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CHROMATIC HOMOTOPY THEORY

CHAPTER 9: FORMAL GROUP LAWS

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See Adams [Ada74, Part II] for an early but standard exposition of Quillen's work on formal group laws and complex bordism. The appendix [Rav86, A2] is another standard reference on formal group laws for algebraic topologists.

For many ring spectra E the computation of the cohomology rings $E^*(\mathbb{C}P^m)$, $E^*(\mathbb{C}P^\infty)$, $E^*(BU(n))$ and $E^*(BU)$, and of the homology algebras $E_*(BU)$ and $E_*(MU)$, follow the same lines as in the case of ordinary cohomology, and the results carry no additional information beyond the coefficient ring $\pi_*(E)$. However, the map $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of complex line bundles, induced by the (abelian) group multiplication $U(1) \times U(1) \to U(1)$, often induces a completed Hopf algebra structure

$$m^*\colon E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}_{E^*}} E^*(\mathbb{C}P^\infty)\,,$$

and it is an insight of Novikov and Quillen that this carries significant additional information about the ring spectrum E. These completed Hopf algebras will corepresent commutative one-dimensional formal groups, and can, with a choice of coordinate, be presented as formal group laws. We can thus draw on the algebraic theory of formal groups to shed light on stable homotopy theory.

1. Complex oriented cohomology theories

((ETC: Cite seminar by Dold.)) Let E be a ring spectrum in the homotopy category, with E^* graded commutative. An E-orientation of a $\mathbb{C}^n$-bundle $\xi\colon E \to X$ is a class

$$U_\xi \in \tilde{E}^{*+2n}(\mathrm{Th}(\xi)) \cong E^{*+2n}(D(\xi), S(\xi))$$

that, for each $x \in X$, restricts to a generator of

$$E^{*+2n}(D(\xi)_x, S(\xi)_x) \cong \tilde{E}^{*+2n}(S^{2n}) \cong E^*$$

as a free E^*-module, i.e., as a unit of the graded commutative ring E^*. If X is connected, it suffices to verify this for one $x \in X$. If the universal line bundle $\gamma^1\colon E(\gamma^1) \to \mathbb{C}P^\infty = BU(1)$ admits an E-orientation

$$U_{\gamma^1} \in \tilde{E}^*(\mathrm{Th}(\gamma^1)) = \tilde{E}^*(MU(1))$$

then so does each other complex line bundle, by pullback, and it turns out that this also determines an E-orientation of each finite-dimensional complex vector bundle. The composite

$$S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^\infty \xrightarrow{z} \mathrm{Th}(\gamma^1)$$

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is homotopic to the inclusion of a slice $S^2 \cong D(\gamma^1)_x/S(\gamma^1)_x \to \mathrm{Th}(\gamma^1)$, since the Euler class $e(\gamma^1)$ generates $H^2(\mathbb{C}P^\infty)$. Moreover, the zero-section map z is a homotopy equivalence, since $S(\gamma^1) = S^\infty \simeq *$. Hence an E-orientation of γ^1 is the same as a Thom class

$$y^E \in \tilde{E}^{*+2}(\mathbb{C}P^\infty)$$

whose restriction to

$$\tilde{E}^{*+2}(\mathbb{C}P^1) \cong \tilde{E}^{*+2}(S^2) \cong E^*$$

is a unit in E^*. Some authors, including Adams [Ada74, §II.2], take this to be the definition of a complex orientation y^E of the cohomology theory E. We shall instead work with strict complex orientations, where we assume that the unit in E^* is the unit element $1 \in E^0$.

Definition 1.1. Let E be a ring spectrum up to homotopy, with E^* graded commutative. A (strict) complex orientation of E is a choice of class

$$y^E \in \tilde{E}^2(\mathbb{C}P^\infty)$$

whose restriction to $\tilde{E}^2(\mathbb{C}P^1) \cong E^0$ is the unit element $1 \in E^0$. A complex oriented ring spectrum is a pair (E, y^E) as above. A ring spectrum is complex orientable if it admits a complex orientation.

Example 1.2. Let R be a commutative ring. Ordinary cohomology with R-coefficients has a unique complex orientation

$$y^{HR} \in \tilde{H}^2(\mathbb{C}P^\infty; R) \cong \tilde{H}^2(\mathbb{C}P^1; R) \cong \tilde{H}^2(S^2; R)$$

corresponding to $\Sigma^2(1) \in \tilde{H}^2(S^2; R)$.

Example 1.3. Let KU denote complex K-theory. The class

$$[\gamma^1] - 1 \in \widetilde{KU}^0(\mathbb{C}P^\infty)$$

restricts to the generator

$$u = [\gamma^1_1] - 1 \in \widetilde{KU}^0(\mathbb{C}P^1) \cong \mathbb{Z}\{u\}$$

and would hence give a complex orientation of KU in the lax sense. We instead normalize it, by setting

$$y^{KU} = u^{-1}([\gamma^1] - 1) \in \widetilde{KU}^2(\mathbb{C}P^\infty),$$

which restricts to the unit $u^{-1}u = 1$ in $\widetilde{KU}^2(\mathbb{C}P^1) \cong \mathbb{Z}$.

Example 1.4. Let MU denote complex bordism. The identity $\mathrm{Th}(\gamma^1) = MU(1) = MU_2$ has left adjoint

$$\omega \colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) = \Sigma^\infty_2 MU(1) \longrightarrow MU$$

whose restriction to $S \simeq \Sigma^{-2}\mathbb{C}P^1$ is homotopic to the unit map $\eta \colon S \to MU$. Its homotopy class defines a tautological class

$$y^{MU} = [\omega] \in MU^0(\Sigma^{-2}MU(1)) \cong \widetilde{MU}^2(\mathbb{C}P^\infty)$$

whose restriction to $\widetilde{MU}^2(\mathbb{C}P^1) \cong MU^0$ is the ring unit.

Example 1.5. Any even ring spectrum, i.e., one with E^* concentrated in even degrees, admits a complex orientation, since the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = H^s(\mathbb{C}P^\infty; E^t) \Longrightarrow_s E^{s+t}(\mathbb{C}P^\infty)$$

collapses at the $\mathcal{E}_2$-page for degree reasons. Any choice of class $y^E \in E^2(\mathbb{C}P^\infty)$ detected by $y \in \mathcal{E}_\infty^{2,0} = \mathcal{E}_2^{2,0} = H^2(\mathbb{C}P^\infty; E^0)$ is then a complex orientation.

Example 1.6. The sphere spectrum S, the real K-theory spectrum KO, and the image-of-J-spectrum $J_p^\wedge$, are not complex orientable. This is because in $\mathbb{C}P^2$ the 4-cell is attached to the 2-cell by the Hopf fibration $\eta\colon S^3 \to S^2$, which is detected by a nontrivial Sq^2 in $\tilde{H}^*(C\eta; \mathbb{F}_2) = \tilde{H}^*(\mathbb{C}P^2; \mathbb{F}_2) = \mathbb{F}_2\{y, y^2\}$, and η is detected in $\pi_1(S)$, $\pi_1(KO)$ and $\pi_1(J_2^\wedge)$, so there is a nonzero Atiyah–Hirzebruch differential

$$d_2(y) = y^2\eta$$

in each of these cases. Hence y does not survive to $\mathcal{E}_\infty$, and cannot detect a complex orientation y^E. For odd primes p the $2p$-cell in $\mathbb{C}P^p$ is (stably only) attached to the 2-cell by a map $\alpha_1\colon S^{2p-1} \to S^2$, which is detected by a nontrivial P^1 in $\tilde{H}^*(\mathbb{C}P^p; \mathbb{F}_p) \to \tilde{H}^*(C\alpha_1; \mathbb{F}_p) = \mathbb{F}_p\{y, y^p\}$, and α_1 is detected in $\pi_{2p-3}(S)$ and $\pi_{2p-3}(J_p^\wedge)$, so there is a nonzero Atiyah–Hirzebruch differential

$$d_{2p-2}(y) = y^p\alpha_1$$

in both of these cases. Hence y does not survive to $\mathcal{E}_\infty$ and cannot detect a complex orientation of (S or) $J_p^\wedge$.

Proposition 1.7. *Let (E, y^E) be complex oriented. The Atiyah–Hirzebruch spectral sequences*

$$\begin{aligned}
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^m; E^*) = \mathbb{Z}[y]/(y^{m+1}) \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^m)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^\infty; E^*) = \mathbb{Z}[y] \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^\infty)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^m \times \mathbb{C}P^n; E^*) = \mathbb{Z}[y_1, y_2]/(y_1^{m+1}, y_2^{n+1}) \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^m \times \mathbb{C}P^n)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; E^*) = \mathbb{Z}[y_1, y_2] \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)\\
\mathcal{E}_2^{*,*} &= H^*(BU(1)^n; E^*) = \mathbb{Z}[y_1, \dots, y_n] \otimes E^* &&\Longrightarrow E^*(BU(1)^n)\\
\mathcal{E}_2^{*,*} &= H^*(BU(n); E^*) = \mathbb{Z}[c_1, \dots, c_n] \otimes E^* &&\Longrightarrow E^*(BU(n))\\
\mathcal{E}_2^{*,*} &= H^*(BU; E^*) = \mathbb{Z}[c_k \mid k \ge 1] \otimes E^* &&\Longrightarrow E^*(BU)
\end{aligned}$$

collapse at the $\mathcal{E}_2$-term, and converge strongly to

$$\begin{aligned}
E^*(\mathbb{C}P^m) &\cong E^*[y^E]/((y^E)^{m+1})\\
E^*(\mathbb{C}P^\infty) &\cong E^*[[y^E]]\\
E^*(\mathbb{C}P^m \times \mathbb{C}P^n) &\cong E^*[y_1^E, y_2^E]/((y_1^E)^{m+1}, (y_2^E)^{n+1})\\
E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) &\cong E^*[[y_1^E, y_2^E]]\\
E^*(BU(1)^n) &\cong E^*[[y_1^E, \dots, y_n^E]]\\
E^*(BU(n)) &\cong E^*[[c_1^E, \dots, c_n^E]]\\
E^*(BU) &\cong E^*[[c_k^E \mid k \ge 1]]\,.
\end{aligned}$$

Proof. Consider the case of $\mathbb{C}P^\infty$, with $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$. The class $y^E \in \tilde{E}^2(\mathbb{C}P^\infty)$ is detected by $y \otimes 1 \in \mathcal{E}_2^{2,0}$, which is therefore an infinite cycle (so that $d_r(y \otimes 1) = 0$ for all $r \ge 2$). The spectral sequence algebra structure implies that $y^m \otimes 1$ is also an infinite cycle, for all $m \ge 0$. Since these generate the $\mathcal{E}_2$-term as an E^*-module, and the differentials are E^*-linear, it follows that $d_r = 0$ for all $r \ge 2$, and the spectral sequence collapses. We then prove by induction on m that

$$\frac{E^*(\mathbb{C}P^\infty)}{F^{2m+1}E^*(\mathbb{C}P^\infty)} \cong E^*[y^E]/((y^E)^{m+1})$$

so that

$$E^*(\mathbb{C}P^\infty) = \lim_m \frac{E^*(\mathbb{C}P^\infty)}{F^{2m+1}E^*(\mathbb{C}P^\infty)} \cong \lim_m E^*[y^E]/((y^E)^{m+1}) = E^*[[y^E]] .$$

In the case of $BU(n)$, recall that

$$i_n^* \colon H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n] \longrightarrow H^*(BU(1)^n) = \mathbb{Z}[y_1, \dots, y_n]$$

is injective (with image the symmetric polynomials). Hence $i_n \colon BU(1)^n \to BU(n)$ induces a morphism of Atiyah–Hirzebruch spectral sequences

$$\mathcal{E}_2^{*,*} = H^*(BU(n); E^*) = \mathbb{Z}[c_1, \dots, c_n] \otimes E^* \\ \longrightarrow {}'\mathcal{E}_2^{*,*} = H^*(BU(1)^n; E^*) = \mathbb{Z}[y_1, \dots, y_n] \otimes E^*$$

that is injective at the $\mathcal{E}_2$-term. Since $d_r = 0$ for all $r \ge 2$ in the target spectral sequence, it follows by induction on r that the same holds in the source spectral sequence, so also the Atiyah–Hirzebruch spectral sequence for $BU(n)$ collapses at the $\mathcal{E}_2$-term. ((ETC: Does it follow that we can choose $c_k^E \in E^{2k}(BU(n))$ to map to the k-th elementary symmetric polynomial in $y_1^E, \dots, y_n^E \in E^*(BU(1)^n)$?)) □

The E-cohomology Chern class $c_n^E \in E^{2n}(BU(n))$ lifts to an orientation class $U_{\gamma^n}^E \in \tilde{E}^{2n}(MU(n))$, hence provides natural E-(co-)homology Thom isomorphisms

$$\Phi_\xi^E \colon E^*(X) \xrightarrow{\cong} \tilde{E}^{*+2n}(\mathrm{Th}(\xi))$$
$$\Phi_\xi^E \colon \tilde{E}_{*+2n}(\mathrm{Th}(\xi)) \xrightarrow{\cong} E_*(X)$$

for all $\mathbb{C}^n$-bundles ξ.

Corollary 1.8. *Let (E, y^E) be complex oriented. The Atiyah–Hirzebruch spectral sequences*

$$\mathcal{E}^2_{*,*} = H_*(\mathbb{C}P^\infty; E_*) = \mathbb{Z}\{\beta_k \mid k \ge 0\} \otimes E_* \Longrightarrow E_*(\mathbb{C}P^\infty)$$
$$\mathcal{E}^2_{*,*} = H_*(BU; E_*) = \mathbb{Z}[b_k \mid k \ge 1] \otimes E_* \Longrightarrow E_*(BU)$$
$$\mathcal{E}^2_{*,*} = H_*(MU; E_*) = \mathbb{Z}[b_k \mid k \ge 1] \otimes E_* \Longrightarrow E_*(MU)$$

collapse at the $\mathcal{E}^2$-term, and converge strongly to

$$E_*(\mathbb{C}P^\infty) \cong E_*\{\beta_k^E \mid k \ge 0\}$$
$$E_*(BU) \cong E_*[b_k^E \mid k \ge 1]$$
$$E_*(MU) \cong E_*[b_k^E \mid k \ge 1] .$$

Here $\langle (y^E)^i, \beta_j^E \rangle = \delta_{ij}$ and $\beta_k^E \mapsto b_k^E$ under $E_*(\mathbb{C}P^\infty) \to E_*(BU) \cong E_*(MU)$. Equivalently, $\beta_{k+1}^E \mapsto b_k^E$ under $\tilde{E}_{*+2}(\mathbb{C}P^\infty) \cong \tilde{E}_{*+2}(MU(1)) \to E_*(MU)$.

Remark 1.9. When (E, y^E) is complex oriented, the tower of graded commutative E^*-algebras

$$E^* = E^*(\mathbb{C}P^0) \longleftarrow \dots \longleftarrow E^*(\mathbb{C}P^m) \longleftarrow \dots \longleftarrow E^*(\mathbb{C}P^\infty)$$

corepresents a sequence of affine schemes

$$\operatorname{Spec}(E^*) \longrightarrow \dots \longrightarrow \operatorname{Spec}(E^*(\mathbb{C}P^m)) \longrightarrow \dots \longrightarrow \operatorname{Spec}(E^*(\mathbb{C}P^\infty))$$

over $\operatorname{Spec}(E^*)$, where

$$\begin{aligned}\operatorname{Spec}(E^*(\mathbb{C}P^m))(R) &= \mathcal{CA}lg_{E^*}(E^*(\mathbb{C}P^m), R) \\ &\cong \mathcal{CA}lg_{E^*}(E^*[y]/(y^{m+1}), R) = \{y \in R \mid y^{m+1} = 0\}\end{aligned}$$

for each $R \in \mathcal{CA}lg_{E^*}$. The colimit of this sequence, in sheaves, is the formal scheme

$$\operatorname{Spf}(E^*(\mathbb{C}P^\infty)) = \operatorname*{colim}_m \operatorname{Spec}(E^*(\mathbb{C}P^m))$$

given by

$$\begin{aligned}\operatorname{Spf}(E^*(\mathbb{C}P^\infty))(R) &= \operatorname*{colim}_m \operatorname{Spec}(E^*(\mathbb{C}P^m))(R) \\ &= \operatorname*{colim}_m \{y \in R \mid y^{m+1} = 0\} = \{y \in R \mid y \text{ is nilpotent}\}.\end{aligned}$$

This formal scheme maps to, but is not isomorphic to the scheme $\operatorname{Spec}(E^*(\mathbb{C}P^\infty))$. See Strickland's notes [Str] for (much) more on formal schemes. By passing to (pre-)sheaves we extend the category of affine schemes by building in additional colimits. Only the colimits given by covers in the topology are preserved.

The affine line $\mathbb{A}^1$ over $\operatorname{Spec}(E^*)$ is the affine scheme $\operatorname{Spec}(E^*[y])$. The ideal $I = (y) \subset E^*[y]$ corresponds to the closed subscheme

$$\operatorname{Spec}(E^*[y]/I) \cong \operatorname{Spec}(E^*),$$

which is viewed as the origin (or zero-section) $0 \in \mathbb{A}^1$. The ideal $I^{m+1} = (y^{m+1}) \subset E^*[y]$ then corresponds to the m-th order infinitesimal neighborhood

$$\operatorname{Spec}(E^*[y]/I^{m+1}) \cong \operatorname{Spec}(E^*(\mathbb{C}P^m))$$

of the origin in $\mathbb{A}^1$. The formal colimit

$$\operatorname*{colim}_m \operatorname{Spec}(E^*[y]/I^{m+1}) \cong \operatorname{Spf}(E^*(\mathbb{C}P^\infty))$$

is the union of all of the m-th order infinitesimal neighborhoods, and is called the formal neighborhood $\hat{\mathbb{A}}^1$ of 0 in $\mathbb{A}^1$ over $\operatorname{Spec}(E^*)$. Hence a choice of complex orientation defines an isomorphism

$$\operatorname{Spf}(E^*(\mathbb{C}P^\infty)) \cong \hat{\mathbb{A}}^1$$

over $\operatorname{Spec}(E^*)$, expressing $\operatorname{Spf}(E^*(\mathbb{C}P^\infty))$ as a formal line over this base.

A complex orientable ring spectrum E will typically admit multiple different choices of complex orientations. Let

$$y, y' \in \tilde{E}^2(\mathbb{C}P^\infty)$$

be two such choices. We can then use y to calculate the right hand side, and write y' in terms of this answer. We find

$$\tilde{E}^*(\mathbb{C}P^\infty) = (y) = yE^*[[y]]$$

inside $E^*(\mathbb{C}P^\infty) = E^*[[y]]$, and

$$y' = \sum_{k \geq 0} b_k y^{k+1}$$

for some sequence of coefficients $b_k \in E^*$. Considering degrees, we find that $b_k \in E^{-2k} = E_{2k}$ for each k. The condition that y' (and y) restricts to the unit element in $\widetilde{E}^2(\mathbb{C}P^1) \cong E^0$ is equivalent to the condition $b_0 = 1$, but otherwise the sequence $\{b_k \in E_{2k}\}_{k\geq 1}$ can be freely chosen. We will often write

$$y' = h(y) = y + \sum_{k\geq 1} b_k y^{k+1} .$$

2. Formal group laws

Definition 2.1. Let R be a (graded) commutative ring. A (commutative, one-dimensional) formal group law over R is a formal power series

$$F(y_1, y_2) \in R[[y_1, y_2]] ,$$

satisfying

(1) $F(0, y) = y = F(y, 0)$,
(2) $F(y_1, y_2) = F(y_2, y_1)$,
(3) $F(F(y_1, y_2), y_3) = F(y_1, F(y_2, y_3))$.

It can be denoted

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j} y_1^i y_2^j$$

with $a_{i,j} = a_{j,i}$ for all $i, j \geq 1$, but further relations between the $a_{i,j}$ are required to ensure that the series will satisfy (3). If R is graded we assume that y_1, y_2 and $F(y_1, y_2)$ are all homogeneous of cohomological degree 2, in which case $a_{i,j}$ has cohomological degree $2(1-i-j)$, or homological degree $2(i+j-1)$. We sometimes write

$$y_1 +_F y_2 = F(y_1, y_2)$$

for the sum of y_1 and y_2 with respect to F.

The group multiplication $U(1) \times U(1) \to U(1)$ induces a map $m\colon BU(1) \times BU(1) \cong B(U(1) \times U(1)) \to BU(1)$. It classifies the tensor product of complex line bundles, so that $m^*(\gamma^1) \cong \gamma^1 \mathbin{\widehat{\otimes}} \gamma^1$, and can also be written as $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.

Proposition 2.2. *Let (E, y^E) be a complex oriented ring spectrum. The homomorphism*

$$E^*[[y]] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{m^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1, y_2]]$$

maps $y = y^E$ to a formal group law

$$m^*(y) = F_E(y_1, y_2) \in E^*[[y_1, y_2]]$$

over E^*.

If need be, we write $F_{(E,y)}$ for this formal group law.

Proof. The external tensor product of complex line bundles is unital, commutative and associative up to isomorphism, so m is unital, commutative and associative up to homotopy. This implies that $F_E(y_1, y_2)$ satisfies the conditions for being a formal group law. □

Lemma 2.3. *For each formal group law $F(y_1, y_2)$ over R there exists a unique formal power series $i(y) = i_F(y) \in R[[y]]$ with $F(y, i(y)) = 0$, called the formal negative. It satisfies $i(y) \equiv -y \mod (y^2)$. We sometimes write*

$$-_F y = i_F(y)$$

for the negative of y with respect to F.

Example 2.4. For a commutative ring R, let $y = y^{HR}$ be the unique complex orientation. Then

$$F_{HR}(y_1, y_2) = m^*(y) = y_1 + y_2$$

in $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; R) = R\{y_1, y_2\}$. Each $a_{i,j} = 0$ for $i, j \geq 0$, since these live in trivial groups. This is equal to the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2$$

over R. It expresses addition in coordinates near 0.

Example 2.5. With $E = KU$, recall that $y^{KU} = y = u^{-1}(\gamma^1 - 1)$, so that $\gamma^1 = 1 + uy$ (with implicit passage to isomorphism classes). Hence

$$m^*(\gamma^1) = \gamma^1 \mathbin{\widehat{\otimes}} \gamma^1 = (1 + uy) \otimes (1 + uy) = 1 + uy_1 + uy_2 + u^2 y_1 y_2$$

in $KU^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, and

$$\begin{aligned} F_{KU}(y_1, y_2) &= m^*(y) = u^{-1}(m^*(\gamma^1) - 1) \\ &= u^{-1}(1 + uy_1 + uy_2 + u^2 y_1 y_2 - 1) = y_1 + y_2 + u y_1 y_2 \end{aligned}$$

in $KU^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Here $a_{1,1} = u$, while the remaining $a_{i,j}$ are zero. This equals the multiplicative formal group law

$$F_m(y_1, y_2) = y_1 + y_2 + u y_1 y_2$$

defined over $KU^* = \mathbb{Z}[u^{\pm 1}]$. It expresses multiplication in coordinates near 1.

Example 2.6. With the notation $e(x) = e^x - 1$, the rewriting $e(x_1 + x_2) = e(x_1) + e(x_2) + e(x_1)e(x_2)$ of $e^{x_1 + x_2} = e^{x_1} e^{x_2}$ is equivalent to the addition formula

$$\int_0^{y_1} \frac{dt}{1+t} + \int_0^{y_2} \frac{dt}{1+t} = \int_0^{F(y_1, y_2)} \frac{dt}{1+t} \,,$$

for $\ell(y) = \int_0^y dt/(1+t) = \log(1+y)$, with

$$F(y_1, y_2) = y_1 + y_2 + y_1 y_2$$

equal to the multiplicative formal group law.

The addition formula

$$\sin(x_1 + x_2) = \sin(x_1)\sqrt{1 - \sin^2(x_2)} + \sin(x_2)\sqrt{1 - \sin^2(x_1)}$$

(for x_1 and x_2 with non-negative cosine) is equivalent to the addition formula

$$\int_0^{y_1} \frac{dt}{\sqrt{1-t^2}} + \int_0^{y_2} \frac{dt}{\sqrt{1-t^2}} = \int_0^{F(y_1, y_2)} \frac{dt}{\sqrt{1-t^2}}$$

for $\arcsin(y) = \int_0^y dt/\sqrt{1-t^2}$, with

$$\begin{aligned} F(y_1, y_2) &= y_1 \sqrt{1 - y_2^2} + y_2 \sqrt{1 - y_1^2} \\ &= y_1 + y_2 - \frac{1}{2}(y_1^2 y_2 + y_1 y_2^2) + \dots . \end{aligned}$$

Euler (written 1751, published 1761) obtained a similar addition theorem

$$\int_0^{y_1} \frac{dt}{\sqrt{1-t^4}} + \int_0^{y_2} \frac{dt}{\sqrt{1-t^4}} = \int_0^{F(y_1, y_2)} \frac{dt}{\sqrt{1-t^4}}$$

for the elliptic integral $\int_0^y dt/\sqrt{1-t^4}$ (related to arc length on ellipses), with

$$\begin{aligned} F(y_1, y_2) &= \frac{y_1\sqrt{1-y_2^4} + y_2\sqrt{1-y_1^4}}{1+y_1^2y_2^2} \\ &= y_1 + y_2 - \frac{1}{2}(y_1^4y_2 + y_1y_2^4) - (y_1^3y_2^2 + y_1^2y_2^3) + \dots . \end{aligned}$$

The formal power series expansions of the latter two expressions $F(y_1, y_2)$ define formal group laws over $\mathbb{Q}$. The latter is an example of an elliptic formal group law. Addition theorems for general elliptic integrals, and even more general hyperelliptic integrals, were among the famous achievements of Abel (ca. 1827), sometimes in competition with Jacobi.

Definition 2.7. Let R be a (graded) commutative ring, and let $F(y_1, y_2)$ and $F'(y_1, y_2)$ be formal group laws defined over R. A homomorphism

$$h\colon F \longrightarrow F'$$

defined over R is a formal power series $h(y) \in R[[y]]$ satisfying

(1) $h(0) = 0$,
(2) $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$.

It can be written

$$h(y) = \sum_{k\geq 0} b_k y^{k+1}$$

with $|b_k| = 2k$. We can rewrite (2) as

$$h(y_1 +_F y_2) = h(y_1) +_{F'} h(y_2) .$$

The identity homomorphism $\mathrm{id}\colon F \to F$ is the formal power series $\mathrm{id}(y) = y$. The composite $h' \circ h = h'h$ of two homomorphisms $h\colon F \to F'$ and $h'\colon F' \to F''$ is the composite formal power series $h'(h(y)) \in R[[y]]$.

Lemma 2.8. *Let R be a (graded) commutative ring. The formal group laws defined over R are the objects of a small category $\mathcal{FGL}(R)$, with morphisms from F to F' given by the homomorphisms defined over R.*

$$\begin{aligned} \mathrm{obj}\,\mathcal{FGL}(R) &= \{F(y_1, y_2) \in R[[y_1, y_2]] \mid F \textit{ is a formal group law}\} \\ \mathcal{FGL}(R)(F, F') &= \{h(y) \in R[[y]] \mid h\colon F \to F' \textit{ is a homomorphism}\} . \end{aligned}$$

Lemma 2.9. *A homomorphism $h\colon F \to F'$ over R, with $h(y) = \sum_{k\geq 0} b_k y^{k+1}$, is an isomorphism if and only if $b_0 = h'(0)$ is a unit in R. In this case F and F' mutually determine one another, by*

$$\begin{aligned} F'(y_1, y_2) &= h(F(h^{-1}(y_1), h^{-1}(y_2))) \\ F(y_1, y_2) &= h^{-1}(F'(h(y_1), h(y_2))) . \end{aligned}$$

Here $h'(0)$ denotes the formal derivative of h at $y = 0$.

Definition 2.10. A strict isomorphism $h\colon F \to F'$ is a homomorphism with $h'(0) = 1$, so that $h(y) \equiv y \mod (y^2)$. Let

$$\mathcal{FGL}_s(R) \subset \mathcal{FGL}_i(R) \subset \mathcal{FGL}(R)$$

be the subcategories of all strict isomorphisms, and all isomorphisms, in $\mathcal{FGL}(R)$. These are both groupoids. ((ETC: These notations are not standardized.))

Proposition 2.11. *Let y and y' be two (strict) complex orientations of the same ring spectrum E, with $y' = h(y)$. Let $F(y_1, y_2) = m^*(y)$ and $F'(y'_1, y'_2) = m^*(y')$ be the associated formal group laws. Then $h\colon F \to F'$ is a strict isomorphism defined over E^*.*

If need be, we can spell out this strict isomorphism as

$$h\colon F_{(E,y)} \xrightarrow{\cong} F_{(E,h(y))} \,.$$

Proof. We saw earlier that $h(y) = y + \sum_{k\ge1} b_k y^{k+1}$ with $b_k \in E^{-2k}$. We calculate

$$\begin{aligned} h(F(y_1, y_2)) &= h(m^*(y)) = m^*(h(y)) \\ &= m^*(y') = F'(y'_1, y'_2) = F'(h(y_1), h(y_2)) \,, \end{aligned}$$

using that m^* is a continuous ring homomorphism. □

3. The Lazard ring

We now consider the functorial dependence of complex orientations on the ring spectrum E, and of formal groups and their homomorphisms on the ring R.

Definition 3.1. Let $g\colon R \to T$ be a homomorphism of (graded) commutative rings. For each formal group law

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge1} a_{i,j} y_1^i y_2^j$$

defined over R we define the pullback g^*F to be the formal group law

$$(g^*F)(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge1} g(a_{i,j}) y_1^i y_2^j$$

defined over T. For each homomorphism $h\colon F \to F'$ between formal group laws defined over R, with

$$h(y) = y + \sum_{k\ge1} b_k y^{k+1} \,,$$

we define $g^*h\colon g^*F \to g^*F'$ to be the homomorphism

$$(g^*h)(y) = y + \sum_{k\ge1} g(b_k) y^{k+1} \,.$$

Here $g(a_{i,j}), g(b_k) \in T$ denote the respective images of $a_{i,j}, b_k \in R$ under g. The terminology and notation is that of algebraic geometry, where we think of g as a map $g\colon \mathrm{Spec}(T) \to \mathrm{Spec}(R)$, so that g^*F is obtained by pulling back an object over $\mathrm{Spec}(R)$ along g to give an object over $\mathrm{Spec}(T)$, and similarly for g^*h.

Lemma 3.2. *Pullback along any ring homomorphism $g\colon R \to T$ defines a function*

$$g^*\colon \mathrm{obj}\,\mathcal{FGL}(R) \longrightarrow \mathrm{obj}\,\mathcal{FGL}(T) \,.$$

Pullback along the identity induces the identity, and

$$k^* \circ g^* = (kg)^*\colon \mathrm{obj}\,\mathcal{FGL}(R) \longrightarrow \mathrm{obj}\,\mathcal{FGL}(U)$$

for any second ring homomorphism $k\colon T \to U$, so

$$\begin{aligned} \mathrm{obj}\,\mathcal{FGL}\colon \mathcal{C}Ring &\longrightarrow \mathcal{S}et \\ R &\longmapsto \mathrm{obj}\,\mathcal{FGL}(R) \end{aligned}$$

is a covariant functor. Writing $\mathcal{A}ff = \mathcal{C}Ring^{op}$*, it defines a presheaf*

$$\begin{aligned} \operatorname{obj}\mathcal{FGL}\colon \mathcal{A}ff^{op} &\longrightarrow \mathcal{S}\mathrm{et} \\ \operatorname{Spec}(R) &\longmapsto \operatorname{obj}\mathcal{FGL}(R)\,. \end{aligned}$$

Proof. This says that g^*F is again a formal group law, that $\mathrm{id}^* F = F$, and that $k^*(g^*(F)) = (kg)^*(F)$, all of which are obvious. □

Passing from sets to small groupoids, we have the following extension of Lemma 3.2, which also accounts for the strict isomorphisms between formal group laws.

Lemma 3.3. *Pullback along any* $g\colon R \to T$ *defines a functor*

$$\begin{aligned} g^*\colon \mathcal{FGL}_s(R) &\longrightarrow \mathcal{FGL}_s(T) \\ F &\longmapsto g^*F \\ h &\longmapsto g^*h\,. \end{aligned}$$

Pullback along the identity induces the identity, and

$$k^* \circ g^* = (kg)^*\colon \mathcal{FGL}(R) \longrightarrow \mathcal{FGL}(U)$$

for any $k\colon T \to U$*, so*

$$\begin{aligned} \mathcal{FGL}_s\colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s(R) \end{aligned}$$

is a covariant functor.

Proof. This says that g^*h is again a strict isomorphism, that $g^*(h'h) = (g^*h')(g^*h)$, that $\mathrm{id}^* h = h$, and that $k^*(g^*(h)) = (kg)^*(h)$, all of which are obvious. □

Definition 3.4. Identifying $\mathcal{C}Ring$ with $\mathcal{A}ff^{op}$, the functor $\mathcal{FGL}_s$ defines a presheaf of small groupoids

$$\begin{aligned} \mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s\colon \mathcal{A}ff^{op} &\longrightarrow \mathcal{G}pd \\ \operatorname{Spec}(R) &\longmapsto \mathcal{FGL}_s(R)\,, \end{aligned}$$

which we call the moduli prestack of formal group laws.

Remark 3.5. To say that $\mathcal{FGL}_s$ is a prestack means that for any two formal group laws F and F' over the same base the set of strict isomorphisms $F \to F'$ satisfies descent. It is not a stack because a local system of formal group laws may not glue together to a global formal group law. We write

$$\mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s$$

when we think of this presheaf of groupoids as a moduli prestack. For each graded commutative ring R the prestack 1- and 2-morphisms

$$\operatorname{Spec}(R) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

constitute the groupoid $\mathcal{FGL}_s(R)$ of formal group laws and strict isomorphisms over R. ((ETC: Working in the ungraded context, one would allow all isomorphisms.))

The pullback function appears naturally in topology. Given a map $g\colon D \to E$ of ring spectra up to homotopy with D^* and E^* graded commutative, with induced ring homomorphism $g\colon D^* \to E^*$, and given a complex orientation $y \in \tilde{D}^2(\mathbb{C}P^\infty)$ of D, the image

$$gy = g^*(y) \in \tilde{E}^2(\mathbb{C}P^\infty)$$

is a complex orientation of E. Here, if y is the homotopy class of $\Sigma^{-2}\mathbb{C}P^\infty \to D$, then gy is the class of the composite

$$\Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y} D \xrightarrow{g} E\,.$$

Example 3.6. Let $n \in \mathbb{Z}_p^\times$. The Adams operation $\psi^n\colon KU_p^\wedge \to KU_p^\wedge$ is a map of ring spectra, taking the complex orientation $y = y^{KU} = u^{-1}(\gamma^1 - 1)$ to

$$\psi^n y = (nu)^{-1}((1+uy)^n - 1)\,,$$

which in this case is a second complex orientation $y' = h^n(y)$ of the same ring spectrum. This defines a strict isomorphism $h^n\colon F_m \to \psi^n F_m$. When composed with the isomorphism $ny\colon \psi^n F_m \to F_m$ it corresponds to the n-series automorphism

$$[n]_{F_m}(y) = u^{-1}((1+uy)^n - 1)$$

of F_m over $KU^* = \mathbb{Z}_p[u^{\pm 1}]$.

Lemma 3.7. *Let $F(y_1, y_2) = m^*(y)$ be the formal group law over D^* associated to (D, y). Then the formal group law over E^* associated to (E, gy) is equal to the pullback $(g^*F)(y_1, y_2)$.*

If $y' = h(y)$ is a second complex orientation of D, then the strict isomorphism over E^* associated to the two complex orientations gy and gy' of E is equal to the pullback $(g^*h)(y)$. □

If need be, we can spell out these identifications as

$$F_{(E,gy)} = g^*F_{(D,y)} \quad \text{and} \quad (g^*h\colon F_{(E,gy)} \overset{\cong}{\rightrightarrows} F_{(E,gy')}) = g^*(h\colon F_{(D,y)} \overset{\cong}{\rightrightarrows} F_{(D,y')})\,.$$

Following Lazard [Laz55], it is not so difficult to see that the set-valued functor

$$\operatorname{obj}\mathcal{FGL}\colon R \longmapsto \{\text{formal group laws } F \text{ over } R\}$$

from Lemma 3.2 is corepresentable, i.e., equal to $\operatorname{Spec}(L)$ for a suitable graded commutative ring L, so that $\operatorname{obj}\mathcal{FGL}(R) \cong \mathcal{C}Ring(L, R)$.

Definition 3.8. Let $\tilde{L} = \mathbb{Z}[\tilde{a}_{i,j} \mid i, j \geq 1]$ and

$$\tilde{F}(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} \tilde{a}_{i,j} y_1^i y_2^j \in \tilde{L}[[y_1, y_2]]\,,$$

define coefficients $b_{i,j,k} \in \tilde{L}$ by

$$\tilde{F}(\tilde{F}(y_1, y_2), y_3) - \tilde{F}(y_1, \tilde{F}(y_2, y_3)) = \sum_{i,j,k\geq 0} b_{i,j,k} y_1^i y_2^j y_3^k \in \tilde{L}[[y_1, y_2, y_3]]\,,$$

and let $\tilde{I} \subset \tilde{L}$ be the ideal generated by $\tilde{a}_{i,j} - \tilde{a}_{j,i}$ for all $i, j \geq 1$ and $b_{i,j,k}$ for all $i, j, k \geq 0$. The ring $\tilde{L}$ is homologically graded with $|\tilde{a}_{i,j}| = 2(i + j - 1)$, and $\tilde{I}$ is a homogeneous ideal with $\tilde{a}_{i,j} - \tilde{a}_{j,i}$ in degree $2(i + j - 1)$ and $b_{i,j,k}$ in degree $2(i + j + k - 1)$. Let

$$L = \tilde{L}/\tilde{I} = \mathbb{Z}[\tilde{a}_{i,j} \mid i, j \geq 1]/\tilde{I}$$

be the (evenly graded) quotient ring, let $a_{i,j} \in L$ be the image of $\tilde{a}_{i,j}$ under the canonical projection, and define

$$F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} a_{i,j} y_1^i y_2^j$$

to be the image of $\tilde{F}(y_1, y_2)$ in $L[[y_1, y_2]]$. Then $F_L(y_1, y_2)$ is a formal group law defined over L. If y_1 and y_2 have homological degree -2 (and cohomological degree 2), then so does $F_L(y_1, y_2)$. We call L the Lazard ring, and $F_L(y_1, y_2)$ the Lazard formal group law.

Proposition 3.9. *The Lazard formal group law F_L over the Lazard ring L is universal, in the sense that*

$$\begin{aligned} \mathcal{CR}ing(L, R) &\xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}(R) \\ (g \colon L \to R) &\longmapsto g^* F_L \end{aligned}$$

defines a natural bijection for all (graded) commutative rings R. Hence F_L represents an isomorphism of sheaves

$$\operatorname{Spec}(L) \xrightarrow{\cong} \operatorname{obj} \mathcal{FGL} .$$

Proof. This asserts that for each formal group law

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} \bar{a}_{i,j} y_1^i y_2^j \in R[[y_1, y_2]]$$

over a ring R there exists a unique ring homomorphism $g \colon L \to R$ such that $F = g^* F_L$. It is obviously given by mapping $\tilde{a}_{i,j} \in \tilde{L}$ to the given $\bar{a}_{i,j} \in R$, and noting that this descends to a ring homomorphism $g \colon L \to R$ because the generators of the ideal $\tilde{I}$ all map to zero, since F is assumed to be a formal group law. The ring homomorphism g thus classifies the formal group law F. □

Remark 3.10. Direct calculation shows that

$$\tilde{I} = (\tilde{a}_{1,2} - \tilde{a}_{2,1}, \tilde{a}_{1,3} - \tilde{a}_{3,1}, 2\tilde{a}_{1,1}\tilde{a}_{1,2} + 3\tilde{a}_{1,3} - 2\tilde{a}_{2,2}, \dots)$$

so that in degrees $* \leq 6$ the Lazard ring is freely generated by $x_1 = a_{1,1}$, $x_2 = a_{1,2}$ and $x_3 = a_{2,2} - a_{1,3}$. These calculations quickly become complicated. Nonetheless, Lazard was able to determine the structure of L.

Theorem 3.11 ([Laz55]). *There exists an isomorphism*

$$L \cong \mathbb{Z}[x_i \mid i \geq 1]$$

of graded commutative rings, with $|x_i| = 2i$.

A proof, following Frölich (1968), is given in [Ada74, Thm. II.7.1]. See also Pstragowski (2021), "Finite height chromatic homotopy theory", Thm. 6.8. We will comment on the proof later, in connection with the Hurewicz homomorphism $\hbar \colon \pi_*(MU) \to H_*(MU)$.

4. Moduli of formal group laws

A strict isomorphism $h\colon F \to F'$ of formal group laws over R is uniquely determined by the formal group law $F(y_1, y_2)$ and the strict isomorphism $h(y)$, since $F'(y_1, y_2) = h(F(h^{-1}(y_1), h^{-1}(y_2))$ as in Lemma 2.9, so the set-valued functor

$$\operatorname{mor}\mathcal{FGL}\colon R \longmapsto \{\text{strict isomorphisms } h\colon F \to F' \text{ over } R\}$$

implicit in Lemma 3.3 is also corepresentable.

Definition 4.1. Let

$$B = \mathbb{Z}[b_k \mid k \geq 1]$$
$$LB = L[b_k \mid k \geq 1] \cong L \otimes B$$

be homologically graded with $|b_k| = 2k$, with canonical inclusions $\eta_L\colon L \to LB$ and $\iota\colon B \to LB$, and let

$$h(y) = y + \sum_{k\geq 1} b_k y^{k+1} \in B[[y]]\,.$$

Let

$$\eta_L^* F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} \eta_L(a_{i,j}) y_1^i y_2^j \in LB[[y_1, y_2]]$$

and

$$\iota^* h(y) = y + \sum_{k\geq 1} \iota(b_k) y^{k+1} \in LB[[y]]$$

be the base changes to LB of F_L and h.

Lemma 4.2. *The target of the strict isomorphism $\iota^* h\colon \eta_L^* F_L \to F'$ is a formal group law defined over LB, hence is equal to $\eta_R^* F_L$ for a well-defined ring homomorphism*

$$\eta_R\colon L \longrightarrow LB\,.$$

Proof. We require that

$$\eta_R^* F_L(y_1, y_2) = (\iota^* h)^{-1}(\eta_L^* F_L((\iota^* h)(y_1), (\iota^* h)(y_2)))\,.$$

Omitting η_L and ι from the notation, this asks that

$$\eta_R^* F_L(y_1, y_2) = h^{-1}(F_L(h(y_1), h(y_2)))\,.$$

Hence $\eta_R\colon L \to LB$ must map $a_{i,j}$ to the coefficient of $y_1^i y_2^j$ in the formal power series expansion of the right hand side. □

Remark 4.3. With x_1, x_2 and x_3 as before, one finds

$$\eta_R(x_1) = x_1 + 2b_1$$
$$\eta_R(x_2) = x_2 + x_1 b_1 + (3b_2 - 2b_1^2)$$
$$\eta_R(x_3) = x_3 + (2x_2 + x_1^2) b_1 + x_1(4b_2 - b_1^2) + (2b_3 + 2b_1 b_2 - 2b_1^3)\,.$$

Again, these calculations quickly become complicated.

Proposition 4.4. *The strict isomorphism $\iota^* h\colon \eta_L^* F_L \to \eta_R^* F_L$ over LB is universal, in the sense that*

$$\begin{aligned} \mathcal{CR}ing(LB, R) &\xrightarrow{\cong} \operatorname{mor} \mathcal{FGL}_s(R) \\ (g\colon LB \to R) &\longmapsto \big(g^*\iota^* h\colon g^*\eta_L^* F_L \to g^*\eta_R^* F_L\big) \\ &\qquad = \big(g^* h\colon g^* F_L \to g^*\eta_R^* F_L\big) \end{aligned}$$

defines a natural bijection for all (graded) commutative rings R. Hence $h\colon F_L \to \eta_R^* F_L$ represents an isomorphism of sheaves

$$\operatorname{Spec}(LB) \xrightarrow{\cong} \operatorname{mor} \mathcal{FGL}_s\ .$$

Proof. Given a strict isomorphism $h\colon F \to F'$ over R there are unique ring homomorphisms $g_0\colon L \to R$ and $g_1\colon B \to R$ classifying F and h, so that

$$\begin{aligned} F(y_1, y_2) &= y_1 + y_2 + \sum_{i,j\ge 1} g_0(a_{i,j}) y_1^i y_2^j \\ h(y) &= y + \sum_{k\ge 1} g_1(b_k) y^{k+1}\ . \end{aligned}$$

Then $g\colon LB \to R$ is characterized by $g\eta_L = g_0$ and $g\iota = g_1$. □

The series expansion $h'(h(y))$ of the composite $h'h\colon F \to F''$ of two strict isomorphisms $h\colon F \to F'$ and $h'\colon F' \to F''$ of formal group laws can be calculated without reference to F, F' or F''. Hence B corepresents a functor to groups, and B acquires the structure of a Hopf algebra.

Definition 4.5. Set $b_0 = 1$. Let

$$\begin{aligned} \epsilon_B\colon B &\longrightarrow \mathbb{Z} \\ \psi_B\colon B &\longrightarrow B \otimes B \\ \chi_B\colon B &\longrightarrow B \end{aligned}$$

be the ring homomorphisms sending b_k to the coefficient of y^{k+1} in $\mathrm{id}(y) = y$, $h''(h'(y))$ and $h^{-1}(y)$, respectively, where $h'(y) = \sum_{i\ge 0}(b_i \otimes 1) y^{i+1}$, $h''(y) = \sum_{j\ge 0}(1 \otimes b_j) y^{j+1}$, and $h(y) = \sum_{k\ge 0} b_k y^{k+1}$.

Lemma 4.6 ([Ada74, Prop. II.7.5, Thm. II.11.3])**.** *$\epsilon_B(b_k) = 0$ for $k \ge 1$,*

$$\psi_B(b_k) = \sum_{j\ge 0} \Big(\sum_{i\ge 0} b_i\Big)^{j+1}_{2(k-j)} \otimes b_j$$

and

$$\chi_B(b_k) = \frac{1}{k+1} \Big(\sum_{i\ge 0} b_i\Big)^{-k-1}_{2k}\,,$$

where $(-)^m_n$ denotes the degree n homogeneous component of $(-)^m$.

Proof. See the proofs in [Ada74, Part II]. □

Remark 4.7. Direct calculation shows that

$$\begin{aligned} \psi(b_1) &= b_1 \otimes 1 + 1 \otimes b_1 \\ \psi(b_2) &= b_2 \otimes 1 + 2b_1 \otimes b_1 + 1 \otimes b_2 \\ \psi(b_3) &= b_3 \otimes 1 + (b_1^2 + 2b_2) \otimes b_1 + 3b_1 \otimes b_2 + 1 \otimes b_3 \end{aligned}$$

and

$$\begin{aligned}
\chi(b_1) &= -b_1 \\
\chi(b_2) &= 2b_1^2 - b_2 \\
\chi(b_3) &= -5b_1^3 + 5b_1 b_2 - b_3 \,.
\end{aligned}$$

Note that this coproduct is different from that on the bipolynomial Hopf algebra $H_*(BU)$, and that the conjugation takes integral values, in spite of the division by $k+1$.

Proposition 4.8. *The pair (L, LB) is a Hopf algebroid corepresenting the functor*

$$\begin{aligned}
\mathcal{FGL}_s^{op} \colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\
R &\longmapsto \mathcal{FGL}_s(R)^{op} \,.
\end{aligned}$$

The left and right units

$$\eta_L \colon L \longrightarrow LB \qquad \text{and} \qquad \eta_R \colon L \longrightarrow LB$$

corepresent the source (= opposite target) and target (= opposite source) of

$$\iota^* h \colon \eta_L^* F_L \xrightarrow{\cong} \eta_R^* F_L \,.$$

The augmentation

$$\epsilon = \mathrm{id} \otimes \epsilon_B \colon LB \longrightarrow L$$

corepresents the identity homomorphism. The coproduct

$$\psi = \mathrm{id} \otimes \psi_B \colon LB = L \otimes B \longrightarrow L \otimes B \otimes B \cong LB \otimes_L LB$$

corepresents composition. The conjugation

$$\chi \colon LB \longrightarrow LB$$

satisfies $\chi\eta_L = \eta_R$ and $\chi\iota = \iota\chi_B$, and corepresents the inverse.

Remark 4.9. This kind of Hopf algebroid is said to be split. It is formed as a semi-direct or twisted tensor product, from a Hopf algebra B and a right B-comodule algebra L, with $G = \mathrm{Spec}(B)$ a group scheme acting from the right on the scheme $X = \mathrm{Spec}(L)$, so that (L, LB) corepresents the "translation" groupoid scheme $\mathcal{B}(X, G)$ from Chapter 3.

Remark 4.10. Writing $h \bullet h' = h' \circ h$ for the opposite composition, the moduli prestack $\mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s \colon \mathcal{A}ff^{op} \to \mathcal{G}pd$ is an affine groupoid scheme, with object scheme $\mathrm{Spec}(L)$, morphism scheme $\mathrm{Spec}(LB)$ and structure maps

$$\mathrm{Spec}(L) \;\underset{\overset{\xleftarrow{\;t\;}}{}}{\overset{\xleftarrow{\;s\;}}{\xrightarrow{\;\mathrm{id}\;}}}\; \mathrm{Spec}(LB) \circlearrowleft^{i} \xleftarrow{\;\bullet\;} \mathrm{Spec}(LB) \times_{\mathrm{Spec}(L)} \mathrm{Spec}(LB)$$

dual to the graded commutative rings and homomorphisms

$$L \;\underset{\overset{\xrightarrow{\;\eta_R\;}}{}}{\overset{\xrightarrow{\;\eta_L\;}}{\xleftarrow{\;\epsilon\;}}}\; LB \circlearrowleft^{\chi} \xrightarrow{\;\psi\;} LB \otimes_L LB \,.$$

((ETC: To avoid the passage to the opposite category, it might be better to corepresent the homomorphism $h^{-1}\colon F' \to F$ with $h^{-1}(y) = y + \sum_{k\geq 1} m_k y^{k+1}$, where $m_k = \chi(b_k)$.))

The R-valued points of the canonical map

$$\pi\colon \operatorname{Spec}(L) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

is the inclusion $\operatorname{obj} \mathcal{FGL}_s(R) \to \mathcal{FGL}_s(R)$, viewing the object set as a subgroupoid with only identity morphisms. There is a 2-categorical pullback square

$$\begin{array}{ccc} \operatorname{Spec}(LB) & \xrightarrow{t} & \operatorname{Spec}(L) \\ {\scriptstyle s}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(L) & \xrightarrow[\pi]{} & \mathcal{M}_{\mathrm{fgl}} \end{array}$$

and the corresponding diagram of nerves (which are simplicial sets, or spaces) is a homotopy pullback square.

5. Quillen's theorem

Recall the tautological complex orientation $y^{MU} \in \widetilde{MU}^2(\mathbb{C}P^\infty)$ represented by the composite

$$\omega\colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) \longrightarrow MU\,.$$

It defines a formal group law $F_{MU}(y_1, y_2)$ over $MU_* = MU^{-*}$. Quillen showed that MU_* (together with the formal group law F_{MU}) has the same universal property in (graded) commutative rings as the Lazard ring.

Theorem 5.1 ([Qui69], [Qui71])**.** *The ring homomorphism*

$$q_0\colon L \xrightarrow{\cong} MU_*$$

classifying the formal group law F_{MU} is an isomorphism. Hence F_{MU} over MU_* is the universal formal group law.

Adams showed that MU (together with the complex orientation y^{MU}) also has a universal property, this time in the category of ring spectra up to homotopy, i.e., of monoids in $(\mathrm{Ho}(\mathcal{S}p), S, \wedge)$.

Lemma 5.2 ([Ada74, Lem. II.4.6])**.** *Let E be a ring spectrum up to homotopy, with E^* graded commutative. The function*

$$\{\textit{ring spectrum maps } g\colon MU \to E\} \xrightarrow{\cong} \{\textit{complex orientations } y \in \tilde{E}^2(\mathbb{C}P^\infty)\}$$
$$g \longmapsto g y^{MU}$$

is a bijection. Hence each complex orientation of E comes from unique ring spectrum map $MU \to E$ in the stable homotopy category.

Proof. If E is not complex orientable, then both of these sets are empty. Otherwise, $E_*(MU) \cong E_*[b_k \mid k \geq 1]$ is free as a left E_*-module, which implies ((ETC: via the universal coefficient theorem or Ext-spectral sequence)) that

$$[MU, E] \cong \operatorname{Hom}_{E_*}(E_*(MU), E_*)$$

(degree-preserving homomorphisms). Similarly, $[S, E] \cong \operatorname{Hom}_{E_*}(E_*, E_*)$ and

$$[MU \wedge MU, E] \cong \operatorname{Hom}_{E_*}(E_*(MU) \otimes_{E_*} E_*(MU), E_*)\,,$$

from which it follows that

$$\begin{aligned}\{\text{ring spectrum maps } MU \to E\} &\cong \mathcal{A}lg_{E_*}(E_*(MU), E_*)\\ &\cong \mathcal{A}lg_{E_*}(E_*[b_k \mid k \geq 1], E_*)\\ &\cong \mathrm{Hom}_{E_*}(E_*\{b_k \mid k \geq 1\}, E_*)\\ &\subset \mathrm{Hom}_{E_*}(E_*(\Sigma^{-2}\mathbb{C}P^\infty), E_*) \cong \tilde{E}^2(\mathbb{C}P^\infty)\end{aligned}$$

corresponds to the subset of (strict) complex orientations of E. Here we use that $y^{MU} \colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) \to MU$ induces $\Sigma^{-2}\beta_{k+1} \mapsto b_k$ in E-homology, and $E_*(\Sigma^{-2}\mathbb{C}P^\infty) = E_*\{\Sigma^{-2}\beta_{k+1} \mid k \geq 0\}$. □

Let (E, y^E) be a complex oriented ring spectrum, temporarily let $\eta_L = \mathrm{id} \wedge \eta \colon E \cong E \wedge S \to E \wedge MU$ and $\eta_R = \eta \wedge \mathrm{id} \colon MU \cong S \wedge MU \to E \wedge MU$, and let

$$y^L = \eta_L y^E \colon \Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y^E} E \xrightarrow{\eta_L} E \wedge MU$$

$$y^R = \eta_R y^{MU} \colon \Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y^{MU}} MU \xrightarrow{\eta_R} E \wedge MU$$

be two complex orientations of $E \wedge MU$. Recall the classes $b_k^E \in E_{2k}(MU) = (E \wedge MU)_{2k}$, coming from $\beta_k^E \in E_{2k}(\mathbb{C}P^\infty) \to E_{2k}(BU) \cong E_{2k}(MU)$, or from $\beta_{k+1}^E \in \tilde{E}_{2k+2}(\mathbb{C}P^\infty) \cong \tilde{E}_{2k+2}(MU(1)) \to E_{2k}(MU)$.

Lemma 5.3 ([Ada74, Lem. II.6.3])**.** *In $(E \wedge MU)^2(\mathbb{C}P^\infty)$ we have $y^R = h(y^L)$ where*

$$h(y) = y + \sum_{k\geq 1} b_k^E y^{k+1} \in (E \wedge MU)_*[[y]] \,.$$

Hence h is a strict isomorphism

$$h \colon F_{(E\wedge MU, y^L)} \xrightarrow{\cong} F_{(E\wedge MU, y^R)}$$

of formal group laws over $(E \wedge MU)_* = E_*(MU)$.

Sketch proof. Chase y^E and y^{MU} through the diagram

$$\begin{array}{ccccc}
[\mathbb{C}P^\infty, E]_* & \xrightarrow{(\eta_L)_*} & [\mathbb{C}P^\infty, E \wedge MU]_* & \xleftarrow{(\eta_R)_*} & [\mathbb{C}P^\infty, MU]_* \\
\downarrow \cong & & \downarrow \cong & \swarrow & \\
\mathrm{Hom}_{E_*}(E_*(\mathbb{C}P^\infty), E_*) & \xrightarrow{(\eta_L)_*} & \mathrm{Hom}_{E_*}(E_*(\mathbb{C}P^\infty), E_*(MU)) & &
\end{array}.$$

□

We apply this in the case $E = MU$. Then $\eta_L \colon MU \to MU \wedge MU$ and $\eta_R \colon MU \to MU \wedge MU$ induce the homomorphisms previously denoted $\eta_L \colon MU_* \to MU_*MU$ and $\eta_R \colon MU_* \to MU_*MU$.

Theorem 5.4. *The ring homomorphism*

$$q \colon LB \xrightarrow{\cong} MU_*MU$$

classifying the strict isomorphism

$$h \colon \eta_L^* F_{MU} \xrightarrow{\cong} \eta_R^* F_{MU}$$

is an isomorphism. Hence h over MU_*MU is the universal strict isomorphism between formal group laws.

Proof. Since the source of h is $\eta_L^* F_{MU}$, the restriction of q over η_L is Quillen's isomorphism $q_0 \colon L \to MU_* \subset MU_*MU$. Moreover, by Lemma 5.3 (in the case $E = MU$), q restricts over ι to the homomorphism

$$\begin{aligned} q_1 \colon B &\longrightarrow \mathbb{Z}[b_k^{MU} \mid k \geq 1] \subset MU_*MU \\ b_k &\longmapsto b_k^{MU} , \end{aligned}$$

which is obviously an isomorphism. This implies that q is an isomorphism. □

Remark 5.5. With this, we have recovered the calculation of the MU-based Steenrod algebra $\mathscr{A}_{MU} = MU^*(MU)$ due to Novikov [Nov67] and Landweber [Lan67], in the dual form of the Hopf algebroid $(MU_*, MU_*MU) \cong (L, LB)$ recommended by Adams, reaching the conclusion that it is the Hopf algebroid corepresenting the functor $R \mapsto \mathcal{FGL}_s(R)^{op}$ taking any commutative ring to (the opposite of) the groupoid of formal group laws and strict isomorphisms defined over R.

The explicit formulas are hard to work with. There is a p-local version of the theory, for each fixed prime p, involving the Brown–Peterson spectrum BP with $H^*(BP; \mathbb{F}_p) = \mathscr{A}//\mathscr{E} = \mathscr{P}$ and p-typical formal group laws, for which more manageable (but still recursive) formulas for η_R, ψ and χ are available.

In the special case $E = H\mathbb{Z}$, Adams' lemma shows that the universal formal group law F_{MU} over MU_* becomes strictly isomorphic to the additive formal group law when base changed along the Hurewicz homomorphism $\hbar \colon MU_* \to H_*(MU) \cong H_*(BU) = \mathbb{Z}[b_k \mid k \geq 1]$. This gives a fairly explicit formula for $\hbar^* F_{MU}$, and since $\hbar \colon MU_* \to H_*(MU)$ is injective, this formula determines $F_{MU}(y_1, y_2) \in MU_*[[y_1, y_2]]$.

Lemma 5.6. *The formal power series*

$$\exp_{MU}(y) = y + \sum_{k \geq 1} b_k y^{k+1} \in H_*(MU)[[y]]$$

defines a strict isomorphism

$$\exp_{MU} \colon F_a \xrightarrow{\cong} \hbar^* F_{MU}$$

over $H_*(MU)$. Letting

$$\log_{MU}(y) = \exp_{MU}^{-1}(y) = y + \sum_{k \geq 1} m_k y^{k+1}$$

denote its inverse, it follows that

$$\hbar^* F_{MU}(y_1, y_2) = \exp_{MU}(\log_{MU}(y_1) + \log_{MU}(y_2))$$

in $H_*(MU)[[y_1, y_2]]$.

Proof. This is the case $E = H\mathbb{Z}$ of Lemma 5.3, noting that $F_{H\mathbb{Z}} = F_a$ remains the additive formal group law after base change to $H_*(MU)$. The logarithm coefficients $m_k = \chi(b_k)$ were calculated in Lemma 4.6. □

Remark 5.7. To prove Lazard and Quillen's theorems, one uses the formula

$$F'(y_1, y_2) = \exp(\log(y_1) + \log(y_2)) ,$$

with $\exp(y) = y + \sum_{k\geq 1} b_k y^{k+1}$ and $\log(y) = \exp^{-1}(y)$, to define a formal group law F' over $B = \mathbb{Z}[b_k \mid k \geq 1]$, which is classified by a ring homomorphism $g\colon L \to B$. The discussion for MU and $H\mathbb{Z} \wedge MU$ gives a commutative square

$$\begin{array}{ccc} L & \xrightarrow{g} & B \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle q'} \\ MU_* & \xrightarrow{\hbar} & H_*(MU)\,. \end{array}$$

Letting $I \subset L$ and $J \subset B$ be the augmentation ideals (= the positive-degree classes), Lazard proves that

$$\mathbb{Z}\{x_k \mid k \geq 1\} = I/I^2 \xrightarrow{g} J/J^2 = \mathbb{Z}\{b_k \mid k \geq 1\}$$

is given by

$$x_k \longmapsto \begin{cases} pb_k & \text{if } k+1 \text{ is a power of } p, \\ b_k & \text{otherwise.} \end{cases}$$

Quillen shows that q' is an isomorphism, and compares with Milnor's calculation of $\hbar$ to deduce that q_0 is also an isomorphism.

Example 5.8. The complex orientation $y^H = y \in \tilde{H}^2(\mathbb{C}P^\infty)$ corresponds to the ring spectrum map $MU \to \tau_{\leq 0} MU \simeq H\mathbb{Z}$. The induced homomorphism $MU_* = L \to \mathbb{Z}$ corepresents the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2$$

over $\mathbb{Z}$.

The complex orientation $y^{KU} = u^{-1}(\gamma^1 - 1) \in \widetilde{KU}^2(\mathbb{C}P^\infty)$ is represented by a map $y^{KU}\colon \Sigma^{-2}\mathbb{C}P^\infty \longrightarrow KU$, corresponding to a ring spectrum map $g\colon MU \longrightarrow KU$ in the stable homotopy category. (Both y^{KU} and g factor uniquely over the connective cover $ku = \tau_{\geq 0} KU \to KU$.) The induced ring homomorphism $g\colon MU_* \cong L \longrightarrow KU_*$ corepresents the multiplicative formal group law

$$F_m(y_1, y_2) = y_1 + y_2 + u y_1 y_2$$

over $KU_* = \mathbb{Z}[u^{\pm 1}]$. Here

$$g(a_{i,j}) = \begin{cases} u & \text{for } (i,j) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Following up on Lemma 5.6, we have the commutative diagram

$$\begin{array}{ccccc} H\mathbb{Z} & \xrightarrow{\eta_L} & H\mathbb{Z} \wedge MU & \xleftarrow{\eta_R} & MU \\ \| & & \downarrow{\scriptstyle \mathrm{id} \wedge g} & & \downarrow{\scriptstyle g} \\ H\mathbb{Z} & \xrightarrow{\eta_L} & H\mathbb{Z} \wedge KU & \xleftarrow{\eta_R} & KU \\ \downarrow & & \downarrow{\scriptstyle \simeq} & & \| \\ H\mathbb{Q} & \xrightarrow{\eta_L} & H\mathbb{Q} \wedge KU & \xleftarrow{\eta_R} & KU \end{array}$$

of ring spectra. Adams shows ((ETC: Reference?)) that the Bott map $u\colon \Sigma^2 ku \to ku$ induces a nilpotent homomorphism $u_*\colon H_*(\Sigma^2 ku; \mathbb{F}_p) \to H_*(ku; \mathbb{F}_p)$ in mod p homology. (In fact $u_*^{p-1} = 0$, for each prime p.) Passing to the colimit along

$$ku \xrightarrow{u} \Sigma^{-2} ku \xrightarrow{u} \Sigma^{-4} ku \longrightarrow \dots \longrightarrow KU$$

we deduce that $H_*(KU;\mathbb{F}_p) = 0$, so that multiplication by p on $H_*(KU)$ is invertible. Hence $H_*(KU)$ is already rational and $H\mathbb{Z} \wedge KU \to H\mathbb{Q} \wedge KU$ is an equivalence. The strict isomorphism

$$\exp_{MU} \colon F_a \xrightarrow{\cong} \hbar^* F_{MU}$$

over $H_*(MU)$ base changes along $H_*(MU) \to H_*(KU)$ to a strict isomorphism

$$g^* \exp_{MU} \colon F_a \xrightarrow{\cong} F_m$$

defined over $H_*(KU) \cong H\mathbb{Q}_*(KU) = KU_* \otimes \mathbb{Q} = \mathbb{Q}[u^{\pm 1}]$. Over any $\mathbb{Q}$-algebra there is a unique strict isomorphism from the additive to the multiplicative formal group law, namely

$$g^* \exp_{MU}(y) = \frac{e^{uy} - 1}{u} = y + \sum_{k \ge 1} \frac{u^k}{(k+1)!} y^{k+1} .$$

Its formal inverse is

$$g^* \log_{MU}(y) = \frac{\log(1 + uy)}{u} = y + \sum_{k \ge 1} (-1)^k \frac{u^k}{k+1} y^{k+1} .$$

Hence $g \colon H_*(MU) \to H_*(KU)$ is given by

$$g(b_k) = \frac{u^k}{(k+1)!} \qquad \text{and} \qquad g(m_k) = (-1)^k \frac{u^k}{k+1}$$

for each $k \ge 1$.

((ETC: Relate $KU \to H\mathbb{Z} \wedge KU \simeq H\mathbb{Q} \wedge KU \simeq \prod_{i \in \mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$ to the Chern character $\mathrm{ch} \colon KU^0(X) \to H^{ev}(X;\mathbb{Q}) = \prod_i H^{2i}(X;\mathbb{Q})$. Relate $\mathrm{ch} \circ g \colon MU \to KU \to \prod_{i \in \mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$ to the Todd genus. Mention Mischenko's theorem $\pm[\mathbb{C}P^k] = (k+1) m_k$, the Conner–Floyd theorem $KU_*(X) \cong KU_* \otimes_{MU_*} MU_*(X)$, and the Hattori–Stong theorem on $KU_*(MU)$.))

6. Formal groups

To each complex orientable ring spectrum E we have assigned the graded commutative E^*-algebra

$$E^*(\mathbb{C}P^\infty) \cong \lim_m E^*(\mathbb{C}P^m)$$

with its augmentation $\epsilon \colon E^*(\mathbb{C}P^\infty) \to E^*$ and completed coproduct

$$m^* \colon E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}}_{E^*} E^*(\mathbb{C}P^\infty) .$$

The corepresented sheaf

$$\hat{G}_E = \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \operatorname*{colim}_m \mathrm{Spec}(E^*(\mathbb{C}P^m))$$

over $\mathrm{Spec}(E^*)$ is an abelian group object in this category, with neutral element

$$\mathrm{Spec}(E^*) \xrightarrow{\mathrm{Spf}(\epsilon)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \hat{G}_E$$

and multiplication

$$\begin{aligned} \hat{G}_E \times_{\mathrm{Spec}(E^*)} \hat{G}_E &= \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) \times_{\mathrm{Spec}(E^*)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) \\ &\xrightarrow{\mathrm{Spf}(m^*)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \hat{G}_E . \end{aligned}$$

This is an example of a formal group (not formal group law) over $\mathrm{Spec}(E^*)$.

Only when we fix a choice of complex orientation $y \in \tilde{E}^2(\mathbb{C}P^\infty)$ do we specify an isomorphism $E^*(\mathbb{C}P^\infty) \cong E^*[[y]]$ and obtain a formal group law $m^*(y) = F_E(y_1, y_2) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1, y_2]]$. Different choices of complex orientations give formal group laws that only agree up to (canonical) strict isomorphism. We therefore want each formal group law to specify a formal group, but also want strictly isomorphic formal group laws to specify the same formal group. A formal group is therefore, roughly, what we obtain from a formal group law by forgetting the choice of coordinate.

Definition 6.1. Let R be a (graded) commutative ring. A (commutative, one-dimensional) formal group $\hat{G}$ over $\mathrm{Spec}(R)$ is an abelian group object in sheaves over $\mathrm{Spec}(R)$ whose underlying object pointed at the unit is locally isomorphic to $\mathrm{Spf}(R[[y]])$ pointed at $y = 0$.

Here "locally isomorphic" means that $\mathrm{Spec}(R)$ is covered by Zariski open subschemes $\mathrm{Spec}(T)$ such that $\hat{G}(T) \cong \mathrm{Spf}(T[[y]])$ is the underlying formal group of a formal group law over T, but also that the local choices of coordinates y need not extend to a global coordinate over R. This means that a formal group over $\mathrm{Spec}(R)$ is a locally defined notion, as is required for these to form the R-valued points of a stack (not prestack) of formal groups. See Naumann [Nau07, Thm. 33(i)] and Goerss [Goe, Thm. 2.34] for expositions of this and related stacks of relevance to algebraic topology.

Theorem 6.2. *The stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups is the stackification of the prestack $\mathcal{M}_{\mathrm{fgl}}$ presented by the Hopf algebroid (L, LB).*

The canonical morphism $\mathcal{M}_{\mathrm{fgl}} \to \mathcal{M}_{\mathrm{fg}}$ extends the class of objects, since not all formal groups admit a global coordinate, and identifies some strictly isomorphic formal group laws by forgetting the choice of coordinate.

We obtain the following diagram of categories and functors, where $U\colon (E, y) \mapsto E$ forgets the complex orientation, V maps $F(y_1, y_2) \in R[[y_1, y_2]]$ to the formal scheme $\mathrm{Spf}(R[[y]])$ with the associated group structure, $F\colon (E, y) \mapsto F_E(y_1, y_2) = m^*(y)$ is the associated formal group law over E^*, and $\hat{G}\colon E \mapsto \hat{G}_E = \mathrm{Spf}(E^*(\mathbb{C}P^\infty))$ is the (Quillen) formal group over $\mathrm{Spec}(E^*)$. Each even ring spectrum E is complex orientable, since the Atiyah–Hirzebruch spectral sequence for $E^*(\mathbb{C}P^\infty)$ collapses.

$$
\begin{array}{ccc}
 & & \text{even ring spectra} \\
 & & \downarrow \\
\text{complex oriented ring spectra} & \xrightarrow{U} & \text{complex orientable ring spectra} \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \hat{G}} \\
\text{formal group laws} & \xrightarrow{V} & \text{formal groups}
\end{array}
$$

The right hand objects are more intrinsic, while the left hand objects may be more amenable to calculation.

It is an interesting question to ask which formal groups can be realized as the Quillen formal group of a complex orientable ring spectrum. A sufficient criterion will be given by Landweber's exact functor theorem [Lan76]. One source of (commutative, one-dimensional) formal groups are the formal completions $\hat{C}$ of the (commutative, one-dimensional) group schemes given by elliptic curves C. These are quite often Landweber exact, and are then realized by complex orientable ring spectra known as elliptic cohomology theories [LRS95].

Other sources of (commutative, one-dimensional) formal groups are given by formal deformations of Brauer groups of $K3$-surfaces, or more general cohomology groups of higher-dimensional Calabi–Yau varieties [Art74], [AM77]. The resulting $K3$-cohomology [Szy10], [Szy11] and Calabi–Yau cohomologies seem not to be well understood.

A more refined question asks which diagrams of formal groups can be realized by diagrams (of the same shape) of complex orientable ring spectra, and whether this realization can take place in ring spectra up to homotopy, orthogonal ring spectra, or commutative orthogonal ring spectra. This includes questions about group actions, since a G-action corresponds to a $\mathcal{B}G$-shaped diagram. Theorems of Hopkins–Miller and Goerss–Hopkins resolve the second and third forms of this question in interesting cases. It is then possible to form the limit of the resulting diagram of (commutative) orthogonal ring spectra, which has led to the construction of topological modular forms and other higher real K-theories.

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