MAT9580: Spectral Sequences

Chapter 6: Cartan–Eilenberg Systems

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Outline

Cartan–Eilenberg systems

- Cohomological Cartan–Eilenberg systems
- Pairings of Cartan–Eilenberg systems
- Filtered differential graded rings
- Multiplicative Serre spectral sequence
- The cohomological Wang and Gysin sequences
- Rational cohomology of integral E–M spaces
- First p-torsion in $\pi_*(S^3)$
- Cohomology of $K(\mathbb{Z}/2, 2)$

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Cohomological Cartan–Eilenberg systems

We adapt Cartan–Eilenberg [CE56].

Definition

A (cohomological) finite Cartan–Eilenberg system (H^*, η, δ) consists of

- graded abelian groups $H^*(i,j)$ for all integers $i \leq j$,
- structure morphisms preserving degree

  $$\eta\colon H^*(i',j') \longrightarrow H^*(i,j)$$

  for all integers $i \leq j$, $i' \leq j'$ with $i \leq i'$ and $j \leq j'$, and
- connecting homomorphisms

  $$\delta\colon H^*(i,j) \longrightarrow H^{*+1}(j,k)$$

  for all integers $i \leq j \leq k$.

Definition (cont.)

These must satisfy

1. Functoriality: $\eta\colon H^*(i,j) \to H^*(i,j)$ equals the identity, and
$$\eta \circ \eta\colon H^*(i'',j'') \to H^*(i',j') \to H^*(i,j)$$
equals $\eta\colon H^*(i'',j'') \to H^*(i,j)$ for all integers $i \le j$, $i' \le j'$ and $i'' \le j''$ with $i \le i' \le i''$ and $j \le j' \le j''$.
2. Naturality: The diagrams
$$\begin{array}{ccc} H^*(i',j') & \xrightarrow{\delta} & H^*(j',k') \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\ H^*(i,j) & \xrightarrow{\delta} & H^*(j,k) \end{array}$$
commutes, for all integers $i \le j \le k$ and $i' \le j' \le k'$ with $i \le i'$, $j \le j'$ and $k \le k'$.
3. Exactness: The sequence
$$\dots \xrightarrow{\delta} H^*(j,k) \xrightarrow{\eta} H^*(i,k) \xrightarrow{\eta} H^*(i,j) \xrightarrow{\delta} H^{*+1}(j,k) \xrightarrow{\eta} \dots$$
is exact, for all integers $i \le j \le k$.

Extended systems

Definition

By an extended integer we mean an element of

$$\{-\infty\} \cup \mathbb{Z} \cup \{\infty\},$$

linearly ordered with $-\infty$ minimal and ∞ maximal.

Definition

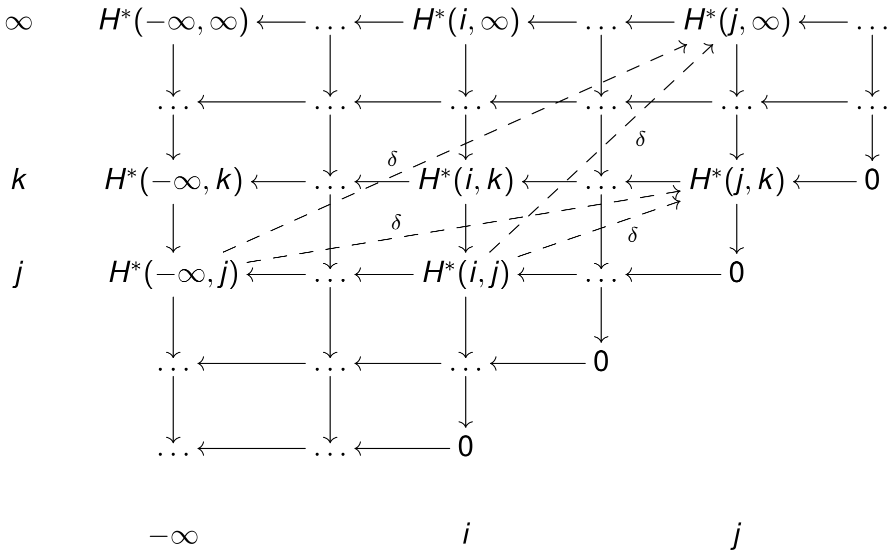
An extended Cartan–Eilenberg system (H^*, η, δ) is defined as a finite Cartan–Eilenberg system, except that all references to "integers" are replaced with "extended integers", and subject to the following additional condition.

4. Colimit: For each extended integer j the canonical homomorphism

$$\operatorname*{colim}_i H^*(i,j) \xrightarrow{\cong} H^*(-\infty, j)$$

is an isomorphism.

Visualization in (i, j)-plane



Filtered cochain complex

- Let $(F^s C^*)_s$ be a decreasing filtration of a cochain complex C^*.
- The associated finite Cartan–Eilenberg system is given by

$$H^*(i,j) = H^*(F^i C^* / F^j C^*)$$

  for integers $i \leq j$, and $\eta\colon H^*(i',j') \to H^*(i,j)$ is induced by the chain map $F^{i'} C^* / F^{j'} C^* \to F^i C^* / F^j C^*$.
- The connecting homomorphism associated to the short exact sequence

$$0 \to F^j C^* / F^k C^* \longrightarrow F^i C^* / F^k C^* \longrightarrow F^i C^* / F^j C^* \to 0$$

  defines $\delta\colon H^*(i,j) \to H^{*+1}(j,k)$.

Filtered cochain complex (cont.)

- Suppose also that the filtration exhausts C^*.
- Letting $F^{-\infty}C^* = C^*$ and $F^{\infty}C^* = 0$, the same expressions define an extended Cartan–Eilenberg system with $H^*(s, \infty) = H^*(F^s C^*)$ and $H^*(-\infty, \infty) = H^*(C^*)$.

Filtered space

- Let $(X_s)_s$ be an increasing filtration of a space X, so that $F^s C^*(X) = C^*(X, X_{s-1})$ defines a decreasing filtration of $C^*(X)$.
- The associated finite Cartan–Eilenberg system is given by

$$H^*(i,j) = H^*(F^i C^*(X)/F^j C^*(X)) = H^*(X_{j-1}, X_{i-1})$$

  for integers $i \leq j$, and $\eta \colon H^*(i', j') \to H^*(i,j)$ is induced by the inclusion of (X_{j-1}, X_{i-1}) into $(X_{j'-1}, X_{i'-1})$.
- The morphism $\delta \colon H^*(i,j) \to H^{*+1}(j,k)$ equals the connecting homomorphism $\delta \colon H^*(X_{j-1}, X_{i-1}) \to H^{*+1}(X_{k-1}, X_{j-1})$ in the long exact cohomology sequence of the triple $(X_{k-1}, X_{j-1}, X_{i-1})$.

Filtered space (cont.)

- Suppose also that $X_{a-1} = \emptyset$ for some finite a, so that $F^a C^*(X) = C^*(X)$.
- Letting $X_{-\infty} = \emptyset$ and $X_\infty = X$ the same expressions define an extended Cartan–Eilenberg system with $H^*(s, \infty) = H^*(X, X_{s-1})$ and $H^*(-\infty, \infty) = H^*(X)$.

Associated exact couple and spectral sequence

- To each cohomological extended Cartan–Eilenberg system (H^*, η, δ) we associate the (top) cohomological exact couple $(A^s, E^s)_s$ given by the diagram

$$
\begin{array}{ccccccc}
\cdots & \xleftarrow{\eta} & H^*(s,\infty) & \xleftarrow{\eta} & H^*(s+1,\infty) & \xleftarrow{\eta} & \cdots \\
 & & \downarrow{\scriptstyle\eta} & & & & \\
 & & H^*(s,s+1) & & \cdots & &
\end{array}
\qquad \delta\colon H^*(s,s+1) \dashrightarrow H^*(s+1,\infty)
$$

  where

$$
\begin{aligned}
(A^s)^* &= H^*(s,\infty) \\
(E^s)^* &= H^*(s,s+1)
\end{aligned}
$$

  with α_s and β_s given by η, while γ_s is given by δ.

- The spectral sequence $(E_r, d_r)_{r\geq 1}$ associated to (H^*, η, δ) is the spectral sequence associated to the exact couple $(A^s, E^s)_s$.

r-th cocycles and r-th coboundaries

Proposition

In the spectral sequence $(E_r, d_r)_{r\geq 1}$ associated to an extended Cartan–Eilenberg system (H^*, η, δ) we have

$$\begin{aligned} Z_r^s &= \delta^{-1}\,\mathrm{im}(\eta\colon H^{*+1}(s+r,\infty) \to H^{*+1}(s+1,\infty)) \\ &= \ker(\delta\colon H^*(s,s+1) \to H^{*+1}(s+1,s+r)) \\ &= \mathrm{im}(\eta\colon H^*(s,s+r) \to H^*(s,s+1)) \end{aligned}$$

and

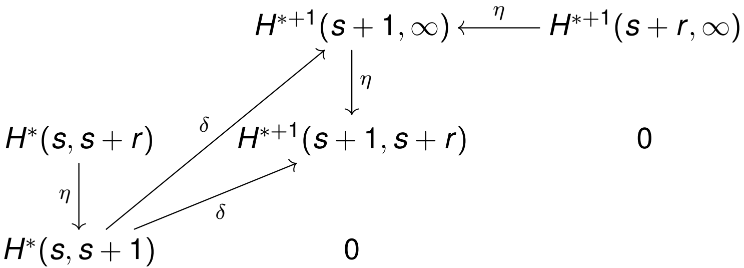
$$\begin{aligned} B_r^s &= \eta\,\ker(\eta\colon H^*(s,\infty) \to H^*(s-r+1,\infty)) \\ &= \mathrm{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,s+1)) \\ &= \ker(\eta\colon H^*(s,s+1) \to H^*(s-r+1,s+1))\,. \end{aligned}$$

Proof

For the r-th cocycles,

$$\begin{aligned}
\delta^{-1}\operatorname{im}(\eta\colon H^{*+1}(s+r,\infty) \to H^{*+1}(s+1,\infty)) \\
= \delta^{-1}\ker(\eta\colon H^{*+1}(s+1,\infty) \to H^{*+1}(s+1,s+r)) \\
= \ker(\delta\colon H^{*}(s,s+1) \to H^{*+1}(s+1,s+r))
\end{aligned}$$

by exactness and naturality.

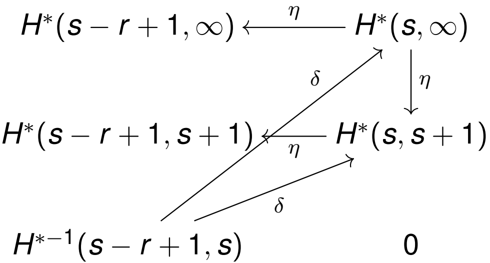


Proof (cont.)

For the r-th coboundaries,

$$\begin{aligned}\eta \ker(\eta\colon H^*(s,\infty) \to H^*(s-r+1,\infty)) \\ = \eta \operatorname{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,\infty)) \\ = \operatorname{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,s+1))\end{aligned}$$

for the same reasons.



□

E_r-term and d_r-differential

Proposition

The map η induces an isomorphism

$$E_r^s \xrightarrow{\cong} \mathrm{im}(\eta\colon H^*(s, s+r) \to H^*(s-r+1, s+1))\,.$$

The d_r-differential is given by

$$\begin{aligned} d_r^s\colon E_r^s &\longrightarrow E_r^{s+r} \\ [x] &\longmapsto [\delta(z)] \end{aligned}$$

where $z \in H^*(s, s+r)$, $x = \eta(z) \in H^*(s, s+1)$ and $\delta(z) \in H^{*+1}(s+r, s+r+1)$.

Proof

Considering the composition $\eta'' \circ \eta'$ (where the primes only serve to keep the two homomorphisms apart),

$$\begin{array}{ccccc}
 & & H^*(s, s+r) & & H^{*+1}(s+1, s+r) \\
 & & \downarrow{\scriptstyle \eta'} & \nearrow{\scriptstyle \delta} & \\
H^*(s-r+1, s+1) & \xleftarrow{\eta''} & H^*(s, s+1) & & 0 \\
 & \nearrow{\scriptstyle \delta} & & & \\
H^{*-1}(s-r+1, s) & & 0 & &
\end{array}$$

the isomorphism

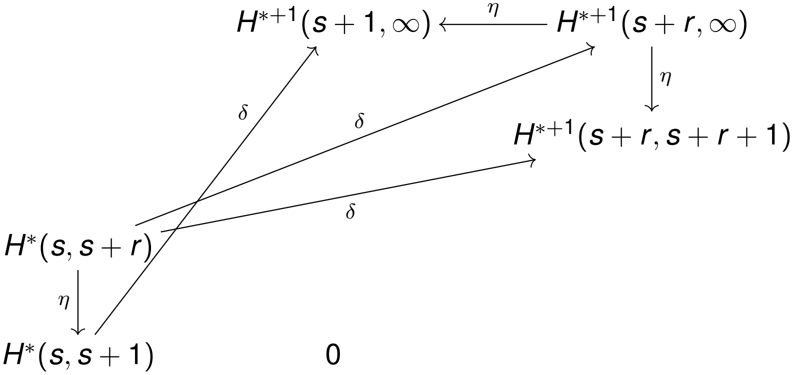
$$\eta'' \colon H^*(s, s+1)/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'')$$

restricts to the asserted isomorphism

$$E_r^s = Z_r^s/B_r^s = \operatorname{im}(\eta')/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'' \circ \eta')\,.$$

Proof (cont.)

If $x = \eta(z) \in Z_r^s \subset H^*(s, s+1)$ with $z \in H^*(s, s+r)$, then $\delta(x) = \eta(y) \in H^{*+1}(s+1, \infty)$ with $y = \delta(z) \in H^{*+1}(s+r, \infty)$, by naturality. Hence $\eta(y) = \delta(z) \in H^{*+1}(s+r, s+r+1)$, also by naturality. Thus $d_r^s([x]) = [\eta(y)] = [\delta(z)]$.



□

Exhaustive filtration

Lemma

The colimit

$$G^* = H^*(-\infty, \infty) \cong \operatorname*{colim}_s H^*(s, \infty)$$

is exhaustively filtered by

$$F^s G^* = \operatorname{im}(\eta \colon H^*(s, \infty) \to H^*(-\infty, \infty)) \,.$$

Proof.

Easy. □

Degreewise discrete filtration

Lemma

Consider an extended (H^*, η, δ) *such that*

$$\dots \xleftarrow{\eta} H^*(s, \infty) \xleftarrow{\eta} H^*(s+1, \infty) \xleftarrow{\eta} \dots$$

is degreewise discrete. Then

$$\begin{aligned} Z^s_\infty &= \ker(\delta\colon H^*(s, s+1) \to H^{*+1}(s+1, \infty)) \\ &= \operatorname{im}(\eta\colon H^*(s, \infty) \to H^*(s, s+1)) \end{aligned}$$

and the filtration $(F^s G^*)_s$ *is degreewise discrete.*

Proof.

If $H^{n+1}(i, \infty) = 0$ for $i > b = b(n+1)$ then

$$\ker(H^n(s, s+1) \xrightarrow{\delta} H^{n+1}(s+1, \infty)) = \ker(H^n(s, s+1) \xrightarrow{\delta} H^{n+1}(s+1, s+r))$$

for all $s + r > b$, i.e., for all $r > b - s$, so $(Z^s_\infty)^n$ equals this common value of $(Z^s_r)^n$. □

Lemma

Consider any extended (H^*, η, δ). *Then*

$$B^s_\infty = \mathrm{im}(\delta\colon H^{*-1}(-\infty, s) \to H^*(s, s+1)) \\ = \ker(\eta\colon H^*(s, s+1) \to H^*(-\infty, s+1))\,.$$

Proof.

The union $B^s_\infty \cong \mathrm{colim}_r\, B^s_r$ equals

$$\operatorname*{colim}_r \ker(\eta\colon H^*(s, s+1) \to H^*(s-r+1, s+1)) \\ \cong \ker(\eta\colon H^*(s, s+1) \to H^*(-\infty, s+1))$$

since $H^*(-\infty, s+1) \cong \mathrm{colim}_r\, H^*(s-r+1, s+1)$. □

Proposition

Let (H^*, η, δ) be an extended cohomological Cartan–Eilenberg system, with associated spectral sequence $(E_r, d_r)_{r\geq 1}$ and filtered target $G^* = H^*(-\infty, \infty)$.

1. *There is always a preferred injective homomorphism*

$$\frac{F^s G^*}{F^{s+1} G^*} \overset{\zeta}{\rightarrowtail} E_\infty^{s,*} ,$$

   which is iso if $Z_\infty^s = \mathrm{im}(\eta\colon H^*(s, \infty) \to H^*(s, s+1))$.

2. *In particular, if the sequence*

$$\dots \overset{\eta}{\longleftarrow} H^*(s, \infty) \overset{\eta}{\longleftarrow} H^*(s+1, \infty) \overset{\eta}{\longleftarrow} \dots$$

   is degreewise discrete, then ζ is an isomorphism and the spectral sequence

$$E_r^{*,*} \Longrightarrow G^*$$

   converges.

Sketch proof.

Consider the following diagram, with $G^* = H^*(-\infty, \infty)$.

$$\begin{array}{ccccc} G^* & \xleftarrow{\; i_s \;} & H^*(s, \infty) & \xleftarrow{\; \alpha_s \;} & H^*(s+1, \infty) \\ & & \big\downarrow \beta_s & \nearrow \gamma_s & \\ & & H^*(s, s+1) & & \end{array}$$

The maps i_s and β_s induce isomorphisms

$$\frac{F^s G^*}{F^{s+1} G^*} \xleftarrow{\cong} \frac{H^*(s, \infty)}{\operatorname{im}(\alpha_s) + \ker(i_s)} \xrightarrow{\cong} \frac{\ker(\gamma_s)}{\beta_s \ker(i_s)}$$

The inclusion $\ker(\gamma_s) \subset Z^s_\infty$ and identity $\beta_s \ker(i_s) = B^s_\infty$ then give the inclusion

$$\frac{\ker(\gamma_s)}{\beta_s \ker(i_s)} \subset \frac{Z^s_\infty}{B^s_\infty} = E^s_\infty \, .$$

□

Lemma

Consider any extended (H^*, η, δ)*. There is a preferred isomorphism*

$$\frac{\operatorname{im}(\eta\colon H^*(s,\infty) \to H^*(s,s+1))}{\ker(\eta\colon H^*(s,s+1) \to H^*(-\infty,s+1))} \cong \frac{F^s G^*}{F^{s+1} G^*}$$

for each $s \in \mathbb{Z}$*.*

$$\begin{array}{ccccc}
 & & H^*(s,\infty) & & \\
 & & \downarrow{\scriptstyle \eta} & & \\
H^*(-\infty,s+1) & \xleftarrow{\eta} & H^*(s,s+1) & & 0 \\
 & & & & \\
 & & 0 & &
\end{array}$$

Outline

Pairings of finite Cartan–Eilenberg systems

We follow Douady's presentation [Dou58] in the Cartan seminar.

Definition

- Let $({}'H^*, \eta, \delta)$, $({}''H^*, \eta, \delta)$ and (H^*, η, δ) be finite cohomological Cartan–Eilenberg systems.
- A pairing $\mu\colon ({}'H^*, {}''H^*) \to H^*$ of finite Cartan–Eilenberg systems is a collection of degree-preserving homomorphisms

$$\mu_r\colon {}'H^*(s, s+r) \otimes {}''H^*(u, u+r) \longrightarrow H^*(s+u, s+u+r)$$

  for $r \geq 1$ and $s, u \in \mathbb{Z}$.
- These are required to satisfy the following two conditions.

Definition (cont.)

(SPP I) Each square

$$\begin{array}{ccc} {}'H^*(s', s'+r') \otimes {}''H^*(u', u'+r') & \xrightarrow{\mu_{r'}} & H^*(s'+u', s'+u'+r') \\ \Big\downarrow{\scriptstyle \eta\otimes\eta} & & \Big\downarrow{\scriptstyle \eta} \\ {}'H^*(s, s+r) \otimes {}''H^*(u, u+r) & \xrightarrow{\mu_r} & H^*(s+u, s+u+r) \end{array}$$

commutes, for $r \geq 1$, $r' \geq 1$, $s \leq s'$, $u \leq u'$, $s+r \leq s'+r'$ and $u+r \leq u'+r'$.

Definition (cont.)

(SPP II) In each (non-commutative) diagram

$$
\begin{array}{ccc}
{}'H^*(s, s+r) \otimes {}''H^*(u, u+r) & \xrightarrow{\eta\otimes\delta} & {}'H^*(s, s+1) \otimes {}''H^*(u+r, u+r+1) \\
\Big\downarrow{\scriptstyle \delta\otimes\eta} & \searrow^{\mu_r} \quad H^*(s+u, s+u+r) \quad \searrow^{\delta} & \Big\downarrow{\scriptstyle \mu_1} \\
{}'H^*(s+r, s+r+1) \otimes {}''H^*(u, u+1) & \xrightarrow{\mu_1} & H^*(s+u+r, s+u+r+1)
\end{array}
$$

with $r \geq 1$ and $s, u \in \mathbb{Z}$, the diagonal composite equals the sum of the two outer composites:

$$\delta\mu_r = \mu_1(\delta \otimes \eta) + \mu_1(\eta \otimes \delta)\,.$$

Remark

- In terms of elements $x \in {}'H^*(s, s+r)$ and $y \in {}''H^*(u, u+r)$, the spectral pairing condition (SPP II) asks that

$$\delta(x \cdot y) = \delta(x) \cdot \eta(y) + (-1)^{|x|}\eta(x) \cdot \delta(y)\,,$$

  where we write $\cdot$ for the pairings μ_r and μ_1, and $|x|$ equals the total degree of x.
- In other words, $|x| = n$ if $x \in {}'H^n(s, s+r)$.
- This follows from how $\delta \otimes \eta$ and $\eta \otimes \delta$ are defined to act on $x \otimes y$, since η has degree 0 and δ has degree 1.

Pairing theorem

Theorem ([Dou58, Thm. II A(a,b,c)])

A pairing $\mu\colon ({}'H^*, {}''H^*) \to H^*$ of finite Cartan–Eilenberg systems induces a pairing $\mu_r\colon ({}'E_r, {}''E_r) \to E_r$ of the associated spectral sequences, with

$$\mu_1\colon {}'E_1^s \otimes {}''E_1^u \longrightarrow E_1^{s+u}$$

equal to

$$\mu_1\colon {}'H^*(s, s+1) \otimes {}''H^*(u, u+1) \longrightarrow H^*(s+u, s+u+1)\,.$$

Remark

This part of Douady's theorem asserts

- $$\mu_r \colon {}'E_r^s \otimes {}''E_r^u \longrightarrow E_r^{s+u}$$

  for each $r \geq 1$ satisfies the Leibniz rule

  $$d_r \mu_r = \mu_r({}'d_r \otimes 1) + \mu_r(1 \otimes {}''d_r)$$
  $$d_r(x \cdot y) = {}'d_r(x) \cdot y + (-1)^{|x|} x \cdot {}''d_r(y)$$

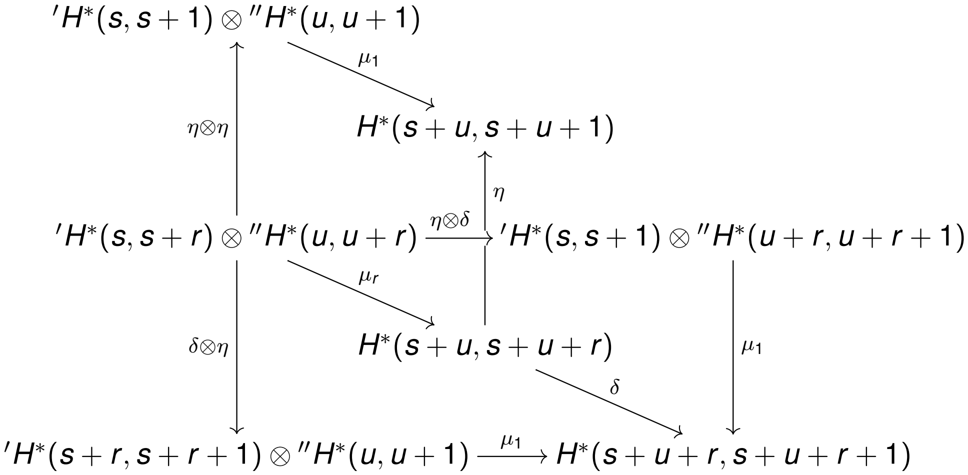
  for $x \in {}'E_r$ and $y \in {}''E_r$, and
- μ_{r+1} is induced by μ_r in the sense that

  $$\mu_{r+1}([x] \otimes [y]) = [\mu_r(x \otimes y)]$$

  in $H(E_r, d_r) \cong E_{r+1}$, where ${}'d_r(x) = 0$ and ${}''d_r(y) = 0$.

Proof

We prove this by induction on $r \geq 1$, using the diagram below.



Proof (cont.)

- Classes $[x] \in {}'E_r^s$ and $[y] \in {}''E_r^u$ are represented by r-th cocycles

$$x = \eta(z) \in {}'Z_r^s \subset {}'H^*(s, s+1)$$
$$y = \eta(w) \in {}''Z_r^u \subset {}''H^*(u, u+1)\,,$$

  with $z \in {}'H^*(s, s+r)$ and $w \in {}''H^*(u, u+r)$.
- Then $\mu_r([x] \otimes [y]) \in E_r^{s+u}$ is the class of

$$\mu_1(x \otimes y) \in Z_r^{s+u} \subset H^*(s+u, s+u+1)\,,$$

  which we can write as $\eta(\mu_r(z \otimes w))$ with $\mu_r(z \otimes w) \in H^*(s+u, s+u+r)$.
- Hence we can calculate $d_r(\mu_r([x] \otimes [y])) \in E_r^{s+u+r}$ as the class of

$$\delta(\mu_r(z \otimes w)) \in Z_r^{s+u+r} \subset H^*(s+u+r, s+u+r+1)\,.$$

Proof (cont.)

- This equals the sum of

$$\mu_1(\delta \otimes \eta)(z \otimes w) = \mu_1(\delta(z) \otimes y)$$

and

$$\mu_1(\eta \otimes \delta)(z \otimes w) = (-1)^{|z|}\mu_1(x \otimes \delta(w)),$$

where $|z| = |[x]|$.

- Here $\delta(z) \in {}'H^*(s+r, s+r+1)$ represents ${}'d_r([x])$, so $\mu_1(\delta(z) \otimes y)$ represents $\mu_r({}'d_r([x]) \otimes [y]) \in E_r^{s+u+r}$.
- Similarly, $\delta(w) \in {}''H^*(u, u+r)$ represents ${}''d_r([y])$, so $\mu_1(x \otimes \delta(w))$ represents $\mu_r([x] \otimes {}''d_r([y])) \in E_r^{s+u+r}$.

Proof (cont.)

- Hence $d_r(\mu_r([x] \otimes [y]))$ equals the sum

$$\mu_r({}'d_r([x]) \otimes [y]) + (-1)^{|[x]|}\mu_r([x] \otimes {}''d_r([y])) \in E_r^{s+u+r} ,$$

as claimed.
- Having proved that μ_1 restricts to define μ_r on E_r-classes for each $r \geq 1$, it follows that μ_r induces μ_{r+1} upon passage to homology with respect to d_r, since both are calculated from μ_1. □

Pairings of extended Cartan–Eilenberg systems

Definition

- Let $('H^*, \eta, \delta)$, $(''H^*, \eta, \delta)$ and (H^*, η, δ) be extended cohomological Cartan–Eilenberg systems.
- A pairing $\mu\colon ('H^*, ''H^*) \to H^*$ of extended Cartan–Eilenberg systems is a pairing (μ_r) of the underlying finite Cartan–Eilenberg systems, together with degree-preserving homomorphisms

$$\mu_\infty\colon {}'H^*(s, \infty) \otimes {}''H^*(u, \infty) \longrightarrow H^*(s+u, \infty)$$

  for $s, u \in \mathbb{Z}$.
- These are required to satisfy the following additional condition, extending (SPP I) to the case $r' = \infty$ and $1 \le r \le \infty$.

Definition (cont.)

(SPP III) The squares

$$\begin{array}{ccc} {}'H^*(s,\infty)\otimes {}''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty) \\ \downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta} \\ {}'H^*(s,s+r)\otimes {}''H^*(u,u+r) & \xrightarrow{\mu_r} & H^*(s+u,s+u+r) \end{array}$$

and

$$\begin{array}{ccc} {}'H^*(s',\infty)\otimes {}''H^*(u',\infty) & \xrightarrow{\mu_\infty} & H^*(s'+u',\infty) \\ \downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta} \\ {}'H^*(s,\infty)\otimes {}''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty) \end{array}$$

commute, for $r \geq 1$, $s \leq s'$ and $u \leq u'$.

Pairing of target groups

Lemma

Given a pairing $\mu\colon ('H^*, ''H^*) \to H^*$ of extended Cartan–Eilenberg systems, with filtered target groups

$$'G^* = 'H^*(-\infty,\infty)\,, \quad ''G^* = ''H^*(-\infty,\infty)\,, \quad G^* = H^*(-\infty,\infty)\,,$$

there is a unique filtration-preserving pairing $\nu\colon 'G^* \otimes ''G^* \longrightarrow G^*$ making the diagrams

$$\begin{array}{ccc}
'H^*(s,\infty) \otimes ''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
F^s{'G^*} \otimes F^u{''G^*} & \xrightarrow{\nu^{s,u}} & F^{s+u}G^* \\
\downarrow & & \downarrow \\
'G^* \otimes ''G^* & \xrightarrow{\nu} & G^*
\end{array}$$

commute for all $s, u \in \mathbb{Z}$.

Proof

- The isomorphisms $\operatorname{colim}_s {}'H^*(s,\infty) \cong {}'G^*$ and $\operatorname{colim}_u {}''H^*(u,\infty) \cong {}''G^*$ induce an isomorphism

$$\operatorname*{colim}_{s,u} {}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \xrightarrow{\cong} {}'G^* \otimes {}''G^* .$$

- Hence ν is the canonical map induced by the composites

$${}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \xrightarrow{\mu_\infty} H^*(s+u,\infty) \longrightarrow G^* ,$$

  which are compatible by the second part of (SPP III).
- This makes the outer rectangle commute.

Proof (cont.)

- The tensor product of the defining surjections ${}'H^*(s,\infty) \to F^{s\prime}G^*$ and ${}''H^*(u,\infty) \to F^{u\prime\prime}G^*$ gives the surjection

$$ {}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \longrightarrow F^{s\prime}G^* \otimes F^{u\prime\prime}G^* $$

  in the left hand column, whose kernel maps to zero in $F^{s+u}G^* \subset G^*$.
- Hence there is a unique homomorphism $\nu^{s,u}$ making the upper square commute.
- It follows that the lower square commutes, by the stated surjectivity. $\square$

Convergence of pairings

Proposition ([Dou58, Thm. II A(d)])

- *Let* $({}'H^*, \eta, \delta)$, $({}''H^*, \eta, \delta)$ *and* (H^*, η, δ) *be extended Cartan–Eilenberg systems with associated spectral sequences* $({}'E_r, {}'d_r)$, $({}''E_r, {}''d_r)$ *and* (E_r, d_r) *converging to* ${}'G^*$, ${}''G^*$ *and* G^*, *respectively.*
- *Let*
$$\mu\colon ({}'H^*, {}''H^*) \longrightarrow H^*$$
be a pairing of extended Cartan–Eilenberg systems.
- *Then the associated spectral sequence pairing*
$$\mu_r\colon ({}'E_r, {}''E_r) \longrightarrow E^r$$
converges to the filtration-preserving pairing
$$\nu\colon {}'G^* \otimes {}''G^* \longrightarrow G^* .$$

Proof

We show that the lower square in the diagram

$$\begin{array}{ccc}
{}'H^*(s,\infty)\otimes {}''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty)\\
\Big\downarrow & & \Big\downarrow\\
F^{s\prime}G^*\otimes F^{u\prime\prime}G^* & \xrightarrow{\nu^{s,u}} & F^{s+u}G^*\\
\Big\downarrow & & \Big\downarrow\\
\dfrac{F^{s\prime}G^*}{F^{s+1\prime}G^*}\otimes\dfrac{F^{u\prime\prime}G^*}{F^{u+1\prime\prime}G^*} & \xrightarrow{\bar{\nu}^{s,u}} & \dfrac{F^{s+u}G^*}{F^{s+u+1}G^*}\\
\Big\downarrow{\scriptstyle \zeta\otimes\zeta} & & \Big\downarrow{\scriptstyle \zeta}\\
{}'E^s_\infty\otimes {}''E^u_\infty & \xrightarrow{\mu_\infty} & E^{s+u}_\infty
\end{array}$$

commutes, where ζ is as before. The upper and middle squares commute by the definition of $\nu^{s,u}$ and $\bar{\nu}^{s,u}$, respectively. By the surjectivity of the upper and middle left hand maps, it suffices to prove that the outer rectangle commutes.

Proof (cont.)

In view of the construction of ζ, the outer rectangle can instead be factored as follows.

$$\begin{array}{ccc}
{}'H^*(s,\infty) \otimes {}''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty) \\
\downarrow & & \downarrow \\
{}'Z^s_\infty \otimes {}''Z^u_\infty & \xrightarrow{\mu_1|} & Z^{s+u}_\infty \\
\twoheaddownarrow & & \twoheaddownarrow \\
{}'E^s_\infty \otimes {}''E^u_\infty & \xrightarrow{\mu_\infty} & E^{s+u}_\infty
\end{array}$$

Here the lower square defines μ_∞ in terms of the restricted pairing $\mu_1|$, and the upper square is part of the following commutative diagram.

$$\begin{array}{ccc}
{}'H^*(s,\infty) \otimes {}''H^*(u,\infty) & \xrightarrow{\mu_\infty} & H^*(s+u,\infty) \\
\downarrow & & \downarrow \\
{}'Z^s_\infty \otimes {}''Z^u_\infty & \xrightarrow{\mu_1|} & Z^{s+u}_\infty \\
\rightarrowtail & & \rightarrowtail \\
{}'H^*(s,s+1) \otimes {}''H^*(u,u+1) & \xrightarrow{\mu_1} & H^*(s+u,s+u+1)
\end{array}$$

□

Remark

In the presence of (SPP I), condition (SPP II) follows from the stronger condition below, which appears in [Nei80].

(SPP II+) In each (non-commutative) diagram

$$\begin{array}{ccc}
{}'H^*(s, s+r) \otimes {}''H^*(u, u+r) & \xrightarrow{1\otimes\delta} & {}'H^*(s, s+r) \otimes {}''H^*(u+r, u+2r) \\
\Big\downarrow{\scriptstyle \delta\otimes 1} \quad \searrow^{\mu_r} & & \Big\downarrow{\scriptstyle \mu_r} \\
& H^*(s+u, s+u+r) & \\
& \searrow^{\delta} & \\
{}'H^*(s+r, s+2r) \otimes {}''H^*(u, u+r) & \xrightarrow{\mu_r} & H^*(s+u+r, s+u+2r)
\end{array}$$

with $r \geq 1$ and $s, u \in \mathbb{Z}$, the diagonal composite equals the sum of the two outer composites:

$$\delta\mu_r = \mu_r(\delta \otimes 1) + \mu_r(1 \otimes \delta)\,.$$

Outline

Tensor product of cochain complexes

Many multiplicative Cartan–Eilenberg systems, with associated multiplicative spectral sequences, arise from filtered differential graded rings.

Definition

The tensor product of two cochain complexes $({}'C^*, {}'\delta)$ and $({}''C^*, {}''\delta)$ is the total complex

$$C^* = {}'C^* \otimes {}''C^*$$

with

$$C^k = \bigoplus_{i+j=k} {}'C^i \otimes {}''C^j\,,$$

equipped with the differential $\delta = {}'\delta \otimes 1 + 1 \otimes {}''\delta$, given by

$$\delta(x \otimes y) = {}'\delta(x) \otimes y + (-1)^{|x|} x \otimes {}''\delta(y)\,,$$

where $|x| = i$ is the total degree of $x \in {}'C^i$.

Symmetric monoidal structure

- The unit cochain complex is $\mathbb{Z}$, concentrated in degree 0.
- The twist isomorphism

$$\tau : {}'C^* \otimes {}''C^* \xrightarrow{\cong} {}''C^* \otimes {}'C^*$$

is the chain isomorphism given by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x .$$

Lemma

The tensor product, unit complex and twist isomorphism define a symmetric monoidal structure on the category of cochain complexes.

Proof

- This means that the tensor product is associative, unital and commutative, up to coherent isomorphisms.
- The associativity isomorphism

$$('C^* \otimes ''C^*) \otimes '''C^* \cong 'C^* \otimes (''C^* \otimes '''C^*)$$

  maps $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$.
- The unitality isomorphisms

$$\mathbb{Z} \otimes C^* \cong C^* \cong C^* \otimes \mathbb{Z}$$

  identify $1 \otimes x$, x and $x \otimes 1$.
- The commutativity isomorphism is given by the twist isomorphism.
- The required coherence diagrams are listed in [ML71, §VII.1 and §VII.7].

□

Differential graded rings

The tensor product lets us define pairings $'C^* \otimes {}''C^* \to C^*$ of two cochain complexes to a third. We concentrate on the case when the three cochain complexes are the same.

Definition

A differential graded ring is a cochain complex (C^*, δ) equipped with a unital and associative cochain homomorphism

$$\mu\colon C^* \otimes C^* \longrightarrow C^* .$$

- μ makes (C^*, δ) a monoid in the monoidal category of cochain complexes.
- μ maps $x \otimes y \in C^n \otimes C^m$ to $\mu(x \otimes y) = x \cdot y \in C^{n+m}$ and satisfies the Leibniz rule

$$\delta(x \cdot y) = \delta(x) \cdot y + (-1)^{|x|} x \cdot \delta(y) .$$

- There is a cocycle $1 \in C^0$ with $x \cdot 1 = x = 1 \cdot x$ for all x, and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and z.

Diagrams

In categorical terms, associativity and unitality ask that the diagrams

$$\begin{array}{ccccc}
(C^* \otimes C^*) \otimes C^* & & \xrightarrow{\cong} & & C^* \otimes (C^* \otimes C^*) \\
\Big\downarrow{\scriptstyle \mu\otimes 1} & & & & \Big\downarrow{\scriptstyle 1\otimes\mu} \\
C^* \otimes C^* & \xrightarrow{\mu} & C^* & \xleftarrow{\mu} & C^* \otimes C^*
\end{array}$$

and

$$\begin{array}{ccccc}
\mathbb{Z} \otimes C^* & \xrightarrow{\eta\otimes 1} & C^* \otimes C^* & \xleftarrow{1\otimes\eta} & C^* \otimes \mathbb{Z} \\
& {\scriptstyle\cong}\searrow & \Big\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\cong} & \\
& & C^* & &
\end{array}$$

commute, where $\eta\colon \mathbb{Z} \to C^*$ maps $1 \in \mathbb{Z}$ to $1 \in C^*$.

Singular cochains

Example

The singular cochains $C^*(X)$ on a space X form a differential graded ring, with respect to the cup product

$$\cup\colon C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$$

given by the Alexander–Whitney formula.

Cohomology ring

Lemma

The cohomology $H^*(C^*)$ of a differential graded ring (C^*, δ, μ) is a graded ring.

Proof.

For cocycles $x \in C^n$ and $y \in C^m$ the product of their cohomology classes $[x] \in H^n(C^*)$ and $[y] \in H^m(C^*)$ is the cohomology class

$$[x] \cdot [y] = [x \cdot y] \in H^{n+m}(C^*)$$

of the product $x \cdot y = \mu(x \cdot y)$.
This is a cocycle by the Leibniz rule, and its cohomology class only depends on the cohomology classes of x and y, by further applications of the Leibniz rule. □

Differential graded algebras

- If C^* is a complex of Λ-modules for some commutative ring Λ, and μ is Λ-bilinear, we say that C^* is a differential graded Λ-algebra, often abbreviated to a “DG algebra”.
- The cohomology $H^*(C^*)$ is then a graded Λ-algebra.

- The further abbreviation “DGA” can be confusing in this context, since a “DGA algebra” means a “differential graded augmented algebra”, in the terminology from the Cartan seminar.
- We will discuss augmentations later, in the context of Hopf algebras.

Massey products

- There is more structure in the cohomology of a differential graded ring than this graded ring structure, including a variety of Massey products.
- If $a = [x]$, $b = [y]$ and $c = [z]$ satisfy $a \cdot b = 0$ and $b \cdot c = 0$ in $H^*(C^*)$, then we can write $x \cdot y = \delta(u)$ and $y \cdot z = \delta(v)$, for some cochains u and v.
- The expression

$$w = u \cdot z - (-1)^{|x|} x \cdot v$$

then defines a cocycle, since

$$\delta(w) = \delta(u) \cdot z - x \cdot \delta(v) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0\,.$$

Massey products (cont.)

- Its cohomology class

$$[w] = [u \cdot z - (-1)^{|x|} x \cdot v] \in \langle a, b, c \rangle$$

  defines an element in the Massey product

$$\langle a, b, c \rangle \subset H^n(C^*)\,,$$

  where $n = |a| + |b| + |c| - 1$.
- Different choices of cobounding classes u and v may give different classes $[w]$, and the Massey product equals the set of all possible such values.
- NB: This is not the most standard sign convention.

Commutative DG rings

Definition

A differential graded ring (C^*, δ, μ) is commutative if the diagram

$$\begin{array}{ccc} C^* \otimes C^* & \xrightarrow[\cong]{\tau} & C^* \otimes C^* \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & C^* & \end{array}$$

commutes, i.e., if $x \cdot y = (-1)^{|x||y|} y \cdot x$ for all $x, y \in C^*$.

- The cohomology of a commutative differential graded ring is a (graded) commutative ring.

Homotopy commutative DG algebras

- There are natural examples of non-commutative differential graded rings, such as the cochains $C^*(X)$ on a space X, whose cohomology is nonetheless (graded) commutative.
- There are more flexible notions of commutativity up to chain homotopy, and higher chain homotopies, that are often more appropriate.
- An E_∞ DG algebra satisfies "homotopy everything" conditions.
- These lead to the construction of power operations in the cohomology of these differential graded rings, or algebras, of which the Steenrod operations in mod p cohomology are prime examples.

Pairings of filtered cochain complexes

We can consider pairings of two filtered cochain complexes to a third. We concentrate on the case when the three filtered cochain complexes are the same.

Definition

A filtered differential graded ring is a cochain complex (C^*, δ) equipped with a decreasing filtration $(F^s C^*)_s$ and an associative and unital cochain morphism

$$\mu\colon C^* \otimes C^* \longrightarrow C^*\,,$$

such that the product preserves the filtration.

In other words, the image of the composite

$$F^s C^* \otimes F^u C^* \longrightarrow C^* \otimes C^* \xrightarrow{\mu} C^*$$

is contained in $F^{s+u}C^*$, for all $s, u \in \mathbb{Z}$.

Lemma

Let C^* be a filtered differential graded ring. There is a unique chain map $\mu^{s,u}$ making the diagram

$$\begin{array}{ccc} F^sC^* \otimes F^uC^* & \xrightarrow{\mu^{s,u}} & F^{s+u}C^* \\ \downarrow & & \downarrow \\ C^* \otimes C^* & \xrightarrow{\mu} & C^* \end{array}$$

commute, for each pair (s, u).

Lemma (cont.)

These induce a unique chain map μ_r making the diagram

$$\begin{array}{ccc} F^s C^* \otimes F^u C^* & \xrightarrow{\mu^{s,u}} & F^{s+u} C^* \\ \downarrow & & \downarrow \\ \dfrac{F^s C^*}{F^{s+r} C^*} \otimes \dfrac{F^u C^*}{F^{u+r} C^*} & \xrightarrow{\mu_r} & \dfrac{F^{s+u} C^*}{F^{s+u+r} C^*} \end{array}$$

commute, for all $r \geq 1$, s and u.

Proof.

Both $\mu^{s+r,u}$ and $\mu^{s,u+r}$ take values in $F^{s+u+r} C^*$. □

A pairing of filtered cochain complexes induces a pairing of finite Cartan–Eilenberg systems and the associated spectral sequences.

Proposition ([Mas54, §7, §9])

Let C^* be a filtered differential graded ring, with associated finite Cartan–Eilenberg system

$$H^*(i,j) = H^*(F^iC^*/F^jC^*)$$

for integers $i \le j$.

- *The pairing μ induces a pairing*

  $$\mu_r \colon H^*(s,s+r) \otimes H^*(u,u+r) \longrightarrow H^*(s+u,s+u+r)$$

  of finite Cartan–Eilenberg systems, and a pairing

  $$\mu_r \colon E_r^s \otimes E_r^u \longrightarrow E_r^{s+u}$$

  of the associated spectral sequences, making $(E_r, d_r)_{r\ge 1}$ a ring spectral sequence.

Proposition (cont.)

- *The E_1-term is given by*

$$E_1^{s,t} = H^{s+t}(F^s C^* / F^{s+1} C^*) \, .$$

- *The E_1-pairing*

$$\mu_1 \colon H^*(F^s C^* / F^{s+1} C^*) \otimes H^*(F^u C^* / F^{u+1} C^*) \longrightarrow H^*(F^{s+u} C^* / F^{s+u+1} C^*)$$

is given by

$$\mu_1 \colon [\pi(\tilde{x})] \otimes [\pi(\tilde{y})] \longmapsto [\pi \mu^{s,u}(\tilde{x} \otimes \tilde{y})] \, ,$$

where $\pi \colon F^s C^* \to F^s C^* / F^{s+1} C^*$, etc.

Proposition (cont.)

- *If the filtration* $(F^s C^*)_s$ *exhausts* C^*, *then* (μ_r) *and*

$$\mu_\infty \colon H^*(s,\infty) \otimes H^*(u,\infty) \longrightarrow H^*(s+u,\infty)$$

define a pairing of extended Cartan–Eilenberg systems, with $H^*(s,\infty) = H^*(F^s C^*)$.

- *The pairing of spectral sequences converges to the filtration-preserving pairing*

$$\mu \colon H^*(C^*) \otimes H^*(C^*) \longrightarrow H^*(C^*)\,,$$

where $G^n = H^n(C^*)$ *is exhaustively filtered by* $F^s G^n = \mathrm{im}(H^n(F^s C^*) \to H^n(C^*))$, *for* $s \in \mathbb{Z}$.

Proof

The chain homomorphism

$$\mu_r \colon F^s C^* / F^{s+r} C^* \otimes F^u C^* / F^{u+r} C^* \longrightarrow F^{s+u} C^* / F^{s+u+r} C^*$$

and the cohomology cross product induce the finite Cartan–Eilenberg system pairing

$$\begin{aligned} \mu_r \colon H^*(F^s C^* / F^{s+r} C^*) &\otimes H^*(F^u C^* / F^{u+r} C^*) \\ &\xrightarrow{\times} H^*(F^s C^* / F^{s+r} C^* \otimes F^u C^* / F^{u+r} C^*) \\ &\xrightarrow{\mu_{r*}} H^*(F^{s+u} C^* / F^{s+u+r} C^*) \,. \end{aligned}$$

In the extended case we set $F^\infty C^* = 0$ and $F^{-\infty} C^* = C^*$, and the chain homomorphism $\mu^{s,u}$ induces

$$\mu_\infty \colon H^*(F^s C^*) \otimes H^*(F^u C^*) \xrightarrow{\times} H^*(F^s C^* \otimes F^u C^*) \xrightarrow{\mu^{s,u}_*} H^*(F^{s+u} C^*) \,.$$

We must confirm conditions (SPP I) and (SPP II) in the finite case, and condition (SPP III) in the extended case.

Proof (cont.)

The diagram

$$\begin{array}{ccc} F^{s'}C^* \otimes F^{u'}C^* & \xrightarrow{\mu^{s',u'}} & F^{s'+u'}C^* \\ \downarrow & & \downarrow \\ F^{s}C^* \otimes F^{u}C^* & \xrightarrow{\mu^{s,u}} & F^{s+u}C^* \end{array}$$

of cochain complexes commutes, for $s \leq s'$ and $u \leq u'$, and induces a commutative diagram

$$\begin{array}{ccc} \dfrac{F^{s'}C^*}{F^{s'+r'}C^*} \otimes \dfrac{F^{u'}C^*}{F^{u'+r'}C^*} & \xrightarrow{\mu_{r'}} & \dfrac{F^{s'+u'}C^*}{F^{s'+u'+r}C^*} \\ \downarrow & & \downarrow \\ \dfrac{F^{s}C^*}{F^{s+r}C^*} \otimes \dfrac{F^{u}C^*}{F^{u+r}C^*} & \xrightarrow{\mu_{r}} & \dfrac{F^{s+u}C^*}{F^{s+u+r}C^*} \end{array}$$

of quotient complexes, for $r \geq 1$, $r' \geq 1$, $s + r \leq s' + r'$ and $u + r \leq u' + r'$. Passing to cohomology, we obtain the square required to commute in (SPP I).

Proof (cont.)

Let $\tilde{x} \in F^s C^*$ and $\tilde{y} \in F^u C^*$ lift cocycles $x \in F^s C^* / F^{s+r} C^*$ and $y \in F^u C^* / F^{u+r} C^*$, representing classes $[x] \in H^*(s, s+r)$ and $[y] \in H^*(u, u+r)$. Note that $\delta(\tilde{x}) \in F^{s+r} C^{*+1}$ and $\delta(\tilde{y}) \in F^{u+r} C^{*+1}$. The product

$$\tilde{z} = \mu^{s,u}(\tilde{x} \otimes \tilde{y}) \in F^{s+u} C^*$$

then lifts

$$z = \mu_r(x \otimes y) \in \frac{F^{s+u} C^*}{F^{s+u+r} C^*}$$

representing $[z] = \mu_r([x] \otimes [y]) \in H^*(s+u, s+u+r)$. Its image

$$\delta([z]) = \delta\mu_r([x] \otimes [y]) \in H^{*+1}(s+u+r, s+u+r+1)$$

under the connecting homomorphism is then given by the class $[\pi\delta(\tilde{z})]$ of the image of the coboundary

$$\delta(\tilde{z}) = \delta\mu^{s,u}(\tilde{x} \otimes \tilde{y}) \in F^{s+u+r} C^{*+1}$$

under the projection
$\pi \colon F^{s+u+r} C^{*+1} \to F^{s+u+r} C^{*+1} / F^{s+u+r+1} C^{*+1}$.

Proof (cont.)

By the Leibniz rule,

$$\delta\mu(\tilde{x} \otimes \tilde{y}) = \mu(\delta(\tilde{x}) \otimes \tilde{y}) + (-1)^{|\tilde{x}|}\mu(\tilde{x} \otimes \delta(\tilde{y}))$$

in C^*, so $[\pi\delta(\tilde{z})]$ equals the sum of

$$[\pi\mu^{s+r,u}(\delta(\tilde{x}) \otimes \tilde{y})] = [\mu_1(\pi\delta(\tilde{x}) \otimes \pi(\tilde{y}))] = \mu_1(\delta([x]) \otimes \eta([y]))$$

and $(-1)^{|\tilde{x}|} = (-1)^{|x|} = (-1)^{|[x]|}$ times

$$[\pi\mu^{s,u+r}(\tilde{x} \otimes \delta(\tilde{y}))] = [\mu_1(\pi(\tilde{x}) \otimes \pi\delta(\tilde{y}))] = \mu_1(\eta([x]) \otimes \delta([y]))\,.$$

This proves that $\delta\mu_r = \mu_1(\delta \otimes \eta) + \mu_1(\eta \otimes \delta)$ when evaluated on any $[x] \otimes [y]$, as demanded by (SPP II).

Proof (cont.)

Letting $F^{\infty}C^* = 0$, the proof of (SPP I) extends as stated to the cases with $r' = \infty$ and $r \geq 1$ or $r = \infty$, where we interpret $n + \infty$ as ∞ for all integers n, and this proves (SPP III). $\square$

Remark

If we redefine π to be the canonical projection $\pi \colon F^sC^* \to F^sC^*/F^{s+r}C^*$, so that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$, then the above proof of (SPP II) proves the stronger form (SPP II+) from an earlier remark.

Outline

- Cartan–Eilenberg systems
  - Cohomological Cartan–Eilenberg systems
  - Pairings of Cartan–Eilenberg systems
  - Filtered differential graded rings
  - Multiplicative Serre spectral sequence
  - The cohomological Wang and Gysin sequences
  - Rational cohomology of integral E–M spaces
  - First p-torsion in $\pi_*(S^3)$
  - Cohomology of $K(\mathbb{Z}/2, 2)$

Serre's filtered DG ring

- We return to the situation of a fiber sequence

$$F \longrightarrow E \xrightarrow{p} B\,.$$

- Serre's original construction [Ser51] of his spectral sequence used singular cubes $\sigma\colon I^n \to E$ to define a cubical chain complex

$$(A_*(E), \partial)$$

  with homology calculating $H_*(E)$, which could be increasingly filtered by saying that σ lies in $F_sA_*(E)$ if $p\sigma\colon I^n \to E \to B$ factors through the projection $I^n \to I^s$ to the s first coordinates.
- Dually, the cubical cochain complex

$$(A^*(E), \delta)$$

  calculating $H^*(E)$ is decreasingly filtered by saying that a cochain lies in $F^sA^*(E)$ if it vanishes on chains of filtration $\leq s - 1$.

Serre's filtered DG ring (cont.)

- There is a cup product making $A^*(E)$ a differential graded ring, and the decreasing filtration $(F^sA^*(E))_s$ respects the product, making $A^*(E)$ a filtered differential graded ring.
- Hence the associated spectral sequence

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E),$$

  which is the cohomology Serre spectral sequence for $p\colon E \to B$, is a ring spectral sequence.
- The pairings of E_1- and E_2-terms are given in terms of the cup products in $A^*(B)$, $H^*(B)$ and $H^*(F)$, and the spectral sequence pairing converges to the cup product in $H^*(E)$.

Filtered singular cochains

- Instead of working with cubical chains and cochains, we will filter the singular cochain complex $C^*(E)$ by the subcomplexes $F^s C^*(E) = C^*(E, E_{s-1})$.
- These are not strictly respected by the cochain level cup product, because the cross product of two cochains vanishing on E_{s-1} and E_{u-1} will vanish on all chains in $E_{s-1} \times E$ and in $E \times E_{u-1}$, but usually not on all chains in $E_{s-1} \times E \cup E \times E_{u-1}$.
- Hence $C^*(E)$ is not a filtered differential graded ring, and we must give a different proof of the multiplicativity of the cohomology Serre spectral sequence.
- For this we will adapt [Whi78, §XIII.8], making use of excision isomorphisms and the formalism of pairings of Cartan–Eilenberg systems.

The Serre Cartan–Eilenberg system

Let $p\colon E \to B$ be a fibration, with B a CW complex. Let $E_s = p^{-1}(B^{(s)})$, with $E_s = \emptyset$ for $-\infty \le s < 0$ and $E_\infty = E$. Define a cohomological extended Cartan–Eilenberg system $H^* = H^*(p)$ by

$$H^*(i,j) = H^*(E_{j-1}, E_{i-1})$$

for $-\infty \le i \le j \le \infty$, with $\delta\colon H^*(i,j) \to H^{*+1}(j,k)$ equal to the connecting homomorphism

$$\delta\colon H^*(E_{j-1}, E_{i-1}) \longrightarrow H^{*+1}(E_{k-1}, E_{j-1})\,.$$

The associated spectral sequence is the cohomological Serre spectral sequence

$$E_r^{s,t} = E_r^{s,t}(p) \Longrightarrow_s H^{s+t}(E)$$

with

$$E_1^{s,t} \cong C^s_{CW}(B; \mathscr{H}^t(F)) \qquad \text{and} \qquad E_2^{s,t} \cong H^s(B; \mathscr{H}^t(F))\,.$$

Proposition

Let $p'\colon E' \to B'$ and $p''\colon E'' \to B''$ be fibrations, where B' and B'' are CW complexes. There is a natural pairing of extended Cartan–Eilenberg systems

$$\mu\colon (H^*(p'), H^*(p'')) \longrightarrow H^*(p' \times p'')$$

with components

$$\begin{aligned}\mu_r\colon\ & H^*(E'_{s+r-1}, E'_{s-1}) \otimes H^*(E''_{u+r-1}, E''_{u-1}) \\ & \xrightarrow{\times} H^*(E'_{s+r-1} \times E''_{u+r-1}, E'_{s-1} \times E''_{u+r-1} \cup E'_{s+r-1} \times E''_{u-1}) \\ & \longrightarrow H^*((E' \times E'')_{s+u+r-1}, (E' \times E'')_{s+u-1})\end{aligned}$$

and

$$\begin{aligned}\mu_\infty\colon\ & H^*(E', E'_{s-1}) \otimes H^*(E'', E''_{u-1}) \\ & \xrightarrow{\times} H^*(E' \times E'', E'_{s-1} \times E'' \cup E' \times E''_{u-1}) \\ & \longrightarrow H^*(E' \times E'', (E' \times E'')_{s+u-1}) .\end{aligned}$$

Proof

To simplify the notation a little we restrict to the case where $p' = p'' = p\colon E \to B$, but the general case is easily recovered by working with p' in the first factor and p'' in the second factor of each product.

The product $B \times B$ has the CW structure with k-skeleton

$$(B \times B)^{(k)} = \bigcup_{i+j=k} B^{(i)} \times B^{(j)} .$$

We lift the skeleton filtration along $p \times p$ to define the filtration on $E \times E$ with

$$(E \times E)_k = \bigcup_{i+j=k} E_i \times E_j .$$

Proof (cont.)

We then have inclusions

$$(B \times B)^{(s+u-1)} \subset B^{(s-1)} \times B \cup B \times B^{(u-1)}$$

and

$$(E \times E)_{s+u-1} \subset E_{s-1} \times E \cup E \times E_{u-1}$$

of subspaces of $B \times B$ and $E \times E$, respectively.
This defines

$$\begin{aligned}\mu_\infty \colon H^*(E, E_{s-1}) \otimes H^*(E, E_{u-1}) \\ \xrightarrow{\times} H^*(E \times E, E_{s-1} \times E \cup E \times E_{u-1}) \\ \longrightarrow H^*(E \times E, (E \times E)_{s+u-1})\end{aligned}$$

as the composite of the cohomology cross product and the (now) evident restriction map.

Proof (cont.)

The definition of μ_r for finite $r \geq 1$ is a little more elaborate. The subcomplexes

$$B^{(s+r-1)} \times B^{(u+r-1)}$$

and

$$(B \times B)^{\wedge}_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} B^{(i)} \times B^{(j)}$$

of $B \times B$ have intersection

$$B^{(s-1)} \times B^{(u+r-1)} \cup B^{(s+r-1)} \times B^{(u-1)}$$

and union

$$B^{(s+r-1)} \times B^{(u+r-1)} \cup (B \times B)^{(s+u+r-1)} .$$

Note that $(B \times B)^{(s+u-1)} \subset (B \times B)^{\wedge}_{s,u,r}$.

Proof (cont.)

Likewise, the subspaces

$$E_{s+r-1} \times E_{u+r-1}$$

and

$$(E \times E)^{\wedge}_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} E_i \times E_j$$

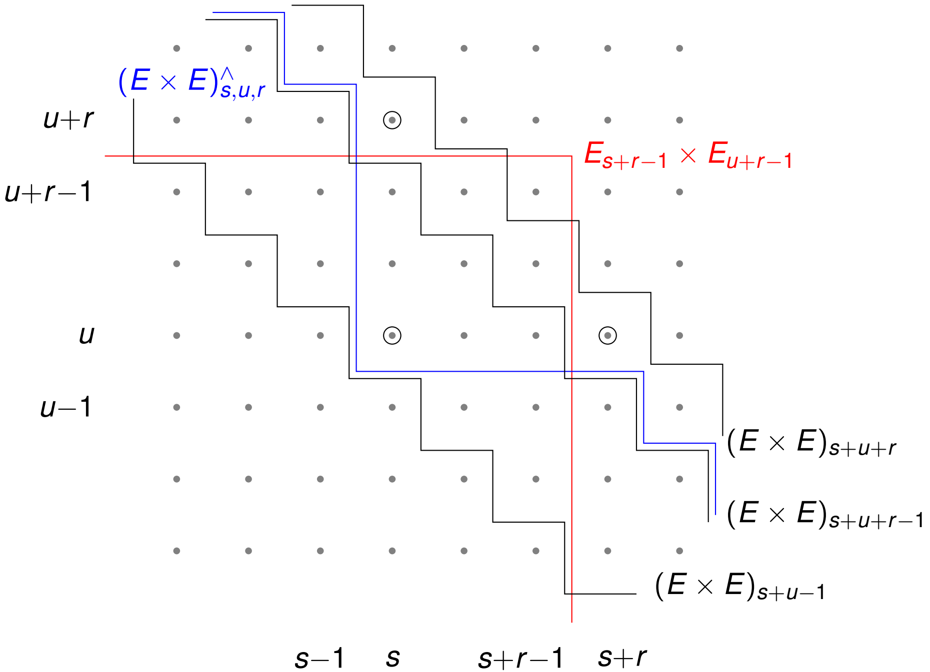
of $E \times E$ have intersection

$$E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1}$$

and union

$$E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1} \, .$$

Furthermore, $(E \times E)_{s+u-1} \subset (E \times E)^{\wedge}_{s,u,r}$.



$(E \times E)^{\wedge}_{s,u,r}$
$u+r$
$E_{s+r-1} \times E_{u+r-1}$
$u+r-1$
u
$u-1$
$(E \times E)_{s+u+r}$
$(E \times E)_{s+u+r-1}$
$(E \times E)_{s+u-1}$
$s-1$
s
$s+r-1$
$s+r$

Proof (cont.)

Hence there is an excision isomorphism

$$H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^{\wedge}_{s,u,r})$$
$$\xrightarrow{\cong} H^*(E_{s+r-1} \times E_{u+r-1}, E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1}),$$

and a restriction homomorphism

$$H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^{\wedge}_{s,u,r})$$
$$\longrightarrow H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}).$$

The pairing μ_r equals the composite

$$H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1})$$
$$\xrightarrow{\times} H^*(E_{s+r-1} \times E_{u+r-1}, E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1})$$
$$\xleftarrow{\cong} H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^{\wedge}_{s,u,r})$$
$$\longrightarrow H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}).$$

Proof (cont.)

Condition (SPP I) follows by naturality of the three homomorphisms composing to μ_r with respect to the inclusions

$$\begin{aligned} E_{s-1} &\subset E_{s'-1} \\ E_{s+r-1} &\subset E_{s'+r'-1} \\ E_{u-1} &\subset E_{u'-1} \\ E_{u+r-1} &\subset E_{u'+r'-1} \\ (E \times E)_{s+u-1} &\subset (E \times E)_{s'+u'-1} \\ (E \times E)_{s+u+r-1} &\subset (E \times E)_{s'+u'+r'-1} \\ (E \times E)^{\wedge}_{s,u,r} &\subset (E \times E)^{\wedge}_{s',u',r'} \end{aligned}$$

for $s \leq s'$, $u \leq u'$, $s + r \leq s' + r'$ and $u + r \leq u' + r'$.

Proof (cont.)

Only the last one requires comment: The inclusion

$$(E \times E)^{\wedge}_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} E_i \times E_j$$
$$\subset \bigcup_{\substack{i'+j'=s'+u'+r'-1 \\ i'<s' \text{ or } j'<u'}} E_{i'} \times E_{j'} = (E \times E)^{\wedge}_{s',u',r'}$$

holds since if $i < s$ and $i + j = s + u + r - 1$ then $E_i \times E_j \subset E_i \times E_{j'}$ with $i < s'$ and $i + j' = s' + u' + r' - 1$, and similarly if $j < u \leq u'$.

Proof (cont.)

Condition (SPP III) holds in the same way, setting $r' = \infty$, and noting that the excision isomorphism in the definition of μ_r is the identity map of

$$H^*(E \times E, E_{s-1} \times E \cup E \times E_{u-1})$$

when $r = \infty$.

To verify condition (SPP II) we consider the composite

$$\begin{aligned}
H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \\
&\xrightarrow{\mu_r} H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \\
&\xrightarrow{\delta} H^{*+1}((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}) \\
&\cong \prod_{i+j=s+u+r} H^{*+1}(E_i \times E_j, E_{i-1} \times E_j \cup E_i \times E_{j-1}),
\end{aligned}$$

where the final isomorphism follows from excision.

Proof (cont.)

We claim that

1. the component with $(i, j) = (s + r, u)$ equals

$$H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \xrightarrow{\delta\otimes\eta} H^{*+1}(E_{s+r}, E_{s+r-1}) \otimes H^*(E_u, E_{u-1}) \xrightarrow{\times} H^{*+1}(E_{s+r} \times E_u, E_{s+r-1} \times E_u \cup E_{s+r} \times E_{u-1}),$$

2. the component with $(i, j) = (s, u + r)$ equals

$$H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \xrightarrow{\eta\otimes\delta} H^{*+1}(E_s, E_{s-1}) \otimes H^*(E_{u+r}, E_{u+r-1}) \xrightarrow{\times} H^{*+1}(E_s \times E_{u+r}, E_{s-1} \times E_{u+r} \cup E_s \times E_{u+r-1}),$$

and

3. the remaining components are zero.

This implies the relation

$$\delta\mu_r = \mu_1(\delta \otimes \eta) + \mu_1(\eta \otimes \delta).$$

Proof (cont.)

For the first claim we use the commutative diagram in the figure on the next page, with the following abbreviations.

$$X = E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r}$$
$$Y = E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}$$
$$Z = E_{s-1} \times E_u \cup E_{s+r} \times E_{u-1}$$

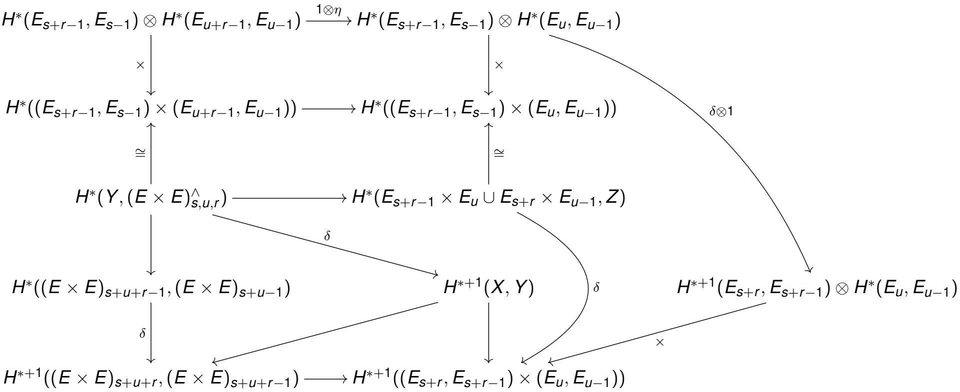
The two quadrangles containing $H^{*+1}(X, Y)$ commute by the naturality of δ with respect to the maps of triples

$$((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \subset (X, Y, (E \times E)^{\wedge}_{s,u,r})$$

and

$$(E_{s+r} \times E_u, E_{s+r-1} \times E_u \cup E_{s+r} \times E_{u-1}, Z) \subset (X, Y, (E \times E)^{\wedge}_{s,u,r}) .$$

The second claim follows from a similar diagram.



The component $(i, j) = (s + r, u)$ of $\delta\mu_r$

Proof (cont.)

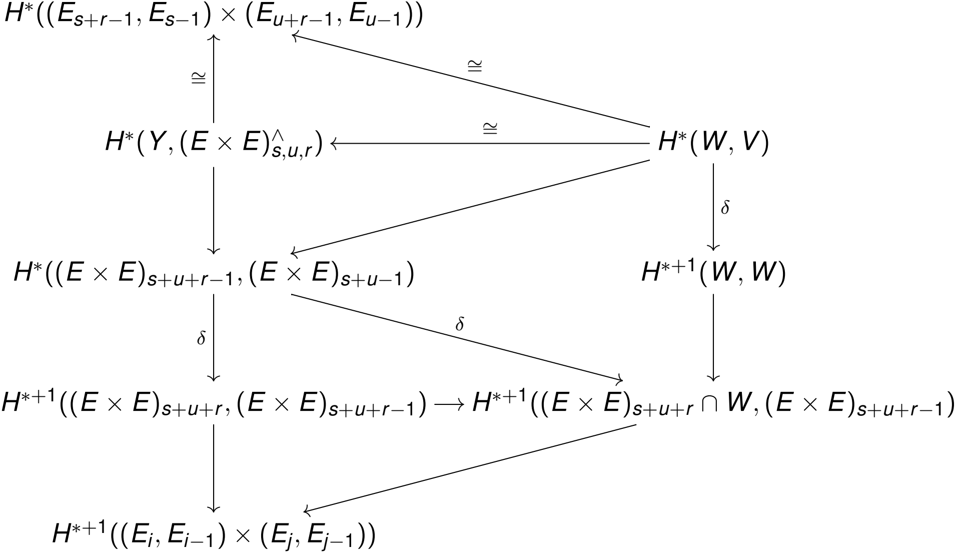
For the third claim we assume $i + j = s + u + r$ with $i \notin \{s, s + r\}$, so that $j \notin \{u, u + r\}$, and use the abbreviations

$$\begin{aligned} V &= E_{s-1} \times E \cup E \times E_{u-1} \\ W &= E_{s-1} \times E \cup E_{s+r-1} \times E_{u+r-1} \cup E \times E_{u-1} \end{aligned}$$

and the commutative diagram on the next page.
The quadrangle commutes by naturality of δ with respect to the map of triples

$$((E \times E)_{s+u+r} \cap W, (E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \subset (W, W, V) .$$

Since $H^{*+1}(W, W)$ is trivial, it follows that the left hand vertical composite is zero. □



The trivial components (i, j) of $\delta\mu_r$

Pairing of Serre spectral sequences

The pairing

$$\mu\colon (H^*(p'), H^*(p'')) \longrightarrow H^*(p' \times p'')$$

of extended Cartan–Eilenberg systems induces a pairing

$$(\mu_r\colon (E_r(p'), E_r(p'')) \to E_r(p' \times p''))$$

of the associated cohomological Serre spectral sequences, converging to a filtration-preserving pairing

$$\nu\colon H^*(E') \otimes H^*(E'') \longrightarrow H^*(E' \times E'')$$

of their abutments. We now make these pairing explicit.

Pairing of Serre E_1-terms

Recall the isomorphism
$E_1^{s,t} = H^{s+t}(E_s, E_{s-1}) \cong C^s_{CW}(B; \mathscr{H}^t(F))$.

Proposition

The pairing of E_1-terms

$$E_1^{s,t}(p') \otimes E_1^{u,v}(p'') = H^{s+t}(E'_s, E'_{s-1}) \otimes H^{u+v}(E''_u, E''_{u-1})$$
$$\xrightarrow{\mu_1} H^{s+u+t+v}((E' \times E'')_{s+u}, (E' \times E'')_{s+u-1}) = E_1^{s+u,t+v}(p' \times p'')$$

corresponds to $(-1)^{tu}$ times the cross product

$$C^s_{CW}(B'; \mathscr{H}^t(F)) \otimes C^u_{CW}(B''; \mathscr{H}^v(F'))$$
$$\xrightarrow{\times} C^{s+u}_{CW}(B' \times B''; \mathscr{H}^{t+v}(F \times F'))\,.$$

Sketch proof

Assume $p' = p'' = p$. The cohomology cross products

$$H^s(B^{(s)}, B^{(s-1)}; \mathscr{H}^t(F)) \otimes H^u(B^{(u)}, B^{(u-1)}; \mathscr{H}^v(F)) \xrightarrow{\times} H^{s+u}(B^{(s+u)}, B^{(s+u-1)}; \mathscr{H}^t(F) \otimes \mathscr{H}^v(F))$$

and

$$\mathscr{H}^t(F) \otimes \mathscr{H}^v(F) \xrightarrow{\times} \mathscr{H}^{t+v}(F \times F)$$

combine to define the cross product of the proposition. The sign $(-1)^{tu}$ arises from the factor

$$H^{s+t}((I^s_\alpha, \partial I^s_\alpha) \times F_{b_\alpha}) \otimes H^{u+v}((I^u_\beta, \partial I^u_\beta) \times F_{b_\beta}) \longrightarrow H^{s+u+t+v}((I^{s+u}_{\alpha,\beta}, \partial I^{s+u}_{\alpha,\beta}) \times F_{b_\alpha} \times F_{b_\alpha})$$

of the pairing μ_1, which sends $(g_{s,\alpha} \times f_\alpha) \otimes (g_{u,\beta} \times f_\beta)$ to $(-1)^{tu} g_{s+u,\alpha,\beta} \times f_\alpha \times f_\beta$, where $t = |f_\alpha|$. $\square$

Pairing of Serre E_2-terms

Lemma

The pairing of E_2-terms

$$\mu_2 \colon E_2^{s,t}(p') \otimes E_2^{u,v}(p') \longrightarrow E_2^{s+u,t+v}(p' \times p'')$$

corresponds to $(-1)^{tu}$ times the cohomology cross product

$$H^s(B'; \mathscr{H}^t(F')) \otimes H^u(B''; \mathscr{H}^v(F'')) \xrightarrow{\times} H^{s+u}(B' \times B''; \mathscr{H}^{t+v}(F' \times F'')) .$$

Proof.

We obtain μ_2 from μ_1 by passing to cohomology with respect to the d_1-differentials. □

Pairing of Serre abutments

Lemma

The filtration-preserving pairing

$$\nu\colon H^*(E') \otimes H^*(E'') \longrightarrow H^*(E' \times E'')$$

equals the cohomology cross product.

Proof.

By definition,

$$\begin{aligned}\mu_\infty\colon H^*(E', E'_{s-1}) \otimes H^*(E'', E''_{u-1}) \\ \longrightarrow H^*(E' \times E'', E'_{s-1} \times E'' \cup E' \times E''_{u-1}) \\ \longrightarrow H^*(E' \times E'', (E' \times E'')_{s+u-1})\end{aligned}$$

is given by the relative cohomology cross product followed by restriction. Passing to the colimit for $s \to -\infty$ and $u \to -\infty$ gives ν, and this colimit is achieved already for $s = u = 0$. □

External cross to internal cup

To pass from the external cross product to the internal cup product, we assume $p' = p'' = p\colon E \to B$ and pull back along a filtration-preserving approximation $D\colon E \to E \times E$ to the diagonal map $\Delta\colon E \to E \times E$.

$$\cup\colon H^*(E) \otimes H^*(E) \xrightarrow{\times} H^*(E \times E) \xrightarrow{D^*=\Delta^*} H^*(E)$$

Let B be a CW complex based at a 0-cell b_0, let $p\colon E \to B$ be a (Hurewicz) fibration, and let $F = p^{-1}(b_0)$ be its fiber.

Proposition

- *There is a homotopy*

$$\bar{H}\colon I \times B \longrightarrow B \times B$$

with $\bar{H}(t, b_0) = (b_0, b_0)$ *for all* t*, from the diagonal map* $\Delta\colon B \to B \times B$ *to a cellular map* $\bar{D}\colon B \to B \times B$*.*

- *It admits a lift*

$$H\colon I \times E \longrightarrow E \times E$$

with $(p \times p)H = \bar{H}(1 \times p)$*, from the diagonal map* $\Delta\colon E \to E \times E$ *to a filtration-preserving map* $D\colon E \to E \times E$*.*

- *This restricts to a homotopy*

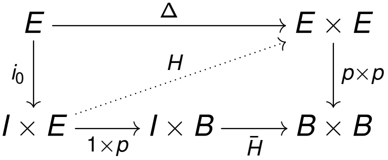
$$\tilde{H}\colon I \times F \longrightarrow F \times F$$

from the diagonal map $\Delta\colon F \to F \times F$ *to a map* $\tilde{D}\colon F \to F \times F$*.*

Proof.

By cellular approximation, the map $\Delta\colon B \to B \times B$ is homotopic to a cellular map $\bar{D}\colon B \to B \times B$, and we may assume that the homotopy $\bar{H}$ is stationary on $\{b_0\}$, since Δ is already cellular on that subspace.

The diagonal map $\Delta\colon E \to E \times E$ lifts $\Delta p\colon E \to B \times B$, so by the homotopy lifting property for $p \times p$ we have a homotopy $H\colon I \times E \to E \times E$ from Δ to $D\colon E \to E \times E$ with $(p \times p)D = \bar{D}p$.



The restriction $H|I \times F$ then factors through $F \times F \subset E \times E$, giving the required homotopy $\tilde{H}$ from $\Delta\colon F \to F \times F$ to $\tilde{D}$. □

Morphism of Cartan–Eilenberg systems

Proposition

- *The filtration-preserving map $D\colon E \to E \times E$ induces a morphism*
$$D^*\colon H^*(p \times p) \longrightarrow H^*(p)$$
of Cartan–Eilenberg systems and a morphism
$$D_r^*\colon E_r^{*,*}(p \times p) \longrightarrow E_r^{*,*}(p)$$
of cohomological Serre spectral sequences.
- *The homomorphism D_1^* corresponds to the restriction*
$$\bar{D}^*\colon C^*_{CW}(B \times B; \mathscr{H}^*(F \times F)) \longrightarrow C^*_{CW}(B; \mathscr{H}^*(F))$$
associated to the cellular map $\bar{D}\colon B \to B \times B$ and the coefficient homomorphism $\tilde{D}^* = \Delta^*\colon \mathscr{H}^*(F \times F) \to \mathscr{H}^*(F)$.

Proposition (cont.)

- *The homomorphism D_2^* corresponds to the restriction homomorphism*

$$\bar{D}^* = \Delta^*\colon H^*(B \times B; \mathscr{H}^*(F \times F)) \longrightarrow H^*(B; \mathscr{H}^*(F))\,.$$

- *The induced morphisms of filtered target groups is*

$$D^* = \Delta^*\colon H^*(p \times p)(-\infty, \infty) = H^*(E \times E) \longrightarrow H^*(E) = H^*(p)(-\infty, \infty)\,.$$

Proof.

The map of pairs $D\colon (E_{j-1}, E_{i-1}) \to ((E \times E)_{j-1}, (E \times E)_{i-1})$ induces

$$D^*\colon H^*(p \times p)(i,j) = H^*((E \times E)_{j-1}, (E \times E)_{i-1}) \longrightarrow H^*(E_{j-1}, E_{i-1}) = H^*(p)(i,j)$$

for all (extended) integers $i \leq j$.

The rest follows by chasing the definitions, and using the homotopies $\bar{H}$, $\tilde{H}$ and H to note that $\bar{D}^* = \Delta^*$, $\tilde{D}^* = \Delta^*$ and $D^* = \Delta^*$, once we have passed to cohomology groups. □

Multiplicative Serre spectral sequence, I

Theorem

Let $p\colon E \to B$ be a Hurewicz fibration, with B a CW complex. Each choice of filtration-preserving lift $D\colon E \to E \times E$ lifting a (cellular) diagonal approximation $\bar{D}\colon B \to B \times B$ induces a pairing of extended Cartan–Eilenberg systems

$$D^*\mu\colon (H^*(p), H^*(p)) \longrightarrow H^*(p)$$

and of cohomological Serre spectral sequences

$$D^*\mu_r\colon (E_r^{*,*}(p), E_r^{*,*}(p)) \longrightarrow E_r^{*,*}(p)\,.$$

Multiplicative Serre spectral sequence, II

Theorem (cont.)

The pairing of E_1-terms

$$E_1^{s,t}(p) \otimes E_1^{u,v}(p) \longrightarrow E_1^{s+u,t+v}(p)$$

corresponds to $(-1)^{tu}$ *times the cochain cup product*

$$C^s_{CW}(B; \mathscr{H}^t(F)) \otimes C^u_{CW}(B; \mathscr{H}^v(F)) \xrightarrow{\cup} C^{s+u}_{CW}(B; \mathscr{H}^{t+v}(F))$$

associated to $\bar{D}$.

Multiplicative Serre spectral sequence, III

Theorem (cont.)

The pairing of E_2-term,

$$E_2^{s,t}(p) \otimes E_2^{u,v}(p) \longrightarrow E_2^{s+u,t+v}(p)$$

corresponds to $(-1)^{tu}$ times the cohomology cup product

$$H^s(B; \mathscr{H}^t(F)) \otimes H^u(B; \mathscr{H}^v(F)) \xrightarrow{\cup} H^{s+u}(B; \mathscr{H}^{t+v}(F)),$$

and is independent of the choice of D and $\bar{D}$.

This pairing of spectral sequences converges to the cup product pairing

$$H^*(E) \otimes H^*(E) \xrightarrow{\cup} H^*(E)$$

in the cohomology of the total space.

Proof.

This follows by composing the external cross product pairing μ with the diagonal approximation morphism D^*. The composites

$$\mathscr{H}^t(F) \otimes \mathscr{H}^v(F) \xrightarrow{\times} \mathscr{H}^{t+v}(F \times F) \xrightarrow{\tilde{D}^*} \mathscr{H}^{t+v}(F)$$

$$H^s(B) \otimes H^u(B) \xrightarrow{\times} H^{s+u}(B \times B) \xrightarrow{\bar{D}^*} H^{s+u}(B)$$

$$H^*(E) \otimes H^*(E) \xrightarrow{\times} H^*(E \times E) \xrightarrow{D^*} H^*(E)$$

are equal to the respective cup products, in view of the homotopies $\tilde{H} \colon \Delta \simeq \tilde{D}$, $\bar{H} \colon \Delta \simeq \bar{D}$ and $H \colon \Delta \simeq D$. □

Outline

Cartan–Eilenberg systems

- Cohomological Cartan–Eilenberg systems
- Pairings of Cartan–Eilenberg systems
- Filtered differential graded rings
- Multiplicative Serre spectral sequence
- The cohomological Wang and Gysin sequences
- Rational cohomology of integral E–M spaces
- First p-torsion in $\pi_*(S^3)$
- Cohomology of $K(\mathbb{Z}/2, 2)$

Wang sequence

Theorem

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence, with $B \simeq S^u$ a 1-connected CW complex. There is a long exact sequence

$$\cdots \to H^{n-1}(F) \xrightarrow{\delta} H^{n-u}(F) \xrightarrow{i^!} H^n(E) \xrightarrow{i^*} H^n(F) \xrightarrow{\delta} H^{n-u+1}(F) \to \dots$$

where i^* is a ring homomorphism and

$$\delta(x \cup y) = \delta(x) \cup y + (-1)^{|x|(u-1)} x \cup \delta(y) .$$

Proof

- The Serre spectral sequence

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow_s H^{s+t}(E)$$

  is a ring spectral sequence with $E_2 = E_u$ and $E_{u+1} = E_\infty$.
- Setting $H^*(B) = \mathbb{Z}\{1, g_u\}$ we can write $d_u(1 \otimes x) = g_u \otimes \delta(x)$ with $\delta\colon H^t(F) \to H^{t-u+1}(F)$.
- The Leibniz rule

$$d_u(1 \otimes x \cup y) = d_u(1 \otimes x) \cup (1 \otimes y) + (-1)^{|x|}(1 \otimes x) \cup d_u(1 \otimes y)$$

  translates to the given derivation rule for δ. □

Divided power and exterior algebras

- Recall the divided power algebra $\Gamma(x) = \mathbb{Z}\{\gamma_i(x) \mid i \geq 0\}$ with $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and

$$\gamma_i(x) \cdot \gamma_j(x) = (i,j)\gamma_{i+j}(x)\,,$$

  graded so that $|\gamma_i(x)| = i|x|$.
- Here

$$(i,j) = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$$

  is the binomial coefficient.
- Let $\Lambda(x) = \mathbb{Z}\{1, x\}$ denote the exterior algebra on x, with $x^2 = 0$.
- Usually $|x|$ is even in the divided power case, and odd in the exterior case.

Loop spaces of spheres

Theorem

Let $u \geq 2$. If u is odd, then

$$H^*(\Omega S^u) \cong \Gamma(x)$$

with $|x| = u - 1$. If u is even, then

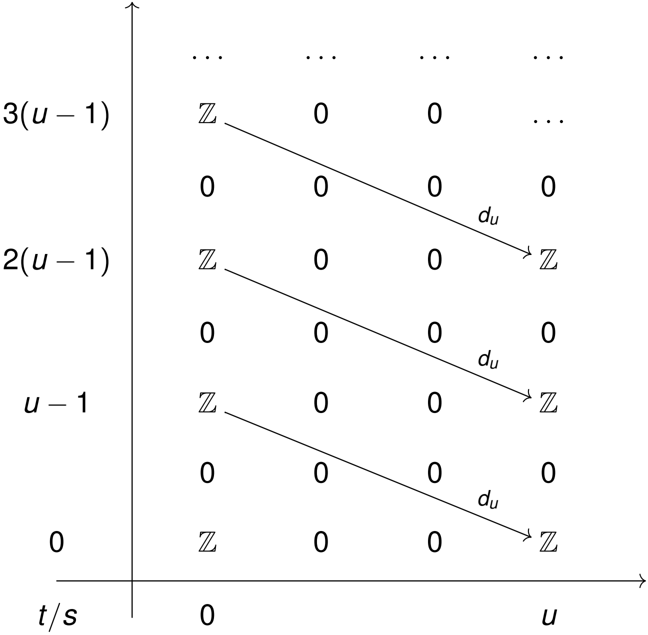
$$H^*(\Omega S^u) \cong \Lambda(x) \otimes \Gamma(y)$$

with $|x| = u - 1$ and $|y| = 2(u - 1)$.

Proof.

The Wang sequence for $\Omega S^u \to PS^u \to S^u$, with PS^u contractible, reduces to isomorphisms

$$\delta \colon \tilde{H}^n(F) \xrightarrow{\cong} H^{n-u+1}(F)\,.$$



3(u − 1)
2(u − 1)
u − 1
0
t/s
0
u
ℤ 0 0 ...
0 0 0 0
ℤ 0 0 ℤ
0 0 0 0
ℤ 0 0 ℤ
0 0 0 0
ℤ 0 0 ℤ
d_u
d_u
d_u

Proof (cont.) for u odd

Suppose first that $u \geq 3$ is odd. Let $\gamma_0(x) = 1$ and inductively set $\gamma_i(x) \in H^{i(u-1)}(\Omega S^u)$ for $i \geq 1$ so that $\delta(\gamma_i(x)) = \gamma_{i-1}(x)$. By induction on i and j,

$$\delta(\gamma_i(x) \cup \gamma_j(x)) = \gamma_{i-1}(x) \cup \gamma_j(x) + \gamma_i(x) \cup \gamma_{j-1}(x)$$

equals $(i-1, j) + (i, j-1) = (i, j)$ times

$$\delta(\gamma_{i+j}(x)) = \gamma_{i+j-1}(x)\,.$$

This proves that $\gamma_i(x) \cup \gamma_j(x) = (i, j)\gamma_{i+j}(x)$.

Proof (cont.) for u even

Next suppose that $u \geq 2$ is even. Fix $x \in H^{u-1}(\Omega S^u)$ so that $\delta(x) = 1$. By graded commutativity, $x^2 = 0$.

Let $\gamma_0(y) = 1$ and inductively set $\gamma_i(y) \in H^{2i(u-1)}(\Omega S^u)$ for $i \geq 1$ so that $\delta(\gamma_i(y)) = x\gamma_{i-1}(y)$. Then

$$\delta(x\gamma_i(y)) = 1 \cup \gamma_i(y) - x \cup x\gamma_{i-1}(y) = \gamma_i(y)\,,$$

so $\gamma_i(y)$ generates $H^{2i(u-1)}(\Omega S^u)$ while $x\gamma_i(y)$ generates $H^{(2i+1)(u-1)}(\Omega S^u)$. By induction on i and j,

$$\delta(\gamma_i(y) \cup \gamma_j(y)) = x\gamma_{i-1}(y) \cup \gamma_j(y) + \gamma_i(y) \cup x\gamma_{j-1}(y)$$

equals $(i-1, j) + (i, j-1) = (i, j)$ times

$$\delta(\gamma_{i+j}(y)) = x\gamma_{i+j-1}(y)\,.$$

Hence $\gamma_i(y) \cup \gamma_j(y) = (i, j)\gamma_{i+j}(y)$. $\square$

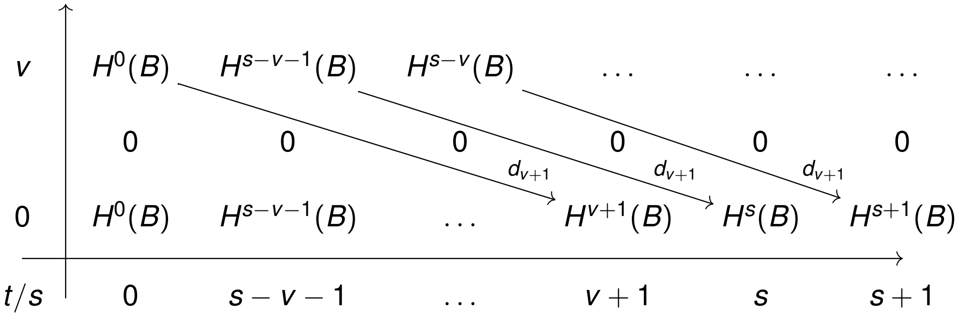
Gysin sequence

Theorem

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence, with $F \simeq S^v$ and B a 1-connected CW complex. There is a long exact sequence

$$\cdots \to H^{n-v-1}(B) \xrightarrow{e\cup} H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{p^!} H^{n-v}(B) \xrightarrow{e\cup} H^{n+1}(B) \to \dots$$

where p^* is a ring homomorphism and $e = \delta(1) \in H^{v+1}(B)$ is the Euler class of the (oriented spherical) fibration.


v
0
t/s
$H^0(B)$
$H^{s-v-1}(B)$
$H^{s-v}(B)$
. . .
. . .
. . .
0
0
0
0
0
0
d_{v+1}
d_{v+1}
d_{v+1}
$H^0(B)$
$H^{s-v-1}(B)$
. . .
$H^{v+1}(B)$
$H^s(B)$
$H^{s+1}(B)$
0
$s-v-1$
. . .
$v+1$
s
$s+1$

Proof

- The Serre spectral sequence

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow_s H^{s+t}(E)$$

  is a ring spectral sequence with $E_2 = E_{v+1}$ and $E_{v+2} = E_\infty$.
- Setting $H^*(F) = \mathbb{Z}\{1, g_v\}$ we can write $d_{v+1}(x \otimes g_v) = \delta(x) \otimes 1$ with $\delta\colon H^{s-v-1}(B) \to H^s(B)$.
- The Leibniz rule

$$\begin{aligned} &d_{v+1}((1 \otimes g_v) \cup (x \otimes 1)) \\ &\quad = d_{v+1}(1 \otimes g_v) \cup (x \otimes 1) + (-1)^v (1 \otimes g_v) \cup d_{v+1}(x \otimes 1) \end{aligned}$$

  translates to $\delta(x) = (-1)^{v|x|} e \cup x$, since $d_{v+1}(x \otimes 1) = 0$ lies in a trivial group.
- We can replace δ with $x \mapsto e \cup x$ without affecting the exactness of the sequence. □

Euler characteristic

Remark

The Euler class vanishes if p admits a section $s\colon B \to E$. If B is a closed, oriented $(v+1)$-manifold with fundamental class

$$[B] \in H_{v+1}(B)\,,$$

and $E = S(TB) \to B$ is the unit sphere bundle in the tangent bundle $TB \to B$, then the Euler class

$$e \in H^{v+1}(B)$$

evaluates on $[B]$ to the Euler characteristic of B:

$$\langle e, [B] \rangle = \chi(B)\,.$$

See [MS74, Cor. 11.12].

In particular, the Euler characteristic vanishes if B admits an everywhere nonzero vector field.

Complex Grassmannians

- Let $U(k)$ denote the rank k unitary group.
- It acts freely on the contractible Stiefel space

$$V_k(\mathbb{C}^\infty) = \{(v_1, \ldots, v_k) \mid v_i^* v_j = \delta_{i,j}\}$$

of unitary k-frames in $\mathbb{C}^\infty = \bigcup_n \mathbb{C}^n$.

- The orbit space is the Grassmannian

$$Gr_k(\mathbb{C}^\infty) = \{V \subset \mathbb{C}^\infty \mid \dim_{\mathbb{C}}(V) = k\}$$

of k-dimensional complex linear subspaces of $\mathbb{C}^\infty$.

Classification of complex vector bundles

- The principal $U(k)$-bundle

$$U(k) \longrightarrow V_k(\mathbb{C}^\infty) \longrightarrow Gr_k(\mathbb{C}^\infty)$$

  is thus universal, and $Gr_k(\mathbb{C}^\infty) \simeq BU(k)$ is a model for the classifying space of $U(k)$.
- We get natural bijections

$$\mathrm{Vect}_k^{\mathbb{C}}(B) \cong \mathrm{Bun}_{U(k)}(B) \cong [B, BU(k)] \cong [B, Gr_k(\mathbb{C}^\infty)]$$

  for all CW complexes B.
- Here $\mathrm{Vect}_k^{\mathbb{C}}(B)$ denotes the set of isomorphism classes of rank k complex vector bundles $E \to B$.

The first Chern class

- When $k = 1$, we have $V_1(\mathbb{C}^\infty) = S(\mathbb{C}^\infty) \cong S^\infty$ and $Gr_1(\mathbb{C}^\infty) \cong \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$.
- Hence

$$[B, BU(1)] \cong [B, \mathbb{C}P^\infty] \cong [B, K(\mathbb{Z}, 2)] \cong H^2(B)$$

  by the Eilenberg–Steenrod representability theorem.
- The class $c_1(L) \in H^2(B)$ corresponding to a complex line bundle $L \to B$ is called the first Chern class of L, and classifies L up to isomorphism.

Characteristic classes

- When $k \geq 2$, the space $BU(k) \simeq Gr_k(\mathbb{C}^\infty)$ is not an Eilenberg–Mac Lane space, so $[B, BU(k)]$ is not naturally identified with a cohomology group of B.
- However, each cohomology class $c \in H^n(BU(k))$ pulls back along the classifying map $f\colon B \to BU(k)$ of any $\mathbb{C}^k$-bundle $E \to B$ to define a class

$$c(E) = f^*(c) \in H^n(B)\,.$$

- This class $c(E)$ depends naturally on $E \to B$, and is called a characteristic class.
- To determine all characteristic classes for complex vector bundles, we calculate $H^*(BU(k))$.

Cohomology of complex Grassmannians

Theorem

For each $k \geq 0$ there are isomorphisms

$$H^*(BU(k)) \cong \mathbb{Z}[c_1, \ldots, c_k]$$

with $|c_i| = 2i$.

Theorem (cont.)

The Gysin sequence associated to the fiber sequence

$$U(k)/U(k-1) \longrightarrow V_k(\mathbb{C}^\infty)/U(k-1) \xrightarrow{p} V_k(\mathbb{C}^\infty)/U(k)\,,$$

with

$$\begin{aligned} F &= U(k)/U(k-1) \cong S^{2k-1} \\ E &= V_k(\mathbb{C}^\infty)/U(k-1) \simeq BU(k-1) \\ B &= V_k(\mathbb{C}^\infty)/U(k) = Gr_k(\mathbb{C}^\infty) \simeq BU(k)\,, \end{aligned}$$

breaks up into short exact sequences

$$0 \to H^{*-2k}(BU(k)) \xrightarrow{c_k \cup} H^*(BU(k)) \xrightarrow{p^*} H^*(BU(k-1)) \to 0\,.$$

Here $p^*(c_i) = c_i$ *for* $1 \le i < k$, *while* $c_k \in H^{2k}(BU(k))$ *is the Euler class of* $p \colon E \to B$.

Proof

- We proceed by induction on k, hence assume that

$$H^*(BU(k-1)) = \mathbb{Z}[c_1, \ldots, c_{k-1}]$$

  where $c_i \in H^{2i}(BU(k-1))$ has been specified for $1 \leq i \leq k-1$.
- We use the fiber sequence $F \to E \to B$, defined as above.
- Here $U(k)$ acts transitively on $S(\mathbb{C}^k) = S^{2k-1}$, with stabilizer $U(k-1)$, which gives the identification $U(k)/U(k-1) \cong S^{2k-1}$.
- The restricted $U(k-1)$-action on $V_k(\mathbb{C}^\infty)$ makes $V_k(\mathbb{C}^\infty) \to V_k(\mathbb{C}^\infty)/U(k-1) = E$ a universal principal $U(k-1)$-bundle, so that $E \simeq BU(k-1)$.

Proof (cont.)

- Since $H^*(BU(k-1))$ is trivial in odd degrees, the Gysin sequence for $F \to E \to B$ breaks up into exact sequences

$$0 \to H^{n-2k}(BU(k)) \xrightarrow{e\cup} H^n(BU(k)) \xrightarrow{p^*} H^n(BU(k-1)) \xrightarrow{p^!} H^{n-2k+1}(BU(k)) \xrightarrow{e\cup} H^{n+1}(BU(k)) \to 0\,,$$

  one for each even integer n.
- Induction on n proves that $H^{n+1}(BU(k)) = 0$ for $n+1$ odd, so the Gysin sequence breaks up into short exact sequences, and $H^*(BU(k))$ is concentrated in even degrees.
- Moreover, $p^* \colon H^n(BU(k)) \to H^n(BU(k-1))$ is an isomorphism for $n < 2k$, so we can uniquely define $c_i \in H^{2i}(BU(k))$ for $1 \le i < k$ by the condition $p^*(c_i) = c_i \in H^{2i}(BU(k-1))$.

Proof (cont.)

- Finally, we set $c_k = e \in H^{2k}(BU(k))$ to be the Euler class of this spherical fibration, so that

$$d_{2k}(1 \otimes g_{2k-1}) = c_k \otimes 1$$

in the cohomological Serre spectral sequence.

- To show that the resulting ring homomorphism

$$h \colon \mathbb{Z}[c_1, \ldots, c_k] \longrightarrow H^*(BU(k))$$

is an isomorphism, we use induction on the degree $*$ and the following vertical map of short exact sequences.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Sigma^{2k}\mathbb{Z}[c_1, \ldots, c_k] & \xrightarrow{c_k \cdot} & \mathbb{Z}[c_1, \ldots, c_k] & \longrightarrow & \mathbb{Z}[c_1, \ldots, c_{k-1}] & \longrightarrow 0 \\ & \Big\downarrow \Sigma^{2k}h & & \Big\downarrow h & & \Big\downarrow \cong & \\ 0 \longrightarrow & H^{*-2k}(BU(k)) & \xrightarrow{c_k \cup} & H^*(BU(k)) & \xrightarrow{p^*} & H^*(BU(k-1)) & \longrightarrow 0 \end{array}$$

$\square$

Chern classes

- We call $c_i \in H^{2i}(BU(k))$ the i-th Chern class.
- For each $\mathbb{C}^k$-bundle $E \to B$ with classifying map $f\colon B \to BU(k)$, we call $c_i(E) = f^*(c_i) \in H^{2i}(B)$ the i-th Chern class of the bundle.
- The Chern classes $c_i(E)$ determine the ring homomorphism

$$\begin{aligned} f^*\colon H^*(BU(k)) &\longrightarrow H^*(B) \\ c_i &\longmapsto c_i(E)\,. \end{aligned}$$

- This is generally less information than the isomorphism class of the vector bundle, i.e., the homotopy class of $f\colon B \to BU(k)$, but characteristic classes often provide conveniently accessible cohomological invariants of this less accessible homotopical datum.

Outline

Integral Eilenberg–MacLane spaces

- Let $n \geq 1$. Recall that $K(\mathbb{Z}, n)$ is a $(n-1)$-connected CW complex, with $\pi_n K(\mathbb{Z}, n) \cong \mathbb{Z}$ and $\pi_i K(\mathbb{Z}, n) = 0$ for $i \neq n$.
- Each homology group $H_i(K(\mathbb{Z}, n))$ is finitely generated of rank equal to the dimension of

  $$H_i(K(\mathbb{Z}, n)) \otimes \mathbb{Q} \xrightarrow{\cong} H_i(K(\mathbb{Z}, n); \mathbb{Q})$$

  over $\mathbb{Q}$.
- The evaluation pairing induces an isomorphism

  $$H^i(K(\mathbb{Z}, n))/(\text{torsion}) \xrightarrow{\cong} \mathrm{Hom}(H_i(K(\mathbb{Z}, n))/(\text{torsion}), \mathbb{Z}) .$$

The universal class

Definition

For $n \geq 1$ let the universal class

$$u_n \in H^n(K(\mathbb{Z}, n)) \cong \mathrm{Hom}(H_n(K(\mathbb{Z}, n)), \mathbb{Z})$$

correspond to the inverse Hurewicz isomorphism

$$h_n^{-1} \colon H_n(K(\mathbb{Z}, n)) \xrightarrow{\cong} \pi_n(K(\mathbb{Z}, n)) \cong \mathbb{Z}\,.$$

Rational cohomology calculation

Theorem

Let $n \geq 1$. If n is odd then

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_n) = \mathbb{Q}\{1, u_n\}$$

with $u_n^2 = 0$.

If n is even then

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[u_n] = \mathbb{Q}\{1, u_n, u_n^2, \dots\}.$$

Rational homology calculation

Finite type and the universal coefficient theorem imply the following consequence, which proves the theorem used earlier.

Corollary

Let $n \geq 1$. If n is odd then

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even then

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } 0 \leq i \equiv 0 \mod n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem

- When $n = 1$, the cohomology of $K(\mathbb{Z}, 1) \simeq S^1$ is well-known to be exterior on $g_1 = u_1$ in degree 1.
- Suppose that the theorem holds for an odd $n \geq 1$.
- We use the cohomology Serre spectral sequence with rational coefficients

$$E_2^{s,t} = H^s(K(\mathbb{Z}, n{+}1); H^t(K(\mathbb{Z}, n); \mathbb{Q})) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}, n{+}1); \mathbb{Q})$$

for the homotopy fiber sequence

$$K(\mathbb{Z}, n) \longrightarrow PK(\mathbb{Z}, n+1) \xrightarrow{p} K(\mathbb{Z}, n+1)$$

- This is isomorphic to the integral spectral sequence tensored with $\mathbb{Q}$, which is still a spectral sequence since $\mathbb{Q}$ is torsion-free, hence flat, so that tensoring with it is exact.

Proof for n odd

- Since $K(\mathbb{Z}, n+1)$ has finite type, we have an isomorphism

$$H^*(K(\mathbb{Z}, n+1); \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{\cong} E_2^{*,*} = H^*(K(\mathbb{Z}, n+1); H^*(K(\mathbb{Z}, n); \mathbb{Q}))\,.$$

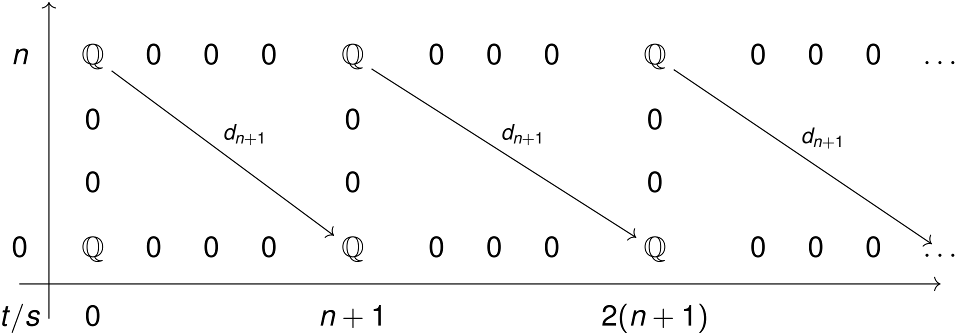
- Since $PK(\mathbb{Z}, n+1)$ is contractible, the abutment is $\mathbb{Q}$ in total degree 0.
- The E_2-term is concentrated in the two rows $t = 0$ and $t = n$, so

$$d_{n+1} \colon H^n(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{\cong} H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q})$$

  must be an isomorphism.
- More precisely, this transgressive differential is an integral isomorphism mapping u_n to

$$d_{n+1}(u_n) = u_{n+1}\,,$$

  by compatibility of the Hurewicz homomorphisms with coboundaries and pullbacks.



$$E_2^{s,t} = H^s(K(\mathbb{Z}, n+1); H^t(K(\mathbb{Z}, n); \mathbb{Q})) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}, n+1); \mathbb{Q})$$

Proof for n odd (cont.)

- We now proceed as for the Gysin sequence. Suppose inductively for a $j \geq 0$ that

$$H^i(K(\mathbb{Z}, n{+}1); \mathbb{Q}) = \begin{cases} \mathbb{Q}\{u_{n+1}^j\} & \text{for } i = j(n+1), \\ 0 & \text{for } j(n+1) < i < (j+1)(n+1). \end{cases}$$

- Then

$$d_{n+1} \colon E_2^{i,n} \longrightarrow E_2^{i+n+1,0}$$

  must be an isomorphism for $j(n+1) \leq i < (j+1)(n+1)$.

- Since

$$d_{n+1}(u_{n+1}^j \cup u_n) = u_{n+1}^j \cup d_{n+1}(u_n) = u_{n+1}^{j+1}$$

  must generate $H^{(j+1)(n+1)}(K(\mathbb{Z}, n+1); \mathbb{Q})$, the inductive claim also holds for $j+1$.

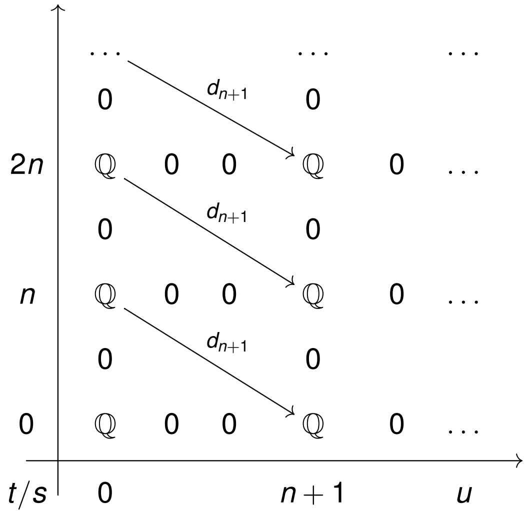
- This proves the theorem for $n+1$ even.

Proof for n even

- Next, suppose that the theorem holds for an even $n \geq 2$.
- We use the same Serre spectral sequence as above, but now the E_2-term is concentrated in the rows $0 \leq t \equiv 0 \mod n$.
- Again the transgressive differential

$$d_{n+1} \colon H^n(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{\cong} H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q})$$

maps u_n to (a unit times) u_{n+1}.



$$E_2^{s,t} = H^s(K(\mathbb{Z}, n+1); H^t(K(\mathbb{Z}, n); \mathbb{Q})) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}, n+1); \mathbb{Q})$$

Proof for n even (cont.)

- It follows from the Leibniz rule that

$$d_{n+1}(u_n^j) = j u_{n+1} \cup u_n^{j-1}$$

  for all $j \geq 1$.
- Since we are working with rational coefficients, $j u_{n+1} \cup u_n^{j-1}$ generates $E_2^{n+1,(j-1)n}$, so that

$$E_{n+2}^{s,t} = \begin{cases} \mathbb{Q} & \text{for } (s,t) = (0,0), \\ 0 & \text{otherwise, for } s \leq n+1. \end{cases}$$

Proof for n even (cont.)

- It remains to confirm that $H^i(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$ for all $i > n+1$.
- Let $u > n+1$ and suppose, inductively, that $H^i(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$ for $n+1 < i < u$.
- Then $E_2^{u,0} \cong H^u(K(\mathbb{Z}, n); \mathbb{Q})$, and we must have $E_\infty^{u,0} = 0$ since the abutment is trivial in total degree n.
- The final differential

$$d_u \colon E_u^{0,u-1} \longrightarrow E_u^{u,0}$$

is trivial, because $E_u^{0,u-1} \subset E_{n+2}^{0,u-1} = 0$.

Proof for n even (cont.)

- Furthermore,

$$d_{u-n-1}\colon E_{u-n-1}^{n+1,u-n-2} \longrightarrow E_{u-n-1}^{u,0}$$

  with $u-n-1 \geq 2$ must also be zero, because $E_{u-n-1}^{n+1,u-n-2}$ is trivial if $0 < u-n-2 < n$ or if $u-n-1 \geq n+2$.
- When $u = 2(n+1)$ the differential

$$d_{n+1}\colon E_{n+1}^{n+1,n} \longrightarrow E_{n+1}^{2(n+1),0}$$

  must be zero because the source is generated by $d_{n+1}(u_n^2) = 2u_{n+1} \cup u_n$ and $d_{n+1}d_{n+1} = 0$.
- Hence we can only have $E_\infty^{u,0} = 0$ of $E_2^{u,0} = 0$, i.e., if $H^u(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$.
- This confirms the claim by induction on n, and proves the theorem for $n+1$ odd. $\square$

The role of the product structure

Remark

For $n \geq 2$ even, the use of the Leibniz rule to calculate

$$d_{n+1}\colon E_{n+1}^{0,jn} \longrightarrow E_{n+1}^{n+1,(j-1)n}$$

relies essentially on knowing the cup product structure of $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ and the fact that the Serre spectral sequence differential d_{n+1} is a derivation.

Furthermore, the presence of the coefficient j in

$$d_{n+1}(u_n^j) = j u_{n+1} \cup u_n^{j-1}$$

means that this argument does not work integrally, since j is usually not an integral unit.

Outline

2-connected cover of S^2

The 2-connected cover of S^2 sits in the Puppe fiber sequence

$$K(\mathbb{Z},1) \longrightarrow \tau_{\geq 3}S^2 \longrightarrow S^2 \xrightarrow{g_2} K(\mathbb{Z},2)\,.$$

Since $\Omega K(\mathbb{Z},2) \simeq K(\mathbb{Z},1) \simeq S^1$ we can recognize this as the Hopf fiber sequence

$$S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2$$

and its classifying map $g_2\colon S^2 \to BS^1 \simeq \mathbb{C}P^\infty$.

3-connected cover of S^3

The 3-connected cover of S^3 is less familiar. We have a Puppe fiber sequence

$$K(\mathbb{Z},2) \longrightarrow \tau_{\geq 4} S^3 \longrightarrow S^3 \xrightarrow{g_3} K(\mathbb{Z},3)\,.$$

The cohomology of $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$ is well known, and allows the following calculation.

Proposition

The Serre spectral sequence

$$E_2^{s,t} = H^s(S^3; H^t(K(\mathbb{Z}, 2))) \Longrightarrow_s H^{s+t}(\tau_{\geq 4} S^3)$$

has E_2-term

$$E_2^{*,*} \cong H^*(S^3) \otimes H^*(\mathbb{C}P^\infty) = \Lambda(g_3) \otimes \mathbb{Z}[y]$$

with $g_3 \in H^3(S^3)$ and $y = u_2 \in H^2(\mathbb{C}P^\infty)$, and nonzero differentials

$$d_3(y^j) = j g_3 y^{j-1}$$

for all $j \geq 1$.

Proposition (cont.)

Hence

$$H^i(\tau_{\geq 4} S^3) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/j & \text{for } i = 2j+1 \geq 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(\tau_{\geq 4} S^3) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/j & \text{for } i = 2j \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

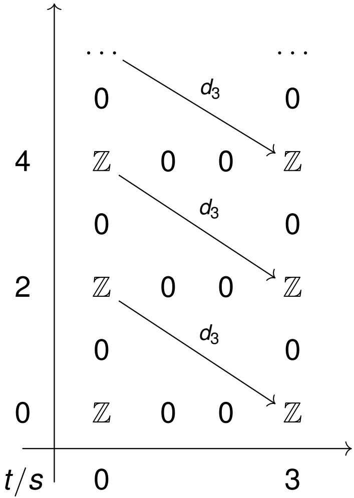
The natural homomorphism

$$H^*(S^3) \otimes H^*(\mathbb{C}P^\infty) \xrightarrow{\cong} H^*(S^3; H^*(\mathbb{C}P^\infty))$$

is an isomorphism.

Proof (cont.)

The $E_2 = E_3$-term thus appears as below, with $g_3 \in E_2^{3,0}$ and $y^j \in E_2^{0,2j}$.



Proof (cont.)

- Since $\tau_{\geq 4} S^3$ is 3-connected, the differential $d_3 \colon \mathbb{Z}\{y\} = E_3^{0,2} \to E_3^{3,0} = \mathbb{Z}\{g_3\}$ is an isomorphism.
- With the right choice of identifications, this implies that

$$d_3(y) = g_3 \,.$$

- The Leibniz rule thus implies

$$d_3(y^j) = j \, g_3 \, y^{j-1}$$

for all $j \geq 0$.

Proof (cont.)

This leaves the following $E_4 = E_\infty$-term, with gy^{j-1} generating a copy of $\mathbb{Z}/j$ in bidegree $(3, 2(j-1))$, for each $j \geq 2$.

	0			$\dots$
	0			0
4	0	0	0	$\mathbb{Z}/3$
	0			0
2	0	0	0	$\mathbb{Z}/2$
	0			0
0	$\mathbb{Z}$	0	0	0
t/s	0			3

Proof (cont.)

This calculates

$$H^*(\tau_{\geq 4} S^3) \cong \mathbb{Z} \oplus \bigoplus_{j \geq 2} \Sigma^{2j+1} \mathbb{Z}/j \, ,$$

and our finite type result and the universal coefficient theorem then determine

$$H_*(\tau_{\geq 4} S^3) \cong \mathbb{Z} \oplus \bigoplus_{j \geq 2} \Sigma^{2j} \mathbb{Z}/j \, .$$

□

$E\eta$ is essential

Corollary

$\pi_4(S^3) \cong \mathbb{Z}/2$ is generated by $E\eta$.

Proof.

We have $\pi_4(\tau_{\geq 4} S^3) \cong H_4(\tau_{\geq 4} S^3) \cong \mathbb{Z}/2$ by the Hurewicz theorem, and $\pi_4(\tau_{\geq 4} S^3) \cong \pi_4(S^3)$ by the long exact sequence in homotopy for the fiber sequence defining $\tau_{\geq 4} S^3$.

We also know that $E \colon \pi_3(S^2) \to \pi_4(S^3)$ is surjective, by Freudenthal's stability theorem, so $E\eta$ must generate $\pi_4(S^3)$. $\square$

First p-torsion

- Let p be a prime.
- Further arguments, with the Serre class of finite abelian groups of order prime to p, shows that

$$\pi_i(S^3) \cong \pi_i(\tau_{\geq i} S^3) \cong H_i(\tau_{\geq i} S^3)$$

  for $3 < i \leq 2p$ maps to

$$H_i(\tau_{\geq 4} S^3)$$

  by a homomorphism with kernel and cokernel finite groups of order prime to p.
- Hence the p-Sylow subgroup of $\pi_i(S^3)$ is trivial for $3 < i < 2p$, and is isomorphic to $\mathbb{Z}/p$ for $i = 2p$.
- A map representing the first p-torsion in $\pi_*(S^3)$ is often denoted $\alpha_1 \colon S^{2p} \longrightarrow S^3$.

Outline

Cartan–Eilenberg systems

Toward $\pi_5(S^3)$

To proceed to calculate $\pi_5(S^3) \cong \pi_5(\tau_{\geq 5} S^3)$ we might study $H_*(\tau_{\geq 5} S^3)$ using the Puppe fiber sequence

$$K(\mathbb{Z}/2, 3) \longrightarrow \tau_{\geq 5} S^3 \longrightarrow \tau_{\geq 4} S^3 \longrightarrow K(\mathbb{Z}/2, 4)$$

and the Serre spectral sequence

$$E_2^{*,*} = H^*(\tau_{\geq 4} S^3; H^*(K(\mathbb{Z}/2, 3))) \Longrightarrow H^*(\tau_{\geq 5} S^3) .$$

For this, we would need to know $H^*(K(\mathbb{Z}/2, 3))$, which we might hope to deduce from $H^*(K(\mathbb{Z}/2, 2))$ using the loop–path fibration

$$K(\mathbb{Z}/2, 2) \longrightarrow PK(\mathbb{Z}/2, 3) \longrightarrow K(\mathbb{Z}/2, 3) .$$

Start with $K(\mathbb{Z}/2, 2)$

To get started with this, we might first deduce $H^*(K(\mathbb{Z}/2, 2))$ from the loop–path fibration

$$K(\mathbb{Z}/2, 1) \longrightarrow PK(\mathbb{Z}/2, 2) \longrightarrow K(\mathbb{Z}/2, 2)\,,$$

where the cohomology of $K(\mathbb{Z}/2, 1) \simeq \mathbb{R}P^\infty$ is well known.

However, in the cohomological Serre spectral sequence with integral coefficients

$$E_2^{s,t} = H^s(K(\mathbb{Z}/2, 2); H^t(\mathbb{R}P^\infty)) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}/2, 2))$$

there are more classes in the E_2-term than those that arise as products of classes on the axes:

$$H^s(K(\mathbb{Z}/2, 2)) \otimes H^t(\mathbb{R}P^\infty) \longrightarrow H^s(K(\mathbb{Z}/2, 2); H^t(K(\mathbb{Z}/2, 1)))\,,$$

due to the presence of Tor-terms.

Field coefficients

Hence it is more convenient to make the calculation with coefficients in the field $\mathbb{F}_2$, and thereafter to use Bockstein arguments to recover the integral information.

Here $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a]$ with $a = u_1 \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$, and the cohomological Serre spectral sequence with $\mathbb{F}_2$-coefficients has the form

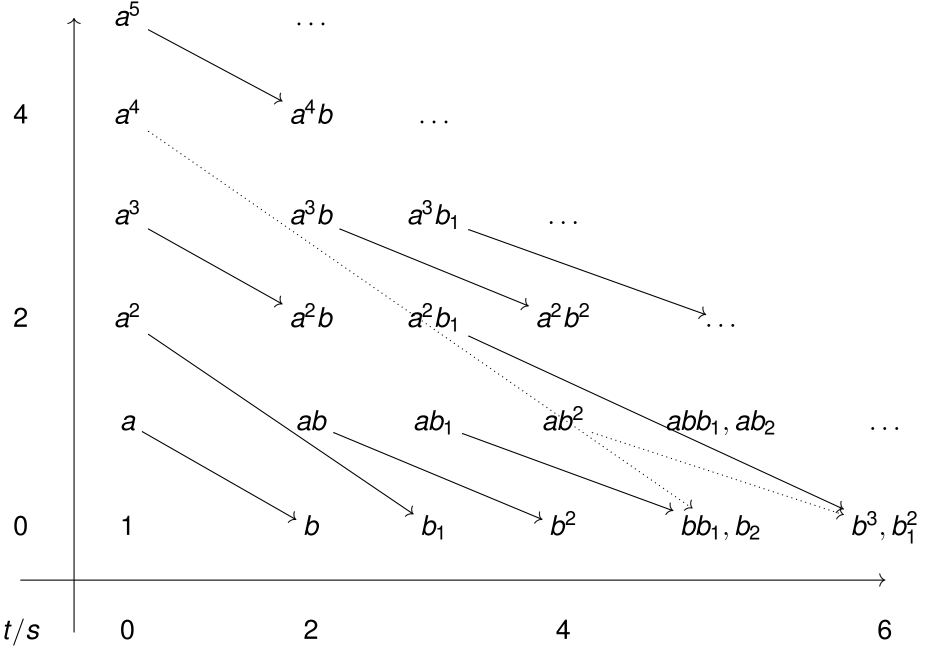
$$E_2^{s,t} = H^s(K(\mathbb{Z}/2, 2); H^t(\mathbb{R}P^\infty; \mathbb{F}_2)) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}/2, 2); \mathbb{F}_2)$$

with

$$H^s(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^t(\mathbb{R}P^\infty; \mathbb{F}_2) \xrightarrow{\cong} E_2^{s,t}\,.$$

The differentials on a and a^2

- As usual, the abutment $H^*(PK(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2$ is known to vanish in positive degrees, and we seek to use this to determine the cohomology of the base.
- Clearly $K(\mathbb{Z}/2,2)$ is 1-connected, and $d_2(a) = b$ with b generating $H^2(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2$.
- Since $d_2(a^2) = ba - ab = 0$, we must have $d_3(a^2) = b_1$ for some nonzero $b_1 \in H^3(K(\mathbb{Z}/2,2);\mathbb{F}_2)$.
- Furthermore, $d_2(ab) = b^2$ must be nonzero, and $d_2(ab_1) = bb_1$ must be nonzero.
- Since $d_3(a^4) = b_1 a^2 + a^2 b_1 = 0$ and $d_2(a^2 b_1) = 0$ we must have $d_3(a^2 b_1) = b_1^2$ nonzero.



a^5
$\dots$
4
a^4
$a^4 b$
$\dots$
a^3
$a^3 b$
$a^3 b_1$
$\dots$
2
a^2
$a^2 b$
$a^2 b_1$
$a^2 b^2$
$\dots$
a
ab
ab_1
ab^2
abb_1, ab_2
$\dots$
0
1
b
b_1
b^2
bb_1, b_2
b^3, b_1^2
t/s
0
2
4
6

The differential on a^4

- At this point we must decide whether
  - $d_2(ab^2) = b^3$ is nonzero in $H^6(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$, so that $d_5(a^4) = b_2$ is nonzero in $H^5(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$, or
  - $b^3 = 0$ and $d_4(a^4) = ab^2$.
- In fact, the former is the case.
- We can see this using the map $f\colon K(\mathbb{Z}, 2) \to K(\mathbb{Z}/2, 2)$ inducing the surjection $\pi_2(f)\colon \mathbb{Z} \to \mathbb{Z}/2$.
- Here $f^*(b) = y$.
- Since $y^3 \neq 0$ in $H^6(K(\mathbb{Z}, 2); \mathbb{F}_2)$, it follows that $b^3 \neq 0$, so that $d_5(a^4) = b_2$ for some nonzero $b_2 \in H^6(K(\mathbb{Z}/2, 1); \mathbb{F}_2)$.

The differential on a^8

- We can continue this argument, up to total degree 8.
- Here we must decide whether
  - $b^2 b_1^2$ and $bb_1 b_2$ are linearly independent in $H^{10}(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$, in which case $d_9(a^8) = b_3$ for a nonzero $b_3 \in H^9(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$, or
  - $d_8(a^8)$ is a nonzero linear combination of abb_1^2 and $ab_1 b_2$.
- Again, some external information in addition to the multiplicative structure of the spectral sequence is needed.

Steenrod squares

- In the next chapter we discuss natural cohomology operations

$$Sq^i\colon H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2)$$

  introduced by Steenrod.
- These were used by Serre [Ser53] to calculate the mod 2 cohomology of Eilenberg–Mac Lane spaces.
- Similar results for mod *p* cohomology, with *p* an odd prime, are due to Cartan [Car54].

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