ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 12: CHROMATIC LOCALIZATION

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1. The chromatic filtration of the stable homotopy category

Implicitly localize at a fixed prime p. The height filtration of formal group laws leads to complementary closed and open substacks

$$\mathcal{M}_{\mathrm{fg}}^{\geq n+1} \xrightarrow{i} \mathcal{M}_{\mathrm{fg}} \xleftarrow{j} \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

and base change (= pullback) functors between their abelian categories of quasi-coherent sheaves

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n+1}) \xleftarrow{i^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xrightarrow{j^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) .$$

These admit right adjoint direct image functors

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n+1}) \xrightarrow{i_*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xleftarrow{j_*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) ,$$

with the adjunction counit $\epsilon \colon j^* j_* \to \mathrm{id}$ being an isomorphism, so that j_* exhibits $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n})$ as a reflective subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. This makes the reflector j^* a localization functor, given algebro-geometrically by restriction to heights $\leq n$, ignoring all difficulties with greater heights. Any choice of Johnson–Wilson theory $E(n)$, with flat Hopf algebroid $(E(n)_*, E(n)_*E(n))$, gives an equivalence

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \xrightarrow{\simeq} E(n)_*E(n){-}\,\mathrm{coMod}$$

such that the composite

$$\mathrm{Ho}(\mathcal{S}p) \xrightarrow{MU_*(-)^\sim} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xrightarrow{j^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \simeq E(n)_*E(n){-}\,\mathrm{coMod}$$

is equal to the composite

$$\mathrm{Ho}(\mathcal{S}p) \xrightarrow{MU_*(-)} LB{-}\,\mathrm{coMod} \xrightarrow{E(n)_* \otimes_L (-)} E(n)_*E(n){-}\,\mathrm{coMod} ,$$

i.e., the $E(n)_*E(n)$-comodule valued homology theory $X \mapsto E(n)_*(X)$. The localization j^* thus annihilates (the quasi-coherent sheaf associated to) all spectra Z with $E(n)_*(Z) = 0$, i.e., the $E(n)$-acyclic spectra. There is a full stable subcategory $L_n\mathcal{S}p \subset \mathcal{S}p$ of so-called $E(n)$-local spectra, and Bousfield constructed a left

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adjoint localization functor $j^*\colon \mathcal{S}p \to L_n\mathcal{S}p$ to the inclusion functor j_*, so that j^* annihilates precisely the $E(n)$-acyclic spectra.

$$\begin{array}{ccc}
\mathrm{Ho}(\mathcal{S}p) & \xrightarrow{\quad j^* \quad} & \mathrm{Ho}(L_n\mathcal{S}p) \\
{\scriptstyle MU_*(-)}\downarrow & & \downarrow{\scriptstyle E(n)_*(-)} \\
LB-\mathrm{coMod} & \xrightarrow{E(n)_*\otimes_L(-)} & E(n)_*E(n)-\mathrm{coMod} \\
{\scriptstyle \simeq}\downarrow & & \uparrow{\scriptstyle \simeq} \\
\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) & \xrightarrow{\quad j^* \quad} & \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})
\end{array}$$

Letting n vary, the resulting tower

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L_n\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_{n-1}\mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L_0\mathcal{S}p) \tag{1.1}$$

of localization functors between the full subcategories

$$\mathrm{Ho}(\mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L_n\mathcal{S}p) \supset \mathrm{Ho}(L_{n-1}\mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L_0\mathcal{S}p) \tag{1.2}$$

defines the chromatic filtration of (p-local) stable homotopy theory. Applied to a spectrum X, this gives the chromatic tower

$$X \longrightarrow \dots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_0X \tag{1.3}$$

in $\mathrm{Ho}(\mathcal{S}p)$.

2. Closed substacks

The stack $\mathcal{M}_{\mathrm{fg}}$ and its closed substack $\mathcal{M}_{\mathrm{fg}}^{\ge n+1}$ are corepresented by the flat Hopf algebroids (L, LB) and $(L/I_{n+1}, LB/I_{n+1})$, respectively, with the closed inclusion i corresponding to the Hopf algebroid homomorphism

$$\pi = \pi_{n+1}\colon (L, LB) \longrightarrow (L/I_{n+1}, LB/I_{n+1})$$

and the base change i^* corresponding to

$$\begin{aligned}
\pi^*\colon LB-\mathrm{coMod} &\longrightarrow LB/I_{n+1}-\mathrm{coMod} \\
M &\longmapsto L/I_{n+1}\otimes_L M = M/I_{n+1}M\,.
\end{aligned}$$

Lemma 2.1. *Let $\nu\colon M \to LB\otimes_L M$ be the LB-coaction on M. Then the LB/I_{n+1}-coaction on $L/I_{n+1}\otimes_L M = M/I_{n+1}M$ is given by the composite*

$$\begin{aligned}
L/I_{n+1}\otimes_L M &\overset{\mathrm{id}\otimes\nu}{\longrightarrow} L/I_{n+1}\otimes_L LB\otimes_L M \\
&\cong LB/I_{n+1}\otimes_L M \\
&\cong LB/I_{n+1}\otimes_{L/I_{n+1}} L/I_{n+1}\otimes_L M\,.
\end{aligned}$$

The following diagram commutes, where U denotes the forgetful functor corresponding to base change along $\mathrm{Spec}(L)\to\mathcal{M}_{\mathrm{fg}}$ or $\mathrm{Spec}(L/I_{n+1})\to\mathcal{M}_{\mathrm{fg}}^{\ge n+1}$.

$$\begin{array}{ccc}
LB/I_{n+1}-\mathrm{coMod} & \xleftarrow{\ \pi^*\ } & LB-\mathrm{coMod} \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
L/I_{n+1}-\mathrm{Mod} & \xleftarrow{\ \pi^*\ } & L-\mathrm{Mod}
\end{array}$$

At the level of modules, the base change π^* admits a right adjoint

$$\begin{aligned} \pi_* \colon L/I_{n+1}-\mathrm{Mod} &\longrightarrow L-\mathrm{Mod} \\ N &\longmapsto N\,, \end{aligned}$$

where the L-action on $\pi_*(N) = N$ is the composite

$$L \otimes N \overset{\pi\otimes\mathrm{id}}{\longrightarrow} L/I_{n+1} \otimes N \longrightarrow N\,.$$

In other words, the L/I_{n+1}-action is restricted to an L-action along $\pi \colon L \to L/I_{n+1}$. This extends to the case of comodules, where

$$\begin{aligned} \pi_* \colon LB/I_{n+1}-\mathrm{coMod} &\longrightarrow LB-\mathrm{coMod} \\ N &\longmapsto N \end{aligned}$$

is right adjoint to the comodule base change functor π^*.

Lemma 2.2. *Let $\nu \colon N \to LB/I_{n+1} \otimes_{L/I_{n+1}} N$ be the LB/I_{n+1}-coaction on N. Then the LB-coaction on $\pi_*(N) = N$ is given by the composite*

$$\begin{aligned} N &\overset{\nu}{\longrightarrow} LB/I_{n+1} \otimes_{L/I_{n+1}} N \\ &\cong LB \otimes_L L/I_{n+1} \otimes_{L/I_{n+1}} N \\ &\cong LB \otimes_L N\,. \end{aligned}$$

The following diagram commutes, where $LB \otimes_L (-)$ denotes the right adjoint of U defining the extended LB-comodule associated to an L-module, and similarly for $LB/I_{n+1} \otimes_{L/I_{n+1}} (-)$.

$$\begin{array}{ccc} LB/I_{n+1}-\mathrm{coMod} & \xrightarrow[\pi_*]{} & LB-\mathrm{coMod} \\ \Big\uparrow {\scriptstyle LB/I_{n+1}\otimes_{L/I_{n+1}}(-)} & & \Big\uparrow {\scriptstyle LB\otimes_L(-)} \\ L/I_{n+1}-\mathrm{Mod} & \xrightarrow[\pi_*]{} & L-\mathrm{Mod}\,. \end{array}$$

A categorical fact called conjugation ensures that any commuting square of left adjoints leads to a commuting square of right adjoints.

Lemma 2.3. *The adjunction counit $\epsilon \colon \pi^*\pi_* \to \mathrm{id}$ is an isomorphism, both in the L/I_{n+1}-module and the LB/I_{n+1}-comodule case. Hence π_* embeds $L/I_{n+1}-\mathrm{Mod}$ as a full subcategory of $L-\mathrm{Mod}$, and embeds $LB/I_{n+1}-\mathrm{coMod}$ as a full subcategory of $LB-\mathrm{coMod}$.*

These are reflective subcategories, in the following sense.

Definition 2.4. Let $G \colon \mathcal{D} \subset \mathcal{C}$ be the inclusion of a full subcategory. We say that $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$ if G admits a left adjoint $F \colon \mathcal{C} \to \mathcal{D}$. In this case, the adjunction counit $\epsilon \colon FG \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. We call F a reflector. The adjunction unit $\eta \colon \mathrm{id}_{\mathcal{C}} \to GF$ defines a natural morphism $\ell_X \colon X \to GFX$ for each X in $\mathcal{C}$.

The left adjoint π^* commutes with colimits, hence is right exact, but has left derived functors $L_s\pi^* = \mathrm{Tor}_s^L(L/I_{n+1}, -)$. ((ETC: At least for L-modules. What happens for LB-comodules?)) The right adjoint π_* is exact.

3. Open substacks

The open substack $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ is not affine, but is covered by affines Spec(R) where $g\colon L \to R$ satisfies $RI_{n+1} = R$. Any choice of Johnson–Wilson theory $E(n)$ is classified by a ring homomorphism $g\colon L = MU_* \to E(n)_*$ satisfying this condition, since $v_n \in I_{n+1}$ is a unit in $E(n)_*$. Hence we have map

$$[\mathrm{Spec}(E(n)_*) \leftleftarrows \mathrm{Spec}(E(n)_*E(n))] \xrightarrow{\tilde{g}} \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

from the stack corepresented by the flat Hopf algebroid $(E(n)_*, E(n)_*E(n))$, and base change along $\tilde{g}$ defines a functor

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \xrightarrow{\tilde{g}^*} E(n)_*E(n)-\mathrm{coMod}\ .$$

Proposition 3.1 (Naumann [Nau07, Thm. 26])**.**

$$\tilde{g}\colon [\mathrm{Spec}(E(n)_*) \leftleftarrows \mathrm{Spec}(E(n)_*E(n))] \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

is an equivalence of stacks, so that

$$\tilde{g}^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \xrightarrow{\simeq} E(n)_*E(n)-\mathrm{coMod}$$

is an equivalence of (tensor) abelian categories.

A key point is that $g\colon L \to E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ admits specializations of all heights $m \leq n$, via $E(n)_* \to v_m^{-1}E(n)_*/I_m$, so that $\tilde{g}$ is surjective on geometric points. The Landweber exactness of $E(n)_*$, or flatness of g, ensures that its image in $\mathcal{M}_{\mathrm{fg}}$ is closed under generalization, from height n to all lesser heights.

The composite inclusion $g = j\tilde{g}$ then corresponds to the Hopf algebroid homomorphism

$$g\colon (L, LB) \longrightarrow (E(n)_*, E(n)_*E(n))$$

associated to the Landweber exact L-algebra $E(n)_*$, and induces a localization functor

$$\begin{aligned} g^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq LB-\mathrm{coMod} &\longrightarrow E(n)_*E(n)-\mathrm{coMod} \\ M &\longmapsto E(n)_* \otimes_L M \end{aligned}$$

that serves as a (non-canonical) replacement for j^*.

Lemma 3.2. *Let $\nu\colon M \to LB \otimes_L M$ be the LB-coaction on M. Then the $E(n)_*E(n)$-coaction on $E(n)_* \otimes_L M$ is given by the composite*

$$\begin{aligned} E(n)_* \otimes_L M &\xrightarrow{\mathrm{id}\otimes\nu} E(n)_* \otimes_L LB \otimes_L M \\ &\cong E(n)_* \otimes_L LB \otimes_L L \otimes_L M \\ &\xrightarrow{\mathrm{id}\otimes g \otimes \mathrm{id}} E(n)_* \otimes_L LB \otimes_L E(n)_* \otimes_L M \\ &\cong E(n)_*E(n)_* \otimes_L M \\ &\cong E(n)_*E(n) \otimes_{E(n)_*} E(n)_* \otimes_L M\ . \end{aligned}$$

The following diagram commutes, where U denotes the forgetful functors.

$$\begin{array}{ccc} LB-\mathrm{coMod} & \xrightarrow{g^*} & E(n)_*E(n)-\mathrm{coMod} \\ \downarrow U & & \downarrow U \\ L-\mathrm{Mod} & \xrightarrow{g^*} & E(n)_*-\mathrm{Mod} \end{array}$$

At the level of modules, the base change g^* admits a right adjoint

$$\begin{aligned} g_*\colon E(n)_*-\mathrm{Mod} &\longrightarrow L-\mathrm{Mod} \\ N &\longmapsto N\,, \end{aligned}$$

where the L-action on $g_*(N) = N$ is the composite

$$L \otimes N \overset{g\otimes \mathrm{id}}{\longrightarrow} E(n)_* \otimes N \longrightarrow N\,.$$

In other words, the $E(n)_*$-action is restricted to an L-action along $g\colon L \to E(n)_*$.

The extension to comodules is now less obvious, but discussed in [MR77, (1.2)] and [Hov04, Prop. 1.2.3]. The tensor product

$$MU_*E(n) \cong LB \otimes_L E(n)_*$$

is simultaneously a left LB-comodule and a right $E(n)_*E(n)$-comodule. For a left $E(n)_*E(n)$-comodule N, the cotensor product

$$MU_*E(n) \,\square_{E(n)_*E(n)}\, N$$

is defined to be the equalizer of the two homomorphisms

$$MU_*E(n) \otimes_{E(n)_*} N \underset{\mathrm{id}\,\otimes \nu}{\overset{\nu'\otimes \mathrm{id}}{\rightrightarrows}} MU_*E(n) \otimes_{E(n)_*} \otimes E(n)_*E(n) \otimes_{E(n)_*} N\,.$$

The left LB-coaction on $MU_*E(n)$ carries over to $MU_*E(n) \,\square_{E(n)_*E(n)}\, N$.

Lemma 3.3. *The comodule direct image functor*

$$\begin{aligned} g_*\colon E(n)_*E(n)-\mathrm{coMod} &\longrightarrow LB-\mathrm{coMod} \\ N &\longmapsto MU_*E(n) \,\square_{E(n)_*E(n)}\, N \end{aligned}$$

is right adjoint to the comodule base change functor g^*.

By conjugation the following diagram commutes, where $LB \otimes_L (-)$ denotes the right adjoint of U defining the extended LB-comodule associated to an L-module, and similarly for $E(n)_*E(n) \otimes_{E(n)_*} (-)$.

$$\begin{array}{ccc} LB-\mathrm{coMod} & \xleftarrow{\;g_*\;} & E(n)_*E(n)-\mathrm{coMod} \\ \uparrow{\scriptstyle LB\otimes_L(-)} & & \uparrow{\scriptstyle E(n)_*E(n)\otimes_{E(n)_*}(-)} \\ L-\mathrm{Mod} & \xleftarrow{\;g_*\;} & E(n)_*-\mathrm{Mod}\,. \end{array}$$

Note that this forces the relation

$$g_*(E(n)_*E(n) \otimes_{E(n)_*} N) \cong LB \otimes_L N \cong MU_*E(n) \otimes_{E(n)_*} N$$

for any $E(n)_*$-module N, which is indeed satisfied by the functor g_* defined in terms of the cotensor product.

Lemma 3.4. *The adjunction counit $\epsilon\colon g^*g_* \to \mathrm{id}$ is an isomorphism, both in the $E(n)_*$-module and the $E(n)_*E(n)$-comodule case. Hence g_* embeds $E(n)_*-\mathrm{Mod}$ as a (full) reflective subcategory of $L-\mathrm{Mod}$, and embeds $E(n)_*E(n)-\mathrm{coMod}$ as a (full) reflective subcategory of $LB-\mathrm{coMod}$.*

Proof. This follows from $E(n)_* \otimes_L N \cong N$ for any $E(n)_*$-module N, and $E(n)_* \otimes_L MU_*E(n) \,\square_{E(n)_*E(n)}\, N \cong N$ for any $E(n)_*E(n)$-comodule N. $\square$

In the case of LB-comodules, the left adjoint g^* is exact, by Landweber's exact functor theorem. The right adjoint g_* commutes with all limits, hence is left exact, but has right derived functors $R^s g_* = \mathrm{Cotor}^s_{E(n)_*E(n)}(MU_*E(n), -)$. ((ETC: Compare with [HS05b].))

In view of the equivalence $\tilde{g}^*$ from Proposition 3.1, the base change

$$j^* \colon \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$$

is an exact left adjoint exhibiting $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$ as a reflective abelian subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. In this case we call j^* a localization functor. ((ETC: Is there a standard general definition?))

4. Hereditary torsion theories

The localization functors

$$j^* \colon \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$$
$$g^* \colon LB-\mathrm{coMod} \longrightarrow E(n)_*E(n)-\mathrm{coMod}$$

are determined up to equivalence by the full subcategories of

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq LB-\mathrm{coMod}$$

that they annihilate, i.e.. map to the zero object. Such full subcategories of abelian categories are known as localizing subcategories, or hereditary torsion theories, and characterize the localization functor (if it exists) up to equivalence. See [HS05a, §1].

Definition 4.1. A localization functor of an abelian category $\mathcal{C}$ is an exact functor $F \colon \mathcal{C} \to \mathcal{D}$ with fully faithful right adjoint $G \colon \mathcal{D} \to \mathcal{C}$. We view G as the inclusion of a reflective abelian subcategory. The adjunction counit $\epsilon \colon FG \to \mathrm{id}_{\mathcal{D}}$ is then a natural isomorphism.

Definition 4.2. A Serre class in an abelian category $\mathcal{C}$ is a full subcategory $\mathcal{T}$ that is closed under subobjects, quotient objects and extensions. In other words, for each short exact sequence

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

the objects M' and M'' lie in $\mathcal{T}$ if and only if M lies in $\mathcal{T}$. A hereditary torsion theory in $\mathcal{C}$ (with arbitrary coproducts) is a Serre class $\mathcal{T}$ that is also closed under coproducts.

((ETC: If $\mathcal{C}$ is graded, with a suspension operator, we also assume that $\mathcal{T}$ is closed under this operator and its inverse.))

Definition 4.3. Let $\mathcal{T}$ be a hereditary torsion theory in an abelian category $\mathcal{C}$. A morphism $f \colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{T}$-equivalence if $\ker(f)$ and $\mathrm{cok}(f)$ are both in $\mathcal{T}$. An object $N \in \mathcal{C}$ is $\mathcal{T}$-local if

$$\mathcal{C}(f, N) \colon \mathcal{C}(Y, N) \xrightarrow{\cong} \mathcal{C}(X, N)$$

is an isomorphism for each $\mathcal{T}$-equivalence $f \colon X \to Y$. Let $L_{\mathcal{T}}\mathcal{C} \subset \mathcal{T}$ denote the full subcategory of $\mathcal{T}$-local objects.

Proposition 4.4. *Let $F: \mathcal{C} \to \mathcal{D}$ be a localization functor. Let*

$$\mathcal{T} = \{Z \in \mathcal{C} \mid F(Z) \cong 0\}$$

be (the full subcategory generated by) the class of objects annihilated by F. Then $\mathcal{T}$ is a hereditary torsion theory. The composite

$$L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is an equivalence, identifying $G: \mathcal{D} \to \mathcal{C}$ with the inclusion $L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C}$. The adjunction counit $\eta: \mathrm{id}_{\mathcal{C}} \to GF$ defines, for each object $M \in \mathcal{C}$, a $\mathcal{T}$-equivalence

$$\eta_M : M \longrightarrow GF(M) = L_{\mathcal{T}}M$$

to a $\mathcal{T}$-local object.

((ETC: Conversely, choices of $\mathcal{T}$-equivalences $M \to L_{\mathcal{T}}M$ to $\mathcal{T}$-local objects determine the localization functor F, and are unique up to isomorphism if they exist.))

Example 4.5. The Landweber exact base change functor

$$g^* : LB\text{-}\,\mathrm{coMod} \longrightarrow E(n)_*E(n)\text{-}\,\mathrm{coMod}$$

is a localization functor, with associated hereditary torsion theory

$$\mathcal{T}_n = \{Z \in LB\text{-}\,\mathrm{coMod} \mid E(n)_* \otimes_L Z = 0\}\,.$$

The LB-comodule L/I_{n+1} lies in $\mathcal{T}_n$, since $v_n \in I_{n+1}$ is a unit in $E(n)_*$, so that $E(n)_* \otimes_L L/I_{n+1} = 0$. ((ETC: Discuss when an LB-comodule M is $\mathcal{T}$-local.))

The hereditary torsion theory $\mathcal{T}_n$ associated to $g: L \to E(n)_*$ also has a different characterization. This coincidence in the current context of abelian categories can be viewed, when lifted to the stable homotopy category, as leading to the (in)famous Telescope Conjecture in [Rav84].

Proposition 4.6 ([HS05a, Prop. 3.2])**.** *The hereditary torsion theory generated by L/I_{n+1} is equal to $\mathcal{T}_n$, when restricted to p-local LB-comodules.*

This is an application of Landweber's work.

The short exact sequence

$$0 \to \Sigma^{|v_n|} L/I_n \longrightarrow L/I_n \longrightarrow L/I_{n+1} \to 0$$

shows that L/I_{n+1} lies in the (Serre class and) hereditary torsion theory generated by L/I_n, so that we have the infinite chain of such full subcategories

$$\{0\} \subset \cdots \subset \mathcal{T}_n \subset \mathcal{T}_{n-1} \subset \cdots \subset \mathcal{T}_0$$

inside p-local LB-comodules, which we denote as $\mathcal{T}_{-1}$. In particular, $E(n)_* \otimes_L Z = 0$ implies that $E(n-1)_* \otimes_L Z = 0$.

Since $\mathcal{T}_n$ is the "kernel" of the $\mathcal{T}_n$-localization functor

$$L_{\mathcal{T}_n} : LB\text{-}\,\mathrm{coMod} \longrightarrow L_{\mathcal{T}_n}(LB\text{-}\,\mathrm{coMod})$$

it follows that we have a similar infinite tower of localization functors between abelian categories

$$LB\text{-}\,\mathrm{coMod} \longrightarrow \ldots \longrightarrow L_{\mathcal{T}_n}(LB\text{-}\,\mathrm{coMod}) \longrightarrow L_{\mathcal{T}_{n-1}}(LB\text{-}\,\mathrm{coMod}) \longrightarrow \ldots \longrightarrow L_{\mathcal{T}_0}(LB\text{-}\,\mathrm{coMod})\,,$$

equivalent to the tower

$$LB\!-\mathrm{coMod} \longrightarrow \dots \longrightarrow E(n)_*E(n)\!-\mathrm{coMod} \longrightarrow E(n-1)_*E(n-1)\!-\mathrm{coMod} \\ \dots \longrightarrow E(0)_*E(0)\!-\mathrm{coMod}\ .$$

Writing $g = g_n \colon L \to E(n)$, the diagrams

$$\begin{array}{ccc} LB\!-\mathrm{coMod} & & \\ {\scriptstyle g_n^*}\downarrow & \searrow^{g_{n-1}^*} & \\ E(n)_*E(n)\!-\mathrm{coMod} & \longrightarrow & E(n-1)_*E(n-1)\!-\mathrm{coMod} \end{array}$$

and

$$\begin{array}{ccc} LB\!-\mathrm{coMod} & & \\ {\scriptstyle g_{n*}}\uparrow & \nwarrow^{g_{n-1*}} & \\ E(n)_*E(n)\!-\mathrm{coMod} & \longleftarrow & E(n-1)_*E(n-1)\!-\mathrm{coMod} \end{array}$$

commute for all $n \ge 1$. We omit to write down formulas for the horizontal functors, since we do not have a direct homomorphism $(E(n)_*, E(n)_*E(n)) \to (E(n-1)_*, E(n-1)_*E(n-1))$ of Hopf algebroids.

Proposition 4.7 ([HS05a, Prop. 3.3]). *If $\mathcal{T}$ is a hereditary torsion theory of p-local LB-comodules, and $L/I_n \notin \mathcal{T}$, then $\mathcal{T} \subset \mathcal{T}_n$.*

The last two propositions imply the following partial classification of hereditary torsion theories in p-local LB-comodules, hence also of localization functors from such LB-comodules onto reflective additive subcategories.

Theorem 4.8 ([HS05a, Thm. 3.1]). *Let $\mathcal{T}$ be a hereditary torsion theory of p-local LB-comodules, containing some nonzero comodule that is coherent, i.e., finitely presented over $L_{(p)}$. Then $\mathcal{T} = \mathcal{T}_n$ for some $n \ge -1$.*

In particular, any two choices of ring homomorphism $g \colon L \to E(n)_*$ specifying a Landweber exact Johnson–Wilson theory give localization functors g^* that annihilate the same hereditary torsion theory $\mathcal{T} = \mathcal{T}_n$, which implies that the associated categories of $\mathcal{T}_n$-local LB-comodules and/or $E(n)_*E(n)$-comodules are independent of those choices.

More generally, for any Landweber exact $g \colon L \to E_*$, Hovey–Strickland define the height of E_* to be the maximal n such that $E_*/I_n \neq 0$. (This is also the maximal height of a specialization k^*F_E of the formal group law F_E, for a homomorphism $k \colon E_* \to R$ to a graded field R.) Then (E_*, E_*E) is a flat Hopf algebroid, $g^* \colon LB\!-\mathrm{coMod} \to E_*E\!-\mathrm{coMod}$ is a localization functor annihilating a hereditary torsion theory $\mathcal{T}_E$, and $L/I_n \notin \mathcal{T}_E$ while $L/I_{n+1} \in \mathcal{T}_E$. This implies $\mathcal{T}_E = \mathcal{T}_n$, by Theorem 4.8, so $\mathcal{T}_E$ and g^* only depend on the height of n.

For $E = E(n)_*$, of height n, this recovers our definition of $\mathcal{T}_n$ as $\mathcal{T}_{E(n)}$.

Applied with $E_* = v_n^{-1}L$, so that $E_*(X) = v_n^{-1}MU_*(X)$, it shows that $\mathcal{T}_n$ is the class of v_n-power torsion LB-comodules, i.e., those LB-comodules M such that for each $x \in M$ there exists an $N \gg 0$ such that $v_n^N x = 0$. Moreover, each v_n-power torsion module (resp. element) is v_m-power torsion for each $0 \le m \le n$, cf. [JY80, Lem. 2.3].

Example 4.9. When $n = 0$, $E(0) = H\mathbb{Q}$ and $(E(0)_*, E(0)_*E(0)) = (\mathbb{Q}, \mathbb{Q})$, so that an $E(0)_*E(0)$-comodule is the same as an $E(0)_*$-module, i.e., a graded $\mathbb{Q}$-vector space. The functor

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow LB\text{-}\mathrm{coMod} \xrightarrow{g_0^*} \mathbb{Q}\text{-}\mathrm{Mod}$$
$$X \longmapsto \mathbb{Q} \otimes_{MU_*} MU_*(X) \cong H_*(X; \mathbb{Q})$$

is given by rational homology.

Example 4.10. When $n = 1$, $E(1) = L \subset KU_{(p)}$ is the Adams summand of p-local complex K-theory. The Hopf algebroid (KU_*, KU_*KU) was determined by Adams and Harris, cf. [AHS71], [Ada74, Part II, §13], and can be used to recast Adams' work [Ada66] on the e-invariant and the image-of-J, cf. [Swi75, Ch. 17, Ch. 19]. Ravenel [Rav84, Thm. 7.6] shows, for p an odd prime, that the category of p-power torsion $E(1)_*E(1)$-comodules is equivalent to that of $\mathbb{Z}/(2p-2)$-graded torsion Λ-modules, where

$$\Lambda = \mathbb{Z}_p[[\mathbb{S}_1^0]] \cong \mathbb{Z}_p[[t]]$$

is the Iwasawa algebra, known from the theory of cyclotomic extensions. Here $\mathbb{S}_1^0 = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ is the strict Morava stabilizer group. The classification of Λ-modules is fairly well understood.

One may now hope to obtain a gradually better understanding of the category of LB-comodules, or quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, by localizing along $g_n \colon L \to E(n)_*$ and studying $E(n)_*E(n)$-comodules or quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}^{\le n}$, for increasing values of n.

5. Bousfield localization

We now aim to lift localizations from the abelian category of LB-comodules to the triangulated category $\mathrm{Ho}(\mathcal{S}p)$. Recall that a triangulated subcategory must be closed under cofibers and desuspensions.

Definition 5.1. A thick subcategory of a triangulated category $\mathcal{C}$ is a full triangulated subcategory $\mathcal{T}$ that is closed under retracts. In other words, any retract of an object in $\mathcal{T}$ is also an object in $\mathcal{T}$. A localizing subcategory of $\mathcal{C}$ (with arbitrary coproducts) is a triangulated subcategory that is also closed under coproducts.

Remark 5.2. Any localizing subcategory is thick, by the Eilenberg swindle: If $X \vee Y \in \mathcal{T}$ with $\mathcal{T}$ localizing, then the distinguished triangle

$$X \longrightarrow \bigvee_{i=1}^{\infty} (X \vee Y) \longrightarrow \bigvee_{j=1}^{\infty} (Y \vee X) \longrightarrow \Sigma X$$

shows that $X \in \mathcal{T}$.

Definition 5.3. Let $\mathcal{T}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. A morphism $f \colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{T}$-equivalence if its cofiber Cf is in $\mathcal{T}$. Here

$$X \longrightarrow Y \longrightarrow Cf \longrightarrow \Sigma X$$

is any distinguished triangle. An object $N \in \mathcal{C}$ is $\mathcal{T}$-local if

$$\mathcal{C}(f, N) \colon \mathcal{C}(Y, N) \xrightarrow{\cong} \mathcal{C}(X, N)$$

is an isomorphism for each $\mathcal{T}$-equivalence $f\colon X \to Y$. Equivalently, N is $\mathcal{T}$-local if $\mathcal{C}(Z, N) = 0$ for each $Z \in \mathcal{T}$. Let $G\colon L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C}$ denote (the inclusion of) the full triangulated subcategory of $\mathcal{T}$-local objects.

Definition 5.4. Let $\mathcal{T}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. A $\mathcal{T}$-localization of an object M in $\mathcal{C}$ is a $\mathcal{T}$-equivalence $\eta\colon M \to N$ to a $\mathcal{T}$-local object N.

Example 5.5. Let E be any spectrum, and let

$$\mathcal{T}_E = \{Z \in \mathrm{Ho}(\mathcal{S}p) \mid E_*(Z) = 0\}$$

be (the full triangulated subcategory generated by) the class of spectra Z with $E_*(Z) = 0$. We call these the E_*-acyclic spectra. Then $\mathcal{T}_E$ is a localizing subcategory of the stable homotopy category. A map $f\colon X \to Y$ is a $\mathcal{T}_E$-equivalence if and only if $f_*\colon E_*(X) \to E_*(Y)$ is an isomorphism, in which case we say that it is an E_*-equivalence. A spectrum N is $\mathcal{T}_E$-local if and only if $[Z, N] = 0$ for each E_*-acyclic spectrum, in which case we say that N is E_*-local. We write

$$G\colon \mathrm{Ho}(L_E\mathcal{S}p) = L_{\mathcal{T}_E} \mathrm{Ho}(\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

for the full triangulated subcategory of E_*-local spectra. (As the notation suggests, $L_{\mathcal{T}_E} \mathrm{Ho}(\mathcal{S}p)$ arises as the homotopy category of a stable model category or stable ∞-category.) A $\mathcal{T}_E$-localization $\eta\colon M \to N$ is an E_*-equivalence to an E_*-local spectrum, and will be called an E_*-localization.

Lemma 5.6. *If a $\mathcal{T}$-localization η exists, it is a terminal $\mathcal{T}$-equivalence out of M and an initial morphism to a $\mathcal{T}$-local object, hence unique up to unique isomorphism.*

Proof. Any $\mathcal{T}$-equivalence $M \to M'$ can be continued with a unique $M' \to N$ to recover η, since $\mathcal{C}(M', N) \cong \mathcal{C}(M, N)$. Any morphism $M \to N'$ to a $\mathcal{T}$-local N' extends uniquely over η since $\mathcal{C}(N, N') \cong \mathcal{C}(M, N')$. □

One might try to construct a $\mathcal{T}$-localization $\eta\colon M \to N$ by forming a colimit over E_*-equivalences out of M, or a limit of E_*-local spectra under M. The difficulty is to show that these (co-)limits (over large indexing categories) exist and agree.

Theorem 5.7 (Bousfield [Bou79, Thm. 1.1])**.** *Let E be any spectrum. Any spectrum X admits an E_*-localization*

$$\eta_X\colon X \longrightarrow L_E X\,.$$

Letting X vary, these choices assemble to a localization functor

$$F\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_E\mathcal{S}p)$$

left adjoint to the full inclusion $G\colon \mathrm{Ho}(L_E\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$, with adjunction unit

$$\eta\colon \mathrm{id} \longrightarrow GF = L_E\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(\mathcal{S}p)$$

and adjunction counit

$$\epsilon\colon FG \xrightarrow{\cong} \mathrm{id}\,.$$

Adams attempted to construct such localizations in [Ada74, Part III, §14], but encountered set-theoretic issues. These were resolved by Bousfield, through working with CW spectra as a model for the stable homotopy category and making cardinality arguments on the number of cells needed to achieve E_*-equivalences and E_*-locality. The problem of realizing general localizing subcategories as the annihilators of localization functors remains closely related to large-cardinal issues [CSS05].

Lemma 5.8. *The functor L_E is exact, idempotent ($L_E L_E \cong L_E$) and lax symmetric monoidal. The class of spectra Z with $L_E Z \simeq *$ is equal to the class of E_*-acyclic spectra.*

Proof. Exactness follows since the left adjoint F preserves cofiber sequences, the right adjoint G preserves fiber sequences, and these are the same (up to sign) in the stable homotopy category.

The spectrum $*$ is always E_*-local, so $Z \to *$ is an E_*-localization if and only if Z is E_*-acyclic.

It follows that $f\colon X \to Y$ induces a stable equivalence $L_E X \to L_E Y$ if and only if f is an E_*-equivalence. In particular, $L_E X \to L_E L_E X$ is a stable equivalence, so L_E is idempotent.

The E_*-localization $X \wedge Y \to L_E(X \wedge Y)$ extends uniquely (in the stable homotopy category) over the E_*-equivalences $X \wedge Y \to L_E X \wedge Y \to L_E X \wedge L_E Y$, and ($X \to L_E X$ and) the resulting map

$$L_E X \wedge L_E Y \longrightarrow L_E(X \wedge Y)$$

defines the lax symmetric monoidal structure. □

In particular, for any (commutative) ring spectrum up to homotopy R, the Bousfield localization $L_E R$ is a (commutative) ring spectrum up to homotopy, with unit $S \to R \to L_E R$ and product

$$L_E R \wedge L_E R \longrightarrow L_E(R \wedge R) \overset{L_E \phi}{\longrightarrow} L_E R\,.$$

For any R-module spectrum M, the localization $L_E M$ is an $L_E R$-module spectrum, in the homotopy category. The following was exhibited by Adams as an example of the convenience of working in a good stable category.

Lemma 5.9 ([Ada71, Prop. 5.2])**.** *If R is a ring spectrum up to homotopy, then any R-module M is R_*-local.*

Proof. If $f \in [Z, M]$, then f factors as

$$Z \cong S \wedge Z \overset{\eta \wedge \mathrm{id}}{\longrightarrow} R \wedge Z \overset{\mathrm{id} \wedge f}{\longrightarrow} R \wedge M \overset{\lambda}{\longrightarrow} M\,,$$

so if $R_*(Z) = 0$ then it factors through $R \wedge Z \simeq *$ and must be zero. □

The converse does not generally hold; not every R-local spectrum is an R-module. For example, the image-of-J spectrum is KU-local but not a KU-module ((ETC: However, this does hold for $R = L_n S$. Give forward reference.))

Remark 5.10. A left Bousfield localization of a given model category $(\mathcal{S}p, \mathcal{W}, \dots)$ of spectra, with $\mathcal{W}$ the subcategory of stable equivalences, is a stable model category $(\mathcal{S}p, \mathcal{V}, \dots)$ with the same cofibrations as before, but with a larger class $\mathcal{V} \supset \mathcal{W}$ of weak equivalences. See [Hir03, §3.3]. The identity functor on $\mathcal{S}p$ is then a left Quillen functor, and induces an adjunction

$$F\colon \mathcal{S}p[\mathcal{W}^{-1}] \rightleftarrows \mathcal{S}p[\mathcal{V}^{-1}]\colon G$$

exhibiting $\mathcal{S}p[\mathcal{V}^{-1}]$ as a reflective subcategory of $\mathrm{Ho}(\mathcal{S}p) = \mathcal{S}p[\mathcal{W}^{-1}]$. Taking $\mathcal{V}$ to be the E_*-equivalences one recovers Bousfield's theorem recalled above.

We often write $L_E\colon \mathrm{Ho}(\mathcal{S}p) \to \mathrm{Ho}(L_E\mathcal{S}p)$ for the unique factorization F of L_E through $G\colon \mathrm{Ho}(L_E\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$

Definition 5.11. For each prime p and $n \geq 0$ let

$$\mathrm{Ho}(L_n\mathcal{S}p) = \mathrm{Ho}(L_{E(n)}\mathcal{S}p)$$

denote the $E(n)_*$-local stable homotopy category and

$$L_n = L_{E(n)}\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_n\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

the $E(n)_*$-localization functor. Let

$$\mathrm{Ho}(\hat{L}_n\mathcal{S}p) = \mathrm{Ho}(L_{K(n)}\mathcal{S}p)$$

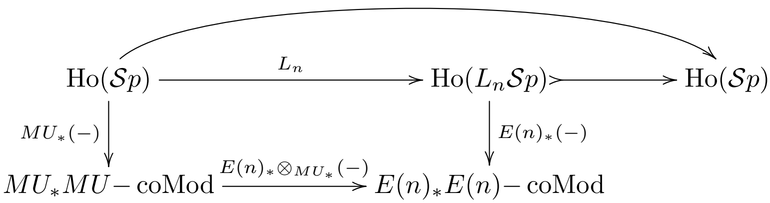
denote the $K(n)_*$-local stable homotopy category and

$$\hat{L}_n = L_{K(n)}\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(\hat{L}_n\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

the $K(n)_*$-localization functor.

The Hovey–Strickland memoir [HS99b] contains a wealth of information about the categories $\mathrm{Ho}(L_n\mathcal{S}p)$ and $\mathrm{Ho}(\hat{L}_n\mathcal{S}p)$ of $E(n)$-local and $K(n)$-local spectra, respectively.

Lemma 5.12. *The diagram*



commutes.

Proof. $E(n)_* \otimes_{MU_*} MU_*(X) \cong E(n)_*(X) \cong E(n)_*(L_nX)$. □

((ETC: Any analogue for $\hat{L}_n\mathcal{S}p$ and $K(n)_*(-)$?))

The unit map $S \to L_ES$ is an E_*-equivalence hence so is $X \cong X \wedge S \to X \wedge L_ES$. The localization map $\eta\colon X \to L_EX$ thus extends uniquely (in the homotopy category) over $X \wedge L_ES$.

Definition 5.13 ([Rav84, Def. 1.28])**.** A (spectrum E or) localization functor L_E is smashing if the natural map

$$X \wedge L_ES \xrightarrow{\simeq} L_EX$$

is an equivalence for each X.

Theorem 5.14 (Hopkins–Ravenel [Rav92, Thm. 7.5.6])**.** $L_n = L_{E(n)}$ *is smashing.*

This smash product theorem was proved for $n = 1$ in [Rav84, Thm. 8.1], conjectured for all n in [Rav84, 10.6] and proved in general in [Rav92, Ch. 8] as a consequence of the Devinatz–Hopkins–Smith nilpotence and thick subcategory theorems. In contrast, $\hat{L}_n = L_{K(n)}$ is not smashing for $n \geq 1$.

((ETC: Compare with p-localization $M \to M \otimes \mathbb{Z}_{(p)} \cong M_{(p)}$ and p-completion $M \to M \otimes \mathbb{Z}_p \to M^\wedge_p$ for abelian groups, keeping in mind that $\mathbb{Z}_{(p)} \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}$ while $\mathbb{Z}_p \otimes \mathbb{Z}_p \ncong \mathbb{Z}_p$.))

6. Bousfield classes

The localization functor L_E is determined by the class of E_*-acyclic spectra, and these classes are partially ordered by (reverse) inclusion.

Definition 6.1. Two spectra D and E are Bousfield equivalent if

$$D_*(X) = 0 \iff E_*(X) = 0$$

for all spectra X. Let $\langle E\rangle$ denote the Bousfield equivalence class of E, so that $\langle D\rangle = \langle E\rangle$ means that the class of D_*-acyclic spectra is equal to the class of E_*-acyclic spectra. We write $\langle D\rangle \le \langle E\rangle$ if

$$D_*(X) = 0 \impliedby E_*(X) = 0\,,$$

i.e., if the class of D_*-acyclic spectra contains the class of E_*-acyclic spectra. This defines a partial ordering on the collection of Bousfield equivalence classes.

In other words, we have a quasi-ordering on spectra, with $D \le E$ if

$$\{X \mid D_*(X) \ne 0\} \subset \{X \mid E_*(X) \ne 0\}\,,$$

and this induces a partial ordering $\langle D\rangle \le \langle E\rangle$ on the associated isomorphism classes. We can view the displayed collections as the support of D and E, respectively, in which case $\le$ denotes inclusion of support.

The relation $\langle D\rangle \le \langle E\rangle$ asserts that $E_*(-)$ is a stronger (or equivalent) homology theory than $D_*(-)$. The Bousfield class of $*$ is initial, while that of S is terminal.

Lemma 6.2. *If D is in the localizing subcategory of* $\mathrm{Ho}(\mathcal{S}p)$ *generated by E, then $\langle D\rangle \le \langle E\rangle$.*

Proof. If D can be built from E by repeated passage to homotopy cofibers, desuspensions, retracts and coproducts, then for any X with $E_*(X) = 0$ we will also have $D_*(X) = 0$. □

Lemma 6.3. *Suppose $\langle D\rangle \le \langle E\rangle$. Then each E_*-equivalence is a D_*-equivalence, and each D_*-local spectrum is E_*-local. For each spectrum X the D_*-localization map $\eta_D \colon X \to L_D X$ factors as*

$$X \xrightarrow{\eta_E} L_E X \longrightarrow L_D X$$

for a unique morphism $L_E X \to L_D X$ in $\mathrm{Ho}(\mathcal{S}p)$*, which is a D_*-equivalence. In particular, $L_D X \simeq L_D L_E X \simeq L_E L_D X$.*

Proof. If $f \colon X \to Y$ is an E_*-equivalence with homotopy cofiber Cf then $E_*(Cf) = 0$, so that $D_*(Cf) = 0$ and f is a D_*-equivalence. If N is D_*-local then $[Z, N] = 0$ for each D_*-acyclic Z. In particular $[Z, N] = 0$ for each E_*-acyclic Z, so that N is E_*-local. The E_*-equivalence $\eta_E \colon X \to L_E X$ is a D_*-equivalence, hence induces a bijection $\eta_E^* \colon [L_E X, L_D X] \cong [X, L_D X]$, so there is a unique morphism $L_E X \to L_D X$ mapping to η_D. It induces an isomorphism on D_*-homology since both η_E and η_D have that property.

In particular, $\eta_E \colon X \to L_E X$ is a D_*-equivalence and induces an equivalence after D_*-localization. Also $L_D X$ is E_*-local so $\eta_E \colon L_D X \to L_E L_D X$ is an equivalence. □

Recall [HS05a, Def. 4.1] that the height of a Landweber exact L-module E_* is the maximal n such that $E_*/I_n \neq 0$. The hereditary torsion theory $\mathcal{T}_E$ of LB-comodules M with $E_* \otimes_L M_* = 0$ is then equal to $\mathcal{T}_n$, by the discussion after Theorem 4.8. Both $E(n)_*$ and $v_n^{-1}MU_*$ have height n.

Proposition 6.4. *If D_* and E_* are Landweber exact of the same height, then $\langle D\rangle = \langle E\rangle$.*

Proof. We write D and E for the spectra representing $D_*(X) = D_* \otimes_{MU_*} MU_*(X)$ and $E_*(X) = E_* \otimes_{MU_*} MU_*(X)$, respectively. If E_* has height n, then $E_*(X) = 0$ if and only if $MU_*(X) \in \mathcal{T}_E$, and $\mathcal{T}_E = \mathcal{T}_n$, so this condition on X only depends on n. It follows that if D also has height n, then $D_*(X) = 0$ if and only if $E_*(X) = 0$, so that $\langle D\rangle = \langle E\rangle$. □

Example 6.5. Any nonzero L-module $E_* \supset \mathbb{Q}$ is Landweber exact of height 0, so that $\langle E\rangle = \langle H\mathbb{Q}\rangle$, and $L_E X = L_0 X \simeq X \wedge S\mathbb{Q} \simeq X \wedge H\mathbb{Q}$ is the rationalization of X, given by inverting every prime. This satisfies $\pi_*(L_0X) = \pi_*(X) \otimes \mathbb{Q}$. The map $X \to X \wedge H\mathbb{Q}$ is an $H\mathbb{Q}_*$-equivalence, since $H\mathbb{Q} \simeq H\mathbb{Q} \wedge H\mathbb{Q}$, and $X \wedge H\mathbb{Q}$ is $H\mathbb{Q}$-local, since it is an $H\mathbb{Q}$-module spectrum.

Example 6.6. Complex K-theory KU, p-local K-theory $KU_{(p)}$, and its Adams summand $E(1)$ are all Landweber exact of height 1, so that $\langle KU_{(p)}\rangle = \langle E(1)\rangle$ and $L_{KU_{(p)}}X = L_1 X$ is KU-localization for p-local spectra X. Ravenel's smash product theorem [Rav84, Thm. 8.1] shows that

$$L_1 X \simeq X \wedge L_1 S$$

for all spectra X. Here the $E(1)$-localization of the sphere spectrum sits in a homotopy cofiber sequence

$$\Sigma^{-2} H\mathbb{Q} \longrightarrow L_1 S \longrightarrow J_{(p)}\,,$$

where (for p an odd prime) the p-local image-of-J ring spectrum $J_{(p)}$ is the homotopy fiber of $\psi^g - 1\colon KU_{(p)} \to KU_{(p)}$ for any integer g generating $(\mathbb{Z}/p^2)^\times$, and $\mathbb{Z}/p^\infty \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Hence

$$\pi_n(L_1S) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{for } n = 0, \\ 0 & \text{for } n = -1, \\ \mathbb{Z}/p^\infty & \text{for } n = -2, \\ \mathbb{Z}/p^{v+1} & \text{for } n+1 = (2p-2)m \text{ with } v = \mathrm{ord}_p(m), \\ 0 & \text{otherwise.} \end{cases}$$

Similar, but more elaborate, results are known for $p = 2$.

Example 6.7. The mod p Moore spectrum S/p is not Landweber exact, but

$$L_{S/p} X \simeq X^\wedge_p$$

for any spectrum X. Here

$$X^\wedge_p = \operatorname*{holim}_n X/p^n \simeq \operatorname*{holim}_n F(S^{-1}/p^n, X) \simeq F(S^{-1}/p^\infty, X)\,,$$

where there is a homotopy cofiber sequence

$$\Sigma^{-1} S\mathbb{Z}/p^\infty = S^{-1}/p^\infty \longrightarrow S \longrightarrow S\mathbb{Z}[1/p]\,.$$

The induced map $X \simeq F(S,X) \to F(S^{-1}/p^\infty, X) \simeq X_p^\wedge$ is a S/p-homology equivalence, since $S/p \wedge S\mathbb{Z}[1/p] \simeq *$, and $F(S^{-1}/p^\infty, X) \simeq X_p^\wedge$ is S/p-local, since $S/p \wedge Z \simeq *$ implies that $Z \simeq Z[1/p]$ so that $Z \wedge S^{-1}/p^\infty \simeq *$ and $[Z, X_p^\wedge] = [Z, F(S^{-1}/p^\infty, X)] \cong [Z \wedge S^{-1}/p^\infty, X] = 0$.

Example 6.8. Mod p complex K-theory KU/p and its Adams summand $K(1)$ are not Landweber exact, but $\langle KU/p\rangle = \langle K(1)\rangle$ and $L_{KU/p}X = \hat{L}_1 X = (L_1 X)_p^\wedge$ is the p-completion of the KU-localization. The map

$$X \wedge \hat{L}_1 S \longrightarrow \hat{L}_1(X)$$

is an equivalence for finite (but not for general) spectra X, and

$$\hat{L}_1 S \simeq J_p^\wedge$$

where (for p an odd prime) the p-complete image-of-J ring spectrum $J_p^\wedge$ is the homotopy fiber of $\psi^g - 1\colon KU_p^\wedge \to KU_p^\wedge$ for any integer g generating $(\mathbb{Z}/p^2)^\times$. One proof uses that

$$0 \leftarrow K(1)^*(S) \leftarrow K(1)^*(KU) \overset{(\psi^g-1)^*}{\longrightarrow} K(1)^*(KU) \leftarrow 0$$

is exact, since $K(1)^*(KU) \cong K(1)^*[[\mathbb{Z}_p^\times]]$, and this can be used to obtain $L_1 S$, as above. Hence

$$\pi_n(\hat{L}_1 S) \cong \pi_n(J_p^\wedge) \cong \begin{cases} \mathbb{Z}_p^\wedge & \text{for } n = 0 \text{ and } n = -1, \\ \mathbb{Z}/p^{v+1} & \text{for } n+1 = (2p-2)m \text{ with } v = \mathrm{ord}_p(m), \\ 0 & \text{otherwise.} \end{cases}$$

Again, there are similar results for $p = 2$.

Proposition 6.9. *(a) $\langle K(n)\rangle \le \langle E(n)\rangle$, so there is a natural $K(n)$-equivalence*

$$L_n X = L_{E(n)}X \overset{\hat{\imath}}{\longrightarrow} L_{K(n)}X = \hat{L}_n X\,.$$

(b) $\langle E(n-1)\rangle \le \langle E(n)\rangle$, so there is a natural $E(n-1)$-equivalence

$$L_n X = L_{E(n)}X \overset{j}{\longrightarrow} L_{E(n-1)}X = L_{n-1} X\,.$$

Proof. (a) We can build $K(n)$ from $E(n)$ using homotopy cofiber sequences

$$\Sigma^{|v_m|} E(n)/I_m \overset{v_m}{\longrightarrow} E(n)/I_m \longrightarrow E(n)/I_{m+1}$$

for $0 \le m < n$, so $K(n)$ is in the (thick or) localizing subcategory generated by $E(n)$, and $\langle K(n)\rangle \le \langle E(n)\rangle$. More explicitly: if $E(n)_*(X) = 0$ then by induction $(E(n)/I_m)_*(X) = 0$ for all $0 \le m \le n$, using the cofiber sequences above. Since $E(n)/I_n = K(n)$ we obtain $K(n)_*(X) = 0$.

(b) We can build $v_{n-1}^{-1}E(n)$ from $E(n)$ using the telescope

$$E(n) \overset{v_{n-1}}{\longrightarrow} \Sigma^{-|v_{n-1}|} E(n) \overset{v_{n-1}}{\longrightarrow} \Sigma^{-2|v_{n-1}|} E(n) \longrightarrow \dots \longrightarrow v_{n-1}^{-1}E(n)\,,$$

so $v_{n-1}^{-1}E(n)$ is in the localizing subcategory generated by $E(n)$, and $\langle v_{n-1}^{-1}E(n)\rangle \le \langle E(n)\rangle$. Here

$$\pi_*(v_{n-1}^{-1}E(n)) = v_{n-1}^{-1}E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-2}, v_{n-1}^{\pm 1}, v_n^{\pm 1}]\,,$$

interpreted as $\mathbb{Q}[v_1^{\pm 1}]$ for $n = 1$. More explicitly: if $E(n)_*(X) = 0$ then by construction $v_{n-1}^{-1}E(n)_*(X) = 0$. Now we use that $v_{n-1}^{-1}E(n)_*$ is Landweber exact of height $(n-1)$, so that $\langle E(n-1)\rangle = \langle v_{n-1}^{-1}E(n)\rangle$. It follows that $\langle E(n-1)\rangle \le \langle E(n)\rangle$. □

It follows that $\langle E(n)\rangle \ge \langle E(m)\rangle \ge \langle K(m)\rangle$ for all $0 \le m \le n$.

Proposition 6.10. *$K(m) \wedge K(n) \simeq *$ for $m \ne n$.*

Proof. We may suppose $m < n$. Then this follows from Chapter 11, Proposition 7.16, since $E(m)_*(K(n)) = 0$ and $\langle E(m)\rangle \ge \langle K(m)\rangle$ implies $K(m)_*(K(n)) = 0$. □

Lemma 6.11. *The wedge $\langle D\rangle \vee \langle E\rangle = \langle D \vee E\rangle$ and smash $\langle D\rangle \wedge \langle E\rangle = \langle D \wedge E\rangle$ only depend on the Bousfield classes of D and E.*

Proof. If $\langle D\rangle = \langle D'\rangle$ and $\langle E\rangle = \langle E'\rangle$ then $(D \vee E)_*(X) = 0$ iff $(D_*(X) = 0$ and $E_*(X) = 0)$ iff $(D'_*(X) = 0$ and $E'_*(X) = 0)$ iff $(D' \vee E')_*(X) = 0$. Moreover, $(D \wedge E)_*(X) = 0$ iff $D_*(E \wedge X) = 0$ iff $D'_*(E \wedge X) = 0$ iff $E_*(D' \wedge X) = 0$ iff $E'_*(D' \wedge X) = 0$ iff $(D' \wedge E')_*(X) = 0$. □

With this notation,

$$\begin{aligned}\langle E(n)\rangle &\ge \langle K(0) \vee K(1) \vee \cdots \vee K(n-1) \vee K(n)\rangle \\ &= \langle K(0)\rangle \vee \langle K(1)\rangle \vee \cdots \vee \langle K(n-1)\rangle \vee \langle K(n)\rangle \,.\end{aligned}$$

In fact, the opposite relation also holds.

Theorem 6.12 ([Rav84, Thm. 2.1(d)])**.**

$$\langle E(n)\rangle = \bigvee_{m=0}^{n} \langle K(m)\rangle \,.$$

Hence $E(n)_*(X) = 0$ if and only if $K(m)_*(X) = 0$ for each $0 \le m \le n$.

Proof. A prototype for this argument is given by Johnson–Wilson in [JW75, §5], and attributed to Morava. We must show that if $K(m)_*(X) = 0$ for each $0 \le m \le n$, then $E(n)_*(X) = 0$. By an outer induction on n we may assume that $E(m)_*(X) = 0$ for each $0 \le m < n$.

Consider the tower of (left hand) distinguished triangles and (right hand) localization maps, in $\mathrm{Ho}(\mathcal{S}p)$.

$$
\begin{array}{ccccc}
E(n) & \xrightarrow{p} & E(n) & \xrightarrow{j} & p^{-1}E(n) \\
 & \nwarrow & \downarrow{\scriptstyle \pi} & & \\
\Sigma^{|v_1|}E(n)/p & \xrightarrow{v_1} & E(n)/p & \xrightarrow{j} & v_1^{-1}E(n)/p \\
 & & \downarrow{\scriptstyle \pi} & & \\
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi} & & \\
\Sigma^{|v_m|}E(n)/I_m & \xrightarrow{v_m} & E(n)/I_m & \xrightarrow{j} & v_m^{-1}E(n)/I_m \\
 & \nwarrow & \downarrow{\scriptstyle \pi} & & \\
\Sigma^{|v_{m+1}|}E(n)/I_{m+1} & \xrightarrow{v_{m+1}} & E(n)/I_{m+1} & \xrightarrow{j} & v_{m+1}^{-1}E(n)/I_{m+1} \\
 & & \downarrow{\scriptstyle \pi} & & \\
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi} & & \\
\Sigma^{|v_{n-1}|}E(n)/I_{n-1} & \xrightarrow{v_{n-1}} & E(n)/I_{n-1} & \xrightarrow{j} & v_{n-1}^{-1}E(n)/I_{n-1} \\
 & \nwarrow & \downarrow{\scriptstyle \pi} & & \\
 & & E(n)/I_n = K(n) & &
\end{array}
$$

We prove by an inner, descending, induction on m that $(E(n)/I_m)_*(X) = 0$. For $m = n$ this holds by the assumption $K(n)_*(X) = 0$. Suppose that $0 \le m < n$ and $(E(n)/I_{m+1})_*(X) = 0$. Then

$$v_m \colon \Sigma^{|v_m|}(E(n)/I_m)_*(X) \xrightarrow{\cong} (E(n)/I_m)_*(X)$$

is an isomorphism by exactness. Hence

$$j \colon (E(n)/I_m)_*(X) \xrightarrow{\cong} v_m^{-1}(E(n)/I_m)_*(X)$$

is a colimit of isomorphisms, and is therefore also an isomorphism. Here $v_m^{-1}E(n)/I_m$ can be built from $v_m^{-1}E(n)$ using cofiber sequences, as in the proof of Proposition 6.9(a), so that $\langle v_m^{-1}E(n)/I_m\rangle \le \langle v_m^{-1}E(n)\rangle$. Moreover,

$$v_m^{-1}E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{m-1}, v_m^{\pm 1}, v_{m+1}, \dots, v_{n-1}, v_n^{\pm 1}]$$

is Landweber exact of height m, so that $\langle v_m^{-1}E(n)\rangle = \langle E(m)\rangle$. By the outer induction on n we know that $E(m)_*(X) = 0$, since $m < n$, so $v_m^{-1}E(n)_*(X) = 0$ and $v_m^{-1}(E(n)/I_m)_*(X) = 0$. The displayed isomorphism j now shows that $(E(n)/I_m)_*(X) = 0$, which completes the inner inductive step from $m+1$ to m. We conclude that $E(n)_*(X) = (E(n)/I_0)_*(X) = 0$, as required. □

Proposition 6.13 ([Rav84, Prop. 1.27])**.** *L_E is smashing if and only if $\langle L_E S\rangle = \langle E\rangle$. In particular, $\langle L_n S\rangle = \langle E(n)\rangle$.*

Proof. If L_E is smashing then $L_E S \wedge L_E S \simeq L_E L_E S \simeq L_E S$ (so $L_E S$ is a solid ring spectrum). Hence $X \to X \wedge L_E S$ is an $L_E S$-homology equivalence. The target is an $L_E S$-module, hence is $L_E S$-local by Adams' Lemma 5.9, so $X \wedge L_E S$ is the $L_E S$-homology localization of X. Since it is also the E_*-localization, it follows that $(L_E S)_*(X) = 0$ if and only if $E_*(X) = 0$, so that $\langle L_E S\rangle = \langle E\rangle$.

Conversely, if $L_E S$ and E are Bousfield equivalent, then since the $L_E S$-module $X \wedge L_E S$ is $L_E S$-local it is also E-local, so that the E_*-equivalence $X \to X \wedge L_E S$ must be the E-localization map. Hence L_E is smashing. □

Proposition 6.14. *$K(n) \wedge L_{n-1} X \simeq *$ for each spectrum X.*

Proof. Since $L_{n-1}X \simeq X \wedge L_{n-1}S$ it suffices to prove that $K(n) \wedge L_{n-1}S \simeq *$, i.e., that $(L_{n-1}S)_*(K(n)) = 0$. Since $L_{n-1}S$ and $E(n-1)$ are Bousfield equivalent, this is equivalent to $E(n-1)_*(K(n)) = 0$, which we proved in Chapter 11, Proposition 7.16. □

7. The chromatic tower

For each spectrum X and prime p we have a chromatic tower

$$X \longrightarrow X_{(p)} \longrightarrow \dots \longrightarrow L_n X \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_1 X \longrightarrow L_0 X \to *$$

in $\mathrm{Ho}(\mathcal{S}p)$, where all but the first object lie in $\mathrm{Ho}(\mathcal{S}p_{(p)})$, and the part from $L_n X$ and to the right lies in $\mathrm{Ho}(L_n\mathcal{S}p)$. The complexity of these categories appears to increase with n, so one can hope for a more complete understanding of $\mathrm{Ho}(L_n\mathcal{S}p)$ than of $\mathrm{Ho}(\mathcal{S}p)$, for gradually increasing values of n.

There is an induced tower of homotopy groups

$$\pi_*(X) \longrightarrow \pi_*(X) \otimes \mathbb{Z}_{(p)} \longrightarrow \dots \longrightarrow \pi_*(L_n X) \longrightarrow \pi_*(L_{n-1}X) \longrightarrow \dots$$
$$\dots \longrightarrow \pi_*(L_1 X) \longrightarrow \pi_*(L_0 X) \cong \pi_*(X) \otimes \mathbb{Q}$$

with potentially interesting behavior on the p-power torsion part of $\pi_*(X)_{(p)} = \pi_*(X) \otimes \mathbb{Z}_{(p)}$.

Definition 7.1. The chromatic filtration of $\pi_*(X)_{(p)}$ is the descending filtration defined by letting

$$F^{n+1}\pi_*(X)_{(p)} = \ker(\pi_*(X_{(p)}) \longrightarrow \pi_*(L_n X))$$

be the graded subgroup of homotopy classes that are not detected at height $\le n$. The filtration quotient

$$\frac{F^n \pi_*(X)_{(p)}}{F^{n+1}\pi_*(X)_{(p)}}$$

is then the subquotient detected at height $= n$, and represents the chromatic height n elements of $\pi_*(X)_{(p)}$.

Remark 7.2. This is understood at height 0 by rational cohomology, at height 1 by topological K-theory and the image-of-J, but only partially at height 2 using topological modular forms and tmf-resolutions. See work by Mark Behrens and coauthors. The elements in $\pi_*(S)_{(p)}$ that are detected in $L_1 S$ are known as the α-family, and there is a β-family of elements detected in $L_2 S$. The non-triviality of the γ-family at height 3 was established by Miller–Ravenel–Wilson in [MRW77]. The construction of an explicit δ-family at height 4 remains an open problem.

Nonetheless, there is the following positive result, known as the chromatic convergence theorem, which tells us that we can in principle recover X from its chromatic localizations L_nX (for all sufficiently high n).

Theorem 7.3 (Hopkins–Ravenel [Rav92, Thm. 7.5.7])**.** *Let X be a finite p-local spectrum. Then the natural map*

$$X \xrightarrow{\simeq} \operatorname*{holim}_n L_nX$$

is an equivalence.

This is proved in [Rav92, Ch. 8] as a consequence of the smash product theorem. It is also true for some other X, but false e.g. for any nontrivial X with $\pi_*(X)$ bounded above and rationally trivial, since for these spectra $L_nX = 0$ for all $n \geq 0$. For $n \geq 1$ this follows from the chromatic fracture square in Theorem 7.5 below, since $K(n)_*(X) = 0$ and $\hat{L}_nX \simeq *$ whenever $\pi_*(X)$ is bounded above.

One might hope to inductively obtain L_nX from $L_{n-1}X$ by building in the height $= n$ information not seen in the latter. For this, one might draw inspiration from number theory. The square of commutative rings

$$\begin{array}{ccc} \mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \end{array}$$

is (both a pushout and) a pullback. It follows that

$$\begin{array}{ccc} M_{(p)} & \longrightarrow & M \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M_p^\wedge & \longrightarrow & (M_p^\wedge) \otimes \mathbb{Q} \end{array}$$

is a pullback for each finitely generated $\mathbb{Z}_{(p)}$-module M. Here $M_p^\wedge = \lim_n M/p^nM$ denotes the algebraic p-completion, and satisfies $M \otimes \mathbb{Z}_p \cong M_p^\wedge$ when M is finitely generated (over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$). This idea was carried over to (simply-connected or nilpotent) spaces by Sullivan (notes from ca. 1970), and to spectra by Bousfield [Bou79, Prop. 2.9].

Theorem 7.4. *For any spectrum X the square*

$$\begin{array}{ccc} X_{(p)} & \longrightarrow & L_0X = X \wedge H\mathbb{Q} \\ \downarrow & & \downarrow \\ X_p^\wedge & \longrightarrow & L_0(X_p^\wedge) = X_p^\wedge \wedge H\mathbb{Q} \end{array}$$

is a homotopy pullback.

This arithmetic fracture square concerns the situation

$$\operatorname{Spec}(\mathbb{F}_p) \subset \operatorname{Spf}(\mathbb{Z}_p) \xrightarrow{\hat{\imath}} \operatorname{Spec}(\mathbb{Z}_{(p)}) \xleftarrow{j} \mathbb{Q}$$

where $\operatorname{Spf}(\mathbb{Z}_p)$ is a formal neighborhood of the closed point $i\colon \operatorname{Spec}(\mathbb{F}_p) \to \operatorname{Spec}(\mathbb{Z}_{(p)})$. The corresponding result for

$$\mathcal{M}_{\mathrm{fg}}^n \xrightarrow{i} \mathcal{M}_{\mathrm{fg}}^{\leq n} \xleftarrow{j} \mathcal{M}_{\mathrm{fg}}^{\leq n-1}$$

is the following chromatic fracture square, presumably due to Hopkins, cf. [Hov95, Proof of Thm. 4.3].

Theorem 7.5. *For any spectrum X the square*

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{n-1} X \\ \downarrow & & \downarrow \\ \hat{L}_n X & \longrightarrow & L_{n-1}(\hat{L}_n X) \end{array}$$

is a homotopy pullback.

Remark 7.6. Hopkins has formulated a chromatic splitting conjecture about the right hand vertical map $L_{n-1}X \to L_{n-1}(\hat{L}_n X)$, which predicts how $L_n X$ is detected by the $\hat{L}_m X$ for $0 \le m \le n$. See [Hov95] for an early paper, and [BGH22] for recent developments.

Here is a common generalization of these theorems (as explained by Neil Strickland on https://mathoverflow.net/q/91057), related to [Hov95, Lem. 4.1]. Note that

$$\langle D \rangle \le \langle D \vee E \rangle \ge \langle E \rangle$$

for any spectra D and E, so we have preferred natural transformations $L_{D \vee E} \to L_D$ and $L_{D \vee E} \to E$.

Theorem 7.7. *Suppose that $D_*(Z) = 0$ implies $D_*(L_E Z) = 0$. Then*

$$\begin{array}{ccc} L_{D \vee E} X & \longrightarrow & L_E X \\ \downarrow & & \downarrow \\ L_D X & \longrightarrow & L_E(L_D X) \end{array}$$

is a homotopy pullback for any spectrum X.

Proof. Let $f \colon X \to P$ denote the map to the homotopy pullback. We must show that P is $(D \vee E)_*$-local and that f is a $(D \vee E)_*$-equivalence. If $(D \vee E)_*(Z) = D_*(Z) \oplus E_*(Z) = 0$ then $[Z, L_D X] = [Z, L_E X] = 0$ and $[\Sigma Z, L_E(L_D X)] = 0$, so $[Z, P] = 0$ by the Mayer–Vietoris sequence for $[Z, -]_*$.

The map $\eta_D \colon X \to L_D X$ is a D_*-equivalence, so $f \colon X \to P$ is a D_*-equivalence if and only if $P \to L_D X$ is a D_*-equivalence, which by the Mayer–Vietoris sequence for $D_*(-)$ is equivalent to $L_E(\eta_D) \colon L_E X \to L_E(L_D X)$ being a D_*-equivalence. The cofiber $Z = C\eta_D$ of $\eta_D \colon X \to L_D X$ is D_*-acyclic, so by assumption $L_E Z$ is D_*-acyclic, which implies that $L_E(\eta_D)$ is a D_*-isomorphism.

Finally, $\eta_E \colon X \to L_E$ is an E_*-equivalence, so $f \colon X \to P$ is an E_*-equivalence if and only if $P \to L_E X$ is one, which by the Mayer–Vietoris sequence for $E_*(-)$ is equivalent to $\eta_E \colon L_D X \to L_E(L_D X)$ being an E_*-equivalence. This is obviously true from the definition of L_E. □

Proof of Theorem 7.4. In the arithmetic case, we apply this to p-local X with $D = S/p$ and $E = H\mathbb{Q}$, in which case $\langle S/p \vee H\mathbb{Q} \rangle = \langle S_{(p)} \rangle$ and $(S/p)_*(Z \otimes H\mathbb{Q}) = 0$ (with no hypothesis on Z). □

Proof of Theorem 7.5. In the chromatic case, we apply it to $E(n)$-local X with $D = K(n)$ and $E = E(n-1)$, so that $\langle D \vee E \rangle = \langle E(n) \rangle$ by Theorem 6.12. We must verify that if $K(n)_*(Z) = 0$, then $K(n)_*(L_{n-1}Z) = 0$. This follows from the smash product theorem $L_{n-1}S \wedge Z \simeq L_{n-1}Z$. $\square$

Remark 7.8. If fact, $K(n)_*(L_{n-1}X) = 0$ for all X by Proposition 6.14, but the proof uses the smash product theorem. For $n \in \{1, 2\}$ we can prove directly that $K(n)_*(L_{n-1}X) = 0$ for all X. Namely, L_0X is rational, so $K(n)_*(L_0X) = 0$ for all $n \geq 1$. This proves the case $n = 1$ of the chromatic fracture square. To prove that $K(n)_*(L_1X) = 0$ for all $n \geq 2$ we use this square to reduce to proving that $K(n)_*(\hat{L}_1X) = 0$. By the Künneth isomorphism, it suffices to prove that $K(n)_*(\hat{L}_1X \wedge S/p) = 0$. The Adams self-map $v_1 \colon \Sigma^{2p-2}S/p \to S/p$ is a $K(1)$-equivalence, hence induces a self-equivalence of the $K(1)$-local spectrum $\hat{L}_1X \wedge S/p = \hat{L}_1X/p$. On the other hand, it induces zero in $K(n)$-homology for $n \geq 2$. This proves that $K(n)_*(\hat{L}_1X/p) = 0$. See Bauer's article [DFHH14, Ch. 6, Thm. 3.6] for this argument.

8. Monochromatic fibers

Definition 8.1. For each spectrum X we define the n-th colocalization C_nX and the n-th monochromatic fiber M_nX by the homotopy (co-)fiber sequences

$$C_nX \longrightarrow X \xrightarrow{\eta} L_nX$$
$$M_nX \longrightarrow L_nX \longrightarrow L_{n-1}X\,.$$

Here $L_{-1}X = *$, so $C_{-1}X = X$ and $M_0X = L_0X$.

Lemma 8.2. *Let $0 \leq m \leq n$.*

(a) Both C_n and M_n are exact, i.e., preserve homotopy (co-)fiber sequences.

(b) The natural maps

$$L_mX \xrightarrow{\simeq} L_mL_nX$$
$$L_mX \xrightarrow{\simeq} L_nL_mX$$

are equivalences.

(c) $L_mC_nX \simeq *$ and $C_nL_mX \simeq *$.

(d) The natural maps

$$C_mC_nX \xrightarrow{\simeq} C_nX$$
$$C_nC_mX \xrightarrow{\simeq} C_nX$$

are equivalences.

(e) There are natural equivalences

$$M_nX \xrightarrow{\simeq} C_{n-1}L_nX$$
$$M_nX \xrightarrow{\simeq} L_nC_{n-1}X$$

Proof. (a) This follows since each L_n is exact.

(b) This follows from $\langle E(m)\rangle \leq \langle E(n)\rangle$ and Lemma 6.3.

(c) The first case uses exactness of L_m, the second holds by definition.

(d) The first holds by definition, the second uses exactness of C_n.

(e) This uses the maps

$$\begin{CD}
M_nX @>>> L_nX @>>> L_{n-1}X \\
@VVV @| @VV{\simeq}V \\
C_{n-1}L_nX @>>> L_nX @>{\eta}>> L_{n-1}L_nX
\end{CD}$$

and

$$\begin{CD}
M_nX @>>> L_nX @>>> L_{n-1}X \\
@VVV @| @VV{\simeq}V \\
L_nC_{n-1}X @>>> L_nX @>{L_n\eta}>> L_nL_{n-1}X
\end{CD}$$

of homotopy cofiber sequences. □

Remark 8.3. By analogy with the associated quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, we think of C_nX as the part of X supported on the closed substack of height $\geq n+1$, and of M_nX as the part of L_nX over the height $\leq n$ open substack that is supported on the height $= n$ closed substack. Equivalently, it is the localization to the height $= n$ open substack of the part $C_{n-1}X$ supported on the closed height $\geq n$ substack.

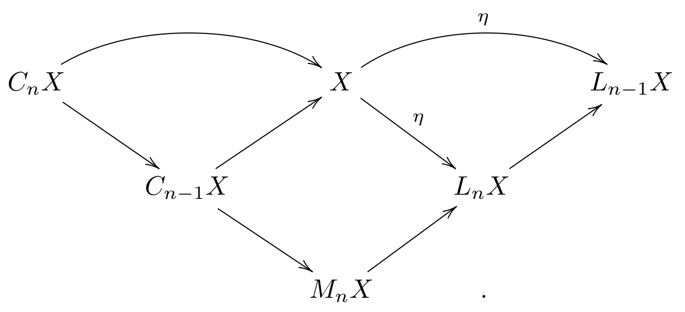
Taking homotopy fibers of the maps from X to the chromatic tower

$$\begin{CD}
 @. M_nX @. M_{n-1}X @. @. M_1X @. M_0X @. \\
 @. @VVV @VVV @. @VVV @VV{\simeq}V @. \\
\dots @>>> L_nX @>>> L_{n-1}X @>>> \dots @>>> L_1X @>>> L_0X @>>> *
\end{CD}$$

(with monochromatic homotopy fibers) we obtain the geometric (= spectrum level) chromatic filtration

$$\begin{CD}
\dots @>>> C_nX @>>> C_{n-1}X @>>> \dots @>>> C_1X @>>> C_0X @>>> X \\
 @. @VV{\eta}V @VV{\eta}V @. @VV{\eta}V @VV{\eta}V @VV{\eta}V \\
 @. M_{n+1}X @. M_nX @. @. M_2X @. M_1X @. M_0X
\end{CD}$$

of X (with monochromatic homotopy cofibers). This follows from the (partial) braid diagram



By Lemma 8.2(e), the maps to the cofibers in the chromatic filtration are the $E(n)$-localization maps

$$\eta\colon C_{n-1}X \longrightarrow L_nC_{n-1}X \simeq M_nX\,.$$

Let $C_{-1}X = X$. We can inductively describe the geometric chromatic filtration by setting $M_nX = L_nC_{n-1}X$ and letting C_nX be the homotopy fiber of the map η displayed above, for each $n \geq 0$.

Theorem 8.4 (Hovey–Strickland [HS99b, Thm. 6.19]). *The natural maps*

$$M_nX \xrightarrow{\simeq} M_n\hat{L}_nX$$
$$\hat{L}_nM_nX \xrightarrow{\simeq} \hat{L}_nL_nX \simeq \hat{L}_nX$$

are equivalences. Hence M_n and $\hat{L}_n$ induce mutually inverse equivalences of categories

$$M_n\colon \operatorname{Ho}(\hat{L}_n\mathcal{S}p) \rightleftarrows \operatorname{Ho}(M_n\mathcal{S}p)\colon \hat{L}_n$$
$$\hat{L}_nX \leftrightarrow M_nX$$

between the $K(n)$-local category and the n-monochromatic category.

Proof. The chromatic fracture square of Theorem 7.5 and the equivalence $\hat{L}_nX \simeq L_n\hat{L}_nX$ induce equivalences

$$M_nX \simeq C_{n-1}\hat{L}_nX \simeq M_n\hat{L}_nX\,.$$

The vanishing of $\hat{L}_nL_{n-1}X$ (which follows from Proposition 6.14) and equivalence $\hat{L}_nX \simeq \hat{L}_nL_nX$ induce equivalences

$$\hat{L}_nM_nX \simeq \hat{L}_nL_nX \simeq \hat{L}_nX\,.$$

□

Remark 8.5. This is reminiscent of a recollement situation, giving an equivalence between sheaves supported on a closed substack and sheaves that are complete along that substack. See Barwick–Glasman (arXiv:1607.02064) for a discussion of this in the context of stable ∞-categories. In their notation, the Hovey–Strickland equivalence corresponds to $\mathbf{X} = L_n\mathcal{S}p$, $\mathbf{U} = L_{n-1}\mathcal{S}p$, $\mathbf{Z}^\wedge = \hat{L}_n\mathcal{S}p$ and $\mathbf{Z}^\vee = M_n\mathcal{S}p$. The inclusion $j_*\colon L_{n-1}\mathcal{S}p \to L_n\mathcal{S}p$ admits the left adjoint $j^*(X) = L_{n-1}S \wedge X$ and the right adjoint $j^\times(X) = F(L_{n-1}S, X)$, so $L_{n-1}\mathcal{S}p$ is reflective and coreflective in $L_n\mathcal{S}p$. The inclusion $i_\wedge\colon \hat{L}_n\mathcal{S}p \to L_n\mathcal{S}p$ has a left adjoint $i^\wedge$ with $i_\wedge i^\wedge = \hat{L}_n$, hence $\hat{L}_n\mathcal{S}p$ is reflective. The inclusion $i_\vee\colon M_n\mathcal{S}p \to L_n\mathcal{S}p$ has a right adjoint $i^\vee$ with $i_\vee i^\vee = M_n$, so $M_n\mathcal{S}p$ is coreflective. The functors $i^\wedge i_\vee\colon M_n\mathcal{S}p \to \hat{L}_n\mathcal{S}p$ and $i^\vee i_\wedge\colon \hat{L}_n\mathcal{S}p \to M_n\mathcal{S}p$ lift the inverse equivalences of Theorem 8.4 to the ∞-category level.

9. The chromatic filtration for MU

For any spectrum X, the Adams–Novikov spectral sequence (or MU-based Adams spectral sequence) has the form

$$\mathcal{E}_2^{s,t} = \operatorname{Ext}_{LB}^{s,t}(L, MU_*(X)) \Longrightarrow_s \pi_{t-s}(X)\,.$$

Here $\operatorname{Ext}_{LB}^{*,*}(L, M)$ denotes Ext formed in the abelian category of LB-comodules. The spectral sequence is strongly convergent if X is bounded below, but convergence for more general X is more subtle. Nonetheless, to study π_*L_nX we are led to study $MU_*(L_nX) = \pi_*(MU \wedge L_nX) \cong \pi_*(L_nMU \wedge X)$, where the isomorphism uses that L_n is smashing.

Definition 9.1. Let R be a ring. For an R-module M and element $x \in R$ we write $\Gamma_x M$ for the x-power torsion and M/x^∞ for the "x-power cotorsion" of M, defined as the kernel and the cokernel, respectively, of the localization homomorphism $\beta \colon M \to x^{-1}M = M[1/x]$ away from x.

$$0 \to \Gamma_x M \longrightarrow M \xrightarrow{\beta} x^{-1}M \longrightarrow M/x^\infty \to 0$$

Definition 9.2. Let R be a ring spectrum. For an R-module spectrum M and element $y \in \pi_*(R)$ we write $\Gamma_y M$ for the y-power torsion and M/y^∞ for the "y-power cotorsion" of M, defined as the homotopy fiber and the homotopy cofiber, respectively, of the localization map $\beta \colon M \to y^{-1}M = M[1/y]$ away from y.

$$\Gamma_y M \longrightarrow M \xrightarrow{\beta} y^{-1}M$$

$$M \xrightarrow{\beta} y^{-1}M \longrightarrow M/y^\infty$$

Clearly $\Sigma\Gamma_y M \simeq M/y^\infty$.

To study the homotopy cofiber sequence

$$C_n S \longrightarrow C_{n-1}S \xrightarrow{\eta} L_n C_{n-1} S = M_n S$$

with associated long exact sequence

$$\dots \to MU_*(C_{n-1}S) \xrightarrow{\eta_*} MU_*(M_n S) \longrightarrow MU_{*-1}(C_n S) \to \dots$$

in MU-homology, we apply $MU \wedge (-)$ to obtain the homotopy cofiber sequence

$$C_n MU \longrightarrow C_{n-1}MU \longrightarrow L_n C_{n-1} MU = M_n MU$$

of MU-module spectra with associated long exact sequence

$$\dots \to \pi_*(C_{n-1}MU) \xrightarrow{\eta_*} \pi_*(M_n MU) \longrightarrow \pi_{*-1}(C_n MU) \to \dots$$

in homotopy. This breaks up into short exact sequences, and can be made explicit using the cotorsion notation above.

Theorem 9.3 ([Rav84, Thm. 6.1])**.** *For each $n \geq 0$ there is an isomorphism*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Sigma^n \pi_*(C_{n-1}MU) & \xrightarrow{\cong} & MU_*/(p^\infty, \dots, v_{n-1}^\infty) \\
\downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \beta} \\
\Sigma^n \pi_*(M_n MU) & \xrightarrow{\cong} & v_n^{-1} MU_*/(p^\infty, \dots, v_{n-1}^\infty) \\
\downarrow & & \downarrow{\scriptstyle \gamma} \\
\Sigma^{n+1} \pi_*(C_n MU) & \xrightarrow{\cong} & MU_*/(p^\infty, \dots, v_{n-1}^\infty, v_n^\infty) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

of short exact sequences of MU_*MU-comodules.

Proof. Let $n \geq 0$ and assume by induction that $\pi_*(C_{n-1}MU)$ is as stated. Once we prove that $E(n)$-localization on the MU-module spectrum $C_{n-1}MU$ induces algebraic localization away from v_n, the formulas for $\pi_*(M_nMU)$ and $\pi_*(C_nMU)$ follow, since β is injective in each case.

For brevity, let $X = C_{n-1}MU$. We must prove that

$$\beta\colon X \longrightarrow v_n^{-1}X$$

is an $E(n)$-localization. This is the colimit of many composites of (desuspensions of) the map

$$X \xrightarrow{v_n} \Sigma^{-|v_n|}X\,,$$

each of which induces the isomorphism

$$v_n\colon E(n)_*(X) \xrightarrow{\cong} E(n)_{*+|v_n|}(X)$$

in $E(n)$-homology (since v_n is a unit in $E(n)_*$). Hence β is an $E(n)$-equivalence. It remains to prove that $v_n^{-1}X$ is $E(n)$-local. Let Z be a spectrum with $E(n)_*(Z) = 0$. Then

$$F(Z, v_n^{-1}X) \simeq F_{v_n^{-1}MU}(v_n^{-1}MU \wedge Z, v_n^{-1}X)$$

since $v_n^{-1}X$ is a $v_n^{-1}MU$-module spectrum. Here $v_n^{-1}MU$ is a Landweber exact theory of height n, so $\langle v_n^{-1}MU\rangle = \langle E(n)\rangle$ by Proposition 6.4. Hence $E(n)_*(Z) = 0$ implies $v_n^{-1}MU \wedge Z \simeq *$, so the function spectra displayed above are trivial. In particular, $[Z, v_n^{-1}X] = 0$, proving $E(n)$-locality. □

Corollary 9.4. *There a short exact sequence*

$$0 \to MU_* \xrightarrow{\eta} MU_*(L_nS) = \pi_*(L_nMU) \longrightarrow \Sigma^{-n}MU_*/(p^\infty, \dots, v_n^\infty) \to 0$$

of MU_*MU-comodules for each $n \geq 0$, which is split as MU_*-modules for $n \geq 1$, and as MU_*MU-comodules for $n \geq 2$.

10. The chromatic spectral sequence

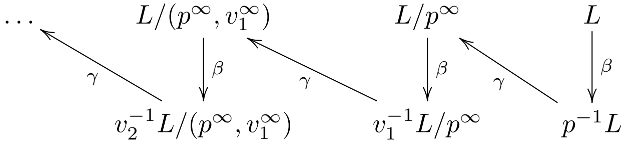
We use the notations

$$L/I_n^\infty = L/(p^\infty, \dots, v_{n-1}^\infty)$$
$$v_n^{-1}L/I_n^\infty = v_n^{-1}L/(p^\infty, \dots, v_{n-1}^\infty)\,.$$

The MU-homology exact couple associated to the chromatic filtration of S, or equivalently, the homotopy exact couple associated to the chromatic filtration of MU, is simply given by the short exact sequences

(10.1) $$0 \to L/I_n^\infty \xrightarrow{\beta} v_n^{-1}L/I_n^\infty \xrightarrow{\gamma} L/I_{n+1}^\infty \to 0$$

for each $n \geq 0$, spliced together in the following diagram.



The resulting long exact sequence

$$0 \to L \xrightarrow{\beta} p^{-1}L \xrightarrow{\beta\gamma} v_1^{-1}L/p^\infty \xrightarrow{\beta\gamma} v_2^{-1}L/(p^\infty, v_1^\infty) \to \dots$$

of LB-comodules is the Cousin complex for L, in the sense of [Har66, Ch. IV, §2], cf. Hopkins–Gross [HG94, Table 1].

This LB-comodule resolution of L was used by Miller–Ravenel–Wilson [MRW77] to construct the chromatic spectral sequence. They were studying the Adams–Novikov spectral sequence

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) = \mathrm{Ext}^{s,t}_{LB}(L, L) \Longrightarrow_s \pi_{t-s}(S)$$

converging (strongly) to the stable homotopy groups of spheres, also known as the MU-based Adams spectral sequence. (To be precise, they worked the the p-local version, based on BP.) The $\mathcal{E}_2$-term is given by Ext groups in the category of LB-comodules. Here

$$\mathcal{E}_2^{0,*} = \mathrm{Hom}_{LB}(L, L) = \mathbb{Z} \cong \pi_0(S)\,,$$

while $\mathcal{E}_2^{1,*}$ was calculated by Novikov [Nov67] and is closely related to $\pi_*(J_p)$ (especially for odd p) and the image-of-J in $\pi_*(S)$. For $p = 2$, $\pi_*(J_2)$ is not entirely accounted for by the Novikov 1-line $\mathcal{E}_2^{1,*}$. However, v_n-periodic phenomena do in a sense only appear in Adams–Novikov filtrations $s \ge n$, in a way we now try to clarify.

For each $n \ge 0$ the short exact sequence (10.1) of LB-modules induces a long exact sequence

$$\begin{aligned} \dots \to \mathrm{Ext}^{s,*}_{LB}(L, L/I_n^\infty) &\xrightarrow{\beta} \mathrm{Ext}^{s,*}_{LB}(L, v_n^{-1}L/I_n^\infty) \\ &\xrightarrow{\gamma} \mathrm{Ext}^{s,*}_{LB}(L, L/I_{n+1}^\infty) \xrightarrow{\delta} \mathrm{Ext}^{s+1,*}_{LB}(L, L/I_n^\infty) \to \dots \end{aligned}$$

in LB-comodule Ext. These combine to an (unrolled) exact couple

$$\begin{array}{ccccc} \dots \longrightarrow & \mathrm{Ext}^{*,*}_{LB}(L, L/I_{n+1}^\infty) & \xrightarrow{\alpha} & \mathrm{Ext}^{*,*}_{LB}(L, L/I_n^\infty) & \longrightarrow \dots \\ & & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} & \\ & & & \mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n^\infty) & \end{array}$$

and a trigraded spectral sequence

$${}^{\mathrm{chrom}}\mathcal{E}_1^{n,s,t} = \mathrm{Ext}^{s,t}_{LB}(L, v_n^{-1}L/I_n^\infty) \Longrightarrow_n \mathrm{Ext}^{s+n,t}_{LB}(L, L)$$

called the chromatic spectral sequence. The filtration n part ${}^{\mathrm{chrom}}\mathcal{E}_1^{n,*,*}$ of its $\mathcal{E}_1$-term consists of v_n-periodic families, and the subquotient ${}^{\mathrm{chrom}}\mathcal{E}_\infty^{n,*,*}$ that survives to the $\mathcal{E}_\infty$-term of the chromatic spectral sequence gives the associated graded of the so-called v_n-periodic part of $\mathrm{Ext}^{*,*}_{LB}(L, L)$, i.e., of the Adams–Novikov $\mathcal{E}_2$-term. In turn, the corresponding subquotient of the p-local Adams–Novikov $\mathcal{E}_\infty$-term defines the v_n-periodic part of $\pi_*(S)_{(p)}$.

In view of Theorem 9.3, the filtration n part of the chromatic $\mathcal{E}_1$-term is also the Adams–Novikov $\mathcal{E}_2$-term for $\Sigma^n M_n S$:

$$\begin{aligned} \mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n^\infty) &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(\Sigma^n M_n S)) \\ &\Longrightarrow \pi_*(\Sigma^n M_n S)\,. \end{aligned}$$

((ETC: Discuss convergence, using Hovey–Sadofsky [HS99a, Thm. 5.3].))

The chromatic resolution, or Cousin complex, of $L = MU_*$ by $LB = MU_*MU$-comodules, can be viewed as a resolution by LB-injective (co-)modules in the sense of [JLY81], i.e., L-modules N such that $\mathrm{Ext}^{s,*}_L(M, N) = 0$ for all LB-comodules M and $s > 0$.

11. The Morava change-of-rings isomorphism

Any morphism of flat Hopf algebroids (or stacks) inducing an equivalence of categories of comodules (or quasi-coherent sheaves) will also induce an isomorphism between Ext-groups formed in these abelian categories. This is the basis for the Morava change-of-rings theorem, various forms of which have been published by Morava [Mor85, §1], Miller–Ravenel [MR77, Thm. 2.10, Thm. 3.10], Hovey–Sadofsky [HS99a, Thm. 3.1], Hovey–Strickland [HS05a, §4] and Naumann [Nau07, §5]. In particular, this applies to the morphism of Hopf algebroids induced by the ring spectrum map $v_n^{-1}MU \to E(n)$.

Theorem 11.1 (Miller–Ravenel [MR77, Thm. 3.10], Hovey–Strickland [HS05a, (4.9)])**.** *There is a natural isomorphism*

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}M) \cong \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_L v_n^{-1}M)$$

for each LB-comodule $v_n^{-1}M$ on which v_n acts invertibly. In particular,

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n^\infty) \cong \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n^\infty)\,.$$

There are short exact sequences of LB-comodules

$$0 \to v_n^{-1}L/(p, \dots, v_m, v_{m+1}^\infty, \dots, v_{n-1}^\infty) \longrightarrow v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty)$$
$$\xrightarrow{v_m} \Sigma^{-|v_m|} v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty) \to 0$$

for $0 \le m < n$, giving rise to long exact sequences connecting the groups

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty))$$
$$\cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty))$$

for $0 \le m \le n$. These can be viewed as a sequence of n algebraic v_m-Bockstein spectral sequences, starting with

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n) \cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \tag{11.1}$$

for $m = n$ and ending with the chromatic $\mathcal{E}_1$-term

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n^\infty) \cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n^\infty)$$

for $m = 0$.

Remark 11.2. A Smith–Toda complex of type n is a finite spectrum $V(n-1)$ with $MU_*(V(n-1)) \cong MU_*/I_n$. Its homology then satisfies $H_*(V(n-1); \mathbb{F}_p) \cong \Lambda(\tau_0, \dots, \tau_{n-1})$. We have $V(-1) = S$ and $V(0) = S/p$ for each prime p. The spectra $V(1)$ exist for $p \ge 3$, the spectra $V(2)$ exist for $p \ge 5$, and the spectra $V(3)$ exist for $p \ge 7$, cf. [Smi71], [Tod71]. No spectra $V(n-1)$ for $n \ge 5$ are known to exist for any prime, cf. [Rav86, (5.6.13)].

When $V(n)$ exists, there exists a map $v_n \colon \Sigma^{2p^n-2}V(n-1) \to V(n-1)$ inducing multiplication by v_n in MU-homology, with homotopy cofiber $V(n)$. We write $v_n^{-1}V(n-1)$ for the mapping telescope. Since $(E(n-1)_*V(n-1) = 0$, so that) $C_{n-1}V(n-1) \simeq V(n-1)$ there is a canonical map

$$v_n^{-1}V(n-1) \longrightarrow M_nV(n-1) \simeq L_nV(n-1)\,,$$

inducing an isomorphism in MU-homology. The starting point (11.1) for the n algebraic Bockstein spectral sequences is thus also the Adams–Novikov $\mathcal{E}_2$-term for

$v_n^{-1}V(n-1)$ and for $L_nV(n-1)$, when these spectra exist:

$$\begin{aligned}\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n) &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(v_n^{-1}V(n-1))) \\ &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(L_nV(n-1)))\,.\end{aligned}$$

Convergence to $\pi_*(L_nV(n-1))$ is known by [HS99a, Thm. 5.3], while convergence to $\pi_*(v_n^{-1}V(n-1)) = v_n^{-1}\pi_*V(n-1)$ is equivalent to the telescope conjecture at height n, which is no longer expected to hold for $n \ge 2$.

In (11.1) we have $E(n)_*/I_n = K(n)_*$, and since $E(n)_*E(n)$ is flat over $E(n)_*$, there is a further change-of-rings isomorphism

$$\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \cong \mathrm{Ext}^{*,*}_{\Sigma(n)_*}(K(n)_*, K(n))\,.$$

Here

$$\Sigma(n)_* := K(n)_*(E(n)) = K(n)_* \otimes_L LB \otimes_L E(n)_* \cong K(n)_* \otimes_L LB \otimes_L K(n)_*$$

since I_n is invariant.

Definition 11.3. Let

$$\Sigma(n)_* = K(n)_*(E(n)) \cong K(n)_* \otimes_L LB \otimes_L K(n)_*$$

be the n-th Morava stabilizer algebra, which is a graded commutative Hopf algebra over $K(n)_*$. (This does not contain the n exterior classes present in $K(n)_*(K(n))$. See Remark 11.6.) Let

$$\Sigma(n)^* = K(n)^*(E(n)) \cong \mathrm{Hom}_{K(n)_*}(\Sigma(n)_*, K(n)_*)$$

be the (Cartier) dual Hopf algebra.

Using formulas from [Rav76a] for the Hopf algebroid structure maps in the p-typical version of (L, LB), modulo the invariant prime ideal I_n, Ravenel made the Hopf algebra structure of $\Sigma(n)$ explicit. It is a sequential colimit of finite étale extensions of the form $A \to A[t_i]/(v_n t_i^{p^n} = v_n^{p^i} t_i)$. ((ETC: Ignoring the grading, and setting $v_n = 1$, this reads $A \to A[t_i]/(t_i^{p^n} = t_i)$, which is étale of degree p^n.))

Proposition 11.4 (Ravenel [Rav76b, Prop. 1.3, Thm. 2.3])**.** *There are algebra isomorphisms*

$$\Sigma(n)_* = K(n)_*[t_i \mid i \ge 1]/(v_n t_i^{p^n} = v_n^{p^i} t_i)$$

and

$$\Sigma(n)^* \otimes \mathbb{F}_{p^n} \cong K(n)^*[[\mathbb{S}_n^0]] \otimes \mathbb{F}_{p^n}$$

(up to grading), where $\mathbb{S}_n^0$ is the strict Morava stabilizer group of H_n.

Remark 11.5. This can be deduced from the Devinatz–Hopkins $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $\hat{L}_nS = L_{K(n)}S \to E_n$, since the sub-extension $\hat{L}_nE(n) \to E_n$ with Galois group $(\mathbb{F}_{p^n})^\times \rtimes \mathrm{Gal}$, and its mod I_n reduction $K(n) \to K_n$, gives isomorphisms

$$\begin{aligned}E_n^*(E_n) &\cong E_n^*\langle\langle \mathbb{G}_n \rangle\rangle \\ E_n^*(E(n)) &\cong E_n^*\langle\langle \mathbb{S}_n^0 \rangle\rangle \\ K_n^*(E(n)) &\cong K_n^*\langle\langle \mathbb{S}_n^0 \rangle\rangle \\ K(n)^*(E(n)) \otimes \mathbb{F}_{p^n} &\cong K(n)^*[[\mathbb{S}_n^0]] \otimes \mathbb{F}_{p^n}\,.\end{aligned}$$

The last step amounts to taking $\mathbb{F}_{p^n}^\times$-invariants, and does not properly preserve the grading.

To summarize: The $\mathcal{E}_2$-term of the Adams–Novikov spectral sequence

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}_{LB}^{s,*}(L, L) \Longrightarrow_s \pi_*(S)$$

is the abutment of the chromatic spectral sequence

$$^{\mathrm{chrom}}\mathcal{E}_1^{n,*,*} = \mathrm{Ext}_{LB}^{*,*}(L, v_n^{-1}L/I_n^\infty) \Longrightarrow_n \mathrm{Ext}_{LB}^{*,*}(L, L)\,.$$

Here layer n of the $\mathcal{E}_1$-term is the abutment of a sequence of n Bockstein spectral sequences starting with

$$\mathrm{Ext}_{LB}^{*,*}(L, v_n^{-1}L/I_n) \cong \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, K(n)_*) \cong \mathrm{Ext}_{\Sigma(n)_*}(K(n)_*, K(n)_*)\,,$$

where $\Sigma(n)_* = K(n)_*E(n)$ is the Morava stabilizer algebra. After a small field extension (and some regrading) this is isomorphic to the continuous group cohomology

$$\mathrm{Ext}_{\Sigma(n)_*}^{*,*}(K(n)_*, K(n)_*) \otimes \mathbb{F}_{p^n} \cong H_c^*(\mathbb{S}_n^0; \mathbb{F}_{p^n}) \otimes K(n)_*$$

of the strict Morava stabilizer group.

((ETC: Truncating the chromatic spectral sequence to the part $^{\mathrm{chrom}}\mathcal{E}_1^{m,*,*}$ with $0 \le m \le n$ calculates the $\mathcal{E}_2$-term $\mathrm{Ext}_{LB}^{*,*}(L, MU_*(L_nS))$ of the Adams–Novikov spectral sequence for $\pi_*(L_nS)$.))

Remark 11.6. Tobias Barthel and Piotr Pstragowski (arXiv:2111.06379) recently proved conditional convergence of the $K(n)$-based Adams spectral sequence

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*(X)) \Longrightarrow_s \pi_{t-s}(\hat{L}_nX)$$

for all spectra X, and strong convergence for $K(n)$-locally (strongly) dualizable X, including $X = S$.

11.1. **Height one.** For $n = 1$, $\mathbb{S}_1^0 = 1 + p\mathbb{Z}_p$, so its group cohomology is easily calculated, recovering Novikov's results for $p > 2$ and for $p = 2$.

Proposition 11.7.

$$H_c^*(\mathbb{S}_1^0; \mathbb{F}_p) \cong H_c^*(1 + p\mathbb{Z}_p; \mathbb{F}_p) \cong \begin{cases} \Lambda(\zeta_1) & \text{for } p \text{ odd,} \\ \Lambda(\zeta_1) \otimes \mathbb{F}_2[\rho_1] & \text{for } p = 2, \end{cases}$$

where ζ_1 and ρ_1 lie in H_c^1, corresponding to homomorphisms $1 + p\mathbb{Z}_p \to \mathbb{F}_p$. Hence

$$\mathrm{Ext}_{\Sigma(1)_*}^{*,*}(K(1)_*, K(1)_*) \cong \begin{cases} \Lambda(\zeta_1) \otimes K(1)_* & \text{for } p \text{ odd,} \\ \Lambda(\zeta_1) \otimes \mathbb{F}_2[\rho_1] \otimes K(1)_* & \text{for } p = 2, \end{cases}$$

with $K(1)_* = \mathbb{F}_p[v_1^{\pm 1}]$, where ζ_1 and ρ_1 lie in $\mathrm{Ext}^{1,0}$ and v_1 lies in $\mathrm{Ext}^{0,2p-2}$.

Corollary 11.8. *For p odd,*

$$\pi_*(L_1S/p) \cong \Lambda(\zeta_1) \otimes \mathbb{F}_p[v_1^{\pm 1}] = \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1^{\pm 1}] \cong \pi_*(J/p)\,,$$

where ζ_1 has degree -1 and $\alpha_1 = \zeta_1 v_1$ has degree $2p - 3$.

((ETC: For $p = 2$ there is an Adams–Novikov differential $d_3(v_1^2) = \eta^3$ leaving

$$\mathcal{E}_\infty = \Lambda(\zeta_1) \otimes \mathbb{F}_2\{1, \eta, \eta^2\} \otimes \mathbb{F}_2\{1, v_1\} \otimes \mathbb{F}_2[v_1^{\pm 4}]\,,$$

with $\eta = \rho_1 v_1$. Draw the chart. This is the associated graded of $\pi_*(L_1S/2) \cong \pi_*(J/2)$. Note the difference in filtrations compared to the Adams spectral sequence for $\pi_*(j/2)$. See Chapter 5, Section 8, Figure 2.))

The passage from $\mathrm{Ext}_{LB}(L, v_1^{-1}L/p) \cong \mathrm{Ext}_{\Sigma(1)_*}(K(1)_*, K(1)_*)$ to

$$\mathrm{Ext}_{LB}(L, v_1^{-1}L/p^\infty) \cong \mathrm{Ext}_{E(1)_*E(1)}(E(1)_*, E(1)_*/p^\infty)$$

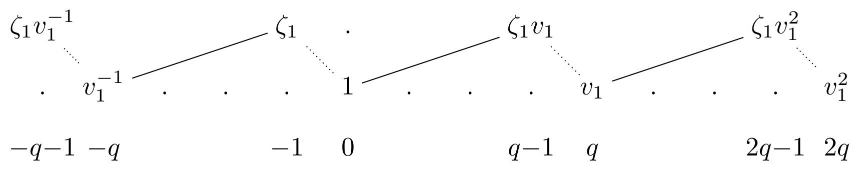


FIGURE 1. Adams–Novikov spectral sequence chart for L_1S/p, with p odd and $q = 2p-2$; multiplications by $\alpha_1 \in \{\zeta_1 v_1\}$ drawn as solid lines

was essentially done by Novikov, suffices to determine $\pi_*(\hat{L}_1 S)$ and $\pi_*(L_1 S)$, and confirms that $\hat{L}_1 S \simeq J_p^\wedge$ at all primes p.

11.2. **Height two.** For $n = 2$, the cohomology of the pro-p-group $\mathbb{S}_2^0$ was calculated in [Rav77, Thms. 3.2, 3.3, 3.4] for the cases $p \geq 5$, $p = 3$ (corrected in the second edition of Ravenel's green book [Rav86, §6.3], following Henn), and $p = 2$ (up to possible multiplicative extensions).

Proposition 11.9. *For $p \geq 5$,*

$$\mathrm{Ext}^{*,*}_{\Sigma(2)_*}(K(2)_*, K(2)_*) \cong \Lambda(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, g_0, g_1, h_{10}g_1 = g_0 h_{11}\} \otimes K(2)_*$$

with $K(2)_* = \mathbb{F}_p[v_2^{\pm 1}]$, where

$$\begin{aligned} \zeta_2 &\in \mathrm{Ext}^{1,0} \\ h_{10} = [t_1] &\in \mathrm{Ext}^{1,2p-2} \\ h_{11} = [t_1^p] &\in \mathrm{Ext}^{1,2p^2-2p} \\ g_0 = \langle h_{10}, h_{11}, h_{10} \rangle &\in \mathrm{Ext}^{2,2p^2+2p-4} \\ g_1 = \langle h_{11}, h_{10}, h_{11} \rangle &\in \mathrm{Ext}^{2,4p^2-2p-2} \\ h_{10}g_1 = g_0 h_{11} &\in \mathrm{Ext}^{3,4p^2-4} \\ v_2 &\in \mathrm{Ext}^{0,2p^2-2} \,. \end{aligned}$$

For odd primes p the passage from

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p, v_1)) \cong \mathrm{Ext}^{s,*}_{\Sigma(2)_*}(K(2)_*, K(2)_*)$$

to

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p, v_1^\infty)) \cong \mathrm{Ext}^{s,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p, v_1^\infty))$$

is carried out by Miller–Ravenel–Wilson [MRW77, §5] for $s = 0$, and partially for $s = 1$, using the LB-comodule extension

$$0 \to v_2^{-1}L/(p, v_1) \longrightarrow v_2^{-1}L/(p, v_1^\infty) \xrightarrow{v_1} \Sigma^{-|v_1|} v_2^{-1}L/(p, v_1^\infty) \to 0 \,.$$

The further passage to

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p^\infty, v_1^\infty)) \cong \mathrm{Ext}^{s,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p^\infty, v_1^\infty))$$

is carried out for $s = 0$ in [MRW77, §6], using the LB-comodule extension

$$0 \to v_2^{-1}L/(p, v_1^\infty) \longrightarrow v_2^{-1}L/(p^\infty, v_1^\infty) \xrightarrow{p} v_2^{-1}L/(p^\infty, v_1^\infty) \to 0 \,.$$

The case $p = 2$ of these calculations is carried out by Shimomura in [Shi81].

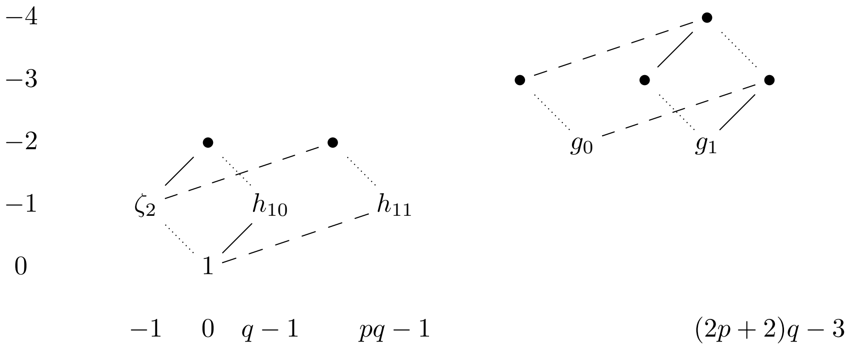


FIGURE 2. Adams–Novikov spectral sequence chart for $L_2V(1)$, with $p \geq 5$ and $q = 2p - 2$, omitting $K(2)_* = \mathbb{F}_p[v_2^{\pm 1}]$; multiplications by $\alpha_1 \in \{h_{10}\}$ are drawn as solid lines, those by $\beta_1' \in \{h_{11}\}$ as dashed lines

For primes $p \geq 5$, Shimomura–Yabe [SY95] determine these Ext groups for all s, which suffices to determine $\pi_*(\hat{L}_2S)$ and $\pi_*(L_2S)$ at these primes. This amazingly complex calculation was revisited by Behrens in [Beh12].

The paper [SW02a] by Shimomoura–Wang obtains these results for $p = 3$. The paper [SW02b] by Shimomoura–Wang obtains the Adams–Novikov $\mathcal{E}_2$-term for $\pi_*(L_2S)$ at $p = 2$. At $p = 2$, recent papers by Beaudry, Bobkova, Goerss and Henn ((ETC: and others?)) make progress towards calculating $\pi_*(L_2S/2)$ and $\pi_*(L_2S)$.

11.3. **Height three.** For $n = 3$ and $p \geq 5$, the cohomology of $\mathbb{S}_3^0$ was additively determined in [Rav77, Thm. 3.8]. Its algebra structure for $p \geq 3$ was calculated by Yamaguchi [Yam92]. Some deductions are made by Kato–Shimomura in [KS12]. See also Gu–Wang–Wu [GWW21].

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