ALGEBRAIC TOPOLOGY III SPRING 2023 CHROMATIC HOMOTOPY THEORY

CHAPTER 11: MORAVA K- AND E-THEORY

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1. Spectral realizations

The following constructions used to rely on Baas–Sullivan theory of bordism with singularities [Baa73a], [Baa73b], but is simplified by working in the module category over a commutative orthogonal ring spectrum. This was first carried out in [EKMM97, Ch. 5].

Definition 1.1. Let R be a commutative orthogonal ring spectrum and let M be an orthogonal R-module. Let $x \in \pi_*(R) = R_*$ have degree $|x|$. Let the R-module M/x be the homotopy cofiber of the multiplication-by-x map, so that there is a homotopy cofiber sequence

$$\Sigma^{|x|} M \xrightarrow{x} M \xrightarrow{i_x} M/x \xrightarrow{j_x} \Sigma^{|x|+1} M \,.$$

Given $x_1, \dots, x_\ell \in R_*$, let

$$M/(x_1, \dots, x_\ell) = M \wedge_R R/x_1 \wedge_R \dots \wedge_R R/x_\ell \,,$$

so that there is a homotopy cofiber sequence

$$\Sigma^{|x_\ell|} M/(x_1, \dots, x_{\ell-1}) \xrightarrow{x_\ell} M/(x_1, \dots, x_{\ell-1}) \longrightarrow M/(x_1, \dots, x_\ell) \longrightarrow \Sigma^{|x_\ell|+1} M/(x_1, \dots, x_{\ell-1}) \,.$$

For a general family of elements $x_\alpha \in R_*$ for $\alpha \in J$, let $M/(x_\alpha \mid \alpha \in J)$ be the colimit over finite subsets $\{\alpha_1, \dots, \alpha_\ell\} \subset J$ of the R-modules $M/(x_{\alpha_1}, \dots, x_{\alpha_\ell})$.

Definition 1.2. An element $x \in R_*$ is not a zero-divisor if multiplication by x acts injectively on R_*. A (finite or infinite) sequence $(x_1, x_2, \dots)$ of elements in R_* is a regular sequence if multiplication by x_i acts injectively on $R_*/(x_1, \dots, x_{i-1})$ for each $i \ge 1$.

Lemma 1.3. *If $x \in R_*$ is not a zero-divisor, then*

$$R_*/(x) \cong \pi_*(R/x) \,,$$

where $(x) = R_* x \subset R_*$. More generally, if $(x_1, x_2, \dots)$ is a regular sequence, then

$$R_*/(x_1, x_2, \dots) \cong \pi_*(R/(x_1, x_2, \dots)) \,,$$

where $(x_1, x_2, \dots) \subset R_*$ is the ideal generated by the listed elements.

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Proof. By induction on ℓ, we can assume that

$$R_*/(x_1, \dots, x_{i-1}) \cong \pi_*(R/(x_1, \dots, x_{i-1}))\,.$$

If x_i acts injectively on this R_*-module, then the long exact sequence in homotopy for the displayed homotopy cofiber sequence simplifies to short exact sequences

$$0 \to \Sigma^{|x_i|} R_*/(x_1, \dots, x_{i-1}) \xrightarrow{x_i} R_*/(x_1, \dots, x_{i-1}) \longrightarrow \pi_*(R/(x_1, \dots, x_i)) \to 0\,.$$

□

Definition 1.4. Let R be a commutative orthogonal ring spectrum and let M be an orthogonal R-module. Let $y \in \pi_*(R) = R_*$ have degree $|y|$. Let the R-module $y^{-1}M = M[1/y] = M[y^{-1}]$ be the homotopy colimit of the sequence

$$M \xrightarrow{y} \Sigma^{-|y|}M \xrightarrow{y} \Sigma^{-2|y|}M \longrightarrow \dots \longrightarrow y^{-1}M\,.$$

Theorem 1.5 ([EKMM97, Thm. VIII.2.2])**.** *The R-module $y^{-1}R$ is equivalent to an essentially unique commutative R-algebra.*

The commutative R-algebra in question is realized as the Bousfield localization of R in commutative R-algebras, with respect to the homology theory in R-modules given by $y^{-1}R$.

Theorem 1.6 ([Str99, Thm. 2.6])**.** *Let R be a commutative orthogonal ring spectrum with $\pi_*(R) = R_*$ concentrated in even degrees. If A_* is a localized regular quotient of R_*, and $1/2 \in A_*$, then there exists a unique (strong realization) homotopy commutative R-ring spectrum A with $\pi_*(A) \cong A_*$.*

((ETC: Recall "strong realization".))

For similar results about localizations of $\mathbb{E}_n$ ring spectra, see Lurie's "Higher Algebra" (for $n = 1$) and Mathew–Naumann–Noel [MNN15, App. A] (for $n \geq 2$). In general, there is extensive literature on the problem of finding $\mathbb{A}_\infty = \mathbb{E}_1$- or higher $\mathbb{E}_n$-realizations of a given (ring) spectrum, or proving that such more structured products do not exist.

We apply Strickland's theorem in the case $R = MU$, in which case $R_* = MU_*$ is integral, so that no $x \neq 0$ divides zero.

Definition 1.7. For each prime p and height $1 \leq n < \infty$ let

$$MU/I_n = MU/(p, v_1, \dots, v_{n-1})$$

be the MU-module with $\pi_*(MU/I_n) \cong \pi_*(MU)/I_n \cong L/I_n$, and similarly for $n = \infty$. Let

$$v_n^{-1}MU/I_n$$

be the localized MU-module with $\pi_*(v_n^{-1}MU/I_n) \cong v_n^{-1}\pi_*(MU)/I_n \cong v_n^{-1}L/I_n$.

By Strickland's theorem, MU/I_n and $v_n^{-1}MU/I_n$ admit unique structures as homotopy commutative MU-ring spectra, as long as $p \neq 2$. ((ETC: For $p = 2$, there are two (opposite) structures as homotopy associative MU-ring spectra.))

Proposition 1.8. *MU/I_n and $v_n^{-1}MU/I_n$ are flat ring spectra, with*

$$(MU/I_n)_*(MU/I_n) \cong LB/I_n \otimes \Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})$$

and

$$(v_n^{-1}MU/I_n)_*(v_n^{-1}MU/I_n) \cong v_n^{-1}LB/I_n \otimes \Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})\,.$$

Here $\bar\tau_i$ in degree $2p^i - 1$ maps under $MU/I_n \to H\mathbb{F}_p$ to the class with the same name in $(H\mathbb{F}_p)_*(H\mathbb{F}_p) = \mathscr{A}_*$.

Remark 1.9. The flat ring spectrum $D = MU/I_n$ is a spectral realization of $\operatorname{obj}\mathcal{FGL}_s^{\geq n}$, but its associated Hopf algebroid (D_*, D_*D) is a nilpotent thickening of the Hopf algebroid $(L/I_n, LB/I_n)$ classifying $\mathcal{FGL}_s^{\geq n}$. Likewise, $E = v_n^{-1}MU/I_n$ is a flat spectral realization of $\operatorname{obj}\mathcal{FGL}_s^n$, but its associated Hopf algebroid (E_*, E_*E) is a nilpotent thickening of the Hopf algebroid $(v_n^{-1}L/I_n, v_n^{-1}LB/I_n)$ classifying $\mathcal{FGL}_s^n$. In other words, the algebraic Hopf algebroids are the reductions (modulo nilpotent elements) of these non-reduced topological Hopf algebroids.

2. Morava K-theory

In the early in 1970s, Morava introduced spectra $K(n)$ giving topological realizations of the Honda formal group law H_n, giving the (unique) geometric point in $\mathcal{M}_{\mathrm{fg}}^n$. Let

$$(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1) = (p, \dots, v_{n-1}, v_{n+1}, \dots, \tilde{b}_m \mid m \neq p^k - 1)$$

be a regular sequence (ordered by degree, say) generating the kernel of the homomorphism $L \to \mathbb{F}_p[v_n] \subset \mathbb{F}_p[v_n^{\pm 1}]$ classifying H_n.

Definition 2.1. For each prime p and height $1 \leq n < \infty$ let the n-th connective and periodic Morava K-theory spectra be the MU-module spectra

$$k(n) = MU/(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1)$$

and

$$K(n) = v_n^{-1}k(n) = v_n^{-1}MU/(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1)$$

with

$$\pi_* k(n) \cong \mathbb{F}_p[v_n] \qquad \text{and} \qquad \pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]\,,$$

respectively. Then

$$H_*(k(n); \mathbb{F}_p) \cong \Lambda(\bar{\tau}_j \mid j \neq n) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]$$

and

$$H^*(k(n); \mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_n) = \mathscr{A}/\mathscr{A}\{Q_n\}\,.$$

By Strickland's theorem, $k(n)$ and $K(n)$ admit unique structures as homotopy commutative MU-ring spectra, as long as $p \neq 2$. ((ETC: For $p = 2$, there are two (opposite) structures as homotopy associative MU-ring spectra.))

Robinson [Rob89, Thm. 2.3] developed an obstruction theory to show that $K(n)$ admits the structure of an $\mathbb{A}_\infty = \mathbb{E}_1$-ring spectrum, and Angeltveit [Ang11] showed that $K(n)$ is uniquely determined up to equivalence in this category, i.e., as an associative orthogonal ring spectrum. For $1 \leq n < \infty$ is does not admit an $\mathbb{E}_2$-ring structure, as can be seen from the Dyer–Lashof operations in its homology.

When $n = 1$, there are splittings

$$ku/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} k(1) \qquad \text{and} \qquad KU/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)\,,$$

so $K(1)$ is a direct summand of mod p complex K-theory. By convention, we let $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, matching the definitions of H_0 and H_∞.

Remark 2.2. There are ring spectrum maps $MU \to K(n)$ inducing the ring homomorphisms $L \cong MU_* \to K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ classifying the Honda formal group law H_n. The corresponding maps

$$\operatorname{Spec}(K(n)_*) \xrightarrow{H_n} \operatorname{Spec}(L) \longrightarrow \mathcal{M}_{\mathrm{fg}}\,,$$

for all p and $0 \le n \le \infty$, then realize all geometric points of $\mathcal{M}_{\mathrm{fg}}$. In particular, those for a fixed p realize all geometric points of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$, and those for $n \ge 1$ realize all geometric points of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{F}_p = \mathcal{M}_{\mathrm{fg}}^{\ge 1}$.

Morava K-theory is about as accessible to calculation as (co-)homology with field coefficients, because of the following Künneth and universal coefficient isomorphisms.

Theorem 2.3. *For any spectra X and Y the canonical maps*

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \xrightarrow{\cong} K(n)_*(X \wedge Y)$$

and

$$K(n)^*(X) \xrightarrow{\cong} \operatorname{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$$

are isomorphisms.

Proof. This follows from the Tor- and Ext-spectral sequences for $K(n) \wedge X \wedge_{K(n)} K(n) \wedge Y \simeq K(n) \wedge X \wedge Y$ and $F_{K(n)}(K(n) \wedge X, K(n)) \simeq F(X, K(n))$, since $K(n)_*$ is a graded field, so that each $K(n)_*$-module is free. □

Remark 2.4. Since $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ is a graded field, each $K(n)_*$-module is free, so (for p odd) $K(n)$ is a flat ring spectrum. ((ETC: Discuss relation of the associated Hopf algebra $(K(n)_*, K(n)_*K(n))$ to the one classifying $\mathcal{B}\operatorname{Aut}_s(H_n/R)$ over $R = \bar{\mathbb{F}}_p$. Also for $E(n)_*E(n)$, later. Cleaner for $(K_n)_*(K_n)$ or $(E_n)_*(E_n)$.))

Remark 2.5. A key feature of $K(n)$ is that its complex orientation, corresponding to a ring spectrum map $MU \to K(n)$ in the homotopy category, defines the Honda formal group law H_n, with p-series

$$[p]_{K(n)}(y) = [p]_{H_n}(y) = v_n y^{p^n} \in K(n)^*[[y]]\,.$$

This means that in the fiber sequence

$$BC_p \longrightarrow \mathbb{C}P^\infty \xrightarrow{[p]} \mathbb{C}P^\infty\,,$$

where $[p]$ classifies $(\gamma^1)^{\otimes p}$, the induced homomorphism

$$K(n)^*(BC_p) \longleftarrow K(n)^*(\mathbb{C}P^\infty) \cong K(n)^*[[y]]$$

maps $v_n y^{p^n}$ to zero. It follows from the Gysin sequence in $K(n)$-cohomology (compare Chapter 4, Thm. 7.1) that

$$K(n)^*(BC_p) \cong K(n)^*[[y]]/(v_n y^{p^n}) \cong K(n)^*[y]/(y^{p^n})$$

is a p^n-dimensional $K(n)^*$-algebra. (For complex cobordism, this calculation goes back to Stong or Landweber around 1970.) On one hand, this illustrates how the formal group law or p-series enters in calculations. It also shows that the structure of $K(n)^*(BC_p)$ depends on the height n, interpolating between

$$K(0)^*(BC_p) = H^*(BC_p; \mathbb{Q}) = \mathbb{Q}$$

and

$$K(\infty)^*(BC_p) = H^*(BC_p;\mathbb{F}_p) = \begin{cases} \mathbb{F}_2[x] & \text{for } p=2, \\ \Lambda(x)\otimes\mathbb{F}_p[y] & \text{for } p \text{ odd.} \end{cases}$$

3. Morava E-theory

In the early 1970s (cf. Morava: "The moduli variety for formal groups", November 22, 1972), Morava interpreted the Lubin–Tate deformation theory [LT66] for formal group laws of finite height as exhibiting a normal bundle, or formal neighborhood, at the point $H_n\colon \operatorname{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\text{fg}}^{\geq n} \subset \mathcal{M}_{\text{fg}}$. This led to a ring spectrum E, now called Morava E-theory, with a map

$$E \longrightarrow K(n)$$

corresponding to the inclusion of H_n in (a universal covering space of) this formal neighborhood. Other mathematicians at the time preferred to reformulate this in more traditional terms, leading to a version $E(n)$ of Morava E-theory with coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1,\dots,v_{n-1},v_n^{\pm1}]$$

having $K(n)_*$ as a residue field at the maximal ideal $I_n = (p,\dots,v_{n-1})$.

The later work of Devinatz–Hopkins and Goerss–Hopkins–Miller led to version E_n of Morava E-theory that is an $\mathbb{E}_\infty$ ring spectrum, i.e., a commutative orthogonal ring spectrum, with

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1,\dots,u_{n-1}]][u^{\pm1}]$$

having the finite extension $\pi_*(K_n) = \mathbb{F}_{p^n}[u^{\pm1}]$ of $K(n)_*$ as its residue field. Here $\pi_0(E_n) = W(\mathbb{F}_{p^n})[[u_1,\dots,u_{n-1}]]$ is the commutative ring classifying Lubin–Tate's universal deformation, and Morava's original E-theory $E \simeq E_n^{\text{Gal}}$ is realized as the homotopy fixed points for an action on E_n by the Galois group $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$. ((ETC: Here we suppress a distinction between 2-periodic and $(2p^n-2)$-periodic theories.))

Since the rings $E(n)_*$ can be presented using only the subset of algebra generators for $\pi_*(MU)_{(p)}$ given by the classes v_m for $m \geq 0$, it is tempting to simplify the algebra by discarding all the other algebra generators. This can be achieved using the Brown–Peterson spectrum BP.

Recall from Chapter 6, Theorem 6.1, that

$$H_*(MU;\mathbb{F}_p) \cong \mathscr{P}_* \otimes \mathbb{F}_p[\tilde{b}_m \mid m \neq p^k-1]$$

and

$$\pi_*(MU)_{(p)} \cong \mathbb{Z}_{(p)}[v_i \mid i \geq 1] \otimes \mathbb{Z}_{(p)}[\tilde{b}_m \mid m \neq p^k-1]\,,$$

where

$$\mathscr{P}_* = \mathbb{F}_p[\xi_i \mid i \geq 1] \subset \mathscr{A}_*$$

is the sub Hopf algebra dual to the quotient algebra $\mathscr{P} = \mathscr{A}//\mathscr{E}$ generated by the Steenrod power operations P^i for $i \geq 1$, and

$$\mathbb{F}_p[\tilde{b}_m \mid m \neq p^k-1] = PH_*(MU;\mathbb{F}_p) \subset H_*(MU;\mathbb{F}_p)$$

is the subalgebra of $\mathscr{A}_*$-comodule primitives. Brown–Peterson [BP66] constructed a spectrum (now denoted) BP such that $H_*(BP;\mathbb{F}_p) \cong \mathscr{P}_*$ as $\mathscr{A}_*$-comodules.

Equivalently, $H^*(BP;\mathbb{F}_p) \cong \mathscr{P} \cong \mathscr{A}//\mathscr{E}$ as $\mathscr{A}$-modules. We can now realize BP as an MU-module by setting

$$BP = MU_{(p)}/(\tilde{b}_m \mid m \neq p^k - 1)\,.$$

Then

$$BP_* = \pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots] =: V\,.$$

It then follows that

$$MU_{(p)} \simeq \bigvee_{\tilde{b}^I} \Sigma^{|\tilde{b}^I|} BP$$

where $\tilde{b}^I$ ranges over a monomial basis for $\mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1]$. In particular, $MU_{(p)*}(X) = 0$ if and only if $BP_*(X) = 0$, for any spectrum X.

By Strickland's theorem, BP is a homotopy commutative ring spectrum, at least for p odd. Quillen gave a more specific construction of BP as the image of a homotopy idempotent ring spectrum map $e\colon MU_{(p)} \to MU_{(p)}$, i.e., as the homotopy colimit of

$$MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} MU_{(p)} \longrightarrow \dots \longrightarrow BP\,.$$

The ring homomorphism $MU_* \to MU_{(p)*} \to BP_* = \pi_*(BP)$ classifies the universal p-typical formal group law, in the sense of Cartier ((ETC: reference)), and $BP_* \to MU_{(p)*}$ classifies the p-typification of the p-localized Lazard formal group law.

Basterra–Mandell [BM13] showed that BP admits a unique $\mathbb{E}_4$ ring structure, hence is an orthogonal ring spectrum that is homotopy commutative, while Lawson [Law18] and Senger ((ETC: arXiv:1710.09822)) showed that BP cannot be realized as an $\mathbb{E}_\infty$ ring spectrum, hence also not as a commutative orthogonal ring spectrum.

((ETC: Discuss Hopf algebroid structure of $(BP_*, BP_*BP) \cong (V, VT)$, classifying the full subgroupoid of $\mathcal{FGL}_s(R)$ generated by p-typical formal group laws over R, for any commutative $\mathbb{Z}_{(p)}$-algebra R. Here $V = \mathbb{Z}_{(p)}[v_i \mid i \geq 1]$, $T = \mathbb{Z}_{(p)}[t_k \mid k \geq 1]$ and $VT = V \otimes T = V[t_k \mid k \geq 1]$, with $|t_k| = 2p^k - 2$.))

The following BP-analogues of MU/I_n and $v_n^{-1}MU/I_n$ were discussed by Johnson–Wilson [JW75]. As a mnemonic, the letter B contains both P and the inverse/upside-down P.

Definition 3.1. Let

$$P(n) = MU/I_n \wedge_{MU} BP \simeq BP/I_n$$

be the MU- and BP-module spectrum with

$$\pi_* P(n) \cong \mathbb{F}_p[v_n, v_{n+1}, \dots]\,.$$

Then

$$H_*(P(n);\mathbb{F}_p) \cong \Lambda(\bar{\tau}_0, \dots, \bar{\tau}_{n-1}) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]$$

and

$$H^*(P(n);\mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_n, Q_{n+1}, \dots)\,.$$

Also let

$$B(n) = v_n^{-1}MU/I_n \wedge_{MU} BP \simeq v_n^{-1}BP/I_n$$

be the MU- and BP-module spectrum with

$$\pi_* B(n) \cong \mathbb{F}_p[v_n^{\pm 1}, v_{n+1}, \dots]\,.$$

The Morava E-theory, complementary to $MU \to v_n^{-1}MU/I_n$ at $MU \to K(n)$, can also be viewed as being complementary to $BP \to v_n^{-1}BP/I_n = B(n)$, and more-or-less realized by the theory $E(n) = v_n^{-1}BP\langle n\rangle$ discussed in [JW73] and [JY80].

Definition 3.2. Let the n-th truncated Brown–Peterson spectrum

$$BP\langle n\rangle = BP/(v_{n+1}, v_{n+2}, \dots)$$

be an MU- and BP-module spectrum with

$$\pi_* BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]\,.$$

Then

$$H_*(BP\langle n\rangle; \mathbb{F}_p) \cong \Lambda(\bar\tau_{n+1}, \bar\tau_{n+2}, \dots) \otimes \mathbb{F}_p[\xi_i \mid i \ge 1]$$

and

$$H^*(BP\langle n\rangle; \mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_0, \dots, Q_n)\,.$$

Let

$$E(n) = v_n^{-1}BP\langle n\rangle = v_n^{-1}BP/(v_{n+1}, v_{n+2}, \dots)$$

be an MU- and BP-module spectrum with

$$\pi_* E(n) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]\,.$$

Again, these are homotopy commutative ring spectra by Strickland's theorem, except for $p = 2$, for which one should see [Nas02].

When $n = 1$, there are splittings

$$ku_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} BP\langle 1\rangle \qquad \text{and} \qquad KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)\,,$$

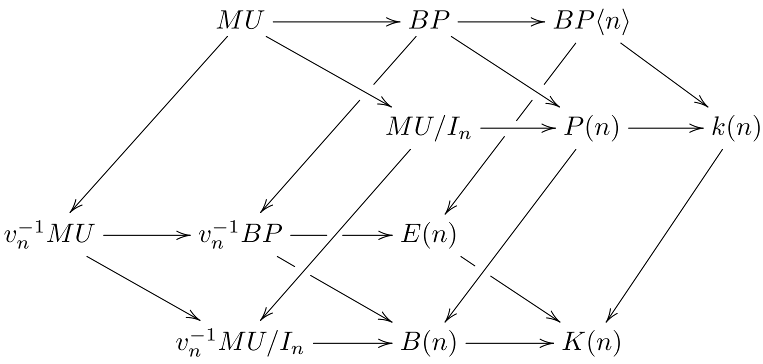
and the p-local Adams summands $\ell = BP\langle 1\rangle$ and $L = E(1)$ of $ku_{(p)}$ and $KU_{(p)}$ all admit unique $\mathbb{E}_\infty$ ring structures [BR05], [BR08].

After p-completion, Angeltveit–Lind [AL17] showed that the spectrum $BP\langle n\rangle$ is uniquely determined by its cohomology $\mathscr{A}$-module.

One should beware that there are many different possible choices of regular sequences $(v_{n+1}, v_{n+2}, \dots)$, so that the spectra $BP\langle n\rangle$ and $E(n)$ are not well-defined, especially as MU- or BP-ring spectra. ((ETC: One might speak of a "form" of $BP\langle n\rangle$ or $E(n)$.))

Hahn–Wilson [HW22] recently proved that for each prime p and height n there exists an $\mathbb{E}_3$ BP-algebra structure on $BP\langle n\rangle$. This makes sense, because BP has an $\mathbb{E}_4$ ring structure. In particular, $BP\langle n\rangle$ admits an $\mathbb{E}_3$ ring structure.

In the following diagram of ring spectra, each square induces a pushout square of (evenly graded) commutative rings after passage to homotopy rings.



4. Nilpotence theorems

Here are two classical theorems about $\pi_*(S)$ as a graded abelian group, and as a graded commutative ring.

Theorem 4.1 (Hurewicz, Serre [Ser51]).

$$\pi_{d+n}(S^n) \cong \begin{cases} 0 & \textit{for } d < 0, \\ \mathbb{Z} & \textit{for } d = 0, \\ \mathbb{Z} \oplus \textit{(finite)} & \textit{for } d = n-1,\ n \textit{ even} \\ \textit{(finite)} & \textit{otherwise.} \end{cases}$$

Hence

$$\pi_d(S) \cong \begin{cases} 0 & \textit{for } d < 0, \\ \mathbb{Z} & \textit{for } d = 0, \\ \textit{(finite)} & \textit{otherwise.} \end{cases}$$

In particular the Hurewicz homomorphism $\pi_*(S) \to \mathbb{Z}$ is a rational isomorphism, with torsion kernel and trivial cokernel.

Serre's proof uses the Serre spectral sequence for fibrations related to the Whitehead covers of S^n.

Theorem 4.2 (Nishida [Nis73]). *Each $f \in \pi_d(S)$ with $d \neq 0$ is nilpotent in $\pi_*(S)$. Hence the kernel of the Hurewicz homomorphism is the nilradical of $\pi_*(S)$, so that $\pi_*(S)_{red} \cong \mathbb{Z}$.*

Nishida's proof uses the structured ($\mathbb{H}_\infty$) commutativity of the sphere spectrum, which shows that suitable extended j-fold powers of spheres admit a retraction to the (ordinary) j-fold smash power of that sphere.

One way to interpret Nishida's theorem is to say that any map $f\colon \Sigma^d S \to S$ that induces zero in integral (or rational) homology is nilpotent with respect to composition, in the sense that

$$f^N = f \circ \cdots \circ f\colon \Sigma^{Nd} S \longrightarrow S$$

is null-homotopic for $N \gg 0$. On the other hand, Adams [Ada66] had exhibited maps

$$v_1\colon \Sigma^{2p-2} S/p \longrightarrow S/p$$

for odd primes p (and $v_1^4\colon \Sigma^8 S/2 \to S/2$ at $p = 2$) that induce zero in integral homology, but induce nonzero isomorphisms

$$v_1^*\colon KU^*(S/p) \xrightarrow{\cong} KU^*(\Sigma^{2p-2}S/p)\,,$$

in topological K-theory, and which are therefore not nilpotent with respect to composition. (This follows, since $(v_1^N)^*$ is a nonzero isomorphism, for each N.)

Based on calculations [MRW77] with the (MU- or BP-based) Adams–Novikov spectral sequence, Ravenel (lecture at 1977 Evanston conference, published as [Rav84, Conj. 10.1]) conjectured that inducing zero in complex bordism would be sufficient to ensure that a map

$$f\colon \Sigma^d X \longrightarrow X\,,$$

with X a finite CW complex or spectrum, is nilpotent.

Several years later, this conjecture was famously proved by Devinatz–Hopkins–Smith. Both of the following two statements generalize Nishida's nilpotence theorem.

Theorem 4.3 (Devinatz–Hopkins–Smith [DHS88, Thm. 1, Cor. 2])**.**
(a) Let R be a ring spectrum (not necessarily associative) in the homotopy category. The kernel of the MU Hurewicz homomorphism

$$h_{MU}\colon \pi_*(R) \longrightarrow MU_*(R)$$

consists of nilpotent elements.
(b) Let $f\colon \Sigma^d X \to X$ be a self-map of a finite spectrum. If $MU_*(f) = 0$ then f is nilpotent.

See also [Rav92, Ch. 9].

Brief outline of thumbnail sketch of proof. Here (b) is deduced from (a) by considering the endomorphism ring spectrum

$$R = F(X, X) \simeq X \wedge DX\,,$$

where $DX = F(X, S)$ denotes the Spanier–Whitehead dual. It suffices to prove (a) when R is an orthogonal ring spectrum that is connective of finite type. In this case, Devinatz–Hopkins–Smith use the Thom ($\mathbb{E}_2$ ring) spectra

$$X(n) = \mathrm{Th}(\xi \downarrow \Omega SU(n))$$

of the virtual complex vector bundles classified by the (double loop) maps

$$\xi\colon \Omega SU(n) \to \Omega SU \simeq BU\,.$$

Here $S = X(1)$ and $X(\infty) = MU$, and the MU Hurewicz homomorphism factors as a chain

$$\pi_*(R) \longrightarrow \dots \longrightarrow X(n)_*(R) \longrightarrow X(n+1)_*(R) \longrightarrow \dots \longrightarrow MU_*(R)\,.$$

There is a Thom isomorphism

$$H_*(\Omega SU(n)) \cong H_*(X(n)) \cong \mathbb{Z}[b_1, \dots, b_{n-1}]\,,$$

compatible with the Thom isomorphism $H_*(BU) \cong H_*(MU) \cong \mathbb{Z}[b_k \mid k \ge 1]$ that we discussed in Chapter 6. Let $f \in \pi_*(R)$. The inductive step is then to prove that $h_{X(n)}(f) \in X(n)_*(R)$ is nilpotent if (and only if) $h_{X(n+1)}(f) \in X(n+1)_*(R)$ is nilpotent. This is then addressed by interpolating between $\Omega SU(n)$ and $\Omega SU(n{+}1)$ by means of homotopy pullbacks

$$\begin{array}{ccccc} \Omega SU(n) & \longrightarrow & \tilde{J}_m S^{2n} & \longrightarrow & \Omega SU(n+1) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & J_m S^{2n} & \longrightarrow & \Omega S^{2n+1} \end{array}$$

over the standard filtration of the James construction model for $\Omega S^{2n+1} \simeq JS^{2n}$, and letting $F_m X(n+1) = \mathrm{Th}(\xi \downarrow \tilde{J}_m S^{2n})$ for $0 \le m \le \infty$. Here

$$H_*(\tilde{J}_m S^{2n}; \mathbb{F}_p) \cong H_*(F_m X(n+1); \mathbb{F}_p) \cong \mathbb{F}_p[b_1, \dots, b_{n-1}]\{1, b_n, \dots, b_n^m\}$$

is coalgebraically best behaved when $m = p^k - 1$ for some $k \ge 0$. Note that $X(n) = F_0 X(n+1)$. The proof proceeds in three steps:

(1) If the image of f in $X(n{+}1)_*(R)$ is nilpotent, then $F_{p^k-1} X(n{+}1) \wedge f^{-1}R \simeq *$ for k sufficiently large. This follows from a vanishing line in the $X(n+1)$-based Adams spectral sequence.

(2) If $F_{p^k-1} X(n+1) \wedge f^{-1}R \simeq *$ then $F_{p^{k-1}-1} X(n+1) \wedge f^{-1}R \simeq *$, for each $k \ge 1$. More precisely, the class of acyclic spectra for $F_{p^k-1}X(n+1)$-homology is the same for all values of k. (This is the hard part, uses the Snaith splitting of $\Omega^2 S^{2m+1}$, and connects to the theory of Bousfield classes.)

(3) If $X(n) \wedge f^{-1}R \simeq *$ then the image of f in $X(n)_*(R)$ is nilpotent. □

The Devinatz–Hopkins–Smith nilpotence theorem expresses how the functor $X \mapsto MU_*(X)$ to MU_*-modules (or MU_*MU-comodules) is almost faithful on (endo-)morphisms on the subcategory of finite spectra

$$\mathrm{Ho}(\mathcal{S}p^\omega) \subset \mathrm{Ho}(\mathcal{S}p) \overset{MU_*(-)}{\longrightarrow} MU_*MU{-}\,\mathrm{coMod} \to MU_*{-}\,\mathrm{Mod}\,,$$

where "almost" means up to nilpotence. ((ETC: Define the full subcategory $\mathrm{Ho}(\mathcal{S}p^\omega) \simeq \mathcal{SW}$ of finite spectra.))

It is often difficult to fully calculate complex bordism groups, while Morava K-groups are easier to compute, mainly because their coefficient rings are graded fields, leading to universal coefficient and Künneth theorems. Recall that $K(0) = H\mathbb{Q}$, $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ and $K(\infty) = H\mathbb{F}_p$. Hence the following extension of the nilpotence theorem can be more effective.

Theorem 4.4 (Hopkins–Smith [HS98, Thm. 3])**.**
(a) Let R be a p-local ring spectrum. An element $f \in \pi_*(R)$ is nilpotent if (and only if) $h_{K(n)}(f) \in K(n)_*(R)$ is nilpotent for each $0 \le n \le \infty$.
(b) Let $f\colon \Sigma^d X \to X$ be a self-map of the p-localization of a finite spectrum. Then f is nilpotent if (and only if) $K(n)_*(f)$ is nilpotent for each $0 \le n \le \infty$.

This has the following cute consequence.

Definition 4.5. A spectral (skew-)field is a non-contractible ring spectrum R such that $R_*(X)$ is a free R_*-module for all spectra X.

Proposition 4.6 ([HS98, Prop. 1.9])**.** *Let R be a spectral field. Then R has the homotopy type of a wedge sum of suspensions of $K(n)$ for some $0 \le n \le \infty$.*

Proof. Since $1 \in \pi_*(R)$ is not nilpotent, there exists a prime p and a height $0 \le n \le \infty$ such that $1 \in K(n)_*(R)$ is not nilpotent. Hence $K(n) \wedge R$ is not contractible.

Since $K(n)$ and R are spectral fields, a suspension of R is a retract of $K(n) \wedge R$, which is a wedge sum of suspensions of $K(n)$. It follows (cf. [HS98, Prop. 1.10]) that R is also such a wedge sum of suspensions. □

In the presence of sufficiently much commutativity, the additional strength of complex bordism over ordinary homology is no longer needed. The following result was conjectured by Peter May in [BMMS86, Conj. II.2.7]. An $\mathbb{H}_\infty$ ring structure is slightly weaker than an $\mathbb{E}_\infty$ ring structure, which is essentially the same as commutativity for orthogonal ring spectra.

Theorem 4.7 (Mathew–Naumann-Noel [MNN15, Thm. A])**.** *Suppose that R is an $\mathbb{H}_\infty$ ring spectrum and $f \in \pi_*(R)$ is in the kernel of the Hurewicz homomorphism $h = h_{\mathbb{Z}} \colon \pi_*(R) \to H_*(R;\mathbb{Z})$. Then f is nilpotent.*

5. Quasi-coherent sheaves

Let A be a commutative ring. Each A-module M determines a quasi-coherent sheaf $M^\sim$ over $\mathrm{Spec}(A)$, with sections over $g \colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ equal to the R-module given by the base change (= pullback)

$$M^\sim(R) = g^*(M) = R \otimes_A M \,.$$

Here A acts (from the right) on R via the ring homomorphism $g \colon A \to R$. It follows that for any A-algebra homomorphism $k \colon R \to T$ the induced T-module homomorphism

$$T \otimes_R M^\sim(R) \xrightarrow{\cong} M^\sim(T)$$

is an isomorphism, which is the defining condition for this module sheaf to be quasi-coherent. Conversely, each quasi-coherent sheaf over $\mathrm{Spec}(A)$ is isomorphic to $M^\sim$ for an A-module M, so there is an equivalence of categories

$$\begin{aligned} A - \mathrm{Mod} &\xrightarrow{\simeq} \mathrm{QCoh}(\mathrm{Spec}(A)) \\ M &\longmapsto M^\sim \,. \end{aligned}$$

Both sides of this equivalence depend covariantly on A, or contravariantly on $\mathrm{Spec}(A)$, so that a ring homomorphism $g \colon A \to B$ takes the A-module M to the B-module $B \otimes_A M$, and $(B \otimes_A M)^\sim \cong g^*(M^\sim)$.

The base change g^* along $g \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is left adjoint to the restriction functor $g_* \colon B - \mathrm{Mod} \to A - \mathrm{Mod}$ (or $\mathrm{QCoh}(\mathrm{Spec}(B)) \to \mathrm{QCoh}(\mathrm{Spec}(A))$) taking a B-module N to the same abelian group with the A-module structure given by the composite

$$A \otimes N \xrightarrow{g \otimes \mathrm{id}} B \otimes N \xrightarrow{\lambda} N \,.$$

The moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ represents the groupoid-valued functor

$$\begin{aligned} \mathcal{A}ff^{op} &\longrightarrow \mathcal{G}pd \\ \mathrm{Spec}(R) &\longmapsto \{\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fgl}}\} \cong \mathcal{FGL}_s(R) \,. \end{aligned}$$

The nerve functor $\mathcal{C} \mapsto N\mathcal{C}$ gives a full and faithful embedding of (categories or) groupoids in simplicial sets, so we can also think about the simplicial set-valued functor

$$\begin{aligned} \mathcal{A}ff^{op} &\longrightarrow s\,\mathcal{S}\mathrm{et} \\ \mathrm{Spec}(R) &\longmapsto N\,\mathcal{FGL}_s(R) \,, \end{aligned}$$

where $N\,\mathcal{FGL}_s(R)$ is isomorphic to the simplicial set

$$\mathrm{Hom}(L,R) \rightleftarrows \mathrm{Hom}(LB,R) \rightleftarrows \mathrm{Hom}(LBB,R) \rightleftarrows \dots .$$

It is represented by the simplicial affine scheme

$$\mathrm{Spec}(L) \rightleftarrows \mathrm{Spec}(LB) \rightleftarrows \mathrm{Spec}(LBB) \rightleftarrows \dots .$$

Here some of the face operators are given by $\eta_L \colon L \to LB$, $\eta_R \colon L \to LB$ and $\psi \colon LB \to LB \otimes_L LB = LBB$, while one of the degeneracy operators is given by $\epsilon \colon LB \to L$. The remaining operators are obtained from these by tensoring with identity morphisms. The nerve construction takes (the moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ or) moduli stack $\mathcal{M}_{\mathrm{fg}}$ to the homotopy colimit of this simplicial scheme. Since the simplicial scheme is generated by the Hopf algebroid structure maps, relating simplicial degrees $q \in \{0,1,2\}$, this homotopy (or ∞-categorical) colimit is in fact a 2-categorical colimit.

Passing to sheaves, we define the category

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

of quasi-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$ to be the corresponding homotopy (or ∞-categorical) limit of the diagram of categories

$$\mathrm{QCoh}(\mathrm{Spec}(L)) \rightleftarrows \mathrm{QCoh}(\mathrm{Spec}(LB)) \rightleftarrows \mathrm{QCoh}(\mathrm{Spec}(LBB)) \rightleftarrows \dots ,$$

which is in fact the 2-categorical limit. In more elementary terms, this is the limit of the diagram of categories

$$L-\mathrm{Mod} \rightleftarrows LB-\mathrm{Mod} \rightleftarrows LBB-\mathrm{Mod} \rightleftarrows \dots .$$

This is a cosimplicial diagram, with some of the coface operators given by base change along η_L, η_R and ψ and one of the codegeneracy operators given by base change along ϵ.

An object in this limit can be given as a sequence of objects

$$\begin{aligned} &M^0 \in L-\mathrm{Mod}\,, \\ &M^1 \in LB-\mathrm{Mod}\,, \\ &M^2 \in LBB-\mathrm{Mod}\,, \dots \end{aligned}$$

together with isomorphisms

$$\begin{aligned} &M^0 \cong \epsilon^* M^1\,, \\ &M^1 \cong \eta_L^* M^0 \overset{\bar\nu}{\cong} \eta_R^* M^0 \cong (\epsilon \otimes \mathrm{id})^* M^2 \cong (\mathrm{id} \otimes \epsilon)^* M^2\,, \\ &M^2 \cong (\eta_L \otimes \mathrm{id})^* M^1 \cong \psi^* M^1 \cong (\mathrm{id} \otimes \eta_R)^* M^1\,, \dots \end{aligned}$$

subject to coherence conditions. The key data here are the L-module $M = M^0$ and the LB-module isomorphism

$$\bar\nu \colon \eta_L^* M \xrightarrow{\cong} \eta_R^* M\,,$$

making the (L- and LBB-module) diagrams

$$\begin{array}{ccc}
\epsilon^*\eta_L^* M & \xrightarrow{\epsilon^*\bar\nu} & \epsilon^*\eta_R^* M \\
\| & & \| \\
M & \xrightarrow{\mathrm{id}} & M
\end{array}$$

and

$$\begin{array}{ccccc}
(\mathrm{id}\otimes\eta_L)^*\eta_L^* M & \xrightarrow{(\mathrm{id}\otimes\eta_L)^*\bar\nu} & (\mathrm{id}\otimes\eta_L)^*\eta_R^* M & = & (\eta_R\otimes\mathrm{id})^*\eta_L^* M \\
\| & & & & \downarrow{\scriptstyle (\eta_R\otimes\mathrm{id})^*\bar\nu} \\
\psi^*\eta_L^* M & \xrightarrow{\psi^*\bar\nu} & \psi^*\eta_R^* M & = & (\eta_R\otimes\mathrm{id})^*\eta_R^* M
\end{array}$$

commute. In other notation, we can write the LB-module isomorphism as

$$\bar\nu\colon M\otimes_L LB \xrightarrow{\cong} LB\otimes_L M$$

and the second coherence condition as

$$\begin{array}{ccccc}
 & & LB\otimes_L M\otimes_L LB & & \\
 & \nearrow{\scriptstyle \bar\nu\otimes\mathrm{id}} & & \searrow{\scriptstyle \mathrm{id}\otimes\bar\nu} & \\
M\otimes_L LB\otimes_L LB & & \xrightarrow{\psi^*\bar\nu} & & LB\otimes_L LB\otimes_L M\,.
\end{array}$$

By the η_L^*–η_{L*} adjunction, the LB-module homomorphism $\bar\nu$ corresponds to a unique L-module homomorphism

$$\nu\colon M \longrightarrow \eta_{L*}\eta_R^* M = LB\otimes_L M\,.$$

Here the tensor product $LB\otimes_L M$ is formed using the right unit $\eta_R\colon L\to LB$, and is viewed as an L-module using the left unit $\eta_L\colon L\to LB$. In these terms, the two coherence conditions are equivalent to the counitality

$$\begin{array}{ccc}
M & \xrightarrow{\nu} & LB\otimes_L M \\
 & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \epsilon\otimes\mathrm{id}} \\
 & & L\otimes_L M
\end{array}$$

and coassociativity

$$\begin{array}{ccccc}
 & & LB\otimes_L M & & \\
 & \nearrow{\scriptstyle\nu} & & \searrow{\scriptstyle \mathrm{id}\otimes\nu} & \\
M & & & & LB\otimes_L LB\otimes_L M \\
 & \searrow{\scriptstyle\nu} & & \nearrow{\scriptstyle \psi\otimes\mathrm{id}} & \\
 & & LB\otimes_L M & &
\end{array}$$

conditions required for ν to define an (L, LB)-coaction on M, i.e., an LB-comodule structure on M.

Recall that $\pi\colon \mathrm{Spec}(L)\to\mathcal{M}_{\mathrm{fgl}}\to\mathcal{M}_{\mathrm{fg}}$ denotes a presentation of the moduli stack of formal groups. Then, to any quasi-coherent sheaf $M^\sim$ over $\mathcal{M}_{\mathrm{fg}}$ we can associate the L-module M corresponding to the quasi-coherent sheaf $\pi^*(M^\sim)$ over

Spec(L). It comes equipped with an LB-module isomorphism $\bar{\nu}\colon \eta_L^* M \cong \eta_R^* M$, which is left adjoint to an LB-coaction $\nu\colon M \to LB \otimes_L M$. This functor

$$\begin{aligned} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) &\xrightarrow{\simeq} LB\!-\mathrm{coMod} \\ M^{\sim} &\longmapsto (M, \nu) \end{aligned}$$

is then the advertised equivalence. ((ETC: Explain why the left adjoint $\bar{\nu}$ of any coaction ν is an isomorphism. This uses the existence of inverses in $\mathcal{FGL}_s(R)$, or the conjugation in B.))

The same argument applies for any Hopf algebroid.

Theorem 5.1 (Hovey [Hov02, Thm. 2.2])**.** *Suppose* (A, Γ) *is a Hopf algebroid. Then there is an equivalence of categories between* Γ*-comodules and quasi-coherent sheaves over* $[\mathrm{Spec}(A) \leftleftarrows \mathrm{Spec}(\Gamma)]$.

We now have the terminology available to formulate the basic object of study in chromatic homotopy theory.

Definition 5.2. To each spectrum X we assign its complex bordism $MU_*(X)$, viewed as an $(MU_*, MU_*MU) \cong (L, LB)$-comodule,

$$\begin{aligned} \mathcal{S}p &\longrightarrow MU_*MU\!-\mathrm{coMod} \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \\ X &\longmapsto \qquad MU_*(X) \qquad \leftrightarrow MU_*(X)^{\sim}\,, \end{aligned}$$

which in turn is equivalent to a quasi-coherent sheaf $MU_*(X)^{\sim}$ over the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups.

6. Invariant ideals and coherent rings

Morava and Landweber [Lan73a], [Lan73b] observed that the (quasi-)coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$ only realize a small subset of all (quasi-)coherent sheaves on Spec(MU_*), i.e., that the (finitely presented) MU_*MU-comodules are quite special among the plethora of (finitely presented) MU_*-modules. After all, every countably generated commutative ring arises as MU_*/I for some ideal $I \subset MU_*$, but fortunately relatively few of these ideals are MU_*MU-comodules.

Recall the Hopf algebroid $(L, LB) \cong (MU_*, MU_*MU)$.

Definition 6.1. Let M be an LB-comodule, with coaction $\nu\colon M \to LB \otimes_L M$. We say that x is LB-comodule primitive if $\nu(x) = 1 \otimes x$, and write $P(M) \subset M$ for the subgroup of LB-comodule primitives. There are canonical isomorphisms

$$P(M) \cong \mathrm{Hom}_{LB-\mathrm{coMod}}(L, M) \cong L \,\square_{LB}\, M\,.$$

Let $\mathrm{Ann}(x) = \{\lambda \in L \mid \lambda x = 0 \in M\} \subset L$ be the annihilator ideal of x. We say that an ideal $I \subset L$ is invariant if it is an LB-subcomodule.

Lemma 6.2. *$I \subset L$ is invariant if and only if $\eta_L(I) \cdot LB = LB \cdot \eta_R(I)$.*

Proof. The ideal is an LB-subcomodule if and only if the composite $\eta_L\colon L \xrightarrow{\nu} LB \otimes_L L \cong LB$ takes I into $LB \otimes_L I \cong LB \cdot \eta_R(I)$, so that $\eta_L(I) \subset LB \cdot \eta_R(I)$, which implies $\eta_L(I) \cdot LB \subset LB \cdot \eta_R(I)$. Applying the conjugation χ then implies the opposite inclusion. □

Lemma 6.3. *Let $x \in M$ have degree d. The L-submodule $\Sigma^d L/\mathrm{Ann}(x) \cong Lx$ of M is an LB-subcomodule if and only if x is LB-comodule primitive and* $\mathrm{Ann}(x)$ *is invariant.*

Proof. If $Lx \subset M$ is an LB-subcomodule, then $\nu(x)$ lies in $LB \otimes_L Lx$, hence is $1 \otimes x$ for degree reasons, so x is LB-comodule primitive. Moreover, $\eta_L(\lambda) \otimes x = \nu(\lambda x) = 0$ in $LB \otimes_L Lx \cong \Sigma^d LB/LB \cdot \eta_R(\mathrm{Ann}(x))$ for $\lambda \in \mathrm{Ann}(x)$ implies $\eta_L(\lambda) \in LB \cdot \eta_R(\mathrm{Ann}(x))$, so $\mathrm{Ann}(x)$ is invariant.

Conversely, if x is LB-comodule primitive then $\lambda \mapsto \lambda x$ defines an LB-comodule homomorphism $\Sigma^d L \to M$, which factors as such over $\Sigma^d L \to Lx$ if $\mathrm{Ann}(x)$ is invariant. □

If M is nonzero and bounded below, then each lowest-degree class is LB-comodule primitive. Recall the ideals $I_{p,n} = (p, v_1, \dots, v_{n-1})$ and $I_{p,\infty} = (p, v_1, \dots, v_n, \dots)$ in L.

Lemma 6.4. *For each prime p and height $1 \le n \le \infty$ the ideal $I_{p,n} \subset L$ is an invariant prime ideal. The zero ideal $(0) \subset L$ is also invariant and prime.*

Proof. For each prime p we have $\eta_L(I_n) \subset LB \cdot \eta_R(I_n)$ by Chapter 10, Lemma 4.12, since (strictly) isomorphic formal group laws have the same height. Hence each I_n is invariant.

The quotient ring

$$L/I_n \cong \mathbb{F}_p[\tilde{v}_m, \tilde{n}_k \mid m \ge n, k+1 \ne p^i]$$

is an integral domain by Chapter 10, Corollary 5.7, so each I_n is prime. □

Definition 6.5. Let R be a (graded) commutative ring. An R-module M is finitely presented if there exists a short exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with F_0 and F_1 finitely generated free R-modules. The finitely presented R-modules are the compact objects in the category of R-modules, i.e., those for which $\mathrm{Hom}_R(M, -)$ commutes with filtered colimits.

A commutative ring R is coherent if each finitely generated ideal $I \subset R$ is finitely presented. A coherent module is a finitely generated module such that (it and) each finitely generated submodule is finitely presented. A module over a coherent ring is coherent if and only if it is finitely presented.

Lemma 6.6. *The Lazard ring $L \cong \mathbb{Z}[x_i \mid i \ge 1] \cong MU_*$ is coherent.*

Proof. Each finitely generated ideal in L is generated over some subring $\mathbb{Z}[x_1, \dots, x_n]$, and is finitely presented over that noetherian subring. The full Lazard ring is flat over that subring, so the finite presentation can be extended up. □

Definition 6.7. We say that an LB-comodule is finitely presented if its underlying L-module is finitely presented (= coherent). Let

$$LB\!-\mathrm{coMod}^{\mathrm{fp}} \subset LB\!-\mathrm{coMod}$$

denote the full subcategory of finitely presented LB-comodules. ((ETC: The category of LB-comodules is abelian, and $LB\!-\mathrm{coMod}^{\mathrm{fp}}$ is a thick abelian subcategory.)) We write

$$\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) \subset \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

for the corresponding full subcategory of coherent sheaves, under the equivalence

$$LB\!-\mathrm{coMod} \cong MU_*MU\!-\mathrm{coMod} \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}).$$

Definition 6.8. Let $\mathcal{S}p^\omega$ denote the category of finite spectra, i.e., the full subcategory of $\mathcal{S}p$ generated by spectra that are equivalent to finite cell (or CW) spectra. Its homotopy category $\mathrm{Ho}(\mathcal{S}p^\omega)$ is equivalent to the Spanier–Whitehead category $\mathcal{SW}$ of formal integer suspensions of finite CW complexes.

The superscript ω indicates the first infinite ordinal, giving the strict upper bound for the number of cells allowed in a CW (or cell) structure on these spectra. Each spectrum is a filtered homotopy colimit of finite spectra, but practically none of the cohomology theories we have discussed so far are represented by finite spectra.

Proposition 6.9 (Conner–Smith [CS69, Thm. 1.3$_*$]). *If X is a finite spectrum, then $MU_*(X)$ is a finitely presented MU_*-module.*

This follows by induction over the number of cells in X, via standard closure properties for coherent modules. We could also say that $MU_*(X)$ is a finitely presented MU_*MU-comodule, for each finite spectrum X.

7. Landweber's exact functor theorem

Theorem 7.1 (Landweber [Lan73b, Thm. 3.3']). *Each finitely presented LB-comodule M (an object in $LB-\mathrm{coMod}^{\mathrm{fp}}$) admits a finite length filtration*

$$0 = M(0) \subset M(1) \subset \cdots \subset M(\ell) = M$$

by finitely presented LB-subcomodules, such that

$$M(s)/M(s-1) \cong \Sigma^{d_s} L/J(s)$$

for each $1 \le s \le \ell$, where $J(s) \subset L$ is some finitely generated invariant prime ideal and d_s is some integer.

The proof uses primary decomposition, as in [AM69, Ch. 4], extended from ideals to modules and from noetherian rings to coherent rings.

Theorem 7.2 (Morava, Landweber [Lan73a, Prop. 2.11]). *The LB-comodule primitives in $L/I_{p,n}$ are*

$$P(L/I_{p,n}) = \mathbb{F}_p[v_n] \subset L/I_{p,n}$$

for each prime p and height $1 \le n < \infty$.

We already know that v_n is LB-comodule primitive in $L/I_{p,n}$, since $v_n \equiv \eta_R(v_n) \mod I_{p,n}$, which implies that each power of v_n is LB-comodule primitive since $L/I_{p,n}$ is an LB-comodule algebra. Seeing that there are no further LB-comodule primitives relies on the strong nontriviality of the coaction, i.e., the significant difference between $\eta_L \colon L \to LB$ and $\eta_R \colon L \to LB$. This requires some detailed calculation. See also [Rav92, Thm. B.5.18]. ((ETC: I believe there are more approaches/references.))

It follows that there are no other invariant prime ideals than the ones we have already discussed, so that the subquotients in a Landweber filtration are always of a familiar kind.

Theorem 7.3 (Morava, Landweber [Lan73a, Prop. 2.7]). *The invariant prime ideals $J \subset L$ are (precisely) the ideals $I_{p,n}$ for primes p and heights $1 \le n \le \infty$, together with the zero ideal (0).*

Proof. If $J \neq (0)$ then $J \cap \mathbb{Z} = (p)$ for some prime p ((ETC: why?)), and then $(p) = I_{p,1} \subset J \subset I_{p,\infty}$. Suppose $I_{p,n} \subset J$ but $v_n \notin J$ for some $1 \leq n < \infty$. Then $v_n^i \notin J$ for each $i \geq 1$, since J is a prime ideal. Hence $J/I_{p,n} \subset L/I_{p,n}$ contains no nonzero LB-comodule primitive elements, by Theorem 7.2, and must therefore be zero. This proves that $I_{p,n} = J$. □

The partially ordered set of invariant prime ideals in L thus matches the set of geometric points of $\mathcal{M}_{\mathrm{fg}}$, partially ordered by specialization.

Let R be a ring spectrum, with coefficient ring $R_* = \pi_*(R)$, and E_* an R_*-module. The functor

$$X \longmapsto E_* \otimes_{R_*} R_*(X)$$

is a homotopy functor with a suspension isomorphism satisfying Milnor's wedge axiom, but it might not be exact, since tensoring E_* over R_* with the long exact sequence

$$\dots \xrightarrow{\partial} R_*(X) \xrightarrow{i} R_*(Y) \xrightarrow{j} R_*(Y/X) \xrightarrow{\partial} \dots$$

might not give an exact sequence. It would suffice that E_* is a flat R_*-module, but from this point of view the following theorem is surprising, since $\mathbb{Z}[u^{\pm 1}] \cong KU_*$ is not a flat MU_*-module.

Theorem 7.4 (Conner–Floyd [CF66, Ch. II])**.** *Let* Td: $MU_* \to \mathbb{Z}[u^{\pm 1}] \cong KU_*$ *be the homomorphism sending the bordism class of an almost complex $2n$-manifold M to its Todd genus times u^n. Then there is a natural isomorphism of (multiplicative) homology theories*

$$KU_* \otimes_{MU_*} MU_*(X) \cong KU_*(X)\,.$$

In particular,

$$KU^* \otimes_{MU^*} MU^*(X) \cong KU^*(X)$$

for all finite spectra X.

The conclusion in cohomology follows from that in homology using Spanier–Whitehead duality, since $MU^{-*}(X) = \pi_* F(X, MU) \cong \pi_*(MU \wedge DX) = MU_*(DX)$ for finite X, and similarly for KU, where $DX = F(X, S)$ is the Spanier–Whitehead dual of X.

The key to this result is the Landweber filtration theorem, telling us that not all MU_*-modules arise as $MU_*(X)$, since the associated prime ideals must all be invariant. Let $I_{p,0} = (0)$.

Definition 7.5. Let E_* be an L-module. We say that $(p, v_1, v_2, \dots)$ is an E_*-regular sequence if all of the homomorphisms

$$\Sigma^{|v_n|} E_*/I_{p,n} \xrightarrow{v_n} E_*/I_{p,n}$$

for $n \geq 0$ are injective.

In particular, we ask that $p\colon E_* \to E_*$ is injective, $v_1\colon \Sigma^{2p-2} E_*/(p) \to E_*/(p)$ is injective, $v_2\colon \Sigma^{2p^2-2} E_*/(p, v_1) \to E_*/(p, v_1)$ is injective, and so on. If at some stage $E_*/I_{p,n} = 0$, then all of the remaining homomorphisms are automatically injective.

Example 7.6. If $E_* = L \otimes \mathbb{Q}$, then $p\colon E_* \to E_*$ is an isomorphism for each p, so $E_*/I_{p,1} = 0$ and $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p.

Example 7.7. If $E_* = \mathbb{Z}[u^{\pm 1}]$ with v_1 acting as multiplication by u^{p-1} for each p, then $p\colon E_* \to E_*$ is injective, $E_*/(p) = \mathbb{F}_p[u^{\pm 1}]$, $v_1\colon \Sigma^{2p-2}\mathbb{F}_p[u^{\pm 1}] \to \mathbb{F}_p[u^{\pm 1}]$ is an isomorphism, and $E_*/I_{p,2} = 0$. Hence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p.

Theorem 7.8 (Landweber [Lan76, Thm. 2.6$_{MU}$]). *Let E_* be an L-module. The functor*

$$\begin{aligned} LB\text{-}\operatorname{coMod}^{\mathrm{fp}} &\longrightarrow gr\mathcal{A}b \\ M &\longmapsto E_* \otimes_L M \end{aligned}$$

is exact if and only if for each prime p the sequence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

Proof. Let $I_0 = (0)$ and $v_0 = p$. The short exact sequences

$$0 \to \Sigma^{|v_n|} L/I_n \xrightarrow{v_n} L/I_n \longrightarrow L/I_{n+1} \to 0$$

for $n \geq 0$ induce long exact sequences

$$\cdots \to \operatorname{Tor}_1^L(E_*, L/I_n) \longrightarrow \operatorname{Tor}_1^L(E_*, L/I_{n+1}) \xrightarrow{\partial} E_* \otimes_L \Sigma^{|v_n|} L/I_n \xrightarrow{\mathrm{id} \otimes v_n} E_* \otimes_L L/I_n \to \dots .$$

Note that $\operatorname{Tor}_1^L(E_*, L) = 0$. Suppose, by induction on $n \geq 0$, that $\operatorname{Tor}_1^L(E_*, L/I_n) = 0$. Then $\operatorname{Tor}_1^L(E_*, L/I_{n+1}) = 0$ if (and only if) $v_n\colon \Sigma^{|v_n|} E_*/I_n \to E_*/I_n$ is injective. Hence $\operatorname{Tor}_1^L(E_*, L/I_n) = 0$ for all $0 \leq n < \infty$, if $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

Consider a Landweber filtration

$$0 = M(0) \subset M(1) \subset \cdots \subset M(\ell) = M\,.$$

The short exact sequences

$$0 \to M(s-1) \longrightarrow M(s) \longrightarrow \Sigma^{d_s} L/J(s) \to 0\,,$$

with $J(s) = I_{n_s}$ for some $0 \leq n_s < \infty$, induce long exact sequences

$$\cdots \to \operatorname{Tor}^1_L(E_*, M(s-1)) \longrightarrow \operatorname{Tor}^1_L(E_*, M(s)) \longrightarrow \operatorname{Tor}^1_L(E_*, \Sigma^{d_s} L/I_{n_s}) \to \dots$$

for $1 \leq s \leq \ell$. Clearly $\operatorname{Tor}^1_L(E_*, M(0)) = 0$. Suppose, by induction on $1 \leq s \leq \ell$, that $\operatorname{Tor}^1_L(E_*, M(s-1)) = 0$. By the assumption of E_*-regularity, $\operatorname{Tor}^1_L(E_*, \Sigma^{d_s} L/I_{n_s}) = 0$, so that $\operatorname{Tor}^1_L(E_*, M(s)) = 0$. Hence $\operatorname{Tor}^1_L(E_*, M) = 0$.

For any short exact sequence

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

in $LB\text{-}\operatorname{coMod}^{\mathrm{fp}}$ we have a long exact sequence

$$\cdots \to \operatorname{Tor}_1^L(E_*, M'') \xrightarrow{\partial} E_* \otimes_L M' \longrightarrow E_* \otimes_L M \longrightarrow E_* \otimes_L M'' \to 0\,.$$

By Theorem 7.1, M'' admits a Landweber filtration, so that $\operatorname{Tor}_1^L(E_*, M'') = 0$. Hence this is in fact a short exact sequence, and $E_* \otimes_L (-)$ defines an exact functor on finitely presented LB-comodules. □

Theorem 7.9 (Landweber [Lan76, Cor. 2.7]). *Let E_* be an MU_*-module. The functor*

$$X \longmapsto E_*(X) := E_* \otimes_{MU_*} MU_*(X)$$

defines a homology theory if and only if for each prime p the sequence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

Proof. We must show that $E_*(-)$ is exact. The composite

$$\mathcal{S}p^\omega \subset \mathcal{S}p \overset{E_*(-)}{\longrightarrow} gr\mathcal{A}b$$

factors as

$$\mathcal{S}p^\omega \overset{MU_*(-)}{\longrightarrow} LB\text{-}\operatorname{coMod}^{\mathrm{fp}} \overset{E_*\otimes_L(-)}{\longrightarrow} gr\mathcal{A}b\,,$$

which is exact by Theorem 7.8. Any spectrum is a filtered homotopy colimit of finite spectra, $E_*(-)$ maps filtered homotopy colimits to filtered colimits, and passage to filtered colimits of graded abelian groups is an exact functor. Hence $E_*(-)$ is also exact. □

Remark 7.10. Miller–Ravenel [MR77, Lem. 2.11] show that each $MU_*MU = LB$-comodule is a filtered colimit of finitely presented LB-comodules, so that Landweber's Theorem 7.8 is also valid if we allow M to range over all LB-comodules, not just the finitely presented ones. (To be precise, these authors work with $BP_*BP = VT$-comodules, but the proof is the same.) Granting this, the proof of Theorem 7.9 becomes even easier.

Remark 7.11. Consider the case where E_* is a commutative L-algebra, via a ring homomorphism $g\colon L \to E_*$. Hopkins (see Miller [Mil19]) and Hollander [Hol09] have explained how Landweber's E_*-regularity condition, and exactness for $M \mapsto E_* \otimes_L M$, are both equivalent to the algebro-geometric assertion that

$$\operatorname{Spec}(E_*) \overset{g}{\longrightarrow} \operatorname{Spec}(L) \overset{\pi}{\longrightarrow} \mathcal{M}_{\mathrm{fg}}$$

is a flat morphism of stacks, even if g alone is far from flat.

Definition 7.12. If E_* is an MU_*-module such that $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p, then we say that E_* and the associated homology theory $X \mapsto E_*(X)$ are Landweber exact.

Corollary 7.13. *Let E_* be Landweber exact. Then*

$$X \longmapsto E_*(X) = E_* \otimes_{MU_*} MU_*(X)$$

is represented by a spectrum E, so that $E_*(X) \cong \pi_*(E \wedge X)$. ((ETC: What more can we say about E? Is it an MU-module spectrum? Is it unique? What is E_* is an MU_*-algebra?))

Lemma 7.14. *If E_* is Landweber exact, then*

$$E_*E \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_* \cong E_* \otimes_L LB \otimes_L E_*$$

is a flat E_*-module. Hence E is flat, if it is a homotopy commutative ring spectrum.

Proof. From

$$E_*(MU) \cong E_* \otimes_{MU_*} MU_*(MU)$$

we obtain $MU_*(E) \cong MU_*MU \otimes_{MU_*} E_*$. Then

$$E_*(E) \cong E_* \otimes_{MU_*} MU_*(E) \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_*\,.$$

To show that E_*E is flat as a (right) E_*-module, we show that

$$M \mapsto E_*E \otimes_{E_*} M \cong E_* \otimes_L LB \otimes_L E_* \otimes_{E_*} M \cong E_* \otimes_L (LB \otimes_L M)$$

is exact as a functor from E_*-modules. Here $M \mapsto LB \otimes_L M$ defines the extended LB-comodule associated to the underlying L-module of M, and is exact because LB is (free, hence) flat as a right L-module. The functor $E_* \otimes_L (-)$ from LB-comodules is exact by Landweber exactness, extended as per Remark 7.10. □

Example 7.15. Let $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ and choose a ring homomorphism $g\colon L \to E(n)_*$ sending (p to p and) $v_m \in L/I_m$ to

$$v_m \in E(n)_*/I_m \cong \mathbb{F}_p[v_m, \dots, v_{n-1}, v_n^{\pm 1}]$$

for each $1 \le m \le n$. Then $(p, v_1, v_2, \dots)$ is an $E(n)_*$-regular sequence, $E(n)_*/I_n \cong \mathbb{F}_p[v_n^{\pm 1}] \cong K(n)_*$, and $E(n)_*/I_{n+1} = 0$. Hence the Johnson–Wilson version $E(n)$ of Morava E-theory is Landweber exact, and can be constructed directly this way. ((ETC: Discuss $E(n)_*E(n)$.))

Proposition 7.16. *$E(m) \wedge K(n) \simeq *$ for $0 \le m < n \le \infty$.*

Proof. Since

$$E(n)_*(MU) \cong E(n)_* \otimes_L LB \cong E(n)_*[b_k \mid k \ge 1]$$

is free as an $E(n)_*$-module, it follows by reduction modulo I_n that $K(n)_*(MU) \cong K(n)_* \otimes_L LB$ and $MU_*(K(n)) \cong LB \otimes_L K(n)_*$. Hence

$$E(m)_*(K(n)) \cong E(m)_* \otimes_{MU_*} MU_*(K(n)) \cong E(m)_* \otimes_L LB \otimes_L K(n)_* \,.$$

If nonzero, this ring would admit a ring homomorphism

$$E(m)_* \otimes_L LB \otimes_L K(n)_* \longrightarrow R$$

to a graded field R, classifying a strict isomorphism $h\colon F \to F'$ with F of height $\le m$ and F' of height n. This is impossible for $m < n$, since (strictly) isomorphic formal group laws have the same height. Thus $E(m)_*(K(n))$ must be the zero ring. □

((ETC: Johnson–Wilson: Only invariant prime ideal in $B(n)_*$ is (0), so

$$B(n)_*(X) \cong B(n)_* \otimes_{K(n)_*} K(n)_*(X)$$

is free and $K(n)_*(X) = K(n)_* \otimes_{B(n)_*} B(n)_*(X)$. Hence $v_m^{-1}(MU/I_m)_*(X) = 0$ iff $B(m)_*(X) = 0$ iff $K(m)_*(X) = 0$.))

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