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CHROMATIC HOMOTOPY THEORY

CHAPTER 5: TOPOLOGICAL K-THEORY

JOHN ROGNES

See [Ati67], [Hus66, Part II], [May99, Ch. 24] and Hatcher (2003).

1. The Grothendieck group of vector bundles

We work in the category $\mathcal{U}$ of unbased topological spaces.

Definition 1.1. For a connected CW complex X let

$$\mathrm{Vect}(X) = \coprod_{n\geq 0} \mathrm{Vect}_n(X)$$

be the set of isomorphism classes of (real or complex) finite-dimensional vector bundles over X. (There is also a story for quaternionic bundles, which we mostly omit to discuss.) Then

$$\mathrm{Vect}^{\mathbb{R}}(X) \cong [X, \coprod_{n\geq 0} BO(n)]$$
$$\mathrm{Vect}^{\mathbb{C}}(X) \cong [X, \coprod_{n\geq 0} BU(n)]\,.$$

For non-connected X we take the right hand side as the definition of $\mathrm{Vect}(X)$. This is the set of isomorphism classes of (real or complex) vector bundles over X, where the fiber dimension is allowed to vary between the components of X. Let

$$V = \coprod_{n\geq 0} BO(n) \qquad \text{or} \qquad V = \coprod_{n\geq 0} BU(n)$$

according to the case.

The Whitney sum of vector bundles defines a pairing

$$\oplus \colon \mathrm{Vect}(X) \times \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X)\,,$$

written additively, making $\mathrm{Vect}(X)$ a commutative monoid (= group without negatives). The neutral element is the class of the 0-dimensional bundle. This pairing is induced by a unital, associative and homotopy commutative map

$$\mu^{\oplus} \colon V \times V \longrightarrow V$$

where $\mu^{\oplus} = \coprod_{n,m} \mu^{\oplus}_{n,m}$.

The tensor product of vector bundles defines another pairing

$$\otimes \colon \mathrm{Vect}(X) \times \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X)\,,$$

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written multiplicatively, making $\mathrm{Vect}(X)$ a commutative semiring (= ring without negatives). The neutral element is the class of the trivial 1-dimensional bundle. This pairing is induced by a unital, associative and homotopy commutative map

$$\mu^{\otimes}\colon V \times V \longrightarrow V$$

where $\mu^{\otimes} = \coprod_{n,m} \mu^{\otimes}_{n,m}$. The homotopy multiplication $\mu^{\otimes}$ distributes over $\mu^{\oplus}$ up to homotopy, making V a commutative ring space up to homotopy.

There is a theory of E_∞ spaces, and E_∞ ring spaces, where coherent choices of these commuting homotopies have been made. These were said to be "homotopy everything", and the E stands for "everything". Calling these E_∞ semiring spaces might have been more consistent. The E_∞ (ring) spaces admitting additive inverses up to homotopy are usually said to be grouplike. See work by Boardman–Vogt and May.

Calculations in commutative monoids or semirings are simplified by the introduction of additive inverses, turning these into abelian groups or rings. This idea was introduced by Grothendieck (1957) in the context of algebraic vector bundles, for his generalization of the (Hirzebruch–)Riemann–Roch theorem. The idea was adapted to topological vector bundles by Atiyah–Hirzebruch [AH59].

Let $\mathcal{C}Mon$ and $\mathcal{A}b$ denote the categories of commutative monoids and abelian groups, respectively.

Lemma 1.2. *The full inclusion $\mathcal{A}b \to \mathcal{C}Mon$ has a left adjoint*

$$\begin{aligned}(-)^{gp}\colon \mathcal{C}Mon &\longrightarrow \mathcal{A}b\\ M &\longmapsto M^{gp}\,,\end{aligned}$$

called group completion, or the Grothendieck construction. The adjunction unit

$$\iota\colon M \longrightarrow M^{gp}$$

is the initial monoid homomorphism from M to an abelian group.

Proof.

$$M^{gp} = (M \times M)/\sim$$

where $(a, b) \sim (c, d)$ if there exists an $e \in M$ with $a + d + e = b + c + e$. We formally write $a - b$ for the class $[a, b]$ of (a, b). The adjunction unit maps a to $a - 0 = [a, 0]$. □

The group completion R^{gp} of a commutative semi-ring R is a commutative ring, with product $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$.

Definition 1.3. For a finite CW complex X let

$$KO(X) = \mathrm{Vect}^{\mathbb{R}}(X)^{gp}$$

$$KU(X) = \mathrm{Vect}^{\mathbb{C}}(X)^{gp}$$

be the commutative ring of virtual (real or complex) vector bundles over X. We write $\xi - \eta = [\xi, \eta]$ for the formal difference between (the classes of) ξ and η.

Many authors write K in place of KU; I prefer to reserve K for algebraic K-theory.

Example 1.4. If $X = *$ is a single point then $\mathrm{Vect}(*) \cong \mathbb{N}_0$ via the vector space dimension, and $KO(*) = \mathbb{Z}$ and $KU(*) = \mathbb{Z}$.

Example 1.5. When $X = S^2 = \mathbb{C}P^1$, we have

$$\mathrm{Vect}_n^{\mathbb{C}}(S^2) \cong [S^2, BU(n)] \cong \pi_1 U(n) \cong \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z} & \text{for } n \geq 1, \end{cases}$$

so that

$$\mathrm{Vect}^{\mathbb{C}}(S^2) \cong \{0\} \sqcup \coprod_{n \geq 1} \mathbb{Z} .$$

For $n = 1$ we use that $U(1) = S^1$. The claim for $n \geq 2$ follows by induction, using the exact sequence

$$\pi_2 S^{2n-1} \longrightarrow \pi_1 U(n-1) \longrightarrow \pi_1 U(n) \longrightarrow \pi_1 S^{2n-1} .$$

A generator for $\mathrm{Vect}_1^{\mathbb{C}}(\mathbb{C}P^1) \cong \mathbb{Z}$ is the class of the Hopf line bundle $\gamma_1^1 = \gamma^1 | \mathbb{C}P^1$. Hence $\mathrm{Vect}_n^{\mathbb{C}}(\mathbb{C}P^1)$ is generated by $\gamma_1^1 + (n-1)$ for each $n \geq 1$. It follows that

$$KU(S^2) \cong \coprod_{n \in \mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z} .$$

is freely generated by the classes of γ_1^1 and $1 = \epsilon^1$.

Proposition 1.6. *$KU(S^2) = \mathbb{Z}[u]/(u^2)$ where $u = \gamma_1^1 - 1$.*

Proof. Additively, $KU(S^2) = \mathbb{Z}\{1, \gamma_1^1\} = \mathbb{Z}\{1, u\}$ with $u \in \widetilde{KU}(S^2)$. Cup products of reduced classes vanish in the (cohomology and) K-theory of any suspension, since

$$S^2 \xrightarrow{\Delta} S^2 \times S^2 \xrightarrow{q} S^2 \wedge S^2$$

is nullhomotopic, so $u^2 = u \cup u = 0$. Alternatively, one can construct an isomorphism

$$\gamma_1^1 \oplus \gamma_1^1 \cong \gamma_1^1 \otimes \gamma_1^1 \oplus \epsilon^1$$

of $\mathbb{C}^2$-bundles over S^2. □

Lemma 1.7. *Let X be a finite CW complex. For each vector bundle η over X there exists a vector bundle ζ over X such that $\eta \oplus \zeta$ is trivial.*

Proof. Suppose X is connected and $n = \dim(\eta)$. We discuss the complex case. The classifying map $f \colon X \to BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$ for η, for which $f^*(\gamma^n) \cong \eta$, factors through $\mathrm{Gr}_n(\mathbb{C}^{n+m})$ for some finite m. The tautological bundle γ_{n+m}^n over $\mathrm{Gr}_n(\mathbb{C}^{n+m})$ is a subbundle of the trivial $\mathbb{C}^{n+m}$ bundle, hence has a unitary complement $(\gamma_{n+m}^n)^\perp$. Let $\zeta = f^*((\gamma_{n+m}^n)^\perp)$. □

Lemma 1.8. *Let X be a finite CW complex. Each element of $KO(X)$ or $KU(X)$ has the form $\xi - k$ for some $\xi \in \mathrm{Vect}(X)$ and $k \geq 0$. Moreover, $\xi - k$ is equal to $\eta - \ell$ if and only if $\xi + \ell + m = \eta + k + m$ for some $m \geq 0$.*

Proof. Here k denotes the class of the trivial bundle ϵ^k. The virtual bundle $\xi - \eta$ is equal to $(\xi + \zeta) - (n + m)$ if $\eta \oplus \zeta \cong \epsilon^{n+m}$. Similarly, $\xi + \ell + \zeta \cong \eta + k + \zeta$ for some ζ if and only if $\xi + \ell + m \cong \eta + k + m$ for some $m \geq 0$. □

Corollary 1.9. *Let X be a finite CW complex. The group completion $\iota \colon \mathrm{Vect}(X) \to \mathrm{Vect}(X)^{gp} = KO(X)$ or $KU(X)$ equals the localization that inverts the stabilization $\xi \mapsto \xi + 1$:*

$$\mathrm{colim}\,(\mathrm{Vect}(X) \xrightarrow{+1} \mathrm{Vect}(X) \xrightarrow{+1} \dots) \cong \mathrm{Vect}(X)^{gp} .$$

Recall the notation $V = \coprod_{n\geq 0} BO(n)$ or $\coprod_{n\geq 0} BU(n)$. The stabilization $\xi \mapsto \xi + 1$ is represented by the map

$$\iota \colon V = \coprod_{n\geq 0} BO(n) \xrightarrow{\coprod \iota} \coprod_{n\geq 0} BO(n+1) = \coprod_{n\geq 1} BO(n) \subset \coprod_{n\geq 0} BO(n) = V$$

or its complex analogue.

Definition 1.10. Let $\mathbb{Z} \times BO$ or $\mathbb{Z} \times BU$ be the (homotopy) colimit

$$\operatorname{colim}\,(V \xrightarrow{\iota} V \xrightarrow{\iota} \dots)\,.$$

The structural maps

$$V = \coprod_{n\geq 0} BO(n) \longrightarrow \mathbb{Z} \times BO$$

$$V = \coprod_{n\geq 0} BU(n) \longrightarrow \mathbb{Z} \times BU$$

are given on the n-th summands by the inclusions $BO(n) \subset BO$ and $BU(n) \subset BU$. The maps $\mu^{\oplus} \colon V \times V \to V$ and $\mu^{\otimes} \colon V \times V \to V$ extend to maps making $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$ grouplike commutative ring spaces up to homotopy, and these structures can be made E_∞ coherent.

Proposition 1.11. *There are natural isomorphisms of commutative rings*

$$KO(X) \cong [X, \mathbb{Z} \times BO]$$

$$KU(X) \cong [X, \mathbb{Z} \times BU]$$

for all finite CW complexes X.

When X is infinite, we shall hereafter take this as the definition of $KO(X)$ and $KU(X)$. This is called represented K-theory.

Remark 1.12. Using the Atiyah–Hirzebruch spectral sequence, we will extend Proposition 1.6 to calculate that

$$KU(\mathbb{C}P^n) \cong \mathbb{Z}[t]/(t^{n+1}) \qquad \text{and} \qquad KU(\mathbb{C}P^\infty) \cong \mathbb{Z}[[t]]\,,$$

where $t = \gamma^1 - 1$.

((ETC: Could also define or describe $\mathbb{Z} \times BO$ as $\Omega B_\oplus V$ where $B_\oplus V$ denotes the bar construction on $V = \coprod_{n\geq 0} BO(n)$ with the additive topological monoid structure given by $\mu^{\oplus} \colon V \times V \to V$. Likewise in the complex case. This uses the group completion theorems of Segal and McDuff.))

Proposition 1.13.

$$\pi_*(\mathbb{Z} \times BU) \cong (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \dots)$$

$$\pi_*(\mathbb{Z} \times BO) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \dots)$$

$$\pi_*(\mathbb{Z} \times BSp) \cong (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots)\,.$$

Proof. Since BU is connected, we have $\pi_0(\mathbb{Z}\times BU) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z}\times BU) \cong \pi_{i-1}U$ for $i \geq 1$. The map $S^1 \cong U(1) \to U$ is 2-connected, so $\pi_0 U = 0$, $\pi_1 U = \mathbb{Z}$ and $\pi_2 U = 0$. The fiber sequence $SU \to U \to U(1)$ admits a section, and $S^3 \cong SU(2) \to SU$ is 4-connected, so $\pi_3 U \cong \mathbb{Z}$.

Since BO is connected, we have $\pi_0(\mathbb{Z} \times BU) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z} \times BO) \cong \pi_{i-1}O$ for $i \geq 1$. The map $O(3) \to O$ is 2-connected. The fiber sequence $SO(3) \to O(3) \to$

$O(1)$ admits a section, and the universal cover $S^3 \cong Spin(3) \to SO(3)$ is a double covering, so $\pi_0 O = \mathbb{Z}/2$, $\pi_1 O = \mathbb{Z}/2$ and $\pi_2 O = 0$.

Since BSp is connected, we have $\pi_0(\mathbb{Z} \times BSp) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z} \times BSp) \cong \pi_{i-1} Sp$ for $i \geq 1$. The map $S^3 \cong Sp(1) \to Sp$ is 6-connected, so $\pi_0 Sp = 0$, $\pi_1 Sp = 0$, $\pi_2 Sp = 0$ and $\pi_3 Sp \cong \mathbb{Z}$. □

The calculation of the homomorphisms

$$\mathbb{Z} \cong \pi_3 O(3) \to \pi_3 O(4) \to \pi_3 O(5) \cong \pi_3 O$$

is interesting; see [Ste51, §23].

2. Bott periodicity

A first glimpse of chromatic periodicity, beyond algebra, is given by the Bott periodicity theorem [Bot57]. In its complex version this is a homotopy equivalence

$$U \simeq \Omega^2 U$$

while in its real and symplectic version it is a pair of homotopy equivalences

$$O \simeq \Omega^4 Sp \qquad \text{and} \qquad Sp \simeq \Omega^4 O\,,$$

which combine to homotopy equivalences

$$O \simeq \Omega^8 O \qquad \text{and} \qquad Sp \simeq \Omega^8 Sp\,.$$

Here is a more definite formulation, in terms of K-theory.

Theorem 2.1 (Bott periodicity)**.** *The external tensor product induces isomorphisms*

$$\hat{\otimes}\colon KU(X) \otimes KU(S^2) \xrightarrow{\cong} KU(X \times S^2)$$
$$\hat{\otimes}\colon KO(X) \otimes KO(S^8) \xrightarrow{\cong} KO(X \times S^8)\,.$$

For based spaces X we define the reduced K-groups by

$$\widetilde{KU}(X) = \ker(KU(X) \to KU(*))$$
$$\widetilde{KO}(X) = \ker(KO(X) \to KO(*))\,.$$

Then $KU(X) \cong \mathbb{Z} \oplus \widetilde{KU}(X)$, $KU(S^2) \cong \mathbb{Z} \oplus \widetilde{KU}(S^2)$ and $KU(X \times S^2) \cong \mathbb{Z} \oplus \widetilde{KU}(X) \oplus \widetilde{KU}(S^2) \oplus \widetilde{KU}(X \wedge S^2)$, since the cofiber sequence

$$X \vee S^2 \longrightarrow X \times S^2 \longrightarrow X \wedge S^2$$

splits after a single suspension. Hence we can rewrite the periodicity theorem as follows.

Theorem 2.2. *The reduced external tensor product induces isomorphisms*

$$\hat{\otimes}\colon \widetilde{KU}(X) \otimes \widetilde{KU}(S^2) \xrightarrow{\cong} \widetilde{KU}(X \wedge S^2)$$
$$\hat{\otimes}\colon \widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \xrightarrow{\cong} \widetilde{KO}(X \wedge S^8)\,.$$

Recall that $KU(S^2) = KU(\mathbb{C}P^1) = \mathbb{Z}\{1, \gamma^1_1\}$, so that $\widetilde{KU}(S^2) = \mathbb{Z}\{\gamma^1_1 - 1\}$. (There are similar results for $KO(S^8)$.)

Definition 2.3. Let $u = \gamma^1_1 - 1 \in \widetilde{KU}(S^2) \cong \mathbb{Z}$ and $B \in \widetilde{KO}(S^8) \cong \mathbb{Z}$ denote generators.

Theorem 2.4. *Product with the generators $u \in \widetilde{KU}(S^2)$ and $B \in \widetilde{KO}(S^8)$ induces isomorphisms*

$$u\colon \widetilde{KU}(X) \xrightarrow{\cong} \widetilde{KU}(\Sigma^2 X)$$
$$B\colon \widetilde{KO}(X) \xrightarrow{\cong} \widetilde{KO}(\Sigma^8 X)\,.$$

Working in based spaces, we have $\widetilde{KU}(X) = [X, \mathbb{Z} \times BU]$ and $\widetilde{KO}(X) \cong [X, \mathbb{Z} \times BO]$, so the theorem asserts that there are natural isomorphisms

$$u\colon [X, \mathbb{Z} \times BU] \xrightarrow{\cong} [\Sigma^2 X, \mathbb{Z} \times BU]$$
$$B\colon [X, \mathbb{Z} \times BO] \xrightarrow{\cong} [\Sigma^8 X, \mathbb{Z} \times BO]\,.$$

The right hand sides are $[X, \Omega^2(\mathbb{Z} \times BU)]$ and $[X, \Omega^8(\mathbb{Z} \times BO)]$, so yet another reformulation of Bott's theorem is that multiplication with $u\colon S^2 \to \mathbb{Z} \times BU$ and $B\colon S^8 \to \mathbb{Z} \times BO$ induce (weak) homotopy equivalences

$$u\colon \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega^2(\mathbb{Z} \times BU)$$
$$B\colon \mathbb{Z} \times BO \xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times BO)\,.$$

Their left adjoints are the composites

$$\bar{u}\colon (\mathbb{Z} \times BU) \wedge S^2 \xrightarrow{1 \wedge u} (\mathbb{Z} \times BU) \wedge (\mathbb{Z} \times BU) \xrightarrow{\mu^{\otimes}} \mathbb{Z} \times BU$$
$$\bar{B}\colon (\mathbb{Z} \times BO) \wedge S^8 \xrightarrow{1 \wedge B} (\mathbb{Z} \times BO) \wedge (\mathbb{Z} \times BO) \xrightarrow{\mu^{\otimes}} \mathbb{Z} \times BO\,.$$

Since $\Omega(\mathbb{Z} \times BU) = \Omega BU \simeq U$ it suffices to prove that $\mathbb{Z} \times BU \simeq \Omega U$. Here $\pi_1 U = \mathbb{Z}$ and the universal cover of U is $SU = \operatorname{colim}_n SU(n)$ where

$$SU(n) = \{A \in U(n) \mid \det(A) = 1\}\,,$$

so the key point in the complex case is to prove that $BU \simeq \Omega SU$. This is what Bott originally proved.

Definition 2.5. Let

$$D(t) = \begin{pmatrix} e^{i\pi t} I_n & 0 \\ 0 & e^{-i\pi t} I_n \end{pmatrix}$$

for $t \in [0,1]$ define a path in $SU(2n)$ from I_{2n} to $-I_{2n}$. Let

$$u_n\colon \operatorname{Gr}_n(\mathbb{C}^{2n}) \cong \frac{U(2n)}{U(n) \times U(n)} \longrightarrow \Omega SU(2n)$$
$$[C] \longmapsto \big(t \mapsto [C, D(t)] = CD(t)C^{-1}D(t)^{-1}\big)$$

map $C \in U(2n)$ to the indicated loop in $SU(2n)$. If $C = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then C and $D(t)$ commute, so the loop $u_n(C)$ only depends on the coset $[C]$ of C in $U(2n)/(U(n) \times U(n))$. The identification with $\operatorname{Gr}_n(\mathbb{C}^{2n})$ takes C to $\mathbb{C}\{Ce_1, \dots, Ce_n\} \subset \mathbb{C}^{2n}$. The maps u_n and u_{n+1} are compatible under suitable stabilization maps.

Theorem 2.6 (Bott)**.** *The map $u_n\colon \operatorname{Gr}_n(\mathbb{C}^{2n}) \to \Omega SU(2n)$ is $(2n+1)$-connected. Hence*

$$u\colon BU \simeq \operatorname{Gr}_\infty(\mathbb{C}^\infty) \xrightarrow{\simeq} \Omega SU$$

is a (weak) homotopy equivalence. Hence

$$\Omega^i(\mathbb{Z} \times BU) \simeq \begin{cases} \mathbb{Z} \times BU & \text{for } i \text{ even,} \\ U & \text{for } i \text{ odd.} \end{cases}$$

Raoul Bott's original proof [Bot59] is an application of Morse theory for the energy functional on a space of piecewise smooth paths in $SU(2n)$. The critical points are given by geodesic curves. This gives a cell complex of the homotopy type of that space of loops, containing $\mathrm{Gr}_n(\mathbb{C}^{2n})$ as a subcomplex, with all remaining cells of dimensions $* \ge 2n+2$. John Milnor's exposition [Mil63] is recommended. The loop space ΩSU has the homotopy type of a CW complex by [Mil59], so u is in fact a homotopy equivalence.

A purely homological proof of Bott's theorem, due to John Moore, was presented by Henri Cartan in his 1959–1960 seminar. Recall that $H_*BU = \mathbb{Z}[b_k \mid k \ge 1]$ is a polynomial algebra on degree $|b_k| = 2k$ generators. Moreover $H_*SU = \Lambda(e_k \mid k \ge 1)$ is a primitively generated exterior algebra on degree $|e_k| = 2k+1$ generators, cf. [Hat02, Prop. 3D.4]. The Eilenberg–Moore spectral sequence for the loop–path fibration $\Omega SU \to PSU \to SU$ collapses at the E^2-term $\mathrm{Cotor}^{H_*SU}_{*,*}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}[\omega e_k \mid k \ge 1]$, with $|\omega e_k| = 2k$. A compatibility check then shows that $BU \to \Omega SU$ is a homology isomorphism of 1-connected spaces, hence is a weak homotopy equivalence. See also [MP12, §21.6].

A more analytic proof (for X compact) was given by Atiyah–Bott [AB64], cf. [Ati67, §2.3] and Hatcher (2003, §2.1). They view vector bundles over $X \times S^2$ as being glued together from bundles $E \times D^2_+ \to X \times D^2_+$ and $E \times D^2_- \to X \times D^2_-$ along $X \times S^1$, where $S^2 = D^2_+ \cup D^2_+$ and $D^2_+ \cap D^2_- = S^1$. The gluing is specified by a continuous clutching function f that assigns to each $(x, z) \in X \times S^1$ a linear automorphism of E_x. By Fejér's theorem one may replace f by a Cesàro mean g of Fourier polynomial approximations, where $g(x, z) = \sum_n a_n(x) z^n$ now is a Laurent polynomial in $z \in S^1$. Linear algebra manipulations lets one reduce to the case where $g(x, z) = b_0(x) + b_1(x)z$ is linear in z. For such clutching functions there is a canonical splitting of E as a sum of vector bundles over X, and this gives the two components in $K(X) \otimes K(S^2)$.

Mark Behrens [Beh02], [Beh04] gave a proof of Bott periodicity using explicit quasi-fibrations.

In the real case, the eight steps in the periodicity are as follows. We have inclusions

$$O \xrightarrow{c} U \longrightarrow Sp$$

induced by $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ and

$$Sp \longrightarrow U \xrightarrow{r} O$$

induced by $\mathbb{H} \cong \mathbb{C}^2$ and $\mathbb{C} \cong \mathbb{R}^2$. The homogeneous spaces U/O, Sp/U, U/Sp and O/U are formed with respect to these inclusions.

Theorem 2.7 (Bott)**.** *There are homotopy equivalences as in Figure 1. Hence*

$$\Omega^i(\mathbb{Z} \times BO) \simeq \begin{cases} \mathbb{Z} \times BO & \text{for } i \equiv 0 \mod 8, \\ O & \text{for } i \equiv 1 \mod 8, \\ O/U & \text{for } i \equiv 2 \mod 8, \\ U/Sp & \text{for } i \equiv 3 \mod 8, \\ \mathbb{Z} \times BSp & \text{for } i \equiv 4 \mod 8, \\ Sp & \text{for } i \equiv 5 \mod 8, \\ Sp/U & \text{for } i \equiv 6 \mod 8, \\ U/O & \text{for } i \equiv 7 \mod 8. \end{cases}$$

$$\begin{aligned}
\mathbb{Z} \times BO &\xrightarrow{\simeq} \Omega(U/O) \\
U/O &\xrightarrow{\simeq} \Omega(Sp/U) \\
Sp/U &\xrightarrow{\simeq} \Omega Sp \\
Sp &\xrightarrow{\simeq} \Omega(\mathbb{Z} \times BSp) \\
\mathbb{Z} \times BSp &\xrightarrow{\simeq} \Omega(U/Sp) \\
U/Sp &\xrightarrow{\simeq} \Omega(O/U) \\
O/U &\xrightarrow{\simeq} \Omega O \\
O &\xrightarrow{\simeq} \Omega(\mathbb{Z} \times BO) .
\end{aligned}$$

FIGURE 1. Bott equivalences

Corollary 2.8. *For $i \geq 0$*

$$\widetilde{KU}(S^i) \cong \pi_i(\mathbb{Z} \times BU) \cong (\mathbb{Z}, 0, \dots)$$

repeats with period 2, so that

$$\pi_*(\mathbb{Z} \times BU) \cong \mathbb{Z}[u]$$

with $|u| = 2$. Similarly,

$$\widetilde{KO}(S^i) \cong \pi_i(\mathbb{Z} \times BO) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \dots)$$

repeats with period 8, so that

$$\pi_*(\mathbb{Z} \times BO) \cong \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$.

Complexification $c \colon \mathbb{Z} \to BO \to \mathbb{Z} \times BU$ takes A to $2u^2$ and B to u^4, which implies the relation $A^2 = 4B$.

3. The Chern character

A first glimpse of transchromatic phenomena, connecting different periodicities, is given by the Chern character [Hir56, §10].

Definition 3.1. For $n \geq 0$ let the Chern character

$$ch = n + \sum_{k \geq 1} \frac{p_k}{k!} \in \prod_{k \geq 1} H^{2k}(BU(n); \mathbb{Q})$$

be the characteristic class specified by

$$i_n^*(ch) = \sum_{i=1}^{n} \sum_{k \geq 0} \frac{y_i^k}{k!} = \sum_{i=1}^{n} e^{y_i} \in \mathbb{Q}[[y_1, \dots, y_n]] .$$

It is represented by a map

$$\{n\} \times BU(n) \xrightarrow{ch} \prod_{k \geq 1} K(\mathbb{Q}, 2k) .$$

Lemma 3.2. *Let ξ and η be complex vector bundles over X. Then*

$$ch(\xi \oplus \eta) = ch(\xi) + ch(\eta)$$
$$ch(\xi \otimes \eta) = ch(\xi) \cup ch(\eta)\,,$$

so the Chern character extends to a natural ring homomorphism

$$KU(X) \xrightarrow{ch} \prod_{k\ge 0} H^{2k}(X;\mathbb{Q})$$

represented by a map

$$\mathbb{Z} \times BU \xrightarrow{ch} \prod_{k\ge 0} K(\mathbb{Q}, 2k)$$

of ring spaces up to homotopy.

Proof. We have $ch(\gamma^n \times \gamma^m) = ch(\gamma^n) \times 1 + 1 \times ch(\gamma^m)$ in

$$\prod_{k\ge 0} H^{2k}(BU(n) \times BU(m))$$

since $\sum_{i=1}^{n+m} e^{y_i} = \sum_{i=1}^{n} e^{y_i} \otimes 1 + 1 \otimes \sum_{j=1}^{m} e^{y_j}$ in

$$\prod_{k\ge 0} H^{2k}(BU(1)^{n+m};\mathbb{Q}) \cong \mathbb{Q}[[y_1,\dots,y_{n+m}]]\,.$$

Hence $ch(\xi \times \eta) = ch(\xi) \times 1 + 1 \times ch(\eta)$ by naturality, and $ch(\xi \oplus \eta) = ch(\xi) + ch(\eta)$ by restriction to the diagonal.

(Recall that $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$.) For line bundles ξ and η we have

$$ch(\xi \otimes \eta) = e^{c_1(\xi\otimes\eta)} = e^{c_1(\xi)+c_1(\eta)} = e^{c_1(\xi)} \cup e^{c_1(\eta)} = ch(\xi) \cup ch(\eta)\,.$$

By additivity of ch, the same formula holds for ξ and η sums of line bundles, and the general case then follows by the splitting principle. □

Proposition 3.3. *For $X = S^{2n}$, the Chern character*

$$ch\colon KU(S^{2n}) \longrightarrow \prod_{k\ge 1} H^{2k}(S^{2n};\mathbb{Q})$$

maps $\widetilde{KU}(S^{2n}) = \mathbb{Z}\{u^n\}$ isomorphically to $\mathbb{Z}\{\iota_{2n}\} = H^{2n}(S^{2n};\mathbb{Z}) \subset H^{2n}(S^{2n};\mathbb{Q})$. Hence the n-th Chern class

$$c_n\colon \widetilde{KU}(S^{2n}) \longrightarrow H^{2n}(S^{2n};\mathbb{Z})$$

maps u^n to $(n-1)!$ times a generator.

Proof. When $n = 1$, $\widetilde{KU}(S^2) = \mathbb{Z}\{u\}$ where $u = \gamma^1_1 - 1$, $c_1(\gamma^1_1) = \iota_2 \in H^2(S^2;\mathbb{Z})$ and $c_1(1) = 0$, so $ch(u) = ch(\gamma^1_1) - ch(1) = e^{\iota_2} - 1 = \iota_2$. The cases $n \ge 2$ follow by multiplicativity of the Chern character. By the Girard–Newton formula

$$n! \cdot ch = p_n = -(-1)^n n \cdot c_n \in H^{2n}(S^{2n};\mathbb{Z})$$

so $n \cdot c_n(u^n)$ is $n!$ times a generator, as claimed. □

An almost complex structure on a smooth manifold M is a complex vector bundle structure on its tangent bundle τ_M. The dimension of M must obviously be even. It is classical that $S^2 = \mathbb{C}P^1$ and S^6, but not S^4, admit almost complex structures. Borel–Serre (1953) showed that there are no further examples. It is a famous open problem whether S^6 admits a complex structure.

Theorem 3.4 ([BS53, Prop. 15.1])**.** *S^{2n} cannot admit an almost complex structure if $n \geq 4$.*

Proof. We have
$$\langle e(\tau_{S^{2n}}), [S^{2n}]\rangle = \chi(S^{2n}) = 2\,.$$
If $\tau_{S^{2n}} = \eta_{\mathbb{R}}$ is the underlying real vector bundle of a complex vector bundle η, then $e(\tau_{S^{2n}}) = \pm c_n(\eta)$, so
$$\langle c_n(\eta), [S^{2n}]\rangle = \pm 2\,.$$
But $c_n(\eta)$ is a multiple of $c_n(u^n)$, hence is divisible by $(n-1)!$, so this is impossible if $n \geq 4$. □

Proposition 3.5. *The Chern character induces an isomorphism*
$$KU(X) \otimes \mathbb{Q} \xrightarrow{\cong} \prod_{k\geq 0} H^{2k}(X;\mathbb{Q})$$
for all finite CW complexes X.

Sketch proof. This follows from the fact that the ring homomorphism
$$\pi_*(ch)\colon \pi_*(\mathbb{Z}\times BU) = \mathbb{Z}[u] \longrightarrow \pi_*(\prod_{k\geq 0} K(\mathbb{Q}, 2k)) = \mathbb{Q}\{\iota_{2k} \mid k \geq 0\}$$
induces an isomorphism upon rationalization, i.e., after tensoring with $\mathbb{Q}$. □

Remark 3.6. It follows by a passage to limits that the map $i_n\colon BU(1)^n \to BU(n)$ induces an injective homomorphism
$$KU(BU(n)) \longrightarrow KU(BU(1)^n)$$
for each n, leading to a splitting principle also for topological K-theory.

Remark 3.7. For each prime p, the first Morava K-theory, $K(1)$, captures the mod p behavior of KU, which is not seen by the Chern character.

4. Topological K-theory spectra

Working in the category $\mathcal{T}$ of based spaces, we can define the negative half of a reduced cohomology theory $\widetilde{KU}^*(X)$ by setting
$$\widetilde{KU}^{-m}(X) = \widetilde{KU}(\Sigma^m X) = [\Sigma^m X, \mathbb{Z}\times KU] \cong [X, \Omega^m(\mathbb{Z}\times KU)]$$
for all $m \geq 0$ and spaces X. By the Bott periodicity theorem, this functor only depends on $m \mod 2$, hence can be extended periodically to a full cohomology theory, as follows.

Definition 4.1. For based spaces X, let
$$\widetilde{KU}^n(X) = \begin{cases} [X, \mathbb{Z}\times BU] & \text{for } n \text{ even,} \\ [X, U] & \text{for } n \text{ odd,} \end{cases}$$
where n ranges over all integers. This defines a contravariant homotopy functor, called the reduced complex K-theory, or K-cohomology, of X. There are suspension isomorphisms
$$\sigma\colon \widetilde{KU}^n(X) \cong \widetilde{KU}^{n+1}(\Sigma X)$$
given by the Bott equivalence
$$[X, \mathbb{Z}\times BU] \cong [X, \Omega U] \cong [\Sigma X, U]$$

for n even, and by the elementary equivalence

$$[X, U] \cong [X, \Omega(\mathbb{Z} \times BU)] \cong [\Sigma X, \mathbb{Z} \times BU]$$

for n odd. For unbased spaces X, let $KU^n(X) = \widetilde{KU}^n(X_+)$ be the unreduced complex K-theory of X. Here X_+ denotes X with a disjoint base point. Note that $KU(X) = KU^0(X)$ and $\widetilde{KU}^0(X) = \widetilde{KU}(X)$. We write $\widetilde{KU}^*(X)$ and $KU^*(X)$ for the combined graded abelian groups.

Definition 4.2. For based spaces X, let

$$\widetilde{KO}^n(X) = [X, \Omega^i(\mathbb{Z} \times BO)]$$

where $n = 8k - i$ with $0 \le i \le 7$ and k an integer. This defines a contravariant homotopy functor, called the reduced real K-theory, or K-cohomology of X. There are suspension isomorphisms

$$\sigma \colon \widetilde{KO}^n(X) \cong \widetilde{KO}^{n+1}(\Sigma X)$$

given by the Bott equivalence

$$[X, \mathbb{Z} \times BO] \cong [X, \Omega^8(\mathbb{Z} \times BO)] = [X, \Omega\Omega^7(\mathbb{Z} \times BO)] \cong [\Sigma X, \Omega^7(\mathbb{Z} \times BO)]$$

for $i = 0$, and by the identification

$$[X, \Omega^i(\mathbb{Z} \times BU)] \cong [X, \Omega\Omega^{i-1}(\mathbb{Z} \times BU)] \cong [\Sigma X, \Omega^{i-1}(\mathbb{Z} \times BU)]$$

for $1 \le i \le 7$. For unbased spaces X, let $KO^n(X) = \widetilde{KO}^n(X_+)$ be the unreduced real K-theory of X. Note that $KO(X) = KO^0(X)$ and $\widetilde{KO}^0(X) = \widetilde{KO}(X)$. We write $\widetilde{KO}^*(X)$ and $KO^*(X)$ for the combined graded abelian groups.

Remark 4.3. The essential data allowing the definition of $KU^*(X)$ is the sequence of spaces KU_n, for $n \ge 0$, with

$$KU_n = \begin{cases} \mathbb{Z} \times BU & \text{for } n \text{ even,} \\ U & \text{for } n \text{ odd,} \end{cases}$$

together with the homotopy equivalences

$$\tilde{\sigma} \colon KU_n \xrightarrow{\simeq} \Omega(KU_{n+1}) \,.$$

The latter correspond to (non-equivalences)

$$\sigma \colon \Sigma(KU_n) \longrightarrow KU_{n+1} \,.$$

This data defines a (sequential) spectrum, which corresponds to a new object in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$. This is the complex K-theory spectrum KU. It is not of the form $\Sigma^\infty X$ or HG for any space X or abelian group G.

Likewise, the real K-theory spectrum KO is the sequence of spaces KO_n, for $n \ge 0$, with

$$KO_n = \Omega^i(\mathbb{Z} \times BO)$$

for $n = 8k - i$ with $0 \le i \le 7$, together with the homotopy equivalences

$$\tilde{\sigma} \colon KO_n \xrightarrow{\simeq} \Omega(KO_{n+1})$$

or their adjoints

$$\sigma \colon \Sigma(KO_n) \longrightarrow KO_{n+1} \,.$$

Remark 4.4. The product structures in $KU(X)$ and $KO(X)$ extend to product structures in $KU^*(X)$ and $KO^*(X)$, which are induced by maps

$$\phi_{n,m}\colon KU_n \wedge KU_m \longrightarrow KU_{n+m}$$
$$\phi_{n,m}\colon KO_n \wedge KO_m \longrightarrow KO_{n+m}$$

that are suitably compatible with the structure maps σ. These define products $\phi\colon KU \wedge KU \to KU$ and $\phi\colon KO \wedge KO \to KO$, making KU and KO into ring spectra. These can be viewed as the objects in $\mathrm{Ho}(\mathcal{S}p)$ that represent multiplicative cohomology theories, i.e., cohomology theories with a natural product, but with the modern categories of spectra they can also be viewed as coherently structured ring spectra, with well-behaved module categories, etc.

Lemma 4.5. *The coefficient groups $\pi_*(KU) = KU_* = KU^{-*}$ form the graded ring*

$$KU_* = \mathbb{Z}[u^{\pm 1}]$$

with $|u| = 2$. The coefficient groups $\pi_*(KO) = KO_* = KO^{-*}$ form the graded ring

$$KO_* = \mathbb{Z}[\eta, A, B^{\pm 1}]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$.

((Chern character as a map $KU \to \bigvee_k \Sigma^{2k} H\mathbb{Q} \simeq \prod_k \Sigma^{2k} H\mathbb{Q}$.))

Remark 4.6. Complexification $V \mapsto cV = \mathbb{C} \otimes_{\mathbb{R}} V$ induces group homomorphisms $O(n) \to U(n)$, maps $BO(n) \to BU(n)$ and $\mathbb{Z} \times BO \to \mathbb{Z} \times BU$ and natural transformations $c\colon \mathrm{Vect}_n^{\mathbb{R}}(X) \to \mathrm{Vect}_n^{\mathbb{C}}(X)$ and $c\colon KO^*(X) \to KU^*(X)$. The latter is represented by a map $c\colon KO \to KU$ of topological K-theory ring spectra. The induced homomorphism of coefficient groups is the ring homomorphism given by

$$\begin{aligned} c\colon KO_* &\longrightarrow KU_* \\ \eta &\longmapsto 0 \\ A &\longmapsto 2u^2 \\ B &\longmapsto u^4 \,. \end{aligned}$$

Realification $W \mapsto rW = W_{\mathbb{R}}$ induces group homomorphisms $U(n) \to O(2n)$, maps $BU(n) \to BO(2n)$ and $\mathbb{Z} \times BU \to \mathbb{Z} \times BO$ and natural transformations $r\colon \mathrm{Vect}_n^{\mathbb{C}}(X) \to \mathrm{Vect}_{2n}^{\mathbb{R}}(X)$ and $r\colon KU^*(X) \to KO^*(X)$. The latter is represented by a map $r\colon KU \to KO$ of (KO-module) spectra. The induced homomorphism of coefficient groups is the KO_*-module homomorphism given by

$$\begin{aligned} r\colon KU_* &\longrightarrow KO_* \\ 1 &\longmapsto 2 \\ u &\longmapsto \eta^2 \\ u^2 &\longmapsto A \\ u^3 &\longmapsto 0 \,. \end{aligned}$$

((Wood's theorem: There is a homotopy cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\Sigma^2 r \circ u^{-1}} \Sigma^2 KO$$

of KO-modules. This is a reinterpretation of the Bott equivalence $\Omega(U/O) \simeq \mathbb{Z} \times BO$.))

((Also mention complex conjugation $V \mapsto tV$, where $z \in \mathbb{C}$ acts on tV as $\bar{z}$ acts on V, inducing group homomorphisms $U(n) \to U(n)$, maps $BU(n) \to BU(n)$ and $\mathbb{Z} \times BU \to \mathbb{Z} \times BU$ and natural transformations $t\colon \mathrm{Vect}_n^{\mathbb{C}}(X) \to \mathrm{Vect}_n^{\mathbb{C}}(X)$ and $t\colon KU^*(X) \to KU^*(X)$. The induced ring homomorphism $t\colon KU_* \to KU_*$ is given by $t(u) = -u$. In each case $t \circ t = \mathrm{id}$. We note that $t \circ c = c$ and $r \circ t = r$. Together with a self-conjugate K-theory $KT(X)$, these form the united K-theory of Bousfield.))

5. Adams operations

Topological K-theory gains in power when enriched by natural operations, much in the same way as mod p cohomology becomes more powerful when viewed as a module or algebra over the mod p Steenrod algebra.

((ETC: We focus on the complex case. The real case is entirely similar, and the natural transformations $c\colon KO(X) \to KU(X)$, $r\colon KU(X) \to KO(X)$ and $t\colon KU(X) \to KU(X)$ are compatible with the (real and complex) Adams operations.))

Definition 5.1. The k-th exterior power of a complex vector space V is the space of coinvariants

$$\Lambda^k V = (V \otimes \cdots \otimes V) \otimes_{\Sigma_k} \mathbb{C}(\mathrm{sgn})$$

where Σ_k acts from the right on $V \otimes \cdots \otimes V = V^{\otimes k}$ by permuting the tensor factors, and from the left on $\mathbb{C}(\mathrm{sgn})$ by the sign representation. We write

$$v_1 \wedge \cdots \wedge v_k$$

for the image of $v_1 \otimes \cdots \otimes v_k \otimes 1$, so that $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sgn}(\sigma)\, v_1 \wedge \cdots \wedge v_k$. If $\{v_1, \dots, v_n\}$ is a basis for V then the $\binom{n}{k}$ elements

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

form a basis for $\Lambda^k V$. In particular, $\Lambda^0 V = \mathbb{C}\{1\}$, $\Lambda^1 V = V$ and $\Lambda^k V = 0$ for $k > n = \dim V$. We call $\Lambda^n V = \mathbb{C}\{v_1 \wedge \cdots \wedge v_n\}$ the determinant line of V, since a linear map $A\colon V \to V$ induces $\det(A)\colon \Lambda^n V \to \Lambda^n V$. The direct sum

$$\Lambda^* V = \bigoplus_{k \ge 0} \Lambda^k V$$

is the exterior algebra on V, of total dimension 2^n.

Lemma 5.2. *There are natural isomorphisms*

$$\Lambda^k(V \oplus W) \cong \bigoplus_{i+j=k} \Lambda^i V \otimes \Lambda^j W$$

and

$$\Lambda^k(V_1 \oplus \cdots \oplus V_n) \cong \bigoplus_{1 \le i_1 < \cdots < i_k \le n} V_{i_1} \otimes \cdots \otimes V_{i_k}\,,$$

where we assume $\dim V_1 = \cdots = \dim V_n = 1$ *in the latter formula.*

Definition 5.3. The k-th exterior power $\lambda^k \xi$ of a $\mathbb{C}^n$-bundle $\pi\colon E \to X$ is given by the fiberwise k-exterior powers, so that

$$E(\lambda^k \xi)_x = \Lambda^k E(\xi)_x$$

for all $x \in X$. This defines a natural operation

$$\lambda^k\colon \mathrm{Vect}_n(X) \longrightarrow \mathrm{Vect}_{\binom{n}{k}}(X)\,.$$

Lemma 5.4. *Let ξ be a $\mathbb{C}^n$-bundle and η a $\mathbb{C}^m$-bundle over the same base space X. Then $\lambda^0\xi = \epsilon^1$, $\lambda^1\xi = \xi$, $\lambda^k\xi = 0$ for $k > n = \dim \xi$,*

$$\lambda^k(\xi \oplus \eta) \cong \bigoplus_{i+j=k} \lambda^i\xi \otimes \lambda^j\eta$$

and

$$\lambda^k(\xi_1 \oplus \dots \oplus \xi_n) \cong \bigoplus_{1\le i_1<\dots<i_k\le n} \xi_{i_1} \otimes \dots \otimes \xi_{i_k} ,$$

where we assume that each ξ_i is a complex line bundle.

One can use this sum formula to extend λ^k to virtual bundles, i.e., to formal differences $\xi - \eta$, giving a non-additive operation

$$\lambda^k : KU(X) \longrightarrow KU(X) .$$

((Grothendieck or earlier?)) Taking a cue from the power-sum polynomials p_k and Chern character ch, expressed in terms of Chern classes c_k, we instead follow Adams [Ada62, §4] and construct an additive operation

$$\psi^k : KU(X) \longrightarrow KU(X)$$

that will also be multiplicative. We use notation from [MS74, §16].

Definition 5.5. For each $1 \le k \le n$ let $s_k(e_1, \dots, e_k)$ be the polynomial determined by

$$p_k = s_k(e_1, \dots, e_k) \in \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} ,$$

where p_k and e_k denote the k-th power-sum and elementary symmetric polynomials, respectively.

The polynomial s_k does not depend on n, as long as $n \ge k$. For example, $s_1(e_1) = e_1$, $s_2(e_1, e_2) = e_1^2 - 2e_2$ and $s_k(e_1, 0, \dots, 0) = e_1^k$ for all k. Note that $s_k(c_1, \dots, c_k) = p_k$ in $H^*BU(n)$.

Definition 5.6. For $k \ge 1$ and ξ any vector bundle over X, let

$$\psi^k(\xi) = s_k(\lambda^1\xi, \dots, \lambda^k\xi) \in KU(X) .$$

This defines an operation

$$\psi^k : \mathrm{Vect}(X) \longrightarrow KU(X) .$$

Lemma 5.7. *$\psi^1(\xi) = \xi$ for any vector bundle ξ. If $\xi = \xi_1 \oplus \dots \oplus \xi_n$, where the ξ_i are line bundles, then*

$$\psi^k\xi = \xi_1^k + \dots + \xi_n^k$$

is the class of $\xi_1^{\otimes k} \oplus \dots \oplus \xi_n^{\otimes k}$.

Proof. In the second case we have $\lambda^k\xi = e_k(\xi_1, \dots, \xi_n)$ for each k, so

$$\psi^k(\xi) = s_k(\lambda^1\xi, \dots, \lambda^k\xi) = p_k(\xi_1, \dots, \xi_n) = \xi_1^k + \dots + \xi_n^k .$$

□

Lemma 5.8. *$\psi^k(\xi \oplus \eta) = \psi^k(\xi) + \psi^k(\eta)$ and $\psi^k(\xi \otimes \eta) = \psi^k(\xi) \cdot \psi^k(\eta)$.*

Proof. For additivity, we appeal to the K-theory splitting principle and assume that $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ and $\eta = \xi_{n+1} \oplus \cdots \oplus \xi_{n+m}$, where each ξ_i is a line bundle. Then

$$\psi^k(\xi \oplus \eta) = \sum_{i=1}^{n+m} \xi_i^k$$

is equal to the sum

$$\psi^k(\xi) + \psi^k(\eta) = \sum_{i=1}^{n} \xi_i^k + \sum_{j=1}^{m} \xi_{n+j}^k \,.$$

For multiplicativity, we may assume ξ and η are line bundles, in which case

$$\psi^k(\xi \otimes \eta) = (\xi \otimes \eta)^k = \xi^k \eta^k = \psi^k(\xi) \cdot \psi^k(\eta) \,.$$

□

Definition 5.9. The k-th Adams operation

$$\psi^k \colon KU(X) \longrightarrow KU(X)$$

is the unique ring homomorphism

$$\psi^k(\xi - \eta) = \psi^k(\xi) - \psi^k(\eta)$$

extending the semi-ring homomorphism $\psi^k\colon \operatorname{Vect}(X) \to KU(X)$ defined above.

Recall from Proposition 1.6 that $KU(S^2) = \mathbb{Z}[u]/(u^2)$.

Proposition 5.10. *The k-th Adams operation satisfies*

$$\psi^k(u) = ku$$

in $\widetilde{KU}(S^2) \cong \pi_2 KU$*, and* $\psi^k(u^n) = k^n u^n$ *in* $\widetilde{KU}(S^{2n}) \cong \pi_{2n} KU$*.*

Proof. Since $u = \gamma_1^1 - 1$ and $u^2 = 0$, we have

$$\begin{aligned} \psi^k(u) &= \psi^k(\gamma_1^1) - \psi^k(1) = (\gamma_1^1)^k - 1 = (u+1)^k - 1 \\ &= u^k + ku^{k-1} + \cdots + ku + 1 - 1 = ku \end{aligned}$$

in $\widetilde{KU}(S^2)$. The case of u^n follows (for all integers n) by multiplicativity. □

6. Hopf invariant one

Let $n = 2m$ be an even positive integer. Recall that the Hopf invariant of a map $f\colon S^{2n-1} \to S^n$ is the integer $H(f)$ defined by

$$a^2 = H(f)b \in H^{2n}(Cf)$$

where $Cf = S^n \cup_f e^{2n}$ is the mapping cone of f, $a \in H^n(Cf)$ restricts to a generator of $H^n(S^n)$ and $b \in H^{2n}(Cf)$ is the image of a (chosen) generator of $H^{2n}(S^{2n})$. This defines a homomorphism

$$H\colon \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z} \,.$$

The cofiber sequence

$$S^n \xrightarrow{j} Cf \xrightarrow{k} S^{2n}$$

and the Chern character induce a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{KU}(S^{2n}) \arrow[r, "k^*"] \arrow[d, "ch"'] & \widetilde{KU}(Cf) \arrow[r, "j^*"] \arrow[d, "ch"'] & \widetilde{KU}(S^n) \arrow[r] \arrow[d, "ch"] & 0 \\ 0 \arrow[r] & H^{2n}(S^{2n};\mathbb{Q}) \arrow[r, "k^*"] & \tilde{H}^*(Cf;\mathbb{Q}) \arrow[r, "j^*"] & H^n(S^n;\mathbb{Q}) \arrow[r] & 0\,. \end{tikzcd}$$

Here $\widetilde{KU}(S^n) = \mathbb{Z}\{u^m\}$ and $\widetilde{KU}(S^{2n}) = \mathbb{Z}\{u^{2m}\}$, so $\widetilde{KU}(Cf) = \mathbb{Z}\{\alpha, \beta\}$ with

$$j^*(\alpha) = u^m \qquad \text{and} \qquad k^*(u^{2m}) = \beta\,.$$

Then $ch(\alpha) \equiv a \mod b$ and $ch(\beta) = b$, so

$$\alpha^2 = H(f)\beta \in \widetilde{KU}(Cf)\,.$$

In other words, $H(f)$ can equally well be computed using topological K-theory.

Lemma 6.1. *$\psi^k\psi^\ell(\xi) = \psi^{k\ell}(\xi)$ and*

$$\psi^p(\xi) \equiv \xi^p \mod p$$

for any prime p.

Proof. Both claims are clear when ξ is a line bundle, and follow in general since all terms are additive in ξ. This uses the congruence

$$(\xi + \eta)^p \equiv \xi^p + \eta^p \mod p$$

which follows from $p \mid \binom{p}{i}$ for $0 < i < p$. $\square$

Here follows the Adams–Atiyah "postcard proof" from [AA66] of the Hopf invariant one theorem, first proved by Adams in [Ada60] using secondary cohomology operations, refining Adem's proof [Ade52] (using primary Steenrod operations) that n must be a power of 2. Topological K-theory, with its product structure and Adams operations, is remarkably useful for this problem.

Theorem 6.2. *Let $f\colon S^{2n-1} \to S^n$. If $H(f) = \pm 1$ then $n \in \{1, 2, 4, 8\}$.*

Proof. If n is odd then $a^2 = 0$ unless $n = 1$. If $n = 2m$ is even then

$$\psi^k(\alpha) = k^m\alpha + \mu_k\beta \qquad \text{and} \qquad \psi^k(\beta) = k^{2m}\beta$$

in $\widetilde{KU}(Cf)$, for some integer μ_k depending on f. If $H(f)$ is odd then μ_2 must be odd, since

$$2^m\alpha + \mu_2\beta = \psi^2(\alpha) \equiv \alpha^2 = H(f)\beta \mod 2\,.$$

For any k we calculate

$$\begin{aligned} \psi^2\psi^k(\alpha) &= \psi^2(k^m\alpha + \mu_k\beta) = k^m(2^m\alpha + \mu_2\beta) + \mu_k 2^{2m}\beta \\ \psi^k\psi^2(\alpha) &= \psi^k(2^m\alpha + \mu_2\beta) = 2^m(k^m\alpha + \mu_k\beta) + \mu_2 k^{2m}\beta\,. \end{aligned}$$

These are both equal to ψ^{2k}, so $k^m\mu_2 + \mu_k 2^{2m} = 2^m\mu_k + \mu_2 k^{2m}$, which we rewrite as

$$2^m(2^m - 1)\mu_k = k^m(k^m - 1)\mu_2\,.$$

If k is odd, it follows that $2^m \mid k^m - 1$, so that $k^m \equiv 1 \mod 2^m$. We may assume $m \geq 2$. Taking $k = 3$ (or $k = 5$), the order of k in $(\mathbb{Z}/2^m)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{m-2}$ is 2^{m-2}, so this implies $2^{m-2} | m$, which only happens for $m \in \{2, 4\}$. $\square$

7. Stable Adams operations

Let $\psi^k \colon \mathbb{Z} \times BU \to \mathbb{Z} \times BU$ be a map of ring spaces representing the natural operation $\psi^k \colon \widetilde{KU}(X) \to \widetilde{KU}(X)$. Since $\psi^k(u) = ku$ the square

$$\begin{tikzcd} S^2 \arrow[r, "u"] \arrow[d, "k"'] & \mathbb{Z} \times BU \arrow[d, "\psi^k"] \\ S^2 \arrow[r, "u"] & \mathbb{Z} \times BU \end{tikzcd}$$

commutes up to homotopy, where $k \colon S^2 \to S^2$ denotes a map of degree k. Hence so do

$$\begin{tikzcd} (\mathbb{Z} \times BU) \wedge S^2 \arrow[r, "\sigma^2"] \arrow[d, "\psi^k \wedge k"'] & \mathbb{Z} \times BU \arrow[d, "\psi^k"] \\ (\mathbb{Z} \times BU) \wedge S^2 \arrow[r, "\sigma^2"] & \mathbb{Z} \times BU \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{Z} \times BU \arrow[r, "\tilde\sigma^2", "\simeq"'] \arrow[d, "\psi^k \cdot k"'] & \Omega^2(\mathbb{Z} \times BU) \arrow[d, "\Omega^2(\psi^k)"] \\ \mathbb{Z} \times BU \arrow[r, "\tilde\sigma^2", "\simeq"'] & \Omega^2(\mathbb{Z} \times BU) \end{tikzcd}$$

In order to extend ψ^k to a natural transformation $KU^*(X) \to KU^*(X)$ of cohomology theories, or a map of spectra $KU \to KU$, we need to be able to divide by k, so that $\psi^k \simeq \Omega^2(\psi^k \cdot 1/k)$, and more generally $\psi^k \simeq \Omega^{2n}(\psi^k \cdot 1/k^n)$ for all $n \geq 0$.

For any abelian group A we call the colimit

$$A[1/k] = \operatorname{colim}\,(A \xrightarrow{k} A \xrightarrow{k} A \to \dots)$$

the localization of A away from k. Since localization away from k is exact, the functor $X \mapsto KU^*(X)[1/k]$ defines a cohomology theory, which is represented by a spectrum $KU[1/k]$ with $\pi_*(KU[1/k]) = (\pi_* KU)[1/k]$. The spaces of this spectrum are localizations

$$KU[1/k]_{2n} = (\mathbb{Z} \times BU)[1/k] \qquad \text{and} \qquad KU[1/k]_{2n-1} = U[1/k]$$

that can be constructed using Postnikov sections (following Sullivan [Sul74]) or by cosimplicial methods (following Bousfield–Kan [BK72]).

Definition 7.1. Let the stable Adams operation

$$\psi^k \colon \widetilde{KU}^*(X)[1/k] \longrightarrow \widetilde{KU}^*(X)[1/k]$$

be the morphism of cohomology theories induced by

$$\psi^k \cdot 1/k^n \colon (\mathbb{Z} \times BU)[1/k] \longrightarrow (\mathbb{Z} \times BU)[1/k]$$

for $* = 2n$ and by $\Omega(\psi^k \cdot 1/k^n)$ for $* = 2n-1$. The corresponding map of spectra

$$\psi^k \colon KU[1/k] \longrightarrow KU[1/k]$$

has components $(\psi^k)_{2n} = \psi^k \cdot 1/k^n$ and $(\psi^k)_{2n-1} = \Omega(\psi^k \cdot 1/k^n)$.

Let p be a prime, and let $A_{(p)} = A[1/q \mid q \neq p]$ be the localization of A at p, i.e., away from all primes $q \neq p$. There are then stable Adams operations

$$\psi^k \colon \widetilde{KU}^*(X)_{(p)} \longrightarrow \widetilde{KU}^*(X)_{(p)}$$

for all $k \geq 1$ relatively prime to p, induced by maps of p-localized spectra

$$\psi^k \colon KU_{(p)} \longrightarrow KU_{(p)} \,,$$

with $\psi^1 = \mathrm{id}$ and $\psi^k \psi^\ell = \psi^{k\ell}$ (at least up to homotopy). Note that ψ^p is not a stable operation at p, so the (essentially 2-local) Adams–Atiyah argument is intrinsically unstable.

For any abelian group A we call the limit

$$A^\wedge_p = \lim\,(\cdots \to A/p^3 A \to A/p^2 A \to A/pA)$$

the p-completion of A. Let $\mathbb{Z}_p = \mathbb{Z}^\wedge_p$ denote the ring of p-adic integers. When A is finitely generated, $A \otimes \mathbb{Z}_p \cong A^\wedge_p$, so

$$X \mapsto KU^*(X) \otimes \mathbb{Z}_p \cong KU^*(X)^\wedge_p$$

behaves as a cohomology theory for finite CW complexes X. To define this cohomology theory for general X we need to perform a construction in the category of spectra. Let

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
KU \arrow[r, "p^{n+1}"] \arrow[d, "p"'] & KU \arrow[r] \arrow[d, "\mathrm{id}"] & KU/p^{n+1} \arrow[r] \arrow[d] & \Sigma KU \arrow[d, "p"'] \\
KU \arrow[r, "p^n"] \arrow[d] & KU \arrow[r] \arrow[d] & KU/p^n \arrow[r] \arrow[d] & \Sigma KU \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

be a tower of Puppe cofiber sequences in spectra, where $p^n \colon KU \to KU$ represents multiplication by p^n. We define the p-completion of KU

$$KU^\wedge_p = \operatorname*{holim}_n KU/p^n$$

as the homotopy limit of this tower. The same construction works to define $E^\wedge_p$ for any spectrum E.

Adams showed that for a fixed n the operation $\psi^k \mod p^n$ only depends on the congruence class of $k \mod p^m$, for some m. Hence there are Adams operations ψ^k for all p-adic integers $k \in \mathbb{Z}_p$, acting compatibly on mod p^n topological K-theory, hence also on p-complete topological K-theory. See e.g. Atiyah–Tall [AT69, §I.5, §III.2]. This gives Adams operations

$$\psi^k \colon KU^\wedge_p(X) \longrightarrow KU^\wedge_p(X)$$

for all $k \in \mathbb{Z}_p$. In particular, $\psi^{-1} = t$ equals the complex conjugation operation. For k relatively prime to p, so that $k \in \mathbb{Z}_p^\times$ is a p-adic unit, these define stable Adams operations, induced by maps of p-completed (ring) spectra

$$\psi^k \colon KU^\wedge_p \longrightarrow KU^\wedge_p \,.$$

This action of ψ^k on $KU_p^\wedge$ for $k \in \mathbb{Z}_p^\times$ equals the action of the first Morava stabilizer group $\mathbb{S}_1 = \mathbb{G}_1$ on the first Morava E-theory = Lubin–Tate spectrum E_1, and is generalized to general heights n by the work of Hopkins–Miller and Goerss–Hopkins.

8. The image-of-J spectrum

For odd p, let g be a topological generator of $\mathbb{Z}_p^\times$. The continuous $\mathbb{Z}_p^\times$-homotopy fixed points of $KU_p^\wedge$ is then the homotopy equalizer

$$J_p^\wedge \longrightarrow KU_p^\wedge \underset{\mathrm{id}}{\overset{\psi^g}{\rightrightarrows}} KU_p^\wedge$$

of $\psi^g \colon KU_p^\wedge \to KU_p^\wedge$ and the identity id $= 1$. We get a homotopy (co-)fiber sequence

$$\Sigma^{-1} KU_p^\wedge \xrightarrow{\partial} J_p^\wedge \longrightarrow KU_p^\wedge \xrightarrow{\psi^g - 1} KU_p^\wedge \,.$$

There is a unit map $S \to J_p^\wedge$, and Adams ((ETC: or Milnor–Kervaire?)) proved that $\pi_*(S_p^\wedge) \to \pi_*(J_p^\wedge)$ is surjective in degrees $* \ge 0$, split by the Whitehead J-homomorphism. For any spectrum X let $X/p = X \wedge S/p$ denote the homotopy cofiber of $p \colon X \to X$. By a theorem of Miller, $\pi_*(S/p) \to \pi_*(J/p) = \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]$ is the localization homomorphism inverting a self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$, so that

$$v_1^{-1} \pi_*(S/p) \xrightarrow{\cong} \pi_*(J/p) \,.$$

When $p = 2$, the group $\mathbb{Z}_2^\times \cong \mathbb{Z}/2 \times \mathbb{Z}_2$ requires two (topological) generators, e.g., -1 and 3. Taking homotopy fixed points for the $\mathbb{Z}/2$-action by ψ^{-1} on $KU_2^\wedge$ gives $KO_2^\wedge$, so the continuous $\mathbb{Z}_2^\times$ homotopy fixed points of $KU_2^\wedge$ is the homotopy equalizer

$$J_2^\wedge \longrightarrow KO_2^\wedge \underset{\mathrm{id}}{\overset{\psi^3}{\rightrightarrows}} KO_2^\wedge$$

of $\psi^3 \colon KO_2^\wedge \to KO_2^\wedge$ and the identity id $= 1$. We get a homotopy (co-)fiber sequence

$$\Sigma^{-1} KO_2^\wedge \xrightarrow{\partial} J_2^\wedge \longrightarrow KO_2^\wedge \xrightarrow{\psi^3 - 1} KO_2^\wedge \,.$$

There is again a unit map $S \to J_2^\wedge$, and the Adams conjecture, proved by Quillen (and Sullivan, Becker–Gottlieb), shows that $\pi_*(S_2^\wedge) \to \pi_*(J_2^\wedge)$ is split surjective in degrees $* \ge 2$. By a theorem of Mahowald, $\pi_*(S/2) \to \pi_*(J/2)$ is the localization homomorphism inverting a self-map $v_1^4 \colon \Sigma^8 S/2 \to S/2$, so that

$$v_1^{-1} \pi_*(S/2) \xrightarrow{\cong} \pi_*(J/2) \,.$$

Here $\pi_*(KO/2) \cong (\dots, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2, \mathbb{Z}/2, 0, 0, 0, \dots)$ starting in degree 0 and repeating 8-periodically, so $\pi_*(J/2) \cong (\dots, \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/2 \oplus \mathbb{Z}/4, \mathbb{Z}/4 \oplus \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/2, 0, 0, \dots)$, starting in degree -1 and also repeating 8-periodically.

These results correspond to the cases $n = 1$ of Ravenel's (overly optimistic) telescope conjecture. At each height $n \ge 1$, the continuous homotopy fixed points for the action of the extended Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n recovers the Bousfield localization $L_{K(n)} S$ of the sphere spectrum S with

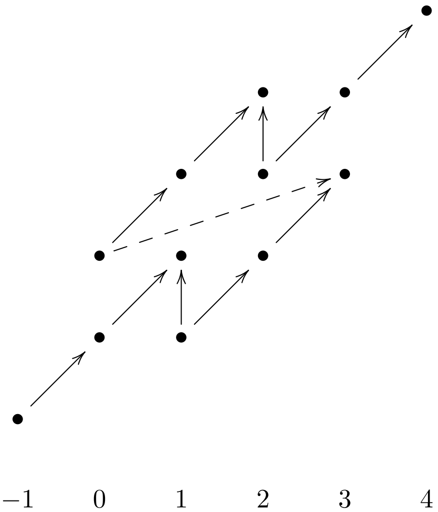

FIGURE 2. Adams spectral sequence chart for the fundamental domain of $\pi_*(J/2)$

respect to the n-th Morava K-theory $K(n)$. The homotopy fiber sequences above turn into a descent spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S$$

and the target provides invariants of the v_n-periodic homotopy of (finite spectra closely related to) S.

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Department of Mathematics, University of Oslo, Norway
Email address: rognes@math.uio.no