)_s}{\operatorname{im}(d^r)_s} \cong \frac{Z_s^{r+1} / B_s^r}{B_s^{r+1} / B_s^r} \cong Z_s^{r+1} / B_s^{r+1} .$$

- This completes the inductive step.

Proof (cont.)

The d^r-differential factors as

$$E^r_s \xrightarrow{\pi} \frac{E^r_s}{\ker(d^r)_s} \xrightarrow{\cong} \operatorname{im}(d^r)_{s-r} \xrightarrow{\iota} E^r_{s-r}$$

and corresponds to the composition

$$Z^r_s/B^r_s \xrightarrow{\pi} Z^r_s/Z^{r+1}_s \xrightarrow{\cong} \frac{Z^r_s/B^r_s}{Z^{r+1}_s/B^r_s} \xrightarrow{\cong} B^{r+1}_{s-r}/B^r_{s-r} \xrightarrow{\iota} Z^r_{s-r}/B^r_{s-r} \,.$$

The composite of the two inner isomorphisms is the required isomorphism from Z^r_s/Z^{r+1}_s to B^{r+1}_{s-r}/B^r_{s-r}, which leads to the asserted expression for d^r_s.

Compatibility

Lemma

When (E^r, d^r) is the E^1-spectral sequence associated to an exact couple (A, E), then

> *the subgroups Z^r and B^r of E associated to the exact couple*

are equal to

> *the subgroups Z^r and B^r of E^1 associated to the spectral sequence.*

Proof.

Chase the definitions. □

Definition of Z^∞ and B^∞

Let (E^r, d^r) be an E^p-spectral sequence.

- For each $s \in \mathbb{Z}$ let the infinite cycles

$$Z_s^\infty = \bigcap_{r \geq p} Z_s^r$$

be the intersection (or limit) of the r-th cycle groups.

- Let the infinite boundaries

$$B_s^\infty = \bigcup_{r \geq p} B_s^r$$

be the union (or colimit) of the r-th boundary groups.

Definition of the E^∞-term

- There are inclusions

$$0 \subset \cdots \subset B_s^r \subset \cdots \subset B_s^\infty \subset Z_s^\infty \subset \cdots \subset Z_s^r \subset \cdots \subset E_s^p$$

  for all $r \geq p$ and $s \in \mathbb{Z}$.
- We define the E^∞-term of the spectral sequence to be the (bi-)graded group

$$E^\infty = (E_s^\infty)_s = E_{*,*}^\infty$$

  with

$$E_s^\infty = Z_s^\infty / B_s^\infty$$
$$E_{s,t}^\infty = Z_{s,t}^\infty / B_{s,t}^\infty$$

  for all $s, t \in \mathbb{Z}$.

A postponed proof

Lemma

If (E^r, d^r) stabilizes in each bidegree, then for each bidegree (s, t) there are isomorphisms $E^\infty_{s,t} \cong E^r_{s,t}$ for all sufficiently large r.

Proof.

Fix a bidegree (s, t). If $d^r_{s,t}$ and $d^r_{s+r,t-r+1}$ are both zero for all $r \geq q(s, t)$ then $Z^r_{s,t}/Z^{r+1}_{s,t} = 0$ and $B^{r+1}_{s,t}/B^r_{s,t} = 0$, so $Z^r_{s,t} = Z^{r+1}_{s,t} = Z^\infty_{s,t}$ and $B^r_{s,t} = B^{r+1}_{s,t} = B^\infty_{s,t}$ for all $r \geq q(s, t)$, and $E^r_{s,t} \cong E^{r+1}_{s,t} \cong E^\infty_{s,t}$ for all $r \geq q(s, t)$. $\square$

Functoriality of B^r- and Z^r-chains

Lemma

A morphism

$$\phi\colon (E^r, d^r)_{r\geq p} \longrightarrow ({}'E^r, {}'d^r)_{r\geq p}$$

of spectral sequences induces compatible morphisms

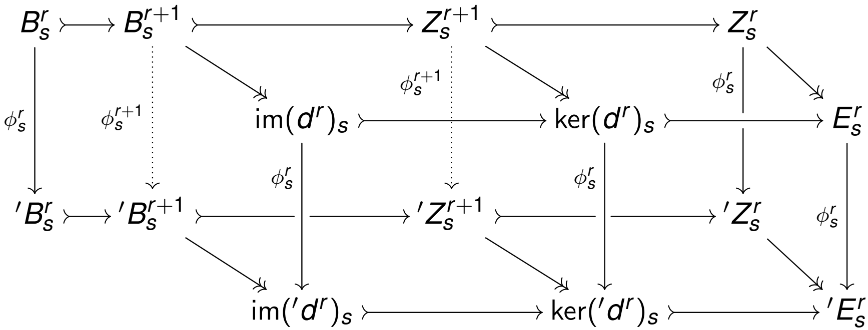
$$\phi^r\colon Z^r \longrightarrow {}'Z^r$$
$$\phi^r\colon B^r \longrightarrow {}'B^r$$

for all $r \geq p$, including $r = \infty$. This also defines a morphism

$$\phi^\infty\colon E^\infty \longrightarrow {}'E^\infty\,.$$

Proof.

By induction on $r \geq p$ we have vertical maps ϕ_s^r, as shown in the following commutative diagram.



There are unique dotted maps ϕ_s^{r+1} making the whole diagram commute, because the lower parallelograms are pullbacks.

Proof (cont.)

- The maps $B_s^{r+1} \to {}'Z_s^r$ and $B_s^{r+1} \to \mathrm{im}('d^r)_s$ with equal composites to ${}'E_s^r$ admit a unique common lift to ${}'B_s^{r+1}$.
- The maps $Z_s^{r+1} \to {}'Z_s^r$ and $Z_s^{r+1} \to \ker('d^r)_s$ with equal composites to ${}'E_s^r$ admit a unique common lift to ${}'Z_s^{r+1}$.
- The map $\phi_s^\infty \colon Z_s^\infty \to {}'Z_s^\infty$ is then given by the intersection (= limit) of the maps $\psi_s^r \colon Z_s^r \to {}'Z_s^r$, and $\phi_s^\infty \colon B_s^\infty \to {}'B_s^\infty$ is given by the union (= colimit) of the maps $\psi_s^r \colon B_s^r \to {}'B_s^r$.
- The induced map of quotient groups is $\phi_s^\infty \colon E_s^\infty \to {}'E_s^\infty$.

□

Another postponed proof

Lemma

If (E^r, d^r) *and* $('E^r, 'd^r)$ *stabilize in each bidegree, then* $\phi^\infty_{s,t} \colon E^\infty_{s,t} \to {}'E^\infty_{s,t}$ *corresponds, for each bidegree* (s,t)*, to* $\phi^r_{s,t} \colon E^r_{s,t} \to {}'E^r_{s,t}$ *for all sufficiently large* r*.*

Proof.

Fix a bidegree (s,t). If $(E^r_{s,t})_r$ and $('E^r_{s,t})_r$ both stabilize for $r \geq q = q(s,t)$, then $Z^\infty_s = Z^r_s$, $B^r_s = B^\infty_s$, $'Z^\infty_s = {}'Z^r_s$ and $'B^r_s = {}'B^\infty_s$ for $r \geq q$, hence $\phi^r_s = \phi^\infty_s$ as maps of infinite cycles, infinite boundaries and E^∞-terms. □

Invariance

The E^∞-term does not depend on where we start indexing the spectral sequence.

Lemma

Let $(E^r, d^r)_{r \geq p}$ be an E^p-spectral sequence, let $q \geq p$, and let $('E^r, 'd^r)_{r \geq q}$ be the E^q-spectral sequence with $E^r = {}'E^r$ and $d^r = {}'d^r$ for $r \geq q$. Then there is a canonical isomorphism

$$E^\infty \cong {}'E^\infty \,.$$

Proof of invariance

The sequence

$$0 = {}'B_s^q \subset \cdots \subset {}'B_s^r \subset {}'B_s^{r+1} \subset \cdots \subset {}'Z_s^{r+1} \subset {}'Z_s^r \subset \cdots \subset {}'Z_s^q = {}'E_s^q$$

equals

$$0 = B_s^q/B_s^q \subset \cdots \subset B_s^r/B_s^q \subset B_s^{r+1}/B_s^q \subset \dots$$
$$\cdots \subset Z_s^{r+1}/B_s^q \subset Z_s^r/B_s^q \subset \cdots \subset Z_s^q/B_s^q = E_s^q$$

so

$$'Z_s^\infty = \bigcap_r Z_s^r/B_s^q \cong Z_s^\infty/B_s^q$$
$$'B_s^\infty = \bigcup_r B_s^r/B_s^q \cong B_s^\infty/B_s^q$$

and

$$'E_s^\infty \cong \frac{Z_s^\infty/B_s^q}{B_s^\infty/B_s^q} \cong E_s^\infty .$$

Commutation of colimits and limits

The only slightly tricky step here is the commutation of quotients (which are colimits) and intersections (which are limits), giving the isomorphism

$$\kappa\colon \Big(\bigcap_r Z_s^r\Big)/B_s^q \xrightarrow{\cong} \bigcap_r (Z_s^r/B_s^q)\,.$$

Preservation of isomorphisms

The following result allows us to make deductions about a morphism between two spectral sequences, even if we are not able to calculate all of their differentials.

Proposition

- *Let* $\phi\colon (E^r, d^r)_{r\geq p} \to ({}'E^r, {}'d^r)_{r\geq p}$ *be a morphism of* E^p*-spectral sequences.*
- *Suppose that there is a* $q < \infty$ *such that*

$$\phi^q\colon E^q_{*,*} \xrightarrow{\cong} {}'E^q_{*,*}$$

  is an isomorphism.
- *Then*

$$\phi^r\colon E^r_{*,*} \xrightarrow{\cong} {}'E^r_{*,*}$$

  is an isomorphism for all $r \geq q$*, including* $r = \infty$*.*

Proof

Ignoring the E^r-terms for $r < q$, we may assume that $p = q$ and that $\phi^p \colon E^p \to {}'E^p$ is an isomorphism. It then follows for each $r \geq p$, by induction, that $\phi^r \colon E^r \to {}'E^r$, $\phi^r \colon \ker(d^r) \to \ker({}'d^r)$ and $\phi^r \colon \operatorname{im}(d^r) \to \operatorname{im}({}'d^r)$ are isomorphisms, in view of the commutative diagrams

$$\begin{array}{ccc} E^r & \xrightarrow[\cong]{\phi^r} & {}'E^r \\ {\scriptstyle d^r}\downarrow & & \downarrow{\scriptstyle {}'d^r} \\ E^r & \xrightarrow[\cong]{\phi^r} & {}'E^r \end{array}$$

and

$$\begin{array}{ccc} H(E^r, d^r) & \xrightarrow[\cong]{\phi^r_*} & H({}'E^r, {}'d^r) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ E^{r+1} & \xrightarrow{\phi^{r+1}} & {}'E^{r+1} \end{array}.$$

Proof (cont.)

Since $\ker(d^r) = Z^{r+1}/B^r$ and $\mathrm{im}(d^r) = B^{r+1}/B^r$ with $0 = B^p \subset Z^p = E^p$, and likewise for ${}'d^r$, it follows that

$$\phi^r \colon Z^r \xrightarrow{\cong} {}'Z^r$$

$$\phi^r \colon B^r \xrightarrow{\cong} {}'B^r$$

are isomorphisms for all $r \geq p$. Passing to intersections and unions, we deduce that

$$\phi^\infty \colon Z^\infty \xrightarrow{\cong} {}'Z^\infty$$

$$\phi^\infty \colon B^\infty \xrightarrow{\cong} {}'B^\infty$$

are isomorphisms, which implies that $\phi^\infty \colon E^\infty \to {}'E^\infty$ is an isomorphism, as claimed. □

Different exact couples, same spectral sequence

- This proposition shows that if $\phi\colon (A, E) \to ({}'A, {}'E)$ is a morphism of exact couples such that $\phi\colon E \to {}'E$ is an isomorphism, then the induced morphism of E^1-spectral sequences $\phi\colon (E^r, d^r) \to ({}'E^r, {}'d^r)$ is an isomorphism.
- This may well happen even if $\phi\colon A \to {}'A$ is not an isomorphism, so different exact couples may give rise to the same spectral sequence.

Outline

Discrete and exhaustive filtrations

We now generalize the definition of convergence, from the degreewise bounded case, by weakening the bounded above condition.

Definition

A filtration $(F_sG_*)_s$ of a graded abelian group G_* is exhaustive if

$$\bigcup_s F_sG_* = G_*\,.$$

It is degreewise discrete if for each total degree n there is an integer $a = a(n)$ such that $F_{a-1}G_n = 0$.

Discrete vs. bounded below

- We might say "bounded below" in place of "discrete", but this may become confusing when we also discuss decreasing filtrations.
- The terminology "degreewise discrete" is suggested by thinking of the subgroups F_sG_n for $s \in \mathbb{Z}$ as forming a neighborhood basis of the origin for a linear topology on G_n.
- The cosets $x + F_sG_n$ for $s \in \mathbb{Z}$ then form a neighborhood basis around x.
- The resulting topology is discrete if and only if $F_sG_n = 0$ for some s.

Convergence

Definition

- Let $(E^r_{*,*}, d^r)_r$ be a spectral sequence and let $(F_s G_*)_s$ be a filtration of a graded abelian group G_*.
- Suppose that the filtration is exhaustive and degreewise discrete.
- Then we say that the spectral sequence converges to G_*, written

  $$E^r_{*,*} \Longrightarrow G_* \,,$$

  if there are isomorphisms

  $$E^\infty_{s,t} \cong \frac{F_s G_{s+t}}{F_{s-1} G_{s+t}}$$

  for all (s, t).

An isomorphism theorem

The next theorem is often used in conjunction with the proposition on preservation of isomorphisms to show that a map of spectral sequences can be used to establish an isomorphism $G_* \cong {}'G_*$, even if we do not know enough about the differentials d^r and ${}'d^r$ in these spectral sequences to actually calculate their abutments.

Theorem

- *Let* $\phi\colon (E^r, d^r)_{r\geq p} \to ({}'E^r, {}'d^r)_{r\geq p}$ *be a morphism of* E^p*-spectral sequences, converging to a morphism* $\psi\colon G_* \to {}'G_*$ *of filtered graded abelian groups.*
- *Suppose that each filtration is degreewise discrete and exhaustive, and suppose that*

$$\phi^\infty\colon E^\infty_{*,*} \xrightarrow{\cong} {}'E^\infty_{*,*}$$

  is an isomorphism.
- *Then*

$$\psi\colon G_* \xrightarrow{\cong} {}'G_*$$

  is an isomorphism.

Proof

- Fix a total degree n. We prove for each s, by induction, that
$$\psi_s \colon F_s G_n \longrightarrow F_s{}' G_n$$
is an isomorphism.
- The assumption that the filtrations $(F_s G_*)_s$ and $(F_s{}' G_*)_s$ are degreewise discrete ensures that there is an integer a with $F_{a-1} G_n = 0$ and $F_{a-1}{}' G_n = 0$.
- Hence ψ_{a-1} is trivially an isomorphism.

Proof (cont.)

Consider the vertical map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{s-1}G_n & \longrightarrow & F_sG_n & \longrightarrow & \dfrac{F_sG_n}{F_{s-1}G_n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{s-1}} & & \downarrow{\scriptstyle \psi_s} & & \downarrow{\scriptstyle \bar{\psi}_s} & & \\
0 & \longrightarrow & F_{s-1}{}'G_n & \longrightarrow & F_s{}'G_n & \longrightarrow & \dfrac{F_s{}'G_n}{F_{s-1}{}'G_n} & \longrightarrow & 0\,.
\end{array}$$

We may assume by induction on s that ψ_{s-1} is an isomorphism. By convergence, the commutative diagram

$$\begin{array}{ccc}
E^{\infty}_{s,n-s} & \xrightarrow[\cong]{\phi^{\infty}} & {}'E^{\infty}_{s,n-s} \\
\cong\downarrow & & \downarrow\cong \\
\dfrac{F_sG_n}{F_{s-1}G_n} & \xrightarrow{\bar{\psi}_s} & \dfrac{F_s{}'G_n}{F_{s-1}{}'G_n}
\end{array}$$

and the assumption that ϕ^{∞} is an isomorphism, we know that $\bar{\psi}_s$ is an isomorphism. It then follows that ψ_s is an isomorphism.

Proof (cont.)

To complete the proof we use that both filtrations are exhaustive to pass to unions over s and conclude that

$$\psi\colon G_n = \bigcup_s F_s G_n \xrightarrow{\cong} \bigcup_s F_s{}' G_n = {}' G_n$$

is an isomorphism. □

Outline

Discrete convergence for exact couples

- We return to the setting of the spectral sequence (E^r, d^r) associated to an exact couple (A, E), where we assume that each α_s preserves the total degree.
- We will show that if the sequence of graded abelian groups

$$\cdots \to A_{s-2} \xrightarrow{\alpha_{s-1}} A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\alpha_{s+1}} A_{s+1} \to \dots$$

  is (degreewise) discrete, then the spectral sequence converges (strongly) to the colimit

$$A_\infty = \operatorname*{colim}_s A_s$$

  of this sequence.
- In a later section we will discuss what happens when the sequence is not discrete.

(Degreewise) discrete sequences

Definition

- The sequence

$$\cdots \to A_{s-2} \xrightarrow{\alpha_{s-1}} A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\alpha_{s+1}} A_{s+1} \to \ldots$$

  is discrete if there is an integer a such that $A_s = 0$ for all $s < a$.
- More generally, it is degreewise discrete if for each total degree n there is an integer $a(n)$ such that $(A_s)_n = 0$ for all $s < a(n)$.

Sequential colimits for abelian groups

Definition

- The colimit $A_\infty = \operatorname{colim}_s A_s$ of the sequence

$$\cdots \to A_{s-2} \xrightarrow{\alpha_{s-1}} A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\alpha_{s+1}} A_{s+1} \to \ldots$$

  is the initial graded abelian group that receives compatible structure morphisms

$$i_s \colon A_s \longrightarrow A_\infty$$

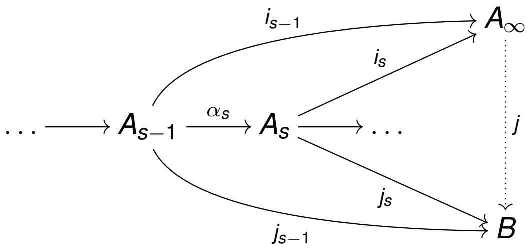
  for each $s \in \mathbb{Z}$.
- Explicitly,

$$A_\infty = \bigoplus_s A_s/(\sim)$$

  where $\sim$ identifies $x \in A_{s-1}$ with $\alpha_s(x) \in A_s$, for all s.

Remarks

- By "compatible" we mean that $i_s \alpha_s = i_{s-1}$ for each s.
- By "initial" we mean that for any other graded abelian group B with compatible homomorphisms $j_s \colon A_s \to B$ there exists a unique homomorphism $j \colon A_\infty \to B$ such that $j_s = j i_s$ for each s.
- This characterizes A_∞, with the structure morphisms i_s, up to unique isomorphism.



Lemma

- *Each element $y \in A_\infty$ is of the form*

$$y = i_s(x)$$

  for some $s \in \mathbb{Z}$ and $x \in A_s$.

- *An element $x \in A_s$ maps to zero in A_∞, meaning that $i_s(x) = 0$, only if there is some $u \geq 0$ with*

$$\alpha^u(x) = \alpha_{s+u} \cdots \alpha_{s+1}(x) = 0$$

  in A_{s+u}.

Proof.

(Easy.) □

No left derived sequential colimit

Lemma

There is a short exact sequence

$$0 \to \bigoplus_s A_s \xrightarrow{1-\alpha} \bigoplus_s A_s \xrightarrow{\pi} A_\infty \to 0\,,$$

where 1 *denotes the identity map and*

$$\alpha\colon (x_s)_s \longmapsto (\alpha_s(x_{s-1}))_s$$

for each sequence $(x_s)_s$ *with only finitely many nonzero terms.*

Proof.

In view of the explicit formula for $A_\infty = \operatorname{colim}_s A_s$, we only need to argue that $1 - \alpha$ is injective. Let $x = (x_s)_s \in \bigoplus_s A_s$, and choose a such that $x_s = 0$ for all $s < a$. If $(1 - \alpha)(x) = 0$ then $x_s = \alpha_s(x_{s-1})$ for all s. It follows by induction on s, starting with $s = a$, that $x_s = 0$ for all s. Hence $x = 0$. □

An exhaustive filtration of the colimit

Definition

For $s \in \mathbb{Z}$ let

$$F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty)\,.$$

This defines an increasing filtration

$$\cdots \subset F_{s-1}A_\infty \subset F_s A_\infty \subset \cdots \subset A_\infty$$

of graded abelian groups.

Lemma (1)

The filtration of $A_\infty = \operatorname{colim}_s A_s$ by $F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty)$ is exhaustive.

Proof.

Each $y \in A_\infty$ has the form $y = i_s(x)$ for some $x \in A_s$, and then $y \in F_s A_\infty$. Hence $\bigcup_s F_s A_\infty = A_\infty$. □

Review about the spectral sequence of an exact couple

Recall the diagram

$$
\begin{array}{ccccccc}
A_{s-r} & \xrightarrow{\alpha^{r-1}} & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{\alpha^{r-1}} & A_{s+r-1} \\
 & & & \nwarrow{\scriptstyle \gamma_s} & \downarrow{\scriptstyle \beta_s} & & \\
 & & & & E_s & &
\end{array}
$$

and the chains

$$
\begin{aligned}
0 = B_s^1 \subset \cdots \subset B_s^r \subset \cdots \subset B_s^\infty \subset \operatorname{im}(\beta_s) \\
= \ker(\gamma_s) \subset Z_s^\infty \subset \cdots \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s^1 = E_s\,.
\end{aligned}
$$

Infinite cycles for discrete sequences

Lemma (2)

- *Consider an exact couple* (A, E) *such that the sequence*

$$\cdots \to A_{s-2} \xrightarrow{\alpha_{s-1}} A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\alpha_{s+1}} A_{s+1} \to \ldots$$

  is degreewise discrete.
- *Then*

$$Z_s^\infty = \ker(\gamma_s)$$

  for each s, and
- *the filtration* $(F_s A_\infty)_s$ *is degreewise discrete.*

Proof

- We always have $\ker(\gamma_s) \subset Z_s^\infty$.
- If $x \in Z_s^\infty$ then $x \in Z_s^r$ for each r, so $\gamma_s(x) \in A_{s-1}$ lies in the image of $\alpha^{r-1}\colon A_{s-r} \to A_{s-1}$ for each r.
- Let n be the total degree of $\gamma_s(x)$.
- By assumption there is an $a(n)$ such that $(A_{s-r})_n = 0$ whenever $s - r < a(n)$.
- It follows that the image of $(A_{s-r})_n$ in $(A_{s-1})_n$ is trivial for all sufficiently large r, which means that $\gamma_s(x) = 0$.
- Hence $x \in \ker(\gamma_s)$.

- If $(A_s)_n = 0$ for all $s < a(n)$ then $(F_s A_\infty)_n = 0$ for $s < a(n)$, so the filtration is degreewise discrete whenever the sequence is. $\square$

Infinite boundaries

Lemma (3)

Let (A, E) *be any exact couple, and set* $A_\infty = \operatorname{colim}_s A_s$*. Then*

$$B_s^\infty = \beta_s \ker(i_s \colon A_s \to A_\infty)$$

for each s.

Proof.

$$\begin{aligned} B_s^\infty &= \bigcup_r B_s^r = \bigcup_r \beta_s \ker(\alpha^{r-1} \colon A_s \to A_{s+r-1}) \\ &= \beta_s \bigcup_r \ker(\alpha^{r-1} \colon A_s \to A_{s+r-1}) = \beta_s \ker(i_s \colon A_s \to A_\infty)\,, \end{aligned}$$

since $x \in A_s$ maps to zero under some α^{r-1} if and only if it maps to zero under i_s. □

The associated graded of $(F_s A_\infty)_s$

Lemma (4)

Let (A, E) be any exact couple, and filter $A_\infty = \operatorname{colim}_s A_s$ by $F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty)$. There is a preferred isomorphism

$$\frac{\ker(\gamma_s)}{\beta_s \ker(i_s \colon A_s \to A_\infty)} \cong \frac{F_s A_\infty}{F_{s-1} A_\infty}$$

for each $s \in \mathbb{Z}$.

We use the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{s-1} & \xrightarrow{\alpha_s} & A_s & \xrightarrow{i_s} & A_\infty \\ & & & \nwarrow{\scriptstyle \gamma_s} & \downarrow{\scriptstyle \beta_s} & & \\ & & & & E_s\,. & & \end{array}$$

Proof

$$\frac{\ker(\gamma_s)}{\beta_s \ker(i_s\colon A_s \to A_\infty)} = \frac{\mathrm{im}(\beta_s)}{\beta_s \ker(i_s\colon A_s \to A_\infty)}$$

receives an isomorphism induced by β_s from

$$\frac{A_s}{\ker(\beta_s) + \ker(i_s\colon A_s \to A_\infty)} = \frac{A_s}{\mathrm{im}(\alpha_s) + \ker(i_s\colon A_s \to A_\infty)}$$

which maps isomorphically by i_s to

$$\frac{\mathrm{im}(i_s\colon A_s \to A_\infty)}{i_s\,\mathrm{im}(\alpha_s)} = \frac{\mathrm{im}(i_s\colon A_s \to A_\infty)}{\mathrm{im}(i_{s-1}\colon A_{s-1} \to A_\infty)} = \frac{F_s A_\infty}{F_{s-1} A_\infty}\,. \quad \square$$

Convergence for discrete exact couples

- Let (A, E) be an exact couple with associated spectral sequence (E^r, d^r) and E^∞-term $(E_s^\infty)_s$.
- Let $A_\infty = \operatorname{colim}_s A_s$ be filtered by $F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty)$.

Proposition

(1) There is always a preferred injective homomorphism

$$\frac{F_s A_\infty}{F_{s-1} A_\infty} \overset{\zeta}{\rightarrowtail} E_s^\infty ,$$

which is an isomorphism if $Z_s^\infty = \ker(\gamma_s)$.
(2) If each α_s preserves total degree and the sequence

$$\cdots \to A_{s-1} \xrightarrow{\alpha_s} A_s \to \dots$$

is degreewise discrete, then ζ is an isomorphism and the spectral sequence $E_s^r \Longrightarrow_s A_\infty$ converges.

Proof.

This summarizes the previous four lemmas, keeping in mind that we always have the inclusion $\ker(\gamma_s) \subset Z_s^\infty$.

$$\frac{F_s A_\infty}{F_{s-1} A_\infty} \cong \frac{\ker(\gamma_s)}{\beta_s \ker(i_s \colon A_s \to A_\infty)} = \frac{\ker(\gamma_s)}{B_s^\infty} \subset \frac{Z_s^\infty}{B_s^\infty} = E_s^\infty$$

□

Remark

For filtrations that are discrete, the notions of weak convergence, convergence and strong convergence coincide. We may therefore replace “convergence” with “strong convergence” in the definition and proposition above.

Outline

Filtrations

We now give examples of how

- filtered chain complexes
- filtered spaces

give rise to exact couples, with associated spectral sequences, through passage to homology or generalizations thereof.

Outline

Filtered chain complexes

An increasing filtration $(F_sC)_s = (F_sC_*, \partial)_s$ of a chain complex $C = (C_*, \partial)$ is a sequence of subcomplexes

$$\cdots \subset (F_{s-1}C_*, \partial) \subset (F_sC_*, \partial) \subset \cdots \subset (C_*, \partial)\,.$$

For each $s \in \mathbb{Z}$ there is a short exact sequence of chain complexes

$$0 \to F_{s-1}C \xrightarrow{i} F_sC \xrightarrow{j} \frac{F_sC}{F_{s-1}C} \to 0\,. \tag{6}$$

We refer to the grading of $C = (C_n)_n$, and of each subcomplex $F_sC = (F_sC_n)_n$, as the total degree, while s is the filtration degree.

Exhaustive, discrete

$$\cdots \subset (F_{s-1}C_*, \partial) \subset (F_sC_*, \partial) \subset \cdots \subset (C_*, \partial)\,.$$

We say that the filtration is exhaustive if

$$\bigcup_s F_sC = C\,.$$

It is degreewise discrete if for each degree n there is an integer $a = a(n)$ such that $F_{a-1}C_n = 0$.

Associated exact couple

The exact couple $(A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ associated to a filtered chain complex $(F_sC)_s$ is the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\alpha_{s-1}} & H_*(F_{s-1}C) & \xrightarrow{\alpha_s} & H_*(F_sC) & \xrightarrow{\alpha_{s+1}} & \cdots \\ & & \nwarrow{\scriptstyle \gamma_s} & & \downarrow{\scriptstyle \beta_s} & & \\ & & \cdots & & H_*(F_sC/F_{s-1}C) & & \cdots, \end{array}$$

where

$$(A_s)_* = H_*(F_sC)$$
$$(E_s)_* = H_*(F_sC/F_{s-1}C)$$

with α_s and β_s induced by i and j, and γ_s equal to the connecting homomorphism associated to the short exact sequence (6).

Bigrading

The bigrading is given by

$$A_{s,t} = H_{s+t}(F_s C)$$
$$E_{s,t} = H_{s+t}(F_s C / F_{s-1} C)\,,$$

so that

- α_s has bidegree $(1, -1)$,
- β_s has bidegree $(0, 0)$ and
- γ_s has bidegree $(-1, 0)$.

Thus

- α_s and β_s preserve the total degree $n = s + t$,
- while γ_s reduces it by 1.

Filtration of $H_*(C)$

Definition

Given a filtration $(F_sC)_s$ of $C = (C_*, \partial)$, let

$$F_sH_*(C) = \mathrm{im}(H_*(F_sC) \to H_*(C))$$

for each s.

Note the two different roles played by the notation "F_s" in this definition. On the left hand side it refers to the filtration of the abutment $H_*(C)$, while on the right hand side it refers to the filtration of the chain complex (C_*, ∂).

Exhaustive

Lemma

If $(F_s C)_s$ exhausts C, then the canonical morphism

$$A_\infty = \operatorname*{colim}_s H_*(F_s C) \xrightarrow{\cong} H_*(C)$$

is an isomorphism, which restricts to isomorphisms

$$F_s A_\infty \cong F_s H_*(C)$$

for all s.

Proof.

This follows from the well-known isomorphism

$$\operatorname*{colim}_s H_*(F_s C) \xrightarrow{\cong} H_*(\operatorname*{colim}_s F_s C) .$$

□

Discrete

Lemma
If $(F_s C)_s$ is degreewise discrete, then the sequence

$$\cdots \to A_{s-1} \xrightarrow{\alpha_s} A_s \to \ldots$$

is degreewise discrete.

Proof.
If $(F_{a-1}C)_n = 0$ for some n and $a = a(n)$, then $(F_s C)_n = 0$ and $H_n(F_s C) = (A_s)_n = 0$ for all $s < a$, which implies the claim. $\square$

Homology sp. seq. of a filtered chain complex

Proposition

Let C be a chain complex with a filtration $(F_sC)_s$.
The associated spectral sequence has E^1-term

$$E^1_{s,*} = H_*(F_sC/F_{s-1}C)$$

and d^1-differential the composite

$$d^1_s = \beta_{s-1}\gamma_s \colon E^1_{s,*} \longrightarrow E^1_{s-1,*}\,,$$

which equals the connecting homomorphism associated to the short exact sequence

$$0 \to F_{s-1}C/F_{s-2}C \xrightarrow{i} F_sC/F_{s-2}C \xrightarrow{j} F_sC/F_{s-1}C \to 0$$

of chain complexes.

Proof

The spectral sequence is the one associated to the exact couple associated to the filtered chain complex.
The vertical map of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{s-1}C & \longrightarrow & F_sC & \longrightarrow & \dfrac{F_sC}{F_{s-1}C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \dfrac{F_{s-1}C}{F_{s-2}C} & \longrightarrow & \dfrac{F_sC}{F_{s-2}C} & \longrightarrow & \dfrac{F_sC}{F_{s-1}C} & \longrightarrow & 0 \end{array}$$

induces a map of long exact homology sequences. The commutative square

$$\begin{array}{ccc} H_*(F_sC/F_{s-1}C) & \xrightarrow{\gamma_s} & H_{*-1}(F_{s-1}C) \\ = \downarrow & & \downarrow \beta_{s-1} \\ H_*(F_sC/F_{s-1}C) & \xrightarrow{\partial} & H_{*-1}(F_{s-1}C/F_{s-2}C) \end{array}$$

shows that $d_s^1 = \beta_{s-1}\gamma_s$ is the stated connecting homomorphism. □

Functoriality

Lemma

Each morphism $\psi\colon (F_sC)_s \to (F_s{}'C)_s$ *of filtered chain complexes induces a morphism* $\phi\colon (A,E) \to ({}'A, {}'E)$ *of exact couples. Hence the associated exact couple defines a functor*

$$\textit{Filtered Chain Complexes} \longrightarrow \textit{Exact Couples}\,.$$

Proof.

$\phi\colon A_s \to {}'A_s$ and $\phi\colon E_s \to {}'E_s$ are induced by the chain maps

$$\psi_s\colon F_sC \longrightarrow F_s{}'C$$

$$\bar{\psi}_s\colon \frac{F_sC}{F_{s-1}C} \longrightarrow \frac{F_s{}'C}{F_{s-1}{}'C}$$

by passage to homology. □

Convergence of the homology spectral sequence

Proposition

Suppose that the filtration $(F_sC)_s$ of the chain complex C is exhaustive and degreewise discrete. Then the spectral sequence

$$E^1_{s,*} = H_{s+*}(F_sC/F_{s-1}C) \Longrightarrow_s H_*(C)$$

converges to $H_*(C)$ with the filtration given by

$$F_sH_*(C) = \mathrm{im}(H_*(F_sC) \to H_*(C)) .$$

Proof.

The (strong) convergence follows since the exact couple is degreewise discrete, with colimit $A_\infty \cong H_*(C)$. □

A case of the isomorphism theorem

- Let C and $'C$ be chain complexes, with filtrations $(F_sC)_s$ and $(F_s{}'C)_s$ that are exhaustive and degreewise discrete.
- Let $\psi\colon C \to {}'C$ be a filtration-preserving map of filtered chain complexes, and suppose that the induced map

$$\phi^r\colon E^r \longrightarrow {}'E^r\,,$$

  of E^r-terms of the associated homology spectral sequences, is an isomorphism for some r.
- Then

$$\psi_*\colon H_*(C) \longrightarrow H_*({}'C)$$

  is an isomorphism.

A case of the isomorphism theorem (cont.)

- For example, it suffices that the map of E^1-terms

$$\phi^1 \colon H_*(F_sC/F_{s-1}C) \longrightarrow H_*(F_s{}'C/F_{s-1}{}'C)$$

  is an isomorphism for each s.
- The expression for d^1 as a connecting homorphism sometimes gives us access to (E^1, d^1) and $({}'E^1, {}'d^1)$ as chain complexes of graded abelian groups.
- It then suffices that

$$\phi^1 \colon (E^1, d^1) \longrightarrow ({}'E^1, {}'d^1)$$

  is a quasi-isomorphism, so that the map ϕ^2 of E^2-term is an isomorphism.

Outline

Strongly filtered spaces

- The singular complexes of filtered spaces provide examples of filtered chain complexes, hence of exact couples and spectral sequences.
- To discuss exhaustion, the following terminology from Neil Strickland's note [Str, Def. 3.4] is useful.

Definition

A space X is strongly filtered by a sequence of subspaces

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

if for each compact subset $K \subset X$ there is an s with $K \subset X_s$.

Strongly filtered implies exhaustive

Lemma

If X is strongly filtered by $(X_s)_s$, then the singular chain complex $(C_*(X), \partial)$ is exhaustively filtered by the subcomplexes

$$\cdots \subset C_*(X_{s-1}) \subset C_*(X_s) \subset \cdots \subset C_*(X) \,.$$

If $X_{a-1} = \emptyset$ for some a, then the filtration $(C_*(X_s))_s$ is discrete.

Proof.

The only thing to prove is that each singular simplex $\sigma \colon \Delta^n \to X$, viewed as an element of $C_n(X)$, lies in the image from some $C_n(X_s)$. Since the image $\sigma(\Delta^n) \subset X$ is compact, this follows from the assumption that the filtration is strong. □

The homology spectral sequence of a filtered space

Proposition

Let X be a space with a filtration $(F_sX)_s$. The associated homology spectral sequence has E^1-term

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$$

and d^1-differential the composite

$$d^1_{s,t} = \beta_{s-1}\gamma_s \colon E^1_{s,t} \longrightarrow E^1_{s-1,t}\,,$$

which equals the connecting homomorphism in the long exact sequence of the triple (X_s, X_{s-1}, X_{s-2}).

Convergence of the homology spectral sequence

Proposition

Suppose that X is strongly filtered by $(X_s)_s$, and that $X_{a-1} = \emptyset$ for some a. Then the spectral sequence

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1}) \Longrightarrow_s H_{s+t}(X)$$

converges to $H_*(X)$ with the filtration given by

$$F_s H_*(X) = \operatorname{im}(H_*(X_s) \to H_*(X))\,.$$

Proof.

The (strong) convergence follows since the exact couple is discrete, with colimit $A_\infty \cong H_*(X)$. □

- The convergence statement tells us that there is an exhaustive filtration

$$0 = F_{a-1}H_n(X) \subset \cdots \subset F_{s-1}H_n(X) \subset F_sH_n(X) \subset \cdots \subset H_n(X)$$

  in each total degree n, with filtration quotients determined by the E^∞-term, through isomorphisms

$$E^\infty_{s,n-s} \cong \frac{F_sH_n(X)}{F_{s-1}H_n(X)}$$

  for all s.
- The components of $E^\infty_{*,*}$ in bidegrees $(s, n-s)$, on a line of slope -1, give the associated graded of this exhaustive filtration.
- By induction on s, starting at $s = a$, we can attempt to determine $F_sH_n(X)$ as an extension of $E^\infty_{s,n-s}$ by $F_{s-1}H_n(X)$. The union of these groups, over all s, gives us $H_n(X)$.

Weak union of closed T_1 subspaces

Many strongly filtered spaces are of the following form.

Lemma ([Ste67, Lem. 9.3])

Let X be filtered by an exhaustive sequence of T_1 subspaces

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

such that X_{s-1} is closed in X_s for each s, and suppose that X has the weak (= colimit) topology. Then X is strongly filtered by these $(X_s)_s$.

We have $X = \bigcup_s X_s$ since the filtration is exhaustive. To be a T_1 space is equivalent to asking that each singleton subset is closed. This is satisfied by all (weak) Hausdorff spaces. The weak topology on X is defined so that a subset $A \subset X$ is closed in X if and only if $A \cap X_s$ is closed in X_s for each s.

Proof

Following Steenrod, we argue that if $K \subset X$ is compact, then $K \subset X_s$ for some s. If not, we can choose a point $x_s \in K \cap (X - X_s)$ for each s. Let

$$A_m = \{x_s \mid s \geq m\} \subset K \cap (X - X_m)\,,$$

so that

$$\cdots \supset A_{m-1} \supset A_m \supset \ldots$$

is a collection of subsets of K, such that each finite subcollection has nonempty intersection $A_{m_1} \cap \cdots \cap A_{m_n} = A_m$ (with $m = \max\{m_1, \ldots, m_n\}$), but the whole collection satisfies $\bigcap_m A_m = \emptyset$.

Proof (cont.)

If we show that each A_m is closed in K, then this contradicts the finite intersection property of compact spaces, and proves that $K \subset X_s$ for some s. To see that each A_m is closed, note that each intersection $A_m \cap X_s \subset \{x_m, \ldots, x_{s-1}\}$ is finite, hence is closed in X_s since this is a T_1 space. By the definition of the weak topology this proves that A_m is closed in X, hence also in the subspace K. □

Cellular homology

- The cellular complex $(C_*^{CW}(X), \partial)$ calculating the homology of a CW complex X is a very special case of this spectral sequence.
- Other notations for the cellular complex are $\Gamma_*(X)$, as in [Whi78, §II.2], or $W_*(X)$.
- Let us write $H_n^{CW}(X) = H_n(C_*^{CW}(X), \partial)$ for the cellular homology groups.
- The usual argument for why cellular homology is isomorphic to singular homology [Whi78, Thm. II.2.19], [Hat02, Thm. 2.35], is contained within our more elaborate algebraic work, as we can now spell out.

Proposition

Let X be a CW complex, with skeleton filtration

$$\emptyset = X^{(-1)} \subset \cdots \subset X^{(s-1)} \subset X^{(s)} \subset \cdots \subset X\,.$$

The associated homology spectral sequence has $(E^1, d^1) = (C_*^{CW}(X), \partial)$, concentrated on the line $t = 0$. Hence

$$E^2_{s,t} = \begin{cases} H_s^{CW}(X) & \text{for } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the spectral sequence collapses at $E^2 = E^\infty$. The filtration of $H_n(X)$ satisfies

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < n, \\ H_n^{CW}(X) & \text{for } s \geq n. \end{cases}$$

Hence $H_*^{CW}(X) \cong H_*(X)$.

Proof

- The CW complex X is strongly filtered by its skeleta.
- By definition, $E^1_{s,t} = H_{s+t}(X^{(s)}, X^{(s-1)})$ equals

$$C^{CW}_s(X) \cong \mathbb{Z}\{n\text{-cells of } X\}$$

  when $t = 0$, and is trivial when $t \neq 0$.
- Likewise, $d^1_{s,t} = \partial_s$ when $t = 0$ and is zero otherwise.

$$\begin{array}{c|ccccccccc}
2 & 0 & & \dots & & 0 & & 0 & & \dots \\
1 & 0 & & \dots & & 0 & & 0 & & \dots \\
0 & C^{CW}_0(X) & \xleftarrow{\partial_1} & \dots & \xleftarrow{\partial_{s-1}} & C^{CW}_{s-1}(X) & \xleftarrow{\partial_s} & C^{CW}_s(X) & \xleftarrow{\partial_{s+1}} & \dots \\
\hline
t/s & 0 & & & & s-1 & & s & &
\end{array}$$

Proof (cont.)

- Hence $E^2_{s,t} = H_s(C^{CW}_*(X), \partial) = H^{CW}_s(X)$ equals the cellular homology of X when $t = 0$, and is trivial otherwise.
- Each d^r-differential for $r \geq 2$ increases t, hence must be zero, so $E^2 = E^\infty$.
- In each total degree n there is only one nonzero group of the form $E^\infty_{s,n-s}$, namely $E^\infty_{n,0} = E^2_{n,0} = H^{CW}_n(X)$.
- The short exact sequences

$$0 \to F_{s-1}H_n(X) \longrightarrow F_sH_n(X) \longrightarrow E^\infty_{s,n-s} \to 0$$

for $s < n$ simplify to

$$0 \to 0 \longrightarrow F_sH_n(X) \longrightarrow 0 \to 0$$

and imply that $F_sH_n(X) = 0$ for $s < n$ by induction on s.

Proof (cont.)

- The short exact sequence sequence for $s = n$ simplifies to an isomorphism

$$0 \to 0 \longrightarrow F_n H_n(X) \xrightarrow{\cong} H_n^{CW}(X) \to 0\,.$$

- Thereafter, for $s > n$ they simplify to isomorphisms

$$0 \to H_n^{CW}(X) \xrightarrow{\cong} F_s H_n(X) \longrightarrow 0 \to 0\,.$$

- Hence $F_s H_n(X) \cong H_n^{CW}(X)$ for $s > n$. □

Outline

Generalized homology theories

- Let G be an abelian group. Singular homology with coefficients in G is an example of a homology theory, sometimes referred to as "ordinary" homology.
- Since ca. 1960 many other "generalized" or "extraordinary" homology theories have come to play important roles in algebraic topology.
- The following definition is close to the axiomatization by Samuel Eilenberg and Norman Steenrod from [ES52, §I.3], but omits their dimension axiom and adds John Milnor's additivity axiom [Mil62].

Definition

A (generalized) homology theory M on the category of CW pairs is a functor assigning to each CW pair (X, A) a graded abelian group

$$M_*(X, A) = (M_n(X, A))_n \, ,$$

and a natural transformation

$$\partial \colon M_*(X, A) \longrightarrow M_{*-1}(A)$$

of degree -1, such that:

Definition (cont.)

1. Exactness: the sequence

$$\cdots \to M_*(A) \xrightarrow{i_*} M_*(X) \xrightarrow{j_*} M_*(X, A) \xrightarrow{\partial} M_{*-1}(A) \to \dots$$

   is long exact.
2. Homotopy invariance: if $f \simeq g \colon (X, A) \to (Y, B)$ are homotopic, then $f_* = g_*$.
3. Excision: if $X = A \cup B$ is a union of subcomplexes, then the inclusion induces an isomorphism

$$M_*(B, A \cap B) \xrightarrow{\cong} M_*(X, A)\,.$$

4. Additivity: the canonical map

$$\bigoplus_\alpha M_*(X_\alpha) \xrightarrow{\cong} M_*(\coprod_\alpha X_\alpha)$$

   is an isomorphism.

Coefficients

Definition

The coefficient groups of a homology theory M is the graded abelian group

$$M_* = (M_n(\text{point}))_n \,.$$

We say that M_* is bounded below if there is an a such that $M_n = 0$ for all $n < a$. We say that M_* is bounded above if there is a b such that $M_n = 0$ for all $n > b$.

Example

Let G be an abelian group. The coefficient groups of ordinary homology with coefficients in G, i.e., the homology theory HG given by

$$HG_n(X) = H_n(X; G)$$

for all n, equals G in degree 0 and 0 in all other degrees. This is the content of the Eilenberg–Steenrod Dimension axiom.

M-homology of discs and spheres

Lemma

For any homology theory M there are isomorphisms

$$M_{s+t}(D^s, \partial D^s) \cong \tilde{M}_{s+t}(S^s) \cong M_t$$

for all $s \geq 0$, $t \in \mathbb{Z}$.

Proof.

This is clear for $s = 0$, and follows by induction for $s \geq 1$ (using exactness, homotopy invariance and excision). □

K-theory and bordism

- For any graded abelian group G_* there is a generalized homology theory with

$$M_n(X) = \bigoplus_{i+j=n} H_i(X; G_j)\,,$$

  but it carries more-or-less the same information as ordinary homology.
- Other important examples of (co-)homology theories include the topological K-theories

$$KO^*(X) \qquad \text{and} \qquad K^*(X) = KU^*(X)$$

  defined by Michael Atiyah and Friedrich Hirzebruch [AH59], following Alexander Grothendick [BS58], and
- the bordism theories

$$N_*(X) = MO_*(X) \qquad \text{and} \qquad \Omega_*(X) = MSO_*(X)$$

  defined by Atiyah [Ati61a], building on the work of René Thom [Tho54].

K-theory and bordism (cont.)

- By construction, these involve vector bundles

$$E \longrightarrow X$$

over X and closed manifolds

$$M^n \longrightarrow X$$

mapping to X, respectively, rather than formal sums of simplices

$$\sigma\colon \Delta^n \longrightarrow X$$

in X, and often turn out to emphasize different information than the ordinary homology of X.

- We will later present generalized (co-)homology theories by the objects, called spectra, of a stable (homotopy) category, and analyze the coefficient groups (and rings) of some of these homology theories.

A generalized homology spectral sequence

Definition

Let X be a CW complex exhaustively filtered by subcomplexes

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X\,,$$

and let M be a homology theory. The associated exact couple is the diagram

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & M_*(X_{s-1}) & \xrightarrow{i_*} & M_*(X_s) & \longrightarrow & \ldots \\
 & & & \nwarrow^{\partial} & \big\downarrow j_* & & \\
 & & & & M_*(X_s, X_{s-1}) & &
\end{array}$$

with

$$(A_s)_* = M_*(X_s)$$
$$(E_s)_* = M_*(X_s, X_{s-1})\,.$$

The abutment

Lemma ([Mil62, Lem. 1])

The canonical homomorphism

$$\operatorname*{colim}_s M_*(X_s) \xrightarrow{\cong} M_*(X)$$

is an isomorphism.

This is the expected abutment

$$A_\infty = \operatorname*{colim}_s A_s = \operatorname*{colim}_s M_*(X_s)$$

of the spectral sequence.

Sketch proof

There is a homotopy cofiber sequence

$$\bigvee_s \Sigma_+ X_s \xrightarrow{1-\alpha} \bigvee_s \Sigma_+ X_s \longrightarrow \Sigma_+ T$$

where $\Sigma_+ Y = \Sigma(Y_+)$, and $T \simeq X$ is the mapping telescope of $(X_s)_s$. In view of our lemma on sequential colimits, the associated long exact sequence in reduced M-homology breaks up into short exact sequences

$$0 \to \bigoplus_s M_*(X_s) \xrightarrow{1-\alpha} \bigoplus_s M_*(X_s) \longrightarrow M_*(T) \to 0$$

that exhibit $M_*(T)$ as $\operatorname{colim}_s M_*(X_s)$. □

Proposition

The spectral sequence associated to $(X_s)_s$ and M has

$$E^1_{s,t} = M_{s+t}(X_s, X_{s-1})$$

and $d^1_{s,t}$ is equal to the composite

$$M_{s+t}(X_s, X_{s-1}) \xrightarrow{\partial} M_{s+t-1}(X_{s-1}) \xrightarrow{j_*} M_{s+t-1}(X_{s-1}, X_{s-2}) \,.$$

If $X_{a-1} = \emptyset$ for some a, then the spectral sequence converges to $M_*(X)$ with the filtration

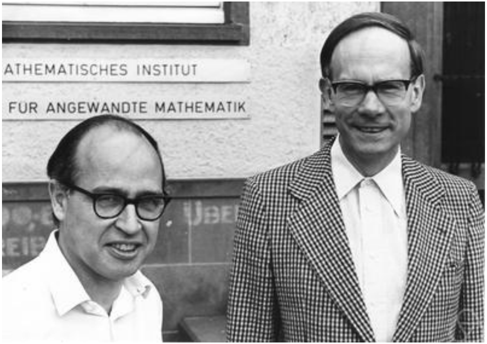
$$F_s M_*(X) = \operatorname{im}(M_*(X_s) \to M_*(X)) \,.$$

Proof.

This follows from the proposition on convergence for discrete exact couples. □

The Atiyah–Hirzebruch spectral sequence

When X is equipped with its skeleton filtration, we can make the E^1- and E^2-term explicit.



Michael Atiyah, Friedrich Hirzebruch

Proposition

Let X be a CW complex filtered by its skeleta

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(s-1)} \subset X^{(s)} \subset \cdots \subset X \,,$$

and let M be a homology theory. The associated spectral sequence

$$E^r_{s,*} \Longrightarrow_s M_*(X)$$

has (E^1, d^1)-term given by the cellular complex $(C^{CW}_*(X; M_*), \partial)$, with

$$E^1_{s,t} \cong C^{CW}_s(X; M_t) = H_s(X^{(s)}, X^{(s-1)}; M_t)$$

and $d^1_{s,t}$ equal to the connecting homomorphism

$$\partial_s \colon H_s(X^{(s)}, X^{(s-1)}; M_t) \longrightarrow H_{s-1}(X^{(s-1)}, X^{(s-2)}; M_t)$$

for homology with coefficients in the group M_t.

Proposition (cont.)

$$
\begin{array}{c|ccccc}
 & \dots & \dots & \dots & \dots & \dots \\
t & C_0^{CW}(X; M_t) \xleftarrow{\partial_1} & \dots \xleftarrow{\partial_{s-1}} & C_{s-1}^{CW}(X; M_t) \xleftarrow{\partial_s} & C_s^{CW}(X; M_t) \xleftarrow{\partial_{s+1}} & \dots \\
 & \dots & \dots & \dots & \dots & \dots \\
0 & C_0^{CW}(X; M_0) \xleftarrow{\partial_1} & \dots \xleftarrow{\partial_{s-1}} & C_{s-1}^{CW}(X; M_0) \xleftarrow{\partial_s} & C_s^{CW}(X; M_0) \xleftarrow{\partial_{s+1}} & \dots \\
\hline
t/s & 0 & & s-1 & s &
\end{array}
$$

Hence

$$E^2_{s,t} \cong H_s^{CW}(X; M_t) \cong H_s(X; M_t)$$

is given by the cellular (or singular) homology of X in degree s, with coefficients in M_t.

Proof

To identify the E^1-term we use the excision and additivity isomorphisms

$$\begin{aligned} E^1_{s,t} &= M_{s+t}(X^{(s)}, X^{(s-1)}) \\ &\cong M_{s+t}(\coprod_\alpha (D^s, \partial D^s)) \cong \bigoplus_\alpha M_{s+t}(D^s, \partial D^s)\,, \end{aligned}$$

where α indexes the s-cells of X. By the lemma on discs and spheres, the right hand side is isomorphic to

$$\bigoplus_\alpha M_t \cong C^{CW}_s(X; M_t)\,.$$

The degree formula for the connecting homomorphism ∂_s implies that $d^1_{s,t}$ corresponds to the cellular boundary homomorphism

$$\partial_s \colon C^{CW}_s(X; M_t) \longrightarrow C^{CW}_{s-1}(X; M_t)\,.$$

Proof (cont.)

Granting this, we can pass to homology to deduce that

$$E^2_{s,t} \cong H^{CW}_s(X; M_t)\,.$$

By the proposition on cellular homology, and its evident analogue for homology with coefficients, we know that this is isomorphic to singular homology with coefficients in M_t. □

Definition

The spectral sequence

$$E^2_{s,t} = H_s(X; M_t) \Longrightarrow_s M_{s+t}(X)$$

is called the Atiyah–Hirzebruch spectral sequence of X for the homology theory M.

- It can be defined for general spaces X by CW approximation.
- It is then natural in the homology theory M and in the space X.

Coefficient isomorphism $\theta\colon M \to N$

Corollary

If $\theta\colon M \to N$ is a morphism of homology theories that induces an isomorphism of coefficient groups, then

$$\theta_*\colon M_*(X) \xrightarrow{\cong} N_*(X)$$

for any CW complex X.

Proof.

The natural transformation θ induces an isomorphism $C_*^{CW}(X; M_*) \cong C_*^{CW}(X; N_*)$ of Atiyah–Hirzebruch E^1-terms, which implies the result by the isomorphism theorem. □

Homology equivalence $f\colon X \to Y$

Corollary

If $f\colon X \to Y$ induces an isomorphism $f_*\colon H_*(X) \cong H_*(Y)$ in integral homology, then it induces an isomorphism

$$f_*\colon M_*(X) \xrightarrow{\cong} M_*(Y)$$

for any generalized homology theory M.

Proof.

The map f induces an isomorphism

$$H_*(X; M_*) \xrightarrow{\cong} H_*(Y; M_*)$$

of Atiyah–Hirzebruch E^2-terms, which implies the result by the isomorphism theorem. □

Eilenberg–Steenrod uniqueness theorem

The dimension axiom characterizes ordinary homology.

Theorem ([ES52, Thm. III.10.1])

Let G be an abelian group and let M be a homology theory with coefficient groups

$$M_t = \begin{cases} G & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then M is naturally isomorphic to HG, so that

$$M_n(X) \cong H_n(X; G)$$

for all n.

Proof

- The Atiyah–Hirzebruch spectral sequence of X for M has E^2-term

$$E^2_{s,t} = \begin{cases} H_s(X;G) & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Since this is concentrated on the line $t = 0$, the d^r-differentials for $r \geq 2$ must vanish, so that $E^2 = E^\infty$ is concentrated on the line $t = 0$.
- Since $E^\infty_{n,0}$ is the only group in total degree n, the extension problems are very easy, and we conclude that

$$M_n(X) \cong E^\infty_{n,0} \cong H_n(X;G)$$

for each n. □

Topological K-theory

According to Whitehead [Whi78, p. 604] the existence of the spectral sequence $H_*(X; M_*) \Longrightarrow M_*(X)$ was folklore by 1955, but Atiyah and Hirzebruch [AH61] were the first to make significant use of it, in the case of topological K-theory.

Example

Complex K-theory is a (co-)homology theory $K = KU$ with coefficient groups

$$KU_n \cong \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

If $H_*(X)$ is concentrated in even degrees, it follows that the E^2-term of the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; KU_t) \Longrightarrow_s KU_{s+t}(X)$$

is concentrated in even total degrees $s + t$.

Topological K-theory (cont.)

Since each d^r-differential reduces the total degree by one, they must all vanish, so the Atiyah–Hirzebruch spectral sequence collapses at the E^2-term. If, furthermore, $H_*(X)$ is free in each degree, then there exists a (non-canonical) sum formula

$$KU_n(X) \cong \bigoplus_{s \equiv n \mod 2} H_s(X)\,,$$

since each extension

$$0 \to F_{s-1}KU_n(X) \longrightarrow F_sKU_n(X) \longrightarrow H_s(X; KU_{n-s}) \to 0$$

satisfies $H_s(X; KU_{n-s}) \cong H_s(X)$ for $n - s$ even and $H_s(X; KU_{n-s}) = 0$ for $n - s$ odd. This applies, for instance, when $X = \mathbb{C}P^\infty$.

[Ada58] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214, DOI 10.1007/BF02564578. MR96219

[Ada60] ______, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104, DOI 10.2307/1970147. MR141119

[Ada66] ______, *A periodicity theorem in homological algebra*, Proc. Cambridge Philos. Soc. **62** (1966), 365–377, DOI 10.1017/s0305004100039955. MR194486

[Ada69] ______, *Lectures on generalised cohomology*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), Springer, Berlin, 1969, pp. 1–138. MR0251716

[Ada72] John Frank Adams, *Algebraic topology—a student's guide*, Cambridge University Press, London-New York, 1972. London Mathematical Society Lecture Note Series, No. 4. MR0445484

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill.-London, 1974. Chicago Lectures in Mathematics. MR0402720

[Ade52] José Adem, *The iteration of the Steenrod squares in algebraic topology*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 720–726, DOI 10.1073/pnas.38.8.720. MR50278

[Ade53] ———, *Relations on iterated reduced powers*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 636–638, DOI 10.1073/pnas.39.7.636. MR56293

[And64] D. W. Anderson, *The real K-theory of classifying spaces*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 634–636.

[ABP69] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson, *Pin cobordism and related topics*, Comment. Math. Helv. **44** (1969), 462–468, DOI 10.1007/BF02564545. MR261613

[AM69] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics, No. 100, Springer-Verlag, Berlin-New York, 1969. MR0245577

[Ati61a] M. F. Atiyah, *Bordism and cobordism*, Proc. Cambridge Philos. Soc. **57** (1961), 200–208, DOI 10.1017/s0305004100035064. MR126856

[Ati61b] ———, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 23–64. MR148722

[AH59] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281, DOI 10.1090/S0002-9904-1959-10344-X. MR110106

[AH61] ———, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38. MR0139181

[AS69] M. F. Atiyah and G. B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18. MR259946

[BJM84] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, J. London Math. Soc. (2) **30** (1984), no. 3, 533–550, DOI 10.1112/jlms/s2-30.3.533. MR810962

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171 (French). MR751966

[Boa82] J. M. Boardman, *The eightfold way to BP-operations or E_*E and all that*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 187–226. MR686116

[Boa99] J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84, DOI 10.1090/conm/239/03597. MR1718076

[BS58] Armand Borel and Jean-Pierre Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136 (French). MR116022

[Bot59] Raoul Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337, DOI 10.2307/1970106. MR110104

[BT82] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982. MR658304

[Bou63] D. G. Bourgin, *Modern algebraic topology*, The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1963. MR0160201

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573

[Bro69] William Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. (2) **90** (1969), 157–186, DOI 10.2307/1970686. MR251736

[Bro62] Edgar H. Brown Jr., *Cohomology theories*, Ann. of Math. (2) **75** (1962), 467–484, DOI 10.2307/1970209. MR138104

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR836132

[Bru] R. R. Bruner, *An Adams Spectral Sequence Primer*. http://www.rrb.wayne.edu/papers/adams.pdf.

[Car84] Gunnar Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. (2) **120** (1984), no. 2, 189–224, DOI 10.2307/2006940. MR763905

[Car48] Henri Cartan, *Sur la cohomologie des espaces où opère un groupe. Notions algébriques préliminaires*, C. R. Acad. Sci. Paris **226** (1948), 148–150 (French). MR23523

[Car50] ______, *Une theórie axiomatique des carrés de Steenrod*, C. R. Acad. Sci. Paris **230** (1950), 425–427 (French). MR35989

[Car54] ______, *Sur les groupes d'Eilenberg-MacLane. II*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 704–707, DOI 10.1073/pnas.40.8.704 (French). MR65161

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR0077480

[Cla81] Mónica Clapp, *Duality and transfer for parametrized spectra*, Arch. Math. (Basel) **37** (1981), no. 5, 462–472, DOI 10.1007/BF01234383. MR643290

[DK01] James F. Davis and Paul Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2001. MR1841974

[DP97] C. T. J. Dodson and Phillip E. Parker, *A user's guide to algebraic topology*, Mathematics and its Applications, vol. 387, Kluwer Academic Publishers Group, Dordrecht, 1997. MR1430097

[DP80] Albrecht Dold and Dieter Puppe, *Duality, trace, and transfer*, Proceedings of the International Conference on Geometric Topology (Warsaw, 1978), PWN, Warsaw, 1980, pp. 81–102. MR656721

[Dou58] Adrien Douady, *La suite spectrale d'Adams : structure multiplicative*, Séminaire Henri Cartan **11** (Unknown Month 1958), no. 2 (fr). talk:19.

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu, *Hopf algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 235, Marcel Dekker, Inc., New York, 2001. An introduction. MR1786197

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126, DOI 10.1016/B978-044481779-2/50003-1. MR1361887

[EM66] Samuel Eilenberg and John C. Moore, *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors*, Comment. Math. Helv. **40** (1966), 199–236, DOI 10.1007/BF02564371. MR203730

[ES52] Samuel Eilenberg and Norman Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, New Jersey, 1952. MR0050886

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR1417719

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR0210125

[Gys42] Werner Gysin, *Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten*, Comment. Math. Helv. **14** (1942), 61–122, DOI 10.1007/BF02565612 (German). MR6511

[Hal65] I. M. Hall, *The generalized Whitney sum*, Quart. J. Math. Oxford Ser. (2) **16** (1965), 360–384, DOI 10.1093/qmath/16.4.360. MR187245

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[Hat] ______, *Spectral Sequences*. https://pi.math.cornell.edu/ hatcher/AT/SSpage.html.

[HW60] P. J. Hilton and S. Wylie, *Homology theory: An introduction to algebraic topology*, Cambridge University Press, New York, 1960. MR0115161

[Hir47] Guy Hirsch, *Sur les groupes d'homologie des espaces fibrés*, Bull. Soc. Math. Belgique **1** (1947/48), 24–33 (1949) (French). MR31716

[Hir48] ______, *Un isomorphisme attaché aux structures fibrées*, C. R. Acad. Sci. Paris **227** (1948), 1328–1330 (French). MR29167

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041

[Hop41] Heinz Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. (2) **42** (1941), 22–52, DOI 10.2307/1968985 (German). MR4784

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262, DOI 10.4007/annals.2016.184.1.1. MR3505179

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134

[Hov04] ———, *Homotopy theory of comodules over a Hopf algebroid*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 261–304, DOI 10.1090/conm/346/06291. MR2066503

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114, DOI 10.1090/memo/0610. MR1388895

[Hur55] Witold Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961, DOI 10.1073/pnas.41.11.956. MR73987

[Kah69] Daniel S. Kahn, *Squaring operations in the Adams spectral sequence*, Bull. Amer. Math. Soc. **75** (1969), 136–138, DOI 10.1090/S0002-9904-1969-12177-4. MR236927

[KN02] Bernhard Keller and Amnon Neeman, *The connection between May's axioms for a triangulated tensor product and Happel's description of the derived category of the quiver* D_4, Doc. Math. **7** (2002), 535–560. MR2015053

[KM63] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537, DOI 10.2307/1970128. MR148075

[Kos47] Jean-Louis Koszul, *Sur les opérateurs de dérivation dans un anneau*, C. R. Acad. Sci. Paris **225** (1947), 217–219 (French). MR22345

[Kra71] D. Kraines, *On excess in the Milnor basis*, Bull. London Math. Soc. **3** (1971), 363–365, DOI 10.1112/blms/3.3.363. MR300271

[Kud50] Tatsuji Kudo, *Homological properties of fibre bundles*, J. Inst. Polytech. Osaka City Univ. Ser. A **1** (1950), 101–114. MR42117

[Kud52] ______, *Homological structure of fibre bundles*, J. Inst. Polytech. Osaka City Univ. Ser. A **2** (1952), 101–140. MR55687

[Kud56] ______, *A transgression theorem*, Mem. Fac. Sci. Kyūsyū Univ. A **9** (1956), 79–81, DOI 10.2206/kyushumfs.9.79. MR79259

[Ler46a] Jean Leray, *L'anneau d'homologie d'une représentation*, C. R. Acad. Sci. Paris **222** (1946), 1366–1368 (French). MR16664

[Ler46b] ______, *Structure de l'anneau d'homologie d'une représentation*, C. R. Acad. Sci. Paris **222** (1946), 1419–1422 (French). MR16665

[Ler46c] ______, *Propriétés de l'anneau d'homologie de la projection d'un espace fibré sur sa base*, C. R. Acad. Sci. Paris **223** (1946), 395–397 (French). MR17529

[Ler46d] ______, *Sur l'anneau d'homologie de l'espace homogène, quotient d'un groupe clos par un sousgroupe abélien, connexe, maximum*, C. R. Acad. Sci. Paris **223** (1946), 412–415 (French). MR17530

[Ler50] ______, *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. Pures Appl. (9) **29** (1950), 1–80, 81–139 (French). MR37505

[LMSM86] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure. MR866482

[Lim58] Elon Lages Lima, *DUALITY AND POSTNIKOV INVARIANTS*, ProQuest LLC, Ann Arbor, MI, 1958. Thesis (Ph.D.)–The University of Chicago. MR2611479

[Lyn48] Roger C. Lyndon, *The cohomology theory of group extensions*, Duke Math. J. **15** (1948), 271–292. MR25468

[ML63] Saunders Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. MR0156879

[ML71] ______, *Categories for the working mathematician*, Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5. MR0354798

[MM79] Ib Madsen and R. James Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, vol. 92, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979. MR548575

[MT67] Mark Mahowald and Martin Tangora, *Some differentials in the Adams spectral sequence*, Topology **6** (1967), 349–369, DOI 10.1016/0040-9383(67)90023-7. MR214072

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512, DOI 10.1112/S0024611501012692. MR1806878

[Mar83] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category. MR738973

[Mas52] W. S. Massey, *Exact couples in algebraic topology. I, II*, Ann. of Math. (2) **56** (1952), 363–396, DOI 10.2307/1969805. MR52770

[Mas53] ______, *Exact couples in algebraic topology. III, IV, V*, Ann. of Math. (2) **57** (1953), 248–286, DOI 10.2307/1969858. MR55686

[Mas54] ______, *Products in exact couples*, Ann. of Math. (2) **59** (1954), 558–569, DOI 10.2307/1969719. MR60829

[May67] J. Peter May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies, No. 11, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. MR0222892

[May70] ______, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, Springer, Berlin, 1970, pp. 153–231. MR0281196

[May77] ______, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics, Vol. 577, Springer-Verlag, Berlin-New York, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. MR0494077

[May80] J. P. May, *Pairings of categories and spectra*, J. Pure Appl. Algebra **19** (1980), 299–346, DOI 10.1016/0022-4049(80)90105-X. MR593258

[May01] ______, *The additivity of traces in triangulated categories*, Adv. Math. **163** (2001), no. 1, 34–73, DOI 10.1006/aima.2001.1995. MR1867203

[MP12] J. P. May and K. Ponto, *More concise algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. MR2884233

[McC85] John McCleary, *User's guide to spectral sequences*, Mathematics Lecture Series, vol. 12, Publish or Perish, Inc., Wilmington, DE, 1985. MR820463

[McC99] ______, *A history of spectral sequences: origins to 1953*, History of topology, North-Holland, Amsterdam, 1999, pp. 631–663, DOI 10.1016/B978-044482375-5/50024-9. MR1721118

[McC69] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273–298, DOI 10.2307/1995173. MR251719

[Mil67] R. James Milgram, *The bar construction and abelian H-spaces*, Illinois J. Math. **11** (1967), 242–250. MR208595

[Mil81] Haynes R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra **20** (1981), no. 3, 287–312, DOI 10.1016/0022-4049(81)90064-5. MR604321

[Mil00] Haynes Miller, *Leray in Oflag XVIIA: the origins of sheaf theory, sheaf cohomology, and spectral sequences*, Gaz. Math. **84, suppl.** (2000), 17–34. Jean Leray (1906–1998). MR1775587

[Mil56] John Milnor, *Construction of universal bundles. II*, Ann. of Math. (2) **63** (1956), 430–436, DOI 10.2307/1970012. MR77932

[Mil58] ______, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171, DOI 10.2307/1969932. MR99653

[Mil59] ______, *On spaces having the homotopy type of a* CW*-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280, DOI 10.2307/1993204. MR100267

[Mil62] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341. MR159327

[Mil64] ______, *Microbundles. I*, Topology **3** (1964), no. suppl, suppl. 1, 53–80, DOI 10.1016/0040-9383(64)90005-9. MR161346

[MM65] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264, DOI 10.2307/1970615. MR174052

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76. MR0440554

[MT68] Robert E. Mosher and Martin C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, Publishers, New York-London, 1968. MR0226634

[Mos68] R. M. F. Moss, *On the composition pairing of Adams spectral sequences*, Proc. London Math. Soc. (3) **18** (1968), 179–192, DOI 10.1112/plms/s3-18.1.179. MR220294

[Mos70] R. Michael F. Moss, *Secondary compositions and the Adams spectral sequence*, Math. Z. **115** (1970), 283–310, DOI 10.1007/BF01129978. MR266216

[Nee91] Amnon Neeman, *Some new axioms for triangulated categories*, J. Algebra **139** (1991), no. 1, 221–255, DOI 10.1016/0021-8693(91)90292-G. MR1106349

[Nee01] ______, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR1812507

[Nei80] Joseph Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. **25** (1980), no. 232, iv+67, DOI 10.1090/memo/0232. MR567801

[Nov67] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951 (Russian). MR0221509

[Pon50] L. S. Pontryagin, *Homotopy classification of the mappings of an* $(n+2)$*-dimensional sphere on an n-dimensional one*, Doklady Akad. Nauk SSSR (N.S.) **70** (1950), 957–959 (Russian). MR0042121

[Pup67] Dieter Puppe, *Stabile Homotopietheorie. I*, Math. Ann. **169** (1967), 243–274, DOI 10.1007/BF01362348 (German). MR211400

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967. MR0223432

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR860042

[Roh51] V. A. Rohlin, *A three-dimensional manifold is the boundary of a four-dimensional one*, Doklady Akad. Nauk SSSR (N.S.) **81** (1951), 355–357 (Russian). MR0048808

[Rud98] Yuli B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller. MR1627486

[Sat99] Hajime Sato, *Algebraic topology: an intuitive approach*, Translations of Mathematical Monographs, vol. 183, American Mathematical Society, Providence, RI, 1999. Translated from the 1996 Japanese original by Kiki Hudson; Iwanami Series in Modern Mathematics. MR1679607

[SS00] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511, DOI 10.1112/S002461150001220X. MR1734325

[SS03] Stefan Schwede and Brooke Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103–153, DOI 10.1016/S0040-9383(02)00006-X. MR1928647

[Ser51] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505, DOI 10.2307/1969485 (French). MR45386

[Ser53] ______, *Cohomologie modulo* 2 *des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198–232, DOI 10.1007/BF02564562 (French). MR60234

[Sha14] Anant R. Shastri, *Basic algebraic topology*, CRC Press, Boca Raton, FL, 2014. With a foreword by Peter Wong. MR3134904

[Spa66] Edwin H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966. MR0210112

[Ste43] N. E. Steenrod, *Homology with local coefficients*, Ann. of Math. (2) **44** (1943), 610–627, DOI 10.2307/1969099. MR9114

[Ste47] ______, *Products of cocycles and extensions of mappings*, Ann. of Math. (2) **48** (1947), 290–320, DOI 10.2307/1969172. MR22071

[Ste51] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, Princeton, N. J., 1951. MR0039258

[Ste52] N. E. Steenrod, *Reduced powers of cohomology classes*, Ann. of Math. (2) **56** (1952), 47–67, DOI 10.2307/1969766. MR48026

[Ste53] ______, *Cyclic reduced powers of cohomology classes*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 217–223, DOI 10.1073/pnas.39.3.217. MR54965

[Ste62] ______, *Cohomology operations*, Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50, Princeton University Press, Princeton, N.J., 1962. MR0145525

[Ste67] ______, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR210075

[Ste68] ______, *Milgram's classifying space of a topological group*, Topology **7** (1968), 349–368, DOI 10.1016/0040-9383(68)90012-8. MR233353

[Sto68] Robert E. Stong, *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968. MR0248858

[Str] N. P. Strickland, *The category of CGWH spaces*. http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf.

[Swi75] Robert M. Switzer, *Algebraic topology—homotopy and homology*, Springer-Verlag, New York-Heidelberg, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212. MR0385836

[Tam99] Hirotaka Tamanoi, *$\mathcal{Q}$-subalgebras, Milnor basis, and cohomology of Eilenberg-MacLane spaces*, J. Pure Appl. Algebra **137** (1999), no. 2, 153–198, DOI 10.1016/S0022-4049(97)00177-1. MR1684268

[Tan70] Martin C. Tangora, *On the cohomology of the Steenrod algebra*, Math. Z. **116** (1970), 18–64, DOI 10.1007/BF01110185. MR266205

[Tho54] René Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86, DOI 10.1007/BF02566923 (French). MR61823

[Tod55] Hirosi Toda, *Le produit de Whitehead et l'invariant de Hopf*, C. R. Acad. Sci. Paris **241** (1955), 849–850 (French). MR72474

[Tod62] ______, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962. MR0143217

[Ver96] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1996), xii+253 pp. (1997) (French, with French summary). With a preface by Luc Illusie; Edited and with a note by Georges Maltsiniotis. MR1453167

[Vog70] Rainer Vogt, *Boardman's stable homotopy category*, Lecture Notes Series, No. 21, Matematisk Institut, Aarhus Universitet, Aarhus, 1970. MR0275431

[Wal60] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311, DOI 10.2307/1970136. MR120654

[Wan49] Hsien-Chung Wang, *The homology groups of the fibre bundles over a sphere*, Duke Math. J. **16** (1949), 33–38. MR28580

[Whi62] George W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283, DOI 10.2307/1993676. MR137117

[Whi78] ———, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978. MR516508

[Yon58] Nobuo Yoneda, *Note on products in* Ext, Proc. Amer. Math. Soc. **9** (1958), 873–875, DOI 10.2307/2033320. MR175957