ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 10: THE HEIGHT FILTRATION

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To understand the Hopf algebroid $(L, LB) \cong (MU_*, MU_*MU)$ corepresenting the moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ of formal group laws and strict isomorphisms, we make a closer study of the latter. Since (L, LB) is defined over $\mathbb{Z}$, we may look at the fibers over the closed points $i\colon \operatorname{Spec}(\mathbb{F}_p) \to \operatorname{Spec}(\mathbb{Z})$, where p ranges over all primes, and the open point $j\colon \operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Z})$.

$$\begin{array}{ccccc}
\mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{Q} & \longrightarrow & \mathcal{M}_{\mathrm{fgl}} & \longleftarrow & \mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{F}_p \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}(\mathbb{Q}) & \xrightarrow{j} & \operatorname{Spec}(\mathbb{Z}) & \xleftarrow{i} & \operatorname{Spec}(\mathbb{F}_p)
\end{array}$$

It can also be convenient to work locally at a single prime, i.e., over $\operatorname{Spec}(\mathbb{Z}_{(p)})$, or completed at that prime, i.e., over $\operatorname{Spec}(\mathbb{Z}_p)$.

Formal group laws in characteristic 0 are canonically isomorphic, via their logarithm, to the additive formal group law. In classical terms they correspond to addition theorems. The classification of formal groups in prime characteristic p is much richer. Each such has a height $n \in \{1, 2, \dots, \infty\}$, and over separably closed fields the height is a perfect invariant.

1. Logarithms

For a formal group law $F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge1} a_{i,j} y_1^i y_2^j$ and homomorphism $h(y) = b_0 y + \sum_{k\ge1} b_k y^{k+1}$ (with no condition on b_0) let us write

$$F_1(y_1, y_2) = \frac{\partial F(y_1, y_2)}{\partial y_1} = 1 + \sum_{i,j\ge1} a_{i,j} i y_1^{i-1} y_2^j$$

for the formal partial derivative with respect to the first variable, and

$$h'(y) = \frac{\partial h(y)}{\partial y} = b_0 + \sum_{k\ge1} b_k (k+1) y^k$$

for the formal derivative.

Lemma 1.1. *Let $h\colon F \to F'$ be a homomorphism of formal group laws over R. If $h'(0) = 0$, then $h'(y) = 0$.*

Date: May 2nd.

Proof. Apply $\frac{\partial}{\partial y_1}\Big|_{(0,y)}$ to $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$ to obtain

$$h'(y)F_1(0, y) = F'_1(0, h(y))h'(0)\,.$$

Since $F_1(0, y) \equiv 1 \mod y$ has a multiplicative inverse in $R[[y]]$, the lemma follows. □

Proposition 1.2. *Suppose $\mathbb{Q} \subset R$ and let F be a formal group law over R. Then*

$$\log_F(y) = \int_0^y \frac{dt}{F_1(0, t)}$$

is the unique strict isomorphism $\log_F : F \to F_a$ to the additive formal group law over R. Hence

$$\int_0^{y_1} \frac{dt}{F_1(0, t)} + \int_0^{y_2} \frac{dt}{F_1(0, t)} = \int_0^{F(y_1,y_2)} \frac{dt}{F_1(0, t)}\,.$$

By analogy with the theory for Lie groups, the expression

$$d\log_F(y) = \frac{dy}{F_1(0, y)}$$

can be interpreted as an invariant differential (= 1-form) on the underlying formal group of F. (The following arguments are probably quite close to those of Euler and Abel, verifying an identity by first passing to derivatives.)

Proof. In order to have a strict isomorphism $h: F \to F_a$ we must have $h(F(y_1, y_2)) = h(y_1) + h(y_2)$. Applying $\frac{\partial}{\partial y_1}$ we obtain

$$h'(F(y_1, y_2))F_1(y_1, y_2) = h'(y_1)\,.$$

Setting $y_1 = 0$ this gives.

$$h'(y_2)F_1(0, y_2) = h'(0) = 1\,.$$

Hence $h'(y_2) = 1/F_1(0, y_2)$, and we must have

$$h(y) = \int_0^y h'(y_2)\, dy_2 = \int_0^y \frac{dy_2}{F_1(0, y_2)}\,,$$

as claimed.

Conversely, apply $\frac{\partial}{\partial y_0}\Big|_{(0,y_1,y_2)}$ to $F(F(y_0, y_1), y_2) = F(y_0, F(y_1, y_2))$ to obtain

$$F_1(y_1, y_2)F_1(0, y_1) = F_1(0, F(y_1, y_2))\,.$$

Hence $h'(y) = 1/F_1(0, y)$ implies

$$h'(F(y_1, y_2))F_1(y_1, y_2) = h'(y_1)\,,$$

and applying $\int_0^y (-)\, dy_1$ we recover

$$h(F(y_1, y_2)) = h(y_1) + h(y_2)\,.$$

We need $\mathbb{Q} \subset R$ in order to be able to formally integrate, since this will typically introduce denominators. □

We write $\exp_F = \log_F^{-1} : F_a \to F$ for the inverse strict isomorphism.

Example 1.3. If $F = F_m$ defined over $\mathbb{Q}[u]$ with $F(y_1, y_2) = y_1 + y_2 + uy_1y_2$ then $F_{m,1}(0, y_2) = 1 + uy_2$ and

$$\log_{F_m}(y) = \int_0^y \frac{dt}{1+ut} = u^{-1}\log(1+uy) = y + \sum_{k\geq 1}(-1)^k \frac{u^k}{k+1} y^{k+1}\,,$$

while

$$\exp_{F_m}(y) = u^{-1}(\exp(uy) - 1) = y + \sum_{k\geq 1} \frac{u^k}{(k+1)!} y^{k+1}\,.$$

Example 1.4. If $F = F_L$ defined over $L \otimes \mathbb{Q}$ then

$$\log_{F_L}(y) = \log_{MU}(y) = y + \sum_{k\geq 1} m_k y^{k+1}$$

and

$$\exp_{F_L}(y) = \exp_{MU}(y) = y + \sum_{k\geq 1} b_k y^{k+1}$$

with $b_k, m_k \in H_*(MU) \subset H_*(MU;\mathbb{Q}) \cong L \otimes \mathbb{Q}$.

The fact that every formal group law over a ring $R \supset \mathbb{Q}$ admits a unique logarithm (or exponential) has the following interpretation in terms of classifying objects.

Corollary 1.5. *The function*

$$m(y) = y + \sum_{k\geq 1} m_k y^{k+1} \longmapsto F(y_1, y_2) = m^{-1}(m(y_1) + m(y_2))$$

is corepresented by $\hbar\colon L \cong \pi_*(MU) \to H_*(MU) = \mathbb{Z}[m_k \mid k \geq 1](= \mathbb{Z}[b_k \mid k \geq 1])$, *and becomes an isomorphism*

$$L \otimes \mathbb{Q} \xrightarrow{\cong} H_*(MU;\mathbb{Q})$$

after rationalization.

An equivalence of Hopf algebroids is defined precisely so as to corepresent a natural equivalence of groupoids, see [Mor85, §1.2] and [Bau08, §2]. It will then induce an equivalence of comodule categories and an isomorphism of comodule Ext groups. ((ETC: Spell this out.))

Proposition 1.6. *For each commutative* $\mathbb{Q}$*-algebra* R *the inclusion*

$$* = \{\mathrm{id}\colon F_a \to F_a\} \xrightarrow{\simeq} \mathcal{FGL}_s(R)$$

is an equivalence of groupoids. Hence there is an equivalence of Hopf algebroids

$$(\mathbb{Q}, \mathbb{Q}) \xleftarrow{\simeq} (L \otimes \mathbb{Q}, LB \otimes \mathbb{Q})$$

and of moduli prestacks

$$\mathrm{Spec}(\mathbb{Q}) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{Q}\,.$$

2. Endomorphism rings

Let F be a formal group law defined over R. Recall that the formal negative $i(y)$ is characterized by $F(y, i(y)) = 0$.

Definition 2.1. The set of homomorphisms $h\colon F \to F$ defined over R forms the (generally non-commutative) endomorphism ring

$$\mathrm{End}(F/R) = \{h\colon F \to F \text{ with } h(y) \in R[[y]]\}\,.$$

Here

$$\begin{aligned}(h_1 + h_2)(y) &= F(h_1(y), h_2(y)) = h_1(y) +_F h_2(y)\\ -h(y) &= i(h(y))\\ (h_1h_2)(y) &= h_1(h_2(y))\,.\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{Aut}(F/R) &= \{h \in \mathrm{End}(F/R) \mid h'(0) \in R^\times\}\\ \mathrm{Aut}_s(F/R) &= \{h \in \mathrm{End}(F/R) \mid h'(0) = 1\}\,.\end{aligned}$$

Definition 2.2. The ring homomorphism

$$\begin{aligned}\mathbb{Z} &\longrightarrow \mathrm{End}(F/R)\\ n &\longmapsto [n]_F(y)\end{aligned}$$

defines the n-series $[n]_F(y) \equiv ny \mod y^2$ for each integer n, so that $[0]_F(y) = 0$ and

$$\begin{aligned}[n]_F(y) &= y +_F \cdots +_F y\\ [-n]_F(y) &= i(y) +_F \cdots +_F i(y)\end{aligned}$$

with n copies of y or $i(y)$, for each $n > 0$.

For example, $[2]_F(y) = F(y, y)$ and $[-1]_F(y) = i(y)$. For any homomorphism $h\colon F \to F'$ the diagram

$$\begin{array}{ccc} F & \xrightarrow{h} & F' \\ {\scriptstyle [n]_F}\downarrow & & \downarrow{\scriptstyle [n]_{F'}} \\ F & \xrightarrow{h} & F' \end{array}$$

commutes.

Lemma 2.3. *Suppose* $\mathbb{Q} \subset R$. *Then*

$$\begin{aligned}\mathrm{End}(F/R) &\xrightarrow{\cong} R\\ h(y) &\longmapsto h'(0)\end{aligned}$$

is a ring isomorphism, so that $\mathrm{Aut}_s(F/R) = \{\mathrm{id}\}$ *is trivial.*

Proof. It is clear that this is a ring homomorphism. To check that it is an isomorphism, we may conjugate by $\log_F$ and assume $F = F_a$, in which case $h(y) = ry$ defines an endomorphism $F_a \to F_a$ with $h'(y) = r$, for each $r \in R$. This characterizes h by Lemma 1.1, since $h'(y) = 0$ implies $h(y) = 0$ when $\mathbb{Q} \subset R$. □

Example 2.4. Let $F = F_m$ be the multiplicative formal group law defined over $\mathbb{Z}[u]$. Its n-series satisfies

$$1 + u[n]_{F_m}(y) = (1 + uy)^n = \sum_{i\geq 0} \binom{n}{i} (uy)^i$$

so that

$$[n]_{F_m}(y) = ny + \sum_{k\geq 1} \binom{n}{k+1} u^k y^{k+1} .$$

If we base change to $\mathbb{Z}_p[u]$, this formula extends to all p-adic integers $n \in \mathbb{Z}_p$, since for each k and e the residue class of $\binom{n}{k+1}$ modulo p^e only depends on the residue class of n modulo some (other) power of p. The extended ring homomorphism

$$\begin{aligned} \mathbb{Z}_p &\xrightarrow{\cong} \mathrm{End}(F_m/\mathbb{Z}_p[u]) \\ n &\longmapsto [n]_{F_m} \end{aligned}$$

is an isomorphism. This follows since

$$\begin{aligned} j^* \colon \mathrm{End}(F_m/\mathbb{Z}_p[u]) &\subset \mathrm{End}(F_m/\mathbb{Q}_p[u]) \\ &\cong \mathrm{End}(F_a/\mathbb{Q}_p[u]) \cong \mathbb{Q}_p . \end{aligned}$$

Here $n \in \mathbb{Q}_p$ corresponds to the endomorphisms $[n]_{F_a}(y) = ny \colon F_a \to F_a$ and $[n]_{F_m}(y) = \exp_{F_m}(n \log_{F_m}(y)) = u^{-1}((1 + uy)^n - 1) \colon F_m \to F_m$, both defined over $\mathbb{Q}_p[u]$, and the latter is defined over $\mathbb{Z}_p[u]$ if and only if $n \in \mathbb{Z}_p$.

The base change homomorphism

$$i^* \colon \mathrm{End}(F_m/\mathbb{Z}_p[u]) \longrightarrow \mathrm{End}(F_m/\mathbb{F}_p[u])$$

is injective, because if $[n]_{F_m}(y) \equiv y \mod p$ then $n \equiv 1 \mod p$ and $\binom{n}{k+1} \equiv 0 \mod p$ for each $k \geq 1$, which implies $n = 1$ by Lucas' theorem. ((ETC: Justify that i^* is also surjective.)) It follows that

$$\mathrm{Aut}(F_m/R) \cong \mathbb{Z}_p^\times \qquad \text{and} \qquad \mathrm{Aut}_s(F_m/R) \cong 1 + p\mathbb{Z}_p$$

for $R = \mathbb{Z}_p[u] = \pi_*(ku_p^\wedge)$ and $\mathbb{F}_p[u] = \pi_*(ku/p)$, and likewise over $R = \mathbb{Z}_p[u^{\pm 1}] = \pi_*(KU_p^\wedge)$ and $\mathbb{F}_p[u^{\pm 1}] = \pi_*(KU/p)$. Lazard [Laz55, Prop. 9] proves that this holds of $\mathbb{F}_p$ is replaced by any field of characteristic p, i.e., that there are no further automorphisms of F_m with coefficients outside of $\mathbb{F}_p$.

3. The height of a formal group law

Definition 3.1. Let p be a prime and suppose that $\mathbb{F}_p \subset R$. Let $\sigma \colon R \to R$ denote the Frobenius (ring) homomorphism, with $\sigma(x) = x^p$. We write $F^{(1)} = \sigma^* F$ for the pullback

$$F^{(1)}(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j}^p y_1^i y_2^j$$

of $F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j} y_1^i y_2^j$ along $\sigma \colon \mathrm{Spec}(R) \to \mathrm{Spec}(R)$. More generally, let $F^{(n)} = (\sigma^n)^* F$ be the pullback along $\sigma^n \colon \mathrm{Spec}(R) \to \mathrm{Spec}(R)$.

((ETC: In the graded case, σ is not degree-preserving, which may cause some confusion here. We only use the copy of R over which F is defined to explicitly grade the coefficients of formal group laws and homomorphisms.))

Lemma 3.2. *Let F be a formal group law defined over R containing $\mathbb{F}_p$. The formula $\varphi(y) = y^p \in R[[y]]$ defines a (relative) Frobenius (formal group law) homomorphism $\varphi\colon F \to F^{(1)} = \sigma^* F$. More generally, $\varphi^n(y) = y^{p^n}$ defines a homomorphism $\varphi^n\colon F \to F^{(n)} = (\sigma^n)^* F$.*

Proof. The identity

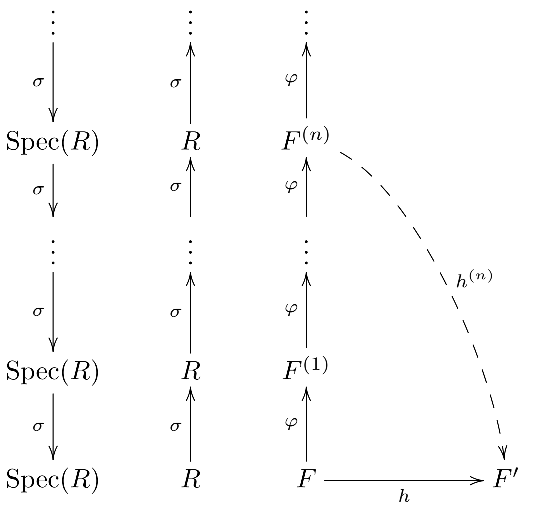
$$\begin{aligned} F(y_1, y_2)^p &= (y_1 + y_2 + \sum_{i,j \geq 1} a_{i,j} y_1^i y_2^j)^p \\ &= y_1^p + y_2^p + \sum_{i,j \geq 1} a_{i,j}^p y_1^{ip} y_2^{jp} = F^{(1)}(y_1^p, y_2^p) \end{aligned}$$

in $R[[y_1, y_2]]$ shows that $\varphi(y) = y^p$ satisfies $\varphi(F(y_1, y_2)) = F^{(1)}(\varphi(y_1), \varphi(y_2))$. □

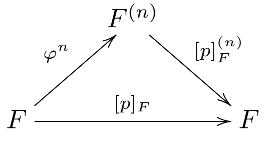
Definition 3.3. Consider F and F' defined over R containing $\mathbb{F}_p$. For $n \geq 0$ we say that a homomorphism $h\colon F \to F'$ has height $\geq n$ if it admits a factorization

$$h = h^{(n)} \circ \varphi^n \colon F \longrightarrow F^{(n)} = (\sigma^n)^* F \longrightarrow F'$$

through φ^n. It has height ∞ if it has height $\geq n$ for all $n \in \mathbb{N}$.



In particular, we say that a formal group law F (defined over $R \supset \mathbb{F}_p$) has height $\geq n$ if its p-series $[p]_F\colon F \to F$ has height $\geq n$. In a factorization



we call $\varphi^n\colon F \to F^{(n)}$ the (n-th) relative Frobenius and $[p]_F^{(n)}\colon F^{(n)} \to F$ the (n-th) Verschiebung, often denoted $F = F_{(n)}$ and $V = V_{(n)}$, respectively.

Lemma 3.4. *Assume $\mathbb{F}_p \subset R$. A homomorphism $h\colon F \to F'$ factors through $\varphi\colon F \to F^{(1)}$ if and only if $h'(0) = 0$.*

Proof. Let $h(y) = b_0 y + \sum_{k\geq 1} b_k y^{k+1}$ with $b_0 = h'(0)$. By Lemma 1.1, $h'(0) = 0$ implies $h'(y) = 0$ in $R[[y]]$. This means that $b_k(k+1) = 0 \in R$ for all $k \geq 0$, so that $b_k = 0$ unless $p \mid k+1$. Hence

$$h(y) = \sum_{i\geq 1} b_{ip-1} y^{ip} = h^{(1)}(\varphi(y)) = h^{(1)}(y^p)$$

for

$$h^{(1)}(y) = \sum_{i\geq 1} b_{ip-1} y^i \,.$$

Here $h^{(1)} \colon F^{(1)} \to F'$ is a homomorphism because

$$\begin{aligned} h^{(1)}(F^{(1)}(y_1^p, y_2^p)) &= h^{(1)}(F(y_1, y_2)^p) = h(F(y_1, y_2)) \\ &= F'(h(y_1), h(y_2)) = F'(h^{(1)}(y_1^p), h^{(1)}(y_2^p)) \end{aligned}$$

in $R[[y_1^p, y_2^p]] \subset R[[y_1, y_2]]$, which implies that

$$h^{(1)}(F^{(1)}(y_1, y_2)) = F'(h^{(1)}(y_1), h^{(1)}(y_2)) \,.$$

Conversely, $\varphi'(y) = py^{p-1} = 0$, so $h = h^{(1)}\varphi$ only if $h'(y) = 0$. □

It follows that the height of a formal group law F defined over $R \supset \mathbb{F}_p$ is never zero, since $[p]_F(y) \equiv py \mod y^2 = 0 \mod y^2$ in $R[[y]]$.

Corollary 3.5. *Let F be defined over $R \supset \mathbb{F}_p$. If F has height $\geq n \geq 1$, then*

$$[p]_F(y) = h^{(n)}(\varphi^n(y)) = h^{(n)}(y^{p^n}) = v_n(F) y^{p^n} + \dots \in R[[y]]$$

where

$$h^{(n)}(y) = v_n(F) y + \dots$$

for a uniquely determined element

$$v_n(F) \in R$$

of degree $2p^n - 2$. Moreover, F has height $\geq n+1$ if and only if $h^{(n)} \colon F^{(n)} \to F$ admits a further factorization through $\varphi \colon F^{(n)} \to F^{(n+1)}$, i.e., if and only if $v_n(F) = 0$.

Definition 3.6. Let F be defined over $R \supset \mathbb{F}_p$. We say that F has height equal to n if it has height $\geq n$ and $v_n(F)$ is a unit in R. This implies that F does not have height $\geq n+1$, and is equivalent to it if R is a graded field.

Example 3.7. The additive formal group law $F_a(y_1, y_2) = y_1 + y_2$ over $R \supset \mathbb{F}_p$ has height ∞, since $[p]_{F_a}(y) = py = 0$.

Example 3.8. The multiplicative formal group law $F_m(y_1, y_2) = y_1 + y_2 + uy_1y_2$ over $R \supset \mathbb{F}_p[u]$ has height ≥ 1, since

$$1 + u[p]_{F_m}(y) = (1+uy)^p = 1 + u^p y^p$$

implies

$$[p]_{F_m}(y) = u^{p-1} y^p \,,$$

so that $v_1(F_m) = u^{p-1} \neq 0$. It has height equal to 1 over $R \supset \mathbb{F}_p[u^{\pm 1}]$.

Example 3.9. Let C be an elliptic curve defined over a field $R \supset \mathbb{F}_p$. A choice of coordinate on the associated formal group $\hat{C}$ defines an elliptic formal group law F_C over R, which has height 1 if C is ordinary and height 2 if C is supersingular. (The projective closure in $\mathbb{P}^2 \supset \mathbb{A}^2$ of) the curve

$$y^2 + y = x^3$$

defined over $\mathbb{F}_2$ is an example of a supersingular elliptic curve.

Example 3.10. The formal Brauer group [Art74], [AM77] of a $K3$ surface is a commutative formal group (law) of height $n \in \{1, 2, \ldots, 9, 10, \infty\}$.

4. The height filtration

Recall that F_L denotes the universal formal group law defined over the Lazard ring $L \cong \mathbb{Z}[x_i \mid i \geq 1]$.

Definition 4.1. Fix a prime p and let $v_0 = p \in L$. Suppose by induction on $n \geq 1$ that

$$\begin{aligned} v_1 &\in L/(p) \\ v_2 &\in L/(p, v_1) \\ &\ldots \\ v_{n-1} &\in L/(p, v_1, \ldots, v_{n-2}) \end{aligned}$$

have been defined so that

$$F_n = \pi_n^* F_L$$

has height $\geq n$, where

$$\pi_n \colon L \longrightarrow L/(p, v_1, \ldots, v_{n-1})$$

is the n-th canonical projection. Then

$$[p]_{F_n}(y) = v_n y^{p^n} + \ldots$$

for a well-defined class $v_n \in L/(p, v_1, \ldots, v_{n-1})$. Moreover, $F_{n+1} = \pi_{n+1}^* F_L$ has height $\geq n+1$, where $\pi_{n+1} \colon L \to L/(p, v_1, \ldots, v_{n-1}, v_n)$ is the next canonical projection, and the induction continues.

It follows that $|v_n| = 2p^n - 2$ for each $n \geq 0$. Let

$$I_n = I_{p,n} = (p, v_1, \ldots, v_{n-1}) \subset L$$

be the ideal generated by the n first classes $v_0 = p, \ldots, v_{n-1}$, so that F_n is defined over L/I_n. Also let

$$I_\infty = I_{p,\infty} = (p, v_1, \ldots, v_n, \ldots) \subset L$$

be the ideal generated by all of the p-primary v_n-classes.

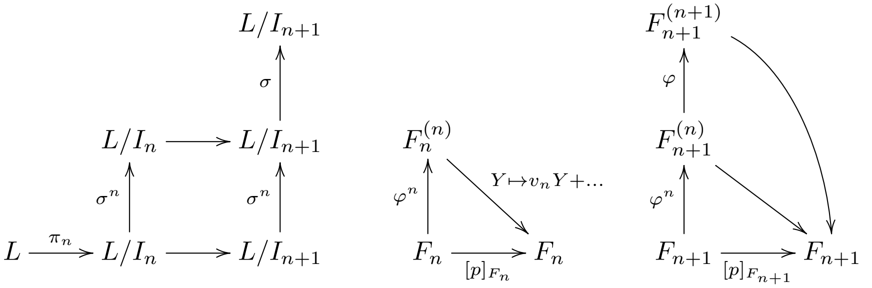
Example 4.2. For the Lazard formal group law we have

$$[2](y) = 2y + a_{1,1}y^2 + 2a_{1,2}y^3 + (2a_{1,3} + a_{2,2})y^4 + \ldots$$

and

$$[3](y) = 3y + 3a_{1,1}y^2 + (a_{1,1}^2 + 8a_{1,2})y^3 + \ldots.$$

With the conventions from ((ETC: Chapter 9, Remark 3.9)) it follows that $v_1 = a_{1,1} = x_1 \mod (2)$ and $v_2 = a_{2,2} \equiv x_3 \mod (2, v_1)$ for $p = 2$, while $v_1 = a_{1,1}^2 + 8a_{1,2} \equiv a_{1,1}^2 - a_{1,2} = x_1^2 - x_2 \mod (3)$ for $p = 3$.

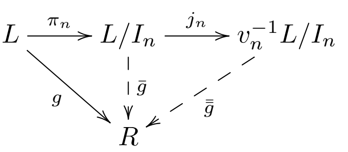


Lemma 4.3. *(a) A formal group law F defined over $R \supset \mathbb{F}_p$ has height $\geq n$ if and only if the classifying ring homomorphism $g\colon L \to R$ factors over $\pi_n\colon L \to L/I_n$ as $g = \bar{g}\pi_n$, i.e., if and only if*

$$g(p) = g(v_1) = \cdots = g(v_{n-1}) = 0$$

in R, in which case $\bar{g}(v_n) = v_n(F)$.

(b) It has height $= n$ if and only if $\bar{g}\colon L/I_n \to R$ factors further over $j_n\colon L/I_n \to v_n^{-1}L/I_n$ as $\bar{g} = \bar{\bar{g}}j_n$, i.e., if and only if $v_n(F)$ is a unit in R.



Proof. (a) We use induction on n. Base change of $[p]_{F_n}(y) = v_n y^{p^n} + \cdots \in L/I_n[[y]]$ along $\bar{g}\colon L/I_n \to R$ gives $[p]_F(y) = \bar{g}(v_n)y^{p^n} + \cdots \in R[[y]]$, so that $\bar{g}(v_n) = v_n(F)$. Hence F has height $\geq n+1$ if and only if $v_n(F) = 0$ if and only if $\bar{g}$ maps v_n to 0 if and only if g factors over π_{n+1}.

Claim (b) is straightforward. □

Lemma 4.4. *A formal group law F of height $\geq n$, classified by $g\colon L \to L/I_n \to R$, admits a restriction k^*F of height $= n$ if $v_n(F) \in R$ is not nilpotent. It admits a restriction k^*F of height $\geq n+1$ if $v_n(F) \in R$ is not a unit.*

Proof. The intersection of all prime ideals in R is the nilradical $\mathrm{Nil}(R)$, consisting of the nilpotent elements in R. The union of the maximal ideals is the set $R \setminus R^\times$ of nonunits in R. Hence there is a ring homomorphism $k\colon R \to T$ with $k(v_n(F))$ a unit if and only if $v_n(F) \notin \mathrm{Nil}(R)$, and a nonzero ring homomorphism $k\colon R \to T$ with $k(v_n(F)) = 0$ if and only if $v_n(F) \notin R^\times$. □

Remark 4.5. There are various strategies (due to Hazewinkel, Araki and others) for specifying elements $v_n \in L$ or $v_n \in L_{(p)} = L \otimes \mathbb{Z}_{(p)}$ that reduce mod I_n to the elements defined above. Note that the ideals $I_n \subset L$ are well-defined, even without a further specification of such choices.

Definition 4.6. (a) For each prime p, height $n \in \{1, 2, \ldots, \infty\}$ and commutative ring $R \supset \mathbb{F}_p$ let

$$\mathcal{FGL}^{\geq n}(R) = \mathcal{FGL}^{p,\geq n}(R) \subset \mathcal{FGL}(R)$$

be the full subcategory generated by the formal group laws F defined over R of height $\ge n$. Let

$$\mathcal{FGL}_s^{\ge n}(R) \subset \mathcal{FGL}_i^{\ge n}(R) \subset \mathcal{FGL}^{\ge n}(R)$$

be the subcategories of strict isomorphisms, and all isomorphisms, in $\mathcal{FGL}^{\ge n}(R)$. These are both groupoids.

(b) Let $\mathcal{FGL}^n(R) \subset \mathcal{FGL}^{\ge n}(R)$ be the full subcategory generated by the F of height $= n$, and let $\mathcal{FGL}_s^n(R) \subset \mathcal{FGL}_i^n(R) \subset \mathcal{FGL}^n(R)$ be the subcategories of strict isomorphisms, and all isomorphisms. Again the latter two are groupoids.

Proposition 4.7. *(a) The height $\ge n$ formal group law $F_n = \pi_n^* F_L$ over L/I_n is universal, in the sense that*

$$\begin{aligned} \mathcal{CA}lg_{\mathbb{F}_p}(L/I_n, R) &\xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{\ge n}(R) \\ \big(\bar g \colon L/I_n \to R\big) &\longmapsto \bar g^* F_n \end{aligned}$$

defines a natural bijection for all (graded) commutative $\mathbb{F}_p$-algebras R. Hence F_n represents an isomorphism of sheaves

$$\operatorname{Spec}(L/I_n) \xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{\ge n} .$$

(b) The height $= n$ formal group law $F_n = j_n^* \pi^* F_L$ over $v_n^{-1} L/I_n$ is universal, in the sense that

$$\begin{aligned} \mathcal{CA}lg_{\mathbb{F}_p}(v_n^{-1} L/I_n, R) &\xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{n}(R) \\ \big(\bar{\bar g} \colon v_n^{-1} L/I_n \to R\big) &\longmapsto \bar{\bar g}^* F_n \end{aligned}$$

defines a natural bijection for all (graded) commutative $\mathbb{F}_p$-algebras R. Hence F_n represents an isomorphism of sheaves

$$\operatorname{Spec}(v_n^{-1} L/I_n) \xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{n} .$$

□

Lemma 4.8. *(a) Let $1 \le n \le \infty$. Any base change of a formal group law of height $\ge n$ has height $\ge n$, so*

$$\begin{aligned} \mathcal{FGL}_s^{\ge n} \colon \mathcal{CA}lg_{\mathbb{F}_p} &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s^{\ge n}(R) \end{aligned}$$

defines a subfunctor of $\mathcal{FGL}_s$ restricted to $\mathcal{CA}lg_{\mathbb{F}_p} \subset \mathcal{CR}ing$. Equivalently, this defines a presheaf

$$\begin{aligned} \mathcal{M}_{\rm fgl}^{\ge n} = \mathcal{FGL}_s^{\ge n} \colon (\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p))^{op} &\longrightarrow \mathcal{G}pd \\ \operatorname{Spec}(R) &\longmapsto \mathcal{FGL}_s^{\ge n}(R) \end{aligned}$$

of small groupoids (in fact, a prestack), which is a sub-presheaf (or sub-prestack) of $\mathcal{M}_{\rm fgl} \otimes \mathbb{F}_p$, i.e., of $\mathcal{M}_{\rm fgl} = \mathcal{FGL}_s$ restricted to $\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p)$.

(b) Any base change of a formal group law of height $= n$ has height $= n$, so

$$\begin{aligned} \mathcal{FGL}_s^{n} \colon \mathcal{CA}lg_{\mathbb{F}_p} &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s^{n}(R) \end{aligned}$$

defines a subfunctor of $\mathcal{FGL}_s^{\ge n}$. Equivalently, this defines a presheaf

$$\begin{aligned} \mathcal{M}_{\rm fgl}^{n} = \mathcal{FGL}_s^{n} \colon (\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p))^{op} &\longrightarrow \mathcal{G}pd \\ \operatorname{Spec}(R) &\longmapsto \mathcal{FGL}_s^{n}(R) \end{aligned}$$

of small groupoids (in fact, a prestack), which is a sub-presheaf (or sub-prestack) of $\mathcal{M}_{\rm fgl}^{\geq n} = \mathcal{FGL}_s^{\geq n}$.

Remark 4.9. For each prime p the chain of ideals

$$(0) \subset I_1 = (p) \subset I_2 = (p, v_1) \subset \cdots \subset I_n = (p, v_1, \dots, v_{n-1}) \subset \cdots \subset I_\infty$$

in L corresponds to a tower of ring homomorphisms

$$L \longrightarrow L/p \longrightarrow L/(p, v_1) \longrightarrow \dots \longrightarrow L/I_n \longrightarrow \dots \longrightarrow L/I_\infty$$

and a sequence of closed subschemes

$$\mathrm{Spec}(L) \supset \mathrm{Spec}(L/p) \supset \mathrm{Spec}(L/(p, v_1)) \supset \cdots \supset \mathrm{Spec}(L/I_n) \supset \cdots \supset \mathrm{Spec}(L/I_\infty)$$

which is isomorphic to the sequence of subsheaves

$$\mathrm{obj}\,\mathcal{FGL} \supset \mathrm{obj}\,\mathcal{FGL}^{\geq 1} \supset \mathrm{obj}\,\mathcal{FGL}^{\geq 2} \supset \cdots \supset \mathrm{obj}\,\mathcal{FGL}^{\geq n} \supset \cdots \supset \mathrm{obj}\,\mathcal{FGL}^{\infty}\ .$$

This defines the height filtration on formal group laws. For each $n \geq 1$, the closed subsheaves $\mathrm{Spec}(L/I_{n+1}) \subset \mathrm{Spec}(L/I_n)$ and $\mathrm{obj}\,\mathcal{FGL}^{\geq n+1} \subset \mathrm{obj}\,\mathcal{FGL}^{\geq n}$ are divisors cut out by the condition $v_n = 0$. The subsheaves $\mathrm{Spec}(v_n^{-1}L/I_n) \subset \mathrm{Spec}(L/I_n)$ and $\mathrm{obj}\,\mathcal{FGL}^n \subset \mathrm{obj}\,\mathcal{FGL}^{\geq n}$ are the open complements of these divisors. This means that

$$\mathrm{Spec}(L/I_n)(R) \cong \mathrm{Spec}(v_n^{-1}L/I_n)(R) \coprod \mathrm{Spec}(L/I_{n+1})(R)$$

as sets if R is a (graded) field, but not for more general R, cf. Lemma 4.4.

((ETC: Add figure of finite codimension subschemes of $\mathrm{Spec}(L/p)$ over $\mathrm{Spec}(\mathbb{F}_p) \subset \mathrm{Spec}(\mathbb{Z})$, with ordinary and supersingular elliptic formal group laws at heights 1 and 2, and heights ≥ 3 at higher codimension. Also show geometric points $\mathrm{Spec}(H_n)$ covering $\mathcal{M}_{\rm fgl} \otimes \mathbb{F}_p$.))

Next, we shall see that the sequence of groupoid presheaves

$$\mathcal{FGL}_s \supset \mathcal{FGL}_s^{\geq 1} \supset \cdots \supset \mathcal{FGL}_s^{\geq n} \supset \cdots \supset \mathcal{FGL}_s^{\infty}\,,$$

also known as the sub-prestacks

$$\mathcal{M}_{\rm fgl} \supset \mathcal{M}_{\rm fgl} \otimes \mathbb{F}_p = \mathcal{M}_{\rm fgl}^{\geq 1} \supset \cdots \supset \mathcal{M}_{\rm fgl}^{\geq n} \supset \cdots \supset \mathcal{M}_{\rm fgl}^{\infty}\,,$$

is corepresented by a tower of Hopf algebroids

$$(L, LB) \longrightarrow (L/p, LB/p) \longrightarrow \dots \longrightarrow (L/I_n, LB/I_n) \longrightarrow \dots \longrightarrow (L/I_\infty, LB/I_\infty)$$

so that each inclusion of prestacks $\mathcal{M}_{\rm fgl}^{\geq n+1} \subset \mathcal{M}_{\rm fgl}^{\geq n}$ is in fact a closed inclusion. Its open complement $\mathcal{M}_{\rm fgl}^{n}$ is corepresented by the localized Hopf algebroid

$$(v_n^{-1}L/I_n, v_n^{-1}LB/I_n)\,.$$

Again, this means that

$$\mathcal{FGL}_s^{\geq n}(R) \cong \mathcal{FGL}_s^{n}(R) \coprod \mathcal{FGL}_s^{\geq n+1}(R)$$

as groupoids when R is a graded field, but not in general. After stackification, we obtain the p-primary height filtration

$$\mathcal{M}_{\rm fg} \supset \mathcal{M}_{\rm fg}^{\geq 1} \supset \cdots \supset \mathcal{M}_{\rm fg}^{\geq n} \supset \cdots \supset \mathcal{M}_{\rm fg}^{\infty}$$

of the moduli stack of formal groups, with $\mathcal{M}_{\rm fg}^{n}$ the complement in $\mathcal{M}_{\rm fg}^{\geq n}$ of $\mathcal{M}_{\rm fg}^{\geq n+1}$. One may say that $\mathcal{M}_{\rm fg} \otimes \mathbb{F}_p = \mathcal{M}_{\rm fg}^{\geq 1}$ is cut out as an effective Cartier divisor in

$\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)} \subset \mathcal{M}_{\mathrm{fg}}$ by p, while $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ is cut out as an effective Cartier divisor in $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ by v_n.

Lemma 4.10. *Let $h\colon F \to F'$ be a strict isomorphism of height $\geq n$ formal group laws defined over $R \supset \mathbb{F}_p$. Then $v_n(F) = v_n(F') \in R$. Hence strictly isomorphic formal group laws have the same height, and $v_n(F) \in R$ only depends on the underlying formal group $\hat{G}_F$ of F.*

Proof. Let $h(y) = b_0 y + \sum_{k\geq 1} b_k y^{k+1}$ specify any isomorphism $h\colon F \xrightarrow{\cong} F'$. The diagram

$$\begin{array}{ccc} F & \xrightarrow{[p]_F} & F \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ F' & \xrightarrow{[p]_{F'}} & F' \end{array}$$

commutes, so

$$\begin{aligned} [p]_{F'}(y) &= h([p]_F(h^{-1}(y))) \equiv h(v_n(F) h^{-1}(y)^{p^n}) \\ &\equiv b_0 v_n(F)(b_0^{-1} y)^{p^n} = b_0^{1-p^n} v_n(F) y^{p^n} \mod (y^{p^n+1})\,. \end{aligned}$$

Hence $v_n(F') = b_0^{1-p^n} v_n(F)$. When h is strict, so that $b_0 = 1$, this is equal to $v_n(F)$. □

Recall the universal strict isomorphism $\iota^* h\colon \eta_L^* F_L \xrightarrow{\cong} \eta_R^* F_L$ defined over LB.

Definition 4.11. Let

$$LB/I_n = LB \otimes_L L/I_n$$

and define $\eta_R\colon L/I_n \to LB/I_n$ and $\epsilon\colon LB/I_n \to L/I_n$ by the pushout squares

$$\begin{array}{ccccc} L & \xrightarrow{\eta_R} & LB & \xrightarrow{\epsilon} & L \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} \\ L/I_n & \xrightarrow{\eta_R} & LB/I_n & \xrightarrow{\epsilon} & L/I_n \end{array}$$

of graded commutative rings.

Lemma 4.12. *There are unique ring homomorphisms*

$$\begin{aligned} \eta_L &\colon L/I_n \longrightarrow LB/I_n \\ \psi &\colon LB/I_n \longrightarrow LB/I_n \otimes_{L/I_n} LB/I_n \\ \chi &\colon LB/I_n \longrightarrow LB/I_n \end{aligned}$$

making the diagrams

$$\begin{array}{ccc} L & \xrightarrow{\eta_L} & LB \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ L/I_n & \xrightarrow{\eta_L} & LB/I_n \end{array} \qquad \begin{array}{ccc} LB & \xrightarrow{\psi} & LB \otimes_L LB \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n \otimes \pi_n} \\ LB/I_n & \xrightarrow{\psi} & LB/I_n \otimes_{L/I_n} LB/I_n \end{array} \qquad \begin{array}{ccc} LB & \xrightarrow{\chi} & LB \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ LB/I_n & \xrightarrow{\chi} & LB/I_n \end{array}$$

commute. In particular

$$LB/I_n \xrightarrow{\cong} L/I_n \otimes_L LB \otimes_L L/I_n\,.$$

Proof. This follows from Lemma 4.10, since in each case one needs to extend some ring homomorphism $g\colon L \to R$ over $\pi_n\colon L \to L/I_n$, and this lemma ensures that the formal group law in question has height $\geq n$. □

Remark 4.13. The defining property of $\eta_L\colon L/I_n \to LB/I_n$ can be rewritten as

$$\begin{array}{ccc} L & \xrightarrow{\nu} & LB \otimes_L L \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \mathrm{id} \otimes \pi_n} \\ L/I_n & \xrightarrow{\nu} & LB \otimes_L L/I_n \,, \end{array}$$

saying that $L \to L/I_n$ is a quotient LB-comodule, or that $I_n \subset L$ is a sub LB-comodule of L. We also say that I_n is an invariant ideal of L.

Proposition 4.14. *(a) The pair $(L/I_n, LB/I_n)$, with structure maps as above, is a Hopf algebroid corepresenting the functor $\mathcal{FGL}_s^{\geq n}$.*

(b) The localized pair $(v_n^{-1}L/I_n, v_n^{-1}LB/I_n)$ is a Hopf algebroid corepresenting $\mathcal{FGL}_s^n$.

Proof. (a) We know that L/I_n corepresents formal group laws of height $\geq n$, and ring homomorphisms $g\colon LB/I_n = LB \otimes_L L/I_n \to R$ corepresent strict isomorphisms $h\colon F \to F'$ with F' of height $\geq n$, which is the same as strict isomorphisms with both F and F' of height $\geq n$. These are the morphisms in $\mathcal{FGL}_s^{\geq n}$.

(b) This follows from the isomorphism

$$v_n^{-1}LB/I_n \cong v_n^{-1}L/I_n \otimes_L LB \otimes_L v_n^{-1}L/I_n \,,$$

with the right hand side corepresenting strict isomorphisms $F \to F'$ where both F and F' have height $= n$. □

Remark 4.15. We can topologically realize the ring L/I_n (resp. $v_n^{-1}L/I_n$) as E_* for a flat ring spectrum $E = MU/I_n$ (resp. $E = v_n^{-1}MU/I_n$) in the homotopy category. Replacing MU by BP this ring spectrum is denoted $P(n) = BP/I_n$ (resp. $B(n) = v_n^{-1}BP/I_n$). The ring LB/I_n (resp. $v_n^{-1}LB/I_n$) is then a subring of E_*E, but the latter will also contain (at least for p odd) an exterior algebra $\Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})$, with $\bar\tau_i$, arising from reducing modulo v_i twice, cf. [JW75], [Wür77] and [Nas02]. The topological realization is thus in a sense richer than the algebraic model, only recovering the latter by reduction modulo nilpotent elements. ((ETC: The construction of MU/I_n, $v_n^{-1}MU/I_n$, $P(n)$ and $B(n)$ used to rely on the Baas–Sullivan theory of bordism with singularities, but is easy in the modern categories of MU-module spectra.))

5. Infinite height

Lazard showed that any formal group law $F(y_1, y_2)$ of height $\geq n$, defined over $R \supset \mathbb{F}_p$, is strictly isomorphic to one that agrees with $F_a(y_1, y_2) = y_1 + y_2$ modulo $(y_1^i y_2^j \mid i + j \geq p^n)$. The following is a special case.

Proposition 5.1 ([Laz55, Prop. 6])**.** *Let F be a formal group law defined over $R \supset \mathbb{F}_p$. The following are equivalent.*

1. *F is strictly isomorphic to F_a.*
2. *$[p]_F = 0$.*
3. *F has infinite height.*

In these cases the ring homomorphism $\mathbb{Z} \to \mathrm{End}(F/R)$ factors through $\mathbb{Z} \to \mathbb{Z}/p$, so we may call such a formal group (law) a formal $\mathbb{Z}/p$-module.

Lemma 5.2. *Let $R \supset \mathbb{F}_p$. The general homomorphism $h\colon F_a \to F_a$ defined over R has the form*

$$h(y) = \sum_{i\ge 0} t_i y^{p^i} = t_0 y + t_1 y^p + t_2 y^{p^2} + \dots$$

with $t_i \in R$ for each $i \ge 0$. Hence

$$\mathrm{End}(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(\mathbb{F}_p[t_i \mid i \ge 0], R)$$

and

$$\mathrm{Aut}_s(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(T, R)\,,$$

where $T = \mathbb{F}_p[t_i \mid i \ge 1]$ with $|t_i| = 2p^i - 2$. The composition of strict automorphisms is corepresented by the coproduct

$$\psi\colon T \longrightarrow T \otimes_{\mathbb{F}_p} T$$
$$\psi(t_k) = \sum_{i+j=k} t_i \otimes t_j^{p^i}\,,$$

where $t_0 = 1$, making T a Hopf algebra over $\mathbb{F}_p$.

Proof. For $h(y) = \sum_{k\ge 0} m_k y^{k+1}$ we have $h(y_1 + y_2) = h(y_1) + h(y_2)$ if and only if $\binom{k+1}{i} m_k = 0$ in R for all $0 < i < k+1$, which is equivalent to $m_k = 0$ for all $k+1$ not a power of p. ((ETC: There is a lemma here about the greatest common divisor of these binomial coefficients.)) □

Remark 5.3. This formula for the coproduct in T should be compared with Milnor's formula

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{p^i}$$

for the coproduct on $\bar\xi_k = \chi(\xi_k)$ in the dual Steenrod algebra $\mathscr{A}_*$ at an odd prime p, cf. Chapter 2, Theorem 8.8. The exterior generators $\bar\tau_k = \chi(\tau_k)$ are not as easy to interpret in terms of formal group laws.

We can identify the full subcategory of $\mathcal{FGL}_s^\infty(R)$ generated by F_a with the one-object groupoid $\mathcal{B}\,\mathrm{Aut}_s(F_a/R)$.

Proposition 5.4. *For each commutative $\mathbb{F}_p$-algebra R the inclusion*

$$\mathcal{B}\,\mathrm{Aut}_s(F_a/R) \xrightarrow{\simeq} \mathcal{FGL}_s^\infty(R)$$

is an equivalence of groupoids. Hence there is an equivalence of Hopf algebroids

$$(\mathbb{F}_p, T) \xleftarrow{\simeq} (L/I_\infty, LB/I_\infty)$$

and of moduli prestacks

$$\mathcal{B}\,\mathrm{Aut}_s(F_a) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fgl}}^\infty\,.$$

Proof. All objects in the groupoid $\mathcal{FGL}_s^\infty(R)$ are isomorphic, so the displayed inclusion is fully faithful and essentially surjective, hence an equivalence. □

In fact, a natural inverse equivalence $\mathcal{FGL}_s^\infty(R) \to \mathcal{B}\,\mathrm{Aut}_s(F_a/R)$ can be chosen, as follows.

Proposition 5.5 ([Qui71, Prop. 7.3], [Mit83, Prop. 1.2])**.** *Every formal $\mathbb{Z}/p$-module F over $R \supset \mathbb{F}_p$ admits a unique (normalized) logarithm $\mathrm{nog}_F \colon F \to F_a$ of the form*

$$\mathrm{nog}_F(y) = y + \sum_{k \geq 1} n_k y^{k+1}$$

with $n_k = 0$ whenever $k+1 = p^i$ is a power of p.

Proof. To each formal power series $\ell(y) = \sum_{k\geq 0} m_k y^{k+1}$ defined over $R \supset \mathbb{F}_p$ we assign its "p-typification"

$$\bar{\ell}(y) = \sum_{j \geq 0} m_{p^j-1} y^{p^j} = m_0 y + m_{p-1} y^p + m_{p^2-1} y^{p^2} + \dots ,$$

which is an endomorphism $\bar{\ell} \colon F_a \to F_a$. For any other endomorphism $h(y) = \sum_{i\geq 0} t_i y^{p^i}$ of F_a we have $\overline{h\ell} = h\bar{\ell}$, since the summands in

$$h(\ell(y)) = \sum_{i\geq 0} t_i (\sum_{k \geq 0} m_k y^{k+1})^{p^i} = \sum_{i,k\geq 0} t_i m_k^{p^i} y^{p^i(k+1)}$$

where $p^i(k+1)$ is a power of p are the same as those where $k+1$ is a power of p, so that

$$\overline{h\ell}(y) = \sum_{i,j\geq 0} t_i m_{p^j-1}^{p^i} y^{p^{i+j}} = h(\bar{\ell}(y)) \,.$$

Letting $\ell \colon F \to F_a$ be any strict isomorphism, we let $\mathrm{nog} = \bar{\ell}^{-1}\ell \colon F \to F_a$, so that $\ell = \bar{\ell}\,\mathrm{nog}$.

$$\begin{array}{ccc} F & \xrightarrow{\ \mathrm{nog}\ } & F_a \\ & {\scriptstyle \ell}\searrow \quad \swarrow{\scriptstyle \bar{\ell}} & \\ & F_a & \end{array}$$

Then $\bar{\ell} = \overline{\bar{\ell}\,\mathrm{nog}} = \bar{\ell}\,\overline{\mathrm{nog}}$, which implies $\overline{\mathrm{nog}} = \mathrm{id}$. This makes nog a normalized logarithm, as claimed.

If $\ell \colon F \to F_a$ is another strict isomorphism with $\bar{\ell} = \mathrm{id}$ then $\ell = h\,\mathrm{nog}$ for some $h \colon F_a \to F_a$, and $\mathrm{id} = \bar{\ell} = \overline{h\,\mathrm{nog}} = h\overline{\mathrm{nog}} = h\,\mathrm{id}$, so that $h = \mathrm{id}$ and $\ell = \mathrm{nog}$. Hence $\mathrm{nog}_F = \mathrm{nog}$ is uniquely defined. □

Proposition 5.6. *Let $N = \mathbb{F}_p[n_k \mid k+1 \neq p^i]$, and define*

$$\mathrm{nog}(y) = y + \sum_{\substack{k\geq 1 \\ k+1 \neq p^i}} n_k y^{k+1}$$

and

$$F_N(y_1, y_2) = \mathrm{nog}^{-1}(\mathrm{nog}(y_1) + \mathrm{nog}(y_2))$$

over N, so that F_N has infinite height and $\mathrm{nog} \colon F_N \to F_a$ is its normalized logarithm. Then the classifying homomorphism

$$\bar{g} \colon L/I_\infty \xrightarrow{\cong} N$$

is an isomorphism.

Proof. For each $R \supset \mathbb{F}_p$, the function

$$\bar{g}^* \colon \mathcal{CAlg}_{\mathbb{F}_p}(N, R) \longrightarrow \mathcal{CAlg}_{\mathbb{F}_p}(L/I_\infty, R)$$

is the bijection, implied by the previous proposition, from the formal group laws over R with a normalized logarithm to the formal group laws over R of infinite height. □

Corollary 5.7. *For any choices of lifts $\tilde{v}_n \in L$ and $\tilde{n}_k \in L$ with $\tilde{v}_n \mapsto v_n \in L/I_n$ and $\tilde{n}_k \mapsto n_k \in L/I_\infty \cong N$, we have*

$$\mathbb{Z}_{(p)}[\tilde{v}_n, \tilde{n}_k \mid n \geq 1, k+1 \neq p^i] \xrightarrow{\cong} L_{(p)}\,.$$

Proof. It suffices to check that the induced homomorphism of $\mathbb{Z}_{(p)}$-algebra indecomposables

$$\mathbb{Z}_{(p)}\{\tilde{v}_n, \tilde{n}_k \mid n \geq 1, k+1 \neq p^i\} \longrightarrow \mathbb{Z}_{(p)}\{x_i \mid i \geq 1\}$$

is an isomorphism, and we know this is true after reduction mod p. □

((ETC: This justifies thinking of the $\tilde{v}_n$ as coordinates on $\mathrm{Spec}(L/p) \to \mathcal{M}_{\mathrm{fg}} \otimes \mathbb{F}_p$, so that the $\mathrm{Spec}(L/I_n)$ are codimension n linear subspaces, rather than more general (higher degree) subvarieties.))

((ETC: Explain how this lets us concentrate on $\mathbb{Z}_{(p)}[\tilde{v}_n \mid n \geq 1] \subset L_{(p)}$.))

((ETC: Note parallel, for $p = 2$, with Thom's calculation of $\mathcal{N}_* = \pi_* MO$.))

Remark 5.8. The normalized logarithm is somewhat related to the Artin–Hasse exponential

$$E_p(y) = \exp\Big(y + \sum_{j \geq 1} \frac{y^{p^j}}{p^j}\Big)\,,$$

defined over $\mathbb{Z}_{(p)}$, where $\sum_{j\geq 0} y^{p^j}/p^j$ is the p-typification of $\sum_{k \geq 0} y^{k+1}/(k+1) = -\log(1-y)$. See also [Hon70, §5.4].

6. Finite height

Fix a prime p and a height $1 \leq n < \infty$, i.e., a finite height. Let $\mathbb{F}_p[v_n]$ denote the polynomial ring over $\mathbb{F}_p$ on a generator in degree $|v_n| = 2p^n - 2$. Its localization $\mathbb{F}_p[v_n^{\pm 1}]$ is a graded field.

Lemma 6.1. *There exists a formal group law F_n defined over $\mathbb{F}_p[v_n]$ with p-series*

$$[p]_{F_n}(y) = v_n y^{p^n} + \dots\,,$$

where the remaining terms lie in (y^{2p^n}).

Proof. With the notation from Corollary 5.7, let

$$g \colon L \subset L_{(p)} \cong \mathbb{Z}_{(p)}[\tilde{v}_m, \tilde{n}_k \mid m \geq 1, k+1 \neq p^i] \longrightarrow \mathbb{F}_p[v_n]$$

be given by mapping $\tilde{v}_n \mapsto v_n$ and sending the other polynomial generators to 0. Then g factors through $\pi_n \colon L \to L/I_n$ and classifies a formal group law F_n with p-series as claimed. □

Hence F_n has height $\geq n$, but not height $\geq n+1$, and its base change to $\mathbb{F}_p[v_n^{\pm 1}]$ has height $= n$. Taira Honda gave a more refined construction, of a formal group law H_n defined over $\mathbb{F}_p$ with p-series exactly $[p]_{H_n}(y) = y^{p^n}$. We state the graded version of his result, introducing the power of v_n needed to make the degrees match.

Theorem 6.2 ([Hon68, Thm. 2]). *Fix a prime p and a finite height n.*

(a) Let

$$\log_{\tilde{H}_n}(y) = \sum_{j\geq 0} \frac{v_n^{\frac{p^{jn}-1}{p^n-1}}}{p^j} y^{p^{jn}} = y + \frac{v_n}{p} y^{p^n} + \frac{v_n^{p^n+1}}{p^2} y^{p^{2n}} + \frac{v_n^{p^{2n}+p^n+1}}{p^3} y^{p^{3n}} + \dots$$

and

$$\tilde{H}_n(y_1, y_2) = \log_{\tilde{H}_n}^{-1}(\log_{\tilde{H}_n}(y_1) + \log_{\tilde{H}_n}(y_2)) = y_1 + y_2 - \frac{v_n}{p} \sum_{i=1}^{p^n-1} \binom{p^n}{i} y_1^i y_2^{p^n-i} + \dots .$$

Then $\tilde{H}_n$ is a formal group law defined over $\mathbb{Z}[v_n]$, and $\log_{\tilde{H}_n} : \tilde{H}_n \to F_a$ is a strict isomorphism defined over $\mathbb{Z}[1/p, v_n]$.

(b) Let $H_n = \pi^* \tilde{H}_n$ be the base change along $\pi\colon \mathbb{Z}[v_n] \to \mathbb{F}_p[v_n]$. Then

$$[p]_{H_n}(y) = v_n y^{p^n} .$$

Honda proves that $\tilde{H}_n$ is in fact defined over $\mathbb{Z}[v_n]$, not just over $\mathbb{Z}[1/p, v_n]$, and that $[p]_{\tilde{H}_n}(y) \equiv v_n y^{p^n} \mod (p)$. ((ETC: Is H_n uniquely determined by being p-typical with the given p-series?))

Remark 6.3. The localization $\mathbb{F}_p[v_n^{\pm 1}]$ is a graded field. The n-th Morava K-theory spectrum $K(n)$ will be defined to be a complex oriented ring spectrum with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ and associated formal group law $F_{K(n)} = H_n$. By convention, $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, with associated formal group laws $F_a = H_0$ over $\mathbb{Q}$ and $F_a = H_\infty$ over $\mathbb{F}_p$.

Theorem 6.4 ([Laz55, Thm. IV]). *Two formal group laws F and F' over the same separably closed (graded) field of characteristic p are isomorphic if and only if they have the same height.*

We have already seen that isomorphic formal group laws have the same height, and that any formal group law over $R \supset \mathbb{F}_p$ of infinite height is strictly isomorphic to F_a. The new assertion is thus that any two formal group laws of finite height $= n$ become isomorphic after base change to a separably closed (graded) field. To construct such an isomorphism $F \cong F'$, Lazard needs to solve algebraic equations [Laz55, (4.29)] over the base ring, which can always be done when the base is algebraically closed. These equations are ((ETC: apparently)) always separable, so it suffices that the base field is separably closed.

Proposition 6.5. *For each separably closed (graded) $\mathbb{F}_p$-algebra R the inclusion*

$$\mathcal{B}\operatorname{Aut}_s(H_n/R) \xrightarrow{\simeq} \mathcal{FGL}_s^n(R) = \mathcal{M}_{\mathrm{fgl}}^n(R)$$

is an equivalence of groupoids, for each $n \geq 1$, so that

$$\mathcal{M}_{\mathrm{fgl}}^{\geq 1}(R) = \mathcal{FGL}_s^{\geq 1}(R) \simeq \coprod_{1\leq n\leq\infty} \mathcal{B}\operatorname{Aut}_s(H_n/R) .$$

((ETC: Can we state this as an equivalence of prestacks, restricted to the subcategory of separably closed $R \supset \bar{\mathbb{F}}_p$?))

7. Morava stabilizer groups

This leads us to study $\mathrm{Aut}_s(H_n/R) \subset \mathrm{End}(H_n/R)$ for (graded) $\mathbb{F}_p$-algebras R. It turns out that the case $R = \mathbb{F}_{p^n}[v_n]$ is the most interesting. We follow Morava's summary [Mor85, §2.1.2].

Remark 7.1. Let $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ denote the field of p-adic numbers. The field extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\omega)$ given by adjoining a primitive (p^n-1)-th root of unity ω is an unramified cyclic Galois extension of degree n. The extension of valuation rings

$$\mathbb{Z}_p \subset \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$$

is given by the ring of Witt vectors of the finite field $\mathbb{F}_{p^n}$, the ideal (p) remains prime in this extension, and $\mathbb{Z}_p[\omega]/(p) = W(\mathbb{F}_{p^n})/(p) \cong \mathbb{F}_{p^n}$. In particular, the group homomorphism $\mathbb{Z}_p[\omega]^\times = W(\mathbb{F}_{p^n})^\times \to \mathbb{F}_{p^n}^\times$ is split surjective, with ω mapping to a generator of $\mathbb{F}_{p^n}^\times \cong \mathbb{Z}/(p^n-1)$, which we also denote as ω. The n Galois conjugates

$$\{\omega, \sigma(\omega) = \omega^p, \dots, \sigma^{n-1}(\omega) = \omega^{p^{n-1}}\}$$

generate $\mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$ as a free $\mathbb{Z}_p$-module, and their images give a basis for $\mathbb{F}_{p^n}$ as an $\mathbb{F}_p$-vector space.

Lemma 7.2. *Consider the base change of $\tilde{H}_n$ along $\mathbb{Z} \to \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$, and the related base change of H_n along $\mathbb{F}_p \to \mathbb{F}_{p^n}$, and their graded analogues. The identity*

$$\log_{\tilde{H}_n}(\omega y) = \omega \log_{\tilde{H}_n}(y)$$

holds over $W(\mathbb{F}_{p^n})[v_n]$, so

$$[\omega]_{\tilde{H}_n}(y) = \omega y$$

defines an endomorphism $[\omega]_{\tilde{H}_n} \colon \tilde{H}_n \to \tilde{H}_n$ over $W(\mathbb{F}_{p^n})[v_n]$. Its base change defines an endomorphism

$$[\omega] = [\omega]_{H_n} \colon H_n \longrightarrow H_n$$

over $\mathbb{F}_{p^n}[v_n]$.

Proof.

$$\log_{\tilde{H}_n}(\omega y) = \sum_{j\geq 0} \frac{v_n^{\frac{p^{jn}-1}{p^n-1}}}{p^j} (\omega y)^{p^{jn}} = \omega \log_{\tilde{H}_n}(y)$$

since $\omega^{p^{jn}} = \omega$ in $W(\mathbb{F}_{p^n})$ for all $j \geq 0$. It follows that the homomorphism $\omega y \colon F_a \to F_a$ defined over $W(\mathbb{F}_{p^n})$ corresponds to the endomorphism

$$[\omega]_{\tilde{H}_n}(y) = \log^{-1}_{\tilde{H}_n}(\omega \log_{\tilde{H}_n}(y)) = \omega y$$

of $\tilde{H}_n$. □

This defines a ring homomorphism

$$\mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n}) \longrightarrow \mathrm{End}(H_n/\mathbb{F}_{p^n}[v_n])$$
$$\omega \longmapsto [\omega]\,,$$

extending the usual homomorphism from $\mathbb{Z}_p$ given by the m-series $m \longmapsto [m] = [m]_{H_n}$.

Since H_n is defined over $\mathbb{F}_p[v_n]$, it is equal to its (ring) Frobenius pullback $\sigma^* H_n = H_n^{(1)}$ along $\sigma = \mathrm{id} \colon \mathbb{F}_p \to \mathbb{F}_p$, so that the (formal group law) Frobenius homomorphism $\varphi \colon H_n \to H_n^{(1)} = H_n$ given by $\varphi(y) = y^p$ is in fact an endomorphism.

Lemma 7.3.
$$\varphi \circ [\omega] = [\omega^p] \circ \varphi \qquad \textit{and} \qquad [p] = \varphi^n$$
in $\mathrm{End}(H_n/\mathbb{F}_{p^n}[v_n])$.

Proof. $(\omega y)^p = \omega^p y^p$ and $[p]_{H_n}(y) = y^{p^n}$. □

Theorem 7.4. *(a) Fix a prime p and finite height n. The natural homomorphisms*
$$W(\mathbb{F}_{p^n})\{1, \varphi, \dots, \varphi^{n-1}\} \xrightarrow{\cong} \mathrm{End}(H_n/\mathbb{F}_{p^n})$$
is an isomorphism of $\mathbb{Z}_p$-algebras, where the (noncommutative) multiplication in the source is given as in Lemma 7.3, so that $\varphi \cdot w = w^p \cdot \varphi$ and $p = \varphi^n$, for each root of unity $w \in \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$.

(b) For any field R containing $\mathbb{F}_{p^n}$, such as the algebraic closure $\bar{\mathbb{F}}_p$, the inclusion
$$\mathrm{End}(H_n/\mathbb{F}_{p^n}) \xrightarrow{\cong} \mathrm{End}(H_n/R)$$
is an isomorphism. Hence $\mathrm{Aut}_s(H_n/\mathbb{F}_{p^n}) \cong \mathrm{Aut}_s(H_n/R)$.

Morava [Mor85, §2.1.2] cites Frölich [Frö68, II §2 Prop. 3] for this fact. Ravenel cites Dieudonné and Lubin, and gives a proof in [Rav86, A2.2.17]. Part (a) says that the endomorphisms we have constructed so far give the whole story over $\mathbb{F}_{p^n}$, while part (b) says that no new endomorphisms appear if the base field is extended further. This is in contrast to the case $n = \infty$, where $\mathrm{Aut}_s(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(T, R)$ varies with R.

Definition 7.5. The profinite group $\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n})$ is called (in topological circles) the Morava stabilizer group at the prime p and finite height n. The subgroup $\mathbb{S}_n^0 = \mathrm{Aut}_s(H_n/\mathbb{F}_{p^n})$ is the strict Morava stabilizer group.
$$1 \to \mathbb{S}_n^0 \longrightarrow \mathbb{S}_n \longrightarrow \mathbb{F}_{p^n}^\times \to 1\,.$$

Definition 7.6. Let
$$\mathbb{D}_n = \mathbb{Q}_p(\omega)\{1, \varphi, \dots, \varphi^{n-1}\}$$
where ω is a primitive $(p^n - 1)$-th root of unity, $\varphi\omega = \omega^p\varphi$ and $\varphi^n = p$. Then $\mathbb{D}_n$ is the central simple $\mathbb{Q}_p$-algebra of Hasse invariant $1/n \in \mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(\mathbb{Q}_p)$. Its left action on itself, with respect to the basis displayed above, defines a faithful representation by $n \times n$ matrices over $\mathbb{Q}_p(\omega) = W(\mathbb{F}_{p^n})[1/p]$. Its determinant defines the (multiplicative, surjective) reduced norm homomorphism
$$\mathrm{Nrd}\colon \mathbb{D}_n \longrightarrow \mathbb{Q}_p\,.$$
Then $\mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p)$ is the maximal $\mathbb{Z}_p$-order in $\mathbb{D}_n$.

Lemma 7.7. *(a)* $\mathrm{Nrd}(p) = p^n$, $\mathrm{Nrd}(\varphi) = (-1)^{n-1}p$ *and*
$$\mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p) = W(\mathbb{F}_{p^n})\{1, \varphi, \dots, \varphi^{n-1}\}\,.$$

(b)
$$\mathbb{O}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) = W(\mathbb{F}_{p^n})^\times\{1\} \oplus W(\mathbb{F}_{p^n})\{\varphi, \dots, \varphi^{n-1}\}$$
is the group of units in the maximal $\mathbb{Z}_p$-order. It is a profinite group, i.e., a filtered limit of finite groups.

(c)
$$\mathbb{D}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Q}_p^\times) = \mathbb{D}_n \setminus \{0\}$$
is the group of (all) units in $\mathbb{D}_n$.

Proposition 7.8. *(a)*

$$\mathrm{End}(H_n/\mathbb{F}_{p^n}) \cong \mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p)$$

is isomorphic as a $\mathbb{Z}_p$*-algebra to the maximal* $\mathbb{Z}_p$*-order in* $\mathbb{D}_n$*.*

(b) The Morava stabilizer group

$$\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) \cong \mathbb{O}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times)$$

is isomorphic to the (profinite) group of units in the maximal $\mathbb{Z}_p$*-order in* $\mathbb{D}_n$*.*

(c) The strict Morava stabilizer group

$$\begin{aligned}\mathbb{S}_n^0 = \mathrm{Aut}_s(H_n/\mathbb{F}_{p^n}) &\cong \mathrm{Nrd}^{-1}(1+p\mathbb{Z}_p) \\ &= (1+pW(\mathbb{F}_{p^n}))\{1\} \oplus W(\mathbb{F}_{p^n})\{\varphi, \dots, \varphi^{n-1}\}\end{aligned}$$

is a pro-p-group, i.e., a filtered limit of finite p-groups.

Remark 7.9. The analysis of $\mathbb{S}_n$ and $\mathbb{S}_n^0$ continues [Rav76, Thm. 2.10] by letting $\mathbb{S}_n^1 = \mathrm{Nrd}^{-1}(1) = \ker(\mathbb{S}_n^0 \to 1+p\mathbb{Z}_p)$, so that there are short exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{S}_n^1 & \longrightarrow & \mathbb{S}_n^0 & \longrightarrow & 1+p\mathbb{Z}_p & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{S}_n^1 & \longrightarrow & \mathbb{S}_n & \longrightarrow & \mathbb{Z}_p^\times & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{F}_p^\times & = & \mathbb{F}_p^\times & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,. & &
\end{array}$$

If p is odd then $1+p\mathbb{Z}_p \cong \mathbb{Z}_p$, while if $p=2$ then $1+2\mathbb{Z}_2 = \mathbb{Z}_2^\times \cong \mathbb{Z}/2 \oplus \mathbb{Z}_2$.

Definition 7.10. Consider the category with objects (k, Γ) where k is a field of characteristic p and Γ is a formal group law of height n defined over k. In this "extended" category a morphism $(g,h)\colon (k,\Gamma) \to (k', \Gamma')$ is a pair (g,h) consisting of a ring homomorphism $g\colon k' \to k$ and a formal group law homomorphism $h\colon \Gamma \to g^*\Gamma'$. Its composite with a second morphism $(g', h')\colon (k', \Gamma') \to (k'', \Gamma'')$ is $(g \circ g', g^*h' \circ h)$. The extended automorphism group $\mathrm{Aut}(k, \Gamma)$ thus consists of pairs (g,h) with $g\colon k \to k$ a ring automorphism and $h\colon \Gamma \to g^*\Gamma$ a formal group law isomorphism. We get a short exact sequence

$$\begin{aligned}1 \to \mathrm{Aut}(\Gamma/k) \longrightarrow \mathrm{Aut}(k,\Gamma) &\longrightarrow \mathrm{Gal}(k/\mathbb{F}_p) \to 1 \\ (g,h) &\longmapsto g^{-1}\,.\end{aligned}$$

When Γ is defined over $\mathbb{F}_p$, this sequence is split by $g \mapsto (g^{-1}, \mathrm{id})$, and

$$\mathrm{Aut}(k,\Gamma) \cong \mathrm{Aut}(\Gamma/k) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$$

is the semidirect product for the left action of $\mathrm{Gal}(k/\mathbb{F}_p)$ on $\mathrm{Aut}(\Gamma/k)$ given by $g \cdot h = g^*h$.

Definition 7.11. The profinite group

$$\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, H_n)$$

is called the extended Morava stabilizer group (at the prime p and finite height n). The short exact sequence

$$1 \to \mathbb{S}_n \longrightarrow \mathbb{G}_n \longrightarrow \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \to 1$$

is split, so that $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ acts on $h \in \mathbb{S}_n \subset \mathbb{F}_{p^n}[[y]]$ by pullback, i.e., via the Galois action on $\mathbb{F}_{p^n}$. We may also consider the fully extended group

$$\mathbb{G}_n^{\mathrm{nr}} = \mathrm{Aut}(\bar{\mathbb{F}}_p, H_n) \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)\,,$$

where $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ is the group of profinite integers.

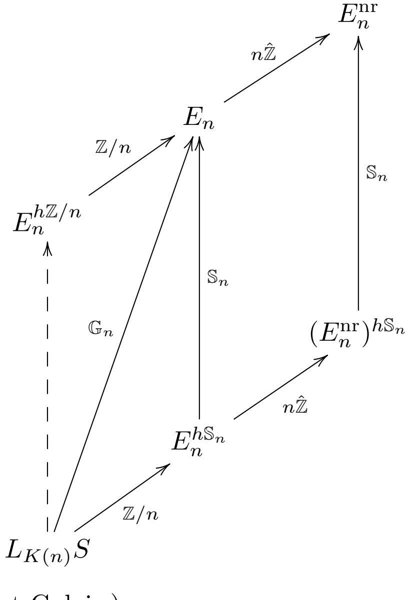
Remark 7.12. The profinite group $\mathbb{G}_n^{\mathrm{nr}}$ is in a sense the absolute (unramified = non ramifié) Galois group of the $K(n)$-local sphere spectrum. Devinatz–Hopkins [DH04] constructed a $K(n)$-local $\mathbb{G}_n^{\mathrm{nr}}$-pro-Galois extension $L_{K(n)}S \to E_n^{\mathrm{nr}}$, in the sense of the author [Rog08]. In particular, continuous homotopy fixed points can be defined so that

$$L_{K(n)}S \simeq E_n^{h\mathbb{G}_n} \simeq (E_n^{\mathrm{nr}})^{h\mathbb{G}_n^{\mathrm{nr}}}$$

and there is a homotopy fixed point spectral sequence

$$\mathcal{E}_2^{s,t} = H_c^s(\mathbb{G}_n; \pi_t(E_n)) \Longrightarrow_s \pi_{t-s}(E_n^{h\mathbb{G}_n}) \cong \pi_{t-s}(L_{K(n)}S)\,.$$

The group action here is discussed in [DH95]. Baker–Richter [BR08] proved that no further connected Galois extensions of E_n^{nr} exist (at least for p odd). This has recently been strengthened into a "chromatic Nullstellensatz" by Burklund–Schlank–Yuan [BSY], for Lubin–Tate spectra such as E_n^{nr}.



(The dashed arrow is not Galois.)

Let $\mathrm{ord}_p \colon \mathbb{Q}_p^\times \to \mathbb{Z}$ denote the p-order homomorphism.

Proposition 7.13 ([Mor85, §2.1.3]). *There is a vertical map of split extensions*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) & \longrightarrow & \mathbb{D}_n^\times & \xrightarrow{\mathrm{ord}_p \mathrm{Nrd}} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \varphi\mapsto\sigma & & \downarrow 1\mapsto\sigma & & \\ 1 & \longrightarrow & \mathbb{S}_n & \longrightarrow & \mathbb{G}_n & \longrightarrow & \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n) & \longrightarrow & 1\,, \end{array}$$

inducing an isomorphism

$$\mathbb{D}_n^\times/p^{\mathbb{Z}} \xrightarrow{\cong} \mathbb{G}_n$$

that extends the isomorphism $\mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) \cong \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) = \mathbb{S}_n$ *by the surjection* $\mathbb{Z} \to \mathbb{Z}/n \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n)$.

Proof. The composite $\mathrm{ord}_p \mathrm{Nrd}$ is split by $1 \mapsto \varphi$, sending n to $\varphi^n = p$, and the conjugation action in $\mathbb{D}_n^\times$ by φ on $\mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times)$ corresponds to the Galois action by σ on $\mathbb{S}_n$, which is the same as the conjugation action in $\mathbb{G}_n$ by σ. ((ETC: Does σ^{-1} appear?)) □

Remark 7.14. It follows that $\mathbb{G}_n^{\mathrm{nr}}$ is the profinite completion of the unit group $\mathbb{D}_n^\times$, hence plays the role of a non-abelian Weil group, analogous to how the group of units $L^\times$ in a p-adic number field $L \supset \mathbb{Q}_p$ is dense in the absolute Galois group $\mathrm{Gal}(\bar{L}/L)$, by local class field theory.

Example 7.15. When $n = 2$,

$$\mathbb{D}_2 = \begin{pmatrix} p, \omega \\ \mathbb{Q}_p \end{pmatrix} \cong \mathbb{Q}_p(\omega)\{1, \varphi\}$$

is the quaternion algebra over $\mathbb{Q}_p$. Here ω is a primitive $(p^2 - 1)$-th root of unity. When also $p = 2$, this is

$$\mathbb{D}_2 \cong \mathbb{Q}_2\{1, i, j, k\}$$

with $i^2 = j^2 = -1$ and $ij = k = -ji$. The maximal $\mathbb{Z}_2$-order is the $\mathbb{Z}_2$-algebra of Hurwitz integers

$$\mathrm{End}(H_2/\mathbb{F}_4) \cong \mathbb{Z}_2\Big\{1, i, j, \frac{1+i+j+k}{2}\Big\}\,,$$

which contains $\mathbb{Z}\{1, i, j, k\}$ as a submodule of index 2. The Morava stabilizer group $\mathbb{S}_2 = \mathrm{Aut}(H_2/\mathbb{F}_4)$ is the profinite group of units in this ring. It has a maximal finite subgroup $Q_8 \rtimes \mathbb{Z}/3 \cong SL_2(\mathbb{F}_3) \cong \hat{A}_4$ of order 24 given by the Hurwitz units

$$\hat{A}_4 = \Big\{\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2}\Big\} \cong A_4 \times_{SO(3)} Spin(3)\,,$$

also known as the binary tetrahedral group, since it is the double cover of the group $A_4 \subset SO(3)$ of orientation-preserving isometries of the regular tetrahedron. This is also the automorphism group of the unique supersingular elliptic curve over a field of characteristic 2, namely $y^2+y = x^3+x$. Let $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$ be the corresponding maximal finite subgroup of the extended stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathbb{Z}/2$, where in both cases $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Hopkins–Miller defined the higher real K-theory spectrum

$$EO_2 = E_2^{G_{48}}$$

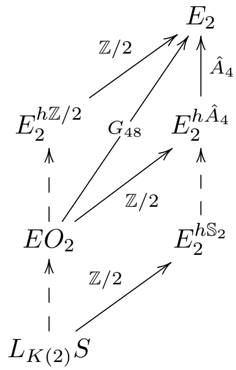
to be the homotopy fixed points for its action on the Lubin–Tate spectrum E_2, and identified this with the $K(2)$-local topological modular forms spectrum

$$EO_2 \simeq L_{K(2)}\,\mathrm{TMF}\ .$$

The homotopy fixed point spectral sequence

$$\mathcal{E}_2^{s,t} = H^s_{gp}(G_{48}; \pi_t(E_2)) \Longrightarrow \pi_{t-s}(EO_2) = \pi_{t-s}(L_{K(2)}\,\mathrm{TMF})$$

is more manageable than that for the full $\mathbb{S}_2$- or $\mathbb{G}_2$-action, and has been analyzed by Henn. ((ETC: Many other contributions along these lines should be mentioned.))



(The dashed arrows are not Galois.)

Remark 7.16. The Morava stabilizer groups $\mathbb{S}_n^0 \subset \mathbb{S}_n$ contain an element of order p^m if and only if $p^{m-1}(p-1)$ divides n. If $p - 1 \mid n$ then $H_c^{2*}(\mathbb{S}_n^0; \mathbb{F}_p)$ has Krull dimension 1, hence is unbounded. If $p - 1 \nmid n$ then $\mathbb{S}_n$ has finite p-cohomological dimension, and is in fact a Poincaré duality group. See [Mor85, §2.2]. This is analogous to properties of absolute Galois groups for global and local number fields.

8. Closed and open substacks

Fix a prime p, and consider the base change $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$ classifying formal group laws over commutative $\mathbb{Z}_{(p)}$-algebras R. For $n \geq 1$ the closed substack $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ is presented by the Hopf algebroid $(L/I_n, LB/I_n)$. A map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ factors through the closed inclusion

$$i \colon \mathcal{M}_{\mathrm{fg}}^{\geq n} \longrightarrow \mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$$

if and only if the classifying homomorphism $g \colon L \to R$ extends over $\pi_n \colon L \to L/I_n$, i.e., if and only if $RI_n = 0$. Note that $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ is covered by a single affine chart $\mathrm{Spec}(L/I_n) \to \mathcal{M}_{\mathrm{fg}}^{\geq n}$.

Let the open substack $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$ be the complement of $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$. A map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$ factors through the open inclusion

$$j \colon \mathcal{M}_{\mathrm{fg}}^{\leq n} \longrightarrow \mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$$

if and only if the base change $L/I_{n+1} \otimes_L R = R/RI_{n+1}$ of R along $\pi_{n+1} \colon L \to L/I_{n+1}$ is zero, i.e., if and only if $RI_{n+1} = R$. In other words, the images of $p, v_1, \dots, v_n$ generate the unit ideal in R. The collection of affine charts

$$F_m \colon \mathrm{Spec}(v_m^{-1} L/I_m) \longrightarrow \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

for $0 \le m \le n$ covers $\mathcal{M}_{\mathrm{fg}}^{\le n}$. The collection of affine charts

$$H_m \colon \operatorname{Spec}(\mathbb{F}_p[v_m^{\pm 1}]) \longrightarrow \mathcal{M}_{\mathrm{fg}}^{\le n}$$

for $0 \le m \le n$ also covers each (geometric) point of $\mathcal{M}_{\mathrm{fg}}^{\le n}$. For $n \ge 1$ there is not a canonical (single) affine chart covering this open substack, but there are non-canonical choices.

((ETC: Discuss how $\operatorname{Spec}(E(n)_*) \to \mathcal{M}_{\mathrm{fg}}^{\le n}$ is a cover, or presentation, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ is the Johnson–Wilson form of Morava's E-theory.))

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