MAT9580: Spectral Sequences

Chapters 4 and 5: The Serre Spectral Sequence and Multiplicative Spectral Sequences

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Outline

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The Leray spectral sequence

- Leray [Ler46a], [Ler46b] was led to spectral sequences by studying the relation between $H^*(B)$ and $H^*(E)$, where $p\colon E \to B$ is a given map.
- To outline the main features we use the modern language of sheaf theory, as it was reworked by Cartan in his 1951 seminar.

(Pre-)sheaves

- For each open $U \subset B$ let $E_U = p^{-1}(U)$ be the part of E above B.
- In each degree t the association

$$U \longmapsto \mathscr{F}^t(U) = H^t(E_U)$$

is a contravariant functor from the category of open subsets of B, partially ordered by inclusions, to the category of abelian groups, i.e., an abelian presheaf on B.
- It is not a sheaf, because

$$H^t(E_{U \cup V}) \longrightarrow H^t(E_U) \oplus H^t(E_V)$$

is not generally injective, but it can be sheafified.

Sheaf cohomology

- For each point $b \in B$ let $F_b = p^{-1}(b)$ be the fiber at b.
- The stalk of this presheaf (and the associated sheaf $\widetilde{\mathscr{F}^t}$) at this point is the colimit

$$\operatorname*{colim}_{b \in U} \mathscr{F}^t(U) = \operatorname*{colim}_{b \in U} H^t(E_U)\,,$$

  which canonically maps to $H^t(F_b)$, and for “nice” $p\colon E \to B$ this map is an isomorphism.
- There results a cohomologically indexed Leray spectral sequence

$$E_2^{s,t} = H^s(B; \widetilde{\mathscr{F}^t}) \Longrightarrow_s H^{s+t}(E)\,,$$

  where the E_2-term is given in terms of sheaf cohomology.

Three in one

- To stay within the realm of topological spaces and their (co-)homology, one would like to replace sheaf cohomology with ordinary cohomology of the base space B,
- and to replace the coefficient sheaf with the ordinary cohomology of the fiber F_b.
- Some hypothesis on the map $p\colon E \to B$ will be needed in order to control how the fiber varies with b.

Fiber bundles

When $p\colon E \to B$ is a fiber bundle with fiber F, so that B is covered by open subsets U for which there are homeomorphisms h_U making the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow[\cong]{h_U} & E_U \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p} & \\ & U & \end{array}$$

commute, this problem was considered by Guy Hirsch [Hir47], [Hir48] and by Tatsuji Kudo [Kud50], [Kud52].

Kudo's filtration

Kudo adapted Leray's algebraic framework to the case where the base space B is a simplicial complex with skeleton filtration

$$\emptyset = B^{(-1)} \subset B^{(0)} \subset \cdots \subset B^{(s-1)} \subset B^{(s)} \subset \cdots \subset B\,.$$

He filtered the total space E by the preimages

$$E_s = p^{-1}(B^{(s)})\,,$$

so that

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E\,.$$

Kudo thus obtained a convergent homological spectral sequence

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) \Longrightarrow H_{s+t}(E)\,.$$

The Kudo E^1-term, I

For each s-simplex $\sigma \subset B$ there is a homeomorphism of pairs

$$h_\sigma \colon (\sigma \times F_b, \partial\sigma \times F_b) \xrightarrow{\cong} (p^{-1}(\sigma), p^{-1}(\partial\sigma))$$

where $F_b = p^{-1}(b)$ for a chosen point $b \in \partial\sigma$. By excision and the Künneth theorem, this induces isomorphisms

$$\begin{aligned} E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) &\cong \bigoplus_\sigma H_{s+t}(p^{-1}(\sigma), p^{-1}(\partial\sigma)) \\ &\cong \bigoplus_\sigma H_{s+t}(\sigma \times F_b, \partial\sigma \times F_b) \\ &\cong \bigoplus_\sigma H_s(\sigma, \partial\sigma) \otimes H_t(F_b)\,, \end{aligned}$$

where σ ranges over the s-simplices in B.

The Kudo E^1-term, II

When B is 1-connected there are preferred isomorphisms $H_t(F_b) \cong H_t(F)$, so that the E^1-term can be identified with

$$\bigoplus_\sigma H_s(\sigma, \partial\sigma) \otimes H_t(F) \cong \bigoplus_\sigma H_t(F) = \Delta_s(B; H_t(F))\,,$$

i.e., the simplicial s-chains of B with coefficients in $H_t(F)$. Moreover, Kudo verified that the d^1-differential

$$d^1_{s,t} \colon H_{s+t}(E_s, E_{s-1}) \longrightarrow H_{s+t-1}(E_{s-1}, E_{s-2})$$

corresponds to the simplicial boundary homomorphism

$$\partial_s \colon \Delta_s(B; H_t(F)) \longrightarrow \Delta_{s-1}(B; H_t(F))$$

under these identifications.

The Kudo E^2-term

- Hence the spectral sequence E^2-term satisfies

$$E^2_{s,t} \cong H^{\Delta}_s(B; H_t(F)) \,.$$

- Since simplicial and singular homology agree for simplicial complexes, this establishes a spectral sequence of the form

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E) \,,$$

  converging to the homology of the total space.
- Kudo also discusses the case non-simply connected bases B, which leads to an E^2-term expressed in terms of Steenrod's homology with local coefficients [Ste43], [Hat02, §3.H].

Fibrations

- There are many geometrically interesting examples of such fiber bundles, arising from the theory of Lie groups and their homogeneous spaces.
- To analyze the (co-)homology of Eilenberg–Mac Lane spaces, Jean–Pierre Serre [Ser51] was led to consider the more general situation of the path–loop fibration $p\colon PX \to X$, with fiber ΩX, which is not a fiber bundle.
- This map has the homotopy lifting property with respect to arbitrary source spaces, hence is a fibration $p\colon E \to B$ in the sense of Witold Hurewicz [Hur55].

Serre's filtration

- Serre recognized that this lifting property allowed him to construct a spectral sequence of the same form

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)$$

  as before, by filtering a version of the singular chain complex $(C_*(E), \partial)$.
- This filtration is different from the one used by Kudo, and does not assume that B has a skeletal filtration.

Singular cubes

- To make this work, Serre uses singular cubes in place of singular simplices.
- A singular n-cube σ has filtration s if $p\sigma$ factors as follows, where $p_s(t_1, \ldots, t_n) = (t_1, \ldots, t_s)$.

$$\begin{array}{ccc} I^n & \xrightarrow{\sigma} & E \\ {\scriptstyle p_s}\downarrow & & \downarrow{\scriptstyle p} \\ I^s & \longrightarrow & B \end{array}$$

Serre fibrations

- For this argument the map $p\colon E \to B$ would only need to satisfy the homotopy lifting condition with respect to maps from compact polyhedra (or finite CW complexes).
- This larger class of maps is now known as Serre fibrations.
- Serre showed that the cup product in cohomology is compatible with the differentials in the cohomological version of his spectral sequence, leading to an algebra spectral sequence in the sense of the “anneau spectral” of Leray.
- This is a key feature needed to make precise calculations with spectral sequences.

Homotopy lifting properties

Definition

- A map $p\colon E \to B$ is a Hurewicz fibration if it has the homotopy lifting property with respect to each space T.
- In other words, for each commutative square of solid arrows

$$\begin{array}{ccc} T & \longrightarrow & E \\ {\scriptstyle i_0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ I \times T & \longrightarrow & B \end{array}$$

  there exists a dotted arrow making both trianges commute.
- The map $p\colon E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to the n-disk D^n for each $n \geq 0$.

Three kinds of fiberings

- Each fiber bundle over a paracompact (e.g., metric) base space is a Hurewicz fibration [Hur55, §4].
- Each Hurewicz fibration is a Serre fibration.
- Pullback preserves fiber bundles, Hurewicz fibrations and Serre fibrations.

Comparing fibers

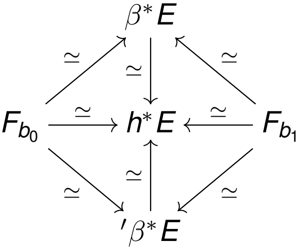
- For a Hurewicz fibration $p\colon E \to B$ with contractible base space, the inclusion $F_b \subset E$ of any fiber $F_b = p^{-1}(b)$ is a homotopy equivalence.
- Let $p\colon E \to B$ be a Hurewicz fibration over a general base.
- For any path $\beta\colon I \to B$ from $\beta(0) = b_0$ to $\beta(1) = b_1$, the pullback $\beta^* p\colon \beta^* E \to I$ is a Hurewicz fibration over a contractible base, so the inclusions

  $$F_{b_0} \xrightarrow{\simeq} \beta^* E \xleftarrow{\simeq} F_{b_1}$$

  are homotopy equivalences.
- This defines a homotopy equivalence $e\colon F_{b_0} \simeq F_{b_1}$, up to homotopy.

Dependence on path

- Let $'\beta$ be another path from b_0 to b_1.
- A path homotopy $h\colon I \times I \to B$ from β to $'\beta$ leads to another Hurewicz fibration over a contractible base, so that all of the inclusions



  are homotopy equivalences.
- It follows that the composite equivalence $e\colon F_{b_0} \simeq \beta^* E \simeq F_{b_1}$ is homotopic to the composite equivalence $'e\colon F_{b_0} \simeq {}'\beta^* E \simeq F_{b_1}$.

Local coefficient systems

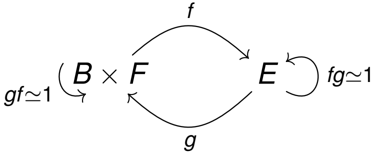
- Passing to homology, a choice of β gives well-defined isomorphisms $e_* \colon H_t(F_{b_0}) \cong H_t(\beta^* E) \cong H_t(F_{b_1})$ for all t, and homotopic paths β and $'\beta$ give the same composite isomorphism $e_* \colon H_t(F_{b_0}) \cong H_t(F_{b_1})$.
- If B is 1-connected, with base point b_0, this gives canonical isomorphisms $H_t(F) \cong H_t(F_b)$ for each $b \in B$.
- In general, it gives a local coefficient system $\mathscr{H}_t(F)$ on B, i.e., a functor from the fundamental groupoid of B to abelian groups.
- If B is 0-connected, with fundamental group $\pi = \pi_1(B, b_0)$, then this structure can equivalently be encoded as an action of π on $H_t(F)$ for $F = F_{b_0}$.

Fiber homotopy equivalence

- The equivalent inclusions of fibers can be made compatible, for varying $b \in B$.
- This follows from the discussion of fiber homotopy equivalence in [Spa66, ?] or [Whi78, Cor. I.7.27].
- We write $(B, A) \times F$ for $(B \times F, A \times F)$.

Proposition

Any Hurewicz fibration $p\colon E \to B$ *over a contractible base space* B *is fiber homotopy trivial, meaning that there are maps* f, g *and homotopies* $gf \simeq 1$, $fg \simeq 1$,



all four of which commute with the projections $p_1\colon B \times F \to B$ *and* $p\colon E \to B$.
In particular, for any $A \subset B$ *with* $E_A = p^{-1}(A)$ *there is a homotopy equivalence of pairs*

$$f\colon (B, A) \times F \xrightarrow{\simeq} (E, E_A)\,.$$

Outline

Filtered fibration

Consider a Hurewicz fibration $p\colon E \to B$, with B a CW complex. Let

$$E_s = p^{-1}(B^{(s)})$$

be the preimage of the s-skeleton of the base B. The total space E is then strongly filtered by the sequence

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E\,,$$

since for any compact $K \subset E$ there is an s with $p(K) \subset B^{(s)}$, and then $K \subset E_s$.

$$\begin{array}{ccccccccccccc}
\emptyset & \rightarrowtail & \dots & \rightarrowtail & E_{s-1} & \rightarrowtail & E_s & \rightarrowtail & \dots & \rightarrowtail & E \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow p \\
\emptyset & \rightarrowtail & \dots & \rightarrowtail & B^{(s-1)} & \rightarrowtail & B^{(s)} & \rightarrowtail & \dots & \rightarrowtail & B
\end{array}$$

The Serre spectral sequence

Definition

The Serre spectral sequence of $p\colon E \to B$ is the homology spectral sequence

$$E^1_{s,t}(p) = H_{s+t}(E_s, E_{s-1}) \Longrightarrow_s H_{s+t}(E)$$

associated to the filtration $(E_s)_s$.

By an earlier proposition, the d^1-differential equals the connecting homomorphism

$$d^1_{s,t} = (\partial_s)_{s+t}\colon H_{s+t}(E_s, E_{s-1}) \longrightarrow H_{s+t-1}(E_{s-1}, E_{s-2})$$

of the triple (E_s, E_{s-1}, E_{s-2}).

Placement, convergence

The E^1-term is concentrated in the right half-plane ($s \geq 0$), and the spectral sequence converges to $H_*(E)$ with the filtration

$$F_s H_*(E) = \mathrm{im}(H_*(E_s) \to H_*(E))\,.$$

We shall see that this is a first quadrant spectral sequence, so that $(F_s H_*(X))_s$ is degreewise bounded.

Two approaches

- This construction is closer to that of Kudo [Kud50] than that of Serre [Ser51], but the E^2-terms will be isomorphic.
- Serre also established multiplicative properties for the cohomology version of his spectral sequence, which led him to stronger conclusions than those that follow from the additive structure.
- The name “Serre spectral sequence” thus reflects the extra versatility and power achieved by Serre’s approach.

0-connected base, fiber at base point

- Since B is the disjoint union of its path components, there is a corresponding sum decomposition of E, and we can assume that B is 0-connected.
- We can assume that B is 0-reduced, in the sense that it only has a single 0-cell, given by the base point $b_0 \in B$.
- The 0-th filtration $E_0 = p^{-1}(B^{(0)})$ is then equal to the fiber

  $$F = p^{-1}(b_0)$$

  of $p\colon E \to B$ at the base point. We write

  $$F \longrightarrow E \xrightarrow{p} B$$

  to refer to this context.
- Beware the double usage of E for total space and spectral sequence terms, and the double usage of F for fiber and various filtrations.

The E^1-term as cellular chains

Proposition ([Whi78, Thm. XIII.4.6])

There are natural isomorphisms

$$H_{s+t}(E_s, E_{s-1}) \cong C_s^{CW}(B; \mathscr{H}_t(F)),$$

where $\mathscr{H}_t(F)$ denotes the local coefficient system on B given by $H_t(F_b)$ at $b \in B$.

If B is 1-connected, then this equals the usual cellular s-chains

$$C_s^{CW}(B; H_t(F))$$

with coefficients in the abelian group $H_t(F)$.

Proof

- Let α run over the s-cells of B, so that we have a pushout square

$$\begin{array}{ccc} \coprod_\alpha \partial D^s_\alpha & \longrightarrow & \coprod_\alpha D^s_\alpha \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ B^{(s-1)} & \longrightarrow & B^{(s)}\,, \end{array}$$

  with attaching maps $\phi = \coprod_\alpha \phi_\alpha$ and characteristic maps $\Phi = \coprod_\alpha \Phi_\alpha$.
- Form the pullbacks of $p\colon E \to B$ along the evident maps to B, to obtain another pushout square

$$\begin{array}{ccc} \coprod_\alpha \phi^*_\alpha E & \longrightarrow & \coprod_\alpha \Phi^*_\alpha E \\ \downarrow & & \downarrow \\ E_{s-1} & \longrightarrow & E_s\,. \end{array}$$

Proof (cont.)

- The proposition on fiber homotopy equivalence applies to the pullback
$$p\colon \Phi_\alpha^* E \longrightarrow D_\alpha^s$$
- Let $d_0 \in \partial D_\alpha^s \subset D_\alpha^s$ be a base point, let $b_\alpha = \phi_\alpha(d_0)$, and let $F_{b_\alpha} = p^{-1}(b_\alpha)$ be the fiber above this point.
- There is then a fiber homotopy equivalence
$$D_\alpha^s \times F_{b_\alpha} \xrightarrow{\simeq} \Phi_\alpha^* E$$
over D_α^s, which restricts to the identity over d_0.
- In particular, there is a homotopy equivalence of pairs
$$(D_\alpha^s, \partial D_\alpha^s) \times F_{b_\alpha} \xrightarrow{\simeq} (\Phi_\alpha^* E, \phi_\alpha^* E)\,.$$

Proof (cont.)

By additivity and excision we obtain isomorphisms

$$\bigoplus_{\alpha} H_{s+t}(\Phi_\alpha^* E, \phi_\alpha^* E) \cong H_{s+t}(\coprod_\alpha \Phi_\alpha^* E, \coprod_\alpha \phi_\alpha^* E) \cong H_{s+t}(E_s, E_{s-1})\,.$$

The homotopy equivalence of pairs induces an isomorphism

$$H_{s+t}((D_\alpha^s, \partial D_\alpha^s) \times F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}(\Phi_\alpha^* E, \phi_\alpha^* E)\,.$$

By the Künneth theorem, the homology cross product induces an isomorphism

$$H_s(D_\alpha^s, \partial D_\alpha^s) \otimes H_t(F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}((D_\alpha^s, \partial D_\alpha^s) \times F_{b_\alpha})\,.$$

Proof (cont.)

- Taken together, we have isomorphisms

$$\bigoplus_\alpha H_s(D^s_\alpha, \partial D^s_\alpha) \otimes H_t(F_{b_\alpha}) \cong H_{s+t}(E_s, E_{s-1})$$

  for all s and t.
- By definition, the left hand side is $C_s^{CW}(B; \mathscr{H}_t(F))$.
- If B is 1-connected, then the canonical isomorphisms $H_t(F_{b_\alpha}) \cong H_t(F)$ identify the direct sum above with

$$C_s^{CW}(B; H_t(F)) \cong \bigoplus_\alpha H_s(D^s_\alpha, \partial D^s_\alpha) \otimes H_t(F) \,.$$

□

The d^1-differential as cellular boundary

Proposition ([Whi78, Thm. XIII.4.8])

The square

$$\begin{array}{ccc} H_{s+t}(E_s, E_{s-1}) & \xrightarrow{d^1_{s,t}} & H_{s+t-1}(E_{s-1}, E_{s-2}) \\ \cong \downarrow & & \downarrow \cong \\ C_s^{CW}(B; \mathscr{H}_t(F)) & \xrightarrow{\partial_s} & C_{s-1}^{CW}(B; \mathscr{H}_t(F)) \end{array}$$

commutes.

Sketch proof

- Let α and β index the s- and $(s-1)$-cells of B, with characteristic maps Φ_α and Ψ_β, respectively.
- We have fiber homotopy equivalences $\Phi_\alpha^* E \simeq D_\alpha^s \times F_{b_\alpha}$ over D_α^s and $\Psi_\beta^* E \simeq D_\beta^{s-1} \times F_{b_\beta}$ over D_β^{s-1}.
- In the cellular complex for B, the boundary ∂_s has components

$$
\begin{aligned}
H_s(D_\alpha^s, \partial D_\alpha^s) &\xrightarrow{\Phi i_\alpha} H_s(B^{(s)}, B^{(s-1)}) \\
&\xrightarrow{\partial} \tilde{H}_{s-1}(B^{(s-1)}) \xrightarrow{q} \tilde{H}_{s-1}(B^{(s-1)}/B^{(s-2)}) \\
&\xrightarrow{p_\beta \Psi^{-1}} \tilde{H}_{s-1}(D_\beta^{s-1}/\partial D_\beta^{s-1}),
\end{aligned}
$$

where we use the isomorphisms

$$
\Phi\colon \bigoplus_\alpha H_s(D_\alpha^s, \partial D_\alpha^s) \xrightarrow{\cong} H_s(B^{(s)}, B^{(s-1)})
$$

$$
\Psi\colon \bigoplus_\beta \tilde{H}_{s-1}(D_\beta^{s-1}/\partial D_\beta^{s-1}) \xrightarrow{\cong} \tilde{H}_{s-1}(B^{(s-1)}/B^{(s-2)}).
$$

Sketch proof (cont.)

This component can also be factored as

$$
\begin{aligned}
H_s(D^s_\alpha, \partial D^s_\alpha) &\overset{\partial}{\cong} \tilde{H}_{s-1}(\partial D^s_\alpha) \\
&\xrightarrow{\phi_\alpha} \tilde{H}_{s-1}(B^{(s-1)}) \xrightarrow{q} \tilde{H}_{s-1}(B^{(s-1)}/B^{\wedge}_\beta) \\
&\overset{\Psi_\beta^{-1}}{\cong} \tilde{H}_{s-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta)\,,
\end{aligned}
$$

where $B^{\wedge}_\beta \subset B^{(s-1)}$ is the complement of $\Psi_\beta(\operatorname{int} D^{s-1}_\beta)$.

$$
\begin{array}{ccc}
\partial D^{s-1}_\beta & \rightarrowtail & D^{s-1}_\beta \\
{\scriptstyle \psi_\beta}\downarrow & & \downarrow{\scriptstyle \Psi_\beta} \\
B^{\wedge}_\beta & \rightarrowtail & B^{(s-1)}
\end{array}
$$

Sketch proof (cont.)

We must identify the corresponding composite

$$\begin{aligned} H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) &\xrightarrow{\Phi i_\alpha} H_{s+t}(E_s, E_{s-1}) \\ &\xrightarrow{\partial} \tilde{H}_{s+t-1}(E_{s-1}) \xrightarrow{q} \tilde{H}_{s+t-1}(E_{s-1}/E_{s-2}) \\ &\xrightarrow{p_\beta \Psi^{-1}} \tilde{H}_{s-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta \wedge F_{b_\beta +}) \,, \end{aligned}$$

where now

$$\Phi \colon \bigoplus_\alpha H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}(E_s, E_{s-1})$$

$$\Psi \colon \bigoplus_\beta \tilde{H}_{s+t-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta \wedge F_{b_\beta +}) \xrightarrow{\cong} \tilde{H}_{s+t-1}(E_{s-1}/E_{s-2}) \,.$$

Sketch proof (cont.)

This can also be factored as

$$\begin{aligned}
H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) &\xrightarrow{\partial} \tilde{H}_{s+t-1}(\partial D^s_\alpha \times F_{b_\alpha}) \\
&\xrightarrow{\phi_\alpha} \tilde{H}_{s+t-1}(E_{s-1}) \xrightarrow{q} \tilde{H}_{s+t-1}(E_{s-1}/E^\wedge_\beta) \\
&\overset{\Psi_\beta^{-1}}{\cong} \tilde{H}_{s+t-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta \wedge F_{b_\beta +}),
\end{aligned}$$

where $E^\wedge_\beta = p^{-1}(B^\wedge_\beta) \subset E_{s-1}$.

$$\begin{array}{ccc}
\partial D^{s-1}_\beta \times F_{b_\beta} & \rightarrowtail & D^{s-1}_\beta \times F_{b_\beta} \\
\downarrow \psi_\beta & & \downarrow \Psi_\beta \\
E^\wedge_\beta & \rightarrowtail & E_{s-1}
\end{array}$$

Sketch proof (cont.)

It thus suffices to verify that the following diagram commutes

$$\begin{array}{ccc} \tilde{H}_{s-1}(\partial D^s_\alpha) \otimes H_t(F_{b_\alpha}) & \xrightarrow{\Psi_\beta^{-1} q\phi_\alpha \otimes e} & \tilde{H}_{s-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta) \otimes H_t(F_{b_\beta}) \\ \times \big\downarrow & & \times \big\downarrow \cong \\ \tilde{H}_{s+t-1}(\partial D^s_\alpha \times F_{b_\alpha}) & \xrightarrow{\Psi_\beta^{-1} q\phi_\alpha} & \tilde{H}_{s+t-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta \wedge F_{b_\beta +}) \,. \end{array}$$

For this, which takes some effort, we refer to
[Whi78, §XIII.5]. □

The Serre spectral sequence E^2-term

Theorem

The Serre spectral sequence

$$E^r_{s,t}(p) \Longrightarrow_s H_{s+t}(E)$$

for $F \to E \xrightarrow{p} B$ *has* E^2*-term*

$$E^2_{s,t}(p) \cong H_s(B; \mathscr{H}_t(F)) .$$

If B *is* 1*-connected, this simplifies to*

$$E^2_{s,t}(p) \cong H_s(B; H_t(F)) .$$

Proof.

This follows from $(E^1_{*,*}, d^1) \cong (C^{CW}_*(B; \mathscr{H}_*(F)), \partial)$ by passage to homology. □

Trinity

- In the context of the Serre spectral sequence, the filtration degree s and complementary degree t are also referred to as the base degree and fiber degree, respectively.
- This makes sense, since the E^2-term in bidegree (s, t) is given in terms of $H_s(B)$, suitably interpreted, and $H_t(F)$.
- The total degree $s + t$ then refers both to the total algebraic degree, and to the grading of the homology $H_*(E)$ of the total space.

In view of the universal coefficient exact sequence

$$0 \to H_s(B) \otimes H_t(F) \longrightarrow H_s(B; H_t(F)) \longrightarrow \mathrm{Tor}(H_{s-1}(B), H_t(F)) \to 0$$

this achieves the aim of obtaining a spectral sequence that connects the ordinary homology groups $H_*(F)$, $H_*(E)$ and $H_*(B)$ in one whole.

Relative cases

There are relative versions of the Serre spectral sequence.

- If $A \subset B$ is a subcomplex, then we can filter E by $p^{-1}(A \cup B^{(s)})$ and obtain a spectral sequence

$$E^2_{s,t} = H_s(B, A; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E, E_A)$$

where $E_A = p^{-1}(A)$.

- If $p' \colon E' \subset E \to B$ is a subfibration with fibers F', then we can filter (E, E') by $(p^{-1}(B^{(s)}), (p')^{-1}(B^{(s)}))$ and obtain a spectral sequence

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F, F')) \Longrightarrow_s H_{s+t}(E, E') \,.$$

We can relax the condition that B be a CW complex, by comparison with a CW approximation.

Outline

- The Serre Spectral Sequence
  - Maps, fiber bundles and fibrations
  - Homology of fiber sequences
  - The Wang and Gysin sequences
  - Edge homomorphisms and the transgression
  - Theorems of Hurewicz and Freudenthal
  - Finite generation and finiteness
- Multiplicative Spectral Sequences
  - Cohomological grading
  - Cohomology of spaces
  - Cohomological Serre spectral sequence
  - Pairings of spectral sequences

The Wang sequence

Hsien-Chung Wang [Wan49] studied fiber bundles with base space a sphere, obtaining a long exact sequence as in the following theorem, which follows in greater generality from the Serre spectral sequence.

Theorem

Let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

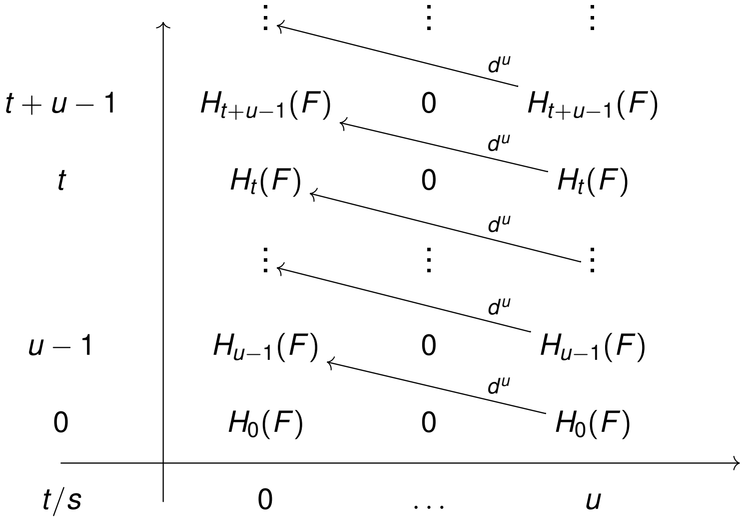
be a fiber sequence, with $B \simeq S^u$ a 1-connected CW complex. There is a long exact sequence

$$\ldots \to H_{n-u+1}(F) \xrightarrow{\partial} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{i_!} H_{n-u}(F) \xrightarrow{\partial} H_{n-1}(F) \to \ldots$$

Proof

- Clearly $u \geq 2$. By the universal coefficient theorem,
$$H_s(B; H_t(F)) \cong \begin{cases} H_t(F) & \text{for } s \in \{0, u\}, \\ 0 & \text{otherwise.} \end{cases}$$
- This shows that the E^2-term of the Serre spectral sequence for $F \to E \to B$ is concentrated in the two columns $s = 0$ and $s = u$.
- For degree reasons $d^r = 0$ except for $r = u$, so $E^2 = E^u$.
- At this stage we have differentials
$$d^u_{u,t} \colon H_t(F) \cong E^u_{u,t} \longrightarrow E^u_{0,t+u-1} \cong H_{t+u-1}(F)$$
leading to an E^{u+1}-term with
$$E^{u+1}_{s,t} = \begin{cases} \operatorname{cok}(d^u_{u,t-u+1}) & \text{for } s = 0, \\ \ker(d^u_{u,t}) & \text{for } s = u, \\ 0 & \text{otherwise.} \end{cases}$$

(E^u, d^u)-term



$t+u-1$
t
$u-1$
0
t/s
0
$\dots$
u
$H_{t+u-1}(F)$
0
$H_{t+u-1}(F)$
$H_t(F)$
0
$H_t(F)$
$H_{u-1}(F)$
0
$H_{u-1}(F)$
$H_0(F)$
0
$H_0(F)$
d^u
d^u
d^u
d^u
d^u

Proof (cont.)

- Since $d^r = 0$ for all $r > u$, the spectral sequence collapses at this term, so that $E^{u+1}_{s,t} = E^{\infty}_{s,t}$ in all bidegrees.
- By the convergence of the spectral sequence, we have isomorphisms
$$F_s H_n(E) \cong E^{\infty}_{0,n}$$
for $0 \leq s < u$, a short exact sequence
$$0 \to F_{u-1}H_n(E) \longrightarrow F_u H_n(E) \longrightarrow E^{\infty}_{u,n-u} \to 0\,,$$
and identities
$$F_s H_n(E) = H_n(E)$$
for $s \geq u$.

Proof (cont.)

In other words, we have a short exact sequence

$$0 \to \mathrm{cok}(d^u_{u,n-u+1}) \xrightarrow{i_*} H_n(E) \xrightarrow{i_!} \ker(d^u_{u,n-u}) \to 0\,.$$

Writing out the definition of the cokernel and kernel gives the exact sequence

$$H_{n-u+1}(F) \xrightarrow{d^u} H_n(F) \longrightarrow H_n(E) \longrightarrow H_{n-u}(F) \xrightarrow{d^u} H_{n-1}(F)\,,$$

as claimed. □

Fibration over a suspension

Remark

For $B = S^u$, and more generally for fibrations over a suspension $B = \Sigma W$, the Wang sequence can be established without spectral sequences, using the Mayer–Vietoris sequence for the covering of E by $p^{-1}(C_+W)$ and $p^{-1}(C_-W)$, where

$$\Sigma W = C_+W \cup_W C_-W$$

is a union of two cones. See [Whi78, §VII.1].

The Gysin sequence

Several years earlier, Werner Gysin [Gys42] studied fiber bundles with fiber a sphere, obtaining the equivalent of a long exact sequence as in the following theorem.
To avoid a discussion of local coefficients or orientability, we restrict to the case where B is 1-connected.

Theorem

Let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fiber sequence, with $H_*(F) \cong H_*(S^v)$ *and* B *a* 1*-connected CW complex. There is a long exact sequence*

$$\ldots \to H_{n+1}(B) \xrightarrow{\partial} H_{n-v}(B) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} H_{n-v-1}(B) \to \ldots$$

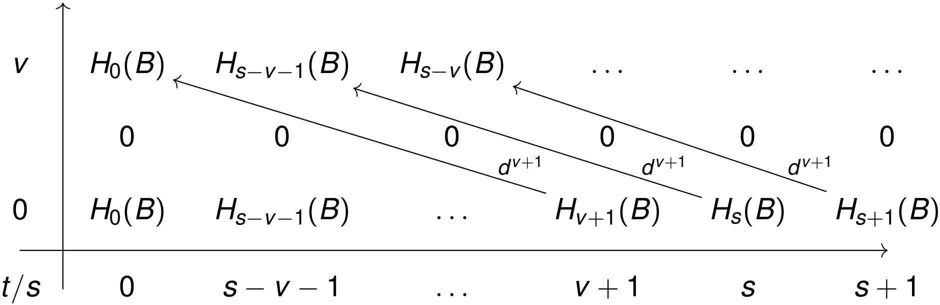
Proof

We assume $v \geq 1$. By the universal coefficient theorem

$$H_s(B; H_t(F)) \cong \begin{cases} H_s(B) & \text{for } t \in \{0, v\}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the E^2-term of the Serre spectral sequence for $F \to E \to B$ is concentrated in the two rows $t = 0$ and $t = v$. For degree reasons $d^r = 0$ except for $r = v + 1$, so $E^2 = E^{v+1}$.

(E^{v+1}, d^{v+1})-term



v
$H_0(B)$ $H_{s-v-1}(B)$ $H_{s-v}(B)$ $\dots$ $\dots$ $\dots$
0 0 0 0 0 0
d^{v+1} d^{v+1} d^{v+1}
0
$H_0(B)$ $H_{s-v-1}(B)$ $\dots$ $H_{v+1}(B)$ $H_s(B)$ $H_{s+1}(B)$
t/s
0 $s-v-1$ $\dots$ $v+1$ s $s+1$

Proof (cont.)

- At this stage we have differentials

$$d^{v+1}_{s,0} \colon H_s(B) \cong E^{v+1}_{s,0} \longrightarrow E^{v+1}_{s-v-1,v} \cong H_{s-v-1}(B)$$

  leading to an E^{v+2}-term with

$$E^{v+2}_{s,t} = \begin{cases} \ker(d^{v+1}_{s,0}) & \text{for } t = 0, \\ \mathrm{cok}(d^{v+1}_{s+v+1,0}) & \text{for } t = v, \\ 0 & \text{otherwise.} \end{cases}$$

- Since $d^r = 0$ for all $r > v + 1$, the spectral sequence collapses at this term, so that $E^{v+2}_{s,t} = E^{\infty}_{s,t}$ in all bidegrees.

Proof (cont.)

- By the convergence of the spectral sequence, we have $F_s H_n(E) = 0$ for $s < n - v$, isomorphisms

$$F_s H_n(E) \cong E^{\infty}_{n-v,v}$$

for $n - v \leq s < n$, and a short exact sequence

$$0 \to F_{n-1} H_n(E) \longrightarrow H_n(E) \longrightarrow E^{\infty}_{n,0} \to 0\,.$$

- In other words, we have a short exact sequence

$$0 \to \mathrm{cok}(d^{v+1}_{n+1,0}) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} \ker(d^{v+1}_{n,0}) \to 0\,.$$

- Writing out the definition of the cokernel and kernel gives the exact sequence

$$H_{n+1}(B) \xrightarrow{\partial} H_{n-v}(B) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} H_{n-v-1}(B)\,,$$

as claimed. □

Thom isomorphism

- The Gysin sequence can be established without spectral sequences, using the Thom isomorphism.
- Let
$$Mp = (I \times E) \cup_E B$$
be the mapping cylinder of $p\colon E \to B$, so that p factors as the inclusion $E \cong \{1\} \times E \subset Mp$ followed by the homotopy equivalence
$$q\colon Mp \xrightarrow{\simeq} B\,.$$
- When $E = S(\xi)$ is the unit sphere bundle in a Euclidean $\mathbb{R}^{v+1}$-bundle $E(\xi) \to B$, the mapping cylinder can be identified with the unit disc bundle $Mp \cong D(\xi)$, so that $(Mp, E) \cong (D(\xi), S(\xi))$.

Thom isomorphism (cont.)

- When B is 1-connected (or the spherical fibration is orientable) there is a Thom isomorphism

$$\Phi : H_n(Mp, E) \xrightarrow{\cong} H_{n-v-1}(B)$$

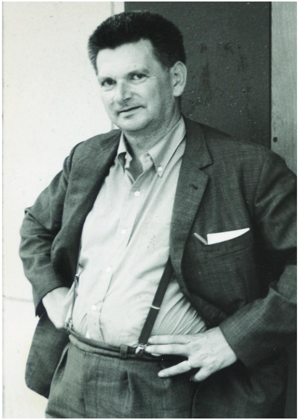
given by the cap product $U \cap (-)$ with a Thom class

$$U \in H^{v+1}(Mp, E) .$$

- In the fiber bundle case, U is characterized by the property that for each $b \in B$ the restriction

$$i_b^* : H^{v+1}(Mp, E) \longrightarrow H^{v+1}(CF_b, F_b) \cong \tilde{H}^v(F_b) \cong \mathbb{Z}$$

maps U to a generator.



René Thom

Thom isomorphism (cont.)

- Here $F_b = p^{-1}(b)$ is the fiber in E over b, and $CF_b \cong q^{-1}(b)$ is the fiber in Mp, which is identified with the cone on F_p.
- In the Euclidean bundle case, $F_b \cong S^v$ is the unit sphere and $CF_b \cong D^{v+1}$ is the unit disc in the fiber of $E(\xi) \to B$ over b.
- Substituting $H_*(Mp) \cong H_*(B)$ and $H_*(Mp, E) \cong H_{*-v-1}(B)$ in the long exact homology sequence

$$\cdots \to H_{n+1}(Mp) \to H_{n+1}(Mp, E) \to H_n(E) \to H_n(Mp) \to H_n(Mp, E) \to \dots$$

of the pair (Mp, E) then gives the Gysin sequence.
See [MS74] or [Whi78, §VII.5].

Reverse engineering

The following examples show that Serre spectral sequences can sometimes be used “in reverse”

- to calculate $H_*(F)$ when $H_*(E)$ and $H_*(B)$ are known, or
- to calculate $H_*(B)$ when $H_*(F)$ and $H_*(E)$ are known.

This is most feasible when $H_*(E)$ is as simple as possible, such as when E is contractible.

The path–loop fibration

Definition

Let $I = [0, 1]$. The path–loop fibration of a based space (X, x_0) is the fiber sequence

$$\Omega X \xrightarrow{i} PX \xrightarrow{p} X\,,$$

where PX is the path space of based maps $\xi\colon (I, 0) \to (X, x_0)$ and p is the Hurewicz fibration with $p(\xi) = \xi(1)$. The fiber ΩX is the loop space of X.

The path–loop fibration

Lemma
The path space PX is contractible.

Proof.
We deform each path $\xi\colon s \mapsto \xi(s)$ to the constant path $s \mapsto x_0$ via the paths $s \mapsto \xi(st)$ for $0 \leq t \leq 1$. □

(c) Disney

Loops on spheres

Proposition

For $u \geq 2$ and $n \geq 0$ there are isomorphisms

$$H_n(\Omega S^u) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \mod u-1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark

Compatible CW structure:

$$\Omega S^u \simeq S^{u-1} \cup e^{2(u-1)} \cup e^{3(u-1)} \cup \dots$$

Proof

- Since $H_*(PS^u) = \mathbb{Z}$ is concentrated in degree 0, the Wang sequence

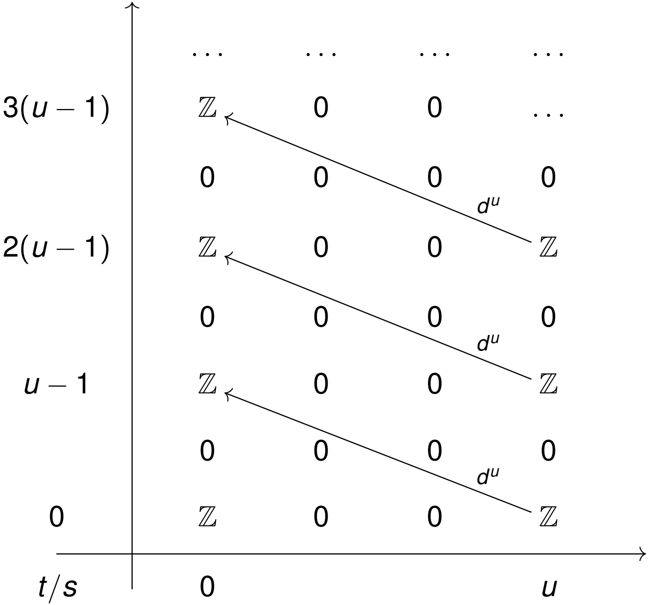
$$\cdots \to H_{n-u+1}(\Omega S^u) \xrightarrow{\partial} H_n(\Omega S^u) \xrightarrow{i_*} H_n(PS^u) \longrightarrow H_{n-u}(\Omega S^u) \to \dots$$

breaks up into isomorphisms

$$H_{n-u+1}(\Omega S^u) \xrightarrow{\cong} \tilde{H}_n(\Omega S^u)\,.$$

- Since $H_n(\Omega S^u)$ is 0 for $n < 0$ and $\mathbb{Z}$ for $n = 0$, the proposition follows by induction on n.
- The differential pattern in the two-column Serre spectral sequence is shown next, with $H_*(S^u)$ on the s-axis and $H_*(\Omega S^u)$ on the t-axis. □

(E^u, d^u)-term



The James construction

- More precise work shows that ΩS^u is equivalent to the James construction $J(S^{u-1})$, see [Hat02, §3.2, §4.J].
- The loop composition induces a Pontryagin product in $H_*(\Omega S^u)$, and
$$H_*(\Omega S^u) \cong \mathbb{Z}[\xi]$$
is the polynomial algebra on ξ, with $|\xi| = u - 1$.
- In other words, ξ^k generates $H_{k(u-1)}(\Omega S^u)$ for each $k \geq 0$, and the remaining homology groups are trivial.

Divided power algebra

- Suppose u is odd. By dualization, the cup product in cohomology satisfies

$$H^*(\Omega S^u) \cong \Gamma(x)$$

with $|x| = u - 1$ even.
- Here

$$\Gamma(x) = \mathbb{Z}\{\gamma_k(x) \mid k \geq 0\}$$

is the divided power algebra on x, with the multiplication

$$\gamma_i(x) \cdot \gamma_j(x) = (i, j)\gamma_{i+j}(x)$$

where $(i, j) = (i + j)!/i!j!$ denotes the binomial coefficient.
- Moreover, $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $|\gamma_k(x)| = k|x|$.
- The terminology comes from the algebra embedding

$$\Gamma(x) \subset \mathbb{Q}[x]$$

sending $\gamma_k(x)$ to the divided k-th power $x^k/k!$.

Deloopings

- The quasi-inverse process to looping a space, $X \mapsto \Omega X$, is called delooping.
- Not every space admits a delooping, and some spaces admit multiple inequivalent deloopings, but for (almost all) topological groups G there is a well-defined space BG with a homotopy equivalence $G \simeq \Omega BG$.

$$X \longmapsto \Omega X$$
$$BG \longmapsto \Omega BG \simeq G$$

- This delooping BG of G is called its classifying space.

Classifying spaces

- Let G be a topological group.
- A map $p\colon P \to B$ is a principal G-bundle if G acts from the right on P and B admits a cover by open subsets U such that there are G-equivariant homeomorphisms $U \times G \cong p^{-1}(U)$ that commute with the projections to U.
- In particular G acts freely on P, and $P/G \cong B$.
- A principal G-bundle $p\colon EG \to BG$ is universal if EG is contractible.
- In this case the base space BG is called a classifying space for G.

A classification theorem

Theorem ([Ste51])

Let $p\colon EG \to BG$ be a universal G-bundle, and let $p\colon P \to B$ be a principal G-bundle with B a CW complex. Then there exists a map $f\colon B \to BG$ and a G-map $\hat{f}\colon P \to EG$ such that the square

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & EG \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & BG \end{array}$$

commutes, and any two such pairs $(f, \hat{f})$ are homotopic. Pullback along f defines a bijection

$$f^*\colon [B, BG] \xrightarrow{\cong} \mathrm{Bun}_G(B)$$

between the homotopy classes of maps $f\colon B \to BG$ and the isomorphism classes of principal G-bundles $p\colon P \to B$.

Example

- Let $G = U(1) = S(\mathbb{C})$ be the circle group, viewed as the complex numbers of unit length.
- It acts freely on $EG = S^\infty = S(\mathbb{C}^\infty)$, viewed as the unit sphere in $\mathbb{C}^\infty$, and the orbit space $BG = EG/G \cong \mathbb{C}P^\infty$ is infinite complex projective space.
- The fiber bundle

$$S^1 \longrightarrow S^\infty \xrightarrow{p} \mathbb{C}P^\infty$$

is, in particular, a (Hurewicz) fibration.

Homology of $BU(1)$

As is well known, the homology of $H_*(\mathbb{C}P^\infty)$ is easily read off from a minimal cell structure on $\mathbb{C}P^\infty$, but the following proposition shows how this can be deduced from the Gysin sequence.

Proposition

For $n \geq 0$ there are isomorphisms

$$H_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Since $H_*(S^\infty) = \mathbb{Z}$ is concentrated in degree 0, the Gysin sequence

$$\cdots \to H_{n-1}(\mathbb{C}P^\infty) \longrightarrow H_n(S^\infty) \xrightarrow{p_*} H_n(\mathbb{C}P^\infty) \xrightarrow{\partial} H_{n-2}(\mathbb{C}P^\infty) \to \dots$$

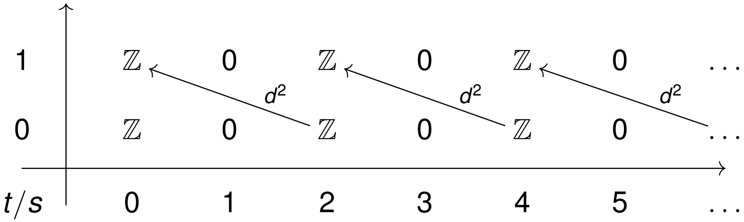
breaks up into isomorphisms

$$\tilde{H}_n(\mathbb{C}P^\infty) \xrightarrow{\cong} H_{n-2}(\mathbb{C}P^\infty)\,.$$

Since $H_n(\mathbb{C}P^\infty)$ is 0 for $n < 0$ and $\mathbb{Z}$ for $n = 0$, the proposition follows by induction on n.

(E^2, d^2)-term

The differential pattern in the two-row Serre spectral sequence is shown below, with $H_*(S^1)$ on the t-axis and $H_*(\mathbb{C}P^\infty)$ on the s-axis.



Outline

- The Serre Spectral Sequence
  - Maps, fiber bundles and fibrations
  - Homology of fiber sequences
  - The Wang and Gysin sequences
  - Edge homomorphisms and the transgression
  - Theorems of Hurewicz and Freudenthal
  - Finite generation and finiteness
- Multiplicative Spectral Sequences
  - Cohomological grading
  - Cohomology of spaces
  - Cohomological Serre spectral sequence
  - Pairings of spectral sequences

Edge homomorphisms

- We continue in the situation

$$F \xrightarrow{i} E \xrightarrow{p} B$$

  with p a Hurewicz fibration, B a 0-reduced CW complex based at the 0-cell b_0, and $F = p^{-1}(b_0)$ the fiber above that point.
- The inclusion i of the fiber and the projection p to the base induce homomorphisms

$$i_* \colon H_*(F) \longrightarrow H_*(E)$$
$$p_* \colon H_*(E) \longrightarrow H_*(B)$$

  called the edge homomorphisms of the Serre spectral sequence.
- They can be factored through the components of the E^∞-term that lie on the vertical and horizontal edges, respectively, of the first quadrant.

The fiber edge

Proposition

The edge homomorphism $i_* \colon H_n(F) \to H_n(E)$ *factors as the surjection*

$$H_n(F) \cong E^1_{0,n} \longrightarrow\!\!\!\!\!\rightarrow E^\infty_{0,n}$$

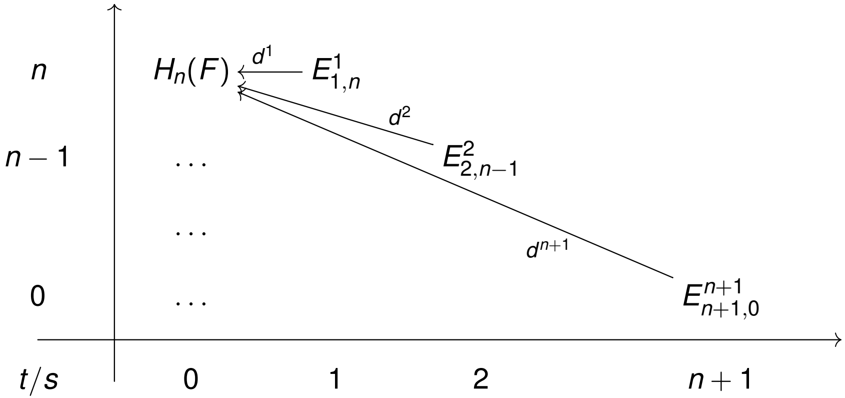
followed by the inclusion

$$E^\infty_{0,n} \cong F_0 H_n(E) \rightarrowtail H_n(E)\,.$$

Proof.

We have $F = E_0$, since B is 0-reduced, so i_* factors as the canonical surjection $H_n(F) \to F_0 H_n(E)$ followed by the inclusion $F_0 H_n(E) \subset H_n(E)$. The first isomorphism follows from $H_n(F) = H_n(E_0) \cong C_0^{CW}(B; \mathscr{H}_n(F)) = E^1_{0,n}$. By convergence, the second isomorphism follows from $E^\infty_{0,n} \cong F_0 H_n(E)/F_{-1} H_n(E)$, since $F_{-1} H_n(E)$ is trivial. □

$H_n(F) \to E^\infty_{0,n} \to H_n(E)$



Remarks

- Every differential $d^r_{0,n}$ lands in a trival group, so $E^1_{0,n}$ consists of infinite cycles.
- There may be differentials $d^r_{r,n-r+1}$ landing in bidegree $(0, n)$, for $1 \leq r \leq n+1$, and their cokernels give a sequence of surjections

$$E^1_{0,n} \twoheadrightarrow E^2_{0,n} \twoheadrightarrow \dots \twoheadrightarrow E^{n+1}_{0,n} \twoheadrightarrow E^{n+2}_{0,n} = E^{\infty}_{0,n}\,.$$

- If B is 1-connected, then $E^1_{1,n} = 0$, the first surjection above is the identity, and $H_n(F) \cong E^2_{0,n}$.

The base edge

Proposition

Suppose that F is 0-connected. The edge homomorphism $p_*\colon H_n(E) \to H_n(B)$ factors as the surjection

$$H_n(E) \twoheadrightarrow E^\infty_{n,0}$$

followed by the inclusion

$$E^\infty_{n,0} \rightarrowtail E^2_{n,0} \cong H_n(B)\,.$$

Proof.

The surjection

$$H_n(E) = F_n H_n(E) \twoheadrightarrow F_n H_n(E)/F_{n-1} H_n(E) \cong E^\infty_{n,0}$$

is given by convergence.

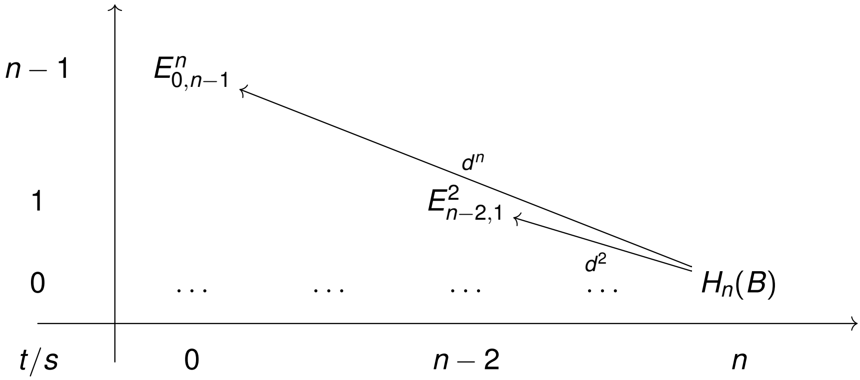
Proof (cont.)

- For $r \geq 2$, every d^r-differential landing in bidegree $(n, 0)$ comes from a trivial group, hence is zero.
- There may be nonzero differentials $d^r_{n,0}$ for $2 \leq r \leq n$, and their kernels give a sequence of inclusions

$$E^\infty_{n,0} = E^{n+1}_{n,0} \subset E^n_{n,0} \subset \cdots \subset E^2_{n,0} \,.$$

- Since F is 0-connected, the coefficient system $\mathscr{H}_0(F) \cong \mathbb{Z}$ is constant, so $E^2_{n,0} = H_n(B; \mathscr{H}_0(F)) \cong H_n(B)$.
- To see that the composite $H_n(E) \to E^\infty_{n,0} \to H_n(B)$ equals p_*, use naturality of Serre spectral sequences with respect to the map from $p \colon E \to B$ to $1 \colon B \to B$. □

$$H_n(E) \to E^{\infty}_{n,0} \to H_n(B)$$



The transgression

Definition

- Let $q\colon (E,F) \to (B,b_0)$ denote the map of pairs induced by p, and suppose $n \geq 1$.
- The additive relation

$$\partial q_*^{-1}\colon H_n(B,b_0) \xleftarrow{q_*} H_n(E,F) \xrightarrow{\partial} H_{n-1}(F)\,,$$

  sending $x = q_*(y)$ to the class of $\partial(y)$, defines a homomorphism

$$\tau_n\colon\ \mathrm{im}(q_*) \longrightarrow H_{n-1}(F)/\partial\,\mathrm{ker}(q_*)$$

  called the transgression.
- The elements of $\mathrm{im}(q_*)$, on which τ_n are defined, are said to be transgressive.

The long differentials

Proposition

The transgression τ_n corresponds to the differential

$$d^n_{n,0} \colon E^n_{n,0} \longrightarrow E^n_{0,n-1}$$

under isomorphisms $E^n_{n,0} \cong \mathrm{im}(q_*)$ and $E^n_{0,n-1} \cong H_{n-1}(F)/\partial \ker(q_*)$.

Proof.

- A relative version of the base edge proposion factors $q_* \colon H_n(E, F) \to H_n(B, b_0)$ as a surjection $H_n(E, F) \to E^n_{n,0}$ followed by an inclusion $E^n_{n,0} \subset H_n(B, b_0)$.
- This gives the isomorphism $\mathrm{im}(q_*) \cong E^n_{n,0}$, and shows that $\ker(q_*) = \ker(H_n(E, F) \to H_n(E, E_{n-1})) = \mathrm{im}(H_n(E_{n-1}, F) \to H_n(E, F))$.

Proof (cont.)

- Hence $\partial \ker(q_*)$ is the image of $\partial\colon H_n(E_{n-1}, F) \to H_{n-1}(F)$, and $H_{n-1}(F)/\partial \ker(q_*)$ is the coimage of $H_{n-1}(F) \to H_{n-1}(E_{n-1})$.
- Consider the following commutative diagram.

$$
\begin{array}{ccccc}
H_n(E_n, F) & \xrightarrow{i_*} & H_n(E, F) & \xrightarrow{\partial} & H_{n-1}(F) \\
\downarrow & & \downarrow & & \downarrow \\
Z^n_{n,0} & \twoheadrightarrow & E^n_{n,0} & \xrightarrow{d^n_{n,0}} & E^n_{0,n-1} \\
\downarrow & & \downarrow & & \downarrow \\
H_n(E_n, E_{n-1}) & \xrightarrow{i_*} & H_n(E, E_{n-1}) & \xrightarrow{\partial} & H_{n-1}(E_{n-1})
\end{array}
$$

- The inclusion $F = E_0 \subset E_{n-1}$ induces the map from the first to the third row, and the second row consists of the images of this vertical map.

Proof (cont.)

- On the left hand side this follows by rewriting the definition of $Z^n_{n,0} = \partial^{-1}\operatorname{im}(H_{n-1}(F) \to H_{n-1}(E_{n-1}))$ as $\ker(H_n(E_n, E_{n-1}) \to H_{n-1}(E_{n-1}, F)) = \operatorname{im}(H_n(E_n, F) \to H_n(E_n, E_{n-1}))$.
- For the middle and right hand sides it follows by consideration of the relative fibration $q\colon (E, F) \to (B, b_0)$ and the restricted fibration $E_{n-1} \to B^{(n-1)}$, respectively.
- Diagram chases then confirm that $Z^n_{n,0} \to E^n_{n,0}$ is the canonical surjection, and that the induced homomorphism $E^n_{n,0} \to E^n_{0,n-1}$ equals the d^n-differential. □

Serre's exact homology sequence for a fibration

Theorem ([Ser51, Prop. III.5])

- *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space.*
- *Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$.*
- *Then there is an exact sequence*

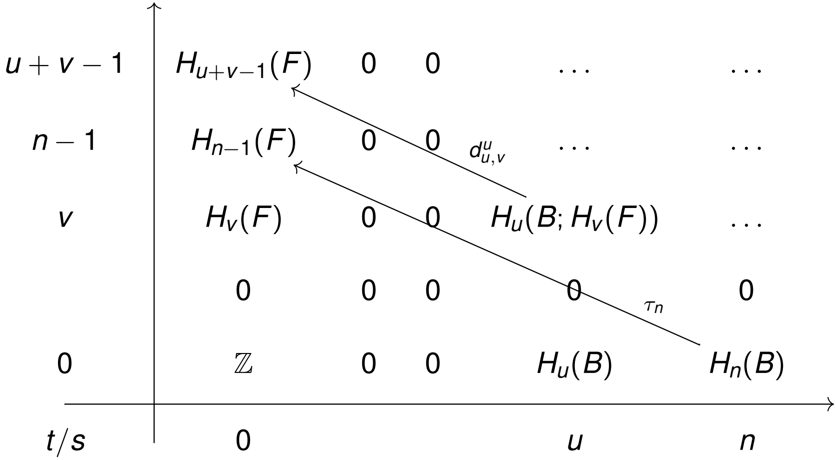
$$H_{u+v-1}(F) \xrightarrow{i_*} H_{u+v-1}(E) \xrightarrow{p_*} H_{u+v-1}(B) \xrightarrow{\tau_{u+v-1}} \dots$$
$$\dots \xrightarrow{\tau_{n+1}} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\tau_n} H_{n-1}(F) \xrightarrow{i_*} \dots$$
$$\dots \xrightarrow{p_*} H_2(B) \xrightarrow{\tau_2} H_1(F) \xrightarrow{i_*} H_1(E) \to 0 \, .$$

Proof

The Serre spectral sequence

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)$$

is concentrated in the region $s = 0$ or $s \geq u$, $t = 0$ or $t \geq v$.



Proof (cont.)

- The only (nonzero) differentials originating in total degree $n = s + t < u + v$ are the transgressions $\tau_n \colon H_n(B) = E^n_{n,0} \to E^n_{0,n-1} = H_{n-1}(F)$.
- The first possible differential from total degree $u + v$ is $d^u_{u,v} \colon E^u_{u,v} \to E^u_{0,u+v-1} = H_{u+v-1}(F)$, where $E^u_{u,v}$ is a quotient of $E^2_{u,v} = H_u(B; H_v(F))$.
- It follows that in total degrees $s + t < u + v$ the E^∞-term is given by

$$E^\infty_{s,t} = \begin{cases} \mathbb{Z} & \text{for } (s,t) = (0,0), \\ \ker(\tau_n) & \text{for } s = n \geq 2 \text{ and } t = 0, \\ \operatorname{cok}(\tau_n) & \text{for } s = 0 \text{ and } t = n-1 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof (cont.)

In total degree $1 \leq n \leq u + v - 1$ we therefore have

$$F_0 H_n(E) = \cdots = F_{n-1} H_n(E) = \mathrm{cok}(\tau_{n+1})$$

and a short exact sequence

$$0 \to F_{n-1} H_n(E) \longrightarrow H_n(E) \longrightarrow \ker(\tau_n) \to 0 \, .$$

This gives an exact sequence

$$H_{n+1}(B) \xrightarrow{\tau_{n+1}} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\tau_n} H_{n-1}(F)$$

for each $n \leq u + v - 2$. When $n = u + v - 1$ the target of τ_{n+1} is a quotient of $H_n(F)$, but i_* nonetheless maps $H_n(F)$ onto its cokernel. Splicing these together we obtain Serre's exact sequence. □

A dual to homotopy excision

Serre's sequence agrees with the long exact homology sequence of the pair (E, F), in the stated range of degrees. The following reformulation is dual to a form of the homotopy excision theorem cf. [Hat02, Prop. 4.28].

Proposition

- *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space.*
- *Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$.*
- *Then*

$$q_* \colon H_n(E, F) \longrightarrow H_n(B, b_0)$$

is an isomorphism for $n \leq u + v - 1$ and is surjective for $n = u + v$.

Proof

- There is a relative Serre spectral sequence

$$E^2_{s,t} = H_s(B, b_0; H_t(F)) \Longrightarrow_s H_{s+t}(E, F)$$

obtained by omitting the edge $s = 0$.

- The differential in lowest possible total degree is

$$d^{v+1} \colon E^{v+1}_{u+v+1,0} \longrightarrow E^{v+1}_{u,v} \,,$$

so when $n \leq u + v$ the edge homomorphism

$$q_* \colon H_n(E, F) \longrightarrow E^\infty_{n,0} = H_n(B)$$

is an isomorphism for $n < u + v$ and a surjection when $n = u + v$. □

Vanishing base homology

Corollary

If $i_*\colon H_n(F) \to H_n(E)$ *is an isomorphism for* $n < k$ *and surjective for* $n = k$*, then* $H_s(B) = 0$ *for* $0 < s \leq k$*.*

Proof.

- We apply the proposition with $v = 1$.
- By the long exact homology sequence for (E, F) we have $H_n(E, F) = 0$ for $n \leq k$.
- Hence, if $H_s(B) = 0$ for $0 < s < u$ then $0 = H_u(E, F) \cong H_u(B)$ as long as $u \leq k$.
- By induction on u it follows that $H_s(B) = 0$ for $0 < s \leq k$.

□

Another dual to homotopy excision

We can also compare Serre's sequence with the long exact homology sequence of the pair (Mp, E), where $Mp \simeq B$ is the mapping cylinder of $p\colon E \to B$.
See Hall [Hal65] or Clapp [Cla81] for the fact that $q\colon Mp \to B$ is a Hurewicz fibration.

Proposition

- *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space.*
- *Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$.*
- *Then*

$$\tilde{H}_{n-1}(F) \cong H_n(CF, F) \xrightarrow{i_*} H_n(Mp, E)$$

is an isomorphism for $n < u + v$ and is surjective for $n = u + v$.

Proof

- There is a relative Serre spectral sequence

$$E^2_{s,t} = H_s(B; H_t(CF, F)) \Longrightarrow_s H_{s+t}(Mp, E)$$

obtained by omitting the edge $t = 0$ and increasing the fiber degrees by 1.

- The differential in lowest possible total degree is

$$d^u \colon E^u_{u,v+1} \longrightarrow E^u_{0,u+v}\,,$$

so when $n \leq u + v$ the edge homomorphism

$$i \colon H_n(CF, F) \longrightarrow E^\infty_{0,n} \cong H_n(Mp, E)$$

is an isomorphism for $n < u + v$ and a surjection for $n = u + v$. □

Vanishing fiber homology

Corollary

If $p_* \colon H_n(E) \to H_n(B)$ is an isomorphism for $n < k$ and surjective for $n = k$, then $H_t(F) = 0$ for $0 < t < k$.

Proof.

- We apply the proposition with $u = 2$.
- By the long exact homology sequence for (Mp, E), and the equivalence $Mp \simeq B$, we have $H_n(Mp, E) = 0$ for $n \leq k$.
- Hence, if $H_t(F) = 0$ for $0 < t < v$ then $H_v(F) \cong H_{v+1}(CF, F) \cong H_{v+1}(Mp, E)$ vanishes as long as $v + 1 \leq k$.
- By induction on v it follows that $H_t(F) = 0$ for $0 < t < k$.

□

Outline

We can deduce

- absolute and
- relative Hurewicz theorems, as well as
- Freudenthal's suspension theorem,

from Serre's exact sequence. Spectral sequences thus give an alternative approach to these results, as opposed to the homotopy excision theorem with its geometric proof, which was used to deduce these results in [Hat02, §4.2].

Hurewicz homomorphisms

Definition

- Let $s_n \in \tilde{H}_n(S^n)$ be a chosen generator, and let X be any based space. The (absolute) Hurewicz homomorphism

$$h_n \colon \pi_n(X) \longrightarrow \tilde{H}_n(X)$$

is given by

$$[f] \longmapsto f_*(s_n) .$$

The elements in the image of h_n are said to be spherical.

- Let $d_{n+1} \in H_{n+1}(D^{n+1}, S^n)$ be a chosen generator, and let (X, A) be any pair of based spaces. The relative Hurewicz homomorphism

$$h_{n+1} \colon \pi_{n+1}(X, A) \longrightarrow H_{n+1}(X, A)$$

is given by

$$[f] \longmapsto f_*(d_{n+1}) .$$

Compatibility

Remark

- With a specified suspension isomorphism

$$E\colon \tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$$

  we can demand that $S^{n+1} = \Sigma S^n$ and $E(s_n) = s_{n+1}$. Then h_n and h_{n+1} are compatible with Freudenthal's suspension $E\colon \pi_n(X) \to \pi_{n+1}(\Sigma X)$ and the isomorphism above.
- We can also demand that

$$\partial\colon H_{n+1}(D^{n+1}, S^n) \longrightarrow \tilde{H}_n(S^n)$$

  maps d_{n+1} to s_n, in which case the relative h_{n+1} and the absolute h_n are compatible with the connecting homomorphisms $\partial\colon \pi_{n+1}(X, A) \to \pi_n(A)$ and $\partial\colon H_{n+1}(X, A) \to \tilde{H}_n(A)$.

Absolute Hurewicz theorem for 1-connected spaces

Theorem

Let X be an $(n-1)$-connected CW complex, with $n \geq 2$. Then

$$h_n\colon \pi_n(X) \xrightarrow{\cong} H_n(X)$$

is an isomorphism.

Proof.

We prove this by induction on n. Consider the homotopy fiber sequence

$$\Omega X \longrightarrow PX \longrightarrow X$$

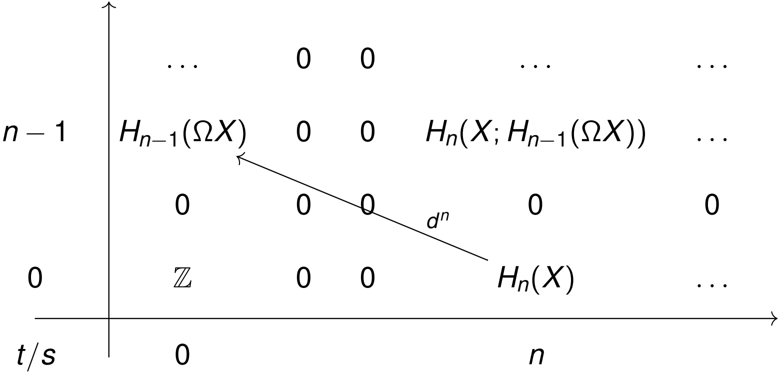
and the associated commutative diagram

$$\begin{array}{ccccc} \pi_{n-1}(\Omega X) & \xleftarrow[\cong]{\partial} & \pi_n(PX, \Omega X) & \xrightarrow[\cong]{q_*} & \pi_n(X) \\ \downarrow{\scriptstyle h_{n-1}} & & \downarrow & & \downarrow{\scriptstyle h_n} \\ H_{n-1}(\Omega X) & \xleftarrow[\cong]{\partial} & H_n(PX, \Omega X) & \xrightarrow{q_*} & H_n(X) \, . \end{array}$$

Proof (cont.)

- The isomorphisms in the upper row follow from the long exact homotopy sequences of a pair and of a fibration, together with the fact that PX is contractible.
- The isomorphism in the lower row follows from the long exact homology sequence of the pair $(PX, \Omega X)$, and the fact just mentioned.
- In particular, ΩX is $(n-2)$-connected, so by induction the left hand homomorphism h_{n-1} is an isomorphism.
- To start the induction, for $n = 2$, we appeal to Poincaré's result [Hat02, Thm. 2A.1] that $h_1 : \pi_1(\Omega X) \to H_1(\Omega X)$ is an isomorphism, where $\pi_1(\Omega X) = \pi_2(X)$ is already abelian.

Proof (cont.)



By the Serre spectral sequence, or the exact sequence deduced from it, the homomorphism $q_* \colon H_n(PX, \Omega X) \to H_n(X)$ is an isomorphism, since $n \leq n + (n-1) - 1$ for $n \geq 2$. Hence $h_n \colon \pi_n(X) \to H_n(X)$ is an isomorphism. $\square$

Absolute connectivity from homology

Corollary

If X is a 1-connected CW complex with $H_m(X) = 0$ for $0 < m < n$ then X is $(n-1)$-connected.

Proof.

- For $m < n$, suppose we have proved that X is $(m-1)$-connected.
- Then $h_m \colon \pi_m(X) \to H_m(X)$ is an isomorphism, so the assumption that $H_m(X) = 0$ implies that $\pi_m(X) = 0$.
- Hence X is m-connected.
- Continue inductively, until $m = n - 1$.

□

Connectivity of maps

Definition

- A map $f\colon X \to Y$ of 0-connected spaces is *n-connected* if $f_*\colon \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$.
- Replacing f by the inclusion $X \subset Mf$ into the mapping cylinder of f, and considering the long exact homotopy sequence

$$\cdots \to \pi_m(X) \longrightarrow \pi_m(Mf) \longrightarrow \pi_m(Mf, X) \xrightarrow{\partial} \pi_{m-1}(X) \to \dots$$

  of the pair (Mf, X), we see that f is n-connected if and only if $\pi_m(Mf, X) = 0$ for each $m \leq n$.

Homotopy fibers

Definition

For any map $f\colon X \to Y$ let $p\colon Ef = X \times_Y Y^I \to Y$ be the associated path space fibration.

$$
\begin{array}{ccccc}
Ff & \longrightarrow & Ef & \longrightarrow & X \\
\downarrow & p & \downarrow & & \downarrow f \\
PY & \longrightarrow & Y^I & \xrightarrow[p_1]{} & Y \\
\downarrow & & \downarrow p_0 & & \\
\{y_0\} & \longrightarrow & Y & &
\end{array}
$$

There is a homotopy equivalence $X \to Ef$, compatible with the two maps to Y. The fiber $p^{-1}(y_0) = Ff = X \times_Y PY$ of this fibration is the homotopy fiber of f at $y_0 \in Y$.

Serre sp. seq. of a homotopy fiber sequence

The Serre spectral sequence

$$E^2_{s,t} = H_s(Y; H_t(Ff)) \Longrightarrow_s H_{s+t}(Ef)$$

for

$$Ff \longrightarrow Ef \xrightarrow{p} Y$$

can be rewritten in the form

$$E^2_{s,t} = H_s(Y; H_t(Ff)) \Longrightarrow_s H_{s+t}(X)\,,$$

in which case we think of it as being associated to the homotopy fiber sequence

$$Ff \longrightarrow X \xrightarrow{f} Y\,.$$

Milnor [Mil59, Thm. 3] proved that if X and Y are CW complexes, then Ff has the homotopy type of a CW complex.

Relative Hurewicz theorem for 1-connected spaces

Theorem

Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $\pi_m(Mf, X) = 0$ for $m \leq n$, where $Mf \simeq Y$ is the mapping cylinder of f. Then

$$h_{n+1}\colon \pi_{n+1}(Mf, X) \xrightarrow{\cong} H_{n+1}(Mf, X)$$

is an isomorphism.

Proof.

- There is only something to prove for $n \geq 1$.
- Using a path space fibration, we may replace $f\colon X \to Y$ with a homotopy equivalent Hurewicz fibration $p\colon E \to B$, with B a CW complex.

Proof (cont.)

- Its fiber $F = p^{-1}(b_0)$ is then the homotopy fiber of f, and $\pi_m(Mp, E) = 0$ for $m \leq n$.
- In the commutative diagram

$$\begin{array}{ccccc}
\pi_n(F) & \xleftarrow[\cong]{\partial} & \pi_{n+1}(CF, F) & \xrightarrow[\cong]{i_*} & \pi_{n+1}(Mp, E) \\
{\scriptstyle h_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle h_{n+1}}\downarrow & & \downarrow{\scriptstyle h_{n+1}} \\
H_n(F) & \xleftarrow[\cong]{\partial} & H_{n+1}(CF, F) & \xrightarrow[\cong]{i_*} & H_{n+1}(Mp, E)
\end{array}$$

  the upper row consists of isomorphisms, because F is equivalent to the homotopy fiber of the inclusion $E \subset Mp$.
- Likewise, $\pi_{m-1}(F) \cong \pi_m(Mp, E) = 0$ for $m \leq n$, so F is $(n-1)$-connected.
- If $n = 1$, then $\pi_1(F)$ is a quotient of $\pi_2(B)$, since $\pi_1(E) = 0$, so $\pi_1(F)$ is abelian.

Proof (cont.)

- The absolute Hurewicz theorem for F thus tells us that the left hand h_n is an isomorphism.
-
$$\begin{array}{ccccc}
\pi_n(F) & \xleftarrow[\cong]{\partial} & \pi_{n+1}(CF, F) & \xrightarrow[\cong]{i_*} & \pi_{n+1}(Mp, E) \\
{\scriptstyle h_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle h_{n+1}}\downarrow & & \downarrow{\scriptstyle h_{n+1}} \\
H_n(F) & \xleftarrow[\cong]{\partial} & H_{n+1}(CF, F) & \xrightarrow[\cong]{i_*} & H_{n+1}(Mp, E)
\end{array}$$
- The lower row consists of isomorphisms by the "second dual homotopy excision" proposition, applied to $F \to E \to B$ with $u = 2$ and $v = n$.
- Hence the right hand h_{n+1} is an isomorphism. □

Relative connectivity from homology

Corollary

- *Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $f_*\colon H_m(X) \to H_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$.*
- *Then $f_*\colon \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$.*

Proof.

An equivalent statement is the following: Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $H_m(Mf, X) = 0$ for $m \leq n$. Then $\pi_m(Mf, X) = 0$ for $m \leq n$.

Proof (cont.)

- For $m \leq n$, suppose we have proved that f is $(m-1)$-connected.
- Then $h_m \colon \pi_m(Mf, X) \to H_m(Mf, X)$ is an isomorphism by the relative Hurewicz theorem, so the assumption that $H_m(Mf, X) = 0$ implies that $\pi_m(Mf, X) = 0$.
- Hence f is m-connected.
- Continue inductively, until $m = n$. □

Freudenthal's suspension homomorphism

Definition

We define the cone and suspension of a based space X to be $CX = I \wedge X$ and $\Sigma X = S^1 \wedge X \cong CX/X = I/\partial I \wedge X$, respectively. We write $[t, x]$ for the image of $(t, x) \in I \times X$ under the quotent map to CX or $CX/X \cong \Sigma X$.

Definition

Freudenthal's suspension homomorphism (Einhängung)

$$E\colon \pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

maps the homotopy class of $f\colon S^n \to X$ to the homotopy class of

$$\Sigma f\colon S^{n+1} = \Sigma S^n \longrightarrow \Sigma X\,.$$

Cones and path spaces

Lemma

Let $\eta\colon X \to \Omega\Sigma X$ map x to the loop $s \mapsto [s,x]$, and let $\bar{\eta}\colon CX \to P\Sigma X$ map $[t,x]$ to the path $s \mapsto [st,x]$. Then the diagram

$$\begin{array}{ccccc}
X & \longrightarrow & CX & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \bar{\eta}} & & \downarrow{\scriptstyle =} \\
\Omega\Sigma X & \longrightarrow & P\Sigma X & \xrightarrow{p} & \Sigma X
\end{array}$$

commutes.

Proof.

Direct from the definitions. □

Homological connectivity

Proposition

Let X be a $(k-1)$-connected CW complex. Then

$$\eta_*\colon H_n(X) \xrightarrow{\cong} H_n(\Omega\Sigma X)$$

is an isomorphism for $n \leq 2k-1$.

Proof.

By the previous lemma we have a commutative diagram

$$\begin{array}{ccccc}
H_n(X) & \xleftarrow[\cong]{\partial} & H_{n+1}(CX, X) & \xrightarrow{\cong} & H_{n+1}(\Sigma X) \\
\downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \bar{\eta}_*} & & \downarrow{\scriptstyle =} \\
H_n(\Omega\Sigma X) & \xleftarrow[\cong]{\partial} & H_{n+1}(P\Sigma X, \Omega\Sigma X) & \xrightarrow{q_*} & H_{n+1}(\Sigma X)\,.
\end{array}$$

Note that ΣX is k-connected and $\Omega\Sigma X$ is $(k-1)$-connected, so q_* in the lower row is an isomorphism for $n+1 \leq 2k$ by the dual of homotopy excision. □

Freudenthal's suspension theorem

Theorem ([Hat02, Cor. 4.24])

Let X be a $(k-1)$-connected CW complex. Then $\eta\colon X \to \Omega\Sigma X$ is $(2k-1)$-connected, meaning that

$$\eta_*\colon \pi_n(X) \longrightarrow \pi_n(\Omega\Sigma X)$$

and

$$E\colon \pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

are isomorphisms for $n < 2k-1$ and surjective for $n = 2k-1$.

Proof

When $k = 1$ we use that X and $\Omega\Sigma X$ are 0-connected, so that in the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\eta_*} & \pi_1(\Omega\Sigma X) \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_1}\,\cong \\ H_1(X) & \xrightarrow[\cong]{\eta_*} & H_1(\Omega\Sigma X) \end{array}$$

the vertical maps are the abelianization homomorphisms, which is surjective for X and an isomorphism for $\Omega\Sigma X$, since $\pi_1(\Omega\Sigma X) \cong \pi_2(\Sigma X)$ is commutative.
The lower homomorphism η_* is an isomorphism by the proposition above, hence the upper homomorphism η_* is surjective.

Proof (cont.)

- For $k \geq 2$ we use that X is 1-connected to deduce that ΣX is 2-connected and $\Omega\Sigma X$ is 1-connected.
- Hence the homological connectivity proposition and relative Hurewicz theorem imply that $\eta\colon X \to \Omega\Sigma X$ is $(2k-1)$-connected.
- The suspension homomorphism E corresponds to η_* under the isomorphism $\pi_{n+1}(\Sigma X) \cong \pi_n(\Omega\Sigma X)$. □

Outline

Finite generation

We discuss some results from [Ser51, Ch. V].

Definition

- An abelian group G is finitely generated if there exists a surjective homomorphism

$$\mathbb{Z}^k \longrightarrow G$$

  for some finite k.
- In this case,

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_s$$

  is isomorphic to a finite direct sum of cyclic groups, i.e., groups of the form $\mathbb{Z}$ or $\mathbb{Z}/m$, where $m \geq 2$.
- Here r is the dimension of $G \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space, which we call the rank of the group G.

Finite type

Definition

- A space X has homology of finite type if each group $H_n(X)$ is finitely generated.
- A 1-connected space X has homotopy of finite type if each homotopy group $\pi_n(X)$ is finitely generated. In this case we also say that X has finite type.

We will show that a 1-connected space has homology of finite type if and only if it has (homotopy of) finite type. This applies, for instance, to $X = S^n$ for $n \geq 2$.

Two-out-of-three: finite type

The following is a special case of [Whi78, Thm. XIII.7.11].

Theorem

Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space.
If two of the following conditions hold, then so does the third.

1. *$H_t(F)$ is finitely generated for each t.*
2. *$H_n(E)$ is finitely generated for each n.*
3. *$H_s(B)$ is finitely generated for each s.*

Proof

There are three cases, which we treat in sequence.

(1. and 3. $\implies$ 2.)

- If F and B have homology of finite type, then each group

$$E^2_{s,n-s} = H_s(B; H_{n-s}(F))$$

  is finitely generated, by the universal coefficient theorem.
- Hence so is each subquotient

$$E^\infty_{s,n-s} \cong F_s H_n(E)/F_{s-1} H_n(E)\,.$$

- It follows by induction on $s \geq 0$ that each $F_s H_n(E)$ is finitely generated.
- When $s = n$, this equals $H_n(E)$.

Proof (cont.)

(1. and 2. $\Longrightarrow$ 3.)

- If F and E have homology of finite type, then we must show that B has homology of finite type.
- Let $n \geq 2$ and assume by induction that $H_s(B)$ is finitely generated for $s < n$.
- Then
$$E^2_{s,t} = H_s(B; H_t(F))$$
is finitely generated for each $s < n$ and t, hence so is each subquotient $E^r_{s,t}$ in this region.
- Since $H_n(E)$ is finitely generated, so is its quotient
$$E^\infty_{n,0} \cong H_n(E)/F_{n-1}H_n(E)\,.$$
- We prove by descending induction on r that $E^r_{n,0}$ is finitely generated.
- This is clear for $r = n + 1$, since $E^{n+1}_{n,0} = E^\infty_{n,0}$.

Proof (cont.)

- Suppose that $E^{r+1}_{n,0}$ is finitely generated, where $r \geq 2$.
- We have an exact sequence

$$0 \to E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d^r_{n,0}} E^r_{n-r,r-1} \, .$$

- Here $E^r_{n-r,r-1}$ is one of the subquotients we have argued must be finitely generated, hence its subgroup $\mathrm{im}(d^r_{n,0})$ is also finitely generated.
- We have assumed inductively that $E^{r+1}_{n,0}$ is finitely generated, so this extension proves that $E^r_{n,0}$ is finitely generated.
- Hence

$$E^2_{n,0} = H_n(B; H_0(F)) \cong H_n(B)$$

is finitely generated, as we wanted to prove.

Proof (cont.)

(2. and 3. $\Longrightarrow$ 1.)

- If E and B have homology of finite type, then we must prove that F has homology of finite type.
- Let $n \geq 1$ and assume by induction that $H_t(F)$ is finitely generated for $t < n$.
- Then
$$E^2_{s,t} = H_s(B; H_t(F))$$
is finitely generated for each s and $t < n$, hence so is each subquotient $E^r_{s,t}$ in this region.
- Since $H_n(E)$ is finitely generated, so is its subgroup
$$E^\infty_{0,n} \cong F_0 H_n(E)\,.$$
- We prove by descending induction on r that $E^r_{0,n}$ is finitely generated.
- This is clear for $r = n + 2$, since $E^{n+2}_{0,n} = E^\infty_{0,n}$.

Proof (cont.)

- Suppose that $E^{r+1}_{0,n}$ is finitely generated, where $r \geq 2$.
- We have an exact sequence

$$E^r_{r,n-r+1} \xrightarrow{d^r_{r,n-r+1}} E^r_{n,0} \longrightarrow E^{r+1}_{n,0} \to 0\,.$$

- Here $E^r_{r,n-r+1}$ is one of the subquotients we have argued must be finitely generated, hence its quotient group $\mathrm{im}(d^r_{r,n-r+1})$ must also be finitely generated.
- We have assumed inductively that $E^{r+1}_{n,0}$ is finitely generated, so this extension proves that $E^r_{n,0}$ is finitely generated.
- Hence

$$E^2_{n,0} = H_0(B; H_n(F)) \cong H_n(F)$$

is finitely generated, as we wanted to prove. □

Two-out-of-three: degreewise finite

Theorem

Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. If two of the following conditions hold, then so does the third.

1. $\tilde{H}_t(F)$ *is finite for each* t.
2. $\tilde{H}_n(E)$ *is finite for each* n.
3. $\tilde{H}_s(B)$ *is finite for each* s.

Sketch proof.

In the proof of the previous theorem replace "finitely generated" by "finite", making allowance for the fact that $H_0(X) = \mathbb{Z}$ for each of the spaces in question. □

Eilenberg–Mac Lane spaces

Definition

Let G be a discrete group, and $n \geq 0$. An Eilenberg–Mac Lane space of type (G, n) is a CW complex $K(G, n)$ such that

$$\pi_i K(G,n) \cong \begin{cases} G & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

- Eilenberg–Mac Lane spaces of type $(G, 1)$ can be constructed by giving a presentation of G in terms of generators and relations, and building a 0-reduced CW complex

$$X = (S^1 \vee \cdots \vee S^1) \cup D^2 \cup \cdots \cup D^2$$

  with one 1-cell for each generator and one 2-cell realizing each relation, so that $\pi_1(X) \cong G$.
- One then attaches k-cells for $k \geq 3$ to kill the higher homotopy groups.

Construction (cont.)

- For $n \geq 2$ and G abelian an Eilenberg–Mac Lane space of type (G, n) can be constructed from a presentation of G by building an $(n-1)$-reduced CW complex

$$X = (S^n \vee \cdots \vee S^n) \cup D^{n+1} \cup \cdots \cup D^{n+1}$$

  with one n-cell for each generator and one $(n+1)$-cell for each relation, so that $\pi_n(X) \cong H_n(X) \cong G$.
- One then attaches k-cells for $k \geq n+2$ to kill the higher homotopy groups.

Uniqueness

- It follows by an obstruction theory argument that any two Eilenberg–Mac Lane spaces of the same type (G, n) are homotopy equivalent, by a map that induces the identity $G = G$ on π_n.
- Hence there is an equivalence

  $$K(G, n-1) \simeq \Omega K(G, n)$$

  whenever $K(G, n)$ is defined.

Fiber sequences

There are homotopy fiber sequences

$$G \longrightarrow PK(G,1) \xrightarrow{p} K(G,1)$$

for any group G, and

$$K(G,n-1) \longrightarrow PK(G,n) \xrightarrow{p} K(G,n)$$

for any abelian group G and $n \geq 1$.
For any universal G-bundle (with G discrete)

$$G \longrightarrow EG \xrightarrow{p} BG$$

the classifying space $BG \simeq K(G,1)$ is an Eilenberg–Mac Lane space of type $(G,1)$.

Homology of $K(G, 1)$

Proposition

- *Let G be a finitely generated abelian group. Then each homology group*

$$H_i(BG) = H_i(K(G, 1))$$

  is finitely generated.
- *If G is finite, then each reduced homology group $\tilde{H}_i(BG)$ is finite.*

Proof

- We can write G as a finite product

$$G \cong \mathbb{Z} \times \cdots \times \mathbb{Z} \times C_{m_1} \times \ldots C_{m_s}$$

  of cyclic groups, and there is then a homotopy equivalence

$$BG \simeq B\mathbb{Z} \times \cdots \times B\mathbb{Z} \times BC_{m_1} \times \cdots \times BC_{m_s} \,,$$

  since both sides are Eilenberg–Mac Lane spaces of type $(G, 1)$.
- Here $B\mathbb{Z} \simeq S^1$ has the homotopy type of the circle, and $BC_m \simeq S^\infty/C_m$ has the homotopy type of an infinite lens space, i.e., the orbit space for the free action by $C_m \subset U(1)$ on the contractible space $S^\infty = S(\mathbb{C}^\infty)$.
- Both S^1 and S^∞/C_m admit CW structures with finitely many cells in each dimension, cf. [Hat02, Ex. 2.43], hence have homology of finite type.

Proof (cont.)

- More precisely,

$$H_i(S^1) \cong \begin{cases} \mathbb{Z} & \text{for } i \in \{0, 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(BC_m) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/m & \text{for } i \geq 1 \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

- By the Künneth theorem (or Serre spectral sequence for the product fibration), it follows that the finite product BG has homology of finite type.
- If G is finite, so that $r = 0$, it also follows that the reduced homology groups of BG are finite. □

Homology of $K(G, n)$

Proposition

Let G be a finitely generated abelian group, and let $n \geq 1$. Then each homology group

$$H_i(K(G, n))$$

is finitely generated.

Proof.

This is was proved in the previous proposition for $n = 1$. The cases $n \geq 2$ follow by induction, by the two-out-of-three theorem for finite type applied to the homotopy fiber sequence

$$K(G, n-1) \longrightarrow PK(G, n) \xrightarrow{p} K(G, n)\,,$$

where we know that $F \simeq K(G, n-1)$ and $E = PK(G, n) \simeq *$ have homology of finite type, while $B = K(G, n)$ is 1-connected. $\square$

Homology of $K(G, n)$ (cont.)

Proposition

Let G be a finite abelian group, and let $n \geq 1$. Then each reduced homology group

$$\tilde{H}_i(K(G, n))$$

is finite.

Proof.

In the previous proof, replace “homology of finite type” by “degreewise finite reduced homology”. □

Postnikov towers

Recall how Postnikov sections and Whitehead covers can be constructed by the method of killing homotopy groups [Hat02, §4.1, §4.3].

Lemma

Let X be a 0-connected CW complex, and let $n \geq 0$. There is a homotopy fiber sequence

$$\tau_{>n}X \xrightarrow{i} X \xrightarrow{p} \tau_{\leq n}X$$

where

$$p_*\colon \pi_m(X) \longrightarrow \pi_m(\tau_{\leq n}X)$$

is an isomorphism for $m \leq n$ and $\pi_m(\tau_{\leq n}X) = 0$ for $m > n$. Equivalently,

$$i_*\colon \pi_m(\tau_{>n}X) \longrightarrow \pi_m(X)$$

is an isomorphism for $m > n$ and $\pi_m(\tau_{>n}X) = 0$ for $m \leq n$.

Proof

- We inductively obtain $\tau_{\leq n}X$ from X by attaching $(k+1)$-cells to kill π_k of the previous stage, for each $k \geq n+1$.
- We let $\tau_{>n}X$ be the homotopy fiber of the map $p\colon X \to \tau_{\leq n}X$. □

Definition

We call $\tau_{\leq n}X = \tau_{<n+1}X$ the *n*-th Postnikov section of X, and refer to $\tau_{>n}X = \tau_{\geq n+1}X$ as the *n*-connected cover of X.

Remark

There are equivalences

$$\tau_{\leq n}(\tau_{\geq n}X) \simeq K(\pi_n(X), n) \simeq \tau_{\geq n}(\tau_{\leq n}X)$$

obtained by passing to the n-th Postnikov section and the $(n-1)$-connected cover, in either order.

Agreement about finite type

Theorem

Let X be a 1-connected CW complex.
Then X has homology of finite type if and only if it has (homotopy of) finite type.

Corollary

Each group $\pi_i(S^n)$ is finitely generated.

Proof of theorem.

We prove the two implications in order.

Proof ($\Longrightarrow$)

- Suppose X is 1-connected with homology of finite type.
- Let $n \geq 2$ and suppose, by induction, that the $(n-1)$-connected cover $\tau_{\geq n}X$ has homology of finite type.
- Then
$$\pi_n(X) \cong \pi_n(\tau_{\geq n}X) \cong H_n(\tau_{\geq n}X)$$
is finitely generated, so $K(\pi_n(X), n)$ has homology of finite type.
- By the two-out-of-three theorem for finite type, applied to the homotopy fiber sequence
$$\tau_{>n}X \longrightarrow \tau_{\geq n}X \longrightarrow K(\pi_n(X), n)\,,$$
it follows that $\tau_{>n}X$ has homology of finite type, completing the inductive step.
- In the course of the proof, we also showed that $\pi_n(X)$ is finitely generated, for each $n \geq 2$, so X has (homotopy of) finite type.

Proof ($\Longleftarrow$)

- Suppose X is 1-connected (with homotopy) of finite type.
- Let $n \geq 2$ and consider the map $p \colon X \to \tau_{\leq n} X$ to the n-th Postnikov section.
- It induces an isomorphism on π_m for $m \leq n$, and a surjection for $m = n + 1$, hence is $(n + 1)$-connected.
- By the relative Hurewicz theorem, it follows that $p_* \colon H_n(X) \to H_n(\tau_{\leq n} X)$ is an isomorphism.
- It therefore suffices to prove that $\tau_{\leq n} X$ has homology of finite type.
- This follows by a finite induction from the two-out-of-three theorem for finite type, applied to the homotopy fiber sequences

$$K(\pi_m(X), m) \longrightarrow \tau_{\leq m} X \longrightarrow \tau_{< m} X \,,$$

since each space $K(\pi_m(X), m)$ has homology of finite type. □

Agreement about degreewise finiteness

Theorem

Let X be a 1-connected space. Then $\tilde{H}_n(X)$ is finite for each n if and only if $\pi_n(X)$ is finite for each n.

Proof.

In the proof of the previous theorem, replace "finitely generated" homology or homotopy by "finite" reduced homology or homotopy. □

Rational homology of $K(\mathbb{Z}, n)$ for n odd

Using the multiplicative structure in the cohomology Serre spectral sequence, we will make the following calculation.

Theorem (Serre)

Let $n \geq 1$ be odd. Then

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is clear for $n = 1$, since $K(\mathbb{Z}, 1) \simeq S^1$.

Homology of $K(\mathbb{Z}, n)$ for n odd

Granting this, we can make the following deductions.

Corollary

Let $n \geq 1$ be odd, and let $f: S^n \to K(\mathbb{Z}, n)$ represent a generator of $\pi_n K(\mathbb{Z}, n) \cong \mathbb{Z}$. Then

1. $H_0(K(\mathbb{Z}, n)) \cong \mathbb{Z}$.
2. $H_i(K(\mathbb{Z}, n)) = 0$ *for* $0 < i < n$.
3. $f_*: H_n(S^n) \to H_n(K(\mathbb{Z}, n))$ *is an isomorphism.*
4. $H_i(K(\mathbb{Z}, n))$ *is finite for each* $i > n$.

Proof.

Cases (1), (2) and (3) follow from the Hurewicz theorem. Case (4) follows from finite generation and the rational calculation, since a finitely generated abelian group of rank 0 is finite. □

Serre's finiteness theorem for odd spheres

Theorem ([Ser51, Prop. V.3])

Let $n \geq 1$ be odd.
Then $\pi_i(S^n)$ is finite for each $i > n$.

Proof

- The case $n = 1$ is well known, so we assume $n \geq 3$.
- Replace the map $f\colon S^n \to K(\mathbb{Z}, n)$ by an equivalent Hurewicz fibration $p\colon E \to B$ with fiber F.
- There is then a homotopy fiber sequence

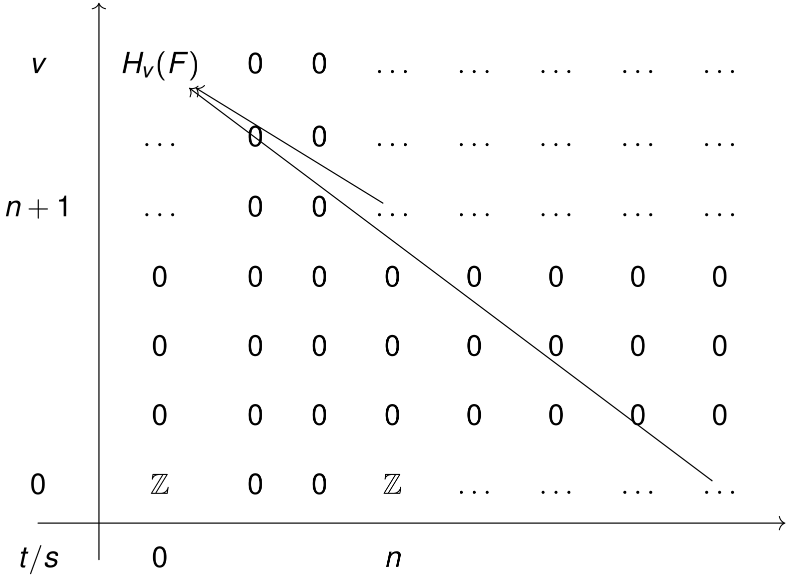
$$F \longrightarrow S^n \xrightarrow{f} K(\mathbb{Z}, n)\,,$$

  where $F = \tau_{>n} S^n$ is the n-connected cover of S^n.
- In particular, $\tilde{H}_t(F) = 0$ for $t \leq n$.
- We claim that $H_t(F)$ is finite for each $t > n$.
- If this is not the case, there is a minimal $v > n$ such that $H_v(F)$ is infinite.
- Consider the Serre spectral sequence

$$E^2_{s,t} = H_s(K(\mathbb{Z}, n); H_t(F)) \Longrightarrow_s H_{s+t}(S^n)$$

  with $E^2_{0,v} \cong H_v(F)$.

Serre sp. seq. for $F \to S^n \to K(\mathbb{Z}, n)$



- By assumption, each group $E^2_{s,t}$ is finite for $t < v$, except when $(s,t) = (0,0)$ or $(n,0)$.
- Each differential

$$d^r_{r,v-r+1} \colon E^r_{r,v-r+1} \longrightarrow E^r_{0,v}$$

  therefore maps from a finite group.
- It follows by a finite induction that $E^\infty_{0,v}$ is infinite.
- Since this group maps injectively to $H_v(S^n) = 0$, we have a contradiction.
- By the agreement theorem for degreewise finiteness, it follows that $\pi_t(F)$ is finite for each $t > n$.
- The conclusion then follows from the isomorphisms $\pi_t(F) \cong \pi_t(S^n)$, valid for this range of values of t. □

Serre's finiteness theorem for even spheres

Theorem ([Ser51, Cor. V.2])

Let $n \geq 2$ be even.
Then $\pi_i(S^n)$ is finite for each $i > n$, except for $i = 2n - 1$, and $\pi_{2n-1}(S^n)$ is the direct sum of $\mathbb{Z}$ and a finite group.

In other words, $\pi_{2n-1}(S^n)$ is finitely generated of rank 1.

Proof

- The Puppe sequence for $f\colon S^n \to K(\mathbb{Z}, n)$ extends to the left, to define a homotopy fiber sequence

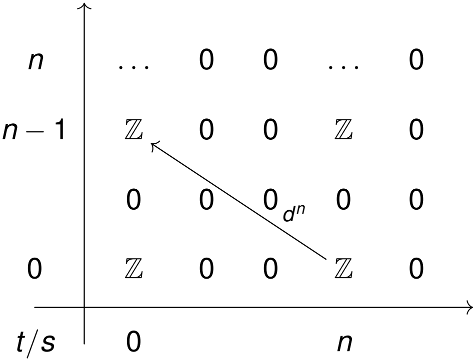
$$K(\mathbb{Z}, n-1) \longrightarrow F \longrightarrow S^n \,,$$

  where $F = \tau_{>n} S^n$ is the n-connected cover of S^n.
- The associated Serre spectral sequence

$$E^2_{s,t} = H_s(S^n; H_t(K(\mathbb{Z}, n-1))) \Longrightarrow_s H_{s+t}(F)$$

  is concentrated in the two columns $s \in \{0, n\}$.
- The entries with $t \in \{0, n-1\}$ are isomorphic to $\mathbb{Z}$, the entries with $0 < t < n-1$ are trivial, and the entries with $t \geq n$ are finite.

Serre sp. seq. for $K(\mathbb{Z}, n-1) \to F \to S^n$



Proof (cont.)

- Since the abutment is n-connected, the differential

$$d^n_{n,0} \colon H_n(S^n) \longrightarrow H_{n-1}(K(\mathbb{Z}, n-1))$$

  is an isomorphism.
- Hence the E^∞-term is $\mathbb{Z}$ in bidegrees $(0, 0)$ and $(n, n-1)$, finite in bidegrees $(0, t)$ and (n, t) for $t \geq n$, and trivial otherwise.
- It follows that $H_i(F)$ is finite for each $i > n$, except for $i = 2n - 1$, and $H_{2n-1}(F)$ the direct sum of $\mathbb{Z}$ and a finite group.

Proof (cont.)

- By the universal coefficient theorem, $H^{2n-1}(F)$ is the direct sum of $\mathbb{Z}$ and a finite group.
- Using the Eilenberg–Mac Lane representability theorem for cohomology, there is a map $f' \colon F \longrightarrow K(\mathbb{Z}, 2n-1)$ representing an element of infinite order in $H^{2n-1}(F)$, so that
$$f'_* \colon H_{2n-1}(F) \longrightarrow H_{2n-1}(K(\mathbb{Z}, 2n-1)) \cong \mathbb{Z}$$
has finite kernel and cokernel.
- (We may arrange that the cokernel is trivial.)
- Note that $2n-1$ is odd, so the corollary on the homology of Eilenberg–Mac Lane spaces applies to $H_*(K(\mathbb{Z}, 2n-1))$.

Proof (cont.)

- Let F' be the homotopy fiber of f', so that we have a homotopy fiber sequence

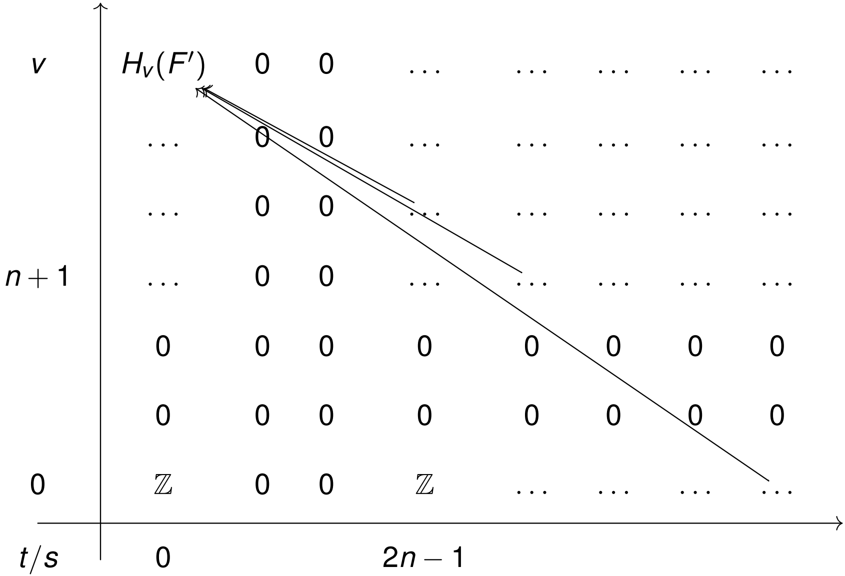
$$F' \longrightarrow F \xrightarrow{f'} K(\mathbb{Z}, 2n-1)\,.$$

- Note that F' is at least n-connected.
- We claim that $H_t(F')$ is finite for each $t > n$.
- If this is not the case, there is a minimal $v > n$ such that $H_v(F')$ is infinite.
- Consider the Serre spectral sequence

$$E^2_{s,t} = H_s(K(\mathbb{Z}, 2n-1); H_t(F')) \Longrightarrow_s H_{s+t}(F)$$

with $E^2_{0,v} \cong H_v(F')$.

Serre sp. seq. for $F' \to F \to K(\mathbb{Z}, 2n-1)$



Proof (cont.)

- Each differential

$$d^r_{r,v-r+1}\colon E^r_{r,v-r+1} \longrightarrow E^r_{0,v}$$

  maps from a finite group, except if $v = 2n - 2$ and $r = 2n - 1$.
- In the exceptional case, the subgroup $E^\infty_{2n-1,0}$ of $E^2_{2n-1,0} \cong H_{2n-1}(K(\mathbb{Z}, 2n-1))$ equals the image of the edge homomorphism f'_*, which has finite index, so also $d^{2n-1}_{2n-1,0}$ must have finite image.
- It follows that the quotient $E^\infty_{0,v}$ of $E^2_{0,v} \cong H_v(F')$ must be infinite.

Proof (cont.)

- Since $E^{\infty}_{0,v}$ is isomorphic to the image of $H_v(F') \to H_v(F)$, which is contained in the kernel of

$$f'_*\colon H_v(F) \longrightarrow H_v(K(\mathbb{Z}, 2n-1)),$$

this contradicts the calculation of $H_*(F)$ and the fact that f'_* has finite kernel.
- By the agreement theorem for degreewise finiteness, it follows that $\pi_t(F')$ is finite for each $t > n$.
- This implies that $\pi_t(F)$ is finite for each $t > n$, except for $t = 2n-1$, and that $\pi_{2n-1}(F)$ is the direct sum of $\mathbb{Z}$ and a finite group.
- The conclusion then follows from the isomorphism $\pi_t(F) \cong \pi_t(S^n)$, valid for $t > n$. □

Outline

Ring structure

- The cohomology groups of a space X come equipped with a cup product, derived from the diagonal map $\Delta\colon X \to X \times X$, which make $H^*(X)$ a graded commutative ring.
- The corresponding “coring” structure in $H_*(X)$ is less familiar, and requires flatness hypotheses to be dealt with in purely algebraic terms.
- We will see that some spectral sequences converging to $H^*(X)$ respect the cup product structure in a suitable manner, and this turns out to be a powerful calculational tool.
- In particular, this ring structure is what Leray referred to when calling the objects he studied “anneau spectral”, or “spectral rings”.

Cohomlogical grading

- Since the first examples of spectral sequences with multiplicative structure arise from cohomology, we first discuss cohomologically graded spectral sequences.
- This amounts to the usual convention of writing a graded abelian group G_* as a cograded abelian group G^*, where $G^s = G_{-s}$.
- If (C_*, ∂) is a chain complex then (C^*, δ) is the cochain complex with $\delta\colon C^s \to C^{s+1}$ given by $\partial\colon C_{-s} \to C_{-s-1}$.
- The r-th term of a spectral sequence will therefore be written in cohomological notation as $E_r^{s,t} = E^r_{-s,-t}$.

Pairings of spectral sequences

- Thereafter we discuss pairings of spectral sequences, and ring spectral sequences.
- These can be seen to arise from pairings of exact couples, but a more useful formalism is a richer structure called a Cartan–Eilenberg system.
- Each Cartan–Eilenberg system gives rise to an exact couple and a spectral sequence, and a pairing of Cartan–Eilenberg systems gives rise to a pairing of (exact couples and) spectral sequences.

Serre and Adams spectral sequences

- This applies, in particular, to the cohomological Serre spectral sequence of a fibration $F \to E \to B$.
- The resulting ring structure implies a close relationship between the graded commutative cohomology rings $H^*(F)$, $H^*(E)$ and $H^*(B)$.
- When we come to the Adams spectral sequence, we will also see that a pairing of spectra gives rise to a pairing of Adams spectral sequences.
- For a ring spectrum representing a multiplicative cohomology theory, the Adams spectral sequence converges to the graded coefficient ring of that cohomology theory.

Outline

Differential cohomologically bigraded abelian groups

Definition

A cohomologically bigraded abelian group $A^{*,*}$ is a doubly-indexed sequence

$$A^{*,*} = (A^{s,t})_{s,t}$$

of abelian groups, where $s, t \in \mathbb{Z}$.

A morphism $f \colon A^{*,*} \to B^{*,*}$ of (cohomological) bidegree (u, v) is a sequence of homomorphisms

$$f^{s,t} \colon A^{s,t} \longrightarrow B^{s+u,t+v} .$$

A morphism $d \colon E^{*,*} \to E^{*,*}$ is a differential if $dd = 0$.

Cohomological spectral sequences

Definition

Let $p \in \mathbb{Z}$. A cohomological E_p-spectral sequence $(E_r, d_r)_{r \geq p}$ is a sequence of bigraded abelian groups $E_r = E_r^{*,*}$ and differentials

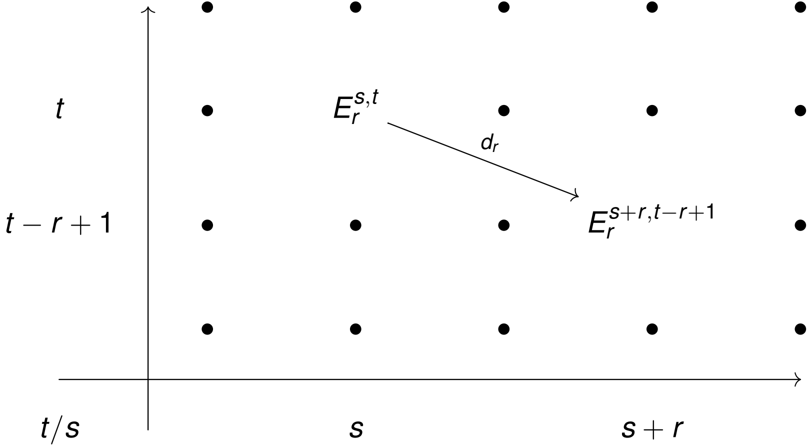
$$d_r \colon E_r^{*,*} \longrightarrow E_r^{*,*}$$

of bidegree $(r, 1-r)$, together with isomorphisms

$$H(E_r, d_r) \cong E_{r+1}$$

of bigraded abelian groups, for all $r \geq p$.

E_r-term and d_r-differential



We call E_r the E_r-term and d_r the d_r-differential.

Filtration, complementary and total degrees

- In $E_r^{s,t}$ we call s the filtration degree, t the complementary degree and $s+t$ the total degree.
  (We could say "codegree", but this gets cumbersome.)
- Note that
  $$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$$
  increases the filtration degree by r and increases the total degree by 1.
- Hence
  $$H^{s,t}(E_r, d_r) = \frac{\ker(d_r^{s,t})}{\operatorname{im}(d_r^{s-r,t+r-1})}$$
  is the cohomology at the center of the diagram
  $$E_r^{s-r,t+r-1} \xrightarrow{d_r^{s-r,t+r-1}} E_r^{s,t} \xrightarrow{d_r^{s,t}} E_r^{s+r,t-r+1} .$$

Homological vs. cohomological indexing

- Each homological spectral sequence $(E^r, d^r)_{r\geq p}$ can be viewed as a cohomological spectral sequence $(E_r, d_r)_{r\geq p}$ with
$$E_r^{s,t} = E^r_{-s,-t}$$
and
$$d_r^{s,t} = d^r_{-s,-t}$$
for all $r \geq p$ and $s, t \in \mathbb{Z}$.
- Note that the sign of r is not reversed.
- Conversely, each cohomological spectral sequence can be viewed as a homological spectral sequence.

Morphisms

A morphism $\phi\colon E \to {}'E$ of cohomological E_p-spectral sequences is a sequence of degree-preserving morphisms

$$\phi_r\colon E_r \longrightarrow {}'E_r$$

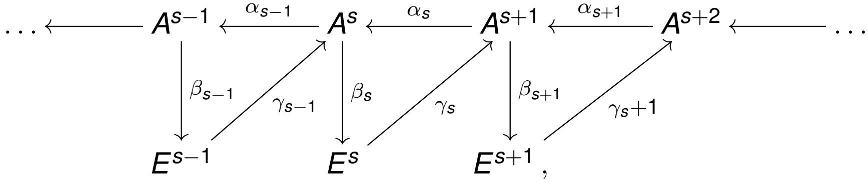
for each $r \geq p$, such that $\phi_r d_r = {}'d_r \phi_r$, and such that the induced morphism $\phi_r^*\colon H(E_r, d_r) \to H({}'E_r, {}'d_r)$ corresponds to $\phi_{r+1}\colon E_{r+1} \to {}'E_{r+1}$.

$$\begin{array}{ccc} H(E_r, d_r) & \xrightarrow{\phi_r^*} & H({}'E_r, {}'d_r) \\ \cong\downarrow & & \downarrow\cong \\ E_{r+1} & \xrightarrow{\phi_{r+1}} & {}'E_{r+1} \end{array}$$

Cohomological exact couple

Definition

A cohomological unrolled exact couple $(A, E) = (A^s, E^s)_s$ is a diagram of the form



in which each triangle forms a long exact sequence

$$\cdots \to A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\beta_s} E^s \xrightarrow{\gamma_s} A^{s+1} \to \cdots .$$

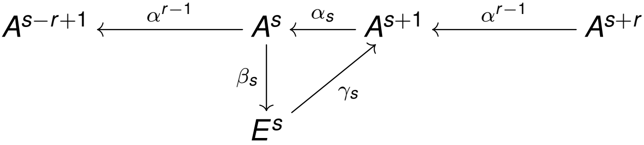
Here each A^s and E^s is a cohomologically graded abelian group, and α_s, β_s and γ_s are graded morphisms of graded abelian groups.

Remarks

- It might be better to write α^s in place of α_s, but we also want to write α^{r-1} for the iterated map, which could then be confusing.
- Each (homological) exact couple $(A_s, E_s)_s$ can be viewed as a cohomological exact couple $(A^s, E^s)_s$ with $A^s = A_{-s}$ and $E^s = E_{-s}$, and vice versa.

Cocycles and coboundaries

The following diagram sits inside the cohomological unrolled exact couple.



Definition

For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$Z_r^s = \gamma_s^{-1} \operatorname{im}(\alpha^{r-1} \colon A^{s+r} \to A^{s+1})$$
$$B_r^s = \beta_s \ker(\alpha^{r-1} \colon A^s \to A^{s+r-1})$$

be the r-th cocycle and coboundary groups in filtration s.

(E_r, d_r) and E_∞

- Let $Z^s_\infty = \bigcap_r Z^s_r$ and $B^s_\infty = \bigcup_r B^s_r$, and let

$$E^s_r = Z^s_r / B^s_r$$

for all $1 \leq r \leq \infty$.

- There are inclusions

$$0 = B^s_1 \subset \cdots \subset B^s_r \subset \cdots \subset B^s_\infty \subset \ker(\gamma_s) \subset Z^s_\infty \subset \cdots \subset Z^s_r \subset \cdots \subset Z^s_1 = E^s$$

- There is a differential

$$\begin{aligned} d^s_r \colon E^s_r &\longrightarrow E^{s+r}_r \\ [x] &\longmapsto [\beta_{s+r}(y)] \end{aligned}$$

where $\gamma_s(x) = \alpha^{r-1}(y)$.

- There are isomorphisms $H^s(E_r, d_r) \cong E^s_{r+1}$. This defines the spectral sequence associated to the exact couple.

Decreasing filtration of colimit

Definition

We give

$$A^{-\infty} = \operatorname{colim}_s A^s$$

the decreasing filtration

$$A^{-\infty} \supset \cdots \supset F^s A^{-\infty} \supset F^{s+1} A^{-\infty} \supset \ldots$$

with

$$F^s A^{-\infty} = \operatorname{im}(A^s \longrightarrow A^{-\infty}) \,.$$

Definition

We say that the exact couple $(A^s, E^s)_s$ is degreewise discrete if each $\alpha_s \colon A^{s+1} \to A^s$ preserves the total degree, and for each n there is an integer $b(n)$ such that $(A^s)^n = 0$ for $s > b(n)$.

(This is where “bounded below” could get confusing.)

Degreewise discrete convergence

Proposition

(1) There is an injective homomorphism

$$\zeta\colon \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \longrightarrow E^s_\infty \,,$$

which is an isomorphism if $Z^s_\infty = \ker(\gamma_s)$*.*

(2) If the exact couple is degreewise discrete, then ζ *is an isomorphism and the spectral sequence*

$$E_r^{s,t} \Longrightarrow_s (A^{-\infty})^{s+t}$$

converges.

Filtered cochain complexes

- A decreasing filtration of a cochain complex $C^* = (C^*, \delta)$ is a sequence of subcomplexes

$$C^* \supset \cdots \supset F^s C^* \supset F^{s+1} C^* \supset \dots .$$

- We refer to the grading n of $C^* = (C^n)_n$ and $F^s C^* = (F^s C^n)_n$ as the total degree, and to s as the filtration degree.
- We set $n = s + t$, where t is the complementary degree.
- For each s there is a short exact sequence of cochain complexes

$$0 \to F^{s+1} C^* \longrightarrow F^s C^* \longrightarrow \frac{F^s C^*}{F^{s+1} C^*} \to 0 .$$

- We call $(F^s C^* / F^{s+1} C^*)_s = (F^s C^n / F^{s+1} C^n)_{s,n}$ the associated (bi-)graded abelian group of the filtration.

Exhaustive, degreewise discrete

Definition

The filtration

$$C^* \supset \cdots \supset F^s C^* \supset F^{s+1} C^* \supset \dots .$$

is exhaustive if

$$C^* = \bigcup_s F^s C^* .$$

It is degreewise discrete if for each n there is a finite $b = b(n)$ such that $F^{b+1} C^n = 0$.

Associated exact couple

The exact couple associated to a filtered cochain complex $(F^s C^*)_s$ is the diagram

$$
\begin{array}{ccccccc}
\cdots & \longleftarrow & H^*(F^s C^*) & \xleftarrow{\alpha_s} & H^*(F^{s+1} C^*) & \longleftarrow & \cdots \\
 & & \Big\downarrow \beta_s & \overset{\gamma_s}{\nearrow} & & & \\
 & & H^*(F^s C^* / F^{s+1} C^*) & & & &
\end{array}
$$

where

$$
\begin{aligned}
(A^s)^* &= H^*(F^s C^*) \\
(E^s)^* &= H^*(F^s C^* / F^{s+1} C^*) \,.
\end{aligned}
$$

Here α_s and β_s preserve the total degree n, while γ_s increases it by 1.

Associated spectral sequence

- The bigrading is given by

$$A^{s,t} = H^{s+t}(F^s C^*)$$
$$E^{s,t} = H^{s+t}(F^s C^*/F^{s+1} C^*)\,.$$

- The associated spectral sequence has

$$E_1^{s,t} = H^{s+t}(F^s C^*/F^{s+1} C^*)\,,$$

and

$$d_1^{s,t} = \beta_{s+1}\gamma_s \colon E_1^{s,t} \longrightarrow E_1^{s+1,t}$$

equals the connecting homomorphism in the long exact cohomology sequence associated to the extension

$$0 \to F^{s+1}C^*/F^{s+2}C^* \longrightarrow F^s C^*/F^{s+2}C^* \longrightarrow F^s C^*/F^{s+1}C^* \to 0$$

of cochain complexes.

Decreasing filtration of cohomology

Definition

Given a filtration $(F^s C^*)_s$ of a cochain complex $C^* = (C^*, \delta)$, let

$$F^s H^*(C^*) = \mathrm{im}(H^*(F^s C^*) \to H^*(C^*))\,.$$

This defines a decreasing filtration

$$\cdots \supset F^s H^*(C^*) \supset F^{s+1} H^*(C^*) \supset \ldots$$

of the graded abelian group $H^*(C^*)$.

Convergence

Definition

- Let $(F^s G^*)_s$ be a decreasing filtration of a graded abelian group $G^* = (G^n)_n$.
- Suppose that the filtration is exhaustive and degreewise discrete.
- A cohomological spectral sequence $(E_r, d_r)_{r \geq p}$ converges to G^*, written
$$E_r^{s,t} \Longrightarrow_s G^{s+t},$$
if there are isomorphisms
$$E_\infty^{s,t} \cong \frac{F^s G^{s+t}}{F^{s+1} G^{s+t}}$$
for all (s, t).

Convergence for degreewise discrete filtered cochain complex

Proposition

If $(F^s C^*)_s$ exhausts $C^* = (C^*, \delta)$ and is degreewise discrete, then the spectral sequence

$$E_1^{s,t} = H^{s+t}(F^s C^* / F^{s+1} C^*) \Longrightarrow_s H^{s+t}(C^*)$$

converges to $H^*(C^*)$ with the decreasing filtration $(F^s H^*(C^*))_s$.

Outline

Towers of spaces

- Given a sequence of spaces

$$Y \longrightarrow \dots \longrightarrow Y_{s-1} \xrightarrow{f_s} Y_s \longrightarrow \dots \longrightarrow$$

  it is a nontrivial hypothesis on the maps f_s that the induced diagram

$$C_*(Y) \longrightarrow \dots \longrightarrow C_*(Y_{s-1}) \longrightarrow C_*(Y_s) \longrightarrow \dots \longrightarrow$$

  consists of surjective homomorphisms.
- This would, however, suffice to ensure that the dual diagram

$$C^*(Y) \supset \cdots \supset C^*(Y_{s-1}) \supset C^*(Y_s) \supset \dots$$

  defines a decreasing filtration of $C^*(Y)$, when suitably indexed.
- A more convenient framework is given by working with relative cochain complexes.

Relative cochains on a filtered space

- Let
$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$
be an increasing filtration of a space X.
- The singular cochain complex $C^*(X)$ then has the decreasing filtration
$$C^*(X) \supset \cdots \supset C^*(X, X_{s-1}) \supset C^*(X, X_s) \supset \ldots$$
with
$$F^s C^*(X) = C^*(X, X_{s-1})$$
$$\frac{F^s C^*(X)}{F^{s+1} C^*(X)} \cong C^*(X_s, X_{s-1})\,.$$
- If $X_{a-1} = \emptyset$ for some a, then $C^*(X) = C^*(X, X_{a-1})$ and the filtration is exhaustive.
- If $X_b = X$ for some b then $C^*(X, X_b) = 0$ and the filtration is (degreewise) discrete.

Remarks

- Note the index shift in the definition of $F^sC^*(X)$, which gives the convenient form

$$0 \to C^*(X, X_s) \longrightarrow C^*(X, X_{s-1}) \longrightarrow C^*(X_s, X_{s-1}) \to 0$$

  for the short exact sequence defining the associated graded of the filtration.
- The hypothesis that $C^n(X, X_{s-1})$ vanishes for sufficiently large s (possibly depending on n) is often not realistic.
- Recall from an earlier proposition that for convergence we only need that the exact couple is degreewise discrete, i.e., that $H^n(X, X_{s-1}) = 0$ for s sufficiently large, and this is satisfied in many cases.

Cohomology spectral sequence of a filtered spaces

Proposition

- *Let* $(X_s)_s$ *be a filtration of* X*.*
- *There is a cohomological spectral sequence*

$$E_1^{s,t} = H^{s+t}(X_s, X_{s-1}) \Longrightarrow_s H^{s+t}(X)$$

with $d_1 \colon E_1^{s,t} \to E_1^{s+1,t}$ *equal to the connecting homomorphism in the long exact cohomology sequence of the triple* (X_{s+1}, X_s, X_{s-1})*.*
- *If* $X_{a-1} = \emptyset$ *for some a, and* $H^n(X, X_{s-1}) = 0$ *for all* $s \geq b(n)$*, for some* $b(n)$ *depending on n, then the spectral sequence converges to* $H^{s+t}(X)$*, with the filtration*

$$\begin{aligned} F^s H^*(X) &= \mathrm{im}(H^*(X, X_{s-1}) \to H^*(X)) \\ &= \ker(H^*(X) \to H^*(X_{s-1}))\,. \end{aligned}$$

Proof

- This is the spectral sequence associated to the exact couple associated to the decreasing filtration of $C^*(X)$ given by $F^s C^*(X) = C^*(X, X_{s-1})$.
- The additional hypotheses ensure that the exact couple is discrete and that $F^a A^{-\infty} = A^{-\infty}$ is exhaustively filtered.
- The second expression for $F^s H^*(X)$ is clear from the long exact cohomology sequence

$$\cdots \to H^*(X, X_{s-1}) \longrightarrow H^*(X) \longrightarrow H^*(X_{s-1}) \to \dots$$

of (X, X_{s-1}). $\square$

The cohomology LES of a pair, as spectral sequence

For a pair of spaces (X, A), we can view the long exact cohomology sequence

$$\cdots \to H^n(X, A) \longrightarrow H^n(X) \longrightarrow H^n(A) \xrightarrow{\delta^n} H^{n+1}(X, A) \to \dots$$

as a cohomological two-column spectral sequence, with

$$E_1^{s,t} = \begin{cases} H^t(A) & \text{for } s = 0, \\ H^{1+t}(X, A) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_1^{0,t} = \delta^t \colon H^t(A) \longrightarrow H^{1+t}(X, A)\,.$$

This corresponds to the bounded filtration with $X_{-1} = \emptyset$, $X_0 = A$ and $X_1 = X$.

(E_1, d_1)-term

The (E_1, d_1)-term is shown below.

$$
\begin{array}{c|ll}
 & \vdots & \vdots \\
t & H^t(A) \xrightarrow{\delta^t} & H^{1+t}(X, A) \\
 & \vdots & \vdots \\
1 & H^1(A) \xrightarrow{\delta^1} & H^2(X, A) \\
0 & H^0(A) \xrightarrow{\delta^0} & H^1(X, A) \\
\hline
t/s & 0 & 1
\end{array}
$$

E_2-term

This leads to the following $E_2 = E_\infty$-term.

	$\vdots$	$\vdots$
t	$\ker(\delta^t)$	$\mathrm{cok}(\delta^t)$
	$\vdots$	$\vdots$
1	$\ker(\delta^1)$	$\mathrm{cok}(\delta^1)$
0	$\ker(\delta^0)$	$\mathrm{cok}(\delta^0)$
t/s	0	1

Extensions

- The groups $E_\infty^{s,n-s}$ give the associated graded of the decreasing filtration

$$H^*(X) = F^0 H^n(X) \supset F^1 H^n(X) \supset 0\,,$$

with $F^1 H^n(X) = \mathrm{im}(H^n(X,A) \to H^n(X)) = \ker(H^n(X) \to H^n(A))$.

- Hence $F^1 H^n(X) \cong E_\infty^{1,n-1}$, and there is a short exact sequence

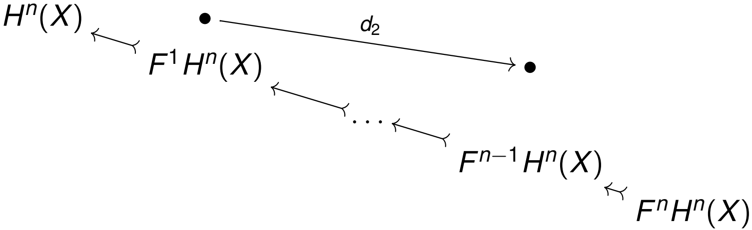
$$0 \to F^1 H^n(X) \longrightarrow H^n(X) \longrightarrow E_\infty^{0,n} \to 0\,.$$

- This is the same extension as that obtained from the long exact cohomology sequence, namely

$$0 \to \mathrm{cok}(\delta^{n-1}) \longrightarrow H^n(X) \longrightarrow \ker(\delta^n) \to 0\,.$$

Directions of differentials and extensions

Note that in the cohomological spectral sequence the differentials map from the left to the right, while the filtration inclusions $F^{s+1}H^n(X) \subset F^sH^n(X)$ map from the right to the left, when we view them as placed in total degree n, with filtration quotients identified with the components of the E_∞-term.



Cellular cohomology

Proposition

- *Let X be a CW complex, equipped with the skeleton filtration.*
- *The associated cohomology spectral sequence*

$$E_1^{s,t} = H^{s+t}(X^{(s)}, X^{(s-1)}) \Longrightarrow_s H^{s+t}(X)$$

  has (E_1, d_1)-term equal to the cellular cocomplex $(C^*_{CW}(X), \partial)$, and E_2-term equal to the cellular cohomology $H^*_{CW}(X)$.
- *Both are concentrated on the line $t = 0$.*
- *It collapses at $E_2 = E_\infty$, and converges to $H^*(X)$.*
- *Hence $H^*_{CW}(X) \cong H^*(X)$.*

Proof

- The d_1-differential equals the connecting homomorphism in the long exact cohomology sequence of the triple $(X^{(s+1)}, X^{(s)}, X^{(s-1)})$, by naturality with respect to the vertical map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(X, X^{(s)}) & \longrightarrow & C^*(X, X^{(s-1)}) & \longrightarrow & C^*(X^{(s)}, X^{(s-1)}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & C^*(X^{(s+1)}, X^{(s)}) & \longrightarrow & C^*(X^{(s+1)}, X^{(s-1)}) & \longrightarrow & C^*(X^{(s)}, X^{(s-1)}) & \longrightarrow & 0 \end{array}$$

  of short exact sequences of cochain complexes.
- Convergence follows from $X^{(-1)} = \emptyset$ and $H^n(X, X^{(s-1)}) = 0$ for all $s > n$, which we can deduce from $H_n(X, X^{(s-1)}) = 0$ for $s > n$ using the universal coefficient theorem. □

Generalized cohomology

Definition

A (generalized) cohomology theory M on the category of CW pairs is a contravariant functor assigning to each CW pair (X, A) a graded abelian group

$$M^*(X, A) = (M^n(X, A))_n \, ,$$

and a natural transformation

$$\delta \colon M^*(A) \longrightarrow M^{*+1}(X, A)$$

of degree $+1$, such that

Definition (cont.)

1. Exactness: the sequence

$$\cdots \to M^*(X,A) \xrightarrow{j^*} M^*(X) \xrightarrow{i^*} M^*(A) \xrightarrow{\delta} M^{*+1}(X,A) \to \dots$$

   is long exact.
2. Homotopy invariance: if $f \simeq g\colon (X,A) \to (Y,B)$ are homotopic, then $f^* = g^*$.
3. Excision: if $X = A \cup B$ is a union of subcomplexes, then the inclusion induces an isomorphism

$$M^*(X,A) \xrightarrow{\cong} M^*(B, A \cap B)\,.$$

4. Additivity: the canonical map

$$M^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha M^*(X_\alpha)$$

   is an isomorphism.

Coefficient groups

The coefficient groups of a cohomology theory M is the (cohomologically) graded abelian group

$$M^* = (M^n(\text{point}))_n \, .$$

There are isomorphisms

$$M^{s+t}(D^s, \partial D^s) \cong \tilde{M}^{s+t}(S^s) \cong M^t$$

for all $s \geq 0$, $t \in \mathbb{Z}$.

Atiyah–Hirzebruch exact couple

Let X be a CW complex. Applying M^* to the triples $(X, X^{(s)}, X^{(s-1)})$ we obtain the exact couple

$$\begin{array}{ccccccc} \cdots \longleftarrow & M^*(X, X^{(s-1)}) & \xleftarrow{\alpha_s} & M^*(X, X^{(s)}) & \longleftarrow \cdots \\ & \downarrow \beta_s & \nearrow \gamma_s & & \\ & M^*(X^{(s)}, X^{(s-1)}) & & & \end{array} \tag{1}$$

with

$$A^{s,t} = M^{s+t}(X, X^{(s-1)})$$
$$E^{s,t} = M^{s+t}(X^{(s)}, X^{(s-1)}) \cong C^s_{CW}(X; M^t)\,,$$

where $d_1^s \colon E_1^{s,t} \to E_1^{s+1,t}$ corresponds to

$$\delta^s \colon C^s_{CW}(X; M^t) \longrightarrow C^{s+1}_{CW}(X; M^t)\,.$$

E_2-term, exhaustive filtration

Hence

$$E_2^{s,t} = H^s_{CW}(X; M^t) \cong H^s(X; M^t) \,.$$

Since $X^{(-1)} = \emptyset$, the map

$$M^*(X) = A^{0,*} \xrightarrow{\cong} A^{-\infty,*}$$

is an isomorphism, and $M^*(X)$ is exhaustively filtered by the graded subgroups

$$\begin{aligned} F^s M^*(X) &= \mathrm{im}(M^*(X, X^{(s-1)}) \to M^*(X)) \\ &= \ker(M^*(X) \to M^*(X^{(s-1)})) \,. \end{aligned}$$

Atiyah–Hirzebruch spectral sequence

Definition

The spectral sequence

$$E_2^{s,t} = H^s(X; M^t) \Longrightarrow_s M^{s+t}(X)$$

associated to the exact couple (1) is the Atiyah–Hirzebruch spectral sequence for X and the cohomology theory M.

For now we only prove convergence when X is a finite-dimensional CW complex or M^* is bounded below, postponing the general case until we have discussed sequential limits and derived limits.

Finite-dimensional convergence

Proposition

If X is finite-dimensional, then the filtration $(F^s M^*(X))_s$ is bounded and the Atiyah–Hirzebruch spectral sequence converges to $M^*(X)$.

Proof.

- By hypothesis, there is a b such that $X = X^{(b)}$.
- For all $s > b$ we then have $A^s = M^*(X, X^{(s-1)}) = 0$ and $F^s A^{-\infty} = 0$.
- This spectral sequence is concentrated in the columns $0 \leq s \leq b$.

□

Bounded below convergence

Proposition

If M^* is bounded below, then the filtration $(F^sM^*(X))_s$ is degreewise bounded and the Atiyah–Hirzebruch spectral sequence converges to $M^*(X)$.

Proof.

- By hypothesis, there is an a such that $M^n = 0$ for all $n < a$.
- Then $M^n(D^s, \partial D^s) = 0$ for all $s > n - a$.
- Fix an n, and let $b \geq n - a$.
- Then $M^n(X^{(s)}, X^{(b)}) = 0$ and $M^{n-1}(X^{(s)}, X^{(b)}) = 0$ for all $s > b$.
- Let $Y = X/X^{(b)}$, so that $Y^{(s)} = X^{(s)}/X^{(b)}$.

Proof (cont.)

- There is a homotopy cofiber sequence

$$\bigvee_{s>b} \Sigma_+ Y^{(s)} \xrightarrow{1-\alpha} \bigvee_{s>b} \Sigma_+ Y^{(s)} \longrightarrow \Sigma_+ T$$

  where $T \simeq Y$ is the mapping telescope of $(Y^{(s)})_{s>b}$.
- The associated long exact sequence in reduced M-cohomology has the form

$$\cdots \to \prod_{s>b} \tilde{M}^{n-1}(Y^{(s)}) \xrightarrow{\delta} \tilde{M}^n(Y) \longrightarrow \prod_{s>b} \tilde{M}^n(Y^{(s)}) \xrightarrow{1-\alpha} \dots$$

  which proves that $\tilde{M}^n(Y) = M^n(X, X^{(b)}) = 0$.
- For all $s > n - a$ we then have $(A^s)^n = M^n(X, X^{(s-1)}) = 0$ and $(F^s A^{-\infty})^n = 0$.
- This spectral sequence is concentrated in the region $s \geq 0$ and $t \geq a$. □

Eilenberg–Steenrod uniqueness theorem

Theorem

Let G be an abelian group and let M be a cohomology theory with coefficient groups $M^0 = G$ and $M^t = 0$ for $t \neq 0$. Then M is naturally isomorphic to HG, so that

$$M^n(X) \cong H^n(X; G)$$

for all n.

Proof

- The coefficients M^* are bounded below (and above).
- The Atiyah–Hirzebruch spectral sequence of X for M has E_2-term
$$E_2^{s,t} = \begin{cases} H^s(X;G) & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$
- Since this is concentrated on the line $t = 0$, the d_r-differentials for $r \geq 2$ must vanish, so that $E_2 = E_\infty$ is concentrated on the line $t = 0$.
- Since $E_\infty^{n,0}$ is the only group in total degree n, the extension problems are very easy, and we conclude that $M^n(X) \cong E_\infty^{n,0} \cong H^n(X;G)$ for each n. □

Outline

Cohomological Serre spectral sequence

Definition

Let $p\colon E \to B$ be a Hurewicz fibration, with B a CW complex. Let $E_s = p^{-1}(B^{(s)})$. The (cohomological) Serre spectral sequence of $p\colon E \to B$ is the spectral sequence

$$E_1^{s,t}(p) = H^{s+t}(E_s, E_{s-1}) \Longrightarrow H^{s+t}(E)$$

associated to the exact couple

$$\begin{array}{ccccccc}
\cdots \longleftarrow & H^*(E, E_{s-1}) & \xleftarrow{\alpha_s} & H^*(E, E_s) & \longleftarrow \cdots \\
 & \downarrow{\scriptstyle \beta_s} & \nearrow{\scriptstyle \gamma_s} & & \\
 & H^*(E_s, E_{s-1}) & & &
\end{array}$$

with $A^{s,t} = H^{s+t}(E, E_{s-1})$ and $E^{s,t} = H^{s+t}(E_s, E_{s-1})$.

The Serre E_1-term

Proposition ([Whi78, XIII.4.6*])

There are natural isomorphisms

$$H^{s+t}(E_s, E_{s-1}) \cong C^s_{CW}(B; \mathscr{H}^t(F))\,,$$

where $\mathscr{H}^t(F)$ denotes the local coefficient system on B given by $H^t(F_b)$ at $b \in B$, with $F_b = p^{-1}(b)$.

If B is 1-connected, with base point b_0, then this equals the cellular s-cochains $C^s_{CW}(B; H^t(F))$ with coefficients in the abelian group $H^t(F)$, with $F = p^{-1}(b_0)$.

Sketch proof

We use the notation from the homological case.
By excision and additivity we have isomorphisms

$$H^{s+t}(E_s, E_{s-1}) \cong H^{s+t}(\coprod_\alpha \Phi_\alpha^* E, \coprod_\alpha \phi_\alpha^* E) \cong \prod_\alpha H^{s+t}(\Phi_\alpha^* E, \phi_\alpha^* E)\,.$$

By fiber homotopy triviality of $\Phi_\alpha^* E \to D_\alpha^s$ we have isomorphisms

$$H^s(D_\alpha^s, \partial D_\alpha^s) \otimes H^t(F_{b_\alpha}) \cong H^{s+t}((D_\alpha^s, \partial D_\alpha^s) \times F_{b_\alpha}) \cong H^{s+t}(\Phi_\alpha^* E, \phi_\alpha^* E)\,.$$

Fixing an isomorphism

$$C_{CW}^s(B; \mathscr{H}^t(F)) \cong \prod_\alpha H^s(D_\alpha^s, \partial D_\alpha^s) \otimes H^t(F_{b_\alpha}) \qquad (2)$$

we obtain the stated E_1-term. □

First quadrant

Proposition

$$(A^s)^n = H^n(E, E_{s-1}) = 0$$

for $s > n$ and

$$(A^s)^n = H^n(E)$$

for $s \leq 0$, so the exact couple $(A^s, E^s)_s$ is degreewise bounded, and the Serre spectral sequence is concentrated in the first quadrant and converges to $H^*(E)$.

Sketch proof.

- Since $\mathscr{H}^t(F)$ is trivial for $t < 0$ we have $H^n(E_s, E_{s-1}) = 0$ for $s > n$, which implies that $H^n(E_u, E_{s-1}) = 0$ for all $u \geq s > n$.
- A mapping telescope argument then shows that $H^n(E, E_{s-1}) = 0$, as claimed.

□

The Serre d_1-differential

Proposition ([Whi78, XIII.4.8*])

The diagram

$$\begin{array}{ccc} H^{s+t}(E_s, E_{s-1}) & \xrightarrow{d_1^{s,t}} & H^{s+t+1}(E_{s+1}, E_s) \\ \downarrow \cong & & \downarrow \cong \\ C^s_{CW}(B; \mathscr{H}^t(F)) & \xrightarrow{\delta^s} & C^{s+1}_{CW}(B; \mathscr{H}^t(F)) \end{array}$$

commutes.

Remark

- Whitehead states this with $(-1)^s\delta^s$ in place of δ^s.
- To give a correct statement one must make the sign conventions more precise than we have done above.

Remark on signs (cont.)

- Working with cubes instead of discs, let us fix generators

$$\gamma_1 \in H_1(I, \partial I) \qquad \text{and} \qquad g_1 \in H^1(I, \partial I)$$

  such that $\langle g_1, \gamma_1 \rangle = 1$.
- Using the cross products

$$H_s(I^s, \partial I^s) \otimes H_u(I^u, \partial I^u) \xrightarrow{\times} H_{s+u}(I^{s+u}, \partial I^{s+u})$$
$$H^s(I^s, \partial I^s) \otimes H^u(I^u, \partial I^u) \xrightarrow{\times} H^{s+u}(I^{s+u}, \partial I^{s+u})$$

  we can define generators

$$\gamma_s = \gamma_1 \times \cdots \times \gamma_1 \in H_s(I^s, \partial I^s)$$
$$g_s = g_1 \times \cdots \times g_1 \in H^s(I^s, \partial I^s)$$

  such that $\gamma_s \times \gamma_u = \gamma_{s+u}$ and $g_s \times g_u = g_{s+u}$.

Remark on signs (cont.)

- In view of the graded commutation rule

$$\langle g_s \times g_u, \gamma_s \times \gamma_u \rangle = (-1)^{su} \langle g_s, \gamma_s \rangle \langle g_u, \gamma_u \rangle$$

  it follows that

$$\langle g_s, \gamma_s \rangle = (-1)^{s(s-1)/2} = \begin{cases} +1 & \text{for } s \equiv 0, 1 \mod 4, \\ -1 & \text{for } s \equiv 2, 3 \mod 4. \end{cases}$$

- It therefore seems best to specify (2) so that a sequence

$$(g_{s,\alpha} \otimes f_\alpha)_\alpha \in \prod_\alpha H^s(I^s_\alpha, \partial I^s_\alpha) \otimes H^t(F_{b_\alpha})$$

  is identified with the cellular cochain

$$\gamma_{s,\alpha} \longmapsto (-1)^{s(s-1)/2} f_\alpha \in C^s_{CW}(B; \mathscr{H}^t(F))$$

  where $\gamma_{s,\alpha} \in H_s(I^s_\alpha, \partial I^s_\alpha) \subset C^{CW}_s(B)$.
- It seems that Whitehead [Whi78, p. 630] instead specifies this isomorphism without the sign $(-1)^{s(s-1)/2}$, mapping $\gamma_{s,\alpha}$ to f_α, which leads to the extra sign $(-1)^s$ in the proposition above.

The Serre E_2-term

Theorem

The Serre spectral sequence

$$E_2^{s,t}(p) \Longrightarrow_s H^{s+t}(E)$$

for $F \to E \xrightarrow{p} B$ *has* E_2*-term*

$$E_2^{s,t}(p) = H^s(B; \mathscr{H}^t(F)) \,.$$

If B *is* 1*-connected, this simplifies to*

$$E_2^{s,t}(p) = H^s(B; H^t(F)) \,.$$

The fiber edge

We suppose that B is 0-reduced, with $i\colon F \to E$ the inclusion of the fiber of $p\colon E \to B$ over the base point $b_0 \in B$.

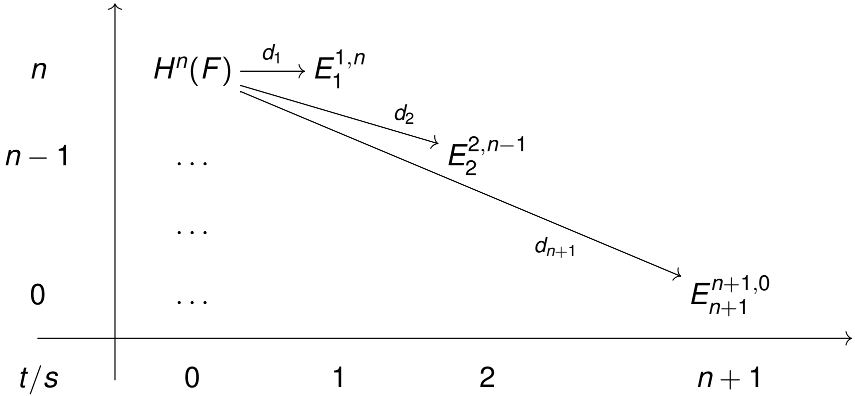
Proposition

The edge homomorphism $i^*\colon H^n(E) \to H^n(F)$ factors as the surjection

$$H^n(E) \twoheadrightarrow E_\infty^{0,n}$$

followed by the inclusion

$$E_\infty^{0,n} \rightarrowtail E_1^{0,n} \cong H^n(F)\,.$$



n
n − 1
0
t/s
0
1
2
n + 1
H^n(F)
d_1
E_1^{1,n}
d_2
E_2^{2,n−1}
d_{n+1}
E_{n+1}^{n+1,0}
. . .
. . .
. . .

The base edge

We also suppose that F is 0-connected.

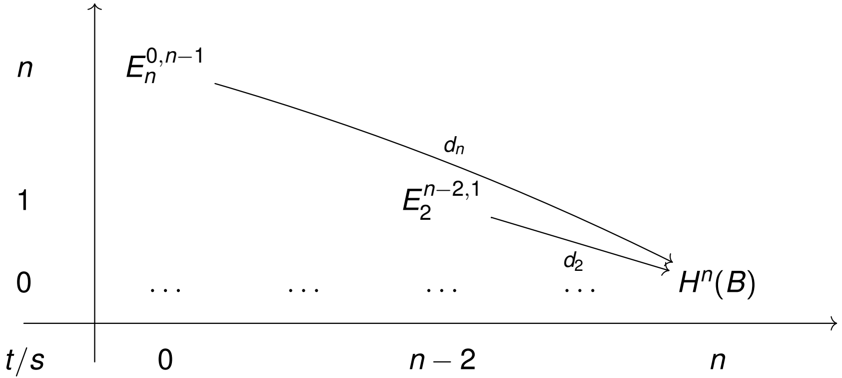
Proposition

The edge homomorphism $p^*\colon H^n(B) \to H^n(E)$ *factors as the surjection*

$$H^n(B) \cong E_2^{n,0} \longrightarrow\!\!\!\!\!\to E_\infty^{n,0}$$

followed by the inclusion

$$E_\infty^{n,0} \rightarrowtail H^n(E)\,.$$



n
1
0
t/s
$E_n^{0,n-1}$
d_n
$E_2^{n-2,1}$
d_2
. . .
. . .
. . .
. . .
$H^n(B)$
0
n − 2
n

Cohomology transgression

Definition
The additive relation

$$(q^*)^{-1}\delta\colon H^{n-1}(F) \xrightarrow{\delta} H^n(E,F) \xleftarrow{q^*} H^n(B,b_0)\,,$$

sending x with $\delta(x) = q^*(y)$ to the class of y, defines a homomorphism

$$\tau^n\colon \delta^{-1}\operatorname{im}(q^*) \longrightarrow H^n(B,b_0)/\ker(q^*)$$

called the cohomology transgression. The elements of $\delta^{-1}\operatorname{im}(q^*)$, on which τ^n are defined, are said to be transgressive.

Transgressive differential

Proposition

The transgression τ^n corresponds to the differential

$$d_n^{0,n-1} \colon E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

under isomorphisms $E_n^{0,n-1} \cong \delta^{-1}\,\mathrm{im}(q^*)$ and $E_n^{n,0} \cong H^n(B, b_0)/\ker(q^*)$.

Serre's exact cohomology sequence

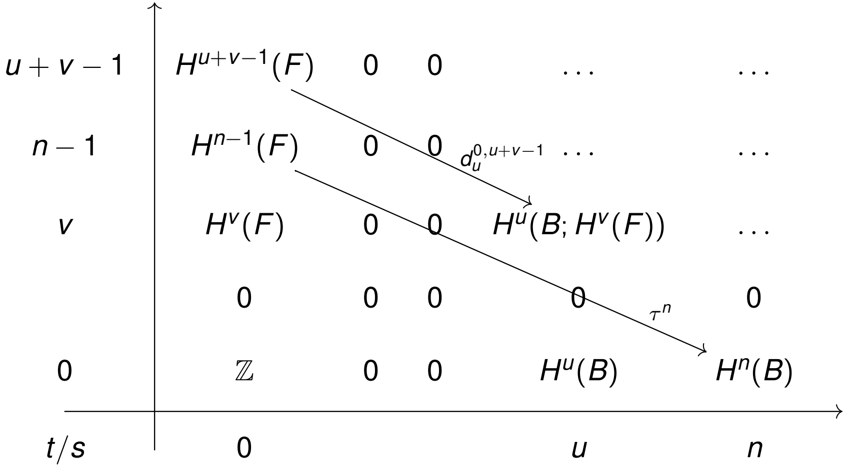
Theorem

- *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space.*
- *Suppose that $H^s(B) = 0$ for $0 < s < u$ and that $H^t(F) = 0$ for $0 < t < v$.*
- *Then there is an exact sequence*

$$0 \to H^1(E) \xrightarrow{i^*} H^1(F) \xrightarrow{\tau^2} H^2(B) \xrightarrow{p^*} \dots$$

$$\dots \xrightarrow{i^*} H^{n-1}(F) \xrightarrow{\tau^n} H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{i^*} H^n(F) \xrightarrow{\tau^{n+1}} \dots$$

$$\dots \xrightarrow{\tau^{u+v-1}} H^{u+v-1}(B) \xrightarrow{p^*} H^{u+v-1}(E) \xrightarrow{i^*} H^{u+v-1}(F)\,.$$


u + v − 1
n − 1
v
0
t/s
H^{u+v−1}(F)
H^{n−1}(F)
H^v(F)
ℤ
0
u
n
d_u^{0,u+v−1}
τ^n
H^u(B; H^v(F))
H^u(B)
H^n(B)

Outline

Comparison of cup products

- Suppose that B is 1-connected, or that $\mathscr{H}^*(F) = H^*(F)$ is a constant coefficient system.
- The cup products for B and F induce a pairing

$$H^s(B; H^t(F)) \otimes H^u(B; H^v(F)) \longrightarrow H^{s+u}(B; H^{t+v}(F)) \,.$$

- Its relationship via the Serre spectral sequence

$$E_2^{*,*} = H^*(B; H^*(F)) \Longrightarrow H^*(E)$$

  to the cup product

$$\cup \colon H^n(E) \otimes H^m(E) \longrightarrow H^{n+m}(E)$$

  can be expressed in terms of a pairing of spectral sequences

$$E_r^{s,t} \otimes E_r^{u,v} \longrightarrow E_r^{s+u,t+v} \,,$$

  making $(E_r^{*,*})_r$ a ring spectral sequence.

Reduction to indecomposables

- Such a pairing reduces the problem of calculating the d_r-differentials in $E_r^{*,*}$ to finding their values on classes that generate $E_r^{*,*}$ under this product, i.e., the ring indecomposables.
- This is often a significant reduction compared to the task of finding the values on classes that generate $E_r^{*,*}$ as a bigraded abelian group.

- We formulate the following definition for a pairing of two spectral sequences to a third, but often all three of these are the same spectral sequence.

Definition

Let $(E_r, d_r)_{r \geq p}$, $('E_r, 'd_r)_{r \geq p}$ and $(''E_r, ''d_r)_{r \geq p}$ be (cohomologically indexed) E_p-spectral sequences. A pairing of spectral sequences

$$\mu_r \colon ('E_r, ''E_r) \longrightarrow E_r$$

is a sequence of pairings

$$\mu_r \colon 'E_r^{*,*} \otimes ''E_r^{*,*} \longrightarrow E_r^{*,*}$$

for $r \geq p$, taking $'E_r^{s,t} \otimes ''E_r^{u,v}$ to $E_r^{s+u,t+v}$ for all (s, t) and (u, v), such that ...

Definition (cont.)

1. … the Leibniz rule

$$d_r(\mu_r(x \otimes y)) = \mu_r({}'d_r(x) \otimes y) + (-1)^{s+t}\mu_r(x \otimes {}''d_r(y))$$

   holds (in $E_r^{s+u+r,t+v-r+1}$) for all $x \in {}'E_r^{s,t}$ and $y \in {}''E_r^{u,v}$, and

2. the induced pairing

$$\begin{aligned} H(\mu_r)\colon H({}'E_r, {}'d_r) \otimes H({}''E_r, {}''d_r) &\longrightarrow H(E_r, d_r) \\ [x] \otimes [y] &\longmapsto [\mu_r(x \otimes y)] \end{aligned}$$

   corresponds to $\mu_{r+1}\colon {}'E_{r+1} \otimes {}''E_{r+1} \to E_{r+1}$ under the isomorphisms $H({}'E_r, {}'d_r) \cong {}'E_{r+1}$, $H({}''E_r, {}''d_r) \cong {}''E_{r+1}$ and $H(E_r, d_r) \cong E_{r+1}$.

Remarks

- The tensor product ${}'E_r^{*,*} \otimes {}''E_r^{*,*}$ of two bigraded abelian groups is itself bigraded, with the group

$$\bigoplus_{s+u=\sigma} \bigoplus_{t+v=\tau} {}'E_r^{s,t} \otimes {}''E_r^{u,v}$$

in bidegree (σ, τ).
- We thus assume that μ_r preserves this bigrading.

- The second condition implies that μ_r determines μ_{r+1}, so a pairing of E_p-spectral sequences is specified by the initial pairing μ_p.
- Not every pairing of bigraded abelian groups will satisfy the Leibniz rule, and inductively induce pairings of later E_r-terms that also satisfy the Leibniz rule, so being part of a pairing $(\mu_r)_{r \geq p}$ of spectral sequences is a significant additional hypothesis on μ_p.

Formula and diagram

- Writing $x \cdot y$ for $\mu_r(x \otimes y)$, $|x| = s + t$ for the total degree of x, and omitting primes, the Leibniz rule takes the form

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y)\,.$$

- In diagrammatic form the diagonal composite

$$\begin{array}{ccccc}
{}'E_r^{s,*} \otimes {}''E_r^{u,*} & & \xrightarrow{\;{}'d_r \otimes 1\;} & & {}'E_r^{s+r,*} \otimes {}''E_r^{u,*} \\
& \searrow^{\mu_r} & & & \\
{\scriptstyle 1 \otimes {}''d_r}\downarrow & & E_r^{s+u,*} & & \downarrow{\scriptstyle \mu_r} \\
& & & \searrow^{d_r} & \\
{}'E_r^{s,*} \otimes {}''E_r^{u+r,*} & & \xrightarrow{\;\mu_r\;} & & E_r^{s+u+r,*}
\end{array}$$

equals the sum of the two peripheral composites, ...

Sign convention

- ... under the assumption that we define $'d_r \otimes 1$ and $1 \otimes {}''d_r$ so that

$$('d_r \otimes 1)(x \otimes y) = {}'d^r(x) \otimes y$$
$$(1 \otimes {}''d_r)(x \otimes y) = (-1)^{|x|} x \otimes {}''d_r(y)\,.$$

- This is in line with the general convention that

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)\,,$$

  for bigraded homomorphisms f and g, and bigraded elements x and y, where $|g|$ denotes the total degree of g.
- In the case at hand, $|1| = 0$ while $|''d_r| = 1$.
- Note that this convention requires access to the internal grading of each object $'E_r^s$, $''E_r^u$ and E_r^{s+u}, or at least to the action by $(-1)^t$ on an element of internal degree t.

Tensor product of spectral sequences

- The sum $D_r = {}'d_r \otimes 1 + 1 \otimes {}''d_r$ defines a differential on ${}'E_r \otimes {}''E_r$, of bidegree $(r, 1-r)$.
- This does not in general make

  $$({}'E_r \otimes {}''E_r, D_r)_{r \geq p}$$

  a spectral sequence, because the cross product

  $${}'E_{r+1} \otimes {}''E_{r+1} \cong H({}'E_r, {}'d_r) \otimes H({}''E_r, {}''d_r) \xrightarrow{\times} H({}'E_r \otimes {}''E_r, D_r)$$

  is not in general an isomorphism.
- In situations where this is an isomorphism, a spectral sequence pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ is the same as a spectral sequence morphism $\mu_r \colon {}'E_r \otimes {}''E_r \to E_r$.
- This happens if $\mathrm{Tor}({}'E_r, {}''E_r) = 0$ for each r, e.g. if each ${}'E_r$ or ${}''E_r$ is torsion-free.

Homological indexing

- A pairing
$$\mu^r \colon ({}'E^r, {}''E^r) \longrightarrow E^r$$
of homologically indexed spectral sequences is defined in the same way, via the identification $E^r_{s,t} = E_r^{-s-t}$.
- The signs $(-1)^{s+t} = (-1)^{|x|} = (-1)^{-s-t}$ in the Leibniz rule then match up, independently of whether we view x as an element in the cohomological or the homological spectral sequence.

Ring spectral sequence

Definition

A ring spectral sequence is a spectral sequence $(E_r, d_r)_{r \geq p}$ equipped with a unital and associative pairing

$$\mu_r \colon (E_r, E_r) \longrightarrow E_r \, .$$

Unitality means that there is an infinite cycle $1 \in E_p^{0,0}$ such that

$$\mu_r(1 \otimes x) = x = \mu_r(x \otimes 1)$$

for all $x \in E_r$. Associativity means that the diagram

$$\begin{array}{ccc} E_r^{*,*} \otimes E_r^{*,*} \otimes E_r^{*,*} & \xrightarrow{\mu_r \otimes 1} & E_r^{*,*} \otimes E_r^{*,*} \\ \downarrow{\scriptstyle 1 \otimes \mu_r} & & \downarrow{\scriptstyle \mu_r} \\ E_r^{*,*} \otimes E_r^{*,*} & \xrightarrow{\mu_r} & E_r^{*,*} \end{array}$$

commutes, for each $r \geq p$.

Commutative ring spectral sequence

Definition

A ring spectral sequence is commutative if the diagram

$$\begin{array}{ccc} E_r^{*,*} \otimes E_r^{*,*} & \xrightarrow{\quad\tau\quad} & E_r^{*,*} \otimes E_r^{*,*} \\ & {\scriptstyle\mu_r}\searrow \quad \swarrow{\scriptstyle\mu_r} & \\ & E_r^{*,*} & \end{array}$$

commutes for each $r \geq p$, where

$$\tau(x \otimes y) = (-1)^{(s+t)(u+v)} y \otimes x$$

for $x \in E_r^{s,t}$ and $y \in E_r^{u,v}$.

Formulas

Writing $x \cdot y$ for $\mu_r(x \otimes y)$, the unitality, associativity and commutativity conditions ask that

$$\begin{aligned} 1 \cdot x = x &= x \cdot 1 \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ x \cdot y &= (-1)^{|x||y|} y \cdot x \,, \end{aligned}$$

where $|x| = s + t$ and $|y| = u + v$ denote the total degrees of x and y.

In the commutative case the Leibniz rule expressions for $d_r(x \cdot y)$ and $d_r((-1)^{|x||y|} y \cdot x)$ are equal.

Morphisms of ring spectral sequences

- Let ${}'\phi_r \colon {}'E_r \to {}'\bar{E}_r$, ${}''\phi_r \colon {}''E_r \to {}''\bar{E}_r$ and $\phi_r \colon E_r \to \bar{E}_r$ be morphisms of spectral sequences, and let $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ and $\bar{\mu}_r \colon ({}'\bar{E}_r, {}''\bar{E}_r) \to \bar{E}_r$ be pairings.
- The morphisms are compatible with the pairings if each diagram

$$\begin{array}{ccc} {}'E_r^{*,*} \otimes {}''E_r^{*,*} & \xrightarrow{\mu_r} & E_r^{*,*} \\ \Big\downarrow {\scriptstyle {}'\phi_r \otimes {}''\phi_r} & & \Big\downarrow {\scriptstyle \phi_r} \\ {}'\bar{E}_r^{*,*} \otimes {}''\bar{E}_r^{*,*} & \xrightarrow{\bar{\mu}_r} & \bar{E}_r^{*,*} \end{array}$$

  commutes, for $r \geq p$.
- A ring morphism

$$\phi_r \colon (E_r, d_r, \mu_r) \longrightarrow ({}'E_r, {}'d_r, {}'\mu_r)$$

  of ring spectral sequences is a morphism $\phi_r \colon E_r \to {}'E_r$ of spectral sequences that is compatible with the pairings μ_r and ${}'\mu_r$, and which satisfies $\phi_p(1) = 1$.

Module spectral sequences

- Let $(E_r, d_r, \mu_r)_{r \geq p}$ be a ring spectral sequence.
- A left module spectral sequence over it is a spectral sequence $({}''E_r, {}''d_r)$ with a unital and associative pairing

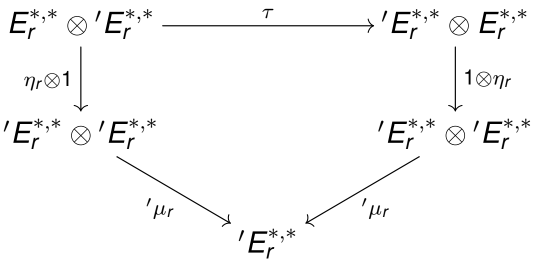
$$\lambda_r \colon (E_r, {}''E_r) \longrightarrow {}''E_r \,.$$

- A right module spectral sequence is a spectral sequence $({}'E_r, {}'d_r)$ with a unital and associative pairing

$$\rho_r \colon ({}'E_r, E_r) \longrightarrow {}'E_r \,.$$

Algebra spectral sequence

- Suppose that $(E_r, d_r, \mu_r)_{r \geq p}$ is commutative.
- An algebra spectral sequence over it is a ring spectral sequence $('E_r, 'd_r, '\mu_r)_{r \geq p}$ with a ring morphism $\eta_r \colon E_r \to {}'E_r$ such that E_r is central in $'E_r$.
- Centrality means that the diagram



  commutes, for each $r \geq p$.
- If $'\mu_r$ is commutative, then η_r is automatically central.

$\Lambda^{*,*}$-module spectral sequences

- Let $\Lambda^{*,*}$ be a bigraded ring. We can view it as a ring spectral sequence with $E_r^{*,*} = \Lambda^{*,*}$, for each $r \geq p$, with all differentials zero.
- A left $\Lambda^{*,*}$-module spectral sequence is a spectral sequence $(''E_r, d_r)_{r \geq p}$ with each $''E_r^{*,*}$ a left $\Lambda^{*,*}$-module and each d_r a $\Lambda^{*,*}$-linear homomorphism. This means that

$$d_r(\lambda \cdot y) = (-1)^{|\lambda|} \lambda \cdot d_r(y)$$

  for each $\lambda \in \Lambda^{*,*}$ and $y \in {}''E_r^{*,*}$. Here $|\lambda|$ denotes the total degree.
- Likewise, a right $\Lambda^{*,*}$-module spectral sequence is a spectral sequence $(E_r', d_r)_{r \geq p}$ with each $'E_r^{*,*}$ a right $\Lambda^{*,*}$-module and each d_r a $\Lambda^{*,*}$-linear homomorphism. This means that

$$d_r(x \cdot \lambda) = d_r(x) \cdot \lambda$$

  for each $\lambda \in \Lambda^{*,*}$ and $x \in {}'E_r^{*,*}$.

$\Lambda^{*,*}$-bilinear pairings

- A pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ is $\Lambda^{*,*}$-bilinear if

$$\mu_r(x \cdot \lambda \otimes y) = \mu_r(x \otimes \lambda \cdot y)$$

  for all $x \in {}'E_r^{*,*}$, $\lambda \in \Lambda^{*,*}$, $y \in {}''E_r^{*,*}$ and $r \geq p$.
- We can then uniquely factor μ_r through the tensor product over $\Lambda^{*,*}$, i.e., over the coequalizer in the following diagram.

$$\begin{array}{ccccc} {}'E_r^{*,*} \otimes \Lambda^{*,*} \otimes {}''E_r^{*,*} & \overset{1\otimes\lambda_r}{\underset{\rho_r\otimes 1}{\rightrightarrows}} & {}'E_r^{*,*} \otimes {}''E_r^{*,*} & \xrightarrow{\pi} & {}'E_r^{*,*} \otimes_{\Lambda^{*,*}} {}''E_r^{*,*} \\ & & & \searrow^{\mu_r} & \downarrow \\ & & & & E_r^{*,*} \end{array}$$

Fewer gradings

- Usually, $\Lambda^{*,*} = \Lambda^*$ is concentrated in filtration degree $s = 0$, so that the total degree equals the internal grading of Λ^*.
- This happens for the Atiyah–Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(X; M^*) \Longrightarrow M^*(X)$$

for a multiplicative cohomology theory M, with $\Lambda^* = M^*$.
- Even more frequently, $\Lambda^* = \Lambda$ is an ungraded ring, concentrated in internal degree $t = 0$, hence in bidegree $(s, t) = (0, 0)$.
- This happens for the Λ-coefficient cohomology spectral sequence

$$E_1^{s,*} = H^*(X_s, X_{s-1}; \Lambda) \Longrightarrow_s H^*(X; \Lambda)$$

for a filtered space.
- In this case, $|\lambda| = 0$, so left Λ-linearity has the usual meaning.

Tensor product of spectral sequences, II

- The sum $D_r = {}'d_r \otimes 1 + 1 \otimes {}''d_r$ defines a differential on ${}'E_r \otimes_{\Lambda^{*,*}} {}''E_r$, of bidegree $(r, 1-r)$.
- This does not in general make

  $$({}'E_r \otimes_{\Lambda^{*,*}} {}''E_r, D_r)_{r \geq p}$$

  a spectral sequence, because the cross product

  $$\begin{aligned} {}'E_{r+1} \otimes_{\Lambda^{*,*}} {}''E_{r+1} &\cong H({}'E_r, {}'d_r) \otimes_{\Lambda^{*,*}} H({}''E_r, {}''d_r) \\ &\xrightarrow{\times} H({}'E_r \otimes_{\Lambda^{*,*}} {}''E_r, D_r) \end{aligned}$$

  is not in general an isomorphism.
- In situations where this is an isomorphism, a $\Lambda^{*,*}$-bilinear spectral sequence pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ is the same as a spectral sequence morphism $\mu_r \colon {}'E_r \otimes_{\Lambda^{*,*}} {}''E_r \to E_r$.
- By the Künneth theorem [ML63, Thm. V.10.1], this is always the case if $\Lambda^{*,*}$ is a bigraded field, e.g. if Λ^* is a graded field, or Λ is a field in the usual sense.

Algebra spectral sequences

- Suppose that $\Lambda^{*,*}$ is (bigraded) commutative.
- A $\Lambda^{*,*}$-algebra spectral sequence is a ring spectral sequence $(E_r, d_r, \mu_r)_{r\geq p}$ such that
  - E_r is a $\Lambda^{*,*}$-algebra,
  - $d_r(\lambda \cdot 1) = 0$ for each $\lambda \in \Lambda^{*,*}$, and
  - the isomorphism $E_{r+1} \cong H(E_r, d_r)$ is $\Lambda^{*,*}$-linear,

  for each $r \geq p$.
- The ring pairing μ_r then factors uniquely through the coequalizer structure morphism π as a $\Lambda^{*,*}$-linear morphism

$$E_r^{*,*} \otimes_{\Lambda^{*,*}} E_r^{*,*} \longrightarrow E_r^{*,*} .$$

Pairing of E_∞-terms

Lemma

Let $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ be a pairing of E_p-spectral sequences. Then

$$\mu_p \colon {}'E_p^s \otimes {}''E_p^u \longrightarrow E_p^{s+u}$$

restricts to pairings

$$\mu_p \colon {}'Z_r^s \otimes {}''Z_r^u \longrightarrow Z_r^{s+u}$$
$$\mu_p \colon {}'B_r^s \otimes {}''Z_r^u \longrightarrow B_r^{s+u}$$
$$\mu_p \colon {}'Z_r^s \otimes {}''B_r^u \longrightarrow B_r^{s+u}$$

for all $p \le r \le \infty$ and (s, u), making the following diagram with exact rows commute.

$$\begin{array}{ccccccc}
{}'B_r^s \otimes {}''Z_r^u \oplus {}'Z_r^s \otimes {}''B_r^u & \longrightarrow & {}'Z_r^s \otimes {}''Z_r^u & \longrightarrow & {}'E_r^s \otimes {}''E_r^u & \longrightarrow & 0 \\
\downarrow {\scriptstyle \mu_p \oplus \mu_p} & & \downarrow {\scriptstyle \mu_p} & & \downarrow {\scriptstyle \mu_r} & & \\
B_r^{s+u} & \rightarrowtail & Z_r^{s+u} & \longrightarrow & E_r^{s+u} & \longrightarrow & 0
\end{array}$$

Proof

- The cases $p \leq r < \infty$ are proved by induction on r.
- The case $r = p$ is clear, since ${}'Z^s_p = {}'E^s_p$ and ${}'B^s_p = 0$, etc.
- Suppose the results holds for some $r \geq p$.
- If $x \in {}'Z^s_{r+1}$ and $y \in {}''Z^u_{r+1}$ then ${}'d_r([x]) = 0$ and ${}''d_r([y]) = 0$ so

$$\begin{aligned} d_r([\mu_p(x \otimes y)]) &= d_r(\mu_r([x] \otimes [y])) \\ &= \mu_r({}'d_r([x]) \otimes [y]) + (-1)^{|x|}\mu_r([x] \otimes {}''d_r([y])) = 0 \end{aligned}$$

  by the Leibniz rule.
- This implies $\mu_p(x \otimes y) \in Z^{s+u}_{r+1} \subset Z^{s+u}_r$, and this defines $\mu_p \colon {}'Z^s_{r+1} \otimes {}''Z^u_{r+1} \to Z^{s+u}_{r+1}$.

Proof (cont.)

- If $x \in {}'B^s_{r+1}$ and $y \in {}''Z^u_{r+1}$ then $[x] = {}'d_r([z])$ and ${}''d_r([y]) = 0$, for some $z \in {}'Z^{s-r}_r$, so

$$d_r([\mu_p(z \otimes y)]) = d_r(\mu_r([z] \otimes [y]))$$
$$= \mu_r({}'d_r([z]) \otimes [y]) + (-1)^{|z|}\mu_r([z] \otimes {}''d_r([y])) = \mu_r({}'d_r([z]) \otimes [y])$$

  by the Leibniz rule.
- This implies

$$[\mu_p(x \otimes y)] = \mu_r([x] \otimes [y]) = \mu_r({}'d_r([z]) \otimes [y]) \in \operatorname{im}(d_r)\,,$$

  so that $\mu_p(x \otimes y) \in B^{s+u}_{r+1}$, and this defines $\mu_p \colon {}'B^s_{r+1} \otimes {}''Z^u_{r+1} \to B^{s+u}_{r+1}$.
- The case $x \in {}'Z^s_{r+1}$ and $y \in {}''B^u_{r+1}$ is very similar.

Proof (cont.)

- The case $r = \infty$, defining the pairing

$$\mu_\infty \colon {}'E^s_\infty \otimes {}''E^u_\infty \longrightarrow E^{s+u}_\infty \,,$$

  follows by passage to (co-)limits.
- If $x \in {}'Z^s_\infty \subset {}'Z^s_r$ and $y \in {}''Z^u_\infty \subset {}''Z^u_r$ then $\mu_p(x \otimes y) \in Z^{s+u}_r$ for all r, hence $\mu_p(x \otimes y) \in Z^{s+u}_\infty$.
- If $x \in {}'B^s_\infty$ and $y \in {}''Z^u_\infty \subset {}''Z^u_r$ then $x \in {}'B^s_r$ for some r, so $\mu_p(x \otimes y) \in B^{s+u}_r \subset B^{s+u}_\infty$.
- The case $x \in {}'Z^s_\infty$ and $y \in {}''B^u_\infty$ is, again, very similar. $\square$

Pairings of filtrations

We formulate the definition of a pairing of filtrations from two such to a third, but often all three filtrations are the same.

Definition

- Let $(F^s{}'G^*)_s$, $(F^s{}''G^*)_s$ and $(F^sG^*)_s$ be (decreasing) filtrations of the graded abelian groups $'G^*$, $''G^*$ and G^*, respectively.
- A bilinear pairing

$$\nu \colon {}'G^* \otimes {}''G^* \longrightarrow G^*$$

  is filtration-preserving if

$$\nu(F^s{}'G^* \otimes F^u{}''G^*) \subset F^{s+u}G^*$$

  for each (s, u).
- More precisely, this means that $\nu(x \otimes y) \in F^{s+u}G^*$ whenever $x \in F^s{}'G^*$ and $y \in F^u{}''G^*$.

Definition (cont.)

Let

$$\nu^{s,u} \colon F^{s\prime} G^* \otimes F^{u\prime\prime} G^* \longrightarrow F^{s+u} G^*$$

be the lift of ν, making the diagram

$$\begin{array}{ccc} F^{s\prime} G^* \otimes F^{u\prime\prime} G^* & \longrightarrow & {}^{\prime}G^* \otimes {}^{\prime\prime}G^* \\ \downarrow{\scriptstyle \nu^{s,u}} & & \downarrow{\scriptstyle \nu} \\ F^{s+u} G^* & \rightarrowtail & G^* \end{array}$$

commute.

(In general, the upper horizontal arrow need not be injective.)

Definition (cont.)

There are then uniquely defined homomorphisms

$$\bar{\nu}^{s,u}\colon \frac{F^{s\prime}G^*}{F^{s+1\prime}G^*} \otimes \frac{F^{u\prime\prime}G^*}{F^{u+1\prime\prime}G^*} \longrightarrow \frac{F^{s+u}G^*}{F^{s+u+1}G^*}$$

making the following diagram (†) with exact rows commute.

$$\begin{array}{ccccccc}
\begin{array}{c} F^{s+1\prime}G^* \otimes F^{u\prime\prime}G^* \\ \oplus \\ F^{s\prime}G^* \otimes F^{u+1\prime\prime}G^* \end{array} & \longrightarrow & F^{s\prime}G^* \otimes F^{u\prime\prime}G^* & \longrightarrow & \dfrac{F^{s\prime}G^*}{F^{s+1\prime}G^*} \otimes \dfrac{F^{u\prime\prime}G^*}{F^{u+1\prime\prime}G^*} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \nu^{s+1,u} \oplus \nu^{s,u+1}} & & {\scriptstyle \nu^{s,u}}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\nu}^{s,u}} & & \\
F^{s+u+1}G^* & \rightarrowtail & F^{s+u}G^* & \longrightarrow & \dfrac{F^{s+u}G^*}{F^{s+u+1}G^*} & \longrightarrow & 0
\end{array}$$

(Exactness of the upper row follows from right exactness of the tensor product.)

Convergence to a pairing

Definition

- Suppose that ${}'E_r$ converges to ${}'G^*$, ${}''E_r$ converges to ${}''G^*$ and E_r converges to G^*.
- A spectral sequence pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ converges to a filtration-preserving pairing $\nu \colon {}'G^* \otimes {}''G^* \to G^*$ if the diagram

$$\begin{array}{ccc}
\dfrac{F^{s\prime}G^*}{F^{s+1\prime}G^*} \otimes \dfrac{F^{u\prime\prime}G^*}{F^{u+1\prime\prime}G^*} & \xrightarrow{\bar{\nu}^{s,u}} & \dfrac{F^{s+u}G^*}{F^{s+u+1}G^*} \\
\cong \updownarrow & & \updownarrow \cong \\
{}'E^s_\infty \otimes {}''E^u_\infty & \xrightarrow{\mu_\infty} & E^{s+u}_\infty
\end{array}$$

commutes for all (s, u).

Filtration shifts

- Suppose that the filtrations of $'G^*$, $''G^*$ and G^* are exhaustive and degreewise discrete.
- Convergence of $(\mu_r)_r$ to ν then lets us recover $\nu \colon {'G^*} \otimes {''G^*} \to G^*$ up to filtration shifts.
- More explicitly, in total degrees n and m we assume that there are integers $a'(n)$ and $a''(m)$ such that $F^s{'G^n} = 0$ for $s > a'(n)$ and $F^u{''G^m} = 0$ for $u > a''(m)$.
- This forms the basis for a descending induction on (s, u), where we may suppose that

$$\nu^{s+1,u} \colon F^{s+1}{'G^n} \otimes F^u{''G^m} \longrightarrow F^{s+u+1}G^{n+m}$$
$$\nu^{s,u+1} \colon F^s{'G^n} \otimes F^{u+1}{''G^m} \longrightarrow F^{s+u+1}G^{n+m}$$

have been determined.

Filtration shifts (cont.)

- Assuming that we have determined

$$\mu_\infty \colon {}'E_\infty^{s,n-s} \otimes {}''E_\infty^{u,m-u} \longrightarrow E_\infty^{s+u,n+m-s-u} ,$$

  which is identified with

$$\bar{\nu}^{s,u} \colon \frac{F^s{}'G^n}{F^{s+1}{}'G^n} \otimes \frac{F^u{}''G^m}{F^{u+1}{}''G^m} \longrightarrow \frac{F^{s+u}G^{n+m}}{F^{s+u+1}G^{n+m}} ,$$

  we can use diagram (†) to determine

$$\nu^{s,u} \colon F^s{}'G^n \otimes F^u{}''G^m \longrightarrow F^{s+u}G^{n+m} ,$$

  up to some indeterminacy.
- More precisely, any two possible choices of $\nu^{s,u}$ differ by a composite of the form

$$F^s{}'G^n \otimes F^u{}''G^m \longrightarrow \frac{F^s{}'G^n}{F^{s+1}{}'G^n} \otimes \frac{F^u{}''G^m}{F^{u+1}{}''G^m} \xrightarrow{f} F^{s+u+1}G^{n+m} \longrightarrow F^{s+u}G^{n+m} ,$$

  where f is any homomorphism.

Filtration shifts (cont.)

- Having determined

$$F^s\,{}'G^n \otimes F^u\,{}''G^m \xrightarrow{\nu^{s,u}} F^{s+u}G^{n+m} \longrightarrow G^{n+m}$$

for all finite s and u, we can then pass to colimits to obtain $\nu\colon {}'G^n \otimes {}''G^m \to G^{n+m}$, since

$$\operatorname*{colim}_s \operatorname*{colim}_u F^s\,{}'G^n \otimes F^u\,{}''G^m \xrightarrow{\cong} {}'G^n \otimes {}''G^m$$

is an isomorphism, by the commutation of sequential colimits with tensor products.

Pairings of exact couples

- Given (cohomological) exact couples $({}'A, {}'E)$, $({}''A, {}''E)$ and (A, E), and a pairing $\mu \colon {}'E \otimes {}''E \to E$, Massey [Mas54, §3, §8] defined conditions that are essentially equivalent to saying that $\mu = \mu_1$ is part of a pairing $(\mu_r)_r$ of the associated spectral sequences.
- These conditions are often not easy to check directly, but in [Mas54, §7, §9], Massey asserts that they can be verified in the case of a filtered differential graded ring.
- In essence, the argument uses that these exact couples arise from Cartan–Eilenberg systems, and the pairing arises from a pairing of Cartan–Eilenberg systems.
- We shall therefore concentrate on this approach to pairings of spectral sequences.

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