ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER X: THE ADAMS SPECTRAL SEQUENCE (INCOMPLETE)

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1. The E-based Adams spectral sequence

We turn to the sequence of spectra $Y_\star$ from Example 1.3 of Chapter 8, and its associated spectral sequence, namely the E-based Adams spectral sequence. Let Y be any orthogonal spectrum, let (E, η, ϕ) be a ring spectrum up to homotopy, and let $\bar{E} = C\eta$, so that we have a homotopy cofiber sequence

$$\Sigma^{-1}\bar{E} \longrightarrow S \xrightarrow{\eta} E \longrightarrow \bar{E} \tag{1.1}$$

(with $I = \Sigma^{-1}\bar{E}$ and $\Sigma I = E$ in the notation of the cited example). We let $Y_0 = Y$ and iteratively define $Y_{s+1} = \Sigma^{-1}\bar{E} \wedge Y_s$ for $s \geq 0$, so that we have homotopy cofiber sequences

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} E \wedge Y_s \xrightarrow{\gamma} \Sigma Y_{s+1}$$

given by smashing (1.1) with Y_s. In particular $Y_{s,1} = C\alpha = E \wedge Y_s$ and $\beta = \eta \wedge \mathrm{id}$. We also let $Y = Y_s$ for $s < 0$, so that

$$Y_s = \begin{cases} (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y & \text{for } s \geq 0, \\ Y & \text{for } s \leq 0, \end{cases}$$

and

$$Y_{s,1} = \begin{cases} E \wedge (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y & \text{for } s \geq 0, \\ * & \text{for } s < 0. \end{cases}$$

Hence the chain of homotopy cofiber sequences

$$\begin{tikzcd}
\dots \arrow[r] & Y_3 \arrow[r, "\alpha"] & Y_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_0 \arrow[r, "="] \arrow[d, "\beta"] & \dots \\
& & Y_{2,1} \arrow[ul, dashed, "\gamma"] & Y_{1,1} \arrow[ul, dashed, "\gamma"] & Y_{0,1} \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

appears as follows.

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\bar{E})^{\wedge 3} \wedge Y \arrow[r, "\alpha"] & (\Sigma^{-1}\bar{E})^{\wedge 2} \wedge Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & \Sigma^{-1}\bar{E} \wedge Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y \arrow[r, "="] \arrow[d, "\beta"] & \dots \\
& & E \wedge (\Sigma^{-1}\bar{E})^{\wedge 2} \wedge Y \arrow[ul, dashed, "\gamma"] & E \wedge \Sigma^{-1}\bar{E} \wedge Y \arrow[ul, dashed, "\gamma"] & E \wedge Y \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

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Replacing Y_s and $Y_{s,1}$ by $\Sigma^s Y_s$ and $\Sigma^s Y_{s,1}$, respectively, we can also draw this as follows.

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & \bar{E}^{\wedge 3} \wedge Y \arrow[r, dashed, "\alpha"] & \bar{E}^{\wedge 2} \wedge Y \arrow[r, dashed, "\alpha"] \arrow[d, "\beta"] & \bar{E} \wedge Y \arrow[r, dashed, "\alpha"] \arrow[d, "\beta"] & Y \arrow[d, "\beta"] \\
& & E \wedge \bar{E}^{\wedge 2} \wedge Y \arrow[ul, "\gamma"] & E \wedge \bar{E} \wedge Y \arrow[ul, "\gamma"] & E \wedge Y \arrow[ul, "\gamma"]
\end{tikzcd}$$

We think of these diagrams as spectrum level resolutions of Y by spectra of the form $E \wedge Z$ for some spectrum Z, which in a sense are injective to the eyes of E-homology, or (in good cases) projective to the eyes on E-cohomology.

Applying homotopy we obtain an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y_3) \arrow[r, "\alpha_*"] & \pi_*(Y_2) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_1) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_0) = \pi_*(Y) \arrow[d, "\beta_*"] \\
& & \pi_*(Y_{2,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{1,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{0,1}) \arrow[ul, dashed, "\gamma_*"]
\end{tikzcd}$$

with

$$\pi_*(Y_s) = \pi_*((\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y)$$
$$\pi_*(Y_{s,1}) = \pi_*(E \wedge (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y)$$

for all $s \geq 0$. The associated spectral sequence is the E-based Adams spectral sequence, which is concentrated in the half-plane $s \geq 0$. Clearly $Y = Y_0 \simeq Y_{-\infty} = \operatorname{hocolim}_s Y_s$, so we take $G = \pi_*(Y)$ as the abutment of the spectral sequence, writing

$$\mathcal{E}_1^{s,*} = \pi_*(Y_{s,1}) \Longrightarrow_s \pi_*(Y)\,.$$

However, $Y_\infty = \operatorname{holim}_s Y_s$ will not generally be trivial, so (conditional) convergence is not guaranteed. Following Bousfield, one way to achieve this is to replace Y by its E-nilpotent completion $Y^\wedge_E$, defined as the homotopy cofiber of $Y_\infty \to Y$, and the convergence problem for the Adams spectral sequence is then to recognize this completion.

In order to obtain an algebraic description of the E-based Adams $\mathcal{E}_1$- and $\mathcal{E}_2$-term, we hereafter assume that E is homotopy commutative and flat, so that E_*E is flat as a (left or right) E_*-module. The pair (E_*, E_*E) is then a Hopf algebroid, and there is a natural left E_*E-coaction

$$\nu \colon E_*(X) \longrightarrow E_*E \otimes_{E_*} E_*(X)$$

for each spectrum X. Let $\operatorname{Hom}^t_{E_*E}(E_*, E_*(X))$ denote the abelian group of E_*E-comodule homomorphisms $\Sigma^t E_* = E_*(S^t) \to E_*(X)$, for each $t \in \mathbb{Z}$, and write $\operatorname{Hom}_{E_*E}(E_*, E_*(X))$ for the resulting graded abelian group.

Lemma 1.1. *The natural homomorphism*

$$\pi_*(X) \xrightarrow{d} \operatorname{Hom}_{E_*E}(E_*, E_*(X))$$
$$[f \colon S^t \to X] \mapsto f_* \colon E_*(S^t) \to E_*(X)$$

is an isomorphism whenever $X \simeq E \wedge Z$ for some spectrum Z.

Proof. There is an equalizer diagram

$$\begin{tikzcd}
\operatorname{Hom}_{E_*E}(E_*, E_*(X)) \arrow[r, "\iota"] & E_*(X) \arrow[r, shift left, "\nu"] \arrow[r, shift right, "\eta_R \otimes \mathrm{id}"'] & E_*E \otimes_{E_*} E_*(X)\,,
\end{tikzcd}$$

where ι evaluates a homomorphism at $1 \in E_*$ and $\eta_R \otimes \mathrm{id}$ maps x to $1 \otimes x$. Hence $\mathrm{Hom}_{E_*E}(E_*, E_*(X)) = E_* \,\square_{E_*E}\, E_*(X) = PE_*(X)$ is the subgroup of E_*E-comodule primitives in $E_*(X)$. The fork diagram

$$\begin{tikzcd} \pi_*(X) \arrow[r, "\iota d"] & E_*(X) \arrow[r, shift left, "\nu"] \arrow[r, shift right, "\eta_R \otimes \mathrm{id}"'] & E_*E \otimes_{E_*} E_*(X) \,, \end{tikzcd}$$

can be rewritten as

$$\begin{tikzcd} \pi_*(X) \arrow[r, "\eta\wedge\mathrm{id}"] & \pi_*(E \wedge X) \arrow[r, shift left, "\mathrm{id}\wedge\eta\wedge\mathrm{id}"] \arrow[r, shift right, "\eta\wedge\mathrm{id}\wedge\mathrm{id}"'] & \pi_*(E \wedge E \wedge X) \,, \end{tikzcd}$$

and when $X = E \wedge Z$ it extends to a split equalizer diagram

$$\begin{tikzcd} \pi_*(E \wedge Z) \arrow[r, "\eta\wedge\mathrm{id}"] & \pi_*(E \wedge E \wedge Z) \arrow[r, shift left, "\mathrm{id}\wedge\eta\wedge\mathrm{id}"] \arrow[r, shift right, "\eta\wedge\mathrm{id}\wedge\mathrm{id}"'] \arrow[l, bend left=60, "\phi\wedge\mathrm{id}"] & \pi_*(E \wedge E \wedge E \wedge Z) \arrow[l, bend left=60, "\mathrm{id}\wedge\phi\wedge\mathrm{id}"] \end{tikzcd}$$

as in [Mac71, §IV.5]. In particular, it is then an equalizer, so that d is an isomorphism. $\square$

Hence we can recover the homotopy groups $\mathcal{E}_1^{s,*} = \pi_*(Y_{s,1}) = \pi_*(E \wedge Y_s)$ from the E_*E-comodules $E_*(Y_{s,1})$. To make use of this, we apply $E_*(-)$ to the chain of homotopy cofiber sequences, and obtain an unrolled exact couple

$$\begin{tikzcd}[column sep=small] \dots \arrow[r] & E_*(Y_3) \arrow[r, "\alpha_*"] & E_*(Y_2) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & E_*(Y_1) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & E_*(Y_0) = E_*(Y) \arrow[d, "\beta_*"] \\ & & E_*(Y_{2,1}) \arrow[ul, dashed, "\gamma_*"] & E_*(Y_{1,1}) \arrow[ul, dashed, "\gamma_*"] & E_*(Y_{0,1}) \,, \arrow[ul, dashed, "\gamma_*"] \end{tikzcd}$$

in the (abelian) category of E_*E-comodules. Here $\beta_* \colon E_*(Y_s) \to E_*(Y_{s,1})$ can be rewritten as

$$\pi_*(E \wedge Y_s) \overset{\mathrm{id}\wedge\eta\wedge\mathrm{id}}{\longrightarrow} \pi_*(E \wedge E \wedge Y_s)$$

and admits the E_*-linear retraction

$$\pi_*(E \wedge E \wedge Y_s) \overset{\phi\wedge\mathrm{id}}{\longrightarrow} \pi_*(E \wedge Y_s) \,,$$

since $\phi(\mathrm{id} \wedge \eta) = \mathrm{id}$ by (right) unitality. Hence each β_* is injective, so by exactness $\alpha_* = 0$ and γ_* is surjective, for each s. We can therefore redraw the diagram above as

$$\begin{tikzcd}[column sep=small] E_*(\Sigma^3 Y_3) \arrow[d, tail] & E_*(\Sigma^2 Y_2) \arrow[d, tail, "\beta_*"] & E_*(\Sigma Y_1) \arrow[d, tail, "\beta_*"] & E_*(Y_0) = E_*(Y) \arrow[d, tail, "\beta_*"] \\ \dots & E_*(\Sigma^2 Y_{2,1}) \arrow[ul, two heads, "\gamma_*"] & E_*(\Sigma Y_{1,1}) \arrow[ul, two heads, "\gamma_*"] & E_*(Y_{0,1}) \,, \arrow[ul, two heads, "\gamma_*"] \end{tikzcd}$$

consisting of the short exact sequences

$$0 \to E_*(Y_s) \overset{\beta_*}{\longrightarrow} E_*(Y_{s,1}) \overset{\gamma_*}{\longrightarrow} E_{*-1}(Y_{s+1}) \to 0$$

of E_*E-comodules. Each underlying short exact sequence of E_*-modules is split by $\phi \wedge \mathrm{id}$, but the splitting is usually not E_*E-(co-)linear. Now we splice these short exact sequences to obtain a long exact sequence

$$\dots \longleftarrow E_*(\Sigma^2 Y_{2,1}) \xleftarrow{\beta_*\gamma_*} E_*(\Sigma Y_{1,1}) \xleftarrow{\beta_*\gamma_*} E_*(Y_{0,1}) \xleftarrow{\beta_*} E_*(Y) \longleftarrow 0$$

of E_*E-comodules. By Lemma 1.1 we now have an isomorphism from the Adams spectral sequence $(\mathcal{E}_1, d_1)$-term

$$\dots \longleftarrow \pi_*(\Sigma^3 Y_{3,1}) \xleftarrow{d_1^2} \pi_*(\Sigma^2 Y_{2,1}) \xleftarrow{d_1^1} \pi_*(\Sigma Y_{1,1}) \xleftarrow{d_1^0} \pi_*(Y_{1,0}) \longleftarrow 0$$

to the cochain complex

$$\dots \longleftarrow \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma^3 Y_{3,1})) \xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma^2 Y_{2,1}))$$
$$\xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma Y_{1,1})) \xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(Y_{1,0})) \longleftarrow 0$$

Letting

$$I^s = E_*(\Sigma^s Y_{s,1}) = E_*(E \wedge Y_s) \cong E_*E \otimes_{E_*} E_*(Y_s)$$

and $\delta = \beta_*\gamma_*$ we have a resolution

$$\dots \longleftarrow I^3 \xleftarrow{\delta} I^2 \xleftarrow{\delta} I^1 \xleftarrow{\delta} I^0 \xleftarrow{\beta_*} E_*(Y) \longleftarrow 0$$

of the E_*E-comodule $E_*(Y)$ by extended E_*E-comodules. These are relatively injective, in the sense that for any diagram of E_*E-comodules

$$\begin{tikzcd} 0 \arrow[r] & M_* \arrow[r, tail] \arrow[d] & N_* \arrow[dl, dashed] \\ & I^s & \end{tikzcd}$$

with $M_* \to N_*$ split injective in the underlying category of E_*-modules, there exists a dashed arrow making the triangle commute. With this notation, the Adams $(\mathcal{E}_1, d_1)$-term is isomorphic to the cochain complex

$$\dots \longleftarrow \mathrm{Hom}_{E_*E}(E_*, I^3) \xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^2)$$
$$\xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^1) \xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^0) \longleftarrow 0$$

obtained by applying the functor $\mathrm{Hom}_{E_*E}(E_*, -)$ the relatively injective resolution $(I^s, \delta)_s$ of $E_*(Y)$. By the comparison theorem in homological algebra, any two relatively injective E_*E-comodule resolutions of $E_*(Y)$ are chain homotopy equivalent, and give chain homotopy equivalent cochain complexes after applying $\mathrm{Hom}_{E_*E}(E_*, -)$. The cohomology of this cochain complex is therefore independent of the choice of resolution, and defines the E_*E-comodule Ext-groups

$$\mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) = H^s(\mathrm{Hom}_{E_*E}(E_*, I^*), \delta)\,.$$

As usual, $\mathrm{Ext}^0_{E_*E}(E_*, E_*(Y)) = \mathrm{Hom}_{E_*E}(E_*, E_*(Y))$.

Theorem 1.2. *The E-based Adams spectral sequence for Y has $\mathcal{E}_2$-term*

$$\mathcal{E}_2^{s,*} = \mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) \Longrightarrow_s \pi_*(Y)\,.$$

More precisely,

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y)) \Longrightarrow_s \pi_{t-s}(Y)$$

with d_r-differentials $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t+r-1}$ of bidegree $(r, r-1)$.

The image groups

$$F^s\pi_*(Y) = \mathrm{im}(\pi_*(Y_s) \longrightarrow \pi_*(Y))$$

define the decreasing Adams filtration

$$\cdots \subset F^{s+1}\pi_*(Y) \subset F^s\pi_*(Y) \subset \cdots \subset F^0\pi_*(Y) = \pi_*(Y)\,,$$

where s is often called the Adams grading (or cohomological degree). To keep track of the grading of $\pi_*(Y)$, we set

$$\begin{aligned}\mathrm{Hom}^t_{E_*E}(E_*, I^s) &= \mathrm{Hom}_{E_*E}(\Sigma^t E_*, I^s)\\ \mathrm{Ext}^{s,t}_{E_*E}(E_*, H_*(Y)) &= H^s(\mathrm{Hom}^t_{E_*E}(E_*, I^s), \delta)\,,\end{aligned}$$

so that

$$\begin{aligned}\pi_n(Y_{s,1}) &= [S^n, Y_{s,1}] \cong [S^{n+s}, \Sigma^s Y_{s,1}]\\ &\cong \mathrm{Hom}_{E_*E}(\Sigma^{n+s}E_*, E_*(\Sigma^s Y_{s,1})) = \mathrm{Hom}^{n+s}_{E_*E}(E_*, I^s)\,.\end{aligned}$$

Letting $t = n + s$ be the internal grading (and $n = t - s$ the topological grading) we denote this group by $\mathcal{E}_1^{s,t}$, so that

$$\begin{aligned}\mathcal{E}_1^{s,t} &= \mathrm{Hom}^t_{E_*E}(E_*, I^s)\\ \mathcal{E}_2^{s,t} &= \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y))\end{aligned}$$

and $\zeta^s\colon F^s\pi_n(Y)/F^{s+1}\pi_n(Y) \to \mathcal{E}_\infty^{s,s+n}$. The d_r-differential is derived from

$$\begin{tikzcd}
\pi_{t-s-1}(Y_{s+r}) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & \pi_{t-s-1}(Y_{s+1}) & \\
\pi_{t-s-1}(Y_{s+r,1}) & & \pi_{t-s}(Y_{s,1}) \arrow[ul, "\gamma"']
\end{tikzcd}$$

hence has components $d_r\colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t+r-1}$, of (s,t)-bidegree $(r, r-1)$, for all s and t.

It is traditional to show the Adams spectral sequence in the $(t-s,s)$-plane, called Adams bigrading, and in these coordinates the d_r-differential has $(t-s,s)$-bidegree $(-1,r)$. This is an upper half-plane spectral sequence with entering differentials. Here is the $(\mathcal{E}_1, d_1)$-term, with $\mathcal{E}_1^{s,t} = \pi_{t-s}(Y_{s,1}) \cong \mathrm{Hom}^t_{E_*E}(E_*, I^s)$.

$$\begin{tikzcd}
\pi_0(Y_{3,1}) & \pi_1(Y_{3,1}) & \pi_2(Y_{3,1}) \\
\pi_0(Y_{2,1}) & \pi_1(Y_{2,1}) \arrow[ul] & \pi_2(Y_{2,1}) \arrow[ul] \\
\pi_0(Y_{1,1}) & \pi_1(Y_{1,1}) \arrow[ul] & \pi_2(Y_{1,1}) \arrow[ul] \\
\pi_0(Y_{0,1}) & \pi_1(Y_{0,1}) \arrow[ul] & \pi_2(Y_{0,1}) \arrow[ul]
\end{tikzcd}$$

Next is the $(\mathcal{E}_2, d_2)$-term, with $\mathcal{E}_2^{s,t} \cong \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y))$, writing Hom in place of Ext^0.

$$
\begin{tikzcd}
\mathrm{Ext}^{3,3}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{3,4}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{3,5}_{E_*E}(E_*, E_*(Y)) \\
\mathrm{Ext}^{2,2}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{2,3}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{2,4}_{E_*E}(E_*, E_*(Y)) \\
\mathrm{Ext}^{1,1}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{1,2}_{E_*E}(E_*, E_*(Y)) \arrow[uul] & \mathrm{Ext}^{1,3}_{E_*E}(E_*, E_*(Y)) \arrow[uul] \\
\mathrm{Hom}^{0}_{E_*E}(E_*, E_*(Y)) & \mathrm{Hom}^{1}_{E_*E}(E_*, E_*(Y)) \arrow[uul] & \mathrm{Hom}^{2}_{E_*E}(E_*, E_*(Y)) \arrow[uul]
\end{tikzcd}
$$

Eventually we come to the $\mathcal{E}_\infty$-term, showing $\mathcal{E}_\infty^{s,t}$ in bidegree $(t-s, s)$.

$$
\begin{tikzcd}
\mathcal{E}_\infty^{3,3} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{3,4} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{3,5} \arrow[d, dash, dashed] \\
\mathcal{E}_\infty^{2,2} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{2,3} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{2,4} \arrow[d, dash, dashed] \\
\mathcal{E}_\infty^{1,1} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{1,2} \arrow[d, dash, dashed] & \mathcal{E}_\infty^{1,3} \arrow[d, dash, dashed] \\
\mathcal{E}_\infty^{0,0} & \mathcal{E}_\infty^{0,1} & \mathcal{E}_\infty^{0,2} \\
n=0 & n=1 & n=2
\end{tikzcd}
$$

Regarding topological degree n, we find the groups $\mathcal{E}_\infty^{s,n-s}$ in the n-th column, for $s \ge 0$. When we have convergence, so that each $\zeta^s \colon F^s\pi_n(Y)/F^{s+1}\pi_n(Y) \cong \mathcal{E}_\infty^{s,n-s}$ is an isomorphism, that column shows the associated graded of the Adams filtration of $\pi_n(Y)$, with the lower filtrations s near the bottom of the chart. The extension problem in degree n is to inductively determine the group extensions

$$0 \to \mathcal{E}_\infty^{s,n-s} \longrightarrow \frac{\pi_n(Y)}{F^{s+1}\pi_n(Y)} \longrightarrow \frac{\pi_n(Y)}{F^s\pi_n(Y)} \to 0\,.$$

When we have strong convergence, that filtration is complete and Hausdorff, so that $\pi_n(Y) = \lim_s \pi_n(Y)/F^s\pi_n(Y)$ can be recovered from the finite stage extensions.

The edge homomorphism

$$\pi_n(Y) = F^0\pi_n(Y) \to F^0\pi_n(Y)/F^1\pi_n(Y) \xrightarrow{\zeta^0} \mathcal{E}_\infty^{0,n} \subset \mathcal{E}_2^{0,n} = \mathrm{Hom}^n_{E_*E}(E_*, E_*(Y))$$

is precisely the natural homomorphism d from Lemma 1.1.

2. Pairings of Adams spectral sequences

Given a pairing $\mu\colon Y \wedge Y' \to Y''$ of orthogonal spectra there is a natural pairing

$$\mu_r\colon \mathcal{E}_r(Y) \otimes \mathcal{E}_r(Y') \longrightarrow \mathcal{E}_r(Y'')$$

of Adams spectral sequences, given at the $\mathcal{E}_2$-term by the algebraic pairing

$$\mu_2\colon \operatorname{Ext}_{E_*E}(E_*, E_*(Y)) \otimes \operatorname{Ext}_{E_*E}(E_*, E_*(Y')) \longrightarrow \operatorname{Ext}_{E_*E}(E_*, E_*(Y'')) ,$$

and with target the pairing

$$\mu_*\colon \pi_*(Y) \otimes \pi_*(Y') \longrightarrow \pi_*(Y'') .$$

To justify this, we assume that the canonical Adams towers $Y_\star$ and $Y'_\star$ of Y and Y' have been cofibrantly replaced (the projective stable model structure on such towers), so that each Y_s and $Y'_{s'}$ is a cell spectrum, and each map $Y_{s+1} \to Y_s$ and $Y'_{s'+1} \to Y'_{s'}$ is a composite of cell attachments. We may then assume that $Y_{-\infty} = \bigcup_s Y_s = \operatorname{colim}_s Y_s$ and $Y'_{-\infty} = \bigcup_{s'} Y'_{s'} = \operatorname{colim}_{s'} Y'_{s'}$. Then the convolution product $(Y \wedge Y')_\star$ is the tower with

$$(Y \wedge Y')_{s''} = \bigcup_{s+s' \ge s''} Y_s \wedge Y'_{s'} = \operatorname*{colim}_{s+s' \ge s''} Y_s \wedge Y'_{s'} \subset Y_{-\infty} \wedge Y'_{-\infty} .$$

This is again cofibrant, with filtration quotients

$$(Y \wedge Y')_{s'',1} = \bigvee_{s+s'=s''} Y_{s,1} \wedge Y'_{s',1} ,$$

and the diagram

$$\begin{tikzcd}
\dots \arrow[r] & (Y \wedge Y')_3 \arrow[r, "\alpha"] & (Y \wedge Y')_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & (Y \wedge Y')_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & (Y \wedge Y')_0 \arrow[d, "\beta"] \\
 & & (Y \wedge Y')_{2,1} \arrow[ul, dashed, "\gamma"] & (Y \wedge Y')_{1,1} \arrow[ul, dashed, "\gamma"] & (Y \wedge Y')_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

is an Adams resolution of $(Y \wedge Y')_0 \simeq Y \wedge Y'$, in a more general sense than the canonical Adams resolutions we have discussed so far. ((ETC/BEWARE: This appears to assume that $E_*(Y \wedge Y') \cong E_*(Y) \otimes_{E_*} E_*(Y'')$, which holds if $E_*(Y)$ or $E_*(Y')$ is flat over E_*.)) This uses that each spectrum $(Y \wedge Y')_{s'',1}$ has the form $E \wedge Z$, and that the cochain complex

$$\dots \longleftarrow E_*(\Sigma^2 (Y \wedge Y')_{2,1}) \xleftarrow{\beta_*\gamma_*} E_*(\Sigma (Y \wedge Y')_{1,1}) \xleftarrow{\beta_*\gamma_*} E_*((Y \wedge Y')_{0,1}) \longleftarrow 0$$

is the tensor product $I^* \otimes_{E_*} {}'I^*$ over E_* of the E_*-split E_*E-comodules resolutions $I^* \simeq E_*(Y)$ and ${}'I^* \simeq E_*(Y')$, with cohomology $E_*(Y \wedge Y')$ concentrated in degree $s'' = 0$. This is equivalent to the condition that $\alpha_*\colon E_*((Y \wedge Y')_{s+1}) \to E_*((Y \wedge Y')_s)$ is zero for each $s \ge 0$.

Moreover, there is a weak map of Adams towers $(Y \wedge Y')_\star \to Y''_\star$, making the diagram

$$\begin{tikzcd}
\dots \arrow[r] & (Y \wedge Y')_3 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_2 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_1 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_0 \arrow[d, "\mu"] \\
\dots \arrow[r] & Y''_3 \arrow[r, "\alpha"] & Y''_2 \arrow[r, "\alpha"] & Y''_1 \arrow[r, "\alpha"] & Y''_0
\end{tikzcd}$$

commute up to homotopy. This is constructed inductively, by noting that

$$(Y \wedge Y')_{s''+1} \xrightarrow{\alpha} (Y \wedge Y')_{s''} \longrightarrow Y''_{s''} \xrightarrow{\beta} Y''_{s'',1} = E \wedge Y''_{s''}$$

is null-homotopic by a generalization of Lemma 1.1.

The strict pairing of towers then gives a pairing of spectral sequences

$$\mathcal{E}_r(Y) \otimes \mathcal{E}_r(Y') \longrightarrow \mathcal{E}_r(Y \wedge Y')$$

as before, while the weak map of towers gives a map of spectral sequences

$$\mathcal{E}_r(Y \wedge Y') \to \mathcal{E}_r(Y'')$$

which combine to the desirect pairing of Adams spectral sequences. The spectral sequence $\mathcal{E}_r(Y \wedge Y')$ is more general than the canonical Adams spectral sequences we have discussed here, but it agrees with the canonical Adams spectral sequence for $Y \wedge Y'$ from the $\mathcal{E}_2$-term and onward.

The first pairing of $\mathcal{E}_1$-terms can be identified with the pairing

$$\mathrm{Hom}_{E_*E}(E_*, I^s) \otimes \mathrm{Hom}_{E_*E}(E_*, I^{s'}) \longrightarrow \mathrm{Hom}_{E_*E}(E_*, (I^* \otimes {}'I^*)^{s+s'})$$

that induces the external pairing

$$\mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) \otimes \mathrm{Ext}_{E_*E}\, s'(E_*, E_*(Y')) \longrightarrow \mathrm{Ext}^{s+s'}_{E_*E}(E_*, E_*(Y \wedge Y'))$$

of $\mathcal{E}_2$-terms. The weak map of Adams towers then induces the standard homomorphism

$$\mathrm{Ext}^{s''}_{E_*E}(E_*, E_*(Y \wedge Y')) \longrightarrow \mathrm{Ext}^{s''}_{E_*E}(E_*, E_*(Y'')) \,,$$

and these combine to the expected pairing of Adams $\mathcal{E}_2$-terms.

((ETC: I believe this result cannot be justify purely within the stable homotopy category.))

3. The cobar resolution

Suppose, until further notice, that E is an orthogonal ring spectrum. The Amitsur complex is the coaugmented cosimplicial diagram

$$\begin{tikzcd}
S \arrow[r, "\eta"] & E \arrow[r, shift left=2, "\eta\wedge \mathrm{id}"] \arrow[r, leftarrow, "\phi" description] \arrow[r, shift right=2, "\mathrm{id}\wedge\eta"'] & E \wedge E \arrow[r, shift left=4, "\eta\wedge\mathrm{id}\wedge\mathrm{id}"] \arrow[r, leftarrow, shift left=2, "\phi\wedge\mathrm{id}" description] \arrow[r, "\mathrm{id}\wedge\eta\wedge\mathrm{id}" description] \arrow[r, leftarrow, shift right=2, "\mathrm{id}\wedge\phi" description] \arrow[r, shift right=4, "\mathrm{id}\wedge\mathrm{id}\wedge\eta"'] & \cdots
\end{tikzcd}$$

of orthogonal spectra, i.e., a functor $\Delta_\eta \to \mathcal{S}p^{\mathbb{O}}$ where Δ_η is the simplex category Δ together with an initial object $[-1]$. The functor maps $[q] = \{0 < 1 \dots < q\}$ to $E \wedge \dots \wedge E$ with $1+q$ copies of E, the face operators/monomorphisms $[p] \to [q]$ induce maps invoving the unit $\eta \colon S \to E$, and the degeneracy operators/epimorphisms $[p] \to [q]$ induce maps involving the product $\phi \colon E \wedge E \to E$. More precisely $\delta^i \colon [q-1] \to [q]$ for $0 \le i \le q$ is given by $\mathrm{id}^{\wedge i} \wedge \eta \wedge \mathrm{id}^{q-i} \colon E^{\wedge q} \to E^{\wedge 1+q}$, while $\sigma^j \colon [q+1] \to [q]$ for $0 \le j \le q$ is given by $\mathrm{id}^{\wedge j} \wedge \phi \wedge \mathrm{id}^{q-j} \colon E^{\wedge 1+q+1} \to E^{\wedge 1+q}$.

The homotopy limit (or totalization) of the unaugmented part of the diagram, i.e., with $q \ge 0$, is called an E-adic completion $S^\wedge_E$ of S, and we obtain a completion map $\eta \colon S \to S^\wedge_E$.

We can smash the diagram (from the right, say) with any given orthogonal spectrum Y and obtain an Amitsur complex

$$\begin{tikzcd}
Y \arrow[r, "\eta\wedge\mathrm{id}"] & E \wedge Y \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & E \wedge E \wedge Y \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & \cdots
\end{tikzcd}$$

with homotopy limit $Y^\wedge_E$, together with a completion map $\eta_Y\colon Y \to Y^\wedge_E$ lifting $\eta \wedge \mathrm{id}$. If we smash either one of these diagrams (from the left, say) with E, then the product ϕ equips the resulting diagram with an extra degeneracy operator σ^{-1}, or cosimplicial contraction, given by $\phi \wedge \mathrm{id}^{\wedge q} \wedge \mathrm{id}\colon E^{\wedge 2+q} \wedge Y \to E^{\wedge 1+q} \wedge Y$ for $q \geq 0$.

$$\begin{tikzcd}
E \wedge Y \arrow[r, "\mathrm{id} \wedge \eta \wedge \mathrm{id}"] & E \wedge E \wedge Y \arrow[l, bend right, "\phi \wedge \mathrm{id}"'] \arrow[r, shift left=2] \arrow[r, shift right=2] & E \wedge E \wedge E \wedge Y \arrow[l] \arrow[l, bend right, "\phi \wedge \mathrm{id} \wedge \mathrm{id}"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \dots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, bend right, "\phi \wedge \mathrm{id} \wedge \mathrm{id} \wedge \mathrm{id}"']
\end{tikzcd}$$

This implies that $E \wedge Y \to (E \wedge Y)^\wedge_E$ is an equivalence.

The corresponding construction at the level of homotopy groups provides a resolution of $\pi_*(E \wedge Y) = E_*(Y)$ by extended E_*E-comodules. To effect this, we allow E to be a ring spectrum up to homotopy, but assume that it is flat, so that (E_*, E_*E) is a Hopf algebroid. For each $q \geq -1$ let

$$\begin{aligned} C^q = C^q_{E_*E}(E_*E, E_*(Y)) &= E_*E \otimes_{E_*} \cdots \otimes_{E_*} E_*E \otimes_{E_*} E_*(Y) \\ &\xrightarrow{\cong} \pi_*(E \wedge E \wedge \cdots \wedge E \wedge Y) \end{aligned}$$

with $1+q$ copies of E_*E, and $2+q$ copies of the spectrum E. Note that $C^{-1} = E_*(Y)$. We get coface operators $\delta^i\colon C^{q-1} \to C^q$ for $0 \leq i \leq q$, given by $\mathrm{id}^{\otimes i} \otimes \psi \otimes \mathrm{id}^{\otimes q-i}$ for $0 \leq i < q$, while δ^q is given by $\mathrm{id}^{\otimes q} \otimes \nu$. Here $\psi\colon E_*E \otimes_{E_*} E_*E$ is the Hopf algebroid coproduct, and $\nu\colon E_*(Y) \to E_*E \otimes_{E_*} E_*(Y)$ is the coaction.

$$\begin{tikzcd}
E_*(Y) \arrow[r, "\delta^0"] & E_*E \otimes_{E_*} E_*(Y) \arrow[r, shift left=2, "\delta^0"] \arrow[r, shift right=2, "\delta^1"'] & E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_*(Y) \arrow[r, shift left=4, "\delta^0"] \arrow[r, "\delta^1" description] \arrow[r, shift right=4, "\delta^2"'] & \dots
\end{tikzcd}$$

((ETC: Get a cosimplicial graded abelian group, an extra codegeneracy, giving a cosimplicial contraction.))

For each $q \geq 0$ we can form the alternating sum

$$d = \sum_{i=0}^{q} (-1)^i \delta^i \colon C^{q-1} \longrightarrow C^q \,.$$

Note that $d\colon C^{-1} \to C^0$ is $\nu\colon E_*(Y) \to E_*E \otimes_{E_*} E_*(Y)$, while $d\colon C^0 \to C^1$ is $\psi \otimes \mathrm{id} - \mathrm{id} \otimes \nu$. The (cosimplicial) relations satisfied by the coface operators imply that $d \circ d = 0$, so that we obtain a cochain complex

$$0 \to E_*(Y) \xrightarrow{\eta} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \longrightarrow \dots \,.$$

Here each C^q with $q \geq 0$ is an extended, hence relatively injective, E_*E-comodule. ((ETC: Get a cochain contraction.))

((ETC: Taking into account the codegeneracies, we may pass to the normalized sub-cocomplex where each of the inner q copies of E_*E is replaced by $\ker(\epsilon\colon E_*E \to E_*)$.))

More generally, $C^*_\Gamma(M, N)$ can be defined for any (flat) Hopf algebroid (A, Γ), right Γ-comodule M and left Γ-comodule N.

((ETC: Might prefer to say all this in terms of monad actions, or comonad coactions.))

((ETC: Give cobar resolution and cobar complex for calculating $\mathrm{Ext}^{*,*}_{E_*E}(E_*, M_*)$ of any E_*E-comodule M_*.

4. The classical Adams spectral sequence

((ETC: Specialize to $E = H\mathbb{F}_p$, with

$$\mathrm{Ext}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(Y;\mathbb{F}_p)) \cong \mathrm{Ext}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p);\mathbb{F}_p)\,,$$

where $\mathrm{Ext}_{\mathscr{A}}(M, \mathbb{F}_p)$ is formed in the category of $\mathscr{A}$-modules, as usual, by applying $\mathrm{Hom}_{\mathscr{A}}(-, \mathbb{F}_p)$ to any projective $\mathscr{A}$-module resolution $P_* \to M$ and passing to cohomology.))

5. The Adams–Novikov spectral sequence

((ETC: Specialize to $E = MU$, with

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*(Y)) \Longrightarrow \pi_*(Y)$$

where Ext is formed in the category of MU_*MU-comodules.))

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