ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 14: GALOIS EXTENSIONS

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1. Lubin–Tate spectra

Let k be a perfect field of prime characteristic $p \neq 0$, and let $\Phi \in k[[y_1, y_2]]$ be a formal group law over k of finite height $n < \infty$. We will eventually focus on the case $k = \mathbb{F}_{p^n}$ and $\Phi = H_n$, the Honda formal group law, which is defined over $\mathbb{F}_p$, with p-series $[p]_{H_n}(y) = y^{p^n}$.

The classifying homomorphism $L \to k$ for Φ corresponds to a point $\mathrm{Spec}(k) \to \mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fgl}} \to \mathcal{M}_{\mathrm{fg}}$ in the moduli stack of formal group laws/groups. Lubin–Tate [LT66] analyzed the formal neighborhood of this point, which is evenly covered by the space of deformations of the formal group law Φ.

Let R be any complete Noetherian local ring. We write $\mathfrak{m} \subset R$ for the maximal ideal and $\pi\colon R \to R/\mathfrak{m}$ for the canonical homomorphism to the residue field. Completeness means that $R \cong \lim_n R/\mathfrak{m}^n$. If $\mathfrak{m}$ is nilpotent then R is an Artinian local ring, and vice versa.

If $h\colon F \to F'$ is a homomorphism of formal group laws over R, with $F, F' \in R[[y_1, y_2]]$ and $h \in R[[y]]$, then the base change $\pi^*h\colon \pi^*F \to \pi^*F'$ is a homomorphism of formal group laws over $R/\mathfrak{m}$, with $\pi^*F, \pi^*F' \in R/\mathfrak{m}[[y_1, y_2]]$ and $\pi^*h \in R/\mathfrak{m}[[y]]$.

Definition 1.1. By a deformation (F, i) of Φ over k to R we mean a field homomorphism $i\colon k \to R/\mathfrak{m}$ and a formal group law F over R such that $i^*\Phi = \pi^*F$ over $R/\mathfrak{m}$.

$$\Phi \overset{i^*}{\longmapsto} i^*\Phi = \pi^*F \overset{\pi^*}{\longleftarrow\!\!\!\dashv} F$$

$$k \xrightarrow{\ i\ } R/\mathfrak{m} \xleftarrow{\ \pi\ } R$$

A morphism $j\colon (F, i) \to (F', i')$ of deformations can exist only if $i = i'$, in which case it is a homomorphism $j\colon F \to F'$ of formal group laws over R that satisfies

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$\pi^* j = \mathrm{id}\colon \pi^* F \to \pi^* F'$. We say that j is a $\star$-isomorphism.

$$\begin{array}{ccc} \pi^* F & \xleftarrow{\ \pi^*\ } & F \\ \mathrm{id} \Big\| & & \cong \Big\downarrow j \\ \pi^* F' & \xleftarrow{\ \pi^*\ } & F' \end{array}$$

$$R/\mathfrak{m} \xleftarrow{\ \pi\ } R$$

Let $\mathcal{DEF}(\Phi, k)(R)$ be the groupoid of deformations of Φ over k to R, and let

$$\mathrm{Def}(\Phi, k)(R) = \pi_0\, \mathcal{DEF}(\Phi, k)(R)$$

be its set of isomorphism classes. We write $[F, i] \in \mathrm{Def}(\Phi, k)(R)$ for the $\star$-isomorphism class of (F, i).

Note that $i = i'$ implies $\pi^* F = i^* \Phi = (i')^* \Phi = \pi^* F'$, so that the displayed identity morphism exists. To see that $\mathcal{DEF}(\Phi, k)(R)$ is a groupoid, note that $\pi^* j = \mathrm{id}$ means that $j(y) \equiv y \mod \mathfrak{m}[[y]]$, so $j'(0) \equiv 1 \mod \mathfrak{m}$ is a unit in the local ring R.

The finite height assumption has the following consequence.

Theorem 1.2 ([LT66, Thm. 3.1])**.** *There is at most one morphism $j\colon F \to F'$ between any two deformations of Φ over k to R. Hence the groupoid $\mathcal{DEF}(\Phi, k)(R)$ is discrete up to homotopy, and is equivalent to the set* $\mathrm{Def}(\Phi, k)(R)$ *of isomorphism classes of deformations to R.*

Example 1.3. The multiplicative formal group law $F = F_m$ over $R = \mathbb{Z}_p$ is a deformation of the multiplicative formal group law $\Phi = F_m$ over $k = \mathbb{F}_p$. The only morphism $j\colon F \to F$ in $\mathcal{DEF}(F_m, \mathbb{F}_p)(\mathbb{Z}_p)$ is the identity, because if $[n]_{F_m}(y) \equiv y \mod p$, then $n = 1$, as we noted in Chapter 10, Example 2.4.

Remark 1.4. For each R there is a pullback square

$$\begin{array}{ccc} \mathcal{DEF}(\Phi, k)(R) & \longrightarrow & \mathcal{FGL}_i(R) \\ \Big\downarrow & & \Big\downarrow \pi^* \\ \mathcal{CRing}(k, R/\mathfrak{m}) & \xrightarrow{\ i \mapsto i^*\Phi\ } & \mathcal{FGL}_i(R/\mathfrak{m}) \end{array}$$

of groupoids, where the set $\mathcal{CRing}(k, R/\mathfrak{m})$ is viewed as a discrete category. Passing to nerves, we obtain a pullback square of simplicial sets. The functor π^* induces a Kan fibration, since for any morphism in $\mathcal{FGL}_i(R/\mathfrak{m})$ and any choice of lift to $\mathcal{FGL}_i(R)$ of its (source or) target, there exists a lifting morphism in $\mathcal{FGL}_i(R)$ with that (source or) target. Hence the pullback square is also a (2-categorical and) homotopy pullback. By Theorem 1.2, each (homotopy) fiber is homotopy discrete, so π^* is a covering space up to homotopy.

Moreover, Lubin–Tate show that the functor

$$R \longmapsto \mathrm{Def}(\Phi, k)(R)$$

is representable, i.e., that there is a universal deformation $F_{LT} = F_{LT(\Phi,k)}$ of Φ over k to a complete Noetherian ring $LT = LT(\Phi, k)$ with residue field k.

Recall that $W(k)$ denotes the Witt vectors of k. Since k is perfect, it has the universal property that each field homomorphism $i\colon k \to R/\mathfrak{m}$ admits a unique lift $\hat{\imath}\colon W(k) \to R$.

Theorem 1.5 ([LT66, Thm. 3.1]). *There is a deformation (F_{LT}, id) of Φ over k to the complete Noetherian ring*

$$LT(\Phi, k) = W(k)[[u_1, \dots, u_{n-1}]]$$

such that the natural function

$$\begin{aligned} \mathcal{C}Ring^{\mathrm{loc}}(LT(\Phi,k), R) &\xrightarrow{\cong} \mathrm{Def}(\Phi, k)(R) \\ g &\longmapsto [g^* F_{LT}, \bar{g}] \end{aligned}$$

is a bijection for all complete Noetherian local rings R.

The local ring $W(k)[[u_1, \dots, u_{n-1}]]$ has maximal ideal $(p, u_1, \dots, u_{n-1})$ and residue field $LT(\Phi, k)/(p, u_1, \dots, u_{n-1}) \cong k$. We suppress the latter canonical isomorphism from the notation. By a local homomorphism $g\colon LT(\Phi, k) \to R$ we mean a ring homomorphism mapping the maximal ideal $(p, u_1, \dots, u_{n-1})$ to the maximal ideal $\mathfrak{m}$, and we write $\bar{g}\colon k \to R/\mathfrak{m}$ for the induced homomorphism of residue fields.

Example 1.6. The Lubin–Tate deformation of $\Phi = F_m$ over $\mathbb{F}_p$ is defined over $LT(F_m, \mathbb{F}_p) = W(\mathbb{F}_p) = \mathbb{Z}_p$, and is equal to $F_{LT} = F_m$ over $\mathbb{Z}_p$. Hence the formal group law associated to the standard complex orientation of $KU_p^\wedge$ is the universal deformation of the formal group law associated to the standard complex orientation of KU/p.

The universal property only specifies the Lubin–Tate deformation ring LT up to isomorphism, and the Lubin–Tate formal group law F_{LT} is only defined up to $\star$-isomorphism. In particular, the deformation parameters $u_1, \dots, u_{n-1}$ are not canonically defined. In the case $\Phi = H_n$, the universal deformation F_{LT} can be constructed so that its p-series satisfies

$$[p]_{F_{LT}}(y) \equiv u_i y^{p^i}$$

modulo terms of degree $> p^i$, for each $1 \le i < n$. Moreover

$$[p]_{F_{LT}}(y) \equiv y^{p^n}$$

modulo terms of degree $> p^n$. Hence the classifying ring homomorphism $g\colon L \to LT$ from the Lazard ring satisfies $v_i \mapsto u_i$ modulo $LT \cdot I_i$ for $1 \le i < n$ and $v_n \mapsto 1$ modulo $LT \cdot I_n$.

Definition 1.7. Let

$$E(\Phi, k)_* = LT(\Phi, k)[u^{\pm 1}]$$

with $|u| = 2$, so that $E(\Phi, k)_0 = LT(\Phi, k) \cong W(k)[[u_1, \dots, u_{n-1}]]$. ((ETC: For some purposes it is better to let $|u| = -2$.))

There is a graded variant of the Lubin–Tate formal group law F_{LT}, defined over $E(\Phi, k)_*$, such that the classifying ring homomorphism $g\colon L = MU_* \to E(\Phi, k)_*$ satisfies

$$\begin{aligned} v_i &\longmapsto u_i u^{p^i - 1} \\ v_n &\longmapsto u^{p^n - 1} \end{aligned}$$

for $1 \leq i < n$. Note that this makes $E(\Phi, k)_*$ satisfy the Landweber exact functor theorem.

Definition 1.8. Let $E(\Phi, k)$ be the spectrum representing the Landweber exact homology theory

$$E(\Phi, k)_*(X) = E(\Phi, k)_* \otimes_{MU_*} MU_*(X)\,.$$

In particular, $\pi_0 E(\Phi, k) = E(\Phi, k)_0 = LT(\Phi, k)$. In the special cases $k = \mathbb{F}_{p^n}$ and $\Phi = H_n$, the height n Honda formal group law, we let

$$E_n = E(H_n, \mathbb{F}_{p^n})\,.$$

In particular, $\pi_0 E_n = LT(H_n, \mathbb{F}_{p^n}) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ and

$$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]\,.$$

These spectra are known as Morava E-theory spectra, completed Johnson–Wilson spectra, or Lubin–Tate spectra.

Example 1.9. $E(F_m, \mathbb{F}_p) = KU_p^\wedge = E_1$.

Proposition 1.10. *Each Lubin–Tate spectrum $E(\Phi, k)$ is $K(n)$-local. In particular, E_n is $K(n)$-local.*

Proof sketch. Being Landweber exact of height n, these spectra are $E(n)$-local. Since $LT(\Phi, k)$ is $(p, u_1, \dots, u_{n-1})$-complete, so that $\pi_* E(\Phi, k)$ is I_n-complete, it follows from [HS99, Prop. 7.10(e)] that these spectra are $K(n)$-local. □

Alan Robinson [Rob89] developed an obstruction theory (in terms of Hochschild cohomology) for the existence of (associative $= \mathbb{A}_\infty =$) $\mathbb{E}_1$ ring structures on spectra, and applied it to prove that each Morava K-theory spectrum $K(n)$ admits such structures.

Andy Baker [Bak91] applied the same obstruction theory to proved that the completed Johnson–Wilson spectra $E(n)^\wedge_{I_n}$ also admit unique $\mathbb{E}_1$ ring structures. These are essentially the same as the Lubin–Tate spectra $E(H_n, \mathbb{F}_p)$.

An obstruction theory for diagrams of $\mathbb{E}_1$ ring spectra was developed by Mike Hopkins and Haynes Miller, see [Rez98], and also shows that each Lubin–Tate spectrum $E(\Phi, k)$ has a unique $\mathbb{E}_1$ ring structure.

Thereafter, an obstruction theory for diagrams of (commutative $=$) $\mathbb{E}_\infty$ ring spectra (in terms of André–Quillen cohomology) was developed by Paul Goerss and Mike Hopkins [GH04]. In particular, this shows that each Lubin–Tate spectrum $E(\Phi, k)$ has a unique $\mathbb{E}_\infty$ ring structure. This is the "E_n is $\mathbb{E}_\infty$" theorem.

((ETC: Also let $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$.))

2. The stabilizer group action

The Lubin–Tate deformation F_{LT} over $LT(\Phi, k)$ depends functorially on Φ over k. Hence the extended Morava stabilizer group, i.e., the profinite automorphism group $\mathrm{Aut}(\Phi, k)$, acts on $LT(\Phi, k)$, and this action lifts to a (continuous!) action on $E(\Phi, k)$. In particular, $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n}) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $E_n = E(H_n, \mathbb{F}_{p^n})$.

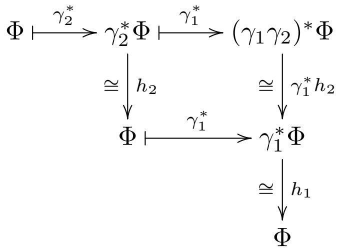
((ETC: Also $\mathbb{G}_n^{\mathrm{nr}} = \mathrm{Aut}(H_n, \bar{\mathbb{F}}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$.))

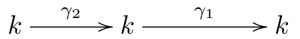
Definition 2.1. An automorphism (h, γ) of (Φ, k) is a field automorphism $\gamma\colon k \to k$ and a formal group law isomorphism $h\colon \gamma^*\Phi \to \Phi$. These form the group $\mathrm{Aut}(\Phi, k)$, with composition law

$$(h_1, \gamma_1) \circ (h_2, \gamma_2) = (h_1 \circ \gamma_1^* h_2, \gamma_1 \circ \gamma_2)\,.$$

If Φ is defined over $\mathbb{F}_p \subset k$, then $\gamma^*\Phi = \Phi$ in each case, and

$$\mathrm{Aut}(\Phi, k) \cong \mathrm{Aut}(\Phi/k) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)\,.$$





Definition 2.2. Let

$$[F, i] = [F/R, i\colon k \to R/\mathfrak{m}]$$

be a deformation of Φ over k to a complete Noetherian local ring R, and let

$$(h, \gamma) = (h\colon \gamma^*\Phi \to \Phi, \gamma\colon k \to k)$$

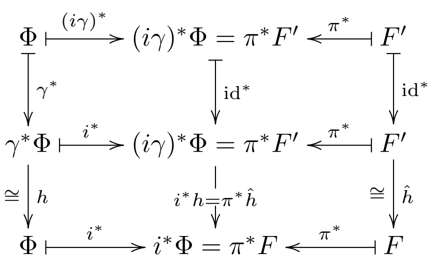
be an automorphism of (Φ, k). The natural (right) action

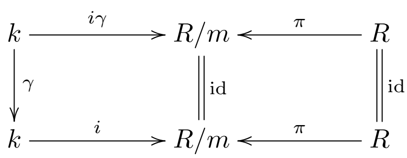
$$\mathrm{Def}(\Phi, k)(R) \times \mathrm{Aut}(\Phi, k) \xrightarrow{\cdot} \mathrm{Def}(\Phi, k)(R)$$

is given by

$$[F, i] \cdot (h, \gamma) = [F', i\gamma]\,,$$

where F' is the source of an isomorphism $\hat{h}\colon F' \to F$ over R such that $i^*h = \pi^*\hat{h}$. (Such lifts $\hat{h}(y) \in R[[y]]$ exist, since $\pi\colon R \to R/\mathfrak{m}$ is surjective. Any two choices of lifts $\hat{h}$ differ by a $\star$-isomorphism, so the deformation class of $(F', i\gamma)$ is well-defined.)





((ETC: Maybe explain the action of $h \in \mathrm{Aut}(\Phi/k)$ and of $\gamma \in \mathrm{Gal}(k/\mathbb{F}_p)$ separately, when Φ is defined over $\mathbb{F}_p$ so that $\gamma^*\Phi = \Phi$.))

The action of $\mathrm{Aut}(\Phi, k)$ on $\mathrm{Def}(\Phi, k)(R) \cong \mathcal{C}Ring^{\mathrm{loc}}(LT, R)$ is natural in R, hence must be induced by an action on the Lubin–Tate ring $LT = LT(\Phi, k)$ through local ring homomorphisms.

More explicitly, $(h, \gamma) \in \mathrm{Aut}(\Phi, k)$ takes the universal deformation $[F_{LT}, \mathrm{id}]$ to LT to the deformation $[F_{LT}, \mathrm{id}] \cdot (h, \gamma) = [F', \gamma]$ where F' is the source of an isomorphism $\hat{h}\colon F' \to F_{LT}$ over LT such that $h = \pi^*\hat{h}$ over k. There is then a unique local ring homomorphism $g\colon LT \to LT$ such that $[g^*F_{LT}, \bar{g}] = [F', \gamma]$. This means that $\bar{g} = \gamma$ (so that $\pi g = \gamma\pi$), and there is a (unique) $\star$-isomorphism $j\colon g^*F_{LT} \to F'$ over LT.

$$
\begin{array}{ccccc}
\Phi & \xmapsto{\mathrm{id}^*} & \Phi = \pi^*F_{LT} & \xleftarrow{\pi^*} & F_{LT} \\
\downarrow{\scriptstyle \mathrm{id}^*} & & \downarrow{\scriptstyle \gamma^* = \bar{g}^*} & & \downarrow{\scriptstyle g^*} \\
\Phi & \xmapsto{\gamma^*} & \gamma^*\Phi = \pi^*g^*F_{LT} & \xleftarrow{\pi^*} & g^*F_{LT} \\
\| {\scriptstyle \mathrm{id}} & & \| {\scriptstyle \mathrm{id}} & & {\scriptstyle \cong}\downarrow{\scriptstyle j} \\
\Phi & \xmapsto{\gamma^*} & \gamma^*\Phi = \pi^*F' & \xleftarrow{\pi^*} & F' \\
\downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \mathrm{id}^*} & & \downarrow{\scriptstyle \mathrm{id}^*} \\
\gamma^*\Phi & \xmapsto{\mathrm{id}^*} & \gamma^*\Phi = \pi^*F' & \xleftarrow{\pi^*} & F' \\
{\scriptstyle \cong}\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h = \pi^*\hat{h}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \hat{h}} \\
\Phi & \xmapsto{\mathrm{id}^*} & \Phi = \pi^*F_{LT} & \xleftarrow{\pi^*} & F_{LT}
\end{array}
$$

$$
\begin{array}{ccccc}
k & \overset{\mathrm{id}}{=\!=\!=} & k & \xleftarrow{\pi} & LT \\
\| {\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma = \bar{g}} & & \downarrow{\scriptstyle g} \\
k & \xrightarrow{\gamma} & k & \xleftarrow{\pi} & LT \\
\downarrow{\scriptstyle \gamma} & & \| {\scriptstyle \mathrm{id}} & & \| {\scriptstyle \mathrm{id}} \\
k & \overset{\mathrm{id}}{=\!=\!=} & k & \xleftarrow{\pi} & LT
\end{array}
$$

Replacing $\hat{h}$ by $\hat{h} \circ j\colon g^*F_{LT} \to F_{LT}$, we may assume that $j = \mathrm{id}$. To each automorphism (h, γ) there thus exists a unique ring automorphism $g\colon LT \to LT$ with $\bar{g} = \gamma$, and a unique formal group law isomorphism $\hat{h}\colon g^*F_{LT} \to F_{LT}$ with $\pi^*\hat{h} = h$.

$$
\begin{array}{ccccc}
\Phi & \xmapsto{\mathrm{id}^*} & \Phi = \pi^*F_{LT} & \xleftarrow{\pi^*} & F_{LT} \\
\downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \gamma^* = \bar{g}^*} & & \downarrow{\scriptstyle g^*} \\
\gamma^*\Phi & \xmapsto{\mathrm{id}^*} & \gamma^*\Phi = \pi^*g^*F_{LT} & \xleftarrow{\pi^*} & g^*F_{LT} \\
{\scriptstyle \cong}\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h = \pi^*\hat{h}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \hat{h}} \\
\Phi & \xmapsto{\mathrm{id}^*} & \Phi = \pi^*F_{LT} & \xleftarrow{\pi^*} & F_{LT}
\end{array}
$$

Theorem 2.3 ([Goe, Thm. 7.16]). *Let $F_{LT}\colon \mathrm{Spf}(LT(\Phi,k)) \to \mathcal{M}_{\mathrm{fg}}$ denote the map representing the Lubin–Tate formal group (law) over the Lubin–Tate ring. There is a homotopy pullback square*

$$\begin{array}{ccc} \mathrm{Spf}(LT(\Phi,k)) \times \mathrm{Aut}(\Phi,k) & \xrightarrow{\cdot} & \mathrm{Spf}(LT(\Phi,k)) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle F_{LT}} \\ \mathrm{Spf}(LT(\Phi,k)) & \xrightarrow{F_{LT}} & \mathcal{M}_{\mathrm{fg}}\,. \end{array}$$

The orbit stack $\mathrm{Spf}(LT(\Phi,k))//\mathrm{Aut}(\Phi,k)$ is the formal neighborhood of Φ/k in $\mathcal{M}_{\mathrm{fg}}$.

Sketch proof. A map from $\mathrm{Spf}(R)$ to the (2-categorical or) homotopy pullback corresponds to two deformations $[F,i]$ and $[F',i']$ of Φ/k to R, and a formal isomorphism $\hat{h}\colon F' \to F$. We may suppose that i and i' are isomorphisms. Let $\gamma = i^{-1}i'$, so that $i\gamma = i'$, and let $h\colon \gamma^*\Phi \to \Phi$ be determined by $i^*h = \pi^*\hat{h}$. Then (h,γ) is the unique automorphism such that $[F,i]\cdot(h,\gamma) = [F',i']$. Hence the map from $\mathrm{Spf}(R)$ corresponds naturally to the pair $([F,i],(h,\gamma))$, mapping under π_1 to $[F,i]$ and under $\cdot$ to $[F',i']$. □

For each $(h,\gamma) \in \mathrm{Aut}(\Phi,k)$, the associated local ring homomorphism $g\colon LT \to LT$ and formal group law isomorphism $\hat{h}\colon g^*F_{LT} \to F_{LT}$ determines a morphism

$$\begin{aligned} E(\Phi,k)_*(X) &= LT \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{g\otimes\nu} LT \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{1\otimes\hat{h}\otimes 1} LT \otimes_{MU_*} LT \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{\phi\otimes 1} LT \otimes_{MU_*} MU_*(X) = E(\Phi,k)_*(X) \end{aligned}$$

of Landweber exact homology theories. (We write MU_* and MU_*MU in place of L and LB, to avoid notational similarity with $LT = LT(\Phi,k) = \pi_0 E(\Phi,k)$.) Here $\nu\colon MU_*(X) \to MU_*MU \otimes_{MU_*} MU_*(X)$ denotes the standard MU_*MU-coaction. The ring homomorphism $\hat{h}\colon MU_*MU \to LT$ represents the isomorphism $\hat{h}\colon g^*F_{LT} \to F_{LT}$. See [Rez98, §6.7] for a discussion of how to arrange that the graded version of $\hat{h}$ is a strict isomorphism.

This morphism of homology theories is represented by a map

$$(h,\gamma)\colon E(\Phi,k) \longrightarrow E(\Phi,k)$$

in the stable homotopy category. This defines an action in $\mathrm{Ho}(\mathcal{S}p)$ of $\mathrm{Aut}(\Phi,k)$ on $E(\Phi,k)$.

Example 2.4. Recall that $\mathrm{Aut}(F_m,\mathbb{F}_p) = \mathrm{Aut}(F_m/\mathbb{F}_p) \cong \mathbb{Z}_p^\times$. For $n \in \mathbb{Z}_p^\times$ the automorphism $[n]_{F_m}$ of $F_m/\mathbb{F}_p$ acts on $E_1 = KU_p^\wedge$ as the p-adic Adams operation ψ^n.

The principal achievement of the Hopkins–Miller and Goerss–Hopkins obstruction theories is to promote this group action in $\mathrm{Ho}(\mathcal{S}p)$ to a group action on (associative =) $\mathbb{E}_1$ ring spectra and (commutative =) $\mathbb{E}_\infty$ ring spectra. The following theorems are usually credited to Goerss–Hopkins–Miller as a group.

Theorem 2.5 (Hopkins–Miller [Rez98, Thm. 7.1]). *For any two Lubin–Tate spectra $E(\Phi,k)$ and $E(\Phi',k')$ the space of $\mathbb{E}_1$ ring maps $E(\Phi,k) \to E(\Phi',k')$ is homotopy equivalent to the (profinite) set of morphisms $(h,\gamma)\colon (\Phi,k) \to (\Phi',k')$, where*

$\gamma\colon k \to k'$ is a field homomorphism and $h\colon \gamma^*\Phi \to \Phi'$ is a formal group law isomorphism.

Hence the action of $\mathrm{Aut}(\Phi, k)$ *in* $\mathrm{Ho}(\mathcal{S}p)$ *on $E(\Phi, k)$ lifts uniquely to a (continuous) action in the category of $\mathbb{E}_1$ ring spectra (= associative orthogonal ring spectra). In particular, $\mathbb{G}_n$ acts (continuously) on E_n through $\mathbb{E}_1$ ring spectrum maps.*

Theorem 2.6 (Goerss–Hopkins [GH04, Cor. 7.7]). *For any two Lubin–Tate spectra $E(\Phi, k)$ and $E(\Phi', k')$ the space of $\mathbb{E}_\infty$ ring maps $E(\Phi, k) \to E(\Phi', k')$ is homotopy equivalent to the (profinite) set of morphisms $(h, \gamma)\colon (\Phi, k) \to (\Phi', k')$, where $\gamma\colon k \to k'$ is a field homomorphism and $h\colon \gamma^*\Phi \to \Phi'$ is a formal group law isomorphism.*

Hence the action of $\mathrm{Aut}(\Phi, k)$ *in* $\mathrm{Ho}(\mathcal{S}p)$ *on $E(\Phi, k)$ lifts uniquely to a (continuous) action in the category of $\mathbb{E}_\infty$ ring spectra (= commutative orthogonal ring spectra). In particular, $\mathbb{G}_n$ acts (continuously) on E_n through $\mathbb{E}_\infty$ ring spectrum maps.*

Remark 2.7. In each case the assertion that the action is continuous requires further work, see work by Daniel G. Davis, Gereon Quick and collaborators. It can now be handled by working over suitable perfect $\mathbb{F}_p$-algebras in place of perfect fields, as in Lurie's account [Lur, §5]. An alternative is to work with "condensed sets", as in the work of Clausen–Scholze. As long as one considers finite (hence discrete) subgroups of $\mathrm{Aut}(\Phi, k)$, continuity is not an issue.

As a consequence of these theorems, any diagram of finite height formal group laws over perfect fields of characteristic p can be lifted to a diagram of (associative or) commutative orthogonal ring spectra. Unlike in $\mathrm{Ho}(\mathcal{S}p)$, it makes good sense to form homotopy limits of such orthogonal ring spectra. For example, for each subgroup $H \subset \mathbb{G}_n$ we may consider the homotopy fixed points

$$E_n^{hH} = F(EH_+, E_n)^H$$

(taking the topology on H into account). There is a conditionally convergent homotopy left half-plane fixed point spectral sequence

$$\mathcal{E}^2_{s,t} = H_c^{-s}(H; \pi_t E_n) \Longrightarrow \pi_{t-s}(E_n^{hH})$$

which is usually (always?) strongly converent.

Example 2.8. Consider $n = 1$ with $\pi_* E_1 = \pi_* KU_p^\wedge = \mathbb{Z}_p[u^{\pm 1}]$.

For p odd the maximal finite subgroup of $\mathbb{G}_1 = \mathbb{Z}_p^\times$ is $\Delta \cong \mathbb{Z}/(p-1)$. The homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(\Delta; \mathbb{Z}_p[u^{\pm 1}]) = \mathbb{Z}_p[u^{\pm(p-1)}] \Longrightarrow \pi_*(E_1^{h\Delta})$$

collapses at the $\mathcal{E}_2$-term, and identifies $E_1^{h\Delta}$ with the p-complete Adams summand $L_p^\wedge = E(1)_p^\wedge$ of $KU_p^\wedge$ with $\pi_* L_p^\wedge = \mathbb{Z}_p[v_1^{\pm 1}]$.

For $p = 2$ the maximal finite subgroup of $\mathbb{G}_1 = \mathbb{Z}_2^\times$ is $\Delta = \{\pm 1\}$, which acts by sign on $\pi_2 E_1 = \mathbb{Z}_2\{u\}$. The homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(\Delta; \mathbb{Z}_2[u^{\pm 1}]) = \mathbb{Z}_2[\eta, u^{\pm 2}]/(2\eta) \Longrightarrow \pi_*(E_1^{h\Delta})$$

has a nonzero differential $d^3(u^2) = \eta^3$, and collapses at

$$\mathcal{E}^4_{*,*} = \mathcal{E}^\infty_{*,*} = \mathbb{Z}_2[\eta, A, B^{\pm 1}]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $A = 2u^2$ and $B = u^4$. This identifies $E_1^{h\Delta}$ with 2-completed real K-theory $KO_2^\wedge$.

For H maximal finite in $\mathbb{G}_n$, the spectra

$$EO_n = E_n^{hH}$$

are sometimes known as higher real K-theory spectra.

Example 2.9. Early calculations with $H \cong \mathbb{Z}/p$ were made by Hopkins–Miller for $n = p - 1$, written out for $n = 2$ and $p = 3$ by Goerss–Henn–Mahowald–Rezk [GHMR05].

For $n = 2$ and $p = 2$ the extended Morava stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathbb{Z}/2$ has the maximal finite subgroup $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$ of order 48, which is also the automorphism group of the unique supersingular elliptic curve over $\mathbb{F}_4$. This leads to the equivalence

$$L_{K(2)}\,\mathrm{TMF} \simeq EO_2 = E_2^{hG_{48}}$$

between $K(2)$-local topological modular forms and this case of higher real K-theory. The structure of the homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(G_{48}; \pi_* E_2) \Longrightarrow \pi_* E_2^{hG_{48}} = \pi_* L_{K(2)}\,\mathrm{TMF}$$

has ((ETC: check)) been documented by Hans–Werner Henn. Another source for this abutment is [BR21].

Remark 2.10. The precise calculation of the action of $\mathrm{Aut}(\Phi, k)$ on $LT(\Phi, k)$, i.e., of the extended Morava stabilizer group $\mathbb{G}_n$ on the coefficient ring $\pi_*(E_n)$ of the n-th Lubin–Tate ring spectrum, is a difficult task. In Devinatz–Hopkins [DH95] the action is compared to a more explicit action on a "divided power envelope" of $\pi_*(E_n)$. In Hopkins–Gross [HG94] this is formulated in terms of a rigid-analytic "crystalline period mapping" to a projective space. Partial results for the action by finite subgroups, or simpler coefficients, are of current computational interest.

3. The Devinatz–Hopkins Galois extensions

By analogy with the Morava change-of-rings theorem and Theorem 2.3, Devinatz–Hopkins [DH04] show that the map

$$h\colon E_n \wedge E_n \longrightarrow \prod_{\mathbb{G}_n} E_n = F(\mathbb{G}_{n+}, E_n)$$
$$b_1 \wedge b_2 \longmapsto (b_1 \cdot g(b_2))_{g \in \mathbb{G}_n}$$

is a $K(n)$-local equivalence. Here the product (or function spectrum) takes the profinite topology on $\mathbb{G}_n$ into account. This implies that the cosimplicial resolution (= Amitsur complex)

$$S \longrightarrow E_n \rightleftarrows E_n \wedge E_n \rightleftarrows E_n \wedge E_n \wedge E_n \qquad \dots$$

is $K(n)$-locally equivalent to the cobar construction

$$E_n^{h\mathbb{G}_n} \longrightarrow E_n \rightleftarrows \prod_{\mathbb{G}_n} E_n \rightleftarrows \prod_{\mathbb{G}_n^2} E_n \qquad \dots$$

for the homotopy fixed points $E_n^{h\mathbb{G}_n} = F(E\mathbb{G}_{n+}, E_n)^{\mathbb{G}_n}$. (There are technical issues here, regarding the continuity of the $\mathbb{G}_n$-action on E_n and how to account for the topology on $\mathbb{G}_n$ in these products, which are resolved in an ad hoc manner in [DH04].) In the framework of [Rog08], this has the following formulation.

Theorem 3.1. *There is a faithful $K(n)$-local $\mathbb{G}_n$-pro-Galois extension*

$$\hat{L}_n S = L_{K(n)} S \simeq E_n^{h\mathbb{G}_n} \longrightarrow E_n$$

of $\mathbb{E}_\infty$ ring spectra.

There is a bijective Galois correspondence [Rog08, Thm. 7.2.3, Thm. 11.2.2] between the separable subextensions of E_n and subgroups of $\mathbb{G}_n$.

Corollary 3.2. *For each finite spectrum F there is a conditionally homotopy fixed point spectral sequence*

$$\mathcal{E}^2_{s,t} = H_c^{-s}(\mathbb{G}_n; \pi_t(E_n \wedge F)) \Longrightarrow \pi_{s+t} L_{K(n)} F\,.$$

When F has type $\geq n$ it agrees with the $E(n)$-based Adams–Novikov spectral sequence for $L_n F \simeq \hat{L}_n F$.

Example 3.3. For $n = 1$ and p odd, the continuous $\mathbb{Z}_p^\times$-homotopy fixed points of $KU_p^\wedge$ agree with the homotopy equalizer of ψ^g and 1, where g is a topological generator of $\mathbb{Z}_p^\times$, so that

$$L_{K(1)} S \simeq (KU_p^\wedge)^{h\mathbb{Z}_p^\times} \simeq J_p^\wedge\,.$$

For $n = 1$ and $p = 2$, the continuous $\mathbb{Z}_2^\times$-homotopy fixed points of $KU_2^\wedge$ agree with the $(1 + 4\mathbb{Z}_2)$-homotopy fixed points of $(KU_2^\wedge)^{h\{\pm 1\}} \simeq KO_2^\wedge$, which in turn agrees with the homotopy equalizer of ψ^5 and 1, so that

$$L_{K(1)} S \simeq (KU_2^\wedge)^{h\mathbb{Z}_2^\times} \simeq J_2^\wedge\,.$$

Recall the notation $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$, with $\mathrm{Aut}(H_n, \bar{\mathbb{F}}_p) = \mathbb{G}_n^{\mathrm{nr}} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. Like Theorem 3.1 above, there is a faithful $K(n)$-local $\mathbb{G}_n^{\mathrm{nr}}$-Galois extension $L_{K(n)} S \to E_n^{\mathrm{nr}}$.

Theorem 3.4 ([BR08]). *Let p be odd and $n \geq 1$. Then E_n^{nr} is separably closed, in the sense that it admits no proper, connected $K(n)$-local Galois extension.*

Hence the profinite completion $\mathbb{G}_n^{\mathrm{nr}}$ of the unit group $\mathbb{D}_n^\times$ (of the central simple $\mathbb{Q}_p$-algebra of invariant $1/n$, see Chapter 10, Remark 7.14) is realized as the absolute Galois group of the $K(n)$-local sphere. It is the fundamental group of the formal neighborhood of H_n over $\mathbb{F}_{p^n}$ in $\mathcal{M}_{\mathrm{fg}}$, with universal cover given by the Lubin–Tate formal group law over $\mathrm{Spf}(E_n^{\mathrm{nr}})$.

4. ((ETC: Unfinished business))

4.1. **Stable comodule categories.** Too little structure in target (abelian, not triangulated) may mean that the chromatic localization is too weak (loses too much information) and that a finer target, giving a stronger (telescopic) localization, is more interesting. Derived or stable ∞-categories; Hovey, Strickland.

4.2. **Elliptic cohomology and topological modular forms.** Map from moduli stack of (generalized) elliptic curves to (finite height) formal groups. Elliptic cohomology.

4.3. **Redshift.** Algebraic K-theory computations using topological cyclic homology of telescopic (rather than chromatic) homotopy groups.

4.4. **Chromatic Nullstellensatz.** Burklund–Schlank–Yuan: The chromatic Nullstellensatz (arXiv:2207.09929).

4.5. **Chromatic Fourier transform.** Barthel–Carmeli–Schlank–Yanovski: The chromatic Fourier transform (arXiv:2210.12822).

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