ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 6: SMOOTH BORDISM

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See [Tho54], [Ati61], [CF64], [Sto68], [MS74], [MM79, Ch. 1], [May99, Ch. 25].

1. Bordism classes of manifolds

Definition 1.1. Let M and N be closed, smooth d-manifolds. A bordism from M to N is a compact, smooth $(d+1)$-manifold W such that

$$\partial W \cong M \sqcup N .$$

If such a bordism exists, we say that M and N are cobordant. This defines an equivalence relation. Let $\mathcal{N}_d = \Omega^O_d$ be the set of cobordism classes of closed, smooth d-manifolds, and let $\mathcal{N}_* = \Omega^O_*$ denote the associated graded set.

Lemma 1.2. *The disjoint union and Cartesian product of manifolds make $\mathcal{N}_* = \Omega^O_*$ a graded commutative $\mathbb{F}_2$-algebra.*

Proof. The sum and product are given by $[M] + [N] = [M \sqcup N]$ and $[M] \cdot [N] = [M \times N]$. Let $I = [0, 1]$. Since $\partial(M \times I) \cong M \sqcup M$ we have $[M] + [M] = 0$ for each M. □

Theorem 1.3 (Thom (1954)). $\mathcal{N}_* \cong \mathbb{F}_2[\tilde{a}_i \mid i \neq 2^j - 1] = \mathbb{F}_2[\tilde{a}_2, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6, \tilde{a}_8, \dots]$ *with* $|\tilde{a}_i| = i$.

We may also consider manifolds with additional structure, such as an orientation, an almost complex structure, or a stable framing. We assume that the boundary of such a manifold again has such a structure, with

$$\partial(M \times I) \cong M \sqcup (-M) .$$

Here $-M$ denotes the opposite structure of that of M. Moreover, we assume that the disjoint union and Cartesian product of two such structured manifolds again has this structure.

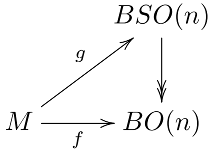
Example 1.4. An orientation of a d-manifold M is equivalent to an orientation of the tangent $\mathbb{R}^d$-bundle τ_M, or of the normal $\mathbb{R}^n$-bundle ν_M for any choice of embedding $M \to \mathbb{R}^{d+n}$. Here

$$E(\nu_M)_x = \mathbb{R}^{d+n}/T_xM .$$

Any two choices of embeddings become isotopic for n sufficiently large, so the stable class of $\nu_M \in \widetilde{KO}(M)$ is well-defined. An orientation of ν_M amounts to a lift of

Date: May 11th.

the classifying map $M \to BO(n)$ through $EO(n)/SO(n) \simeq BSO(n)$.

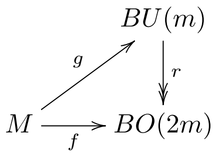


We write $\Omega_d = \Omega_d^{SO}$ for the group of cobordism classes of closed, oriented, smooth d-manifolds, with additive inverse $-[M] = [-M]$, and $\Omega_* = \Omega_*^{SO}$ for the associated graded commutative ring.

Theorem 1.5 (Thom, Milnor, Averbuch). *$\Omega_*[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_i \mid i \geq 1]$ with $|y_i| = 4i$.*

The precise structure of the 2-torsion was determined by Wall (1960).

Example 1.6. An almost complex structure on a manifold M is given by a complex structure on the normal bundle ν_M, for any choice of embedding $M \to \mathbb{R}^{d+n}$. Here $n = 2m$ must be even, so $\nu_M = r(\eta) = \eta_{\mathbb{R}}$ for some $\mathbb{C}^m$-bundle η over M. A complex structure on ν_M corresponds to a lift of the classifying map $M \to BO(2m)$ through $EO(2m)/U(m) \simeq BU(m)$.

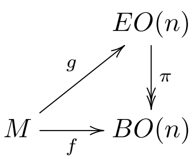


We write Ω_d^U for the group of cobordism classes of almost complex d-manifolds, and Ω_*^U for the associated graded commutative ring. Every (smooth, closed) complex manifold is almost complex, but the converse does not hold for $d = 4$. Shing-Tung Yau has conjectured that for even $d \geq 6$ each almost complex d-manifold admits a complex structure. This is unknown for $M = S^6$.

Theorem 1.7 (Milnor (1960), Novikov (1960)). *$\Omega_*^U \cong \mathbb{Z}[x_i \mid i \geq 1]$ with $|x_i| = 2i$.*

In particular, each odd-dimensional almost complex manifold is a boundary.

Example 1.8. A stable framing of M is given by a trivialization $\nu_M \cong \epsilon_M^n$ of the normal bundle of any embedding $M \to \mathbb{R}^{d+n}$. This is equivalent to giving a stable trivialization $\tau_M \oplus \epsilon^n \cong \epsilon^{d+n}$ for some n. A stable framing of M is equivalent to giving a nullhomotopy of the classifying map $M \to BO(n)$, or a lift through the contractible space $EO(n) \simeq B\{e\}$.



We write Ω_d^{fr} for the group of cobordism classes of stably framed d-manifolds, and Ω_*^{fr} for the associated graded commutative ring.

Theorem 1.9 (Pontryagin (1936/1950)). *$\Omega_*^{\mathrm{fr}} \cong \pi_*(S) = (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \dots)$.*

((ETC: Other bordism theories. h- and s-cobordism theorems. Exotic spheres.))

2. Bordism theories

Following Atiyah (1961) we can realize the rings Ω_*^O, Ω_*^{SO}, Ω_*^U, Ω_*^{fr} etc. as coefficient rings of multiplicative homology theories $\Omega_*^O(-)$, $\Omega_*^{SO}(-)$, $\Omega_*^U(-)$, $\Omega_*^{\mathrm{fr}}(-) = \pi_*^S((-)_+)$ etc.

Definition 2.1. For a space X, consider maps

$$\sigma \colon M \longrightarrow X \qquad \text{and} \qquad \tau \colon N \longrightarrow X$$

from closed, smooth unoriented (resp. oriented, almost complex, stably framed, etc.) d-manifolds M and N to X, and say that (M, σ) is cobordant to (N, τ) if there exists a map

$$\phi \colon W \longrightarrow X$$

from a compact, smooth $(d+1)$-manifold unoriented (resp. oriented, almost complex, stably framed, etc.) W to X, such that $\partial W \cong M \sqcup N$ and $\phi|\partial W \cong \sigma \sqcup \tau$. Let $\Omega_d^O(X)$ (resp. $\Omega_d^{SO}(X)$, $\Omega_d^U(X)$, $\Omega_d^{\mathrm{fr}}(X)$, etc.) be the set of cobordism classes $[M, \sigma]$ of such maps $\sigma \colon M \to X$. Given $f \colon X \to Y$ let $f_* \colon \Omega_d^O(X) \to \Omega_d^O(Y)$ map $[M, \sigma]$ to $[M, f\sigma]$.

For a pair (X, A) consider maps of pairs

$$\sigma \colon (M, \partial M) \longrightarrow (X, A) \qquad \text{and} \qquad \tau \colon (N, \partial N) \longrightarrow (X, A)$$

from compact, smooth unoriented (resp. oriented, almost complex, stably framed, etc.) d-manifolds M and N to X, and say that these are cobordant if there exists a map of pairs

$$\phi \colon (W, \partial W) \longrightarrow (X, A)$$

where $\partial W \cong M \cup_{\partial M} V \cup_{\partial N} N$ with $\phi|\partial W \cong \sigma \cup \psi \cup \tau$. Let $\Omega_d^O(X, A)$ (resp. $\Omega_d^{SO}(X, A)$, $\Omega_d^U(X, A)$, $\Omega_d^{\mathrm{fr}}(X, A)$, etc.) be the set of cobordism classes of such maps of pairs. Given $f \colon (X, A) \to (Y, B)$ let $f_* \colon \Omega_d^O(X, A) \to \Omega_d^O(Y, B)$ map $[M, \sigma]$ to $[M, f\sigma]$. Let $\partial \colon \Omega_d^O(X, A) \to \Omega_{d-1}^O(A)$ map the bordism class of $\sigma \colon (M, \partial M) \to (X, A)$ to the bordism class of $\sigma|\partial M \colon \partial M \to A$.

Proposition 2.2. *The functor $(X, A) \mapsto \Omega_*^O(X, A)$ (resp. $\Omega_d^{SO}(X, A)$, $\Omega_d^U(X, A)$, $\Omega_d^{\mathrm{fr}}(X, A)$, etc.) defines a multiplicative homology theory, called unoriented (resp. oriented, almost complex, stably framed, etc.) bordism.*

Proof. The operations $[M, \sigma] + [N, \tau] = [M \sqcup N, \sigma \sqcup \tau]$ and $-[M, \sigma] = [-M, \sigma]$ give $\Omega_d^O(X)$ a group structure. To prove homotopy invariance use $W = M \times I$. Transversality for smooth maps implies that there is a natural isomorphism

$$\Omega_d^O(X, A) \cong \Omega_d^O(X \cup CA, *)\,,$$

which implies excision.

For $\tau \colon N \to Y$ the operation $[M, \sigma] \cdot [N, \tau] = [M \times N, \sigma \times \tau]$ defines a bilinear pairing $\Omega_d^O(X) \times \Omega_e^O(Y) \to \Omega_{d+e}^O(X \times Y)$. In the case $Y = *$, this makes $\Omega_*^O(X)$ a (right or left) Ω_*^O-module. There are also relative pairings, compatible with the boundary homomorphisms, making $\Omega_*^O(-)$ a multiplicative homology theory.

The oriented, almost complex, stably framed, etc. cases work the same way. □

3. Thom spectra

Recall that $\mathrm{Th}(\xi) = D(\xi)/S(\xi)$ denotes the Thom complex of a Euclidean vector bundle $\xi\colon E \to X$, and that

$$\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta)$$

if $\eta\colon F \to Y$ is a second Euclidean vector bundle. In the special case $\eta = \epsilon^1$ over $Y = *$ we have $\xi \times \eta = \xi \oplus \epsilon^1$ and $\mathrm{Th}(\eta) = D^1/S^0 \cong S^1$, so

$$\mathrm{Th}(\xi \oplus \epsilon^1) \cong \mathrm{Th}(\xi) \wedge S^1 = \Sigma\,\mathrm{Th}(\xi)\,.$$

For a bundle map

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\hat{f}} & E(\eta) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y\,, \end{array}$$

with $\xi \cong f^*\eta$, we write $\mathrm{Th}(f)\colon \mathrm{Th}(\xi) \to \mathrm{Th}(\eta)$ for the induced map of Thom complexes.

Definition 3.1. Let $\gamma^n = \gamma^n_{\mathbb{R}}$ denote the tautological $\mathbb{R}^n$-bundle

$$\pi\colon E(\gamma^n) = EO(n) \times_{O(n)} \mathbb{R}^n \longrightarrow BO(n)\,.$$

Recall that $\gamma^{n+1}|BO(n) \cong \gamma^n \oplus \epsilon^1$, where we view $\iota\colon BO(n) \to BO(n+1)$ as the inclusion of a subspace. Let

$$MO(n) = \mathrm{Th}(\gamma^n) = \frac{EO(n) \times_{O(n)} D^n}{EO(n) \times_{O(n)} S^{n-1}} \cong EO(n)_+ \wedge_{O(n)} S^n\,.$$

Here $O(n)$ acts on $D^n/S^{n-1} \cong S^n$ as on the one-point compactification $\mathbb{R}^n \cup \{\infty\}$. Let MO denote the unoriented Thom spectrum, with n-th space $MO_n = MO(n)$ and n-th structure map $\Sigma MO_n \to MO_{n+1}$ given by the composite

$$\sigma\colon \Sigma\,\mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\gamma^n \oplus \epsilon^1) \cong \mathrm{Th}(\gamma^{n+1}|BO(n)) \xrightarrow{\mathrm{Th}(\iota)} \mathrm{Th}(\gamma^{n+1})\,.$$

Definition 3.2. Let $\tilde{\gamma}^n$ denote the tautological oriented $\mathbb{R}^n$-bundle

$$\pi\colon E(\tilde{\gamma}^n) = ESO(n) \times_{SO(n)} \mathbb{R}^n \longrightarrow BSO(n)\,.$$

Let

$$MSO(n) = \mathrm{Th}(\tilde{\gamma}^n) \cong ESO(n)_+ \wedge_{SO(n)} S^n\,.$$

Let MSO denote the oriented Thom spectrum, with n-th space $MSO_n = MSO(n)$ and n-th structure map $\Sigma MSO_n \to MSO_{n+1}$ given by the composite

$$\sigma\colon \Sigma\,\mathrm{Th}(\tilde{\gamma}^n) \cong \mathrm{Th}(\tilde{\gamma}^n \oplus \epsilon^1) \cong \mathrm{Th}(\tilde{\gamma}^{n+1}|BSO(n)) \xrightarrow{\mathrm{Th}(\iota)} \mathrm{Th}(\tilde{\gamma}^{n+1})\,.$$

Definition 3.3. Let $\gamma^n = \gamma^n_{\mathbb{C}}$ denote the tautological $\mathbb{C}^n$-bundle

$$\pi\colon E(\gamma^n) = EU(n) \times_{U(n)} \mathbb{C}^n \longrightarrow BU(n)\,.$$

Recall that $\gamma^{n+1}|BU(n) \cong \gamma^n \oplus \epsilon^1$, where $\epsilon^1 = \epsilon^1_{\mathbb{C}}$ and we view $\iota\colon BU(n) \to BU(n+1)$ as the inclusion of a subspace. Let

$$MU(n) = \mathrm{Th}(\gamma^n) = \frac{EU(n) \times_{U(n)} D^{2n}}{EU(n) \times_{U(n)} S^{2n-1}} \cong EU(n)_+ \wedge_{U(n)} S^{2n}\,.$$

Here $U(n)$ acts on $D^{2n}/S^{2n-1} \cong S^{2n}$ as on the one-point compactification $\mathbb{C}^n \cup \{\infty\}$. Let MU denote the complex Thom spectrum, with $2n$-th space $MU_{2n} = MU(n)$, $(2n+1)$-th space $MU_{2n+1} = \Sigma MU(n)$, $2n$-th structure map the identity $\Sigma MU_{2n} = MU_{2n+1}$, and $(2n+1)$-th structure map $\Sigma MU_{2n+1} = \Sigma^2 MU_{2n} \to MU_{2n+2}$ given by the composite

$$\sigma\colon \Sigma^2 \operatorname{Th}(\gamma^n) \cong \operatorname{Th}(\gamma^n \oplus \epsilon^1) \cong \operatorname{Th}(\gamma^{n+1}|BU(n)) \xrightarrow{\operatorname{Th}(\iota)} \operatorname{Th}(\gamma^{n+1}) .$$

Definition 3.4. The tautological $\mathbb{R}^n$-bundle over $B\{e\} = *$ is $\pi\colon \mathbb{R}^n \to *$, with Thom complex $D^n/S^{n-1} \cong S^n$. The framed bordism Thom spectrum $M\{e\}$ has n-th space $M\{e\}_n = S^n$ and n-th structure map $\Sigma M\{e\}_n \to M\{e\}_{n+1}$ equal to the identity $\Sigma S^n = S^{n+1}$. Hence $M\{e\} = S$ is equal to the sphere spectrum.

The Thom spectrum MO (resp. MSO, MU, S, etc.) defines a reduced homology theory $MO_*(-)$ by

$$\widetilde{MO}_d(X) = \operatorname*{colim}_n \pi_{d+n}(MO_n \wedge X) ,$$

where the colimit is formed over the homomorphisms

$$\pi_{d+n}(MO_n \wedge X) \xrightarrow{\Sigma} \pi_{d+n+1}\Sigma(MO_n \wedge X) \cong \pi_{d+n+1}(\Sigma MO_n \wedge X) \xrightarrow{\sigma_*} \pi_{d+n+1}(MO_{n+1} \wedge X) .$$

The suspension isomorphism $\Sigma\widetilde{MO}_d(X) \cong \widetilde{MO}_{d+1}(\Sigma X)$ is given by

$$\operatorname*{colim}_n \pi_{d+n}(MO_n \wedge X) \xrightarrow{\cong} \operatorname*{colim}_n \pi_{d+n+1}\Sigma(MO_n \wedge X) \cong \operatorname*{colim}_n \pi_{d+1+n}(MO_n \wedge \Sigma X) .$$

The associated unreduced homology theory is defined by $MO_d(X) = \widetilde{MO}_d(X_+)$ and $MO_d(X, A) = \widetilde{MO}_d(X \cup CA)$.

The bundle map

$$\begin{array}{ccc} E(\gamma^n) \times E(\gamma^m) & \xrightarrow{\hat{\mu}_{n,m}} & E(\gamma^{n+m}) \\ \downarrow & & \downarrow \\ BO(n) \times BO(m) & \xrightarrow{\mu_{n,m}} & BO(n+m) \end{array}$$

induces a pairing

$$MO_n \wedge MO_m = \operatorname{Th}(\gamma^n) \wedge \operatorname{Th}(\gamma^m) \xrightarrow{\operatorname{Th}(\mu_{n,m})} \operatorname{Th}(\gamma^{n+m}) = MO_{n+m}$$

that makes MO into a ring spectrum.

Likewise, the Thom spectra MSO, MU, $M\{e\} = S$, etc. are ring spectra that define multiplicative homology theories $MSO_*(-)$, $MU_*(-)$, $S_*(-)$, etc. Note that

$$S_d(X) = \operatorname*{colim}_n \pi_{d+n}(S^n \wedge X_+) \cong \pi_d^S(X_+) ,$$

so that $S_*(-)$ is given by the unreduced stable homotopy groups.

4. The Pontryagin–Thom construction and transversality

Theorem 4.1. *There are natural isomorphisms of multiplicative homology theories*

$$\Omega^O_*(X, A) \cong MO_*(X, A)$$
$$\Omega^{SO}_*(X, A) \cong MSO_*(X, A)$$
$$\Omega^U_*(X, A) \cong MU_*(X, A)$$
$$\Omega^{\mathrm{fr}}_*(X, A) \cong S_*(X, A)$$

etc. In particular

$$\mathcal{N}_* = \Omega^O_* \cong \pi_*(MO)$$
$$\Omega_* = \Omega^{SO}_* \cong \pi_*(MSO)$$
$$\Omega^U_* \cong \pi_*(MU)$$
$$\Omega^{\mathrm{fr}}_* \cong \pi_*(S)\,.$$

The case of framed bordism is due to Pontryagin (ca. 1936), that of unoriented and oriented bordism is due to Thom [Tho54].

Proof. We discuss the case $(X, A) = (*, \emptyset)$ for complex bordism.

Let $[M] \in \Omega^U_d$ be represented by an almost complex d-manifold $M \subset \mathbb{R}^{d+2n}$. Its normal bundle ν_M is classified by a map $g\colon M \to BU(n)$, which is covered by a bundle map

$$\begin{array}{ccc} E(\nu_M) & \xrightarrow{\hat{g}} & E(\gamma^n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & BU(n)\,. \end{array}$$

The disc bundle can be embedded as a tubular neighborhood $D(\nu_M) \subset \mathbb{R}^{d+2n} \subset S^{d+2n}$ of M. Let

$$S^{d+2n} \xrightarrow{\wp} \frac{S^{d+2n}}{S^{d+2n} \setminus (D(\nu_M) \setminus S(\nu_M))} \cong \frac{D(\nu_M)}{S(\nu_M)} = \mathrm{Th}(\nu_M)$$

be the Pontryagin–Thom collapse map, taking the complement of the open disc bundle $D(\nu_M) \setminus S(\nu_M)$ to the base point. The composite

$$S^{d+2n} \xrightarrow{\wp} \mathrm{Th}(\nu_M) \xrightarrow{\mathrm{Th}(g)} \mathrm{Th}(\gamma^n) = MU_{2n}$$

determines a homotopy class in

$$\pi_d(MU) = \operatorname*{colim}_n \pi_{d+2n} MU_{2n}\,.$$

Conversely, let $[f] \in \pi_d(MU)$ be represented by a map $f\colon S^{d+2n} \to MU_{2n} = \mathrm{Th}(\gamma^n)$. It may be deformed slightly to become transverse to the zero-section

$$z = qs_0\colon BU(n) \xrightarrow{s_0} D(\gamma^n) \xrightarrow{q} \mathrm{Th}(\gamma^n)\,,$$

whose normal bundle is isomorphic to γ^n. Let

$$M = f^{-1}(BU(n)) \subset \mathbb{R}^{d+2n} \subset S^{d+2n}$$

be the preimage of this zero-section, which is then a closed, smooth d-manifold (by a generalization of the regular level set theorem). Moreover, there is a bundle map

$$\begin{array}{ccc} E(\nu_M) & \xrightarrow{\hat{f}} & E(\gamma^n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f|M} & BU(n)\,, \end{array}$$

which implies that $\nu_M \cong (f|M)^*(\gamma^n)$ has a complex structure. Hence M is almost complex, and determines a bordism class in Ω^U_d.

To complete the proof, one verifies that these two constructions define mutual inverses

$$\Omega^U_d \overset{\cong}{\longleftrightarrow} \pi_d(MU)\,.$$

□

Remark 4.2. Thom worked with smooth (DIFF) manifolds, in order to have transversality available. For piecewise-linear (PL) manifolds, or topological (TOP) manifolds in dimension $d \neq 4$, transversality will hold in sufficiently large codimension by results of Williamson (1966) and Kirby–Siebenmann (1977).

See [Swi75, Lem. 14.40] or [May99, §25.5] for the proof that $\wp$ has degree 1, which we can state as follows. (In the unoriented case, this must be interpreted with $\mathbb{F}_2$-coefficients.)

Proposition 4.3. *The Hurewicz image of the Pontryagin–Thom collapse map corresponds under the Thom isomorphism to the fundamental class of M:*

$$\pi_{d+2n}(\mathrm{Th}(\nu_M)) \xrightarrow{h} \tilde{H}_{d+2n}(\mathrm{Th}(M)) \overset{\Phi_\nu}{\cong} H_d(M)$$
$$[\wp] \longmapsto \Phi_\nu h([\wp]) = [M]\,.$$

5. Unoriented bordism

To calculate the commutative $\mathbb{F}_2$-algebra $\mathcal{N}_* = \Omega^O_* \cong \pi_*(MO)$, Thom compared the homology of MO with the homology of spectra X such that $\pi_*(X)$ is known, namely (wedge sums of suspensions of) Eilenberg–MacLane spectra. The argument was streamlined by Liulevicius, using the multiplicative structure. Note that [Liu62, (3.27)] is corrected in [Liu68, Prop. 9] and improved by [Swi73, Thm. 1(i)].

Recall that $\mathscr{A}_* = \mathbb{F}_2[\zeta_k \mid k \geq 1]$ with $|\zeta_k| = 2^k - 1$. Let

$$H_*(MO;\mathbb{F}_2) = \operatorname*{colim}_n H_{*+n}(MO_n;\mathbb{F}_2)\,,$$

with the induced $\mathscr{A}_*$-coaction. The $\mathbb{F}_2$-linear dual

$$H^*(MO;\mathbb{F}_2) = \lim_n H^{*+n}(MO_n;\mathbb{F}_2)$$

has the dual $\mathscr{A}$-action.

Theorem 5.1 ([Tho54], [Liu62]). *The $\mathscr{A}_*$-comodule algebra*

$$H_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[a_m \mid m \geq 1]$$

is isomorphic to $\mathscr{A}_* \otimes PH_*(MO;\mathbb{F}_2)$, where $PH_*(MO;\mathbb{F}_2) \subset H_*(MO;\mathbb{F}_2)$ is the subalgebra of $\mathscr{A}_*$-comodule primitives. Here

$$PH_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k - 1]\,,$$

with $\tilde{a}_m \equiv a_m$ *modulo algebra decomposables for all* $m \neq 2^k - 1$*.*

Proof. Recall that

$$H_*(BO; \mathbb{F}_2) = \mathbb{F}_2[a_m \mid m \geq 1]$$

is generated as a commutative algebra by the images of the additive generators α_m of $\tilde{H}_*(BO(1); \mathbb{F}_2) = \mathbb{F}_2\{\alpha_m \mid m \geq 1\}$ under the inclusion $\mathbb{R}P^\infty \simeq BO(1) \to BO$. The colimit over n of the Thom isomorphisms

$$U_{\gamma^n} \cap -\colon \tilde{H}_{*+n}(MO_n; \mathbb{F}_2) = \tilde{H}_{*+n}(\mathrm{Th}(\gamma^n); \mathbb{F}_2) \xrightarrow{\cong} H_*(BO(n); \mathbb{F}_2)$$

defines a stable Thom isomorphism

$$\Phi\colon H_*(MO; \mathbb{F}_2) \xrightarrow{\cong} H_*(BO; \mathbb{F}_2)\,.$$

We first calculate the $\mathscr{A}_*$-coaction on $\tilde{H}_{*+1}(MO_1; \mathbb{F}_2)$. Note that $S(\gamma^1) = EO(1) \times_{O(1)} S^0 \cong EO(1) \simeq *$ is contractible, so in the homotopy cofiber sequence

$$S(\gamma^1) \xrightarrow{\pi} BO(1) \xrightarrow{z} \mathrm{Th}(\gamma^1) = MO_1$$

the zero-section z is a homotopy equivalence. It follows that z_* maps $\alpha_{m+1} \in \tilde{H}_{m+1}(BO(1); \mathbb{F}_2)$ to the generator $z_*(\alpha_{m+1})$ of $\tilde{H}_{m+1}(MO_1; \mathbb{F}_2)$ that corresponds to $\alpha_m \in H_m(BO(1); \mathbb{F}_2)$ under the Thom isomorphism $U_{\gamma^1} \cap -$, and which therefore stabilizes to $a_m \in H_m(MO; \mathbb{F}_2)$.

$$\begin{array}{ccccc}
\tilde{H}_{*+1}(BO(1); \mathbb{F}_2) & \xrightarrow[\cong]{z_*} & \tilde{H}_{*+1}(MO_1; \mathbb{F}_2) & \xrightarrow[\cong]{U_{\gamma^1} \cap -} & H_*(BO(1); \mathbb{F}_2) \\
 & & \downarrow & & \downarrow \\
 & & \tilde{H}_{*+n}(MO_n; \mathbb{F}_2) & \xrightarrow[\cong]{U_{\gamma^n} \cap -} & H_*(BO(n); \mathbb{F}_2) \\
 & & \downarrow & & \downarrow \\
 & & H_*(MO; \mathbb{F}_2) & \xrightarrow[\cong]{\Phi} & H_*(BO; \mathbb{F}_2)
\end{array}$$

From [Swi73], see Chapter 2, Lemma 8.3, we know that $\nu\colon H_*(BO(1); \mathbb{F}_2) \to \mathscr{A}_* \otimes H_*(BO(1); \mathbb{F}_2)$ satisfies

$$\nu(\alpha_{m+1}) = \sum_{n=0}^{m} (Z^{n+1})_{m-n} \otimes \alpha_{n+1}\,,$$

where $Z = 1 + \zeta_1 + \zeta_2 + \dots$ is a formal sum in $\mathscr{A}_*$. This implies that $\nu\colon H_*(MO; \mathbb{F}_2) \to \mathscr{A}_* \otimes H_*(MO; \mathbb{F}_2)$ satisfies

$$\nu(a_m) = \sum_{n=0}^{m} (Z^{n+1})_{m-n} \otimes a_n\,,$$

where $a_0 = 1$. Modulo decomposable products, this equals

$$\nu(a_m) \equiv \begin{cases} \zeta_k \otimes 1 + 1 \otimes a_m & \text{if } m = 2^k - 1, \\ 1 \otimes a_m & \text{otherwise.} \end{cases}$$

Let $f\colon H_*(MO; \mathbb{F}_2) \to \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1]$ be the algebra homomorphism given by

$$f(a_m) = \begin{cases} 0 & \text{if } m = 2^k - 1, \\ \bar{a}_m & \text{otherwise.} \end{cases}$$

The composite

$$\phi\colon H_*(MO;\mathbb{F}_2) \xrightarrow{\nu} \mathscr{A}_* \otimes H_*(MO;\mathbb{F}_2) \xrightarrow{1\otimes f} \mathscr{A}_* \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k-1]$$

is then a left $\mathscr{A}_*$-comodule algebra homomorphism

$$\mathbb{F}_2[a_m \mid m \geq 1] \longrightarrow \mathbb{F}_2[\zeta_k \mid k \geq 1] \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k-1]$$

satisfying

$$\phi(a_m) \equiv \begin{cases} \zeta_k \otimes 1 & \text{if } m = 2^k-1, \\ 1 \otimes \bar{a}_m & \text{otherwise} \end{cases}$$

modulo decomposables, and is therefore an isomorphism. Let

$$PH_*(MO;\mathbb{F}_2) = \{x \in H_*(MO;\mathbb{F}_2) \mid \nu(x) = 1 \otimes x\}$$

be the subalgebra of $\mathscr{A}_*$-comodule primitives. It maps isomorphically by $P\phi$ to

$$P(\mathscr{A}_* \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k-1]) = \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k-1]\,,$$

hence has the form

$$PH_*(MO;\mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k-1] \subset H_*(MO;\mathbb{F}_2)$$

where $\tilde{a}_m \equiv a_m$ modulo decomposables, for each $m \neq 2^k-1$. □

Corollary 5.2. *$H^*(MO;\mathbb{F}_2) \cong \mathscr{A} \otimes PH^*(MO;\mathbb{F}_2)^\vee$ is a free $\mathscr{A}$-module of finite type, with basis dual to the monomial basis for $PH_*(MO;\mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k-1]$.*

Theorem 5.3 ([Tho54])**.** *The mod 2 Hurewicz homomorphism*

$$h\colon \pi_*(MO) \longrightarrow H_*(MO;\mathbb{F}_2)$$

maps the $\mathbb{F}_2$-algebra $\pi_*(MO) \cong \Omega^O_*$ isomorphically to

$$PH_*(MO;\mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k-1]\,.$$

Proof. Let $\{\tilde{a}^I\}_I$ be the monomial basis for $PH_*(MO;\mathbb{F}_2)$, and let $\{\tilde{a}_I^\vee\}_I$ be the dual basis, corresponding to an $\mathscr{A}$-module basis for $H^*(MO;\mathbb{F}_2)$. For each I let $|I|$ denote the degree of $\tilde{a}_I^\vee$, and let

$$g_I\colon MO \longrightarrow \Sigma^{|I|} H\mathbb{F}_2$$

be a map of spectra representing $\tilde{a}_I^\vee$. Let

$$\prod_I g_I\colon MO \longrightarrow \prod_I \Sigma^{|I|} H\mathbb{F}_2$$

be the product of these maps. Since there are only finitely many basis elements below any given degree, the inclusion

$$\bigvee_I \Sigma^{|I|} H\mathbb{F}_2 \xrightarrow{\simeq} \prod_I \Sigma^{|I|} H\mathbb{F}_2$$

is an equivalence of spectra. The resulting chain of maps

$$g\colon MO \longrightarrow \prod_I \Sigma^{|I|} H\mathbb{F}_2 \simeq \bigvee_I \Sigma^{|I|} H\mathbb{F}_2$$

induces an isomorphism of $\mathscr{A}$-modules

$$H^*(g;\mathbb{F}_2)\colon \bigoplus_I H^*(\Sigma^{|I|} H\mathbb{F}_2) \cong \prod_I H^*(\Sigma^{|I|} H\mathbb{F}_2) = H^*(\bigvee_I \Sigma^{|I|} H\mathbb{F}_2;\mathbb{F}_2)$$
$$\xrightarrow{g^*} H^*(MO;\mathbb{F}_2)\,,$$

and can therefore be shown to be an equivalence. It must therefore also induce an isomorphism in homotopy

$$\pi_*(g)\colon \pi_*(MO) \xrightarrow{\cong} \pi_*(\bigvee_I \Sigma^{|I|} H\mathbb{F}_2)$$

$$\cong \bigoplus_I \pi_*(\Sigma^{|I|} H\mathbb{F}_2) = \mathbb{F}_2\{\tilde{a}^I\}_I = PH_*(MO;\mathbb{F}_2)\,.$$

□

6. Complex bordism

To calculate the graded commutative ring $\Omega^U_* = \pi_*(MU)$, Milnor [Mil60] and Novikov [Nov60] again compared the homology of MU with the homology of spectra X such that $\pi_*(X)$ is known. More precisely, they follow Adams [Ada58] and resolve MU by a tower of spectra

$$\dots \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} \dots \xrightarrow{\alpha} Y_0 \simeq MU$$

such that each cofiber

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} K_s \xrightarrow{\gamma} \Sigma Y_{s+1}$$

is a wedge sum of suspensions of Eilenberg–MacLane spectra. This leads to a case of the Adams spectral sequence. A posteriori, this amounts to a comparison with (wedge sums of suspensions of) the Brown–Peterson spectra BP, one for each prime p.

We discuss the odd-primary case (the case $p = 2$ is similar), so that

$$\mathscr{A}_* = \Lambda(\tau_i \mid i \ge 0) \otimes \mathbb{F}_p[\xi_i \mid i \ge 1]$$

with $|\tau_i| = 2p^i - 1$ and $|\xi_i| = 2p^i - 2$. Note that

$$\mathscr{E}_* = \Lambda(\tau_i \mid i \ge 0)$$

is a primitively generated quotient bialgebra of $\mathscr{A}_*$, and

$$\mathscr{P}_* = \mathbb{F}_p[\xi_i \mid i \ge 1] = \mathscr{A}_* \,\square_{\mathscr{E}_*}\, \mathbb{F}_p$$

is a sub bialgebra of $\mathscr{A}_*$. Dually,

$$\mathscr{E} = \Lambda(Q_i \mid i \ge 0)$$

is a primitively generated sub bialgebra of $\mathscr{A}$, and

$$\mathscr{P} = \mathscr{A} \otimes_{\mathscr{E}} \mathbb{F}_p$$

is a quotient bialgebra, sometimes denoted $\mathscr{P} = \mathscr{A}//\mathscr{E}$. The classes $Q_i \in \mathscr{E} \subset \mathscr{A}$ are called the Milnor primitives, and can be iteratively defined by $Q_0 = \beta$ (the Bockstein homomorphism) and

$$Q_{i+1} = [P^{p^i}, Q_i] = P^{p^i} Q_i - Q_i P^{p^i}$$

for $i \ge 0$.

Let

$$H_*(MU;\mathbb{F}_p) = \operatorname*{colim}_n H_{*+n}(MU_n;\mathbb{F}_p)$$

with the induced $\mathscr{A}_*$-coaction. The $\mathbb{F}_p$-linear dual

$$H^*(MU;\mathbb{F}_p) = \lim_n H^{*+n}(MU_n;\mathbb{F}_p)$$

has the dual $\mathscr{A}$-action.

Theorem 6.1. *The $\mathscr{A}_*$-comodule algebra*

$$H_*(MU;\mathbb{F}_p) \cong \mathbb{F}_p[b_m \mid m \geq 1]$$

is isomorphic to $\mathscr{P}_* \otimes PH_*(MU;\mathbb{F}_p)$, where $PH_*(MU;\mathbb{F}_p) \subset H_*(MU;\mathbb{F}_p)$ is the subalgebra of $\mathscr{A}_*$-comodule primitives. Here

$$PH_*(MU;\mathbb{F}_p) \cong \mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1]\,,$$

with $\tilde{b}_m \equiv b_m$ modulo algebra decomposables for all $m \neq p^k - 1$.

Proof. Recall that

$$H_*(BU;\mathbb{F}_p) = \mathbb{F}_p[b_m \mid m \geq 1]$$

is generated as a commutative algebra by the images of the additive generators β_m of $\tilde{H}_*(BU(1);\mathbb{F}_p) = \mathbb{F}_p\{\beta_m \mid m \geq 1\}$ under the inclusion $\mathbb{C}P^\infty \simeq BU(1) \to BU$. The colimit over n of the Thom isomorphisms

$$U_{\gamma^n} \cap -\colon \tilde{H}_{*+2n}(MU_{2n};\mathbb{F}_p) = \tilde{H}_{*+2n}(\mathrm{Th}(\gamma^n);\mathbb{F}_p) \xrightarrow{\cong} H_*(BU(n);\mathbb{F}_p)$$

defines a stable Thom isomorphism

$$\Phi\colon H_*(MU;\mathbb{F}_p) \xrightarrow{\cong} H_*(BU;\mathbb{F}_p)\,.$$

We first calculate the $\mathscr{A}_*$-coaction on $\tilde{H}_{*+2}(MU_2;\mathbb{F}_p)$. Note that $S(\gamma^1) = EU(1) \times_{U(1)} S^1 \cong EU(1) \simeq *$ is contractible, so in the homotopy cofiber sequence

$$S(\gamma^1) \xrightarrow{\pi} BU(1) \xrightarrow{z} \mathrm{Th}(\gamma^1) = MU_2$$

the zero-section z is a homotopy equivalence. It follows that z_* maps $\beta_{m+1} \in \tilde{H}_{2m+2}(BU(1);\mathbb{F}_p)$ to the generator $z_*(\beta_{m+1})$ of $\tilde{H}_{2m+2}(MU_2;\mathbb{F}_p)$ that corresponds to $\beta_m \in H_{2m}(BU(1);\mathbb{F}_p)$ under the Thom isomorphism $U_{\gamma^1}\cap-$, and which therefore stabilizes to $b_m \in H_{2m}(MU;\mathbb{F}_p)$.

$$\begin{array}{ccccc}
\tilde{H}_{*+2}(BU(1);\mathbb{F}_p) & \xrightarrow[\cong]{z_*} & \tilde{H}_{*+2}(MU_2;\mathbb{F}_p) & \xrightarrow[\cong]{U_{\gamma^1}\cap-} & H_*(BU(1);\mathbb{F}_p) \\
 & & \downarrow & & \downarrow \\
 & & \tilde{H}_{*+2n}(MU_{2n};\mathbb{F}_p) & \xrightarrow[\cong]{U_{\gamma^n}\cap-} & H_*(BU(n);\mathbb{F}_p) \\
 & & \downarrow & & \downarrow \\
 & & H_*(MU;\mathbb{F}_p) & \xrightarrow[\cong]{\Phi} & H_*(BU;\mathbb{F}_p)
\end{array}$$

From [Swi73, Thm. 1(ii)] we know that $\nu\colon H_*(BU(1);\mathbb{F}_p) \to \mathscr{A}_* \otimes H_*(BU(1);\mathbb{F}_p)$ satisfies

$$\nu(\beta_{m+1}) = \sum_{n=0}^{m} (X^{n+1})_{2m-2n} \otimes \beta_{n+1}\,.$$

where $X = 1 + \xi_1 + \xi_2 + \dots$. This implies that $\nu\colon H_*(MU;\mathbb{F}_p) \to \mathscr{A}_* \otimes H_*(MU;\mathbb{F}_p)$ satisfies

$$\nu(b_m) = \sum_{n=0}^{m} (X^{n+1})_{2m-2n} \otimes b_n\,,$$

where $b_0 = 1$. Modulo decomposable products, this equals

$$\nu(b_m) \equiv \begin{cases} \xi_k \otimes 1 + 1 \otimes b_m & \text{if } m = p^k - 1, \\ 1 \otimes b_m & \text{otherwise.} \end{cases}$$

In particular, the $\mathscr{A}_*$-coaction factors as

$$H_*(MU;\mathbb{F}_p) \xrightarrow{\tilde\nu} \mathscr{P}_* \otimes H_*(MU;\mathbb{F}_p) \subset \mathscr{A}_* \otimes H_*(MU;\mathbb{F}_p)\,,$$

making $H_*(MU;\mathbb{F}_p)$ a $\mathscr{P}_*$-comodule algebra.

Let $f\colon H_*(MU;\mathbb{F}_p) \to \mathbb{F}_p[\bar b_m \mid m \ne p^k-1]$ be the algebra homomorphism given by

$$f(b_m) = \begin{cases} 0 & \text{if } m = p^k-1, \\ \bar b_m & \text{otherwise.} \end{cases}$$

The composite

$$\phi\colon H_*(MU;\mathbb{F}_p) \xrightarrow{\tilde\nu} \mathscr{P}_* \otimes H_*(MU;\mathbb{F}_p) \xrightarrow{1\otimes f} \mathscr{P}_* \otimes \mathbb{F}_p[\bar b_m \mid m \ne p^k-1]$$

is then a left $\mathscr{P}_*$-comodule algebra homomorphism

$$\mathbb{F}_p[b_m \mid m \ge 1] \longrightarrow \mathbb{F}_p[\xi_k \mid k \ge 1] \otimes \mathbb{F}_p[\bar b_m \mid m \ne p^k-1]$$

satisfying

$$\phi(b_m) \equiv \begin{cases} \xi_k \otimes 1 & \text{if } m = p^k-1, \\ 1 \otimes \bar b_m & \text{otherwise} \end{cases}$$

modulo decomposables, and is therefore an isomorphism. Let

$$PH_*(MU;\mathbb{F}_p) = \{x \in H_*(MU;\mathbb{F}_p) \mid \nu(x) = 1 \otimes x\}$$

be the subalgebra of $\mathscr{A}_*$-comodule primitives, which is equal to the subalgebra of $\mathscr{P}_*$-comodule primitives. It maps isomorphically by $P\phi$ to

$$P(\mathscr{P}_* \otimes \mathbb{F}_p[\bar b_m \mid m \ne p^k-1]) = \mathbb{F}_p[\bar b_m \mid m \ne p^k-1]\,,$$

hence has the form

$$PH_*(MU;\mathbb{F}_p) = \mathbb{F}_p[\tilde b_m \mid m \ne p^k-1] \subset H_*(MU;\mathbb{F}_p)$$

where $\tilde b_m \equiv b_m$ modulo decomposables, for each $m \ne p^k-1$. □

Recall that $\mathscr{P} = \mathscr{A} \otimes_{\mathscr{E}} \mathbb{F}_p = \mathscr{A}//\mathscr{E}$ is a cyclic $\mathscr{A}$-module algebra.

Corollary 6.2. *$H^*(MU;\mathbb{F}_p) \cong \mathscr{P} \otimes PH^*(MU;\mathbb{F}_p)^\vee$ is a free $\mathscr{P}$-module of finite type, with basis dual to the monomial basis for $PH_*(MU;\mathbb{F}_p) = \mathbb{F}_p[\tilde b_m \mid m \ne p^k-1]$.*

Theorem 6.3.

$$\pi_*(MU_p^\wedge) \cong \mathbb{Z}_p[v_i \mid i \ge 1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\tilde b_m \mid m \ne p^k-1]$$

where $|v_i| = 2p^i-2$ for each $i \ge 1$, and the mod p Hurewicz homomorphism $h\colon \pi_*(MU) \to H_*(MU;\mathbb{F}_p)$ maps $\pi_*(MU_p^\wedge)$ onto $PH_*(MU;\mathbb{F}_p)$.

Proof. This is easiest seen using the mod p Adams spectral sequence. Let $\{\tilde b^I\}_I$ be the monomial basis for $PH_*(MU;\mathbb{F}_p)$, and let $\{\tilde b_I^\vee\}_I$ be the dual basis. We obtain isomorphisms of $\mathscr{A}_*$-comodule algebras

$$H_*(MU;\mathbb{F}_p) \xrightarrow{\cong} \bigoplus_I \Sigma^{|I|}\mathscr{P}_*$$

and of $\mathscr{A}$-module coalgebras

$$\bigoplus_I \Sigma^{|I|}\mathscr{P} \xrightarrow{\cong} H^*(MU;\mathbb{F}_p)\,.$$

Hence the Adams spectral sequence, in its homological form

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*(MU;\mathbb{F}_p)) \Longrightarrow_s \pi_{t-s}(MU_p^\wedge)$$

or its cohomological form

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(MU;\mathbb{F}_p), \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(MU_p^\wedge)$$

is an algebra spectral sequence with E_2-term

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{P}_*) \otimes PH_*(MU;\mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{P}, \mathbb{F}_p) \otimes PH_*(MU;\mathbb{F}_p)\,.$$

Since $\mathscr{A}$ is a bialgebra and $\mathscr{E}$ a sub bialgebra, [MM65, Thm. 4.4, Thm. 4.7] imply that $\mathscr{A}$ is free a left $\mathscr{E}$-module, and $\mathscr{A}_*$ is cofree as a left $\mathscr{E}_*$-comodule, so there are change-of-rings isomorphisms

$$\begin{aligned}\mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{P}_*) &= \mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{A}_* \,\Box_{\mathscr{E}_*}\, \mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{E}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)\\ \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{P}, \mathbb{F}_p) &= \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}/\!/\mathscr{E}, \mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{E}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)\,.\end{aligned}$$

Since $\mathscr{E}_* = \Lambda(\tau_i \mid i \ge 0)$ and $\mathscr{E}_* = \Lambda(Q_i \mid i \ge 0)$, standard homological algebra shows that

$$\mathrm{Ext}_{\mathscr{E}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) = \mathrm{Ext}_{\mathscr{E}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[q_i \mid i \ge 0]$$

with $q_i \in \mathrm{Ext}^{1,2p^i-1}(\mathbb{F}_p, \mathbb{F}_p)$ representing an extension detected by Q_i. Hence

$$E_2^{*,*} \cong \mathbb{F}_p[q_i \mid i \ge 0] \otimes PH_*(MU;\mathbb{F}_p)$$

is concentrated in even topological degrees $t-s$. There is therefore no room for non-zero differentials, since these decrease the topological degree by 1. Hence $E_2^{*,*} = E_\infty^{*,*}$. Since the E_∞-term is free as a graded commutative $\mathbb{F}_p$-algebra, there can only be additive extensions, with multiplication by p in the abutment being represented by multiplication by q_0 in the E_∞-term, and it follows that

$$\pi_*(MU_p^\wedge) \cong \mathbb{Z}_p[v_i \mid i \ge 1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\tilde{b}_m \mid m \ne p^k - 1]$$

with v_i in degree $|v_i| = 2p^i - 2$ being detected by q_i, for each $i \ge 1$. □

Note that as a $\mathbb{Z}_p$-algebra, $\pi_*(MU_p^\wedge)$ has one polynomial generator in each positive even degree $2m$, which is of the form v_i if $2m = 2p^i - 2$, and of the form $\tilde{b}_m$ otherwise. Serre proved that $\pi_*(S) \otimes \mathbb{Q} \cong \mathbb{Q}$, so

$$\pi_*(MU_{\mathbb{Q}}) = \pi_*(MU) \otimes \mathbb{Q} \cong H_*(MU;\mathbb{Q}) \cong H_*(BU;\mathbb{Q}) \cong \mathbb{Q}[b_k \mid k \ge 1]$$

is also polynomial on one generator in each positive even degree. Further work with the arithmetic square

$$\begin{array}{ccc} MU & \longrightarrow & MU_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ MU^\wedge & \longrightarrow & (MU^\wedge)_{\mathbb{Q}}\,, \end{array}$$

where $MU_{\mathbb{Q}} = MU[1/2, \dots, 1/p, \dots]$ denotes the rationalization of MU and $MU^\wedge = \prod_p MU_p^\wedge$ denotes its profinite completion, leads to the following integral result.

Theorem 6.4 ([Mil60], [Nov60]).

$$\Omega_*^U = \pi_*(MU) \cong \mathbb{Z}[x_i \mid i \ge 1]$$

where $|x_i| = 2i$ *for each* $i \ge 1$.

Theorem 6.5. *The Hurewicz homomorphism*

$$h\colon \pi_*(MU) \longrightarrow H_*(MU)$$

satisfies

$$h(x_m) \equiv \begin{cases} pb_m & \text{if } m = p^i - 1 \text{ for some prime } p, \\ b_m & \text{otherwise,} \end{cases}$$

modulo decomposables, for each $m \geq 1$.

Note that $m+1 \geq 2$ can be equal to a prime power p^i for at most one prime p.

7. Framed bordism

The $\mathscr{A}_*$-comodule algebra $H_*(S;\mathbb{F}_p) = \mathbb{F}_p$ has the trivial coaction (via the coaugmentation $\eta\colon \mathbb{F}_p \to \mathscr{A}_*$), and dually the $\mathscr{A}$-module coalgebra $H^*(S;\mathbb{F}_p) = \mathbb{F}_p$ has the trivial action (via the augmentation $\epsilon\colon \mathscr{A} \to \mathbb{F}_p$).

Theorem 7.1. *The mod* p *Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, \mathbb{F}_p) = \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(S^\wedge_p)$$

converges to the p*-completion of* $\Omega^{\mathrm{fr}}_* = \pi_*(S)$.

This spectral sequence is only partially understood.

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