MAT9580: Spectral Sequences

Chapter 9: Stable Homotopy Theory

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Outline

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Stable Homotopy Theory

- Smooth bordism and stable homotopy groups
- Sequential spectra
- Triangulated structure
- Spectral homology and cohomology
- Orthogonal spaces
- Orthogonal spectra
- Closed symmetric monoidal structure
- Monoidal model structure
- Multiplicative (co-)homology theories

Smooth bordism and stable homotopy groups

Lev Pontryagin [Pon50] (and earlier?) and René Thom [Tho54] developed the close connection between

- the bordism classification of manifolds and
- the stable range homotopy groups of certain spaces.

See also Milnor–Stasheff [MS74], Stong [Sto68] and Rudyak [Rud98].

Transversality

- We can view the k-sphere S^k as the one-point compactification $\mathbb{R}^k \cup \{\infty\}$, based at infinity, or as the quotient space $D^k/\partial D^k$, based at the image of the boundary.
- Any map $f\colon S^{n+k} \to S^k$ is homotopic to a smooth map with $0 \in \mathbb{R}^k \subset S^k$ as a regular value, i.e., a map that is transverse to 0, and the preimage $M = f^{-1}(0)$ is then a closed smooth n-dimensional submanifold of $\mathbb{R}^{n+k} \subset S^{n+k}$.
- The stabilization $f \wedge S^1 = f \wedge 1\colon S^{n+k} \wedge S^1 \to S^k \wedge S^1$ then has the same preimage

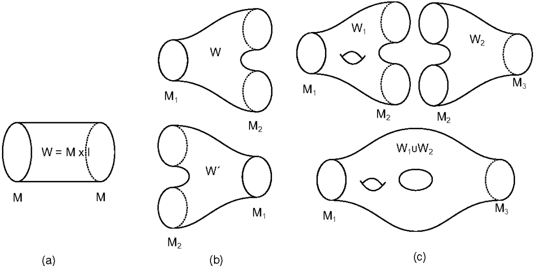
$$(f \wedge 1)^{-1}(0) \cong f^{-1}(0)\,,$$

but is now realized as a submanifold of $\mathbb{R}^{n+k+1}$.

Cobordism

- If $F \colon I_+ \wedge S^{n+k} \to S^k$ is a homotopy from f_0 to f_1, with both f_0 and f_1 transverse to 0, then F can be deformed relative to $\partial I_+ \wedge S^{n+k}$ to a smooth map that is transverse to 0.
- The preimage $W = F^{-1}(0)$ is then a compact smooth $(n+1)$-dimensional submanifold of $I \times \mathbb{R}^{n+k} \subset I_+ \wedge S^{n+k}$, with boundary
$$\partial W \cong M_0 \coprod M_1 \,.$$
- We call W a bordism from $M_0 = f_0^{-1}(0)$ to $M_1 = f_1^{-1}(0)$, and say that M_0 and M_1 are cobordant.

Bordism classes



- This defines an equivalence relation, and we write $[M]$ for the bordism class of M.
- The set of all bordism classes of closed (always smooth) n-manifolds is denoted $\mathscr{N}_n$.

Ring structures

Lemma

The rule $[f] \mapsto [M]$ *with* $M = f^{-1}(0)$ *defines a homomorphism of graded (commutative) rings*

$$\pi_*(S) \longrightarrow \mathscr{N}_* \,.$$

Proof.

We have seen that $f \mapsto f^{-1}(0)$ defines a function $\pi_{n+k}(S^k) \to \mathscr{N}_n$ that is compatible with stabilization, hence factors uniquely through the stable homotopy group

$$\pi_n(S) = \operatorname*{colim}_k \pi_{n+k}(S^k) \,.$$

Proof (cont.)

- The disjoint union of manifolds defines a sum

$$+\colon \mathscr{N}_n \times \mathscr{N}_n \longrightarrow \mathscr{N}_n$$

and the Cartesian product of manifolds defines a product

$$\cdot\colon \mathscr{N}_n \times \mathscr{N}_m \longrightarrow \mathscr{N}_{n+m}\,,$$

making $\mathscr{N}_* = (\mathscr{N}_n)_n$ a graded commutative $\mathbb{F}_2$-algebra, known as the unoriented bordism ring.

Proof (cont.)

- The sum in $\pi_{n+k}(S^k)$ takes $[f]$ and $[g]$ to the class of

$$f + g\colon S^{n+k} \longrightarrow S^{n+k} \vee S^{n+k} \xrightarrow{f \vee g} S^k ,$$

so that $(f+g)^{-1}(0) \cong f^{-1}(0) \coprod g^{-1}(0)$.

- The smash product of $f\colon S^{n+k} \to S^k$ and $g\colon S^{m+\ell} \to S^\ell$ defines a map

$$S^{n+m+k+\ell} \cong S^{n+k} \wedge S^{m+\ell} \xrightarrow{f \wedge g} S^k \wedge S^\ell = S^{k+\ell} ,$$

with $(f \wedge g)^{-1}(0) \cong f^{-1}(0) \times g^{-1}(0)$.

- It follows that the sum and product in $\pi_*(S)$ are mapped to the sum and product in $\mathscr{N}_*$. □

Refinements

To be useful, this ring homomorphism must be refined, by either

- restricting the manifolds $M \subset \mathbb{R}^{n+k}$ studied to account for special structure on their normal bundles, which arises from their construction as transverse preimages, or
- by extending the targets of the maps $f\colon S^{n+k} \to S^k$ to allow for more general normal bundles,

or both.

Normal framing

- A smooth embedding $M \subset \mathbb{R}^{n+k}$ induces an embedding of the tangent bundle $\tau \colon TM \to M$ into the trivial bundle $\epsilon^{n+k} \colon M \times \mathbb{R}^{n+k} \to M$, with normal complement the normal bundle $\nu \colon NM \to M$.
- For each $x \in M$, the fiber $N_x M \subset \mathbb{R}^{n+k}$ is the orthogonal complement of $T_x M \subset \mathbb{R}^{n+k}$.
- If $M = f^{-1}(0)$ is the preimage of the regular value $0 \in \mathbb{R}^k \subset S^k$, then the derivative $f_* \colon T\mathbb{R}^{n+k}|_M \to T\mathbb{R}^k|_0$ of f along M induces a bundle isomorphism

$$\theta \colon NM \xrightarrow{\cong} M \times \mathbb{R}^k .$$

- This is a trivialization, or framing, of the normal bundle of M.

Stable normal framing

- If we replace f with $f \wedge 1$, then the normal bundle of $M \subset \mathbb{R}^{n+k+1}$ is $\nu \oplus \epsilon^1 \colon NM \times \mathbb{R} \to M$, with trivialization

$$\theta \times \mathbb{R} \colon NM \times \mathbb{R} \cong M \times \mathbb{R}^{k+1} .$$

- We say that θ and $\theta \times \mathbb{R}$ define the same stable framing, and that (M, θ) is stably framed.

Framed cobordism

- If $F\colon I_+ \wedge S^{n+k} \to S^k$ is a smooth homotopy from f_0 to f_1, all of which are transverse to 0, then the derivative F_* of F along the compact $(n+1)$-manifold $W = F^{-1}(0) \subset I \times \mathbb{R}^{n+k}$ induces a trivialization

$$\Theta\colon NW \xrightarrow{\cong} W \times \mathbb{R}^k .$$

- It restricts to the trivializations θ_0 and θ_1 of the normal bundles of $M_0 = f_0^{-1}(0) \subset \mathbb{R}^{n+k}$ and $M_1 = f_1^{-1}(0) \subset \mathbb{R}^{n+k}$, respectively.
- We say that M_0 and M_1 are stably framed cobordant.
- This defines an equivalence relation, and we write Ω_n^{fr} for the set of all stably framed bordism classes of stably framed closed n-manifolds.

A theorem of Pontryagin

Theorem ([Pon50])

The rule $[f] \mapsto [(M, \theta)]$ *with* $M = f^{-1}(0)$ *and* $\theta \colon NM \cong M \times \mathbb{R}^k$ *defines an isomorphism of graded commutative rings*

$$\pi_*(S) \xrightarrow{\cong} \Omega^{fr}_* .$$

Sketch proof

- To construct the inverse, consider a stably framed, closed n-manifold M.
- There exists an embedding $M \subset \mathbb{R}^{n+k}$, with a trivialization $\theta\colon NM \cong M \times \mathbb{R}^k$, and any two such become isotopic if we enlarge k.
- Choosing a Euclidean metric, we get a homeomorphism

$$D(\theta)\colon D(NM) \xrightarrow{\cong} M \times D^k$$

  of unit disc bundles over M.
- Let $S(\theta)\colon S(NM) \cong M \times \partial D^k$ denote its restriction to the unit sphere bundles.
- We can view M as a subspace of $D(NM)$ by the zero section.

Sketch proof (cont.)

- By the tubular neighborhood theorem there is an embedding $D(NM) \subset \mathbb{R}^{n+k}$ that extends the inclusion $M \subset \mathbb{R}^{n+k}$, such that the open disc bundle $D(NM) - S(NM) = \operatorname{int} D(NM) \subset \mathbb{R}^{n+k}$ is an open neighborhood of M.
- We can then form the composite map

$$f\colon S^{n+k} \to \frac{S^{n+k}}{S^{n+k} - \operatorname{int} D(NM)} \xleftarrow{\cong} \frac{D(NM)}{S(NM)} \xrightarrow{\cong} \frac{M \times D^k}{M \times \partial D^k} \to \frac{D^k}{\partial D^k} \cong S^k .$$

- It has $0 \in D^k \to S^k$ as a regular value, with preimage $f^{-1}(0) \cong M \times \{0\} \cong M$, which is normally framed by θ.
- Hence the stable class of $[f] \in \pi_{n+k}(S^k)$ in $\pi_n(S)$ maps to the stably framed bordism class of (M, θ), and these are mutually inverse correspondences. □

Geometric calculation of $\pi_2(S)$

- Pontryagin used this construction, and the classification of stably framed closed surfaces, to prove that $\pi_2(S) \cong \mathbb{Z}/2$, generated by the stable class η^2 of the composite

$$\eta \circ E\eta \colon S^4 \longrightarrow S^3 \longrightarrow S^2 .$$

- This $\mathbb{Z}/2$ detects the Arf invariant of a quadratic form that refines the bilinear intersection form on $H_1(-;\mathbb{F}_2)$ of the framed surface.
- In particular, not every framed closed surface is framed cobordant to a sphere.
- Pontryagin thereby rectified an earlier mistake he had made (in 1938) concerning this problem.

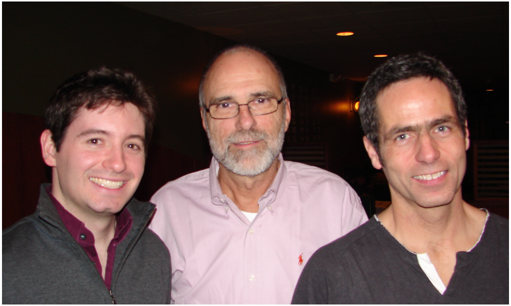
Surgery

- Similar work shows that the stable homotopy classes $\nu^2 \in \pi_6(S)$ and $\sigma^2 \in \pi_{14}(S)$, where ν and σ are the stable classes of the Hopf fibrations $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$, correspond to 6- and 14-dimensional framed manifolds, respectively, that are not framed cobordant to homotopy spheres.
- Work by Kervaire–Milnor [KM63] addressed the question whether each framed n-manifold can be modified, by a process now called "surgery", so as to be framed cobordant to a homotopy sphere.
- This is can always be done unless $n = 4m - 2$, in which case there is a possible obstruction in $\mathbb{Z}/2$, known as the Kervaire invariant of the framed bordism class, given by the Arf invariant of a quadratic form on the middle homology $H_{2m-1}(-;\mathbb{F}_2)$ of the manifold.

The Kervaire invariant one problem

- Browder [Bro69] showed that the Kervaire invariant vanishes for each n not of the form $2(2^j - 1)$.
- The Kervaire invariant one problem then asks: For which $n = 2(2^j - 1)$ does there exist a class $\theta_j \in \pi_n(S) \cong \Omega_n^{fr}$ with nontrivial Arf–Kervaire invariant?
- The squared Hopf fibration examples show that such classes exists for $j \in \{1, 2, 3\}$.
- Mahowald–Tangora [MT67] showed that $\theta_4 \in \pi_{30}(S)$ exists, and Barratt–Jones–Mahowald [BJM84] proved that $\theta_5 \in \pi_{62}(S)$ exists, by hard calculations with the mod 2 Adams spectral sequence for the sphere spectrum.
- The next problem, concerning the existence of $\theta_6 \in \pi_{126}(S)$ lies outside our current computational range.

Hill–Hopkins–Ravenel



- It was a great surprise when Hopkins–Hill–Ravenel [HHR16] proved, using an equivariant form of complex bordism, that θ_j does not exist for any $j \geq 7$.
- The case $j = 6$ remains open.

The Gauss map

- For a general smooth embedding $M \subset \mathbb{R}^{n+k}$ there need not exist
- a (stable) trivialization θ of the normal bundle $\nu \colon NM \to M$. However, there exists a Gauss map

$$g \colon M \longrightarrow Gr_k(\mathbb{R}^{n+k}) \subset Gr_k(\mathbb{R}^\infty) \simeq BO(k)$$

  to the Grassmann manifold of k-dimensional real subspaces of $\mathbb{R}^{n+k}$, given by $g(x) = N_x M \subset \mathbb{R}^{n+k}$ for all $x \in M$.
- By including $\mathbb{R}^{n+k}$ in $\mathbb{R}^\infty \oplus \mathbb{R}^k \cong \mathbb{R}^\infty$ we can continue this map to the Grassmannian of k-dimensional subspaces of $\mathbb{R}^\infty$, which is a classifying space for principal $O(k)$-bundles.

The universal case

- The universal principal $O(k)$-bundle

$$O(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow Gr_k(\mathbb{R}^\infty)\,,$$

  where $V_k(\mathbb{R}^\infty)$ is the contractible Stiefel space of orthonormal k-frames in $\mathbb{R}^\infty$, has an associated "tautological" $\mathbb{R}^k$-bundle $\gamma^k \colon E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$, whose fiber over $V \in Gr_k(\mathbb{R}^\infty)$ is the k-dimensional vector space $V \subset \mathbb{R}^\infty$.
- The identity maps on the $N_x M$, for $x \in M$, define a bundle map

$$\begin{array}{ccc} NM & \xrightarrow{\hat{g}} & E(\gamma^k) \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & Gr_k(\mathbb{R}^\infty) \end{array}$$

  covering the Gauss map.
- Equivalently, there is an isomorphism $\nu \cong g^*(\gamma^k)$, expressing the normal bundle of M as the pullback along g of the tautological bundle over $Gr_k(\mathbb{R}^\infty)$.

Thom complexes

Definition

For a Euclidean vector bundle $\xi\colon E(\xi) \to B$, with unit disc bundle $D(\xi) \to B$ and unit sphere bundle $S(\xi) \to B$, let the Thom complex be the quotient space

$$Th(\xi) = D(\xi)/S(\xi)\,.$$

In particular, let $Th(\gamma^k)$ denote the Thom complex of the tautological $\mathbb{R}^k$-bundle $\gamma^k\colon E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$.

- If ξ is associated to the principal $O(k)$-bundle $p\colon P \to B$, then
$$E(\xi) \cong P \times_{O(k)} \mathbb{R}^k\,,$$
so that
$$Th(\xi) \cong \frac{P \times_{O(k)} D^k}{P \times_{O(k)} \partial D^k} \cong P_+ \wedge_{O(k)} D^k/\partial D^k \cong P_+ \wedge_{O(k)} S^k\,.$$
- In particular, $Th(\gamma^k) \simeq MO(k) = EO(k)_+ \wedge_{O(k)} S^k$.

One-point compactification

- If B is a compact Hausdorff space, then

$$Th(\xi) \cong E(\xi) \cup \{\infty\}$$

  can be characterized as the one-point compactification of the total space $E(\xi)$.
- In general, $Th(\xi)$ is the quotient of the fiberwise one-point compactification

$$P \times_{O(k)} S^k$$

  of $E(\xi)$ by the section $P \times_{O(k)} \{\infty\} \cong B$ at infinity.

Functoriality and stabilization

Lemma

The Thom complex is functorial, and there is a natural homeomorphism $Th(\xi \oplus \epsilon^1) \cong Th(\xi) \wedge S^1$.

Proof.

- A bundle map $\xi \to \eta$ induces maps $D(\xi) \to D(\eta)$, $S(\xi) \to S(\eta)$ and $Th(\xi) \to Th(\eta)$, so the Thom complex is functorial.
- The Whitney sum bundle $\xi \oplus \epsilon^1$ has total space $E(\xi \oplus \epsilon^1) \cong E(\xi) \times \mathbb{R}$, so $D(\xi \oplus \epsilon^1) \cong D(\xi) \times D^1$ and $S(\xi \oplus \epsilon^1) \cong S(\xi) \times D^1 \cup D(\xi) \times \partial D^1$.
- Hence

$$Th(\xi \oplus \epsilon^1) \cong \frac{D(\xi) \times D^1}{S(\xi) \times D^1 \cup D(\xi) \times \partial D^1} \cong Th(\xi) \wedge S^1 .$$

□

The Pontryagin–Thom construction

- Returning to the context of the normal bundle $NM \to M$ and the Gauss map $g\colon M \to Gr_k(\mathbb{R}^\infty)$, we can now use the bundle map $\hat{g}\colon NM \to E(\gamma^k)$ to form the Pontryagin–Thom construction

$$f\colon S^{n+k} \to \frac{S^{n+k}}{S^{n+k} - \operatorname{int} D(NM)} \overset{\cong}{\leftarrow} \frac{D(NM)}{S(NM)} = Th(\nu) \overset{\hat{g}}{\to} Th(\gamma^k) \simeq MO(k)\,,$$

representing a homotopy class

$$[f] \in \pi_{n+k}(Th(\gamma^k)) \cong \pi_{n+k}(MO(k))\,.$$

- In general, two embeddings $M \to \mathbb{R}^{n+k}$ and $M \to \mathbb{R}^{n+\ell}$ become isotopic if we increase k and ℓ to a sufficiently large common value, and isotopic embeddings induce homotopic Pontryagin–Thom maps f.

The structure map σ

- Replacing $M \subset \mathbb{R}^{n+k}$ with $M \subset \mathbb{R}^{n+k+1}$ has the effect of replacing $f \colon S^{n+k} \to Th(\gamma^k)$ with the composite

$$S^{n+k+1} \cong S^{n+k} \wedge S^1 \xrightarrow{f \wedge 1} Th(\gamma^k) \wedge S^1 \xrightarrow{\sigma} Th(\gamma^{k+1}) \,.$$

- Here

$$\sigma \colon Th(\gamma^k) \wedge S^1 \cong Th(\gamma^k \oplus \epsilon^1) \longrightarrow Th(\gamma^{k+1})$$

is the map of Thom complexes induced by the bundle map

$$\begin{array}{ccc} E(\gamma^k \oplus \epsilon^1) & \longrightarrow & E(\gamma^{k+1}) \\ \downarrow & & \downarrow \\ Gr_k(\mathbb{R}^\infty) & \rightarrowtail & Gr_{k+1}(\mathbb{R}^\infty) \end{array}$$

covering the inclusion taking $V \subset \mathbb{R}^\infty$ to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$.

Transversality to the zero section

- To each M we associate a well-defined class in

$$\operatorname*{colim}_k \pi_{n+k}(Th(\gamma^k)) \cong \operatorname*{colim}_k \pi_{n+k}(MO(k)) = \pi_n(MO)\,.$$

- Conversely, given $f\colon S^{n+k} \to Th(\gamma^k) \simeq MO(k)$ we can deform f to be transverse to the zero section $Gr_k(\mathbb{R}^\infty) \subset Th(\gamma^k)$, in which case the preimage

$$M = f^{-1}(Gr_k(\mathbb{R}^\infty))$$

is a smooth and closed submanifold of S^{n+k} of codimension k, i.e., a closed n-manifold.

A theorem of Thom

Theorem ([Tho54])

The rule $[f] \mapsto [M]$ *defines an isomorphism of graded commutative rings*

$$\pi_*(MO) \cong \mathscr{N}_* \,.$$

Sketch proof.

The Pontryagin–Thom construction defines the inverse isomorphism $[M] \mapsto [f]$. □

Stable range homotopy groups

- More generally, we are lead to study sequences of spaces $(M_k)_k$ and their homotopy groups $\pi_{n+k}(M_k)$ in a “stable” range of degrees n that grows to infinity with k.
- We can compare these groups for different k if we are given maps $\sigma\colon M_k \wedge S^1 \to M_{k+1}$, inducing homomorphisms

$$\pi_{n+k}(M_k) \xrightarrow{E} \pi_{n+k+1}(M_k \wedge S^1) \xrightarrow{\sigma_*} \pi_{n+k+1}(M_{k+1}) \,.$$

- The stable range groups are then given by the sequential colimit

$$\pi_n(M) = \operatorname*{colim}_k \pi_{n+k}(M_k) \,.$$

- These objects $M = (M_k, \sigma)_k$ are the (sequential) spectra of algebraic topology, and a key feature of stable homotopy theory is to view a spectrum M as an object that gives an undivided presentation of the sequence of abelian groups $\pi_*(M) = (\pi_n(M))_n$.

Cohomology of *BO* and *MO*

- Proceeding from the theorem above, and knowledge of the cohomology

$$H^*(BO; \mathbb{F}_2) \cong \lim_k H^*(BO(k); \mathbb{F}_2) \cong \mathbb{F}_2[w_i \mid i \geq 1]$$

with $|w_i| = i$, as a module over the mod 2 Steenrod algebra A, Thom went on to calculate the cohomology

$$H^*(MO; \mathbb{F}_2) \cong \lim_k H^{*+k}(MO(k); \mathbb{F}_2)$$

as an A-module, finding it to be free on specific generators.

Thom's theorem on unoriented bordism

Using a trivial case of the Adams spectral sequence, this led to the following conclusion.

Theorem ([Tho54])

$$\pi_*(MO) \cong \mathbb{F}_2[x_n \mid n \neq 2^i - 1] = \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

is the graded polynomial ring over $\mathbb{F}_2$ *on one generator* x_n *in each positive degree* n *not of the form* $2^i - 1$*.*

Example

$\mathscr{N}_3 \cong \pi_3(MO) \cong 0$, so each closed 3-manifold is the boundary $M \cong \partial W$ of a compact 4-manifold.

Outline

- Stable Homotopy Theory
  - Smooth bordism and stable homotopy groups
  - Sequential spectra
  - Triangulated structure
  - Spectral homology and cohomology
  - Orthogonal spaces
  - Orthogonal spectra
  - Closed symmetric monoidal structure
  - Monoidal model structure
  - Multiplicative (co-)homology theories

Approaches to the stable category

- Building on the work of Lima [Lim58] and Boardman [Vog70], Adams' Chicago lectures from 1971 [Ada74] gave a construction of the stable (homotopy) category as a (closed) symmetric monoidal category, based on an underlying category of sequential CW spectra without precise monoidal properties.
- Around 1995 several categories of spectra with closed symmetric monoidal properties were discovered. We focus on "orthogonal spectra", and use the paper [MMSS01] by Mandell, May, Schwede and Shipley to give a parallel development of sequential and orthogonal spectra.
- We work in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces and basepoint preserving maps.

Sequential spectra

Definition

A sequential spectrum M is a sequence of spaces M_k for $k \geq 0$ and structure maps

$$\sigma \colon M_k \wedge S^1 \longrightarrow M_{k+1} \,.$$

A map of sequential spectra $f \colon M \to N$ is a sequence of maps $f_k \colon M_k \to N_k$ such that the diagram

$$\begin{array}{ccc} M_k \wedge S^1 & \xrightarrow{\sigma} & M_{k+1} \\ \downarrow{\scriptstyle f_k \wedge 1} & & \downarrow{\scriptstyle f_{k+1}} \\ N_k \wedge S^1 & \xrightarrow{\sigma} & N_{k+1} \end{array}$$

commutes, for each $k \geq 0$. Let $Sp^{\mathbb{N}}$ denote the category of sequential spectra.

The homotopy groups of a spectrum

Definition

The graded homotopy groups $\pi_*(M)$ of a sequential spectrum M are given in degree n by the colimit

$$\pi_n(M) = \operatorname*{colim}_k \pi_{n+k}(M_k)$$

of the sequence of homomorphisms

$$\dots \longrightarrow \pi_{n+k}(M_k) \longrightarrow \pi_{n+k+1}(M_{k+1}) \longrightarrow \dots ,$$

for $n+k \ge 2$, each mapping the homotopy class of $x \colon S^{n+k} \to M_k$ to the homotopy class of the composite

$$\sigma(x \wedge 1) \colon S^{n+k+1} \cong S^{n+k} \wedge S^1 \xrightarrow{x \wedge 1} M_k \wedge S^1 \xrightarrow{\sigma} M_{k+1} .$$

Functoriality

Any map $f\colon M \to N$ induces compatible homomorphisms $\pi_{n+k}(f_k)\colon \pi_{n+k}(M_k) \to \pi_{n+k}(N_k)$ with colimit $\pi_n(f)\colon \pi_n(M) \to \pi_n(N)$, for all integers n, making

$$\pi_*\colon Sp^{\mathbb{N}} \longrightarrow grAb$$

a functor from sequential spectra to graded abelian groups. We often write f_* for $\pi_n(f)$ or $\pi_*(f)$.

Stable equivalences

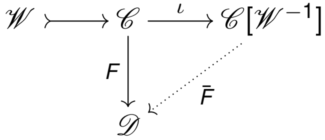
Definition

- A map $f\colon M \to N$ is a stable equivalence if the induced homomorphism $f_*\colon \pi_*(M) \to \pi_*(N)$ is an isomorphism.
- We may then write $f\colon M \xrightarrow{\sim} N$ or $M \sim N$.
- The stable equivalences form a subcategory $\mathscr{W} \subset Sp^{\mathbb{N}}$, which properly contains the homotopy equivalences.

Localization of categories

Definition

A localization $\mathscr{C}[\mathscr{W}^{-1}]$ of a category $\mathscr{C}$ at a subcategory $\mathscr{W}$ is a category with a functor $\iota\colon \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ mapping each morphism in $\mathscr{W}$ to an isomorphism in $\mathscr{C}[\mathscr{W}^{-1}]$, such that for any functor $F\colon \mathscr{C} \to \mathscr{D}$ mapping each morphism in $\mathscr{W}$ to an isomorphism in $\mathscr{D}$ there is a unique functor $\bar{F}\colon \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ such that $F = \bar{F} \circ \iota$.



The localization $\mathscr{C}[\mathscr{W}^{-1}]$, if it exists, is well-defined up to unique isomorphism as a category under $\mathscr{C}$.

Existence of localizations

- In general there can be set-theoretical hindrances to the existence of a localization.
- Quillen's theory of model categories provides one approach to showing that a localization exists, which will let us make sense of the following definition.

Definition

The stable category $\mathrm{Ho}(Sp^{\mathbb{N}})$ is the localization

$$Sp^{\mathbb{N}} \xrightarrow{\iota} Sp^{\mathbb{N}}[\mathscr{W}^{-1}] = \mathrm{Ho}(Sp^{\mathbb{N}})$$

of the category $Sp^{\mathbb{N}}$ of sequential spectra with respect to the subcategory $\mathscr{W}$ of stable equivalences. Let

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{N}})(M, N)$$

denote the set of morphisms in the stable category from M to N.

Stable model, triangulated and abelian categories

- We get a factorization of functors

$$\pi_* \colon Sp^{\mathbb{N}} \xrightarrow{\iota} \mathrm{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\bar{\pi}_*} grAb$$

  from a stable model category via a triangulated category to a graded abelian category.
- We shall later give an equivalent definition of the stable category as a localization $\mathrm{Ho}(Sp^{\mathbb{O}})$ of a category $Sp^{\mathbb{O}}$ of orthogonal spectra.
- The latter stable model category has better (closed symmetric) monoidal properties than $Sp^{\mathbb{N}}$, compatible with the "tensor triangulated" structure on the stable category and the "abelian monoidal" structure on graded abelian groups.

Level ℓ evaluation and free functors

- For each $\ell \geq 0$ let
$$Ev_\ell \colon Sp^{\mathbb{N}} \longrightarrow \mathscr{T}$$
be the level ℓ evaluation functor mapping $M = (M_k, \sigma)_k$ to M_ℓ.
- Let the level ℓ free functor
$$F_\ell \colon \mathscr{T} \longrightarrow Sp^{\mathbb{N}}$$
be its left adjoint, so that there is a natural bijection
$$Sp^{\mathbb{N}}(F_\ell X, N) \cong \mathscr{T}(X, Ev_\ell(N)) .$$
- Explicitly,
$$(F_\ell X)_k = \begin{cases} X \wedge S^{k-\ell} & \text{for } k \geq \ell, \\ * & \text{otherwise.} \end{cases}$$
The structure maps $\sigma \colon (F_\ell X)_k \wedge S^1 \to (F_\ell X)_{k+1}$ are the identities when $k \geq \ell$, and the base point inclusion otherwise.

Suspension and sphere spectra

Definition

- In particular, $F_0 X = \Sigma^\infty X$ is the suspension spectrum of X, with $(\Sigma^\infty X)_k = X \wedge S^k$ for each $k \geq 0$.
- For each integer n we define the n-sphere spectrum S^n by

$$S^n = \begin{cases} F_0 S^n & \text{for } n \geq 0, \\ F_{-n} S^0 & \text{for } n < 0, \end{cases}$$

  so that $(S^n)_k = S^{n+k}$ for $n + k \geq 0$ and $*$ otherwise.
- In particular, $S^0 = S$ with $S_k = S^k$ for each $k \geq 0$ is the sphere spectrum.

Bounded below spectra

- A spectrum M is $(n-1)$-connected, or n-connective, if $\pi_*(M) = 0$ for all $* < n$.
- It is bounded below if it is n-connective for some integer n.
- We abbreviate (-1)-connected (or 0-connective) to connective.

The stable stems

- The homotopy groups

$$\pi_n(S) = \operatorname*{colim}_k \pi_{n+k}(S^k)$$

  of the sphere spectrum are the stable homotopy groups of spheres, also known as the stable stems.
- By the Hurewicz theorem they are trivial for $n < 0$, and isomorphic to $\mathbb{Z}$ for $n = 0$, so the sphere spectrum is connective.
- By Freudenthal's suspension theorem, $\pi_{n+k}(S^k) \cong \pi_n(S)$ for all $k \geq n + 2$.
- By Serre's finiteness theorem, these abelian groups are finite for all $n > 0$.
- The determination of $\pi_n(S)$ for moderately large n is an ongoing field of study.

Relative cell spectra

- Let I be the set of inclusions $i\colon S^{n-1}_+ \to D^n_+$ for $n \geq 0$, where $S^{-1} = \emptyset$.
- Let $FI = F^{\mathbb{N}} I$ be the set of maps of sequential spectra $F_\ell i\colon F_\ell S^{n-1}_+ \to F_\ell D^n_+$ for $\ell \geq 0$ and $n \geq 0$.
- A map $i\colon M \to N$ of sequential spectra is a relative cell spectrum if N is the colimit of a sequence of maps

$$M = N(0) \longrightarrow \dots \longrightarrow N(j) \longrightarrow N(j+1) \longrightarrow \dots \longrightarrow N$$

where each $N(j) \to N(j+1)$ is obtained by cobase change

$$\begin{array}{ccc} \bigvee_\alpha S(\alpha) & \longrightarrow & \bigvee_\alpha D(\alpha) \\ \phi \downarrow & & \downarrow \Phi \\ N(j) & \longrightarrow & N(j+1) \end{array}$$

from a sum of maps $S(\alpha) \to D(\alpha)$ in FI.

Quillen cofibrations

Definition

A map $i \colon M \to N$ in $Sp^{\mathbb{N}}$ is a Quillen cofibration if it is a retract of a relative cell spectrum $i' \colon M' \to N'$, meaning that there is a commutative diagram

$$\begin{array}{ccccc} M & \longrightarrow & M' & \longrightarrow & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ N & \longrightarrow & N' & \longrightarrow & N \end{array}$$

where the horizontal composites are the identity maps.

- We say that N is a cell spectrum if $* \to N$ is a relative cell spectrum.
- We say that N is Quillen cofibrant if $* \to N$ is a Quillen cofibration.
- Any retract of a cell spectrum is Quillen cofibrant.

Cofibrant replacement

- Cobase changes and colimits are created levelwise, so for a cell spectrum N each space N_k is a cell complex, and

$$\sigma \colon N_k \wedge S^1 \to N_{k+1}$$

  is the inclusion of a subcomplex.
- If $q \colon M^c \xrightarrow{\sim} M$ is a stable equivalence, and M^c is Quillen cofibrant, then we say that M^c is a cofibrant replacement for M.
- Cofibrant replacements can be constructed by CW approximation.

Adjoint structure maps

Definition

Let M be a sequential spectrum. The adjoint structure map

$$\tilde{\sigma}\colon M_k \longrightarrow \Omega M_{k+1}$$

is the right adjoint, for the loop–suspension adjunction, of the structure map $\sigma\colon M_k \wedge S^1 \to M_{k+1}$.

Weak homotopy pullbacks

Definition

A commutative square of based spaces

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

in which p and q are Serre fibrations, is a weak homotopy pullback if the induced map

$$D \longrightarrow A \times_B E$$

is a weak homotopy equivalence or, equivalently, if $g\colon q^{-1}(a) \to p^{-1}(f(a))$ is a weak homotopy equivalence for each $a \in A$.

Stable fibrations

Definition

A map $p\colon M \to N$ of sequential spectra is a stable fibration if and only if $p_k\colon M_k \to N_k$ is a Serre fibration and the diagram

$$\begin{array}{ccc} M_k & \xrightarrow{\tilde{\sigma}} & \Omega M_{k+1} \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle \Omega p_{k+1}} \\ N_k & \xrightarrow{\tilde{\sigma}} & \Omega N_{k+1} \end{array}$$

is a weak homotopy pullback, for each $k \geq 0$.

- We say that M is stably fibrant if $M \to *$ is a stable fibration.
- If $j\colon N \xrightarrow{\sim} N^f$ is a stable equivalence, and N^f is stably fibrant, then we say that N^f is a fibrant replacement for N.

Ω-spectra

Lemma

M is stably fibrant if and only if it is an Ω-spectrum, i.e., if each adjoint structure map

$$\tilde{\sigma} \colon M_k \xrightarrow{\simeq} \Omega M_{k+1}$$

is a weak homotopy equvalence.

Proof.

Each map $M_k \to *$ is a Serre fibration. □

Fibrant replacement by mapping telescope

Example

A fibrant replacement $M \sim M^f$ can be constructed by setting M^f_k equal to the mapping telescope (or homotopy colimit) of the sequence of maps

$$M_k \xrightarrow{\tilde{\sigma}} \Omega M_{k+1} \xrightarrow{\Omega\tilde{\sigma}} \Omega^2 M_{k+2} \xrightarrow{\Omega^2\tilde{\sigma}} \Omega^3 M_{k+3} \longrightarrow \dots .$$

Model categories

Definition

A model category $\mathscr{C}$ is a category with all small limits and colimits, together with a model structure. A model structure on $\mathscr{C}$ is three subcategories, of weak equivalences, cofibrations and fibrations, with the following four properties.

1. If $f\colon L \to M$ and $g\colon M \to N$ are composable morphisms, and two of f, g and $gf\colon L \to N$ are weak equvalences, then so is the third of these.
2. If $f\colon M \to N$ is a retract of $f'\colon M' \to N'$, and f' is a weak equivalence, cofibration or fibration, then f is a weak equivalence, cofibration or fibration, respectively.

Model categories (cont.)

3. If $i\colon K \to L$ is a cofibration, $p\colon M \to N$ is a fibration, the square diagram

$$\begin{array}{ccc} K & \longrightarrow & M \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

commutes, and i or p is a weak equivalence, then there exists a map $L \to M$ making both triangles commute.

4. Each map $K \to N$ admits factorizations

$$K \xrightarrow{j} M \xrightarrow{p} N \quad \text{and} \quad K \xrightarrow{i} L \xrightarrow{q} N$$

where j and q are weak equivalences, i and j are cofibrations, and p and q are fibrations.

The stable model structure

Theorem

The category $Sp^{\mathbb{N}}$ of sequential spectra is a model category with respect to the classes of stable equivalences, Quillen cofibrations and stable fibrations.

We refer to [MMSS01, Thm. 9.2] for the proof.

Tensored and cotensored structure over spaces

Definition

For each sequential spectrum M and space X we define

$$X \wedge M\ ,\ M \wedge X\ ,\ \mathrm{Map}(X, M)$$

to be the sequential spectra with k-th spaces

$$X \wedge M_k\ ,\ M_k \wedge X\ ,\ \mathrm{Map}(X, M_k)\,,$$

respectively, and with structure maps

$$X \wedge M_k \wedge S^1 \xrightarrow{1\wedge\sigma} X \wedge M_{k+1}\,,$$

$$M_k \wedge X \wedge S^1 \xrightarrow{1\wedge\tau} M_k \wedge S^1 \wedge X \xrightarrow{1\wedge\sigma} M_{k+1} \wedge X$$

$$\mathrm{Map}(X, M_k) \wedge S^1 \longrightarrow \mathrm{Map}(X, M_k \wedge S^1) \xrightarrow{\mathrm{Map}(1,\sigma)} \mathrm{Map}(X, M_{k+1})\,.$$

Cylinders, cones, etc.

In particular, let

$$I_+ \wedge M$$
$$CM = I \wedge M$$
$$\Sigma M = S^1 \wedge M$$
$$\mathrm{Map}(I_+, M)$$
$$PM = \mathrm{Map}(I, M)$$
$$\Omega M = \mathrm{Map}(S^1, M)$$

denote the cylinder, cone, suspension, free paths, paths and loops on M, respectively.

Homotopic maps

Lemma

If M is Quillen cofibrant then $I_+ \wedge M$ is a cylinder object for M, meaning that

$$M \vee M \cong \partial I_+ \wedge M \xrightarrow{i_0 \vee i_1} I_+ \wedge M \xrightarrow{\simeq} M$$

is a factorization of the fold map $M \vee M \to M$ through a Quillen cofibration followed by a stable equivalence (in fact, a homotopy equivalence).

Two maps $f, g \colon M \to N$ are homotopic, denoted $f \simeq g$, if there exists a map

$$H \colon I_+ \wedge M \longrightarrow N$$

such that $Hi_0 = f$ and $Hi_1 = g$.

Homotopy equivalence vs. stable equivalence

Homotopic maps induce identical homomorphisms of stable homotopy groups, so any homotopy equivalence is a stable equivalence.

Proposition ([Hov99])

If M is Quillen cofibrant and N is stably fibrant, then a map $f\colon M \to N$ is a stable equivalence if and only if it is a homotopy equivalence.

This is a formal consequence of the model category structure.

The subcategory of cofibrant and fibrant objects

Let

$$Sp^{\mathbb{N}}_{cf} \subset Sp^{\mathbb{N}}$$

denote the full subcategory of simultaneously Quillen cofibrant and stably fibrant spectra, and let $Sp^{\mathbb{N}}_{cf}/\simeq$ denote the quotient category with the same objects, but with morphism sets the homotopy classes

$$(Sp^{\mathbb{N}}_{cf}/\simeq)(M, N) = Sp^{\mathbb{N}}(M, N)/\simeq$$

of maps $M \to N$.

The stable category exists

Theorem ([Hov99])

The induced functor

$$Sp^{\mathbb{N}}_{cf}/{\simeq} \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}}) = Sp^{\mathbb{N}}[\mathscr{W}^{-1}]$$

is an equivalence of categories. If $q\colon M^c \xrightarrow{\sim} M$ *and* $j\colon N \xrightarrow{\sim} N^f$ *are cofibrant and fibrant replacements, respectively, then*

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{N}})(M, N) \cong Sp^{\mathbb{N}}(M^c, N^f)/{\simeq}\,.$$

In particular, the stable category $\mathrm{Ho}(Sp^{\mathbb{N}}) = Sp^{\mathbb{N}}[\mathscr{W}^{-1}]$ exists as a category (with sets, not proper classes, of morphisms).

Presentation of stable homotopy groups

Corollary

For each integer n there is a natural isomorphism

$$[S^n, M] \cong \pi_n(M)\,.$$

Proof.

- Any fibrant replacement $j\colon M \xrightarrow{\sim} M^f$ induces isomorphisms $j_*\colon \pi_*(M) \cong \pi_*(M^f)$ and $j_*\colon [S^n, M] \cong [S^n, M^f]$, so we may assume that M is stably fibrant, i.e., an Ω-spectrum.
- Furthermore, S^n is a cell spectrum, hence Quillen cofibrant, so

$$[S^n, M] \cong Sp^{\mathbb{N}}(S^n, M)/\simeq$$

  is given by the homotopy classes of spectrum maps $x\colon S^n \to M$.

Proof (cont.)

- When $n \geq 0$, this is the homotopy classes of maps $x_0 \colon S^n \to M_0$. Since M is an Ω-spectrum, each homomorphism

$$\pi_n(M_0) \longrightarrow \pi_{n+1}(M_1) \longrightarrow \ldots \longrightarrow \pi_n(M)$$

  is an isomorphism, so $[S^n, M] \cong \pi_n(M)$.

- When $n \leq 0$, we are instead considering the homotopy classes of maps $x_{-n} \colon S^0 \to M_{-n}$. Again, each homomorphism

$$\pi_0(M_{-n}) \longrightarrow \pi_1(M_{-n+1}) \longrightarrow \ldots \longrightarrow \pi_n(M)$$

  is an isomorphism, so $[S^n, M] \cong \pi_n(M)$. □

Outline

Loop isomorphism

We now prove that the stable category is, indeed, stable.

Lemma

There is a natural isomorphism

$$\pi_n(\mathrm{Map}(S^1, M)) \cong \pi_{1+n}(M)\,.$$

Proof.

This is the colimit of the isomorphisms

$$\pi_{n+k}(\mathrm{Map}(S^1, M_k)) \cong \pi_{1+n+k}(M_k)\,,$$

matching $S^{n+k} \to \mathrm{Map}(S^1, M_k)$ to its left adjoint $S^1 \wedge S^{n+k} \to M_k$. □

Suspension isomorphism

Proposition

There is a natural isomorphism

$$E\colon \pi_n(M) \xrightarrow{\cong} \pi_{1+n}(S^1 \wedge M)$$

mapping the class of $x\colon S^{n+k} \to M_k$ *to the class of* $1 \wedge x\colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$.

Proof of injectivity

- Suppose that $x\colon S^{n+k} \to M_k$ is such that $1 \wedge x\colon S^{1+n+k} \to S^1 \wedge M_k$ represents zero in $\operatorname{colim}_k \pi_{1+n+k}(S^1 \wedge M_k)$.
- By first increasing k, we may assume that $1 \wedge x\colon S^{1+n+k} \to S^1 \wedge M_k$ is null-homotopic.
- It follows that

$$x \wedge 1 = \tau(1 \wedge x)\tau\colon S^{n+k} \wedge S^1 \longrightarrow M_k \wedge S^1$$

  and $\sigma(x \wedge 1)\colon S^{n+k+1} \to M_{k+1}$ are null-homotopic.
- Hence $\sigma(x \wedge 1)$ represents the zero class in $\pi_n(M) = \operatorname{colim}_k \pi_{n+k}(M_k)$.
- Since x and $\sigma(x \wedge 1)$ represent the same class in this colimit, E is injective.

$$\begin{array}{ccccccc}
\pi_{n+k}(M_k) & \longrightarrow & \pi_{n+k+1}(M_{k+1}) & \longrightarrow & \pi_{n+k+2}(M_{k+2}) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_{1+n+k}(S^1 \wedge M_k) & \longrightarrow & \pi_{1+n+k+1}(S^1 \wedge M_{k+1}) & \longrightarrow & \pi_{1+n+k+2}(S^1 \wedge M_{k+2}) & \longrightarrow & \dots
\end{array}$$

Proof of surjectivity

- Consider an element in $\operatorname{colim}_k \pi_{1+n+k}(S^1 \wedge M_k)$ represented by the homotopy class of a map $y \colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$, as well as by its stabilization $\sigma^2(y \wedge 1 \wedge 1)$.
- Let

$$x = \tau y \tau \wedge 1 \colon S^{n+k} \wedge S^1 \wedge S^1 \longrightarrow M_k \wedge S^1 \wedge S^1 .$$

- Then $1 \wedge x \colon S^1 \wedge S^{n+k} \wedge S^1 \wedge S^1 \to S^1 \wedge M_k \wedge S^1 \wedge S^1$ is homotopic to $y \wedge 1 \wedge 1$, since a cyclic permutation of $S^1 \wedge S^1 \wedge S^1$ is homotopic to the identity.
- Hence $\sigma^2(y \wedge 1 \wedge 1)$ is homotopic to $(1 \wedge \sigma^2)(1 \wedge x) = 1 \wedge \sigma^2(x) \colon S^{1+n+k+2} \to S^1 \wedge M_{k+2}$.
- It therefore represents the same class as the image of $\sigma^2(x) \colon S^{n+k+2} \to M_{k+2}$, which proves that E is surjective. $\square$

Unit equivalence

Proposition

The adjunction unit

$$\eta \colon M \longrightarrow \mathrm{Map}(S^1, S^1 \wedge M)$$

is a stable equivalence.

Proof.

The homomorphism η_* factors as the composite

$$\eta_* \colon \pi_n(M) \cong \pi_{1+n}(S^1 \wedge M) \cong \pi_n(\mathrm{Map}(S^1, S^1 \wedge M))$$

of the isomorphisms from the previous two lemmas, hence is an isomorphism for each n. □

Counit equivalence

Proposition

The adjunction counit

$$\epsilon\colon S^1 \wedge \mathrm{Map}(S^1, M) \longrightarrow M$$

is a stable equivalence.

Proof.

The composite

$$\mathrm{Map}(S^1, M) \xrightarrow{\eta_{\mathrm{Map}(S^1,M)}} \mathrm{Map}(S^1, S^1 \wedge \mathrm{Map}(S^1, M)) \xrightarrow{\mathrm{Map}(S^1,\epsilon)} \mathrm{Map}(S^1, M)$$

is the identity [ML71], and $\pi_n(\eta)$ is an isomorphism for each n, so $\pi_n(\mathrm{Map}(S^1, \epsilon))$ is an isomorphism. Hence $\pi_{1+n}(\epsilon)$ is an isomorphism. □

Suspension is an equivalence

The stable model structure makes $Sp^{\mathbb{N}}$ a stable model category in the sense of [SS03], which implies that its homotopy category is triangulated.

Theorem

The suspension functor $M \mapsto S^1 \wedge M$ is an equivalence of categories

$$S^1 \wedge - \colon \operatorname{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\simeq} \operatorname{Ho}(Sp^{\mathbb{N}}) \, .$$

In other words, for all sequential spectra M and N the function

$$S^1 \wedge - \colon [M, N] \xrightarrow{\cong} [S^1 \wedge M, S^1 \wedge N]$$

is a bijection, and each M is stably equivalent to a spectrum of the form $S^1 \wedge N$.

Proof

- $S^1 \wedge -$ preserves Quillen cofibrant objects and $\mathrm{Map}(S^1, -)$ preserves stably fibrant objects.
- Hence the adjunction
$$Sp^{\mathbb{N}}(S^1 \wedge M^c, N^f) \cong Sp^{\mathbb{N}}(M^c, \mathrm{Map}(S^1, N^f))$$
passes to a natural bijection
$$\begin{aligned}[S^1 \wedge M, N] &\cong Sp^{\mathbb{N}}(S^1 \wedge M^c, N^f)/\simeq \\ &\cong Sp^{\mathbb{N}}(M^c, \mathrm{Map}(S^1, N^f))/\simeq \cong [M, \mathrm{Map}(S^1, N)]\,.\end{aligned}$$
- Replacing N by $S^1 \wedge N$, the composite
$$[M, N] \xrightarrow{S^1 \wedge -} [S^1 \wedge M, S^1 \wedge N] \cong [M, \mathrm{Map}(S^1, S^1 \wedge N)]$$
is induced by the unit $\eta \colon N \to \mathrm{Map}(S^1, S^1 \wedge N)$.
- Since this is a stable equivalence, the induced function is a bijection.
- The counit $\epsilon \colon S^1 \wedge \mathrm{Map}(S^1, M) \to M$ exhibits M as being stably equivalent to $S^1 \wedge N$ for $N = \mathrm{Map}(S^1, M)$. □

Puppe sequences

Definition

For a map $f\colon M \to N$ of spectra let

$$Cf = N \cup CM$$

be the mapping cone (= homotopy cofiber) of $f\colon M \to N$. There are canonical maps

$$M \xrightarrow{f} N \xrightarrow{i} Cf \xrightarrow{q} S^1 \wedge M$$

with i and q induced by $M \to CM$ and $N \to *$, respectively. We call this the (homotopy cofiber) Puppe sequence generated by f.

Exactness for stable homotopy groups

Proposition

The Puppe sequence of $f\colon M \to N$ induces a long exact sequence of stable homotopy groups

$$\cdots \to \pi_n(M) \xrightarrow{f_*} \pi_n(N) \xrightarrow{i_*} \pi_n(Cf) \xrightarrow{\partial} \pi_{n-1}(M) \to \dots,$$

where $E \circ \partial = q_*\colon \pi_n(Cf) \to \pi_n(S^1 \wedge M)$.

Proof.

It suffices to prove exactness at $\pi_n(N)$, and $i_* f_* = 0$ is clear.

Proof (cont.)

$$
\begin{array}{ccccccc}
 & & S^{n+k} & \xrightarrow{i} & CS^{n+k} & \xrightarrow{q} & S^1 \wedge S^{n+k} \\
 & & \downarrow y & & \downarrow z & & \downarrow x' \\
M_k & \xrightarrow{f_k} & N_k & \xrightarrow{i_k} & Cf_k & \xrightarrow{q_k} & S^1 \wedge M_k
\end{array}
$$

- If $y\colon S^{n+k} \to N_k$ corresponds to a class $[y]$ in $\ker(i_*)$, then we may increase k and assume that $i_k y$ is null-homotopic, hence extends to a map $z\colon CS^{n+k} \to Cf_k$.
- The induced map of quotients $x'\colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$ then represents a class $[x']$ in $\pi_{1+n}(S^1 \wedge M)$ which corresponds, under the isomorphism E, to a class $[x] \in \pi_n(M)$ with $f_*([x]) = [y]$. □

Additive categories

Puppe introduced axioms for a triangulated category (at a 1962 conference in Aarhus), and Verdier added the octahedral axiom (in his 1967 PhD thesis).

Definition

- An *Ab*-category is a category $\mathscr{C}$ in which each morphism set $\mathscr{C}(X, Y)$ is an abelian group and composition is bilinear.
- An additive category is an *Ab*-category with all finite sums and products, such that these are canonically isomorphic.

We write 0 for the zero object, i.e., the empty sum and product.

Triangulated categories

We adopt May's formulation of the definition below. Axiom (3) is the braid form of Verdier's octahedral axiom.

Definition

A triangulated category is an additive category $\mathscr{C}$ equipped with an additive functor

$$\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$$

called suspension, and a collection Δ of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X\,,$$

called distinguished triangles, briefly denoted (f, g, h). We assume that $\Sigma\colon \mathscr{C} \to \mathscr{C}$ is an equivalence of categories.

Definition (cont.)

Furthermore, we assume that:

1. For any object X the triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

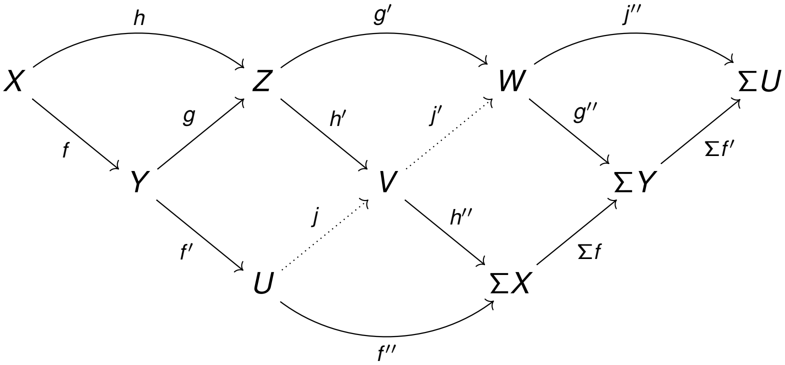
   is distinguished, for any morphism $f\colon X \to Y$ there exists a distinguished triangle (f, g, h), and any triangle that is isomorphic to a distinguished triangle is itself distinguished.

2. If (f, g, h) is distinguished, then so is its rotation

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y\,.$$

Definition (cont.)

3. Consider the following braid diagram.



Assume that $h = gf$ and $j'' = (\Sigma f')g''$, and that (f, f', f''), (g, g', g'') and (h, h', h'') are distinguished. Then there are maps j and j' such that the diagram commutes and (j, j', j'') is distinguished.

Lemma ([BBD82], [May01])

Assume that $jf = f'i$ and the two top rows and two left columns are distinguished in the following diagram.

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
\downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j'} & & \vdots{\scriptstyle k'} & & \downarrow{\scriptstyle \Sigma i'} \\
X'' & \overset{f''}{\cdots\to} & Y'' & \overset{g''}{\cdots\to} & Z'' & \overset{h''}{\cdots\to} & \Sigma X'' \\
\downarrow{\scriptstyle i''} & & \downarrow{\scriptstyle j''} & & \vdots{\scriptstyle k''} & (-) & \downarrow{\scriptstyle \Sigma i''} \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma g} & \Sigma Z & \xrightarrow{\Sigma h} & \Sigma^2 X
\end{array}
$$

Then there is an object Z'' and maps f'', g'', h'', k, k' and k'' such that the diagram is commutative, except for its bottom right hand square, which commutes up to the sign -1, and all four rows and columns are distinguished.

The fill-in axiom

In particular, the fill-in axiom of Puppe and Verdier follows from those above. We state it as a lemma.

Lemma ([May01])

If the rows are distinguished and the left hand square commutes in the following diagram, then there is a morphism k that makes the remaining two squares commute.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \vdots\,{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

Exactness and coexactness of distinguished triangles

Proposition

For (f, g, h) distinguished and T any object, the sequences

$$\dots \longrightarrow \mathscr{C}(T, X) \xrightarrow{f_*} \mathscr{C}(T, Y) \xrightarrow{g_*} \mathscr{C}(T, Z) \xrightarrow{h_*} \mathscr{C}(T, \Sigma X) \longrightarrow \dots$$

and

$$\dots \longleftarrow \mathscr{C}(X, T) \xleftarrow{f^*} \mathscr{C}(Y, T) \xleftarrow{g^*} \mathscr{C}(Z, T) \xleftarrow{h^*} \mathscr{C}(\Sigma X, T) \longleftarrow \dots$$

are exact.

Proof

We show that $\operatorname{im}(f_*) = \ker(g_*)$. Given $i\colon T \to X$ in $\mathscr{C}(T, X)$ we have

$$\begin{array}{ccccccc}
T & \xrightarrow{1} & T & \longrightarrow & 0 & \longrightarrow & \Sigma T \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array}$$

with $j = fi$, and there is a fill-in map k. Hence $gfi = 0$, so $\operatorname{im}(f_*) \subset \ker(g_*)$.
Conversely, given $j\colon T \to Y$ in $\mathscr{C}(T, Y)$ with $gj = 0$ we have

$$\begin{array}{ccccccc}
T & \longrightarrow & 0 & \longrightarrow & \Sigma T & \xrightarrow{-\Sigma 1} & \Sigma T \\
\downarrow{\scriptstyle j} & & \downarrow & & \vdots{\scriptstyle \Sigma i} & & \downarrow{\scriptstyle \Sigma j} \\
Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y
\end{array}$$

and there is a fill-in map Σi. Hence $\Sigma j = \Sigma(fi)$, so $j = fi$, and $\ker(g_*) \subset \operatorname{im}(f_*)$. $\square$

The stable category is triangulated

Puppe [Pup67] proved the following theorem for a smaller category than $\mathrm{Ho}(Sp^{\mathbb{N}})$, while in its present form the result is due to Boardman (unpublished, but see [Vog70] or [Ada74]).

Theorem

The stable category $\mathrm{Ho}(Sp^{\mathbb{N}})$ *is triangulated, with suspension functor*

$$\Sigma M = S^1 \wedge M$$

and distinguished triangles the diagram that are isomorphic to Puppe sequences

$$M \xrightarrow{f} N \xrightarrow{i} Cf \xrightarrow{q} S^1 \wedge M\,.$$

Proof

- We first argue that $\mathrm{Ho}(Sp^{\mathbb{N}})$ is an *Ab*-category.
- For any L and N there are spectra L' and N' and stable equivalences $L \sim S^2 \wedge L'$ and $N \sim \mathrm{Map}(S^2, N')$.
- The homotopy commutative cogroup structure on S^2 induces abelian group structures

$$[L, M] \cong [S^2 \wedge L', M] \quad \text{and} \quad [M, N] \cong [M, \mathrm{Map}(S^2, N')]$$

so that the composition pairing

$$[M, N] \times [L, M] \cong [M, \mathrm{Map}(S^2, N')] \times [S^2 \wedge L', M] \xrightarrow{\circ} [S^2 \wedge L', \mathrm{Map}(S^2, N')] \cong [L, N]$$

is bilinear.

Proof (cont.)

- The stable category has all sums and products, defined levelwise.
- The *Ab*-category structure implies that the finite sums are canonically isomorphic to the finite products, see [ML71], so that $\mathrm{Ho}(Sp^{\mathbb{N}})$ is additive.
- In particular,

$$M \vee N \xrightarrow{\sim} M \times N$$

  is a stable equivalence, and

$$\pi_*(M) \oplus \pi_*(N) \cong \pi_*(M \vee N) \cong \pi_*(M \times N) \cong \pi_*(M) \times \pi_*(N) .$$

- We showed earlier that the suspension functor $M \mapsto \Sigma M = S^1 \wedge M$ is an equivalence.

Proof (cont.)

Axiom (1) is straightforward, and Axiom (2) follows from the known fact that the Puppe sequence

$$N \xrightarrow{i} Cf \xrightarrow{i'} Ci \xrightarrow{q'} \Sigma N$$

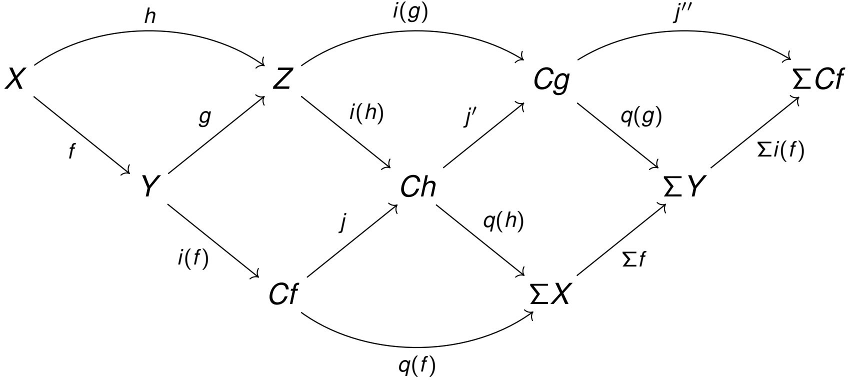
is isomorphic, in the stable category, to

$$N \xrightarrow{i} Cf \xrightarrow{q} \Sigma M \xrightarrow{-\Sigma f} \Sigma N\,.$$

To verify the braid/octahedral axiom (3), we follow [May01].

Proof (cont.)

We may assume that the distinguished triangles (f, f', f''), (g, g', g'') and (h, h', h'') are Puppe cofiber sequences, so that there is a commutative diagram



with $j\colon Y \cup CX \to Z \cup CX$ induced by $g\colon Y \to Z$ and $j'\colon Z \cup CX \to Z \cup CY$ induced by $Cf\colon CX \to CY$.

Proof (cont.)

It remains to verify that (j, j', j'') is distinguished, which amounts to constructing an explicit (stable) equivalence $Cj \sim Cg$ that is compatible with j' and j''. □

Outline

Spectra represent (co-)homology theories

- The Eilenberg–MacLane representability of ordinary cohomology readily extends to show that any spectrum M represents a generalized cohomology theory.
- Dually, George Whitehead [Whi62] showed how spectra also give rise to generalized homology theories.
- Since we are working with based spaces these theories will always be reduced.

Absolute M-homology

Definition

Let $M = (M_k, \sigma)_k$ be a sequential spectrum. For each space X let

$$M_n(X) = \pi_n(M \wedge X)$$

be equal to $\operatorname{colim}_k \pi_{n+k}(M_k \wedge X)$, and let

$$\sigma\colon M_n(X) \xrightarrow{\cong} M_{1+n}(S^1 \wedge X)$$

be the composite of the isomorphisms

$$\pi_n(M \wedge X) \xrightarrow{E} \pi_{1+n}(S^1 \wedge M \wedge X) \xrightarrow{(\tau\wedge 1)_*} \pi_{1+n}(M \wedge S^1 \wedge X)\,.$$

Relative M-homology

Definition (cont.)

For any pair (X, A) let

$$M_n(X, A) = M_n(X \cup CA)$$

and let

$$\partial \colon M_n(X, A) \longrightarrow M_{n-1}(A)$$

be the composite

$$M_n(X \cup CA) \xrightarrow{q_*} M_n(S^1 \wedge A) \xrightarrow{\sigma^{-1}} M_{n-1}(A)$$

induced by $q \colon X \cup CA \to S^1 \wedge A$ and the inverse of σ.

Note that if (X, A) is a CW pair, or $A \to X$ is a Hurewicz cofibration, then $X \cup CA \simeq X/A$ so $M_n(X, A) \cong M_n(X/A)$.

A (generalized) homology theory

Proposition

The functor $(X, A) \mapsto M_*(X, A)$ *and the natural transformation* $\partial\colon M_*(X, A) \to M_{*-1}(A)$ *define a (generalized) homology theory for pairs* (X, A)*.*
Its coefficient groups are $M_* \cong \pi_*(M)$*.*

Proof.

We check the axioms for CW pairs (X, A). Functoriality and naturality are clear.

Proof (cont.)

By exactness for stable homotopy groups, the Puppe sequence

$$A \wedge M \xrightarrow{f \wedge 1} X \wedge M \xrightarrow{i \wedge 1} Cf \wedge M \xrightarrow{q \wedge 1} S^1 \wedge A \wedge M$$

of $f \wedge 1$, where $f \colon A \to X$ is the inclusion and $Cf = X \cup CA$, induces a long exact sequence of homotopy groups. It is isomorphic to the sequence

$$M \wedge A \xrightarrow{1 \wedge f} M \wedge X \xrightarrow{1 \wedge i} M \wedge Cf \xrightarrow{1 \wedge q} M \wedge S^1 \wedge A$$

by way of the evident twist maps, and this proves exactness. Homotopy invariance and excision are likewise clear.

Proof (cont.)

Additivity for infinite sums requires a bit more effort. Let $(X_\alpha)_{\alpha \in J}$ be a collection of (based) spaces, where J is some indexing set. Letting F range over the filtering (= directed and nonempty) poset of finite subsets of J, we can form the inclusions

$$\bigvee_{\alpha \in F} M_k \wedge X_\alpha \longrightarrow \bigvee_{\alpha \in J} M_k \wedge X_\alpha$$

and consider the canonical homomorphism

$$\operatorname*{colim}_F \pi_{n+k}\Big(\bigvee_{\alpha \in F} M_k \wedge X_\alpha\Big) \longrightarrow \pi_{n+k}\Big(\bigvee_{\alpha \in J} M_k \wedge X_\alpha\Big)\,.$$

We claim that this is an isomorphism, because S^{n+k} and the cylinder $I_+ \wedge S^{n+k}$ are both compact. For this we need that $\bigvee_{\alpha \in J} M_k \wedge X_\alpha$ is strongly filtered by the $\bigvee_{\alpha \in F} M_k \wedge X_\alpha$. See [Str, Lem. 3.6] for a proof.

Proof (cont.)

These isomorphisms are compatible for increasing k, and passing to sequential colimits we deduce that

$$\operatorname*{colim}_F M_n(\bigvee_{\alpha\in F} X_\alpha) \xrightarrow{\cong} M_n(\bigvee_{\alpha\in J} X_\alpha)$$

is an isomorphism. By finite additivity,

$$\bigoplus_{\alpha\in F} M_n(X_\alpha) \xrightarrow{\cong} M_n(\bigvee_{\alpha\in F} X_\alpha)$$

and

$$\operatorname*{colim}_F \bigoplus_{\alpha\in F} M_n(X_\alpha) \xrightarrow{\cong} \bigoplus_{\alpha\in J} M_n(X_\alpha)$$

are isomorphisms. Stringing these together, we have confirmed the additivity axiom for the homology theory $X \mapsto M_*(X)$. Finally, the coefficients groups of this theory are $M_* = M_*(S^0) = \pi_*(M \wedge S^0) \cong \pi_*(M)$. □

Absolute M-cohomology

Definition

Let $M = (M_k, \sigma)_k$ be a sequential spectrum. Let $M^f = (M^f_k, \sigma)_k$ be a fibrant replacement of M, i.e., an Ω-spectrum with a stable equivalence $M \sim M^f$. For each space X let

$$M^n(X) = \pi_{-n} \operatorname{Map}(X, M^f)$$

be equal to $\operatorname{colim}_k \pi_{-n+k} \operatorname{Map}(X, M^f_k)$, and let

$$\sigma\colon M^n(X) \xrightarrow{\cong} M^{1+n}(S^1 \wedge X)$$

be the composite of the isomorphisms

$$\pi_{-n} \operatorname{Map}(X, M^f) \cong \pi_{-1-n} \operatorname{Map}(S^1, \operatorname{Map}(X, M^f)) \cong \pi_{-1-n}(S^1 \wedge X, M^f) \,.$$

Relative M-cohomology

Definition (cont.)

For any pair (X, A) let

$$M^n(X, A) = M^n(X \cup CA)$$

and let

$$\delta \colon M^n(A) \longrightarrow M^{1+n}(X, A)$$

be the composite

$$M^n(A) \xrightarrow{\sigma} M^{1+n}(S^1 \wedge A) \xrightarrow{q^*} M^{1+n}(X \cup CA)$$

induced by the isomorphism σ and $q \colon X \cup CA \to S^1 \wedge A$.

A (generalized) cohomology theory

Proposition

The contravariant functor $(X, A) \mapsto M^*(X, A)$ *and the natural transformation* $\delta\colon M^*(A) \to M^{1+*}(X, A)$ *define a (generalized) cohomology theory for pairs* (X, A)*.*
Its coefficient groups are $M^* = \pi_{-*}(M)$*.*

Proof.

We check the axioms for CW pairs (X, A). Contravariant functoriality and naturality are clear.

Proof (cont.)

For each k the Puppe fiber sequence

$$\cdots \to \Omega\,\mathrm{Map}(A, M_k^f) \xrightarrow{\mathrm{Map}(q,1)} \mathrm{Map}(X \cup CA, M_k^f) \xrightarrow{\mathrm{Map}(i,1)} \mathrm{Map}(X, M_k^f) \xrightarrow{\mathrm{Map}(f,1)} \mathrm{Map}(A, M_k^f)$$

induces a long exact sequence of homotopy groups

$$\cdots \to \pi_{1-n+k}\,\mathrm{Map}(A, M_k^f) \xrightarrow{\delta} \pi_{-n+k}\,\mathrm{Map}(X \cup CA, M_k^f) \xrightarrow{i^*} \pi_{-n+k}\,\mathrm{Map}(X, M_k^f) \xrightarrow{f^*} \pi_{-n+k}\,\mathrm{Map}(A, M_k^f)$$

for $k > n$ (and some weaker form of exactness for $k = n$). Passing to sequential colimits over k confirms the exactness axiom. Homotopy invariance and excision are straightforward.

Proof (cont.)

Additivity requires that the canonical exchange map

$$M^n(\bigvee_\alpha X_\alpha) \cong \operatorname*{colim}_k \prod_\alpha \pi_{-n+k} \operatorname{Map}(X_\alpha, M^f_k)$$
$$\xrightarrow{\kappa} \prod_\alpha \operatorname*{colim}_k \pi_{-n+k} \operatorname{Map}(X_\alpha, M^f_k) \cong \prod_\alpha M^n(X_\alpha)$$

is an isomorphism, which holds because M^f is an Ω-spectrum, so that each colimit is achieved at a finite stage, i.e., for any $k \geq n$.
The coefficient groups of this cohomology theory are $M^* = M^*(S^0) = \pi_{-*} \operatorname{Map}(S^0, M) \cong \pi_{-*}(M)$. □

Spectra to cohomology theories

- Each morphism $f \in [M, N]$ in the stable category induces morphisms

$$f_*\colon M_n(X) \cong [S^n, M \wedge X] \longrightarrow [S^n, N \wedge X] \cong N_n(X)$$
$$f_*\colon M^n(X) \cong [S^{-n} \wedge X, M] \longrightarrow [S^{-n} \wedge X, N] \cong N^n(X)$$

  of homology and cohomology theories.
- In particular, we have a functor

$$\mathrm{Ho}(Sp^{\mathbb{N}}) \longrightarrow \text{Cohomology theories}$$

  from the stable category to the category of cohomology theories (defined on CW complexes, or spaces).
- By Brown's representability theorem [Bro62], each cohomology theory $X \mapsto M^*(X)$ arises in this way from some spectrum M, so this functor is essentially surjective.

Superphantom maps

- The functor $M \mapsto M^*(-)$ is also full, in the sense that each morphism of cohomology theories comes from a morphism in the stable category.
- In general it is not faithful, meaning that there may be nontrivial morphisms $f \in [M, N]$ that induce the zero morphism $f_* \colon M^*(X) \to N^*(X)$ for each space X.
- These are called superphantom maps by Margolis [Mar83, p. 81], and their existence shows that the stable category is not equivalent to the category of cohomology theories.
- Goodwillie (MathOverflow, 2013) notes that there are nonzero superphantom maps $KU \to \Sigma H\mathbb{Z}$.

Ordinary (co-)homology

- Let G be an abelian group. The Eilenberg–MacLane spectrum HG has k-th space

$$HG_k = K(G, k)$$

and adjoint structure maps given by homotopy equivalences

$$\tilde{\sigma}\colon K(G, k) \xrightarrow{\simeq} \Omega K(G, k+1)\,.$$

- Its coefficient groups are $\pi_*(HG) = G$ concentrated in degree 0, and this characterizes HG up to stable equivalence.
- It represents ordinary homology and cohomology with G-coefficients, so that

$$\tilde{H}_n(X; G) \cong HG_n(X) = \operatorname*{colim}_k \pi_{n+k}(HG_k \wedge X)$$
$$\tilde{H}^n(X; G) \cong HG^n(X) = [X, HG_n]$$

for $n \geq 0$. These groups are trivial for $n < 0$.

Stable (co-)homotopy

- The sphere spectrum S has k-th space $S_k = S^k$ and structure maps given by the identifications $\sigma \colon S^k \wedge S^1 \cong S^{k+1}$.
- There is a fibrant replacement $S \sim S^f$ with

$$S^f_k = Q(S^k) = \operatorname*{colim}_\ell \Omega^\ell S^{k+\ell}$$

for each k, such that each adjoint structure map

$$\tilde{\sigma} \colon Q(S^k) \xrightarrow{\cong} \Omega Q(S^{k+1})$$

is a homeomorphism.
- The sphere spectrum represents stable homotopy and cohomotopy, so that

$$\pi^S_n(X) \cong S_n(X) \cong \operatorname*{colim}_k \pi_{n+k}(X \wedge S^k)$$
$$\pi^n_S(X) \cong S^n(X) \cong \operatorname*{colim}_\ell [X \wedge S^\ell, S^{n+\ell}] .$$

Pontryagin–Thom / Segal's Burnside ring conjecture

- The Pontryagin–Thom construction extends to an isomorphism
$$\Omega^{fr}_*(X) \cong \pi^S_*(X_+)$$
where $\Omega^{fr}_n(X)$ is given by framed bordism classes of framed n-manifolds $M^n \to X$, equipped with structure maps to X.
- When $X = BG$ is the classifying space of a finite group G, the proven Segal conjecture [Car84] implies that
$$\pi^0_S(BG_+) \cong A(G)^\wedge_{I(G)}$$
is the completion of the Burnside ring $A(G)$ of G at its augmentation ideal, while
$$\pi^n_S(BG_+) = 0$$
for $n > 0$. The precise statement also determines $\pi^n_S(BG_+) = 0$ for $n < 0$, in terms of stable homotopy groups.

Thom spectra

- The Thom spectra MO and MSO have k-th spaces

$$MO_k = Th(\gamma^k) \simeq EO(k)_+ \wedge_{O(k)} S^k$$
$$MSO_k = Th(\tilde{\gamma}^k) \simeq ESO(k)_+ \wedge_{SO(k)} S^k$$

  the Thom complexes of the tautological vector bundles $E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$ and $E(\tilde{\gamma}^k) \to \widetilde{Gr}_k(\mathbb{R}^\infty)$.

- The structure maps

$$\sigma\colon\ Th(\gamma^k) \wedge S^1 \cong Th(\gamma^k \oplus \epsilon^1) \longrightarrow Th(\gamma^{k+1})$$
$$\sigma\colon\ Th(\tilde{\gamma}^k) \wedge S^1 \cong Th(\tilde{\gamma}^k \oplus \epsilon^1) \longrightarrow Th(\tilde{\gamma}^{k+1})$$

  are induced by the vector bundle maps covering the inclusions $Gr_k(\mathbb{R}^\infty) \subset Gr_{k+1}(\mathbb{R}^\infty)$ and $\widetilde{Gr}_k(\mathbb{R}^\infty) \subset \widetilde{Gr}_{k+1}(\mathbb{R}^\infty)$.

Bordism theories

- Their coefficient groups are $\pi_*(MO) \cong \mathscr{N}_*$ and $\pi_*(MSO) \cong \Omega_*$.
- The associated homology theories are precisely (unoriented) bordism and oriented bordism, so that

$$\mathscr{N}_n(X) \cong MO_n(X_+) = \operatorname*{colim}_k \pi_{n+k}(MO(k) \wedge X_+)$$

$$\Omega_n(X) \cong MSO_n(X_+) = \operatorname*{colim}_k \pi_{n+k}(MSO(k) \wedge X_+)\,.$$

Spectrum level presentations

- There are (ring) spectrum maps

$$\begin{array}{ccccc} S & \longrightarrow & MSO & \longrightarrow & MO \\ & & \downarrow & & \downarrow \\ & & H\mathbb{Z} & \longrightarrow & H\mathbb{F}_2 \end{array}$$

  inducing (multiplicative) morphisms

$$\begin{array}{ccccc} \Omega_*^{fr}(X) & \longrightarrow & \Omega_*(X) & \longrightarrow & \mathcal{N}_*(X) \\ & & \downarrow & & \downarrow \\ & & H_*(X;\mathbb{Z}) & \longrightarrow & H_*(X;\mathbb{F}_2) \end{array}$$

  of homology theories.
- The vertical maps take an oriented (resp. unoriented) bordism class $[f]$ with $f\colon M^n \to X$ to the image $f_*[M]$ of the integral (resp. mod 2) fundamental class of M.

The Steenrod problem

$$\begin{array}{ccccc} S & \longrightarrow & MSO & \longrightarrow & MO \\ & & \downarrow & & \downarrow \\ & & H\mathbb{Z} & \longrightarrow & H\mathbb{F}_2 \end{array}$$

- The map $MO \to H\mathbb{F}_2$ admits a section, so each mod 2 homology class can be represented by a closed manifold.
- The map $MSO \to H\mathbb{Z}$ does not admit a section, and not every integral homology class can be represented by a closed oriented manifold.
- Thom [Tho54] showed than for $n \leq 6$ every integral homology class can be represented by a closed oriented manifold, but for each $n \geq 7$ there exist integral homology classes that cannot be so represented.

Complex K-theory

- The classification of rank k complex vector bundles
$$\mathrm{Vect}_k^{\mathbb{C}}(X) \cong [X, BU(k)]$$
extends to a classification of bundles of arbitrary rank
$$\mathrm{Vect}^{\mathbb{C}}(X) \cong [X_+, \coprod_{k \geq 0} BU(k)] \,.$$
- The Whitney sum $\xi \oplus \eta$ of vector bundles induces a commutative monoid structure.
- We can localize, to make the operation $- \oplus \epsilon^1$ invertible, and obtain an isomorphism
$$KU(X) \cong [X_+, \mathbb{Z} \times BU]$$
where BU is the classifying space for the infinite unitary group $U = \bigcup_k U(k)$.
- When X is a finite-dimensional CW complex, the left hand side is the group completion of $\mathrm{Vect}^{\mathbb{C}}(X)$.

The complex K-theory spectrum KU

- The (complex) Bott periodicity theorem [Bot59] asserts that

  $$\mathbb{Z} \times BU \simeq \Omega U \,,$$

  while

  $$U \simeq \Omega(\mathbb{Z} \times BU)$$

  is clear from the existence of a principal U-bundle $p\colon EU \to BU$ with contractible total space.
- Following Atiyah and Hirzebruch [AH61], we can therefore define an Ω-spectrum KU with

  $$KU_k = \begin{cases} \mathbb{Z} \times BU & \text{for } k \text{ even,} \\ U & \text{for } k \text{ odd,} \end{cases}$$

  having adjoint structure maps given by the two homotopy equivalences above.

Real K-theory

- Working with real vector bundles, we have the classification

$$\mathrm{Vect}^{\mathbb{R}}(X) \cong [X_+, \coprod_{k \geq 0} BO(k)] .$$

- Inverting $- \oplus \epsilon^1$ gives

$$KO(X) \cong [X_+, \mathbb{Z} \times BO] ,$$

  where BO is the classifying space of the infinite orthogonal group $O = \bigcup_k O(k)$.
- When X is finite-dimensional, this is the group completion of $\mathrm{Vect}^{\mathbb{R}}(X)$, i.e., the real K-group of X.

Real Bott periodicity

- The (real) Bott periodicity theorem [Bot59] asserts that

$$\begin{aligned} \mathbb{Z} \times BO &\simeq \Omega(U/O) \\ U/O &\simeq \Omega(Sp/U) \\ Sp/U &\simeq \Omega Sp \\ Sp &\simeq \Omega(\mathbb{Z} \times BSp) \\ \mathbb{Z} \times BSp &\simeq \Omega(U/Sp) \\ U/Sp &\simeq \Omega(O/U) \\ O/U &\simeq \Omega O \\ O &\simeq \Omega(\mathbb{Z} \times BO)\,. \end{aligned}$$

  Here $Sp = \bigcup_k Sp(k)$ denotes the infinite symplectic group.
- It follows that

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO)\,.$$

The real K-theory spectrum KO

- We can therefore define an Ω-spectrum KO with

$$KO_k = \begin{cases} \mathbb{Z} \times BO & \text{for } k \equiv 0 \mod 8, \\ U/O & \text{for } k \equiv 1 \mod 8, \\ Sp/U & \text{for } k \equiv 2 \mod 8, \\ Sp & \text{for } k \equiv 3 \mod 8, \\ \mathbb{Z} \times BSp & \text{for } k \equiv 4 \mod 8, \\ U/Sp & \text{for } k \equiv 5 \mod 8, \\ O/U & \text{for } k \equiv 6 \mod 8, \\ O & \text{for } k \equiv 7 \mod 8, \end{cases}$$

  having adjoint structure maps given by the eight homotopy equivalences above.
- It follows that $KO_k \simeq \Omega^\ell(\mathbb{Z} \times BO)$ where $k + \ell \equiv 0 \mod 8$, and we may assume that $0 \leq \ell < 8$.

Topological K-cohomology

- The associated cohomology theories are complex and real (topological) K-theory, with

$$KU^n(X) \cong [X_+, KU_n]$$
$$KO^n(X) \cong [X_+, KO_n]$$

  (unreduced theories) for $n \geq 0$, extended 2- and 8-periodically, respectively, for $n < 0$.
- The coefficient groups of these theories are

$$\pi_*(KU) = \begin{cases} \mathbb{Z} & \text{for } * \text{ even,} \\ 0 & \text{for } * \text{ odd,} \end{cases}$$

  and

$$\pi_*(KO) = \begin{cases} \mathbb{Z} & \text{for } * \equiv 0, 4 \mod 8, \\ \mathbb{Z}/2 & \text{for } * \equiv 1, 2 \mod 8, \\ 0 & \text{for } * \equiv 3, 5, 6, 7 \mod 8. \end{cases}$$

Ring structure

- The external tensor product of vector bundles induces pairings

$$KU^n(X) \otimes KU^m(Y) \longrightarrow KU^{n+m}(X \times Y)$$
$$KO^n(X) \otimes KO^m(Y) \longrightarrow KO^{n+m}(X \times Y)$$

turning these coefficient groups into graded rings.

- Their structures are

$$\pi_*(KU) = \mathbb{Z}[u, u^{-1}]$$

with $|u| = 2$, and

$$\begin{aligned}\pi_*(KO) &= \mathbb{Z}[\eta, A, B, B^{-1}]/(2\eta, \eta^3, \eta A, A^2 - 4B) \\ &= (\dots, \mathbb{Z}\{1\}, \mathbb{Z}/2\{\eta\}, \mathbb{Z}/2\{\eta^2\}, 0, \mathbb{Z}\{A\}, 0, 0, 0, \mathbb{Z}\{B\}, \dots)\end{aligned}$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$.

K-theory and representations

- When $X = BG$ is the classifying space of a finite group G, Atiyah [Ati61b] proved that

$$KU^0(BG) \cong R(G)^\wedge_{I(G)}$$

is the completion of the complex representation ring $R(G)$ at its augmentation ideal, while

$$KU^1(BG) = 0 .$$

- Since $KU^n(BG)$ is 2-periodic, this calculates $KU^*(BG)$ in all degrees.
- The corresponding result for connected compact Lie groups G is due to Atiyah and Hirzebruch [AH61], while the result for general compact Lie groups is part of the Atiyah–Segal completion theorem [AS69], and motivated the Segal conjecture for stable cohomotopy, mentioned above.

Extensions from spaces to spectra

The expressions for $M_*(X)$ and $M^*(X)$ for spaces X, in terms of morphisms in the stable category, suggest that we can extend these homology and cohomology theories over the suspension spectrum functor

$$\begin{aligned} \Sigma^\infty \colon \mathrm{Ho}(\mathscr{T}) &\longrightarrow \mathrm{Ho}(Sp^{\mathbb{N}}) \\ X &\longmapsto \Sigma^\infty X = F_0 X \,, \end{aligned}$$

so as to define the M-homology

$$M_n(X) = \pi_n(M \wedge X) = [S^n, M \wedge X]$$

and M-cohomology

$$M^n(X) = [X, S^n \wedge M]$$

of a spectrum X.

Cohomology of spectra

- For cohomology, this makes sense as stated, and extends the previous definition.
- In particular, with $H = H\mathbb{F}_p$ the mod p Eilenberg–MacLane spectrum, we see that

$$A^n \cong H^n(H) = [H, S^n \wedge H]$$

recovers the stable cohomology operations of type $(\mathbb{F}_p; \mathbb{F}_p, n)$, i.e., the degree n part of the Steenrod algebra.
- Hence there is an algebra isomorphism

$$A \cong H^*(H)\,.$$

Homology of spectra

- For homology, we have not yet made sense of $M \wedge X$ when M and X are both sequential spectra.
- This can be done [Ada74], but a more satisfactory construction can be given in the context of orthogonal spectra, which we turn to in the next section.
- This will then lead to the formula

  $$A_n \cong H_n(H) = [S^n, H \wedge H]$$

  for the degree n part of the dual Steenrod algebra, and there is a Hopf algebra isomorphism

  $$A_* \cong H_*(H).$$

Outline

The stable category, modeled by orthogonal spectra

- We give a different model for the stable category, namely as the homotopy category $\mathrm{Ho}(Sp^{\mathbb{O}}) = Sp^{\mathbb{O}}[\mathscr{W}^{-1}]$ obtained by inverting the stable equivalences in a category $Sp^{\mathbb{O}}$ of orthogonal spectra.
- The categories $Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$ are not equivalent, but their associated homotopy categories

$$\mathrm{Ho}(Sp^{\mathbb{N}}) \simeq \mathrm{Ho}(Sp^{\mathbb{O}})$$

  are, so that we may replace our earlier use of $\mathrm{Ho}(Sp^{\mathbb{N}})$ with $\mathrm{Ho}(Sp^{\mathbb{O}})$.
- This has the advantage that $Sp^{\mathbb{O}}$ is closed symmetric monoidal, with the (orthogonal) sphere spectrum S as unit object, a symmetric monoidal smash product $L \wedge M$ as monoidal pairing, and a function spectrum $F(M, N)$ as the closed structure.
- Furthermore, these data induce a closed symmetric monoidal structure on $\mathrm{Ho}(Sp^{\mathbb{O}})$.

$\mathscr{I}_*$-prespectra, symmetric spectra

- Orthogonal spectra were defined in [May80], under the name of $\mathscr{I}_*$-prespectra.
- Orthogonal ring spectra were defined even earlier in [May77], under the name of $\mathscr{I}_*$-prefunctors.
- The good properties mentioned above first became apparent with the introduction by Jeff Smith of symmetric spectra in 1994 ("Specters of symmetry", unpublished), and the unification of the two ideas in [MMSS01].
- Following Schwede index orthogonal spectra on a minimal subcategory of the category of all finite-dimensional inner product spaces and isometries used by Mandell–May–Schwede–Shipley.
- See also "model structure on orthogonal spectra" on https://ncatlab.org/nlab/show/HomePage .

Orthogonal groups

Definition

- Let $O(k)$ denote the group of orthogonal $k \times k$ matrices.
- It acts linearly on $\mathbb{R}^k$ and its one-point compactification $S^k = \mathbb{R}^k \cup \{\infty\}$.
- We consider $O(k) \times O(\ell)$ as a subgroup of $O(k + \ell)$, via the block sum of matrices.

We continue to work in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces and basepoint preserving maps.

Orthogonal spectra

Definition

An orthogonal spectrum M consists of a left $O(k)$-space M_k and a map

$$\sigma\colon M_k \wedge S^1 \longrightarrow M_{k+1}$$

for each $k \geq 0$, such that the composite

$$\sigma^\ell\colon M_k \wedge S^\ell \xrightarrow{\sigma} M_{k+1} \wedge S^{\ell-1} \xrightarrow{\sigma} \dots \xrightarrow{\sigma} M_{k+\ell-1} \wedge S^1 \xrightarrow{\sigma} M_{k+\ell}$$

is $O(k) \times O(\ell)$-equivariant for every $k, \ell \geq 0$.

Spectra as S-modules

- To justify this definition, we take a step back and define a closed symmetric monoidal category of orthogonal spaces.
- The sphere spectrum S is a commutative monoid in this category, and the category of right S-modules becomes the category of orthogonal spectra.
- This is closed symmetric monoidal because S is commutative.
- There is an analogous story for sequential spectra, which are the right S-modules in a category of sequential spaces, but in this case S is not commutative, so the module category does not inherit the monoidal structure.

The symmetric monoidal category $\mathbb{O}$

Definition

Let $\mathbb{O}$ be the topological category with objects the integers $k \geq 0$, and with morphism spaces

$$\mathbb{O}(k, \ell) = \begin{cases} O(k) & \text{for } k = \ell, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is symmetric monoidal, with unit object 0 and monoidal pairing

$$+ \colon (k, \ell) \longmapsto k + \ell$$
$$+ \colon (A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ,$$

where $A \in O(k)$ and $B \in O(\ell)$.

Definition (cont.)

The symmetry isomorphism

$$\tau \colon k + \ell \xrightarrow{\cong} \ell + k$$

equals the block permutation matrix

$$\chi_{k,\ell} = \begin{pmatrix} 0 & I_\ell \\ I_k & 0 \end{pmatrix}$$

where $I_k \in O(k)$ and $I_\ell \in O(\ell)$ are the identity matrices.

The object k may also be viewed as $\mathbb{R}^k$ with the standard inner product, in which case the monoidal pairing is the direct sum of inner product spaces, and the symmetry is the usual twist isomorphism $\mathbb{R}^k \oplus \mathbb{R}^\ell \cong \mathbb{R}^\ell \oplus \mathbb{R}^k$.

Orthogonal spaces

Definition

An orthogonal space is a continuous functor

$$M\colon \mathbb{O} \longrightarrow \mathscr{T} .$$

A map of orthogonal spaces is a continuous natural transformation $f\colon M \to N$. We write $\mathscr{T}^{\mathbb{O}}$ for the topological category of orthogonal spaces.

Explicitly, M maps each $k \geq 0$ to a space $M_k \in \mathscr{T}$, and for each $A \in O(k)$ we have a map $M(A)\colon M_k \to M_k$, which defines a continuous left group action

$$\begin{aligned} \lambda\colon O(k)_+ \wedge M_k &\longrightarrow M_k \\ (A, x) &\longmapsto Ax . \end{aligned}$$

Tensored and cotensored structure

Lemma

The category $\mathscr{T}^{\mathbb{O}}$ is tensored and cotensored over $\mathscr{T}$, by setting

$$\begin{aligned}(X \wedge M)_k &= X \wedge M_k \\ (M \wedge X)_k &= M_k \wedge X \\ F(X, M)_k &= \mathrm{Map}(X, M_k)\end{aligned}$$

for $M \in \mathscr{T}^{\mathbb{O}}$ and $X \in \mathscr{T}$, with the evident $O(k)$-actions. There are natural homeomorphisms

$$\mathrm{Map}(X, \mathscr{T}^{\mathbb{O}}(M, N)) \cong \mathscr{T}^{\mathbb{O}}(M \wedge X, N) \cong \mathscr{T}^{\mathbb{O}}(M, F(X, N)) \,.$$

All limits and colimits

Lemma

The category $\mathscr{T}^{\mathbb{O}}$ has all small limits and colimits, given for any diagram $\alpha \mapsto M_\alpha$ by

$$(\lim_\alpha M_\alpha)_k = \lim_\alpha (M_\alpha)_k$$
$$(\operatorname{colim}_\alpha M_\alpha)_k = \operatorname{colim}_\alpha (M_\alpha)_k \,,$$

with the evident $O(k)$-actions.

Closed symmetric monoidal structure

Lemma

The category $(\mathscr{T}^{\mathbb{O}}, U, \otimes, \mathrm{Hom})$ *of orthogonal spaces is closed symmetric monoidal, with unit object given by* $U_0 = S^0$ *and* $U_k = *$ *for* $k \geq 1$*, with monoidal pairing given by the Day convolution product*

$$(L \otimes M)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j$$

and with closed structure given by

$$\mathrm{Hom}(M, N)_i = \prod_{i+j=k} \mathrm{Map}(M_j, N_k)^{O(j)} \,.$$

Lemma (cont.)

The symmetry $\tau \colon L \otimes M \xrightarrow{\cong} M \otimes L$ maps

$$C \wedge x \wedge y \in O(k)_+ \wedge_{O(i) \times O(j)} L_i \wedge M_j$$

to

$$C\chi_{j,i} \wedge y \wedge x \in O(k)_+ \wedge_{O(j) \times O(i)} M_j \wedge L_i$$

for $i + j = k$. There is a natural homeomorphism

$$\mathscr{T}^{\mathbb{O}}(L \otimes M, N) \cong \mathscr{T}^{\mathbb{O}}(L, \mathrm{Hom}(M, N))\,.$$

Proof

The Day convolution can also be written as the topological colimit

$$(L \otimes M)_k = \operatorname*{colim}_{i,j,i+j\to k} L_i \wedge M_j \,.$$

where the indexing category is the left fiber $+/k$ of $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ at k.

The symmetry is well defined, because

$$C \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \wedge x \wedge y = C \wedge Ax \wedge By$$

is mapped to

$$C \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \chi_{j,i} \wedge y \wedge x = C\chi_{j,i} \wedge By \wedge Ax \,.$$

It would not be well-defined without the factor $\chi_{j,i}$.

Proof (cont.)

The adjunction homomorphism for L, M and N can be expanded to

$$\prod_k \mathscr{T}((L \otimes M)_k, N_k)^{O(k)} \cong \prod_{i,j} \mathscr{T}(L_i \wedge M_j, N_{i+j})^{O(i) \times O(j)}$$

$$\cong \prod_{i,j} \mathscr{T}(L_i, \mathrm{Map}(M_j, N_{i+j})^{O(j)})^{O(i)} \cong \prod_i \mathscr{T}(L_i, \mathrm{Hom}(M, N)_i)^{O(i)}$$

□

The orthogonal sphere spectrum

Definition

The (orthogonal) sphere spectrum S has underlying orthogonal space

$$S\colon k \longmapsto S_k = S^k = \mathbb{R}^k \cup \{\infty\}$$

based at ∞, with $A \in O(k)$ acting on S^k by its linear action on $\mathbb{R}^k$.

A commutative monoid

The following lemma makes the whole theory work.

Lemma

The sphere spectrum is a commutative monoid in orthogonal spaces, with unit $\eta\colon U \to S$ *given by the identity* $\eta_0\colon U_0 = S^0 = S_0$*, and with multiplication* $\mu\colon S \otimes S \to S$ *given by the* $O(k)$*-equivariant maps*

$$\mu_k\colon (S \otimes S)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j \longrightarrow S^k = S_k$$

that are left adjoint to the $O(i) \times O(j)$*-equivariant identifications*

$$S^i \wedge S^j \xrightarrow{\cong} S^k$$

for $i + j = k$*.*

Proof.

Associativity and unitality are clear, so the key thing to check is commutativity, which amounts to the commutativity of the diagrams

$$\begin{array}{ccc} \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j & \xrightarrow{\tau} & \bigvee_{j+i=k} O(k)_+ \wedge_{O(j)\times O(i)} S^j \wedge S^i \\ & \searrow^{\mu_k} \quad \swarrow^{\mu_k} & \\ & S^k & \end{array}$$

for all $k \geq 0$. This follows from the observation that $I_k \wedge x \wedge y$ for $x \in S^i$ and $y \in S^j$ maps via $\chi_{j,i} \wedge y \wedge x$ to $\chi_{j,i} \cdot (y \wedge x) = x \wedge y$ along the upper and right hand route, and maps directly to $x \wedge y$ along the left hand route. □

Level ℓ evaluation and free functors

Lemma

For each $\ell \geq 0$ the evaluation functor $Ev_\ell \colon \mathscr{T}^{\mathbb{O}} \longrightarrow \mathscr{T}$ given by $Ev_\ell(M) = M_\ell$ has left adjoint $G_\ell \colon \mathscr{T} \longrightarrow \mathscr{T}^{\mathbb{O}}$ given by

$$G_\ell(X)_k = \begin{cases} O(\ell)_+ \wedge X & \text{for } k = \ell, \\ * & \text{otherwise.} \end{cases}$$

Lemma

There is a natural isomorphism

$$G_i(X) \otimes G_j(Y) \cong G_{i+j}(X \wedge Y)$$

for $X, Y \in \mathscr{T}$ and $i, j \geq 0$.

Outline

Right S-modules

Definition ([MMSS01])

- An orthogonal spectrum M is a right S-module in orthogonal spaces.
- A map $f\colon M \to N$ of orthogonal spectra is a map of right S-modules.
- We write $Sp^{\mathbb{O}}$ for the topological category of orthogonal spectra.

Compatibility

- This agrees with the earlier definition, because the right S-action $\rho\colon M \otimes S \to M$ is given by $O(k) \times O(\ell)$-equivariant maps

$$\rho_{k,\ell}\colon M_k \wedge S^\ell \longrightarrow M_{k+\ell}$$

  that satisfy unitality and associativity, and which are therefore determined by the components $\sigma = \rho_{k,1}\colon M_k \wedge S^1 \to M_{k+1}$, for all $k \geq 0$.
- Conversely, the latter determine the right S-action when the equivariance condition for $\sigma^\ell\colon M_k \wedge S^\ell \to M_{k+\ell}$ is satisfied.

Tensored and cotensored structure

Lemma

The category $Sp^{\mathbb{O}}$ is tensored and cotensored over $\mathscr{T}$, by setting

$$
\begin{aligned}
(X \wedge M)_k &= X \wedge M_k \\
(M \wedge X)_k &= M_k \wedge X \\
F(X, M)_k &= \mathrm{Map}(X, M_k)
\end{aligned}
$$

for $M \in Sp^{\mathbb{O}}$ and $X \in \mathscr{T}$.

There are natural homeomorphisms

$$
\mathrm{Map}(X, Sp^{\mathbb{O}}(M, N)) \cong Sp^{\mathbb{O}}(M \wedge X, N) \cong Sp^{\mathbb{O}}(M, F(X, N)) .
$$

All limits and colimits

Lemma

The category $Sp^{\mathbb{O}}$ has all small limits and colimits, given for any diagram $\alpha \mapsto M_\alpha$ by

$$(\lim_{\alpha} M_\alpha)_k = \lim_{\alpha}(M_\alpha)_k$$
$$(\operatorname{colim}_{\alpha} M_\alpha)_k = \operatorname{colim}_{\alpha}(M_\alpha)_k \, .$$

Level ℓ forgetful and free functors

Lemma

The forgetful functor $U\colon Sp^{\mathbb{O}} \to \mathscr{T}^{\mathbb{O}}$ has left adjoint $L \mapsto L \otimes S$ and right adjoint $N \mapsto \mathrm{Hom}(S, N)$.

The evaluation functor $Ev_\ell\colon Sp^{\mathbb{O}} \to \mathscr{T}$ given by $Ev_\ell(M) = M_\ell$ has left adjoint $F_\ell\colon \mathscr{T} \to Sp^{\mathbb{O}}$ given by

$$F_\ell(X) = G_\ell(X) \otimes S\,,$$

so that

$$F_\ell(X)_k = \begin{cases} O(k)_+ \wedge_{O(k-\ell)} X \wedge S^{k-\ell} & \text{for } k \geq \ell, \\ * & \text{otherwise.} \end{cases}$$

Proof

The left adjoint of the composite $Ev_\ell U \colon Sp^{\mathbb{O}} \to \mathscr{T}^{\mathbb{O}} \to \mathscr{T}$ is the composite of left adjoints $G_\ell(-) \otimes S \colon \mathscr{T} \to \mathscr{T}^{\mathbb{O}} \to Sp^{\mathbb{O}}$.

This evaluates on X to the orthogonal spectrum $F_\ell(X)$ given at level k by

$$F_\ell(X)_k = (G_\ell(X) \otimes S)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} G_\ell(X)_i \wedge S^j \,,$$

which equals

$$O(k)_+ \wedge_{O(\ell)\times O(k-\ell)} O(\ell)_+ \wedge X \wedge S^{k-\ell} \cong O(k)_+ \wedge_{O(k-\ell)} X \wedge S^{k-\ell}$$

for $k \geq \ell$, and is $*$ for $k < \ell$. □

Orthogonal suspension spectra

Definition

The orthogonal suspension spectrum $\Sigma^\infty X = F_0 X$ of a space X is given by

$$(\Sigma^\infty X)_k = X \wedge S^k$$

with the standard $O(k)$-action on S^k, for each $k \geq 1$.

Orthogonal n-sphere spectra

Definition

For each integer n we define the orthogonal n-sphere spectrum S^n by

$$S^n = \begin{cases} F_0 S^n & \text{for } n \geq 0, \\ F_{-n} S^0 & \text{for } n < 0, \end{cases}$$

so that

$$(S^n)_k = \begin{cases} S^{n+k} & \text{for } n \geq 0, \\ O(k)_+ \wedge_{O(n+k)} S^{n+k} & \text{for } n < 0 \text{ and } n+k \geq 0, \\ * & \text{for } n+k < 0, \end{cases}$$

with the evident $O(k)$-action.

Underlying sequential spectra

Definition

Given an orthogonal spectrum $M = (M_k, \sigma)_k$, the underlying sequential spectrum $UM = (M_k, \sigma)_k$ is obtained by

- forgetting the $O(k)$-action on M_k, and
- ignoring the $O(k) \times O(\ell)$-equivariance condition on σ^ℓ.

Let

$$U\colon Sp^{\mathbb{O}} \longrightarrow Sp^{\mathbb{N}}$$

denote the forgetful functor.

Homotopy groups of orthogonal spectra

Definition

The homotopy groups $\pi_*(M) = (\pi_n(M))_n$ of an orthogonal spectrum M are the homotopy groups of its underlying sequential spectrum:

$$\pi_n(M) = \pi_n(UM) = \operatorname{colim}_k \pi_{n+k}(M_k)\,.$$

A map $f\colon M \to N$ of orthogonal spectra is a stable equivalence if the induced homomorphism $f_*\colon \pi_*(M) \to \pi_*(N)$ is an isomorphism, which is equivalent to asking that the underlying map of sequential spectra is a stable equivalence.

The prolongation to orthogonal spectra

Proposition ([MMSS01])

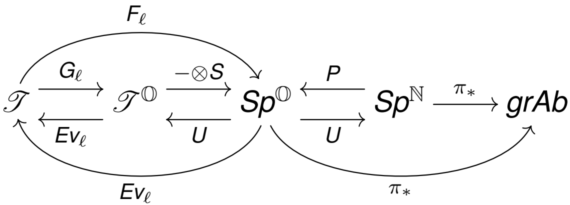
The forgetful functor $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ *admits a left adjoint, called the* *prolongation* *functor*

$$P\colon Sp^{\mathbb{N}} \longrightarrow Sp^{\mathbb{O}},$$

which satisfies

$$P(F_\ell^{\mathbb{N}} X) = F_\ell X$$

for each $\ell \geq 0$ *and space* X*, and which commutes with colimits.*



The stable equivalence $\lambda\colon F_1(S^1) \to S$

- $F_0(S^0) = S$ equals the sphere spectrum, while $F_1(S^1)$ has k-th space

$$F_1(S^1)_k = O(k)_+ \wedge_{O(k-1)} (S^1 \wedge S^{k-1}) \cong Th(\epsilon^1 \oplus \tau_{S^{k-1}})$$

  for each $k \geq 1$.
- The left adjoint of the identity $S^1 = Ev_1(S)$ is a map of orthogonal spectra

$$\lambda\colon F_1(S^1) \longrightarrow S$$

  given at levels $k \geq 1$ by the $O(k)$-equivariant extension

$$\lambda_k\colon O(k)_+ \wedge_{O(k-1)} S^k \longrightarrow S^k$$

  of the $O(k-1)$-action on $S^1 \wedge S^{k-1} \cong S^k$.
- This is $2(k-1)$-connected, so λ is a stable equivalence.

Stable equivalences $\lambda\colon F_{\ell+1}(S^{\ell+1}) \to F_\ell(S^\ell)$

More generally [MMSS01], the left adjoint

$$\lambda^\ell\colon F_{\ell+1}(S^1) \longrightarrow F_\ell(S^0)$$

of the canonical inclusion

$$S^1 \longrightarrow O(\ell+1)_+ \wedge_{O(1)} S^1 = F_\ell(S^0)_{\ell+1}$$

is a stable equivalence for each $\ell \geq 0$.

This is the feature of orthogonal spectra that allows us to define the stable equivalences as the π_*-isomorphisms.

Symmetric spectra

- In the parallel theory of symmetric spectra, based on the symmetric groups Σ_k in place of the orthogonal groups $O(k)$, the corresponding maps λ^ℓ are not π_*-isomorphisms.
- They must nonetheless be taken to be stable equivalences, hence invertible in the stable category, to ensure that the stably fibrant objects are the Ω-spectra.
- By working with orthogonal spectra we do not need to distinguish between stable equivalences and π_*-isomorphisms, which simplifies the exposition.

The stable category, redefined

Definition

The stable category $\mathrm{Ho}(Sp^{\mathbb{O}})$ is the localization

$$Sp^{\mathbb{O}} \xrightarrow{\iota} Sp^{\mathbb{O}}[\mathscr{W}^{-1}] = \mathrm{Ho}(Sp^{\mathbb{O}})$$

of the category of orthogonal spectra with respect to the subcategory $\mathscr{W}$ of stable equivalences.

For orthogonal spectra M and N, let

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{O}})(M, N)$$

denote the set of morphisms in the stable category from M to N.

Equivalence of definitions

The new definition does not conflict with our earlier usage, because of the following theorem. It follows from a Quillen equivalence of model categories, as we will soon explain.

Theorem

The functor $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ *preserves stable equivalences and induces an equivalence of categories*

$$U\colon \mathrm{Ho}(Sp^{\mathbb{O}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}}) .$$

Quillen cofibrations in $Sp^{\mathbb{O}}$

Definition ([MMSS01])

- Let I be the set of inclusions $i\colon S^{n-1}_+ \to D^n_+$ for $n \geq 0$.
- Let $FI = F^{\mathbb{O}} I$ be the set of maps of orthogonal spectra $F_\ell i\colon F_\ell S^{n-1}_+ \to F_\ell D^n_+$, for $\ell \geq 0$ and $n \geq 0$.
- A map $i\colon M \to N$ of orthogonal spectra is a relative cell spectrum if N is the colimit of a sequence of maps starting with M, where each map is obtained by cobase change from a sum of maps in FI.
- A map $i\colon M \to N$ in $Sp^{\mathbb{O}}$ is a Quillen cofibration if it is a retract of a relative cell spectrum.

Quillen cofibrant orthogonal spectra

- We say that an orthogonal spectrum N is a cell spectrum if $* \to N$ is a relative cell spectrum, and that N is Quillen cofibrant if $* \to N$ is a Quillen cofibration.
- If $q \colon M^c \stackrel{\sim}{\to} M$ is a stable equivalence of orthogonal spectra, and M^c is Quillen cofibrant, then we say that M^c is a cofibrant replacement for M.

Stable fibrations in $Sp^{\mathbb{O}}$

Definition ([MMSS01])

A map $p \colon M \to N$ of orthogonal spectra is a stable fibration if and only if the underlying map $Up \colon UM \to UN$ of sequential spectra is a stable fibration.

This means that $p_k \colon M_k \to N_k$ is a (non-equivariant) Serre fibration and

$$\begin{array}{ccc} M_k & \xrightarrow{\tilde{\sigma}} & \Omega M_{k+1} \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle \Omega p_{k+1}} \\ N_k & \xrightarrow{\tilde{\sigma}} & \Omega N_{k+1} \end{array}$$

is a (non-equivariant) weak homotopy pullback, for each $k \geq 0$.

Stably fibrant orthogonal spectra

- We say that an orthogonal spectrum M is stably fibrant if $M \to *$ is a stable fibration.
- If $j\colon N \xrightarrow{\sim} N^f$ is a stable equivalence of orthogonal spectra, and N^f is stably fibrant, then we say that N^f is a fibrant replacement for N.

The stable model structure on orthogonal spectra

Theorem

The category $Sp^{\mathbb{O}}$ of orthogonal spectra is a model category with respect to the classes of stable equivalences, Quillen cofibrations and stable fibrations.

- We refer to [MMSS01, Thm. 9.2] for the proof.
- The prolongation of a (relative) sequential cell spectrum is a (relative) orthogonal cell spectrum, with the same cell filtration.
- An orthogonal spectrum is stably fibrant if and only if it is an Ω-spectrum, i.e., if each adjoint structure map is a weak homotopy equivalence.

Quillen adjunctions

To compare the stable model structures on $Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$ we discuss Quillen adjunctions and Quillen equivalences.

Definition ([Hov99])

Let $\mathscr{C}$ and $\mathscr{D}$ be model categories, and let $F\colon \mathscr{C} \to \mathscr{D}$ be left adjoint to $G\colon \mathscr{D} \to \mathscr{C}$, so that there is a natural bijection

$$\mathscr{D}(F(X), Y) \cong \mathscr{C}(X, G(Y))\,.$$

We say that the adjoint pair (F, G) is a Quillen adjunction if

- F preserves cofibrations, and
- G preserves fibrations.

Total derived functors

Definition ([Hov99])

Given a Quillen adjunction (F, G), let the total left derived functor $LF\colon \operatorname{Ho}(\mathscr{C}) \to \operatorname{Ho}(\mathscr{D})$ be defined by

$$(LF)(X) = F(X^c)$$

where $X^c \sim X$ is a (functorially defined) cofibrant replacement.

Let the total right derived functor $RG\colon \operatorname{Ho}(\mathscr{D}) \to \operatorname{Ho}(\mathscr{C})$ be defined by

$$(RG)(Y) = G(Y^f)$$

where $Y \sim Y^f$ is a (functorially defined) fibrant replacement.

Total derived adjuction

Lemma ([Hov99])

Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and (F, G) a Quillen adjunction. Then

$$LF\colon \operatorname{Ho}(\mathscr{C}) \longrightarrow \operatorname{Ho}(\mathscr{D})$$

is left adjoint to

$$RG\colon \operatorname{Ho}(\mathscr{D}) \longrightarrow \operatorname{Ho}(\mathscr{C})$$

so that (LF, RG) form an adjoint pair.

Quillen equivalences

Definition ([Hov99])

A Quillen adjunction (F, G) is called a Quillen equivalence when, for each cofibrant X in $\mathscr{C}$ and each fibrant Y in $\mathscr{D}$,

- a map $f\colon F(X) \to Y$ is a weak equivalence in $\mathscr{D}$

if and only if

- its right adjoint $g\colon X \to G(Y)$ is a weak equivalence in $\mathscr{C}$.

Total derived equivalences

Proposition ([Hov99])

Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ be a Quillen adjunction. The following are equivalent:

1. (F, G) *is a Quillen equivalence.*
2. $LF\colon \operatorname{Ho}(\mathscr{C}) \to \operatorname{Ho}(\mathscr{D})$ *and* $RG\colon \operatorname{Ho}(\mathscr{D}) \to \operatorname{Ho}(\mathscr{C})$ *are adjoint equivalences of categories.*
3. *G reflects weak equivalences between fibrant objects and, for every cofibrant X in $\mathscr{C}$ the map $\eta\colon X \to G((FX)^f)$ is a weak equivalence.*

$Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$ are Quillen equivalent

Theorem ([MMSS01])

The adjoint pair (P, U)*, with* $P\colon Sp^{\mathbb{N}} \to Sp^{\mathbb{O}}$ *and* $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$*, is a Quillen equivalence. Hence*

$$LP\colon \mathrm{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{O}})$$
$$RU\colon \mathrm{Ho}(Sp^{\mathbb{O}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}})$$

are adjoint equivalences of categories.

Sketch proof.

$U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ preserves stable equivalences and stable fibrations, so (P, U) is a Quillen adjunction. Furthermore, U reflects stable equivalences and, for every Quillen cofibrant M in $Sp^{\mathbb{N}}$ the map $\eta\colon M \to UPM \sim U((PM)^f)$ is a stable equivalence. This verifies the third condition of the previous proposition. □

Outline

Tensor and Hom for right S-modules

- Since S is commutative, the category of right S-modules is isomorphic to the category of left S-modules.
- A right S-action $\rho\colon M \otimes S \to M$ determines a left S-action

$$\lambda\colon S \otimes M \xrightarrow{\tau} M \otimes S \xrightarrow{\rho} M$$

  and vice versa.
- Furthermore, we can form the tensor product $L \otimes_S M$ and the function object $\mathrm{Hom}_S(M, N)$ of right S-modules L, M and N, and these remain right S-modules.
- These define the smash product and function spectrum, respectively, for orthogonal spectra.

Closed symmetric monoidal structure on $Sp^{\mathbb{O}}$

Theorem

The category $Sp^{\mathbb{O}}$ of orthogonal spectra is closed symmetric monoidal, with

- *unit object the sphere spectrum S, and*
- *monoidal pairing the smash product given by the coequalizer*

$$L \otimes S \otimes M \underset{1 \otimes \lambda}{\overset{\rho \otimes 1}{\rightrightarrows}} L \otimes M \xrightarrow{\pi} L \otimes_S M = L \wedge M .$$

- *The symmetry $\tau \colon L \wedge M \xrightarrow{\cong} M \wedge L$ is induced by $\tau \colon L \otimes M \xrightarrow{\cong} M \otimes L$.*

Theorem (cont.)

- The closed structure is the function spectrum given by the equalizer

$$F(M,N) = \mathrm{Hom}_S(M,N) \xrightarrow{\iota} \mathrm{Hom}(M,N) \underset{\rho_*}{\overset{\rho^*}{\rightrightarrows}} \mathrm{Hom}(M \otimes S, N)\,.$$

- There is a natural homeomorphism

$$Sp^{\mathbb{O}}(L \wedge M, N) \cong Sp^{\mathbb{O}}(L, F(M,N))\,.$$

Explicit coequalizer

The coequalizer defining $(L \wedge M)_k$ can be expanded as follows.

$$\begin{array}{c} \bigvee_{a+b+c=k} O(k)_+ \wedge_{O(a)\times O(b)\times O(c)} L_a \wedge S^b \wedge M_c \\ \rho\wedge 1 \downarrow \; \downarrow 1\wedge\lambda \\ \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j \\ \downarrow \pi \\ (L \wedge M)_k \end{array}$$

The identifications for $b = 1$ generate the remaining ones, and set the composite

$$L_a \wedge S^1 \wedge M_c \xrightarrow{\sigma\wedge 1} L_{a+1} \wedge M_c \longrightarrow (L \wedge M)_{a+1+c}$$

equal to the composite

$$L_a \wedge S^1 \wedge M_c \xrightarrow{1\wedge\sigma\tau} \{\chi_{c,1}\}_+ \wedge L_a \wedge M_{c+1} \longrightarrow (L \wedge M)_{a+1+c}\,,$$

for all $a \geq 0$ and $c \geq 0$.

Explicit equalizer

The equalizer defining $F(M,N)_i$ can be expanded as below.

$$\begin{array}{c} F(M,N)_i \\ \Big\downarrow \iota \\ \prod_{i+j=k} \mathrm{Map}(M_j, N_k)^{O(j)} \\ \rho^* \Big\downarrow \Big\downarrow \rho_* \\ \prod_{i+a+b=c} \mathrm{Map}(M_a \wedge S^b, N_c)^{O(a)\times O(b)} \end{array}$$

The conditions for $b = 1$ generate the remaining ones, and demand that the composite

$$F(M,N)_i \longrightarrow \mathrm{Map}(M_{a+1}, N_{i+a+1}) \xrightarrow{\mathrm{Map}(\sigma,1)} \mathrm{Map}(M_a \wedge S^1, N_{i+a+1})$$

is equal to the composite

$$F(M,N)_i \longrightarrow \mathrm{Map}(M_a, N_{i+a}) \xrightarrow{\mathrm{Map}(-\wedge S^1,\sigma)} \mathrm{Map}(M_a \wedge S^1, N_{i+a+1})\,.$$

Compatibility of smash products

The smash product of spectra extends the smash product of spaces, in the following sense.

Lemma

There is a natural isomorphism

$$F_i(X) \wedge F_j(Y) \cong F_{i+j}(X \wedge Y).$$

for $X, Y \in \mathscr{T}$ *and* $i, j \geq 0$*.*

Proof.

$G_i(X) \otimes S \otimes_S G_j(Y) \otimes S \cong G_{i+j}(X \wedge Y) \otimes S$. □

Suspension vs. desuspension

Example

The isomorphism

$$S^1 \wedge S^{-1} = F_0 S^1 \wedge F_1 S^0 \cong F_1 S^1$$

followed by the stable equivalence $\lambda \colon F_1 S^1 \to S$ define a stable equivalence

$$S^1 \wedge S^{-1} \xrightarrow{\sim} S\,.$$

Extension of (co-)tensored to closed structure

The smash product of spectra also generalizes the smash product of a space with a spectrum.

Lemma

For $X \in \mathscr{T}$ and $M \in Sp^{\mathbb{O}}$ there are natural isomorphisms

$$\Sigma^\infty X \wedge M \cong X \wedge M$$
$$M \wedge \Sigma^\infty X \cong M \wedge X$$
$$F(\Sigma^\infty X, M) \cong F(X, M)\,.$$

π_* is lax symmetric monoidal

The homotopy group functor π_* is compatible with the smash product of orthogonal spectra and the tensor product of graded abelian groups, in a lax sense.

Proposition

There is a natural homomorphism

$$\cdot\colon \pi_*(L) \otimes \pi_*(M) \longrightarrow \pi_*(L \wedge M)$$

and a homomorphism

$$\mathbb{Z} \longrightarrow \pi_*(S)$$

that make $\pi_*\colon Sp^{\mathbb{O}} \longrightarrow grAb$ a lax symmetric monoidal functor.

Monoidal functors

- For π_* to be a lax monoidal functor means that the two evident composite pairings

$$\pi_*(L) \otimes \pi_*(M) \otimes \pi_*(N) \longrightarrow \pi_*(L \wedge M \wedge N)$$

  are equal, together with two unitality conditions, see [ML71, §XI.2].
- To be symmetric then means that the square

$$\begin{array}{ccc} \pi_*(L) \otimes \pi_*(M) & \xrightarrow{\cdot} & \pi_*(L \wedge M) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_*} \\ \pi_*(M) \otimes \pi_*(L) & \xrightarrow{\cdot} & \pi_*(M \wedge L) \end{array}$$

  commutes.
- Lax monoidal functors send monoids to monoids, and lax symmetric monoidal functors take commutative monoids to commutative monoids.

Sketch proof of monoidality

- Given $f\colon S^{\ell+i} \to L_i$ and $g\colon S^{m+j} \to M_j$ we form the composite

$$f{\cdot}g\colon S^{\ell+m+i+j} \xrightarrow{\tau'} S^{\ell+i+m+j} \cong S^{\ell+i}{\wedge}S^{m+j} \xrightarrow{f\wedge g} L_i{\wedge}M_j \xrightarrow{\iota_{i,j}} (L{\wedge}M)_{i+j}\,,$$

  where τ' is any map of degree $(-1)^{mi}$.
- If $m \geq 0$ we can let $\tau' = 1 \wedge \tau_{S^m,S^i} \wedge 1$, but we should also allow $m < 0$ in this construction.
- Then $[f \cdot g] \in \pi_{\ell+m+i+j}((L \wedge M)_{i+j})$ only depends on $[f]$ and $[g]$.

Sketch proof of monoidality (cont.)

- Furthermore, one can check that the stable class of $[f \cdot g]$ in $\pi_{\ell+m}(L \wedge M)$ only depends on the stable classes of $[f]$ in $\pi_\ell(L)$ and of $[g]$ in $\pi_m(M)$, so that we obtain a well-defined pairing
$$\pi_\ell(L) \times \pi_m(M) \longrightarrow \pi_{\ell+m}(L \wedge M)\,.$$
- This is bilinear, and hence factors uniquely through the tensor product, as asserted.
- One can also check that
$$\tau_*([f] \cdot [g]) = (-1)^{\ell m}[g] \cdot [f]$$
for $\ell = |f|$ and $m = |g|$, so that the lax monoidal functor π_* is symmetric. □

Graded ring and module structures

- The pairing
$$\cdot\colon \pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S \wedge S) \cong \pi_*(S)$$
makes $\pi_*(S)$ a graded commutative ring.
- For each orthogonal spectrum M the pairing
$$\cdot\colon \pi_*(M) \otimes \pi_*(S) \longrightarrow \pi_*(M \wedge S) \cong \pi_*(M)$$
makes $\pi_*(M)$ a right $\pi_*(S)$-module.
- The lax monoidal structure homomorphism factors uniquely through
$$\pi_*(L) \otimes_{\pi_*(S)} \pi_*(M) \longrightarrow \pi_*(L \wedge M)\,.$$

Orthogonal ring spectra

An (orthogonal) ring spectrum is an orthogonal spectrum E equipped with a multiplication $\mu\colon E \wedge E \to E$ and a unit $\eta\colon S \to E$ such that

$$
\begin{array}{ccc}
E \wedge E \wedge E & \xrightarrow{1\wedge\mu} & E \wedge E \\
{\scriptstyle \mu\wedge 1}\downarrow & & \downarrow{\scriptstyle \mu} \\
E \wedge E & \xrightarrow{\mu} & E
\end{array}
\qquad
\begin{array}{ccccc}
S \wedge E & \xrightarrow{\eta\wedge 1} & E \wedge E & \xleftarrow{1\wedge\eta} & E \wedge S \\
& {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\
& & E & &
\end{array}
$$

commute in $Sp^{\mathbb{O}}$.
It is commutative if the diagram

$$
\begin{array}{ccc}
E \wedge E & \xrightarrow{\tau} & E \wedge E \\
& {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\
& E &
\end{array}
$$

commutes.

Orthogonal module spectra

An (orthogonal) left E-module spectrum is an orthogonal spectrum M with a pairing $\lambda \colon E \wedge M \to M$ such that

$$
\begin{array}{ccc}
E \wedge E \wedge M & \xrightarrow{1 \wedge \lambda} & E \wedge M \\
\downarrow{\scriptstyle \mu \wedge 1} & & \downarrow{\scriptstyle \lambda} \\
E \wedge M & \xrightarrow{\lambda} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
S \wedge M & \xrightarrow{\eta \wedge 1} & E \wedge M \\
& {\scriptstyle \cong} \searrow & \downarrow{\scriptstyle \lambda} \\
& & M
\end{array}
$$

commute in $Sp^{\mathbb{O}}$.

There are similar definitions of right module spectra, bimodule spectra, algebra spectra and commutative algebra spectra.

S-modules and S-algebras

Example

- The sphere spectrum S is a commutative orthogonal ring spectrum.
- Any orthogonal spectrum M is an orthogonal (left and right) S-module spectrum.
- A (commutative) orthogonal ring spectrum E is a (commutative) S-algebra spectrum.

Ring and module spectra up to homotopy

Remark

There are also weaker notions, of ring spectra and module spectra up to homotopy, for which the structure maps μ, η and λ diagrams above are only required to exist in the stable category $\mathrm{Ho}(Sp^{\mathbb{O}})$, and (more significantly) the diagrams are only required to commute in that stable category.

Lemma

For each ring spectrum E (orthogonal, or up to homotopy), the homotopy groups $\pi_*(E)$ form a graded $\pi_*(S)$-algebra, which is graded commutative if E is commutative.

For each left E-module spectrum M (orthogonal, or up to homotopy) the homotopy groups $\pi_*(M)$ form a graded left $\pi_*(E)$-module.

Outline

Stable Homotopy Theory

- Smooth bordism and stable homotopy groups
- Sequential spectra
- Triangulated structure
- Spectral homology and cohomology
- Orthogonal spaces
- Orthogonal spectra
- Closed symmetric monoidal structure
- Monoidal model structure
- Multiplicative (co-)homology theories

The pushout-product map

Definition

Let $\mathscr{C}$ be a closed symmetric monoidal category, with monoidal pairing $\otimes$. Let $i\colon A \to X$ and $j\colon B \to Y$ be morphisms in $\mathscr{C}$.

Their pushout-product map

$$i \,\square\, j\colon A \otimes Y \cup_{A\otimes B} X \otimes B \longrightarrow X \otimes Y$$

is the canonical morphism from the pushout to the lower right hand corner in the following commutative square.

$$\begin{array}{ccc} A \otimes B & \xrightarrow{1\otimes j} & A \otimes Y \\ {\scriptstyle i\otimes 1}\downarrow & & \downarrow{\scriptstyle i\otimes 1} \\ X \otimes B & \xrightarrow{1\otimes j} & X \otimes Y \end{array}$$

The pushout-product and unit axioms

Definition

Let $\mathscr{C}$ be a closed symmetric monoidal category with a model structure. The pushout-product axiom requires that:

- If $i\colon A \to X$ and $j\colon B \to Y$ are cofibrations, then so is their pushout-product $i \,\square\, j$.
- If, furthermore, (i or) j is a weak equivalence, then so is $i \,\square\, j$.

The unit axiom requires that:

- The canonical map $q \otimes 1\colon U^c \otimes Y \to U \otimes Y$ is a weak equivalence for each cofibrant Y, where $q\colon U^c \sim U$ is a cofibrant replacement of the unit.

Remark

The unit axiom is automatically satisfied when the unit object U is cofibrant, which is the case for the stable model structure on $Sp^{\mathbb{O}}$.

Monoidal model categories

Definition

A monoidal model category $\mathscr{C}$ is a a closed symmetric monoidal category with a model structure satisfying the pushout-product axiom and the unit axiom.

Definition

Let $\mathscr{C}$ be a monoidal model category. The derived pairing

$$\otimes^L\colon \operatorname{Ho}(\mathscr{C}) \times \operatorname{Ho}(\mathscr{C}) \longrightarrow \operatorname{Ho}(\mathscr{C})$$

maps (X, Y) to $X^c \otimes Y^c$.

The derived closed structure

$$\operatorname{Hom}^R\colon \operatorname{Ho}(\mathscr{C})^{op} \times \operatorname{Ho}(\mathscr{C}) \longrightarrow \operatorname{Ho}(\mathscr{C})$$

maps (X, Y) to $\operatorname{Hom}(X^c, Y^f)$.

Monoidal structure on the homotopy category

Theorem ([Hov99, Thm. 4.3.2])

- *Let $\mathscr{C}$ be a monoidal model category.*
- *The*
  - *derived pairing $\otimes^L$,*
  - *unit object U,*
  - *symmetry τ and*
  - *derived closed structure Hom^R*

  define a closed symmetric monoidal structure on $\mathrm{Ho}(\mathscr{C})$.
- *In particular, there is an adjunction*

$$\mathrm{Ho}(\mathscr{C})(X \otimes^L Y, Z) \cong \mathrm{Ho}(\mathscr{C})(X, \mathrm{Hom}^R(Y, Z)) .$$

The stable model structure on $Sp^{\mathbb{O}}$ is monoidal

Theorem

The closed symmetric monoidal category $Sp^{\mathbb{O}}$ of orthogonal spectra, with the stable model structure, is a monoidal model category.

Corollary

The stable category $\mathrm{Ho}(Sp^{\mathbb{O}})$ *of orthogonal spectra is a closed symmetric monoidal category.*

It is traditional to write $L \wedge M$ for the total left derived smash product

$$L \wedge^L M = L^c \wedge M^c$$

and to write $F(M, N)$ for the total right derived function spectrum

$$F^R(M, N) = F(M^c, N^f)\,,$$

omitting the superscripts L and R from the notation.

(Co-)homology of spectra

Definition

Given spectra M and X, we define

$$M_n(X) = [S^n, M \wedge X]$$

$$\sigma\colon M_n(X) \xrightarrow{\cong} M_{1+n}(S^1 \wedge X)\,,$$

thereby extending the homology theory M over the functor $\Sigma^\infty\colon \mathscr{T} \to Sp^{\mathbb{O}}$ from based spaces to orthogonal spectra.

Definition

Given spectra M and X, we define

$$M^n(X) = [X, S^n \wedge M]$$

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thereby extending the cohomology theory M over the functor $\Sigma^\infty\colon \mathscr{T} \to Sp^{\mathbb{O}}$ from based spaces to orthogonal spectra.

Sketch proof of monoidality theorem

- We must verify the pushout-product axiom.
- To verify the first part, it suffices to consider pairs of maps

$$i\colon F_k S^{n-1}_+ \longrightarrow F_k D^n_+$$
$$j\colon F_\ell S^{m-1}_+ \longrightarrow F_\ell D^m_+$$

  in the set FI generating the relative cell spectra.
- The pushout-product map $i \,\square\, j$ then has the form

$$F_{k+\ell}(S^{n-1} \times D^m \cup_{S^{n-1} \times S^{m-1}} D^n \times S^{m-1})_+ \longrightarrow F_{k+\ell}(D^n \times D^m)_+ ,$$

  which is a Quillen cofibration.
- The second part is proved in [MMSS01, Prop. 12.6], and relies on the proposition below.

□

Cofibrant spectra are flat

Proposition ([MMSS01, Prop. 12.3])

For any Quillen cofibrant orthogonal spectrum L, the functor

$$L \wedge - \colon M \longmapsto L \wedge M$$

preserves stable equivalences.

This is first proved for $L = F_\ell S^n$, from which the general case follows.

Informally, the proposition says that Quillen cofibrant orthogonal spectra are flat.

Additivity of connectivity

Proposition

If L is ℓ-connective and M is m-connective, with L (or M) Quillen cofibrant, then $L \wedge M$ is $(\ell + m)$-connective and

$$\cdot \colon \pi_\ell(L) \otimes \pi_m(M) \xrightarrow{\cong} \pi_{\ell+m}(L \wedge M)$$

is an isomorphism.

Sketch proof

- There exists a stable equivalence $M^c \sim M$, where M^c is built from $*$ by attaching n-cells of the form (CS^{n-1}, S^{n-1}) with $n \geq m$, and $L \wedge M^c \sim L \wedge M$.
- There is also a stable equivalence $L^c \sim L$, where L^c is built from n-cells with $n \geq \ell$, and $L^c \wedge M^c \sim L \wedge M^c$.
- Here $L^c \wedge M^c$ is built from n-cells with $n \geq \ell + m$, which implies that $L^c \wedge M^c$ is $(\ell + m)$-connective.
- A more precise account of the m- and $(m+1)$-cells of M^c, and of the ℓ- and $(\ell+1)$-cells of L^c, shows that the $(m+\ell)$- and $(m+\ell+1)$-cells of $M^c \wedge L^c$ give a presentation of $\pi_{m+\ell}(M^c \wedge L^c)$ as the tensor product $\pi_m(M^c) \otimes \pi_\ell(L^c)$.

$\square$

Tensor triangulated structure

Since the stable model structure on $Sp^{\mathbb{O}}$ is both

- monoidal and
- stable,

the homotopy category $\mathrm{Ho}(Sp^{\mathbb{O}})$ is both

- closed symmetric monoidal and
- triangulated,

and several compatibility conditions between the latter structures are satisfied.

Two of these are given in [HPS97] and [May01]. Let $\Sigma X = S^1 \wedge X$.

1. The composite

$$\Sigma S^1 = S^1 \wedge S^1 \xrightarrow{\tau} S^1 \wedge S^1 = \Sigma S^1$$

is multiplication by -1.

Tensor triangulated structure (cont.)

2. For each distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and object W in $\mathrm{Ho}(Sp^{\mathbb{O}})$, the following triangles are distinguished.

$$W \wedge X \xrightarrow{1\wedge f} W \wedge Y \xrightarrow{1\wedge g} W \wedge Z \xrightarrow{1\wedge h} \Sigma(W \wedge X)$$

$$X \wedge W \xrightarrow{f\wedge 1} Y \wedge W \xrightarrow{g\wedge 1} Z \wedge W \xrightarrow{h\wedge 1} \Sigma(X \wedge W)$$

$$F(W,X) \xrightarrow{F(1,f)} F(W,Y) \xrightarrow{F(1,g)} F(W,Z) \xrightarrow{F(1,h)} \Sigma F(W,X)$$

$$\Sigma^{-1} F(X,W) \xrightarrow{-F(h,1)} F(Z,W) \xrightarrow{F(g,1)} F(Y,W) \xrightarrow{F(f,1)} F(X,W)$$

Further compatability

In (2) we use fixed identifications $W \wedge \Sigma X \cong \Sigma(W \wedge X)$, $\Sigma X \wedge W \cong \Sigma(X \wedge W)$, $F(W, \Sigma X) \cong \Sigma F(W, X)$ and $F(\Sigma X, W) \cong \Sigma^{-1} F(X, W)$, coming from the closed structure.

May [May01] gives three more compatibility conditions that are also satisfied, but these are not the full story, as explained by Keller and Neeman [KN02]. We will make use of the following Leibniz rule for the connecting homomorphism in homotopy.

Pushout smash product

Let $i\colon A \to X$ and $j\colon B \to Y$ be Quillen cofibrations and let

$$W = A \wedge Y \cup_{A \wedge B} X \wedge B$$

be the pushout in the commutative square

$$\begin{array}{ccc} A \wedge B & \xrightarrow{i \wedge 1} & X \wedge B \\ {\scriptstyle 1 \wedge j}\downarrow & & \downarrow{\scriptstyle 1 \wedge j} \\ A \wedge Y & \xrightarrow{i \wedge 1} & X \wedge Y\,. \end{array}$$

By the pushout-product axiom the canonical map $i \,\square\, j\colon W \to X \wedge Y$ is a Quillen cofibration, and we have isomorphisms

$$\frac{X \wedge Y}{W} \cong \frac{X}{A} \wedge \frac{Y}{B} \quad \text{and} \quad \frac{W}{A \wedge B} \cong A \wedge \frac{Y}{B} \vee \frac{X}{A} \wedge B\,.$$

A Leibniz rule in homotopy

Proposition

Let $x \in \pi_n(X/A)$ *and* $y \in \pi_m(Y/B)$. *Then*

$$\partial(x \cdot y) = \partial x \cdot y + (-1)^n x \cdot \partial y$$

in

$$\pi_{-1+n+m}(W/(A \wedge B)) \\ \cong \pi_{-1+n+m}(A \wedge (Y/B)) \oplus \pi_{-1+n+m}((X/A) \wedge B)\,.$$

Here $\partial x \in \pi_{-1+n}(A)$ and $\partial y \in \pi_{-1+m}(B)$ are given by the composites

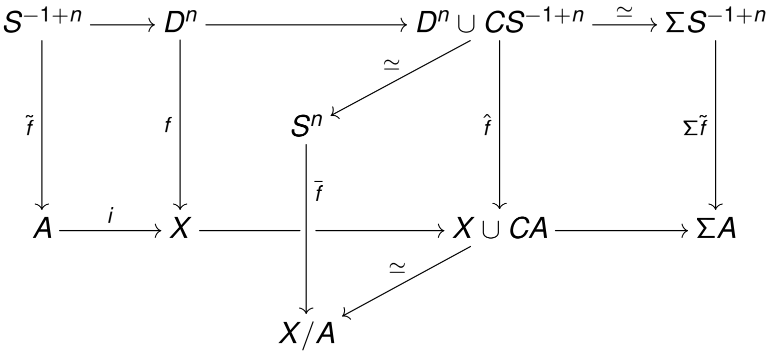
$$\pi_n(X/A) \xleftarrow{\cong} \pi_n(X, A) \xrightarrow{\partial} \pi_{-1+n}(A)$$
$$\pi_m(Y/B) \xleftarrow{\cong} \pi_m(Y, B) \xrightarrow{\partial} \pi_{-1+m}(B)\,.$$

Leibniz rule (cont.)

Moreover, $\partial(x \cdot y)$ is calculated using the following diagram.

$$
\begin{array}{ccc}
\pi_n(X, A) \otimes \pi_m(Y, B) & \xrightarrow{\cong} & \pi_n(X/A) \otimes \pi_m(Y/B) \\
\Big\downarrow{\scriptstyle \cdot} & & \Big\downarrow{\scriptstyle \cdot} \\
\pi_{n+m}(X \wedge Y, W) & \xrightarrow{\cong} & \pi_{n+m}(X/A \wedge Y/B) \\
\Big\downarrow{\scriptstyle \partial} & & \\
\pi_{-1+n+m}(W, A \wedge B) & \longrightarrow & \pi_{-1+n+m}(W/(A \wedge B)) \\
 & & \Big\uparrow{\scriptstyle \cong} \\
 & & \pi_{-1+n+m}(A \wedge (Y/B)) \oplus \pi_{-1+n+m}((X/A) \wedge B)
\end{array}
$$

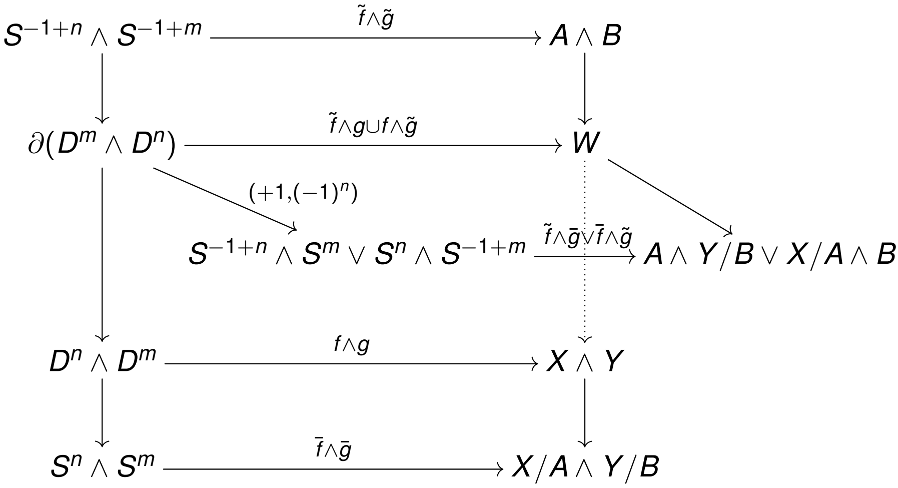
Sketch proof



$$f \wedge g \colon D^n \wedge D^m \longrightarrow X \wedge Y$$

$$\tilde{f} \wedge g \cup f \wedge \tilde{g} \colon S^{-1+n} \wedge D^m \cup D^n \wedge S^{-1+m} \longrightarrow W\,.$$

Sketch proof (cont.)



□

Outline

Orthogonal Eilenberg–MacLane spectra

To exhibit the Eilenberg–MacLane spectra HG as orthogonal spectra, we use a functorial construction of Eilenberg–MacLane spaces due to McCord [McC69, §5, §6], who works in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces.

Definition

Let G be a commutative topological monoid and X a based space. Let

$$B(G,X) = \coprod_{j\geq 0} (G \times X)^j / \sim$$

be the space of formal sums

$$u = \sum_{i=1}^{j} (g_i, x_i)$$

with $g_i \in G$ and $x_i \in X$, subject to the relations

$$\begin{aligned}(g', x') + (g'', x'') &= (g'', x'') + (g', x') \\ (g', x) + (g'', x) &= (g' + g'', x) \\ (g, x_0) &= 0\,,\end{aligned}$$

where $x_0 \in X$ is the base point.

Topology on $B(G, X)$

- We think of u as a finite set of points in X, labeled with elements in G, ignoring any label at the base point.
- We give the image $B_j(G, X) \subset B(G, X)$ of $(G \times X)^j$ the quotient topology, and give $B(G, X) = \bigcup_{j \geq 0} B_j(G, X)$ the (weak) colimit topology.
- In particular, there is a closed inclusion

$$G_+ \wedge X = B_1(G, X) \longrightarrow B(G, X) .$$

- The construction is clearly natural in the commutative topological monoid G and the based space X.

Tensored and monoidal structure

- There is a natural map

$$\rho\colon B(G,X)\wedge Y\longrightarrow B(G,X\wedge Y)$$

mapping $(\sum_i (g_i,x_i))\wedge y$ to

$$\sum_i (g_i, x_i\wedge y)\,,$$

so homotopic maps $X\to X'$ induce homotopic maps $B(G,X)\to B(G,X')$.

- There is also a natural pairing

$$\iota\colon B(G,X)\wedge B(H,Y)\longrightarrow B(G\otimes H, X\wedge Y)$$

sending $\sum_i (g_i,x_i)\wedge \sum_j (h_j,y_j)$ to

$$\sum_{i,j} (g_i\otimes h_j)x_i\wedge y_j\,.$$

Cofiber sequences become fiber sequences

McCord [McC69, Thm. 8.8] proves that $B(G, -)$ maps (some) cofiber sequences to fiber sequences.

Theorem

If G is a discrete abelian group, and (X, A) is a based triangulable pair, then

$$B(G, X) \longrightarrow B(G, X/A)$$

is a numerable principal $B(G, A)$-bundle. In particular, it is a Hurewicz fibration with fiber $B(G, A)$.

Spheres become Eilenberg–MacLane spaces

Corollary

Let G be a discrete abelian group. Then $B(G, D^n)$ is contractible and $B(G, S^n)$ is a $K(G, n)$-space, for each $n \geq 0$.

Proof.

- The homotopy equivalence $D^n \to *$ induces a homotopy equivalence $B(G, D^n) \simeq B(G, *) = *$.
- The Hurewicz fibration

$$B(G, S^{n-1}) \longrightarrow B(G, D^n) \longrightarrow B(G, S^n)$$

  exhibits $B(G, S^n)$ as a (connected) delooping of $B(G, S^{n-1})$.
- Since $B(G, S^0) \cong G$ is a $K(G, 0)$-space, it follows by induction that $B(G, S^n)$ is a $K(G, n)$-space.

□

Orthogonal Eilenberg–MacLane spectra

For each (discrete) abelian group G let the Eilenberg–MacLane spectrum HG be the orthogonal spectrum with

$$(HG)_k = B(G, S^k)$$

having the $O(k)$-action induced by the linear $O(k)$-action on $S^k = \mathbb{R}^k \cup \{\infty\}$, and with structure maps

$$\sigma\colon (HG)_k \wedge S^1 = B(G, S^k) \wedge S^1 \xrightarrow{\rho} B(G, S^k \wedge S^1) \cong (HG)_{k+1}$$

for each $k \geq 0$.

This is an Ω-spectrum with $\pi_n(HG) = \begin{cases} G & \text{for } n = 0, \\ 0 & \text{otherwise} \end{cases}$.

Hence there are natural isomorphisms

$$HG_*(X) \cong \tilde{H}_*(X; G) \quad \text{and} \quad HG^*(X) \cong \tilde{H}^*(X; G)$$

for all based X of the homotopy type of a CW complex.

H is lax symmetric monoidal

Let $\mu\colon G' \otimes G'' \to G$ be a pairing of (discrete) abelian groups. The induced maps

$$\mu\colon (HG')_i \wedge (HG'')_j = B(G', S^i) \wedge B(G'', S^j) \xrightarrow{\iota} B(G' \otimes G'', S^i \wedge S^j) \xrightarrow{\mu} B(G, S^{i+j}) = (HG)_{i+j}$$

are $O(i) \times O(j)$-equivariant and compatible with the right S-module structures, hence induce a map

$$\mu\colon HG' \wedge HG'' \longrightarrow HG$$

of orthogonal spectra.

Eilenberg–MacLane ring and module spectra

- If $\mu\colon R \otimes R \to R$ is a ring multiplication, then

$$\mu\colon HR \wedge HR \longrightarrow HR$$

  makes HR an orthogonal ring spectrum, which is commutative if R is commutative.
- If $\lambda\colon R \otimes N \to N$ is a left R-module action, then

$$\lambda\colon HR \wedge HN \longrightarrow HN$$

  makes HN an orthogonal left HR-module spectrum.
- In particular, $H\mathbb{Z}$ is a commutative orthogonal ring spectrum, and HG is an orthogonal (left and right) $H\mathbb{Z}$-module spectrum, for each abelian group G.

Homology smash product

- Let $\mu \colon L \wedge M \to N$ be a map of orthogonal spectra.
- For based spaces or spectra X and Y the homology smash product pairing

$$L_*(X) \otimes M_*(Y) \xrightarrow{\wedge} N_*(X \wedge Y)$$

is given by the composition

$$[S^\ell, L \wedge X] \otimes [S^m, M \wedge Y] \xrightarrow{\cdot} [S^\ell \wedge S^m, L \wedge X \wedge M \wedge Y]$$
$$\xrightarrow{1\wedge\tau\wedge 1} [S^{\ell+m}, L \wedge M \wedge X \wedge Y] \xrightarrow{\mu\wedge 1\wedge 1} [S^{\ell+m}, N \wedge X \wedge Y] \,.$$

Cohomology smash and cup products

- The cohomology smash product pairing

$$L^*(X) \otimes M^*(Y) \xrightarrow{\wedge} N^*(X \wedge Y)$$

is given by the composition

$$[X, S^\ell \wedge L] \otimes [Y, S^m \wedge M] \xrightarrow{\cdot} [X \wedge Y, S^\ell \wedge L \wedge S^m \wedge M]$$
$$\xrightarrow{1 \wedge \tau \wedge 1} [X \wedge Y, S^\ell \wedge S^m \wedge L \wedge M] \xrightarrow{1 \wedge 1 \wedge \mu} [X \wedge Y, S^{\ell+m} \wedge N] \,.$$

- If $X = Y$ are spaces, the cup product pairing is the composition

$$\cup \colon L^*(X) \otimes M^*(X) \xrightarrow{\wedge} N^*(X \times X) \xrightarrow{\Delta^*} N^*(X) \,.$$

Further products

- There are also slant and cap products, including Kronecker pairings, satisfying various associative laws.
- The interaction between the closed symmetric monoidal and the triangulated structures lead to compatibility of products and connecting homomorphisms.
- The Leibniz rule for homotopy leads to Leibniz rules for (co-)homology, and to pairings of Atiyah–Hirzebruch spectral sequences.

Operations in E-cohomology

We now follow Adams [Ada69] to discuss Steenrod operations and cooperations for (generalized) E-cohomology and E-homology.

Proposition

Let E be a spectrum. The composition pairing

$$\phi\colon E^*(E) \otimes E^*(E) \longrightarrow E^*(E)$$

makes $E^*(E)$ a graded ring. For each spectrum Y the composition pairing

$$\lambda\colon E^*(E) \otimes E^*(Y) \longrightarrow E^*(Y)$$

makes $E^*(Y)$ a graded left $E^*(E)$-module.

The classical example

When $E = H = H\mathbb{F}_p$, this algebra equals the mod p Steenrod algebra

$$A \cong H^*(H)\,,$$

and the action

$$\lambda\colon H^*(H) \otimes H^*(Y) \longrightarrow H^*(Y)$$

agrees with the natural left A-action

$$\lambda\colon A \otimes H^*(Y;\mathbb{F}_p) \longrightarrow H^*(Y;\mathbb{F}_p)\,.$$

An isomorphism for flat $\pi_*(E)$-modules

Proposition

- *Let E be a ring spectrum (orthogonal, or up to homotopy).*
- *The map*

$$X \wedge E \wedge E \wedge Y \xrightarrow{1 \wedge \mu \wedge 1} X \wedge E \wedge Y$$

  induces a pairing

$$m \colon \pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y)$$

  for all spectra X and Y.
- *If X and E are such that $\pi_*(X \wedge E)$ is flat as a right $\pi_*(E)$-module then m is an isomorphism for all Y.*

Proof

- The composite

$$\pi_*(X \wedge E) \otimes \pi_*(E \wedge Y) \xrightarrow{\cdot} \pi_*(X \wedge E \wedge E \wedge Y) \xrightarrow{1 \wedge \mu \wedge 1} \pi_*(X \wedge E \wedge Y)$$

  equalizes the two homomorphisms from $\pi_*(X \wedge E) \otimes \pi_*(E) \otimes \pi_*(E \wedge Y)$ by (homotopy) associativity.
- If $\pi_*(X \wedge E)$ is flat over $\pi_*(E)$, then $\pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y)$ and $\pi_*(X \wedge E \wedge Y)$ both define homology theories for CW complexes Y, or cell spectra Y.
- Since m is an isomorphism for $Y = S$, it follows by induction that it is an isomorphism for all cell spectra Y, hence for all Y when the smash products are interpreted in the total left derived sense. □

A Hopf algebra with two units

Suppose that E is a commutative ring spectrum (orthogonal, or up to homotopy), and that $E_*E = \pi_*(E \wedge E)$ is flat as a right (or left) module over $E_* = \pi_*(E)$. Let

$$\eta_L = (\eta \wedge 1)_* \colon E_* = \pi_*(S \wedge E) \longrightarrow \pi_*(E \wedge E) = E_*E$$
$$\eta_R = (1 \wedge \eta)_* \colon E_* = \pi_*(E \wedge S) \longrightarrow \pi_*(E \wedge E) = E_*E$$
$$\phi = \wedge \colon E_*E \otimes E_*E \longrightarrow E_*E$$
$$\epsilon = \mu_* \colon E_*E = \pi_*(E \wedge E) \longrightarrow \pi_*(E) = E_*$$
$$\chi = \tau_* \colon E_*E = \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E) = E_*E$$

denote the left unit, right unit, product, counit and conjugation. Furthermore, let

$$\psi = (1 \wedge \eta \wedge 1)_* \colon E_*E = \pi_*(E \wedge S \wedge E) \longrightarrow \pi_*(E \wedge E \wedge E) \overset{m}{\cong} E_*E \otimes_{E_*} E_*E$$

define the coproduct.

Cooperations

Proposition

- *The pair* (E_*E, E_*)*, with the structure maps above, form a graded Hopf algebroid.*
- *If* $\eta_L = \eta_R$ *then* E_*E *is a graded commutative Hopf algebra over* E_**.*
- *When* $E = H = H\mathbb{F}_p$*, this Hopf algebra over* $E_* = \mathbb{F}_p$ *equals the dual Steenrod algebra*

$$A_* \cong H_*(H)\,.$$

Hopf algebroids

- The terminology “Hopf algebroid” is due to Haynes Miller.
- It means that E_* and E_*E are graded commutative rings that corepresent the object set and morphism set of a functor from graded commutative rings to small groupoids, i.e., to small categories in which each morphism is invertible.
- The homomorphisms η_L and η_R corepresent the target (= codomain) and source (= domain), ϵ corepresents the identity morphism, ψ corepresents composition, and χ expresses the existence of inverses.

Cooperations in E-homology

For each space or spectrum Y, let

$$\nu = (1 \wedge \eta \wedge 1)_* \colon E_*(Y) = \pi_*(E \wedge S \wedge Y) \longrightarrow \pi_*(E \wedge E \wedge Y) \overset{m}{\cong} E_*E \otimes_{E_*} E_*(Y)$$

define the E_*E-coaction on $E_*(Y)$.

Lemma

- *The coaction ν makes $E_*(Y)$ a left E_*E-comodule.*
- *When $E = H = H\mathbb{F}_p$, the coaction*

$$\nu \colon H_*(Y) \longrightarrow H_*H \otimes_{H_*} H_*(Y)$$

agrees with the natural left A_*-coaction

$$\nu \colon H_*(Y; \mathbb{F}_p) \longrightarrow A_* \otimes H_*(Y; \mathbb{F}_p) .$$

Note that this construction does not presume that $H_*(Y; \mathbb{F}_p)$ is of finite type.

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