MAT9580: Spectral Sequences

Chapter 7: The Steenrod Algebra

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Outline

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A universal class

Eilenberg and Mac Lane proved a representability theorem for cohomology.

Definition

For $n \geq 1$ and G any abelian group let the universal class

$$u_n \in H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)),G)$$

correspond to the inverse Hurewicz isomorphism

$$h_n^{-1} \colon H_n(K(G,n)) \xrightarrow{\cong} \pi_n(K(G,n)) \cong G\,.$$

For $n = 0$, with $K(G,0) = G$, we let $u_0 \in \tilde{H}^0(K(G,0);G)$ be the class of the 0-cocycle that takes $g \in K(G,0)$ to $g \in G$.

Representability of cohomology

Recall that $[X, Y]$ denotes the based homotopy classes of base-point preserving maps from a CW complex X to a space Y.

Theorem (Eilenberg–MacLane, [Hat02, Thm. 4.57])

There is a natural isomorphism

$$\begin{aligned}[X, K(G,n)] &\xrightarrow{\cong} \tilde{H}^n(X;G)\\ [f] &\longmapsto f^*(u_n)\end{aligned}$$

for all based CW complexes X.

Sketch proof

Fix a homotopy equivalence

$$\tilde{\sigma}\colon K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$$

and let

$$\sigma\colon \Sigma K(G,n) \longrightarrow K(G,n+1)$$

be the adjoint map.

We define a generalized cohomology theory M on CW pairs (X,A) by

$$M^n(X,A) = [X/A, K(G,n)]\,,$$

with $\delta\colon M^n(A) \longrightarrow M^{n+1}(X,A)$ sending the homotopy class of $f\colon A \to K(G,n)$ to the homotopy class of the composite

$$X/A \simeq X \cup CA \longrightarrow \Sigma A \xrightarrow{\Sigma f} \Sigma K(G,n) \xrightarrow{\sigma} K(G,n+1)\,.$$

Proof (cont.)

The abelian group structure on $M^n(X, A)$, and the additivity of δ, can be deduced from the fact that $K(G, n) \simeq \Omega^2 K(G, n+2)$ is a double loop space.

The coexactness of the Puppe cofiber sequence

$$A \longrightarrow X \longrightarrow X \cup CA \longrightarrow \Sigma A \longrightarrow \dots$$

proves exactness, while homotopy invariance, excision and additivity are straightforward.

Proof (cont.)

The coefficients groups of this cohomology theory are $M^t = M^t(\text{point}) = [S^0, K(G,t)]$, which equals G for $t = 0$ and 0 for $t \neq 0$.

Hence the hypotheses of the Eilenberg–Steenrod uniqueness theorem are satisfied, and $M^*(X,A) \cong H^*(X,A;G)$.

For based CW complexes X we deduce that there is a natural isomorphism

$$[X, K(G,n)] = M^n(X, \{x_0\}) \cong H^n(X, \{x_0\}; G) \cong \tilde{H}^n(X; G)\,.$$

By the Yoneda lemma, the isomorphism must be induced by the class

$$y_n \in \tilde{H}^n(K(G,n); G)$$

that corresponds to the identity map of $X = K(G,n)$, and more careful check of definitions shows that $y_n = u_n$ is the universal class. □

Cohomology operations

A cohomology operation is a natural transformation between (possibly generalized) cohomology groups. We concentrate on the case of ordinary cohomology theories.

Definition

A cohomology operation of type $(G, n; G', n')$ is a natural transformation

$$\theta_X \colon \tilde{H}^n(X; G) \longrightarrow \tilde{H}^{n'}(X; G')$$

of functors from CW complexes to sets.

The sum (or difference) of two cohomology operations of type $(G, n; G', n')$ is another cohomology operation of the same type, so the set of such cohomology operations is an abelian group.

Cohomology classification of operations

Lemma

The abelian group of cohomology operations of type $(G, n; G', n')$ is isomorphic to

$$[K(G,n), K(G',n')] \cong \tilde{H}^{n'}(K(G,n); G') .$$

Proof.

This is the Yoneda lemma classifying natural transformations from a represented functor.

A map $\theta\colon K(G,n) \to K(G',n')$ corresponds to the natural transformation θ with components θ_X taking the homotopy class of $f\colon X \to K(G,n)$ to the homotopy class of $\theta f\colon X \to K(G',n')$.

Conversely, the natural transformation θ corresponds to the homotopy class of a map $\theta\colon K(G,n) \to K(G',n')$ representing $\theta_{K(G,n)}(u_n)$ in $\tilde{H}^{n'}(K(G,n); G')$. □

k-th power operations

Computing the cohomology of $K(G, n)$ is thus equivalent to determining the cohomology operations from $H^n(X; G)$.

By the Hurewicz theorem, there are only nontrivial cohomology operations of type $(G, n; G', n')$ when $n' \geq n$.

Example

For $k \geq 1$ and R a commutative ring, let the k-th power operation

$$\xi^k = \xi^k_X \colon H^n(X; R) \longrightarrow H^{kn}(X; R)$$

be the cohomology operation of type $(R, n; R, kn)$ given by

$$\xi^k(x) = x^k = x \cup \cdots \cup x$$

(with k copies of x).

This operation is additive if $k = p$ is a prime and $p = 0$ in R.

Outline

Reduced power operations

- Let p be a prime. Steenrod [Ste47], [Ste52], [Ste53] introduced cohomology operations in mod p cohomology, i.e., cohomology with coefficients in the field $\mathbb{F}_p = \mathbb{Z}/p$. which in a sense generate all other such cohomology operations.
- These are "reduced power operations", meaning that they are linked to the p-th power operation

$$\xi^p \colon H^n(X; \mathbb{F}_p) \longrightarrow H^{pn}(X; \mathbb{F}_p)\,,$$

  but generally land in $H^{n'}(X; \mathbb{F}_p)$ with $n \leq n' \leq pn$.
- See Steenrod–Epstein [Ste62], May [May70] and Hatcher [Hat02, §4.L] for more detailed expositions.

Steenrod squares

We start with $p = 2$, when the reduced power operations are called reduced squaring operations, or Steenrod squares.

The following theorem can be taken as the basis for an axiomatic development of the theory.

Theorem ([Ste62, §I.1])

There are natural transformations

$$Sq^i : \tilde{H}^n(X; \mathbb{F}_2) \longrightarrow \tilde{H}^{n+i}(X; \mathbb{F}_2)$$

for all $i \geq 0$ and $n \geq 0$. These satisfy

1. $Sq^0(x) = x$ *for all* x*;*
2. $Sq^n(x) = x \cup x$ *for* $n = |x|$*;*
3. $Sq^i(x) = 0$ *for* $i > |x|$*;*
4. $$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)\,.$$

Remarks

- Note that Sq^i increases cohomological degree by i.
- By the first three items, the only "new" operations are the $Sq^i(x)$ for $0 < i < n$.
- The fourth item

$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$

is the Cartan formula from [Car50].

Definition of the Sq^i

- To define the $Sq^i(x)$ for $x \in \tilde{H}^n(X; \mathbb{F}_2)$ represented by the homotopy class of a map $f\colon X \to K(\mathbb{F}_2, n)$, we will construct maps

$$\mathbb{R}P^\infty_+ \wedge X \xrightarrow{1\wedge f} \mathbb{R}P^\infty_+ \wedge K_n \xrightarrow{1\wedge\Delta} S^\infty_+ \wedge_{C_2} K_n \wedge K_n \xrightarrow{\theta} K_{2n} .$$

- Here $\mathbb{R}P^\infty = S^\infty/C_2$ and we write $K_n = K(\mathbb{F}_2, n)$ and $K_{2n} = K(\mathbb{F}_2, 2n)$ to simplify the notation.
- The homotopy class of the composite represents an element

$$y = [\theta(1 \wedge \Delta)(1 \wedge f)] \in \tilde{H}^{2n}(\mathbb{R}P^\infty_+ \wedge X; \mathbb{F}_2) .$$

Definition of the Sq^i (cont.)

- By the Künneth theorem,

$$\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge X; \mathbb{F}_2) \cong H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes \tilde{H}^*(X; \mathbb{F}_2)$$

  where $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[a]$ with $|a| = 1$.
- Hence we can write

$$y = \sum_{i=0}^{n} a^{n-i} \otimes Sq^i(x)$$

  for a unique sequence of elements $Sq^i(x) \in \tilde{H}^{n+i}(X; \mathbb{F}_2)$.
- This defines the (potentially) nonzero $Sq^i(x)$.

The quadratic construction

- To explain θ, we must first introduce the quadratic construction

$$D_2(X) = S^\infty_+ \wedge_{C_2} X \wedge X \,.$$

- Here $C_2 = \{e, t\}$ is the group of order 2, with unit element e.
- It acts freely from the right on the unit sphere $S^\infty = S(\mathbb{R}^\infty)$, with $v \cdot t = -v$ for each unit vector v, and the orbit space is $S^\infty/C_2 = \mathbb{R}P^\infty$.

Balanced smash product

- For a based CW complex X the group C_2 acts from the left on the smash product

$$X \wedge X = \frac{X \times X}{X \vee X}$$

by the twist isomorphism $\tau \colon X \wedge X \longrightarrow X \wedge X$, with $t \cdot (x \wedge y) = y \wedge x$.

- The quadratic construction is the balanced product

$$S^\infty_+ \wedge_{C_2} X \wedge X = (S^\infty_+ \wedge X \wedge X)/(\sim)$$

where $\sim$ denotes the relation

$$(-v, x \wedge y) = (v \cdot t, x \wedge y) \sim (v, t \cdot (x \wedge y)) = (v, y \wedge x)$$

for $v \in S^\infty$, $x \in X$ and $y \in Y$.

Filtration of the quadratic construction

- Let $S^i = S(\mathbb{R}^{i+1}) \subset S^\infty$.
- The action of C_2 respects this subspace, so we can filter $D_2(X)$ by the subspaces

$$\cdots \subset D_2^{i-1}(X) \subset D_2^i(X) = S_+^i \wedge_{C_2} X \wedge X \subset \cdots \subset D_2(X)\,.$$

- There are homeomorphisms $X \wedge X \cong S_+^0 \wedge_{C_2} X \wedge X = D_2^0(X)$ and

$$I_+ \wedge X \wedge X/(\sim) \cong S_+^1 \wedge_{C_2} X \wedge X = D_2^1(X)$$

  where $(0, x \wedge y) \sim (1, y \wedge x)$ at the left hand side.
- Hence there is a long exact cohomology sequence

$$\to \tilde{H}^{*-1}(X\wedge X;\mathbb{F}_2) \xrightarrow{\delta} \tilde{H}^*(D_2^1(X);\mathbb{F}_2) \to \tilde{H}^*(X\wedge X;\mathbb{F}_2) \xrightarrow{1-\tau} H^*(X\wedge X;\mathbb{F}_2) \to\ .$$

The extension θ_1

- We now specialize to the case $X = K_n = K(\mathbb{F}_2, n)$ and degree $* = 2n$.
- By the Künneth theorem, $K_n \wedge K_n$ is $(2n-1)$-connected, and
$$\tilde{H}^{2n}(K_n \wedge K_n; \mathbb{F}_2) = \mathbb{F}_2\{u_n \wedge u_n\}$$
where $u_n \in \tilde{H}^n(K_n; \mathbb{F}_2)$ is the universal class.
- Furthermore,
$$(1-\tau)(u_n \wedge u_n) = u_n \wedge u_n - (-1)^{n^2} u_n \wedge u_n = 0\,,$$
since we are working with $\mathbb{F}_2$-coefficients, so $\theta_0 = u_n \wedge u_n$ admits a unique extension $\theta_1 \in \tilde{H}^{2n}(D_2^1(K_n); \mathbb{F}_2)$.

The extension θ

- Moreover, $D_2^1(K_n) \to D_2(K_n)$ is $(2n+1)$-connected, so the restriction homomorphism

$$\tilde{H}^{2n}(D_2(K_n); \mathbb{F}_2) \xrightarrow{\cong} \tilde{H}^{2n}(D_2^1(K_n); \mathbb{F}_2)$$

  is an isomorphism, and θ_1 admits a unique extension $\theta \in \tilde{H}^{2n}(D_2(K_n); \mathbb{F}_2)$.
- It is represented by a map

$$\theta \colon D_2(K_n) = S_+^\infty \wedge_{C_2} K_n \wedge K_n \longrightarrow K_{2n}$$

  whose restriction

$$\theta_0 \colon D_2^0(K_n) \cong K_n \wedge K_n \longrightarrow K_{2n}$$

  represents the smash product $\wedge \colon \tilde{H}^n(X; \mathbb{F}_2) \otimes \tilde{H}^n(Y; \mathbb{F}_2) \to \tilde{H}^{2n}(X \wedge Y; \mathbb{F}_2)$.

The extended diagonal map

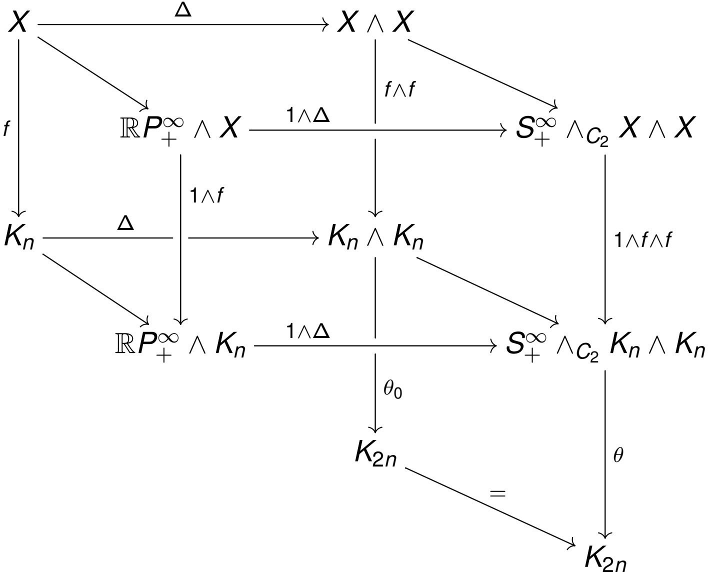
- The (reduced) diagonal map $\Delta\colon X \to X \wedge X$ satisfies $t \cdot \Delta(x) = \Delta(x) = x \wedge x$, hence induces a map

$$1 \wedge \Delta\colon \mathbb{R}P^\infty_+ \wedge X \longrightarrow S^\infty_+ \wedge_{C_2} X \wedge X = D_2(X)$$

  sending $([v], x)$ to $[v \wedge x \wedge x]$, for $v \in S^\infty$ and $x \in X$.
- Its restriction to $v \in S^0 \subset S^\infty$ is identified with the diagonal map

$$\Delta\colon X \cong \mathbb{R}P^0_+ \wedge X \longrightarrow D^0_2(X) \cong X \wedge X\,.$$

Given a class $x \in \tilde{H}^n(X; \mathbb{F}_2)$, represented by a map $f\colon X \to K_n$, we can form the following commutative diagram.



Definition of Sq^i

- The composite

$$\theta(1 \wedge \Delta)(1 \wedge f) = \theta(1 \wedge f \wedge f)(1 \wedge \Delta) \colon \mathbb{R}P^\infty_+ \wedge X \longrightarrow K_{2n}$$

defines the cohomology class we write as

$$\sum_{i=0}^{n} a^{n-i} \otimes Sq^i(x) \in H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes \tilde{H}^*(X; \mathbb{F}_2) \cong \tilde{H}^*(\mathbb{R}P^\infty_+ \wedge X; \mathbb{F}_2)\,.$$

- Its restriction to $\tilde{H}^*(X; \mathbb{F}_2)$, corresponding to $i = n$, is the pullback along Δ of $x \wedge x \in \tilde{H}^{2n}(X \wedge X; \mathbb{F}_2)$, represented by $\theta_0(f \wedge f)$, which equals $x^2 = x \cup x \in \tilde{H}^{2n}(X; \mathbb{F}_2)$.
- This defines the natural transformations Sq^i, satisfying conditions (2) and (3) in the theorem.

The Cartan formula, I

The Cartan formula (4) can be deduced from the following diagram.

$$
\begin{array}{ccc}
D_2(K_n \wedge K_m) & \xrightarrow{D_2(\wedge)} & D_2(K_{n+m}) \\
\downarrow{\scriptstyle \delta} & & \\
D_2(K_n) \wedge D_2(K_m) & & \downarrow{\scriptstyle \theta} \\
\downarrow{\scriptstyle \theta\wedge\theta} & & \\
K_{2n} \wedge K_{2m} & \xrightarrow{\wedge} & K_{2(n+m)}
\end{array}
$$

It commutes up to homotopy, as can be verified by comparing the two composites after restriction to $(K_n \wedge K_m) \wedge (K_n \wedge K_n) = D_2^0(K_n \wedge K_m)$.

The Cartan formula, II

If $f\colon X \to K_n$ and $g\colon Y \to K_m$ represent $x \in \tilde{H}^n(X; \mathbb{F}_2)$ and $y \in \tilde{H}^m(Y; \mathbb{F}_2)$, respectively, then the composite

$$\mathbb{R}P^\infty_+ \wedge X \wedge Y \xrightarrow{1\wedge\Delta} D_2(X \wedge Y) \xrightarrow{D_2(f\wedge g)} D_2(K_n \wedge K_m) \longrightarrow K_{2(n+m)}$$

can be expanded in two ways, to yield the identity

$$\sum_{k=0}^{n+m} a^{n+m-k} \otimes Sq^k(x \wedge y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a^{n-i} \cup a^{m-j} \otimes Sq^i(x) \cup Sq^j(y)\,.$$

Comparing terms gives the Cartan formula.

Cup, smash and cross

By naturality, the Cartan formula also holds for relative and unreduced cohomology, as well as for the external smash product and cross product pairings.

For example,

$$Sq^k(x \wedge y) = \sum_{i+j=k} Sq^i(x) \wedge Sq^j(y)$$

in $\tilde{H}^*(X \wedge Y; \mathbb{F}_2)$.

Sq^0 is the identity

- Property (1), that $Sq^0(x) = x$, is not obvious.
- The statement for $n = 1$ follows by naturality from the case $x = u_1 \in H^1(K_1; \mathbb{F}_2)$, which is an assertion about the composite

$$\mathbb{R}P^\infty_+ \wedge K_1 \xrightarrow{1\wedge\Delta} S^\infty_+ \wedge_{C_2} K_1 \wedge K_1 \xrightarrow{\theta} K_2 \,.$$

- By naturality with respect to $g_1 \colon S^1 \to K_1$, it suffices to check that

$$\mathbb{R}P^1_+ \wedge S^1 \xrightarrow{1\wedge\Delta} S^1_+ \wedge_{C_2} S^1 \wedge S^1$$

  induces the nonzero homomorphism (an isomorphism) in $H^2(-; \mathbb{F}_2)$, which can be seen from an explicit cellular model. See [Hat02, p. 505].
- This shows that $Sq^0(g_1) = g_1$ in $\tilde{H}^*(S^1; \mathbb{F}_2)$.

- When combined with the Cartan formula for $\Sigma X = S^1 \wedge X$, it follows that each reduced squaring operation commutes with the suspension isomorphisms

$$\sigma \colon \tilde{H}^n(X; \mathbb{F}_2) \xrightarrow{\cong} \tilde{H}^{n+1}(\Sigma X; \mathbb{F}_2)$$

  given by $\sigma(x) = g_1 \wedge x$, since

$$Sq^i(g_1 \wedge x) = Sq^0(g_1) \wedge Sq^i(x) = g_1 \wedge Sq^i(x)\,.$$

- It then follows, by naturality with respect to $X \cup CA \to \Sigma A$, that each Sq^i commutes with the connecting homomorphisms

$$\delta \colon H^n(A; \mathbb{F}_2) \longrightarrow H^{n+1}(X, A; \mathbb{F}_2)\,.$$

- It also follows that each Sq^i is additive, i.e., is an $\mathbb{F}_2$-linear homomorphism.

End of proof of theorem

- Finally, to verify that $Sq^0(x) = x$ for $x \in H^n(X; \mathbb{F}_2)$ it suffices, by naturality, to check the case $x = u_n \in H^n(K_n; \mathbb{F}_2)$.
- Since $g_n \colon S^n \to K_n$ induces an isomorphism $g_n^* \colon H^n(K_n; \mathbb{F}_2) \to H^n(S^n; \mathbb{F}_2)$, it suffices to treat the case $x = g_n \in H^n(S^n; \mathbb{F}_2)$.
- This now follows from the case $x = g_1 \in H^1(S^1; \mathbb{F}_2)$, by commutation of Sq^0 with the suspension isomorphism. $\square$

Bockstein homomorphisms

The operation Sq^1 had also been previously considered.

Definition

Let

$$0 \to G' \longrightarrow G \longrightarrow G'' \to 0$$

be a short exact sequence of abelian groups. The induced short exact sequence

$$0 \to C^*(X; G') \longrightarrow C^*(X; G) \longrightarrow C^*(X; G'') \to 0$$

of cochain complexes induces a long exact sequence in cohomology, with connecting homomorphisms

$$\beta \colon H^n(X; G'') \longrightarrow H^{n+1}(X; G')$$

called the cohomology Bockstein homomorphism associated to the extension $0 \to G' \to G \to G'' \to 0$.

Bockstein composition

The Bockstein homomorphism is a cohomology operation of type $(G'', n; G', n+1)$.

Lemma

Let $0 \to G' \to G_1 \to G'' \to 0$ *and* $0 \to G'' \to G_2 \to G''' \to 0$ *be extensions of abelian groups. Then the composite of Bockstein homomorphisms*

$$H^n(X; G''') \xrightarrow{\beta_2} H^{n+1}(X; G'') \xrightarrow{\beta_1} H^{n+2}(X; G')$$

is zero.

Proof

There exists a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & G' & \xrightarrow{=} & G' & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & G''' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & G'' & \longrightarrow & G_2 & \longrightarrow & G''' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns. The Bockstein for $G'' \to G_2 \to G'''$ factors as

$$\beta_2 = j\beta \colon H^n(X; G''') \xrightarrow{\beta} H^{n+1}(X; G_1) \xrightarrow{j} H^{n+1}(X; G''),$$

and the composite

$$\beta_1 j \colon H^{n+1}(X; G_1) \xrightarrow{j} H^{n+1}(X; G'') \xrightarrow{\beta_1} H^{n+2}(X; G')$$

is zero. □

Sq^1 is the Bockstein

Proposition

$Sq^1 = \beta\colon H^n(X;\mathbb{F}_2) \to H^{n+1}(X;\mathbb{F}_2)$ equals the cohomology Bockstein for the extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0\,.$$

In particular, $Sq^1 Sq^1 = \beta\beta = 0$.

Proof.

- By naturality it suffices that $Sq^1(u_n) = \beta(u_n) \in H^{n+1}(K_n;\mathbb{F}_2)$ for $u_n \in H^n(K_n;\mathbb{F}_2)$.
- Consider the Moore space $M_n = S^n \cup_2 e^{n+1}$, which admits an $(n+1)$-connected map $f\colon M_n \to K_n$.
- Since $f^*\colon H^{n+1}(K_n;\mathbb{F}_2) \to H^{n+1}(M_n;\mathbb{F}_2)$ is an isomorphism, it suffices to check that $Sq^1(a) = \beta(a)$ for $a = [f]$.

Proof (cont.)

- Since Sq^1 and β both commute with suspension isomorphisms, it suffices to verify this when $n = 1$ and $M_1 = S^1 \cup_2 e^2 \cong \mathbb{R}P^2$.
- Here $Sq^1(a) = a^2$ generates $H^2(\mathbb{R}P^2; \mathbb{F}_2)$, and a direct calculation with $\tilde{H}^*(\mathbb{R}P^2; \mathbb{Z}/4)$ shows that $\beta(a) = a^2$.
- The composite $\beta\beta$ is trivial, by the previous lemma with $G' = G'' = G''' = \mathbb{Z}/2$, $G_1 = G_2 = \mathbb{Z}/4$ and $G = \mathbb{Z}/8$. $\square$

Steenrod squares on powers

Lemma

The Steenrod squares on the powers of any $a \in H^1(X; \mathbb{F}_2)$ are given by

$$Sq^i(a^j) = \binom{j}{i} a^{i+j} \, .$$

The binomial coefficient can be read mod 2, since the expression takes place in $H^*(X; \mathbb{F}_2)$. Hence Lucas' theorem (see below) is helpful.

Lucas' theorem

Binomial coefficents mod p can be conveniently calculated from base p expansions. See [Ste62, Lem. 2.6] or [Hat02, Lem. 3C.6] for a proof.

Lemma (Lucas)

Let p be a prime, and write $n = \sum_i n_i p^i$ and $k = \sum_i k_i p^i$ with $n_i, k_i \in \{0, 1, \ldots, p-1\}$. Then

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \mod p\,.$$

For $p = 2$, this reduces the calcuation of $\binom{n}{k}$ to the cases $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and $\binom{0}{1} = 0$.

Hence $\binom{n}{k} \equiv 0 \mod 2$ if and only if there is a 1 below a 0 when n and k are written in base 2.

Proof of lemma

Let the inhomogeneous sum

$$Sq(x) = \sum_i Sq^i(x) \in \bigoplus_n H^n(X; \mathbb{F}_2)$$

denote the total squaring operation on x. The Cartan formula then reads

$$Sq(xy) = Sq(x)Sq(y)$$

and $Sq(a) = a + a^2 = a(1 + a)$ in $H^*(X; \mathbb{F}_2)$. Hence

$$Sq(a^j) = Sq(a)^j = (a + a^2)^j = a^j(1 + a)^j$$

so that

$$Sq^i(a^j) = a^j \cdot \binom{j}{i} a^i = \binom{j}{i} a^{i+j}$$

for $0 \leq i \leq j$, and $Sq^i(a^j) = 0$ otherwise. $\square$

Outline

Steenrod square composites

Let $Sq^i Sq^j$ denote the composite operation

$$\tilde{H}^n(X;\mathbb{F}_2) \xrightarrow{Sq^j} \tilde{H}^{n+j}(X;\mathbb{F}_2) \xrightarrow{Sq^i} \tilde{H}^{n+i+j}(X;\mathbb{F}_2)\,.$$

These satisfy the Adem relations.

Theorem ([Ade52], [Ste62, §I.1])

The identity

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

holds, for $i < 2j$.

Sample relations

- The binomial coefficients can be read mod 2.
- The summation limits can be omitted, given the convention that $\binom{n}{k} = 0$ for $k < 0$ and $k > n$.
- In particular,

$$\begin{aligned} Sq^1 Sq^{2j} &= Sq^{2j+1} \\ Sq^1 Sq^{2j+1} &= 0 \\ Sq^{2j+1} Sq^{j+1} &= 0 \end{aligned}$$

for all $j \geq 0$.

Adem relations in degrees $* \leq 8$

$$Sq^1 Sq^1 = 0$$
$$Sq^1 Sq^3 = 0$$
$$Sq^1 Sq^4 = Sq^5$$
$$Sq^3 Sq^2 = 0$$
$$Sq^2 Sq^4 = Sq^6 + Sq^5 Sq^1$$
$$Sq^1 Sq^6 = Sq^7$$
$$Sq^3 Sq^4 = Sq^7$$
$$Sq^1 Sq^7 = 0$$
$$Sq^3 Sq^5 = Sq^7 Sq^1$$
$$Sq^5 Sq^3 = 0$$
$$Sq^1 Sq^2 = Sq^3$$
$$Sq^2 Sq^2 = Sq^3 Sq^1$$
$$Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1$$
$$Sq^1 Sq^5 = 0$$
$$Sq^3 Sq^3 = Sq^5 Sq^1$$
$$Sq^2 Sq^5 = Sq^6 Sq^1$$
$$Sq^4 Sq^3 = Sq^5 Sq^2$$
$$Sq^2 Sq^6 = Sq^7 Sq^1$$
$$Sq^4 Sq^4 = Sq^7 Sq^1 + Sq^6 Sq^2$$

Adem relations in degrees $9 \leq * \leq 11$

$$Sq^1 Sq^8 = Sq^9$$
$$Sq^3 Sq^6 = 0$$
$$Sq^5 Sq^4 = Sq^7 Sq^2$$
$$Sq^2 Sq^8 = Sq^{10} + Sq^9 Sq^1$$
$$Sq^4 Sq^6 = Sq^{10} + Sq^8 Sq^2$$
$$Sq^6 Sq^4 = Sq^7 Sq^3$$
$$Sq^2 Sq^9 = Sq^{10} Sq^1$$
$$Sq^4 Sq^7 = Sq^{11} + Sq^9 Sq^2$$
$$Sq^6 Sq^5 = Sq^9 Sq^2 + Sq^8 Sq^3$$

$$Sq^2 Sq^7 = Sq^9 + Sq^8 Sq^1$$
$$Sq^4 Sq^5 = Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2$$
$$Sq^1 Sq^9 = 0$$
$$Sq^3 Sq^7 = Sq^9 Sq^1$$
$$Sq^5 Sq^5 = Sq^9 Sq^1$$
$$Sq^1 Sq^{10} = Sq^{11}$$
$$Sq^3 Sq^8 = Sq^{11}$$
$$Sq^5 Sq^6 = Sq^{11} + Sq^9 Sq^2$$
$$Sq^7 Sq^4 = 0$$

Biquadratic construction

We consider the universal case of $Sq^i Sq^j(x)$ for $x = u_n$ in $H^n(X; \mathbb{F}_2)$ with $X = K_n$, and apply the quadratic construction twice.

$$\begin{array}{ccccc}
\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \wedge K_n & \xrightarrow{1\wedge 1\wedge\Delta} & \mathbb{R}P^\infty_+ \wedge D_2(K_n) & \xrightarrow{1\wedge\theta} & \mathbb{R}P^\infty_+ \wedge K_{2n} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle 1\wedge\Delta} & & \Big\downarrow{\scriptstyle 1\wedge\Delta} \\
 & & D_2(D_2(K_n)) & \xrightarrow{D_2(\theta)} & D_2(K_{2n}) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \theta} \\
B\Sigma_{4+} \wedge K_n & \xrightarrow{1\wedge\Delta} & D_4(K_n) & \xrightarrow{\theta'} & K_{4n}
\end{array}$$

Here

$$D_2(D_2(X)) = S^\infty_+ \wedge_{C_2} (S^\infty_+ \wedge_{C_2} X^{\wedge 2})^{\wedge 2} \cong (S^\infty \times (S^\infty)^2)_+ \wedge_{C_2 \ltimes (C_2)^2} X^{\wedge 4},$$

where $C_2 \ltimes (C_2)^2$ denotes the semi-direct product.

Sketch proof

In the upper part of the diagram,

$$(1 \wedge \Delta)^* \theta^*(u_{4n}) = \sum_k a^{2n-k} \otimes Sq^k(u_{2n})$$

in $\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge K_{2n}; \mathbb{F}_2) \cong \mathbb{F}_2[a] \otimes \tilde{H}^*(K; \mathbb{F}_2)$, which maps to

$$\begin{aligned}
z = (1 \wedge 1 \wedge \Delta)^*(1 \wedge \theta)^*(\sum_k a^{2n-k} \otimes Sq^k(u_{2n}))& \\
&= \sum_k a^{2n-k} \otimes (1 \wedge \Delta)^* \theta^*(Sq^k(u_{2n})) \\
&= \sum_k a^{2n-k} \otimes Sq^k(\sum_\ell b^{n-\ell} \otimes Sq^\ell(u_n)) \\
&= \sum_{i,j} a^{2n-i-j} \otimes \sum_\ell Sq^i(b^{n-\ell}) \otimes Sq^j(Sq^\ell(u_n)) \\
&= \sum_{i,j,\ell} \binom{n-\ell}{i} a^{2n-i-j} \otimes b^{n+i-\ell} \otimes Sq^j Sq^\ell(u_n)
\end{aligned}$$

in $\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \wedge K_n; \mathbb{F}_2) \cong \mathbb{F}_2[a] \otimes \mathbb{F}_2[b] \otimes \tilde{H}^*(K_n; \mathbb{F}_2)$.

Proof (cont.)

- We claim that z is invariant under the twist map $\tau \wedge 1$ that interchanges the two copies of $\mathbb{R}P^\infty_+$.
- This implies an identity among the composite operations $Sq^j Sq^\ell(u_n)$, for varying j and ℓ
- The Adem relations can be extracted from this with some effort.
- See [Ste62, p. 119] or [Hat02, p. 508]. □

Proof of claim

- To prove the claim, we use the extended power

$$D_4(X) = E\Sigma_{4+} \wedge_{\Sigma_4} (X \wedge X \wedge X \wedge X)\,,$$

  where Σ_4 denotes the symmetric group on four letters and $p\colon E\Sigma_4 \to B\Sigma_4$ is a universal principal Σ_4-bundle.
- The group Σ_4 acts freely from the right on $E\Sigma_4$, and acts from the left on $X^{\wedge 4} = X \wedge X \wedge X \wedge X$ by permuting the factors.
- When $X = K_n$ the map $\theta'_0\colon K_n^{\wedge 4} \to K_{4n}$ representing the fourfold smash product extends, uniquely up to homotopy, to a map $\theta'\colon D_4(K_n) \to K_{4n}$.
- An inclusion $G = C_2 \ltimes (C_2 \times C_2) \subset \Sigma_4$ induces $\beta\colon D_2(D_2(X)) \to D_4(X)$, so that $\theta'\beta \simeq \theta D_2(\theta)$.

Proof of claim (cont.)

- The diagonal map $\Delta\colon K_n \to K_n^{\wedge 4}$ is Σ_4-equivariant, and leads to the map $1 \wedge \Delta\colon B\Sigma_{4+} \wedge K_n \to D_4(K_n)$.
- The inclusion $1 \times \Delta\colon H = C_2 \times C_2 \subset C_2 \ltimes (C_2 \times C_2) = G \subset \Sigma_4$ now induces $\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \cong B(C_2 \times C_2)_+ \to B\Sigma_{4+}$ and the left hand vertical map, making the whole diagram commute up to homotopy.
- Hence z can also be calculated as the pullback of $(1 \wedge \Delta)^*(\theta')^*(u_{4n}) \in H^*(B\Sigma_4; \mathbb{F}_2) \otimes \tilde{H}^*(K_n; \mathbb{F}_2)$.
- There is an inner automorphism of Σ_4 that maps $H = C_2 \times C_2$ to itself by the twist map τ.
- Since inner automorphisms induce the identity map on group cohomology, i.e., on $H^*(B\Sigma_4; \mathbb{F}_2)$, the claim that z is invariant under τ follows. $\square$

Outline

Generators and relations

Definition

The mod 2 Steenrod algebra is the graded (unital and associative) $\mathbb{F}_2$-algebra

$$A = \mathscr{A}(2)$$

generated by the symbols Sq^i for $i \geq 0$, subject to the Adem relations

$$Sq^i Sq^j = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

for $i < 2j$, and $Sq^0 = 1$.

Natural representations

Lemma

For each space X the mod 2 cohomology $H^*(X; \mathbb{F}_2)$ is naturally a graded left A-module, where $A = \mathscr{A}(2)$.

Proof.

For $p = 2$, each symbol Sq^i in A acts on $H^*(X; \mathbb{F}_2)$ as the Steenrod operation of the same name. This defines a left action by A, since the Steenrod operations satisfy the Adem relations and Sq^0 acts as the identity. □

Length, degree, admissibility

- Let $I = (i_1, i_2, \ldots, i_\ell)$ be a finite sequence of positive integers.
- We call $\ell = \ell(I)$ the length of I,
- write

$$|I| = \sum_{s=1}^{\ell} i_s$$

  for the degree of I,
- and say that I is admissible if

$$i_s \geq 2i_{s+1}$$

  for each $1 \leq s < \ell$.
- Let

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdot \ldots \cdot Sq^{i_\ell}$$

  denote the product in A, as well as the corresponding composite of Steenrod operations.

Admissible basis

Theorem ([Ste62, Thm. I.3.1])

The admissible monomials Sq^I form a vector space basis for $A = \mathscr{A}(2)$.

Sketch proof.

- The monomials Sq^I clearly generate A.
- If I is not admissible, meaning that $i_s < 2i_{s+1}$ for some s, then we can rewrite Sq^I by means of the Adem relation for $Sq^{i_s} Sq^{i_{s+1}}$.
- This replaces I with sequences of lower moment $\sum_{s=1}^{\ell} s i_s$, so the process eventually halts.
- This proves that the admissible monomials generate A.

Proof (cont.)

- To prove that the admissible monomials form a basis, recall the action

$$Sq^i(a^j) = \binom{j}{i} a^{i+j}$$

of the Steenrod operations on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a]$.

- By the Cartan formula, this determines the action of Sq^I on

$$H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a_1, \ldots, a_n]\,,$$

where the product contains n copies of $\mathbb{R}P^\infty$.

- A proof by induction on n shows that the elements

$$Sq^I(a_1 \cdot \ldots \cdot a_n) \in \mathbb{F}_2[a_1, \ldots, a_n]$$

for I admissible of degree $|I| \leq n$ are linearly independent.

- Since n can be chosen to be arbitrarily large, this proves that the admissible Sq^I are linearly independent. □

Admissible basis for *A* in degrees $* \leq 11$

0. 1
1. Sq^1
2. Sq^2
3. Sq^3, $Sq^2 Sq^1$
4. Sq^4, $Sq^3 Sq^1$
5. Sq^5, $Sq^4 Sq^1$
6. Sq^6, $Sq^5 Sq^1$, $Sq^4 Sq^2$
7. Sq^7, $Sq^6 Sq^1$, $Sq^5 Sq^2$, $Sq^4 Sq^2 Sq^1$
8. Sq^8, $Sq^7 Sq^1$, $Sq^6 Sq^2$, $Sq^5 Sq^2 Sq^1$
9. Sq^9, $Sq^8 Sq^1$, $Sq^7 Sq^2$, $Sq^6 Sq^2 Sq^1$, $Sq^6 Sq^3$
10. Sq^{10}, $Sq^9 Sq^1$, $Sq^8 Sq^2$, $Sq^7 Sq^2 Sq^1$, $Sq^7 Sq^3$, $Sq^6 Sq^3 Sq^1$
11. Sq^{11}, $Sq^{10} Sq^1$, $Sq^9 Sq^2$, $Sq^8 Sq^2 Sq^1$, $Sq^8 Sq^3$, $Sq^7 Sq^3 Sq^1$

Augmentation ideal and indecomposable quotient

- Let the augmentation $\epsilon\colon A \to \mathbb{F}_2$ be the graded ring homomorphism given by $\epsilon(1) = 1$.
- Its kernel is the augmentation ideal

$$I(A) = \ker(\epsilon)$$

which equals the positive degree part of A.
- The classes in the image $I(A)^2 \subset I(A)$ of the pairing

$$I(A) \otimes I(A) \subset A \otimes A \overset{\cdot}{\longrightarrow} A$$

are said to be decomposable.
- The quotient

$$Q(A) = I(A)/I(A)^2$$

is the graded vector space of (algebra) indecomposables of A.

Indecomposables of *A*

Theorem ([Ade52, Thm. 1.5], [Ste62, Thm. 4.3])

The operation Sq^k is decomposable if and only if k is not a power of 2. Hence

$$Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^i}, \ldots$$

generate A as an algebra, and

$$Q(A) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\} .$$

Proof

- If k is not a power of 2, we can write $k = i + 2^\ell$ with $0 < i < 2^\ell$.
- The Adem relation

$$Sq^i Sq^{2^\ell} = \binom{2^\ell - 1}{i} Sq^{i+2^\ell} + \text{(decomposable terms)}$$

and the case $\binom{2^\ell - 1}{i} = 1$ of Lucas' theorem show that $Sq^k = Sq^{i+2^\ell}$ is decomposable.

Proof (cont.)

- Conversely, to see that Sq^k is not decomposable for $k = 2^\ell$, consider the A-module action on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a]$.
- From
$$Sq^i(a^{2^\ell}) = \begin{cases} a^{2^\ell} & \text{for } i = 0, \\ a^{2^{\ell+1}} & \text{for } i = 2^\ell, \\ 0 & \text{otherwise} \end{cases}$$
we see that any operation of degree $0 < * < 2^\ell$ acts trivially on a^{2^ℓ}.
- Hence any decomposable operation of degree 2^ℓ must also map a^{2^ℓ} to zero.
- Since Sq^{2^ℓ} instead maps a^{2^ℓ} to $a^{2^{\ell+1}}$, it cannot be decomposable. $\square$

Spaces with polynomial cohomology

Proposition

If X is a space with

$$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]$$

or

$$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{h+1})$$

with $h \geq 2$, and $|x| = n$, then n is a power of 2.

Proof.

- Since $H^{n+i}(X;\mathbb{F}_2) = 0$ for $0 < i < n$ the operation $Sq^n(x)$ must be trivial if Sq^n is decomposable.
- Since $Sq^n(x) = x^2$ is assumed to be nontrivial, it must instead be the case that Sq^n is indecomposable.

□

Hopf invariant one, I

Proposition

If $f\colon S^{2n-1} \to S^n$ has odd Hopf invariant, then n is a power of 2.

Proof.

If f has odd Hopf invariant, then its mapping cone

$$Cf = S^n \cup_f e^{2n}$$

is a space with

$$H^*(Cf; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$$

with $|x| = n$. □

Polynomial cohomology mod 3

Using the reduced power operations for mod 3 cohomology, one can prove:

Proposition

If X is a space with

$$H^*(X;\mathbb{F}_3) \cong \mathbb{F}_3[x]$$

or

$$H^*(X;\mathbb{F}_3) \cong \mathbb{F}_3[x]/(x^{h+1})$$

with $h \geq 3$, and $|x| = n$ is a power of 2, then $n \in \{2, 4\}$.

Theorem

If X is a space of finite type with

$$H^*(X) \cong \mathbb{Z}[x] \quad \textit{or} \quad H^*(X) \cong \mathbb{Z}[x]/(x^{h+1})$$

with $h \geq 3$, then $n = |x|$ is 2 or 4. If

$$H^*(X) \cong \mathbb{Z}[x]/(x^3)$$

then $n = 2^i \geq 2$ is a power of 2.

Proof.

- The finite type assumption ensures that $H^*(X; \mathbb{F}_p) \cong H^*(X) \otimes \mathbb{F}_p$.
- Suppose that $H^*(X) \cong \mathbb{Z}[x]$ or $\mathbb{Z}[x]/(x^{h+1})$ with $h \geq 2$.
- By graded commutativity, $n = |x|$ is even.
- The case $p = 2$ implies that n is a power of 2.
- If $h \geq 3$, then the case $p = 3$ implies that $n \in \{2, 4\}$.

□

Projective spaces

- The complex and quaternionic projective spaces $\mathbb{C}P^\infty$, $\mathbb{C}P^h$, $\mathbb{H}P^\infty$ and $\mathbb{H}P^h$ show that $\mathbb{Z}[x]$ and $\mathbb{Z}[x]/(x^{h+1})$ with $|x| = n$ are realized as the integral cohomology of spaces for $n \in \{2, 4\}$ and any $h \geq 0$.
- The octonionic projective plane $\mathbb{O}P^2 = S^8 \cup_\sigma e^{16}$ realizes the case $n = 8$ and $h = 2$.
- There is no space $\mathbb{O}P^3$ realizing the case $n = 8$ and $h = 3$.

Hopf invariant one, II

- The question remains whether $\mathbb{Z}[x]/(x^3)$ can be realized as the cohomology of a space when $|x| = n = 2^i$ with $i \geq 4$.
- This is equivalent to the Hopf invariant one problem, of deciding whether there exists a map $f\colon S^{2n-1} \to S^n$ with $H^*(Cf) \cong \mathbb{Z}[x]/(x^3)$.
- This was famously decided in the negative for all $i \geq 4$ by Adams [Ada60].
- The case $i = 4$ was excluded earlier by Toda.
- We will see later that Adams' result corresponds to nonzero differentials in the Adams spectral sequence for the sphere spectrum.

Outline

Transgressions and Steenrod operations

Using Steenrod operations, we can resolve the question from the previous chapter about the mod 2 cohomology Serre spectral sequence for the loop–path fibration of $K(\mathbb{Z}/2, 2)$.

Lemma

The mod 2 *cohomology transgression*

$$d_n^{0,n-1} \colon E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

commutes with the Steenrod squares in $H^*(F; \mathbb{F}_2)$ *and* $H^*(B; \mathbb{F}_2)$.

Proof

- Recall that $\tau^n = d_n^{0,n-1}$ is given by the additive relation

$$(q^*)^{-1}\delta \colon H^{n-1}(F;\mathbb{F}_2) \xrightarrow{\delta} H^n(E,F;\mathbb{F}_2) \xleftarrow{q^*} H^n(B,b_0;\mathbb{F}_2)\,.$$

- Any cohomology operation commutes with q^*, and the Steenrod operations commute with δ.
- Hence if $\tau^n(x) = y$ then $\tau^{n+i}(Sq^i(x)) = Sq^i(y)$, since $\delta(Sq^i(x)) = Sq^i(\delta(x)) = Sq^i(q^*(y)) = q^*(Sq^i(y))$. □

Cohomology of $K(\mathbb{Z}/2, 2)$

Proposition

Let $M_i = (2^{i-1}, 2^{i-2}, \ldots, 2, 1)$ *for* $i \geq 1$*. Then*

$$H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[b, b_1, b_2, \ldots]$$

with $b = u_2$ *and* $b_i = Sq^{M_i}(b) \in H^{2^i+1}(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ *for* $i \geq 1$*.*
The Serre spectral sequence

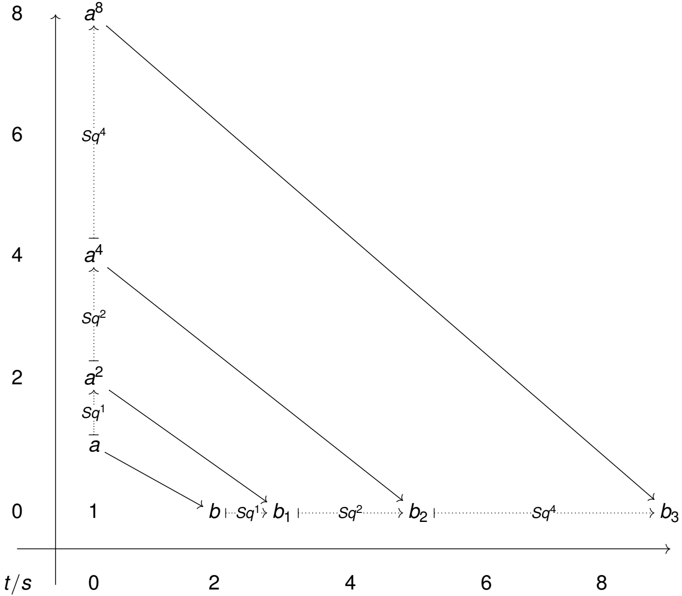
$$\begin{aligned} E_2^{*,*} &\cong H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \otimes H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \\ &\cong \mathbb{F}_2[b, b_1, b_2, \ldots] \otimes \mathbb{F}_2[a] \Longrightarrow H^*(PK(\mathbb{Z}/2, 2); \mathbb{F}_2) = \mathbb{F}_2 \end{aligned}$$

has transgressive differentials $d_2(a) = b$ *and*

$$d_{2^i+1}(a^{2^i}) = b_i$$

for each $i \geq 1$*.*

Transgressive differentials for $K(\mathbb{Z}/2, 2)$



Sketch proof

- By induction on i, we have $Sq^{M_i}(a) = a^{2^i}$, for each $i \geq 1$.
- Hence each a^{2^i} is transgressive, with $d_{2^i+1}(a^{2^i}) = d_{2^i+1}(Sq^{M_i}(a)) = Sq^{M_i}(d_2(a)) = Sq^{M_i}(b) = b_i$.
- It follows by an induction on $u \geq 0$, using a theorem of Borel, that the $\mathbb{F}_2$-algebra homomorphism

$$\mathbb{F}_2[b, b_i \mid i \geq 1] \otimes \mathbb{F}_2[a] \longrightarrow H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \otimes \mathbb{F}_2[a] \cong E_2^{*,*}$$

is an isomorphism in base degrees $s \leq u$. $\square$

Excess

This was generalized by Serre to calculate $H^*(K(G,n);\mathbb{F}_2)$ for all finitely generated abelian G.
The role of the collection $\{M_i\}_i$ is replaced by a condition on the excess of an admissible sequence.

Definition

If $I = (i_1, \ldots, i_\ell)$ is an admissible sequence, so that $i_s \geq 2i_{s+1}$ for each $1 \leq s < \ell$, we define its excess to be

$$e(I) = (i_1 - 2i_2) + \cdots + (i_{\ell-1} - 2i_\ell) + i_\ell = i_1 - i_2 - \cdots - i_\ell = 2i_1 - |I| \,.$$

This is a non-negative integer. The only admissible sequence with $e(I) = 0$ is $I = ()$, and the only admissible sequences with $e(I) = 1$ are the M_i for $i \geq 1$.

Cohomology of mod 2 Eilenberg-MacLane spaces

Theorem ([Ser53, Thm. 2])

Suppose $n \geq 1$. Then

$$H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \mathbb{F}_2[Sq^I(u_n) \mid e(I) < n] \,.$$

The mod 2 cohomology of $K(\mathbb{Z}/2, n)$ is the polynomial algebra generated by the classes $Sq^I(u_n)$, where I ranges over all admissible sequences of excess less than n.

Stable range cohomology, I

Serre's result includes the following stable range calculation.

Corollary

The homomorphism

$$\Sigma^n A \longrightarrow \tilde{H}^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$$
$$\Sigma^n Sq^I \longmapsto Sq^I(u_n)$$

is an isomorphism in degrees $* < 2n$, i.e., for $|I| < n$.

Proof.

- Each admissible I of degree $|I| < n$ has excess $e(I) < n$.
- Hence the $Sq^I(u_n)$ with I admissible of degree $|I| < n$ range over the algebra generators of $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ in degrees $* < 2n$.
- There are no decomposables in that range of degrees.

□

Cohomology of integral Eilenberg–MacLane spaces

Let $\bar{u}_n \in H^n(K(\mathbb{Z}, n); \mathbb{F}_2)$ denote the unique nonzero class.
Note that $\beta(\bar{u}_n) = 0$, so that $Sq^1(\bar{u}_n) = 0$.

Theorem ([Ser53, Thm. 3])

Suppose $n \geq 2$. Then

$$H^*(K(\mathbb{Z}, n); \mathbb{F}_2) \cong \mathbb{F}_2[Sq^I(\bar{u}_n) \mid e(I) < n, i_\ell > 1] \, .$$

The mod 2 cohomology of $K(\mathbb{Z}, n)$ is the polynomial algebra generated by the classes $Sq^I(\bar{u}_n)$, where $I = (i_1, \ldots, i_\ell)$ ranges over all admissible sequences of excess less than n, except those with final term $i_\ell = 1$.

Stable range cohomology, II

Corollary

Let $n \geq 2$. The homomorphism

$$\begin{aligned} \Sigma^n A/ASq^1 &\longrightarrow \tilde{H}^*(K(\mathbb{Z}, n); \mathbb{F}_2) \\ \Sigma^n Sq^I &\longmapsto Sq^I(\bar{u}_n) \end{aligned}$$

is an isomorphism in degrees $* < 2n$, i.e., for $|I| < n$.

Proof.

- By ASq^1 we mean the left ideal in A generated by Sq^1.
- It has a basis consisting of the admissible Sq^I with $I = (i_1, \dots, i_\ell)$ where $i_\ell = 1$.
- Hence the $Sq^I(\bar{u}_n)$ with I admissible of degree $|I| < n$ and $i_\ell > 1$ (if $\ell \geq 1$) range over the algebra generators of $H^*(K(\mathbb{Z}, n); \mathbb{F}_2)$ in degrees $* < 2n$.
- There are no decomposables in that range of degrees.

□

Example

Write $H^*X = H^*(X; \mathbb{F}_2)$. The exact Serre sequence

$$0 \to H^nK(\mathbb{Z}/2, n) \xrightarrow{i^*} H^nK(\mathbb{Z}, n) \xrightarrow{\tau^{n+1}} H^{n+1}K(\mathbb{Z}, n+1) \xrightarrow{p^*} \dots$$
$$\dots \xrightarrow{\tau^{2n}} H^{2n}K(\mathbb{Z}, n+1) \xrightarrow{p^*} H^{2n}K(\mathbb{Z}/2, n) \xrightarrow{i^*} H^{2n}K(\mathbb{Z}, n)$$

associated to the homotopy fiber sequence

$$K(\mathbb{Z}, n) \xrightarrow{i} K(\mathbb{Z}/2, n) \xrightarrow{p} K(\mathbb{Z}, n+1)$$

satisfies $i^*(u_n) = \bar{u}_n$, so that $i^*(Sq^I(u_n)) = Sq^I(\bar{u}_n)$, by naturality. Hence i^* is surjective, and $\tau^m = 0$ for $n < m \leq 2n$, It follows that $p^*(\bar{u}_{n+1}) = Sq^1 u_n$, since this is the only nonzero class in its degree, so that $p^*(Sq^I \bar{u}_{n+1}) = Sq^I Sq^1 u_n$.

Example (cont.)

In particular, the Serre sequence splits up into the short exact sequences

$$0 \to \Sigma^{n+1} A/ASq^1 \xrightarrow{p^*} \Sigma^n A \xrightarrow{i^*} \Sigma^n A/ASq^1 \to 0$$

in degrees $n \leq * < 2n$. Here

$$p^*(\Sigma^{n+1} Sq^I) = \Sigma^n Sq^I Sq^1 \,,$$

while

$$i^*(\Sigma^n Sq^I) = \Sigma^n Sq^I \mod ASq^1 \,.$$

This is a (nontrivial) extension of A-modules.

Outline

Stable operations

The Steenrod operations Sq^I are stable, in the following sense.

Definition

A stable cohomology operation $\theta = (\theta_k)_k$ of type $(G; G', n)$ is a sequence of cohomology operations θ_k of type $(G, k; G', n+k)$ such that each diagram

$$\begin{array}{ccc} \tilde{H}^k(X;G) & \xrightarrow{\theta_k} & \tilde{H}^{n+k}(X;G') \\ \sigma \downarrow \cong & & \cong \downarrow \sigma \\ \tilde{H}^{k+1}(\Sigma X;G) & \xrightarrow{\theta_{k+1}} & \tilde{H}^{n+k+1}(\Sigma X;G') \end{array}$$

commutes, where σ denotes the suspension isomorphism.

Cohomology suspensions

Definition

The cohomology suspension

$$\omega\colon \tilde{H}^{m+1}(Y;G') \longrightarrow \tilde{H}^{m}(\Omega Y;G')$$

maps the homotopy class of $f\colon Y \to K(G',m+1)$ to the homotopy class of $\Omega f\colon \Omega Y \to \Omega K(G',m+1) \simeq K(G',m)$.

Remark

The standard notation for the cohomology suspension is σ, not ω, but for this argument is seems clearer to reserve $\tilde{\sigma}$ to denote the equivalence $K(G,k) \simeq \Omega K(G,k+1)$ and the suspension isomorphism represented by it.

Lemma

A sequence $(\theta_k)_k$ of cohomology operations is stable if and only if

$$\omega(\theta_{k+1}) = \theta_k$$

for each k, where

$$\omega\colon \tilde{H}^{n+k+1}(K(G,k+1);G') \longrightarrow \tilde{H}^{n+k}(K(G,k);G')$$

is the cohomology suspension.

Proof.

Each condition is equivalent to asking that

$$\begin{array}{ccc} K(G,k) & \xrightarrow{\theta_k} & K(G',n+k) \\ {\scriptstyle \tilde{\sigma}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \tilde{\sigma}} \\ \Omega K(G,k+1) & \xrightarrow{\Omega\theta_{k+1}} & \Omega K(G',n+k+1) \end{array}$$

commutes up to homotopy, for each k. □

Stable operations as a limit

In other words, the abelian group of stable cohomology operations of type $(G; G', n)$ is isomorphic to the sequential limit

$$\lim_{k} \tilde{H}^{n+k}(K(G,k); G')$$

of the tower

$$\ldots \xrightarrow{\omega} \tilde{H}^{n+k+1}(K(G,k+1); G') \xrightarrow{\omega} \tilde{H}^{n+k}(K(G,k); G') \xrightarrow{\omega} \ldots .$$

Stable operations as a graded ring

- The composite of a stable operation of type $(G; G', n)$ followed by a stable operation of type $(G'; G'', m)$ is a stable operation of type $(G; G'', n + m)$
- The collection of all stable cohomology operations of type $(G; G, n)$ for $n \in \mathbb{Z}$ forms a graded (usually non-commutative) ring.
- When $G = \mathbb{F}_2$, this ring is the mod 2 Steenrod algebra, as we can now deduce from the calculations of Serre.

The Steenrod operations give all stable operations

Proposition

Let $A^n \subset A = \mathscr{A}(2)$ denote the degree n part of the mod 2 Steenrod algebra. The homomorphism

$$A^n \xrightarrow{\cong} \lim_k \tilde{H}^{n+k}(K(\mathbb{F}_2, k); \mathbb{F}_2)$$
$$\theta \longmapsto (\theta(u_k))_k$$

is an isomorphism. Hence A is isomorphic to the graded ring of stable cohomology operations of type $(\mathbb{F}_2; \mathbb{F}_2, n)$ for arbitrary n.

Proof

- The homomorphisms

$$\Sigma^k A^n \longrightarrow \tilde{H}^{n+k}(K(\mathbb{F}_2, k); \mathbb{F}_2)$$
$$\Sigma^k \theta \longmapsto \theta(u_k)$$

  are compatible with the cohomology suspensions ω, and are isomorphisms for $k > n$.
- Hence they combine to map A^n isomorphically to the group of compatible sequences $(\theta_k)_k$.
- In particular, each morphism ω (in the earlier tower) is an isomorphism, for $k > n$.
- The product in A corresponds to the composition of (stable) cohomology operations. □

Outline

Milnor's view on the Cartan formula

- The mod 2 cohomology of any space $H^*(X;\mathbb{F}_2)$, is naturally an A-module and a commutative $\mathbb{F}_2$-algebra, satisfying the Cartan formula

$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$

  and the instability condition $Sq^i(x) = 0$ for $i > |x|$.
- Following Milnor [Mil58, Lem. 1], there is an algebra homomorphism

$$\begin{aligned} \psi\colon A &\longrightarrow A \otimes A \\ Sq^k &\longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j \,, \end{aligned}$$

  and each $A \otimes A$-module can be viewed as an A-module by restriction along ψ.

Milnor's view on the Cartan formula (cont.)

- The Cartan formula then says that the cup product

$$H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \xrightarrow{\cup} H^*(X;\mathbb{F}_2)$$

  is an A-module homomorphism, where the A-module structure in the source is obtained by restriction in this way.
- We also say that $H^*(X;\mathbb{F}_2)$ is a A-module algebra.
- The coproduct ψ makes A a cocommutative Hopf algebra, and we shall now review this algebraic structure.
- The paper [MM65] by Milnor and Moore is a standard reference.

Closed structure on graded R-modules

- Let R be a commutative ring, which will be the field $\mathbb{F}_2$ in our main application.
- For R-modules L and M we write $L \otimes M = L \otimes_R M$ for the tensor product over R and $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M, N)$ for the R-linear homomorphisms.
- If L, M and N are (homologically) graded, then

$$(L \otimes M)_k = \bigoplus_{i+j=k} L_i \otimes M_j$$

and

$$\mathrm{Hom}(M, N)_i = \prod_{i+j=k} \mathrm{Hom}(M_j, N_k)\,.$$

Closed symmetric monoidal structure (cont.)

- The twist isomorphism

$$\tau\colon L \otimes M \longrightarrow M \otimes L$$

  maps $x \otimes y$ to $(-1)^{ij} y \otimes x$, for $x \in L_i$ and $y \in N_j$.
- There is a natural isomorphism

$$\mathrm{Hom}(L \otimes M, N) \cong \mathrm{Hom}(L, \mathrm{Hom}(M, N))$$

  taking $f\colon L \otimes M \to N$ to $g\colon L \to \mathrm{Hom}(M, N)$, with $f(x \otimes y) = g(x)(y)$.
- Here f is left adjoint to g and g is right adjoint to f.

Adjunction counit and unit

- The natural evaluation homomorphism (= adjunction counit)

$$\epsilon\colon \operatorname{Hom}(M, N) \otimes M \longrightarrow N$$

  is left adjoint to the identity on $\operatorname{Hom}(M, N)$.
- The natural homomorphism (= adjunction unit)

$$\eta\colon L \longrightarrow \operatorname{Hom}(M, L \otimes M)$$

  is right adjoint to the identity on $L \otimes M$.
- We say that (graded) R-modules form a closed symmetric monoidal category, cf. [ML63, §VII.7].

Algebras

A (graded) R-algebra is a (graded) R-module A with a product $\phi\colon A \otimes A \to A$ and a unit $\eta\colon R \to A$ such that

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{1 \otimes \phi} & A \otimes A \\ {\scriptstyle \phi \otimes 1}\downarrow & & \downarrow{\scriptstyle \phi} \\ A \otimes A & \xrightarrow{\phi} & A \end{array}$$

$$\begin{array}{ccccc} R \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A & \xleftarrow{1 \otimes \eta} & A \otimes R \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

commute. It is commutative if the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ {\scriptstyle \phi}\searrow & & \swarrow{\scriptstyle \phi} \\ & A & \end{array}$$

commutes.

Tensor product of algebras

- The ring R is the initial R-algebra.
- The product $\phi\colon R \otimes R \to R$ is the canonical isomorphism and the unit $\eta\colon R \to R$ is the identity.

- The tensor product of two R-algebras A and B is the R-algebra $A \otimes B$ with product given by the composite

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\phi \otimes \phi} A \otimes B$$

  and unit

$$R \cong R \otimes R \xrightarrow{\eta \otimes \eta} A \otimes B\,.$$

- In the full subcategory of commutative R-algebras, the tensor product is the categorical sum.

Augmented algebras

- An R-algebra (A, ϕ, η) is augmented if it comes equipped with an algebra morphism $\epsilon \colon A \to R$.
- Let
$$I(A) = \ker(\epsilon \colon A \to R)$$
be the augmentation ideal.
- Let the R-module of indecomposables $Q(A)$ be the cokernel
$$I(A) \otimes I(A) \xrightarrow{\phi} I(A) \xrightarrow{\pi} Q(A) \to 0$$
of the restricted product.

Indecomposables and generators

- A subset $S \subset I(A)$ that generates A as an R-algebra will map to a subset $\pi(S) \subset Q(A)$ that generates $Q(A)$ as an R-module.
- The converse often holds.
- If $A = R[[x]]$ is a formal power series algebra, with $\epsilon(x) = 0$, then $Q(A) \cong R\{x\}$, but x does not generate A algebraically.
- The elements in $I(A)^2 = \phi(I(A) \otimes I(A))$ are said to be (algebra) decomposable, and an element $x \in I(A)$ with $\pi(x) \neq 0$ is (algebra) indecomposable.

Left modules

Definition

A left A-module is a (graded) R-module M with a pairing $\lambda\colon A \otimes M \to M$ such that

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{1 \otimes \lambda} & A \otimes M \\ {\scriptstyle \phi \otimes 1}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A \otimes M & \xrightarrow{\lambda} & M \end{array}$$

and

$$\begin{array}{ccc} R \otimes M & \xrightarrow{\eta \otimes 1} & A \otimes M \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \lambda} \\ & & M \end{array}$$

commute.

Right modules

Definition

A right A-module is a (graded) R-module L with a pairing $\rho\colon L \otimes A \to L$ such that

$$\begin{CD}
L \otimes A \otimes A @>{\rho \otimes 1}>> L \otimes A \\
@V{1 \otimes \phi}VV @VV{\rho}V \\
L \otimes A @>{\rho}>> L
\end{CD}$$

and

$$\begin{array}{ccc}
L \otimes A & \xleftarrow{1 \otimes \eta} & L \otimes R \\
\Big\downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \cong} & \\
L & &
\end{array}$$

commute.

Tensor and Hom over A

Given a right A-module L and a left A-module M, the tensor product $L \otimes_A M$ is the coequalizer

$$L \otimes A \otimes M \underset{\rho \otimes 1}{\overset{1 \otimes \lambda}{\rightrightarrows}} L \otimes M \xrightarrow{\pi} L \otimes_A M$$

where $1 \otimes \lambda$ and $\rho \otimes 1$ are given by the left and right action maps, respectively.

Given two left A-modules M and N, the R-module of A-linear homomorphisms $\mathrm{Hom}_A(M, N)$ is the equalizer

$$\mathrm{Hom}_A(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \underset{\lambda_*}{\overset{\lambda^*}{\rightrightarrows}} \mathrm{Hom}(A \otimes M, N)\,,$$

where $\lambda^*(f) = f\lambda\colon A \otimes M \to N$ and
$\lambda_*(f) = \lambda(1 \otimes f)\colon A \otimes M \to N$ for $f\colon M \to N$.

Pontryagin product

Example

Let G be a topological group, with multiplication $m\colon G \times G \to G$. The Pontryagin product

$$\phi\colon H_*(G;R) \otimes H_*(G;R) \xrightarrow{\times} H_*(G \times G;R) \xrightarrow{m_*} H_*(G;R)$$

and the homomorphisms $\eta\colon R \to H_*(G;R)$ and $\epsilon\colon H_*(G;R) \to R$ induced by $\{e\} \subset G$ and $G \to \{e\}$ make $H_*(G;R)$ an augmented R-algebra.

Likewise, if X is a topological space with a left G-action, then $M = H_*(X;R)$ is a left $H_*(G;R)$-module.

Cup product

Example

For any space X the cup product

$$\cup\colon H^*(X;R) \otimes H^*(X;R) \xrightarrow{\times} H^*(X \times X;R) \xrightarrow{\Delta^*} H^*(X;R)$$

and the homomorphism $\eta\colon R \to H^*(X;R)$ induced by $X \to \{x_0\}$ make $H^*(X;R)$ a (graded) commutative R-algebra.

A choice of base point $x_0 \in X$ determines an augmentation $\epsilon\colon H^*(X;R) \to R$, induced by $\{x_0\} \subset X$.

Extended modules

- If V is an R-module, then the left action

$$\lambda\colon A \otimes A \otimes V \xrightarrow{\phi\otimes 1} A \otimes V$$

  makes $A \otimes V$ a left A-module, known as an extended A-module.
- There is a natural isomorphism

$$\mathrm{Hom}_A(A \otimes V, N) \cong \mathrm{Hom}(V, UN)\,,$$

  where N is any A-module and UN its underlying R-module.
- Hence the extended A-module functor $V \mapsto A \otimes V$ is left adjoint to the forgetful functor U from left A-modules to R-modules.

Coalgebras

The dual theory of coalgebras and comodules is developed in [MM65] and [EM66].

Definition

A (graded) *R-coalgebra* is a (graded) R-module C with a coproduct $\psi\colon C \to C \otimes C$ and a counit $\epsilon\colon C \to R$ such that

$$\begin{CD} C @>{\psi}>> C \otimes C \\ @V{\psi}VV @VV{\psi\otimes 1}V \\ C \otimes C @>{1\otimes\psi}>> C \otimes C \otimes C \end{CD}$$

and

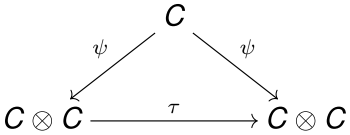
$$\begin{array}{ccccc} & & C & & \\ & \swarrow{\scriptstyle\cong} & \downarrow{\scriptstyle\psi} & {\scriptstyle\cong}\searrow & \\ R \otimes C & \xleftarrow[\epsilon\otimes 1]{} & C \otimes C & \xrightarrow[1\otimes\epsilon]{} & C \otimes R \end{array}$$

commute.

Cocommutativity

Definition (cont.)

It is cocommutative if the diagram



commutes.

Notation for coproducts

We can write
$$\psi(x) = \sum_\alpha x'_\alpha \otimes x''_\alpha$$
for suitable $x'_\alpha, x''_\alpha \in C$. Then
$$\sum_{\alpha,\beta} (x'_\alpha)'_\beta \otimes (x'_\alpha)''_\beta \otimes x''_\alpha = \sum_{\alpha,\beta} x'_\alpha \otimes (x''_\alpha)'_\beta \otimes (x''_\alpha)''_\beta$$
by coassociativity, and
$$\sum_\alpha \epsilon(x'_\alpha) x''_\alpha = x = \sum_\alpha x'_\alpha \epsilon(x''_\alpha)$$
by counitality. Cocommutativity asks that
$$\sum_\alpha x'_\alpha \otimes x''_\alpha = \sum_\alpha (-1)^{|x'_\alpha||x''_\alpha|} x''_\alpha \otimes x'_\alpha \, .$$

Notation (cont.)

We often omit the summation indices in these formulas, and write

$$\psi(x) = \sum x' \otimes x''$$

$$\sum (x')' \otimes (x')'' \otimes x'' = \sum x' \otimes (x'')' \otimes (x'')''$$

$$\sum \epsilon(x')x'' = x = \sum x'\epsilon(x'')$$

$$\sum x' \otimes x'' = \sum (-1)^{|x'||x''|} x'' \otimes x' .$$

Tensor product of coalgebras

- The ring R is the terminal R-coalgebra.
- The coproduct $\psi\colon R \to R \otimes R$ is the inverse of the canonical isomorphism and the counit $\epsilon\colon R \to R$ is the identity.
- The tensor product of two R-coalgebras C and D is the R-coalgebra $C \otimes D$ with coproduct given by the composite

$$C \otimes D \xrightarrow{\psi \otimes \psi} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D$$

  and counit

$$C \otimes D \xrightarrow{\epsilon \otimes \epsilon} R \otimes R \cong R\,.$$

- In the full subcategory of cocommutative R-coalgebras, the tensor product is the categorical product.

Coaugmented coalgebras

- An R-coalgebra (C, ψ, ϵ) is coaugmented if it comes equipped with a coalgebra morphism $\eta \colon R \to C$.
- Let
$$J(C) = \mathrm{cok}(\eta \colon R \to C)$$
be the coaugmentation coideal, also known as the unit coideal.
- Let the R-module of primitives $P(C)$ be the kernel
$$0 \to P(C) \xrightarrow{\iota} J(C) \xrightarrow{\psi} J(C) \otimes J(C)$$
of the corestricted coproduct.
- In terms of elements,
$$P(C) \cong \{x \in C \mid \psi(x) = x \otimes 1 + 1 \otimes x\} ,$$
and an element $x \in C$ with $\psi(x) = x \otimes 1 + 1 \otimes x$ is said to be (coalgebra) primitive.

Notation (revisited)

Remark

In the coaugmented case, we can write

$$\psi(x) = x \otimes 1 + \sum_{\alpha} x'_{\alpha} \otimes x''_{\alpha} + 1 \otimes x$$

for $x \in I(C) = \ker(\epsilon) \cong J(C)$, with $x'_{\alpha}, x''_{\alpha} \in I(C)$, and this often gets abbreviated to

$$\psi(x) = x \otimes 1 + \sum x' \otimes x'' + 1 \otimes x\,.$$

Left comodules

Definition

A left C-comodule is a (graded) R-module M with a coaction $\nu\colon M \to C \otimes M$ such that

$$\begin{CD} M @>{\nu}>> C \otimes M \\ @V{\nu}VV @VV{\psi \otimes 1}V \\ C \otimes M @>{1 \otimes \nu}>> C \otimes C \otimes M \end{CD}$$

and

$$\begin{array}{ccc} & & M \\ & \swarrow^{\cong} & \downarrow{\scriptstyle \nu} \\ R \otimes M & \xleftarrow[\epsilon \otimes 1]{} & C \otimes M \end{array}$$

commute.

Right comodules

Definition

A right C-comodule is a (graded) R-module L with a coaction $\sigma\colon L \to L \otimes C$ such that

$$\begin{array}{ccc} L & \xrightarrow{\sigma} & L \otimes C \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle 1\otimes\psi} \\ L \otimes C & \xrightarrow{\sigma\otimes 1} & L \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccc} L & & \\ {\scriptstyle\sigma}\downarrow & \overset{\cong}{\searrow} & \\ L \otimes C & \xrightarrow[1\otimes\epsilon]{} & L \otimes R \end{array}$$

commute.

Cotensor product

Definition

Given a right C-comodule L and a left C-comodule M, the cotensor product $L \square_C M$ is the equalizer

$$L \square_C M \xrightarrow{\iota} L \otimes M \underset{\sigma \otimes 1}{\overset{1 \otimes \nu}{\rightrightarrows}} L \otimes C \otimes M$$

where $1 \otimes \nu$ and $\sigma \otimes 1$ are given by the left and right coaction maps, respectively.

Comodule Hom

Definition

Given two left C-comodules M and N, the R-module of comodule homomorphisms $\mathrm{Hom}_C(M, N)$ is the equalizer

$$\mathrm{Hom}_C(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \underset{\nu_*}{\overset{\nu^*}{\rightrightarrows}} \mathrm{Hom}(M, C \otimes N)\,,$$

where $\nu^*(f) = (1 \otimes f)\nu\colon M \to C \otimes N$ and $\nu_*(f) = \nu f\colon M \to C \otimes N$ for $f\colon M \to N$.

Module vs. comodule Hom

- We write $\mathrm{Hom}_B(M, N)$ to denote
  - the B-module homomorphisms $f\colon M \to N$ when B is an algebra and M and N are B-modules,
  - and to denote the B-comodule homomorphisms $f\colon M \to N$ when B is a coalgebra and M and N are B-comodules.
- This will also apply to the derived functors $\mathrm{Ext}_B^s(M, N)$.
- We may say "module Ext" or "comodule Ext" to distinguish the two cases.

Pontryagin coproduct

Example

Let G be a topological group, with multiplication $m\colon G \times G \to G$. Suppose that $H^*(G; R)$ is finitely generated and projective over R in each degree, so that the cross product

$$H^*(G; R) \otimes H^*(G; R) \xrightarrow{\times} H^*(G \times G; R)$$

is an isomorphism. (Recall that $\otimes = \otimes_R$.) Then the Pontryagin coproduct

$$\psi\colon H^*(G; R) \xrightarrow{m^*} H^*(G \times G; R) \xrightarrow{\times^{-1}} H^*(G; R) \otimes H^*(G; R)$$

and the homomorphisms $\epsilon\colon H^*(G; R) \to R$ and $\eta\colon R \to H^*(G; R)$ induced by $\{e\} \subset G$ and $G \to \{e\}$ make $H^*(G; R)$ a coaugmented R-coalgebra.

Pontryagin comodule

Example

Likewise, if X is a topological space with a left G-action, then $M = H^*(X; R)$ is a left $H^*(G; R)$-comodule.
The hypothesis on G ensures that

$$H^*(G; R) \otimes H^*(X; R) \xrightarrow{\times} H^*(G \times X; R)$$

is also an isomorphism.

Diagonal coproduct

Example

Dually, for any space X with $H_*(X; R)$ flat over R in each degree, the diagonal coproduct

$$H_*(X; R) \xrightarrow{\Delta_*} H_*(X \times X; R) \xrightarrow{\times^{-1}} H_*(X; R) \otimes H_*(X; R)$$

and the homomorphism $\epsilon \colon H_*(X; R) \to R$ induced by $X \to \{x_0\}$ make $H_*(X; R)$ a (graded) cocommutative R-coalgebra.

A choice of base point $x_0 \in X$ determines a coaugmentation $\eta \colon R \to H_*(X; R)$, induced by $\{x_0\} \subset X$.

Extended comodules

- If V is an R-module, then the left coaction

$$\nu\colon C \otimes V \xrightarrow{\psi\otimes 1} C \otimes C \otimes V$$

  makes $C \otimes V$ a left C-comodule, known as an extended C-comodule.
- There is a natural isomorphism

$$\mathrm{Hom}(UM, V) \cong \mathrm{Hom}_C(M, C \otimes V)\,,$$

  where M is any C-comodule and UM its underlying R-module.
- Hence the extended C-comodule functor $V \mapsto C \otimes V$ is right adjoint to the forgetful functor U from left C-comodules to R-modules.

Bialgebras

Definition

A (graded) *R*-bialgebra is a (graded) *R*-module *B* with

- a product $\phi\colon B \otimes B \to B$,
- unit $\eta\colon R \to B$,
- coproduct $\psi\colon B \to B \otimes B$ and
- counit $\epsilon\colon B \to R$

such that

1. (B, ϕ, η) is an *R*-algebra,
2. (B, ψ, ϵ) is an *R*-coalgebra, and
3. ψ and ϵ are *R*-algebra homomorphisms.

Lemma

The following are equivalent:

- ψ *and* ϵ *are R-algebra homomorphisms.*
- ϕ *and* η *are R-coalgebra homomorphisms.*

Proof

The conditions that ψ and ϵ are R-algebra homomorphisms ask that the diagrams

$$\begin{array}{ccccc}
B \otimes B & \xrightarrow{\psi\otimes\psi} & B \otimes B \otimes B \otimes B & & \\
 & & & \searrow^{1\otimes\tau\otimes 1} & \\
\Big\downarrow{\scriptstyle \phi} & & & & B \otimes B \otimes B \otimes B \\
 & & & & \Big\downarrow{\scriptstyle \phi\otimes\phi} \\
B & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & B \otimes B
\end{array}$$

and

$$\begin{CD}
R @>{\cong}>> R \otimes R \\
@V{\eta}VV @VV{\eta\otimes\eta}V \\
B @>{\psi}>> B \otimes B
\end{CD}
\qquad
\begin{CD}
B \otimes B @>{\epsilon\otimes\epsilon}>> R \otimes R \\
@V{\phi}VV @VV{\cong}V \\
B @>{\epsilon}>> R
\end{CD}
\qquad
\begin{CD}
R @>{=}>> R \\
@V{\eta}VV @VV{=}V \\
B @>{\epsilon}>> R
\end{CD}$$

commute. These are also the conditions that ϕ and η are R-coalgebra homomorphisms. $\square$

Primitively generated bialgebras

Definition

There are natural homomorphisms

$$P(B) \rightarrowtail J(B) \xleftarrow{\cong} I(B) \twoheadrightarrow Q(B)$$

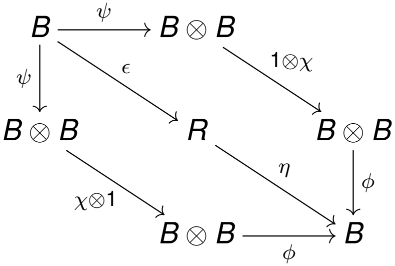
for each bialgebra B.
If $P(B) \to Q(B)$ is surjective, then B is primitively generated.

This terminology is most appropriate when a set of module generators for $Q(B)$ also generates B as an algebra.

Hopf algebras

Definition

A Hopf algebra over R is an R-bialgebra B equipped with an R-linear conjugation $\chi\colon B \to B$ such that



commutes.

If $\psi(b) = \sum b' \otimes b''$ then the condition is

$$\sum b' \cdot \chi(b'') = \eta\epsilon(b) = \sum \chi(b') \cdot b''\,.$$

Lemma

A bialgebra admits at most one conjugation.

Hence being a Hopf algebra is a property, not a structure, for bialgebras.

Lemma

The conjugation $\chi \colon B \to B$ is an anti-homomorphism of algebras, and an anti-homomorphism of coalgebras, so that

$$\chi\phi = \phi\tau(\chi \otimes \chi)$$
$$\psi\chi = (\chi \otimes \chi)\tau\psi \,.$$

Lemma

Let B be a commutative or cocommutative Hopf algebra. Then $\chi^2 = 1$, so

$$\chi = \chi^{-1} \colon B \longrightarrow B \,.$$

See [MM65, §8] or [DNR01, §4.2] for proofs.

Homology of topological groups

Examples studied by Heinz Hopf [Hop41]:

Example

Let G be a topological group. Suppose that $H_*(G;R)$ is flat over R in each degree, so that the unit $\eta\colon R \to H_*(G;R)$, Pontryagin product

$$\phi\colon H_*(G;R) \otimes H_*(G;R) \longrightarrow H_*(G;R)\,,$$

counit $\epsilon\colon H_*(G;R) \to R$ and diagonal coproduct

$$\psi\colon H_*(G;R) \longrightarrow H_*(G;R) \otimes H_*(G;R)$$

make $H_*(G;R)$ an R-bialgebra. The inverse map $i\colon G \to G$ induces the conjugation

$$\chi = i_*\colon H_*(G;R) \longrightarrow H_*(G;R)$$

making $H_*(G;R)$ a cocommutative Hopf algebra over R.

Cohomology of topological groups

Example

Suppose instead that $H^*(G; R)$ is finitely generated and projective over R in each degree, so that the unit $\eta\colon R \to H^*(G; R)$, cup product

$$\phi\colon H^*(G; R) \otimes H^*(G; R) \longrightarrow H^*(G; R)\,,$$

counit $\epsilon\colon H^*(G; R) \to R$ and Pontryagin coproduct

$$\psi\colon H^*(G; R) \longrightarrow H^*(G; R) \otimes H^*(G; R)$$

make $H^*(G; R)$ an R-bialgebra. The inverse map $i\colon G \to G$ induces the conjugation

$$\chi = i^*\colon H^*(G; R) \longrightarrow H^*(G; R)$$

making $H^*(G; R)$ a commutative Hopf algebra over R.

Diagonal action on $\otimes_R$ of B-modules

Definition

Let B be a Hopf algebra over R. For left B-modules L and M we give the tensor product

$$L \otimes M$$

the "diagonal" B-module structure with left action $\lambda \colon B \otimes L \otimes M \to L \otimes M$ given by the composition

$$B \otimes L \otimes M \xrightarrow{\psi} B \otimes B \otimes L \otimes M \xrightarrow{1 \otimes \tau \otimes 1} B \otimes L \otimes B \otimes M \xrightarrow{\lambda \otimes \lambda} L \otimes M \, .$$

Margolis [Mar83, §12.1] writes $L \wedge M$ for this tensor product of B-modules.

Conjugate action on Hom_R of B-modules

Definition

For left B-modules M and N we give

$$\mathrm{Hom}(M, N)$$

the "conjugate" B-module structure with left action $\lambda\colon B \otimes \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N)$ given by the right adjoint of the composition

$$\begin{aligned} B \otimes \mathrm{Hom}(M, N) \otimes M &\xrightarrow{\psi\otimes 1\otimes 1} B \otimes B \otimes \mathrm{Hom}(M, N) \otimes M \\ \xrightarrow{1\otimes\tau\otimes 1} B \otimes \mathrm{Hom}(M, N) \otimes B \otimes M &\xrightarrow{1\otimes 1\otimes\chi\otimes 1} B \otimes \mathrm{Hom}(M, N) \otimes B \otimes M \\ \xrightarrow{1\otimes 1\otimes\lambda} B \otimes \mathrm{Hom}(M, N) \otimes M &\xrightarrow{1\otimes\epsilon} B \otimes N \xrightarrow{\lambda} N\,. \end{aligned}$$

Closed symmetric monoidal structure

- There is a natural isomorphism

$$\mathrm{Hom}_B(L \otimes M, N) \cong \mathrm{Hom}_B(L, \mathrm{Hom}(M, N)) \,,$$

  so that $f\colon L \otimes M \to N$ is B-linear if and only if its right adjoint $g\colon L \to \mathrm{Hom}(M, N)$ is B-linear.
- If B is cocommutative, then the twist isomorphism

$$\tau\colon L \otimes M \longrightarrow M \otimes L$$

  is B-linear, and the left B-modules form a closed symmetric monoidal category.

Functional dual

Example

The left B-action on the functional dual $DM = \mathrm{Hom}(M, R)$ of a left B-module M is adjoint to the composition

$$B \otimes DM \otimes M \xrightarrow{\tau \otimes 1} DM \otimes B \otimes M$$
$$\xrightarrow{1 \otimes \chi \otimes 1} DM \otimes B \otimes M \xrightarrow{1 \otimes \lambda} DM \otimes M \xrightarrow{\epsilon} R\,.$$

Explicit formulas

- For $b \in B$ with $\psi(b) = \sum b' \otimes b''$, $\ell \in L$ and $m \in M$ we have
$$b \cdot (\ell \otimes m) = \sum (-1)^{|b''||\ell|} b' \cdot \ell \otimes b'' \cdot m \,.$$
- For $f \in \mathrm{Hom}(M, N)$ we have
$$(b \cdot f)(m) = \sum (-1)^{|b''||f|} b' \cdot f(\chi(b'') \cdot m) \,.$$
- In particular, for $b \in B$ and $f \in \mathrm{Hom}(M, R)$, we have
$$(b \cdot f)(m) = (-1)^{|b||f|} f(\chi(b) \cdot m) \,.$$

Codiagonal coaction on $\otimes_R$ of B-comodules

Definition

Let B be a Hopf algebra over R. For left B-comodules L and M we give the tensor product

$$L \otimes M$$

the "codiagonal" B-comodule structure with left coaction $\nu \colon L \otimes M \to B \otimes L \otimes M$ given by the composition

$$L \otimes M \xrightarrow{\nu \otimes \nu} B \otimes L \otimes B \otimes M \xrightarrow{1 \otimes \tau \otimes 1} B \otimes B \otimes L \otimes M \xrightarrow{\phi \otimes 1 \otimes 1} B \otimes L \otimes M\,.$$

If B is commutative, then the twist isomorphism $\tau \colon L \otimes M \to M \otimes L$ is B-colinear, and the left B-comodules form a symmetric monoidal category.

No coconjugate coaction on Hom_R for B-comodules

- For left B-comodules M and N we cannot generally give the R-module

$$\mathrm{Hom}(M, N)$$

  a natural "coconjugate" B-comodule structure such that $f\colon L \otimes M \to N$ is B-colinear if and only if its right adjoint $g\colon L \to \mathrm{Hom}(M, N)$ is B-colinear.

- If $M = \mathrm{colim}_i M_i$ and $\nu_i\colon \mathrm{Hom}(M_i, N) \to B \otimes \mathrm{Hom}(M_i, N)$ is a suitable coaction, then

$$\lim_i \nu_i\colon \mathrm{Hom}(M, N) \longrightarrow \lim_i B \otimes \mathrm{Hom}(M_i, N)$$

  will not generally factor through $B \otimes \lim_i \mathrm{Hom}(M_i, N) \cong B \otimes \mathrm{Hom}(M, N)$.

Hovey's approach

- When B is flat as an R-module there is, however, a different internal function object $F(M, N)$ with a natural B-comodule structure, and a natural isomorphism

$$\mathrm{Hom}_B(L \otimes M, N) \cong \mathrm{Hom}_B(L, F(M, N))$$

  so that $f\colon L \otimes M \to N$ is B-colinear if and only if $g\colon L \to F(M, N)$ is B-colinear.
- See Hovey's paper [Hov04, Thm. 1.3.1] for a construction, which satisfies $F(M, B \otimes V) \cong B \otimes \mathrm{Hom}(M, V)$ when $N = B \otimes V$ is a coextended B-comodule. Here V is any left R-module.
- There is a natural homomorphism $F(M, N) \to \mathrm{Hom}(M, N)$, which is injective if M is finitely generated over R, and an isomorphism if M is finitely presented over R, cf. [Hov04, Prop. 1.3.2]. We can think of $F(M, N)$ as the elements of $\mathrm{Hom}(M, N)$ with algebraic B-coaction.

Other approaches

- A second approach [Boa82] is to consider B-comodules as a subcategory of B^*-modules, where B^* is the (non-commutative) ring of (right) R-module homomorphisms $B \to R$.
- A third approach is to consider $\mathrm{Hom}(M, N)$ as a "completed" B-comodule, with coaction $\mathrm{Hom}(M, N) \to B \mathbin{\widehat{\otimes}} \mathrm{Hom}(M, N)$ landing in a completed tensor product.

Behavior under dualization

Lemma

Let M be a graded R-module, with functional dual $DM = \mathrm{Hom}(M, R)$.

- *If M is bounded below then DM is bounded above, while if M is bounded above then DM is bounded below.*
- *If M is finitely generated and projective over R in each degree, then DM is also finitely generated and projective over R in each degree, and the canonical homomorphism*

  $$\rho\colon M \longrightarrow DDM$$

  is an isomorphism.

Dual of tensor product

Lemma

Let L and M be graded R-modules.

- *If L and M are both bounded below (or both are bounded above, or one of them is bounded above and below), and*
- *L (or M) is finitely generated projective over R in each degree,*

then the canonical homomorphism

$$DL \otimes DM \xrightarrow{\otimes} D(L \otimes M)$$

is an isomorphism. Here

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \cdot g(y)$$

for $f \in DL$, $g \in DM$, $x \in L$ and $y \in M$.

Dual of algebra is often a coalgebra

Lemma

Let A be a graded R-algebra that is bounded below (or bounded above) and finitely generated projective over R in each degree. Then DA with the coproduct

$$\psi \colon DA \xrightarrow{D\phi} D(A \otimes A) \xrightarrow{\otimes^{-1}} DA \otimes DA$$

and counit

$$\epsilon \colon DA \xrightarrow{D\eta} DR \cong R$$

is a graded R-coalgebra.

Dual of coalgebra is always an algebra

Lemma

If C is a graded R-coalgebra, then DC with the product

$$\phi\colon DC \otimes DC \xrightarrow{\otimes} D(C \otimes C) \xrightarrow{D\psi} DC$$

and the unit

$$\eta\colon R \cong DR \xrightarrow{D\epsilon} DC$$

is a graded R-algebra.

Dual of indecomposables and primitives of dual

Lemma

Let A be an augmented graded R-algebra that is bounded below (or bounded above) and finitely generated projective over R in each degree.

Then DA is coaugmented by

$$\eta\colon R \cong DR \xrightarrow{D\epsilon} DA\,,$$

and the isomorphism $J(DA) \cong DI(A)$ restricts to an isomorphism

$$P(DA) \cong DQ(A)\,.$$

Dual of primitives and indecomposables of dual

Lemma

If C is a coaugmented graded R-coalgebra, then DC is augmented by

$$\epsilon\colon DC \xrightarrow{D\eta} DR \cong R\,,$$

and the isomorphism $I(DC) \cong DJ(C)$ induces a homomorphism

$$Q(DC) \longrightarrow DP(C)\,.$$

If R is a field, then this is a surjection. If, furthermore, C is bounded below (or bounded above) and finitely generated over the field R in each degree, then this is an isomorphism.

Proof

$$\begin{array}{ccccccc}
I(DC) \otimes I(DC) & \xrightarrow{\phi} & I(DC) & \xrightarrow{\pi} & Q(DC) & \longrightarrow & 0 \\
\downarrow \cong & & & & & & \\
DJ(C) \otimes DJ(C) & & \downarrow \cong & & \downarrow & & \\
\downarrow \otimes & & & & & & \\
D(J(C) \otimes J(C)) & \xrightarrow{D\psi} & DJ(C) & \xrightarrow{D\iota} & DP(C) & \longrightarrow & 0
\end{array}$$

$\square$

Dual of module is often a comodule

Lemma

Let M be a left A-module, with A and M both bounded below (or both bounded above, or A bounded above and below), and with A finitely generated projective over R in each degree.

Then DM with the left coaction

$$\nu \colon DM \xrightarrow{D\lambda} D(A \otimes M) \xrightarrow{\otimes^{-1}} DA \otimes DM$$

is a left DA-comodule.

The result for right A-modules is similar.

Dual of comodule is always a module

Lemma

If C is a graded R-coalgebra and M is a left C-comodule, then DM with the left action

$$\lambda\colon DC \otimes DM \xrightarrow{\otimes} D(C \otimes M) \xrightarrow{D\nu} DM$$

is a left DC-module.

The result for right C-comodules is similar.

Dual of tensor over A

Lemma

Let L and M be right and left A-modules, respectively, with L, M and A all bounded below (or all bounded above, or two of them bounded above and below), and with A finitely generated projective over R in each degree.

Then the isomorphism $DL \otimes DM \cong D(L \otimes M)$ restricts to an isomorphism

$$DL \,\square_{DA}\, DM \cong D(L \otimes_A M)\,.$$

Dual of module homomorphism is often a comodule homomorphism

Lemma

Let M and N be left A-modules, with M, N and A all bounded below (or all bounded above, or A bounded above and below), and with A finitely generated projective over R in each degree.

Then $f \mapsto Df$ defines a homomorphism

$$D\colon \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{DA}(DN, DM)\,.$$

If, furthermore, M and N are finitely generated projective over R in each degree, then D is an isomorphism.

Dual of comodule homomorphism is always a module homomorphism

Lemma

If M and N are left C-comodules, then $f \mapsto Df$ defines a homomorphism

$$D\colon \operatorname{Hom}_C(M, N) \longrightarrow \operatorname{Hom}_{DC}(DN, DM)\,.$$

If M, N and C are all bounded below (or all bounded above, or C is bounded above and below), and they are all finitely generated projective over R in each degree, then D is an isomorphism.

Dual of bialgebra

Proposition

Let B be a graded R-bialgebra that is bounded below (or bounded above) and finitely generated projective over R in each degree. Then DB with

product

$$\phi\colon DB \otimes DB \xrightarrow{\otimes} D(B \otimes B) \xrightarrow{D\psi} DB\,,$$

unit

$$\eta\ : R \cong DR \xrightarrow{D\epsilon} DB\,,$$

coproduct

$$\psi\colon DB \xrightarrow{D\phi} D(B \otimes B) \xrightarrow{\otimes^{-1}} DB \otimes DB$$

counit

$$\epsilon\colon DB \xrightarrow{D\eta} DR \cong R$$

is a graded R-bialgebra.

Dual of Hopf algebra

Proposition (cont.)

If B is commutative (resp. cocommutative), then DB is cocommutative (resp. commutative).
If B is a Hopf algebra, then DB is a Hopf algebra with conjugation

$$\chi\colon DB \xrightarrow{D\chi} DB\,.$$

Example: Polynomial ring $B = \mathbb{Z}[\xi]$

- Let $R = \mathbb{Z}$. There is a bicommutative Hopf algebra $B = \mathbb{Z}[\xi]$, with underlying algebra the polynomial ring on one generator ξ in nonzero even degree.
- The product is given by $\phi(\xi^i \otimes \xi^j) = \xi^{i+j}$.
- For degree reasons, the coproduct on ξ can only be $\psi(\xi) = \xi \otimes 1 + 1 \otimes \xi$, which implies that

$$\psi(\xi^k) = \sum_{i+j=k} (i,j)\, \xi^i \otimes \xi^j$$

  by the binomial theorem.
- The conjugation satisfies $\chi(\xi) = -\xi$.
- The coalgebra primitives and algebra indecomposables of B are

$$\mathbb{Z}\{\xi\} \cong P(B) \xrightarrow{\cong} Q(B) \cong \mathbb{Z}\{\xi\}\,,$$

  so B is primitively generated.

Example: Divided power ring $DB = \Gamma(x)$

- The dual Hopf algebra $DB = \Gamma(x)$ has underlying algebra the divided power ring on one generator x in a nonzero even degree.
- Here $\Gamma(x) = \mathbb{Z}\{\gamma_k(x) \mid k \geq 0\}$ with $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_k(x)$ dual to ξ^k.
- The product is given by $\phi(\gamma_i(x) \otimes \gamma_j(x)) = (i,j)\,\gamma_{i+j}(x)$, and the coproduct is given by

$$\psi(\gamma_k(x)) = \sum_{i+j=k} \gamma_i(x) \otimes \gamma_j(x)\,.$$

- The conjugation satisfies $\chi(\gamma_k(x)) = (-1)^k \gamma_k(x)$.

Example: Divided power ring $DB = \Gamma(x)$ (cont.)

- The coalgebra primitives of DB are

$$P(DB) = \mathbb{Z}\{x\}$$

while the algebra indecomposables are

$$Q(DB) \cong \mathbb{Z}\{x\} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}/p\{\gamma_{p^n}(x) \mid n \geq 1\}\,.$$

- This uses the number-theoretic fact that

$$\gcd\{\binom{k}{i} \mid 0 < i < k\} = \begin{cases} p & \text{if } k = p^n \text{ with } n \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

- In other words, $\gamma_k(x)$ is indecomposable if and only if $k = p^n$ is a prime power, and in this case $p\gamma_k(x)$ is decomposable.

Comparison of primitives and indecomposables

The general theory ensures that

$$\mathbb{Z}\{x\} = P(DB) \cong DQ(B) \cong D(\mathbb{Z}\{\xi\})$$

while in this example, the homomorphism

$$\mathbb{Z}\{x\} \oplus \bigoplus_{p,n} \mathbb{Z}/p\{\gamma_{p^n}(x)\} \cong Q(DB) \longrightarrow DP(B) = D(\mathbb{Z}\{\xi\})$$

is not an isomorphism.

Homological realization of polynomial ring

- For $|\xi| = u - 1 \geq 2$, the primitively generated Hopf algebra $B = \mathbb{Z}[\xi]$ is homologically realized by $B \cong H_*(\Omega S^u)$ with $DB \cong H^*(\Omega S^u)$.
- Here ΩS^u is equivalent as an A_∞ space (in particular, as a homotopy associative H-space) to a topological group G.
- The problem of realizing B cohomologically is more subtle, and was discussed earlier in relation to the Hopf invariant.

Outline

Coproduct on A

Theorem ([Mil58, Lem. 1], [Ste62, Thm. II.1.1])

Let $A = \mathscr{A}(2)$ be the mod 2 Steenrod algebra. The assignment

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

extends uniquely to a ring homomorphism

$$\psi\colon A \longrightarrow A \otimes A$$

so that

$$\theta(x \cup y) = \sum \theta'(x) \cup \theta''(y)$$

for each $\theta \in A$, $x, y \in H^*(X; \mathbb{F}_2)$ and $\psi(\theta) = \sum \theta' \otimes \theta'' \in A \otimes A$.

Sketch proof

- Let R be the set of $\theta \in A$ for which there exists an element $\rho \in A \otimes A$ such that

$$\theta\phi = \phi\rho \colon H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \longrightarrow H^*(X;\mathbb{F}_2)$$

  for all spaces X.
- Then R is closed under sum and product in A, and contains the Sq^k, hence is equal to the whole of A.
- To prove uniqueness of ρ, evaluate $\theta\phi$ on $H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2)$ for a space X that faithfully detects the action by A in a large range of degrees.
- If $|\theta| < n$, one can let $X = K(\mathbb{Z}/2, n)$ or $X = K(\mathbb{Z}/2, 1)^n$.
- Letting $\psi(\theta) = \rho$ then defines the ring homomorphism ψ. $\square$

Connected algebra of finite type, I

- The admissible basis

$$\{Sq^I \mid I \text{ admissible}\}$$

  shows that A is concentrated in non-negative cohomological degrees, and is finite-dimensional over $\mathbb{F}_2$ in each degree.
- Moreover, $\mathbb{F}_2\{1\}$ equals the degree 0 part of A, so we say that A is a connected algebra.
- This implies that there is a unique augmentation $\epsilon \colon A \to \mathbb{F}_2$.

Hopf algebra structure on A

Theorem ([Mil58, Thm. 1], [Ste62, Thm. II.1.2])

The Steenrod algebra A, with the coproduct $\psi \colon A \to A \otimes A$ and the augmentation $\epsilon \colon A \to \mathbb{F}_2$, is a cocommutative Hopf algebra over $\mathbb{F}_2$.

Proof.

The known formula for $\psi(Sq^k)$ implies that ψ is coassociative and counital. The existence of the conjugation χ follows from the fact that A is connected [MM65, Def. 8.4]. It satisfies

$$\sum_{i+j=k} Sq^i \chi(Sq^j) = 0$$

for $k \geq 1$. □

The dual Steenrod algebra A_*

Definition

Let the (mod 2) dual Steenrod algebra $A_* = DA = \mathrm{Hom}(A, \mathbb{F}_2)$ be the function dual of the mod 2 Steenrod algebra.

Corollary ([Mil58, Cor. 1])

The dual Steenrod algebra A_* is a commutative Hopf algebra over $\mathbb{F}_2$.

Connected algebra of finite type, II

- The finite type results for A imply that A_* is concentrated in non-negative homological degrees, and is finite-dimensional over $\mathbb{F}_2$ in each degree.
- Hence $DA_* \cong A$.
- Moreover, $\mathbb{F}_2\{1\}$ equals the degree 0 part of A_*, so A_* is connected.

Four out of eight (co-)actions

- Milnor determined the structure of A_* as an algebra, with product dual to the coproduct $\psi\colon A \to A \otimes A$, as well as its coproduct, dual to the product $\phi\colon A \otimes A \to A$.
- Let X be any space. For brevity we set $H_*(X) = H_*(X; \mathbb{F}_2)$ and $H^*(X) = H^*(X; \mathbb{F}_2)$.
- There are natural left and right A-module and A^*-comodule structures on $H_*(X)$ and $H^*(X)$, for a total of eight combinations, as explained by Boardman in his paper [Boa82].
- Four of these were discussed by Milnor in [Mil58], and we review these below. The remaining four are then obtained by use of the conjugation $\chi\colon A \to A$, or its dual.

Left A-action on cohomology

First, the cup product

$$\cup\colon H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

and the Steenrod operations

$$\lambda\colon A \otimes H^*(X) \longrightarrow H^*(X)$$

naturally give the cohomology $H^*(X)$ the structure of a (commutative) left A-module algebra.

Diagrams, I

This means that the diagrams

$$\begin{array}{ccc} A \otimes A \otimes H^*(X) & \xrightarrow{1\otimes\lambda} & A \otimes H^*(X) \\ \downarrow{\scriptstyle \phi\otimes 1} & & \downarrow{\scriptstyle \lambda} \\ A \otimes H^*(X) & \xrightarrow{\lambda} & H^*(X) \end{array}$$

and

$$\begin{array}{ccc} A \otimes H^*(X) \otimes H^*(X) & \xrightarrow{1\otimes\cup} & A \otimes H^*(X) \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \lambda} \\ A \otimes A \otimes H^*(X) \otimes H^*(X) & & H^*(X) \\ \downarrow{\scriptstyle 1\otimes\tau\otimes 1} & & \uparrow{\scriptstyle \cup} \\ A \otimes H^*(X) \otimes A \otimes H^*(X) & \xrightarrow{\lambda\otimes\lambda} & H^*(X) \otimes H^*(X) \end{array}$$

commute, together with unitality conditions.

Left A_*-coaction on homology

Second, applying $\mathrm{Hom}(-, \mathbb{F}_2)$ to the left A-module action λ defines a homomorphism

$$\mathrm{Hom}(\lambda, 1)\colon\ \mathrm{Hom}(H^*(X), \mathbb{F}_2) \longrightarrow \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_2)\,.$$

When $H_*(X)$ has finite type, there are natural isomorphisms

$$H_*(X) \xrightarrow{\cong} \mathrm{Hom}(H^*(X), \mathbb{F}_2)$$
$$A_* \otimes H_*(X) \xrightarrow{\cong} \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_2)$$

and the composite

$$H_*(X) \cong \mathrm{Hom}(H^*(X), \mathbb{F}_2) \longrightarrow \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_2) \cong A_* \otimes H_*(X)$$

defines a natural left A_*-coaction

$$\nu\colon\ H_*(X) \longrightarrow A_* \otimes H_*(X)\,.$$

General spaces

- Using CW approximation and commutation of homology with strongly filtered colimits, one can show that the coaction ν is well-defined and natural for all spaces X, not just those with mod 2 homology of finite type.
- The cup product is dual to the homomorphism

$$\Delta_* \colon H_*(X) \longrightarrow H_*(X \times X) \cong H_*(X) \otimes H_*(X)$$

  induced by the diagonal map $\Delta \colon X \to X \times X$.
- The homology $H_*(X)$ is naturally a (cocommutative) left A_*-comodule coalgebra.

Diagrams, II

It follows that the diagrams

$$
\begin{array}{ccc}
H_*(X) & \xrightarrow{\nu} & A_* \otimes H_*(X) \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \psi\otimes 1} \\
A_* \otimes H_*(X) & \xrightarrow{1\otimes\nu} & A_* \otimes A_* \otimes H_*(X)
\end{array}
$$

and

$$
\begin{array}{ccc}
H_*(X) \otimes H_*(X) & \xrightarrow{\nu\otimes\nu} & A_* \otimes H_*(X) \otimes A_* \otimes H_*(X) \\
\uparrow{\scriptstyle \Delta_*} & & \downarrow{\scriptstyle 1\otimes\tau\otimes 1} \\
H_*(X) & & A_* \otimes A_* \otimes H_*(X) \otimes H_*(X) \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \phi\otimes 1\otimes 1} \\
A_* \otimes H_*(X) & \xrightarrow{1\otimes\Delta_*} & A_* \otimes H_*(X) \otimes H_*(X)
\end{array}
$$

commute.

Right A-action on homology

Third, we can give $H_*(X)$ the structure of a right A-module, with action

$$\rho\colon H_*(X) \otimes A \longrightarrow H_*(X)$$

taking $\xi \in H_n(X)$ and $\theta \in A^k$ to $\rho(\xi \otimes \theta) = \xi \cdot \theta \in H_{n-k}(X)$. Here $\xi \cdot \theta$ is characterized by the condition

$$\langle \theta \cdot x, \xi \rangle = \langle x, \xi \cdot \theta \rangle$$

for each $x \in H^*(X)$, where $\theta \cdot x = \lambda(\theta \otimes x) = \theta(x)$. In other words,

$$\theta \cdot : H^*(X) \longrightarrow H^*(X)$$
$$x \mapsto \theta \cdot x$$

corresponds to the dual of the homomorphism

$$\cdot \theta\colon H_*(X) \longrightarrow H_*(X)$$
$$\xi \longmapsto \xi \cdot \theta$$

under the identification $H^*(X) \cong \mathrm{Hom}(H_*(X), \mathbb{F}_2)$.

Sq_*^I-notation

- It is traditional to write

$$Sq_*^I(\xi) = \xi \cdot Sq^I$$

  for this right action.
- Beware that this means that

$$Sq_*^J Sq_*^I = Sq_*^{IJ}$$

  where *IJ* denotes the concatenation of *I* and *J*.
- The homology $H_*(X)$ is a (cocommutative) right *A*-module coalgebra.

Diagrams, III

Direct calculation shows that the diagrams

$$
\begin{CD}
H_*(X) \otimes A \otimes A @>{\rho \otimes 1}>> H_*(X) \otimes A \\
@V{1 \otimes \phi}VV @VV{\rho}V \\
H_*(X) \otimes A @>{\rho}>> H_*(X)
\end{CD}
$$

and

$$
\begin{CD}
H_*(X) \otimes A @>{\rho}>> H_*(X) \\
@V{\Delta_* \otimes 1}VV @VV{\Delta_*}V \\
H_*(X) \otimes H_*(X) \otimes A @. H_*(X) \otimes H_*(X) \\
@V{1 \otimes 1 \otimes \psi}VV @AA{\rho \otimes \rho}A \\
H_*(X) \otimes H_*(X) \otimes A \otimes A @>{1 \otimes \tau \otimes 1}>> H_*(X) \otimes A \otimes H_*(X) \otimes A
\end{CD}
$$

commute.

Right A_*-coaction on cohomology

Fourth, applying $\mathrm{Hom}(-, \mathbb{F}_2)$ to the right A-module action ρ defines a homomorphism

$$\mathrm{Hom}(\rho, 1) \colon \mathrm{Hom}(H_*(X), \mathbb{F}_2) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_2) \,.$$

The natural homomorphism

$$H^*(X) \otimes A_* \cong \mathrm{Hom}(H_*(X), \mathbb{F}_2) \otimes \mathrm{Hom}(A, \mathbb{F}_2) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_2)$$

is an isomorphism if $H^*(X)$ is bounded above, in which case the composite

$$H^*(X) \cong \mathrm{Hom}(H_*(X), \mathbb{F}_2) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_2) \cong H^*(X) \otimes A_*$$

defines a natural right A_*-coaction

$$\lambda^* \colon H^*(X) \longrightarrow H^*(X) \otimes A_* \,.$$

(The notation λ^* is the one used by Milnor in [Mil58, §4].)

Completed coaction

- In general, there is an isomorphism

$$\mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_2) \cong H^*(X) \mathbin{\widehat{\otimes}} A_* ,$$

  where the right hand side denotes the completed tensor product with

$$\prod_n H^{n+k}(X) \otimes A_n$$

  in cohomological degree k.
- We then have a completed right A_*-coaction

$$\lambda^* \colon H^*(X) \longrightarrow H^*(X) \mathbin{\widehat{\otimes}} A_*$$

  and this is an algebra homomorphism.
- The cohomology $H^*(X)$ is a (commutative) completed right A_*-comodule algebra.

Diagrams, IV

The diagrams

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\lambda^*} & H^*(X)\,\widehat{\otimes}\,A_* \\ {\scriptstyle\lambda^*}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes\psi} \\ H^*(X)\,\widehat{\otimes}\,A_* & \xrightarrow{\lambda^*\otimes 1} & H^*(X)\,\widehat{\otimes}\,A_*\,\widehat{\otimes}\,A_* \end{array}$$

and

$$\begin{array}{ccc} H^*(X)\,\widehat{\otimes}\,A_*\,\widehat{\otimes}\,H^*(X)\,\widehat{\otimes}\,A_* & \xrightarrow{1\otimes\tau\otimes 1} & H^*(X)\,\widehat{\otimes}\,H^*(X)\,\widehat{\otimes}\,A_*\,\widehat{\otimes}\,A_* \\ {\scriptstyle\lambda^*\otimes\lambda^*}\big\uparrow & & \big\downarrow{\scriptstyle 1\otimes 1\otimes\phi} \\ H^*(X)\,\widehat{\otimes}\,H^*(X) & & H^*(X)\,\widehat{\otimes}\,H^*(X)\,\widehat{\otimes}\,A_* \\ {\scriptstyle\cup}\big\downarrow & & \big\downarrow{\scriptstyle\cup\otimes 1} \\ H^*(X) & \xrightarrow{\lambda^*} & H^*(X)\,\widehat{\otimes}\,A_* \end{array}$$

commute.

(Co-)homology of $\mathbb{R}P^\infty$

- Recall the admissible sequences

$$M_i = (2^{i-1}, \ldots, 4, 2, 1)$$

  for $i \geq 1$.
- We set $M_0 = ()$.
- Recall also that $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$ and

$$H^*(\mathbb{R}P^\infty) \cong \mathbb{F}_2[a]\,,$$

  with a in degree 1 corresponding to the universal class u_1 in mod 2 cohomology.
- We let $\alpha_j \in H_j(\mathbb{R}P^\infty)$ be dual to a^j, so that $H_*(\mathbb{R}P^\infty) \cong \mathbb{F}_2\{\alpha_j \mid j \geq 0\}$.

The left A-action on $H^*(\mathbb{R}P^\infty)$

Lemma

$$Sq^I(a) = \begin{cases} a^{2^i} & \text{if } I = M_i,\ i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for I admissible.

Proof.

This follows by induction on the length of I, using the formula

$$Sq^k(a^{2^i}) = \binom{2^i}{k} a^{k+2^i} = \begin{cases} a^{2^i} & \text{for } k = 0, \\ a^{2^{i+1}} & \text{for } k = 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

□

The Milnor generators ξ_i

Definition

For $i \geq 1$ let the Milnor generator

$$\xi_i \in A_{2^i-1}$$

be characterized by

$$\langle Sq^I, \xi_i \rangle = \begin{cases} 1 & \text{for } I = M_i, \\ 0 & \text{otherwise,} \end{cases}$$

for each admissible I of degree $2^i - 1$. Furthermore, let $\xi_0 = 1$.

Remark

Milnor actually writes ζ_i for this class in A_{2^i-1}. Other authors instead write ζ_i for the conjugate $\chi(\xi_i)$ of this class, which can be confusing. Another notation for the conjugate is $\bar{\xi}_i$.

Alternative characterization of ξ_i

Lemma

The homomorphism

$$\tilde{H}_j(\mathbb{R}P^\infty) \longrightarrow \operatorname*{colim}_n \tilde{H}_{j-1+n}(K(\mathbb{Z}/2, n)) \cong A_{j-1}$$

with Hom*-dual*

$$A^{j-1} \cong \lim_n \tilde{H}^{j-1+n}(K(\mathbb{Z}/2, n)) \longrightarrow \tilde{H}^j(\mathbb{R}P^\infty)$$

is given by

$$\alpha_j \longmapsto \begin{cases} \xi_i & \text{for } j = 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof

- The homomorphism
$$\begin{aligned} A^{j-1} &\longrightarrow \tilde{H}^j(\mathbb{R}P^\infty) \\ \theta &\longmapsto \theta(a) \end{aligned}$$
maps Sq^{M_i} to a^j for $i \geq 0$ and $j = 2^i$ and sends the remaining admissible Sq^I to zero.
- Hence the dual homomorphism $\tilde{H}_j(\mathbb{R}P^\infty) \to A_{j-1}$ maps α_j to ξ_i for $j = 2^i$ with $i \geq 0$, and to zero for the remaining j. □

A_* is a polynomial $\mathbb{F}_2$-algebra

Since A is cocommutative, A_* is a commutative $\mathbb{F}_2$-algebra.

Theorem ([Mil58, Thm. 2, App. 1])

There is an algebra isomorphism

$$A_* \cong \mathbb{F}_2[\xi_i \mid i \geq 1]\,,$$

with $|\xi_i| = 2^i - 1$*.*

Sketch proof

- The monomials
$$\xi^R = \xi_1^{r_1}\xi_2^{r_2}\cdot\dots\cdot\xi_\ell^{r_\ell}$$
where $R = (r_1, r_2, \dots, r_\ell, 0, \dots)$ ranges over all finite length sequences of non-negative integers, form a basis for $\mathbb{F}_2[\xi_i \mid i \geq 1]$, which maps to A_*.
- Milnor checks [Mil58, Lem. 8] that in each degree n, a matrix with entries
$$\langle Sq^I, \xi^R\rangle \in \mathbb{F}_2$$
is lower triangular with no zeros on the diagonal, hence is invertible, where I ranges over the admissible sequences of degree n and R ranges over the sequences of degree $\sum_i (2^i - 1)r_i$ equal to n.
- Since these Sq^I form a basis for A^n, it follows that these monomials ξ^R form a basis for A_n. □

The right A_*-coaction on $H^*(\mathbb{R}P^\infty)$

The algebra homomorphism

$$\lambda^*\colon H^*(\mathbb{R}P^\infty) \longrightarrow H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} A_*$$

is determined by its value on $a \in H^1(\mathbb{R}P^\infty)$.

Proposition

$$\lambda^*(a) = \sum_{i \geq 0} a^{2^i} \otimes \xi_i$$

in $H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} A_*$*.*

Proof

- The right A-module action

$$H_j(\mathbb{R}P^\infty) \otimes A^{j-1} \longrightarrow H_1(\mathbb{R}P^\infty)$$

  is zero unless $j = 2^i$, in which case

$$\rho(\alpha_{2^i} \otimes Sq^I) = \begin{cases} \alpha_1 & \text{if } I = M_i, \\ 0 & \text{otherwise,} \end{cases}$$

  for I admissible of degree $2^i - 1$.
- Dually, the right A^*-coaction

$$H^1(\mathbb{R}P^\infty) \longrightarrow H^j(\mathbb{R}P^\infty) \otimes A_{j-1}$$

  is zero unless $j = 2^i$, in which case it maps a to $a^{2^i} \otimes \xi_i$.
- Collecting terms for all j, we obtain the stated formula for $\lambda^*(a)$. □

The coproduct in A_*

Since A is non-commutative, A_* is not cocommutative. The coproduct for A_* encodes much the same information as the Adem relations do for A, but the following formula is often easier to work with for theoretical purposes.

Theorem ([Mil58, Thm. 3, App. 1])

The coproduct $\psi\colon A_* \to A_* \otimes A_*$ is given by

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j \,,$$

where $\xi_0 = 1$.

Proof

- The multiplicative right A_*-coaction λ^* satisfies

$$\lambda^*(a^{2^j}) = \lambda^*(a)^{2^j} = \Big(\sum_{i\geq 0} a^{2^i} \otimes \xi_i\Big)^{2^j} = \sum_{i\geq 0} a^{2^{i+j}} \otimes \xi_i^{2^j} .$$

- It is also coassociative, so that

$$(\lambda^* \otimes 1)(\lambda^*(a)) = (\lambda^* \otimes 1)(\sum_{j\geq 0} a^{2^j} \otimes \xi_j)$$
$$= \sum_{j\geq 0} \lambda^*(a^{2^j}) \otimes \xi_j = \sum_{i\geq 0} \sum_{j\geq 0} a^{2^{i+j}} \otimes \xi_i^{2^j} \otimes \xi_j$$

is equal to

$$(1 \otimes \psi)(\lambda^*(a)) = (1 \otimes \psi)(\sum_{k\geq 0} a^{2^k} \otimes \xi_k) = \sum_{k\geq 0} a^{2^k} \otimes \psi(\xi_k)$$

as an element in $H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} A_* \mathbin{\widehat{\otimes}} A_*$.

- Comparing coefficients of a^{2^k} gives the stated formula for $\psi(\xi_k)$, for each $k \geq 0$. □

Indecomposables and primitives

- The indecomposable quotient $Q(A) = \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$ is dual to the primitives

$$P(A_*) = \mathbb{F}_2\{\xi_1^{2^i} \mid i \geq 0\}\,.$$

- Furthermore, the indecomposable quotient $Q(A_*) = \mathbb{F}_2\{\xi_i \mid i \geq 1\}$ is dual to the primitives

$$P(A) = \mathbb{F}_2\{Q_j \mid j \geq 0\}\,,$$

  with Q_j in degree $2^{j+1} - 1$ dual to ξ_{j+1}.
- Here the Milnor primitives are $Q_0 = Sq^1$ and

$$Q_j = [Sq^{2^j}, Q_{j-1}] = Sq^{2^j} Q_{j-1} + Q_{j-1} Sq^{2^j}$$

  for $j \geq 1$.

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