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CHROMATIC HOMOTOPY THEORY

CHAPTER 7: SEQUENTIAL AND ORTHOGONAL SPECTRA

JOHN ROGNES

Stable homotopy theory was developed by Spanier and J.H.C. Whitehead [SW53], [SW55], and expressed in terms of spectra, in the sense of algebraic topology, by Lima [Lim59] and G. Whitehead [Whi60], [Whi62]. The Spanier–Whitehead homotopy category $\mathcal{SW}$ was extended by Boardman (1965, cf. Vogt [Vog70]) to contain representing objects for all cohomology theories, cf. Brown [Bro62]. A popular exposition of Boardman's homotopy category $\mathcal{B}$ was given by Adams [Ada74, Part III]. The resulting homotopy category is triangulated by Puppe cofiber sequences, cf. Verdier's 1967 thesis [Ver96], and has a symmetric monoidal smash product. This allows the study of ring spectra up to homotopy, and module spectra up to homotopy over these, but is not sufficient to give a triangulated structure on these module categories. More structured versions of ring and module spectra were studied by May and collaborators [May77], [May80] under the names of $\mathscr{I}_*$-prefunctors and $\mathscr{I}_*$-prespectra, but these were then only viewed as a source of examples, rather than as a fully fledged model for the stable homotopy category. Instead, coherent structures were expressed in terms of operad actions, e.g. in the context of Lewis–May spectra [LMSM86].

This changed with the insight by Jeff Smith (1994, see Hovey–Shipley–Smith [HSS00]) that by adding symmetric group actions to the Lima–Whitehead (sequential) spectra, one obtains a stable and symmetric monoidal model category Sp^Σ of symmetric spectra, whose homotopy category $\mathrm{Ho}(\mathcal{S}p^\Sigma)$ is equivalent to Boardman's. It was soon realized that one could equally well use orthogonal groups in place of symmetric groups, and that this would recover May's $\mathscr{I}_*$-prespectra. Another approach refining Lewis–May spectra was developed at the same time by Elmendorf–Kriz–Mandell–May [EKMM97]. The different theories were compared by Mandell–May–Schwede–Shipley [MMSS01]. In the orthogonal case, the stable equivalences are the same as the π_*-isomorphisms, whereas this relationship is more subtle for symmetric spectra. Hence we shall focus on the category $\mathcal{S}p^\mathbb{O}$ of orthogonal spectra as our stable and closed symmetric monoidal model for the stable homotopy category.

1. Sequential and orthogonal spectra

We work in the category $\mathcal{T}$ of based (compactly generated weak Hausdorff) spaces and basepoint-preserving maps.

Definition 1.1. A sequential spectrum X is a sequence of spaces X_n for $n \ge 0$ and structure maps $\sigma\colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$. A map $f\colon X \to Y$ of sequential

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spectra is a sequence of maps $f_n\colon X_n \to Y_n$ such that $f_{n+1}\sigma = \sigma(f_n \wedge S^1)$ for all $n \ge 0$. Let $\mathcal{S}p^{\mathbb{N}}$ be the topological category of sequential spectra.

Let $O(n)$ denote the n-th orthogonal group, acting on $\mathbb{R}^n$ by isometries, which extend to the one-point compactification $S^n = \mathbb{R}^n \cup \{\infty\}$. We view $O(n) \times O(m)$ as a subgroup of $O(n+m)$, compatibly with the isometry $\mathbb{R} \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$ and homeomorphism $S^n \wedge S^m \cong S^{n+m}$. (A coordinate-free approach, using isometries between Euclidean inner product spaces, is often more convenient for equivariant applications.)

Definition 1.2. An orthogonal spectrum X is a sequence of $O(n)$-spaces X_n for $n \ge 0$ and structure maps $\sigma\colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$, such that the m-fold iterate

$$\sigma^m\colon \Sigma^m X_n = X_n \wedge S^m \longrightarrow X_{n+m}$$

is $O(n) \times O(m)$-equivariant, for all $n, m \ge 0$. A map $f\colon X \to Y$ of orthogonal spectra is a sequence of $O(n)$-equivariant maps $f_n\colon X_n \to Y_n$ such that $f_{n+1}\sigma = \sigma(f_n \wedge S^1)$ for all $n \ge 0$. Let $\mathcal{S}p^{\mathbb{O}}$ be the topological category of orthogonal spectra.

We shall see that sequential spectra are the same as right S-modules in a symmetric monoidal category $(\mathcal{T}^{\mathbb{N}}, U, \otimes, \gamma)$ of sequential spaces, while orthogonal spectra are the same as right S-modules in a symmetric monoidal category $(\mathcal{T}^{\mathbb{O}}, U, \otimes, \gamma)$ of orthogonal spaces. In the sequential case S is a non-commutative monoid, while in the orthogonal case it is commutative. This is why we cannot expect $X \wedge Y = X \otimes_S Y$ to be an S-module in the sequential setting, while it will be an S-module in the orthogonal context.

((ETC: Consider writing $\boxtimes$ in place of $\otimes$ for the convolution products in the categories $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$ $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$, so that $X \wedge Y = X \boxtimes_S Y$.))

Definition 1.3. The homotopy groups $\pi_*(X)$ of a sequential spectrum X is the graded abelian group with

$$\pi_k(X) = \operatorname*{colim}_n \pi_{k+n}(X_n)$$

in degree $k \in \mathbb{Z}$. Here $\pi_{k+n}(X_n) \to \pi_{k+n+1}(X_{n+1})$ maps the homotopy class of $g\colon S^{k+n} \to X_n$ to the class of $\sigma(g \wedge S^1)$. The homomorphism $f_*\colon \pi_k(X) \to \pi_k(Y)$ maps the homotopy class of g to the class of $f_n g$. This defines a functor

$$\pi_*\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow gr\mathcal{A}b\,.$$

There is a forgetful functor

$$\mathbb{U}\colon \mathcal{S}p^{\mathbb{O}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$$

and the homotopy groups $\pi_*(X)$ of an orthogonal spectrum are defined to be the homotopy groups of the underlying sequential spectrum.

Definition 1.4. A map $f\colon X \to Y$ of sequential or orthogonal spectra is a π_*-isomorphism if the induced homomorphism $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Let $\mathcal{W}_{\mathbb{N}} \subset \mathcal{S}p^{\mathbb{N}}$ be the subcategory of π_*-isomorphisms. The stable homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ of sequential spectra is the localization of $\mathcal{S}p^{\mathbb{N}}$ away from the π_*-isomorphisms, i.e., the target of the initial functor

$$\mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}[\mathcal{W}_{\mathbb{N}}^{-1}] = \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$$

from $\mathcal{S}p^{\mathbb{N}}$ that maps each π_*-isomorphism to an isomorphism.

Likewise, let $\mathcal{W}_{\mathbb{O}} \subset \mathcal{S}p^{\mathbb{O}}$ be the subcategory of π_*-isomorphisms. The stable homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$ of orthogonal spectra is the localization of $\mathcal{S}p^{\mathbb{O}}$ away from the π_*-isomorphisms, i.e., the target of the initial functor

$$\mathcal{S}p^{\mathbb{O}} \longrightarrow \mathcal{S}p^{\mathbb{O}}[\mathcal{W}_{\mathbb{O}}^{-1}] = \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$$

from $\mathcal{S}p^{\mathbb{O}}$ that maps each π_*-isomorphism to an isomorphism.

It is not obvious that such initial functors exist, but if they do then they are uniquely determined up to unique isomorphism, by the usual argument involving a universal property. Quillen's theory of model categories [Qui67], [Hov99] provides a way of exhibiting such initial functors, both for sequential and orthogonal spectra. Moreover, the forgetful functor $\mathbb{U}$ is part of a Quillen equivalence, so that the (total right derived) induced functor

$$R\mathbb{U}\colon \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$$

is an equivalence of categories. By the stable homotopy category we shall mean either one of these two equivalent categories.

2. Sequential and orthogonal spaces

Definition 2.1. A symmetric monoidal category is a category $\mathcal{C}$ with a unit object U and a pairing

$$\begin{aligned} \otimes\colon \mathcal{C} \times \mathcal{C} &\longrightarrow \mathbb{C} \\ X, Y &\longmapsto X \otimes Y\,, \end{aligned}$$

together with natural unitality, associativity and commutativity isomorphisms

$$\begin{aligned} U \otimes Y \cong Y &\cong Y \otimes U \\ (X \otimes Y) \otimes Z &\cong X \otimes (Y \otimes Z) \\ \gamma\colon X \otimes Y &\cong Y \otimes X\,, \end{aligned}$$

satisfying some coherence axioms, including $\gamma^2 = \mathrm{id}$. We call γ the symmetry isomorphism. The category is closed if there is a functor

$$\begin{aligned} \mathrm{Hom}\colon \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ X, Y &\longmapsto \mathrm{Hom}(X, Y) \end{aligned}$$

and a natural bijection

$$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \mathrm{Hom}(Y, Z))\,,$$

i.e., if the functor $(-) \otimes Y$ admits a right adjoint $\mathrm{Hom}(Y, -)$, for each Y in $\mathcal{C}$. The adjunction counit $\epsilon\colon \mathrm{Hom}(Y, Z) \otimes Y \to Z$ is called evaluation.

See e.g. [Mac71, Ch. VII] for the coherence diagrams.

Definition 2.2. A monoid in $\mathcal{C}$ is an object R with unit and product maps $\eta\colon U \to R$ and $\phi\colon R \otimes R \to R$ such that unitality and associativity diagrams commute. It is commutative if

$$\begin{array}{ccc} R \otimes R & \xrightarrow{\gamma} & R \otimes R \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & X & \end{array}$$

commutes. A right R-module in $\mathcal{C}$ is then an object M with an action map $\rho\colon M \otimes R \to M$ such that unitality and associativity diagrams commute.

Definition 2.3. Let $\mathbb{N}$ be the discrete category with objects the integers $n \geq 0$ and only identity morphisms. The usual pairing

$$\begin{aligned} \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ m, n &\mapsto m + n \end{aligned}$$

is symmetric monoidal, with unit element $0 \in \mathbb{N}$.

Let $\mathbb{O}$ be the topological category with objects the integers $n \geq 0$ and morphism spaces

$$\mathbb{O}(m, n) = \begin{cases} O(n) & \text{for } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition is given by matrix multiplication. The block sum pairing

$$\begin{aligned} \mathbb{O} \times \mathbb{O} &\longrightarrow \mathbb{O} \\ m, n &\longmapsto m + n \\ A, B &\longmapsto A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{aligned}$$

is symmetric monoidal, with symmetry isomorphism $\chi_{m,n}\colon m + n \to n + m$ given by

$$\chi_{m,n} = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$$

This is natural, because

$$\begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} .$$

Definition 2.4. Let

$$\mathcal{T}^{\mathbb{N}} = \mathrm{Fun}(\mathbb{N}, \mathcal{T})$$

be the topological category of $\mathbb{N}$-spaces, i.e., sequences of based spaces $X = (X_n)_{n \geq 0}$. A map $f\colon X \to Y$ is a sequence of base-point preserving maps $(f_n\colon X_n \to Y_n)_{n \geq 0}$. Let

$$\begin{aligned} \mathcal{T}^{\mathbb{N}} \times \mathcal{T}^{\mathbb{N}} &\longrightarrow \mathcal{T}^{\mathbb{N}} \\ X, Y &\longmapsto X \otimes Y \end{aligned}$$

be the Day convolution product, given by

$$(X \otimes Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j$$

for each $n \geq 0$. It is the left Kan extension of

$$\mathbb{N} \times \mathbb{N} \xrightarrow{X \times Y} \mathcal{T} \times \mathcal{T} \xrightarrow{\wedge} \mathcal{T}$$

along $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $U \in \mathcal{T}^{\mathbb{N}}$ be given by $U_0 = S^0$ and $U_n = *$ for $n > 0$. Then $(\mathcal{T}^{\mathbb{N}}, U, \otimes, \gamma)$ is closed symmetric monoidal, with symmetry given by

$$\begin{aligned} \gamma_n\colon (X \otimes Y)_n &\xrightarrow{\cong} (Y \otimes X)_n \\ x \wedge y &\longmapsto y \wedge x \end{aligned}$$

for $i+j=n=j+i$, $x \in X_i$ and $y \in Y_j$. Let the sphere $\mathbb{N}$-space $S \in \mathcal{T}^{\mathbb{N}}$ be given by $S_n = S^n$ for each $n \geq 0$, let $\eta \colon U \to S$ be given by $\eta_0 = \mathrm{id}$, and let $\phi \colon S \otimes S \to S$ be given by

$$\phi_n \colon \bigvee_{i+j=n} S^i \wedge S^j \longrightarrow S^n$$
$$x \wedge y \longmapsto x \wedge y$$

for $i+j=n$, $x \in S^i$, $y \in S^j$ and $x \wedge y \in S^i \wedge S^j = S^n$. The internal Hom functor is given by

$$\mathrm{Hom}(Y,Z)_i = \prod_{i+j=n} \mathrm{Map}(Y_j, Z_n)\,.$$

Lemma 2.5. *(S, η, ϕ) is a non-commutative monoid in $\mathcal{T}^{\mathbb{N}}$.*

Proof. Unitality and associativity is straightforward. The pairings ϕ and $\phi\gamma \colon S \otimes S \to S$ map $x \wedge y \in S^i \wedge S^j \subset (S \otimes S)_n$, for $i+j=n$, to $x \wedge y$ and $y \wedge x$ in $S^n = S_n$, which are not generally equal, so S is not commutative. □

Lemma 2.6. *The category $\mathcal{S}p^{\mathbb{N}}$ of sequential spectra is isomorphic to the category of right S-modules in $\mathbb{N}$-spaces.*

Proof. Let X be a sequential spectrum. The underlying $\mathbb{N}$-space has the right S-module structure

$$\sigma \colon X \otimes S \longrightarrow X$$

given in degree n by the map

$$\sigma_n \colon (X \otimes S)_n = \bigvee_{i+j=n} X_i \wedge S^j \longrightarrow X_n$$

given by the composite structure maps

$$\sigma^j \colon X_i \wedge S^j \overset{\sigma \wedge \mathrm{id}}{\longrightarrow} \dots \overset{\sigma}{\longrightarrow} X_{i+j} = X_n\,.$$

Each right S-module arises this way, by the associativity of the right action. □

Definition 2.7. Let

$$\mathcal{T}^{\mathbb{O}} = \mathrm{Fun}(\mathbb{O}, \mathcal{T})$$

be the topological category of $\mathbb{O}$-spaces, i.e., sequences $X = (X_n)_{n\geq 0}$, where X_n is a based $O(n)$-space for each $n \geq 0$. A map $f \colon X \to Y$ is a sequence of base-point preserving maps $(f_n \colon X_n \to Y_n)_{n \geq 0}$, where $f_n \colon X_n \to Y_n$ is $O(n)$-equivariant for each $n \geq 0$. Let

$$\mathcal{T}^{\mathbb{O}} \times \mathcal{T}^{\mathbb{O}} \longrightarrow \mathcal{T}^{\mathbb{O}}$$
$$X, Y \longmapsto X \otimes Y$$

be the Day convolution product, given by

$$(X \otimes Y)_n = \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge Y_j$$

for each $n \geq 0$. It is the (continuous) left Kan extension of

$$\mathbb{O} \times \mathbb{O} \overset{X \times Y}{\longrightarrow} \mathcal{T} \times \mathcal{T} \overset{\wedge}{\longrightarrow} \mathcal{T}$$

along $+\colon \mathbb{O}\times\mathbb{O}\to\mathbb{O}$. Let $U\in\mathcal{T}^{\mathbb{O}}$ be given by $U_0=S^0$ and $U_n=*$ for $n>0$, with the only possible $O(n)$-actions. Then $(\mathcal{T}^{\mathbb{O}},U,\otimes,\gamma)$ is closed symmetric monoidal, with symmetry given by

$$\begin{aligned}\gamma_n\colon (X\otimes Y)_n &\xrightarrow{\cong} (Y\otimes X)_n\\ A\wedge x\wedge y &\longmapsto A\chi_{j,i}\wedge y\wedge x\end{aligned}$$

for $A\in O(n)$, $x\in X_i$, $y\in Y_j$ and $i+j=n=j+i$. Let the sphere $\mathbb{O}$-space $S\in\mathcal{T}^{\mathbb{O}}$ be given by $S_n=S^n=\mathbb{R}^n\cup\{\infty\}$ with the $O(n)$-action extending the action by isometries on $\mathbb{R}^n$ for each $n\geq 0$. Let $\eta\colon U\to S$ be given by $\eta_0=\mathrm{id}$, and let $\phi\colon S\otimes S\to S$ be given by the $O(n)$-equivariant map

$$\begin{aligned}\phi_n\colon \bigvee_{i+j=n} O(n)_+\wedge_{O(i)\times O(j)} S^i\wedge S^j &\longrightarrow S^n\\ A\wedge x\wedge y &\longmapsto A(x\wedge y)\end{aligned}$$

for $i+j=n$, $A\in O(n)$, $x\in S^i$, $y\in S^j$ and $x\wedge y\in S^i\wedge S^j=S^n$. The internal Hom functor is given by

$$\mathrm{Hom}(Y,Z)_i=\prod_{i+j=n}\mathrm{Map}(Y_j,Z_n)^{O(j)}\,,$$

with the $O(i)$-action from $O(i)\to O(i)\times O(j)\subset O(n)$.

Lemma 2.8. *(S,η,ϕ) is a commutative monoid in $\mathcal{T}^{\mathbb{O}}$.*

Proof. Unitality and associativity is straightforward. The pairings ϕ and $\phi\gamma\colon S\otimes S\to S$ map

$$A\wedge x\wedge y\in O(n)_+\wedge_{O(i)\times O(j)} S^i\wedge S^j\subset (S\otimes S)_n\,,$$

for $i+j=n$, to $A(x\wedge y)$ and $A\chi_{j,i}(y\wedge x)$ in $S^n=S_n$, which are exactly equal. Hence S is commutative. □

Lemma 2.9. *The category $\mathcal{S}p^{\mathbb{O}}$ of orthogonal spectra is isomorphic to the category of right S-modules in $\mathbb{O}$-spaces.*

Proof. Let X be an orthogonal spectrum. The underlying $\mathbb{O}$-space has the right S-module structure

$$\sigma\colon X\otimes S\longrightarrow X$$

given in degree n by the $O(n)$-equivariant map

$$\sigma_n\colon (X\otimes S)_n=\bigvee_{i+j=n} O(n)_+\wedge_{O(i)\times O(j)} X_i\wedge S^j\longrightarrow X_n$$

with components

$$O(n)_+\wedge_{O(i)\times O(j)} X_i\wedge S^j\longrightarrow X_n$$

that are left adjoint to the $O(i)\times O(j)$-equivariant composite structure maps

$$\sigma^j\colon X_i\wedge S^j\xrightarrow{\sigma\wedge\mathrm{id}}\dots\xrightarrow{\sigma} X_{i+j}=X_n\,.$$

Each right S-module arises this way, by the associativity of the right action. □

3. Model category structures

Let $\mathcal{C}$ be a category with all colimits and limits, and let $\mathcal{W}$ be a subcategory of weak equivalences. A model structure [Qui67], [Hov99] on $\mathcal{C}$ is given by two additional subcategories, of cofibrations and fibrations, satisfying a list of axioms. These ensure that the localization $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$ can be constructed with morphism sets

$$[X, Y] = \{\text{morphisms } X^c \to Y^f \text{ in } \mathcal{C}\}/\sim$$

where $X^c \to X$ and $Y \to Y^f$ are so-called cofibrant and fibrant replacements, and $\sim$ denotes homotopy classes of maps.

Lemma 3.1. *The categories $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ have all (small) colimits and limits.*

Proof. Any diagram $\alpha \mapsto X(\alpha)$ of $\mathbb{N}$-spaces, resp. $\mathbb{O}$-spaces, has colimit and limit

$$(\operatorname*{colim}_\alpha X(\alpha))_n = \operatorname*{colim}_\alpha (X(\alpha)_n)$$
$$(\lim_\alpha X(\alpha))_n = \lim_\alpha (X(\alpha)_n)$$

formed "pointwise" in spaces, resp. $O(n)$-spaces. If this is a diagram of right S-modules, then the colimit and limit have right S-module structures given by

$$(\operatorname*{colim}_\alpha X(\alpha)) \otimes S \cong (\operatorname*{colim}_\alpha X(\alpha) \otimes S) \overset{\operatorname{colim}_\alpha \sigma}{\longrightarrow} \operatorname*{colim}_\alpha X(\alpha)$$

and

$$(\lim_\alpha X(\alpha)) \otimes S \overset{\kappa}{\longrightarrow} (\lim_\alpha X(\alpha) \otimes S) \overset{\lim_\alpha \sigma}{\longrightarrow} \operatorname*{colim}_\alpha X(\alpha)$$

for a canonical exchange map κ. □

Lemma 3.2. *The topological categories $\mathcal{C} = \mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ are tensored and cotensored over $\mathcal{T}$. There are natural homeomorphisms*

$$\mathrm{Map}(T, \mathcal{C}(X, Y)) \cong \mathcal{C}(T \wedge X, Y) \cong \mathcal{C}(X \wedge T, Y) \cong \mathcal{C}(X, \mathrm{Map}(T, Y)) .$$

Proof. Given an $\mathbb{N}$-space, resp. $\mathbb{O}$-space, X and a space $T \in \mathcal{T}$ define $T \wedge X$, $X \wedge T$ and $\mathrm{Map}(T, X)$ so that

$$(T \wedge X)_n = T \wedge X_n$$
$$(X \wedge T)_n = X_n \wedge T$$
$$\mathrm{Map}(T, X)_n = \mathrm{Map}(T, X_n)$$

in spaces, resp. $O(n)$-spaces. If X is a right S-module, then these have right S-module structures given by

$$(T \wedge X) \otimes S \cong T \wedge (X \otimes S) \overset{T \wedge \sigma}{\longrightarrow} T \wedge X$$
$$(X \wedge T) \otimes S \cong (X \otimes S) \wedge T \overset{\sigma \wedge T}{\longrightarrow} X \wedge T$$
$$\mathrm{Map}(T, X) \otimes S \overset{\kappa}{\longrightarrow} \mathrm{Map}(T, X \otimes S) \overset{\mathrm{Map}(T,\sigma)}{\longrightarrow} \mathrm{Map}(T, X)$$

for a canonical exchange map κ. □

Definition 3.3. For X in $\mathcal{C} = \mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$, let $CX = X \wedge I$, $\Sigma X = X \wedge S^1$ and $\Omega X = \mathrm{Map}(S^1, X)$ be the cone, suspension and loop space or spectrum. There are natural homeomorphisms

$$\Omega \mathcal{C}(X, Y) \cong \mathcal{C}(S^1 \wedge X, Y) \cong \mathcal{C}(\Sigma X, Y) \cong \mathcal{C}(X, \Omega Y) .$$

For $f\colon X \to Y$ a map of diagram spaces or spectra, let the mapping cone of f be the pushout

$$Cf = Y \cup_X CX\,.$$

We call the diagram

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X$$

the Puppe cofiber sequence generated by f.

Lemma 3.4. *For each $m \ge 0$ there are free functors*

$$\begin{aligned} F_m &\colon \mathcal{T} \longrightarrow \mathcal{T}^{\mathbb{N}} \\ F_m &\colon \mathcal{T} \longrightarrow \mathcal{T}^{\mathbb{O}} \\ \Sigma^\infty_m &\colon \mathcal{T} \longrightarrow \mathcal{S}p^{\mathbb{N}} \\ \Sigma^\infty_m &\colon \mathcal{T} \longrightarrow \mathcal{S}p^{\mathbb{O}} \end{aligned}$$

that are left adjoint to the forgetful functors from $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ mapping X to the (non-equivariant) space X_m.

Proof. Let

$$F_m(T)_n = \begin{cases} T & \text{for } m = n, \\ * & \text{otherwise} \end{cases}$$

in the sequential case, and let

$$F_m(T)_n = \begin{cases} O(n)_+ \wedge T & \text{for } m = n, \\ * & \text{otherwise} \end{cases}$$

in the orthogonal case. In either case, let $\Sigma^\infty_m T = F_m(T) \otimes S$ with the evident right S-module structure, so that

$$(\Sigma^\infty_m T)_n = \begin{cases} T \wedge S^{n-m} & \text{for } n \ge m \\ * & \text{for } n < m \end{cases}$$

in the sequential case, and

$$(\Sigma^\infty_m T)_n = \begin{cases} O(n)_+ \wedge_{O(n-m)} (T \wedge S^{n-m}) & \text{for } n \ge m \\ * & \text{for } n < m \end{cases}$$

in the orthogonal case. □

Definition 3.5. Let $\Sigma^\infty = \Sigma^\infty_0$ denote the suspension spectrum functor, from $\mathcal{T}$ to $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$. Then

$$(\Sigma^\infty T)_n = T \wedge S^n = \Sigma^n T\,,$$

with the standard $O(n)$-action on S^n in the orthogonal case. The structure maps

$$\sigma\colon \Sigma(\Sigma^\infty T)_n \longrightarrow (\Sigma^\infty T)_{n+1}$$

are the identity maps.

Definition 3.6. For $m \ge 0$ let $S^m = \Sigma^\infty S^m$ and $S^{-m} = \Sigma^\infty_m S^0$ as sequential or orthogonal spectra. For $m = 0$ these definitions agree, and $S^0 = \Sigma^\infty S^0 = S$ is the sphere spectrum.

Lemma 3.7. *The canonical maps $\Sigma S^m \to S^{m+1}$ are isomorphisms for $m \ge 0$, and π_*-isomorphisms for $m < 0$.*

Proof. This is easy in the sequential case, and amounts to a key calculation in the orthogonal case. As a representative case, consider $\lambda\colon \Sigma S^{-1} = \Sigma_1^\infty S^1 \to S$, given at level n by the $O(n)$-map

$$\lambda_n \colon O(n)_+ \wedge_{O(n-1)} S^1 \wedge S^{n-1} \longrightarrow S^n$$

left adjoint to the $O(n-1)$-equivariant identity $S^1 \wedge S^{n-1} = S^n$. The source is the Thom complex of an $\mathbb{R}^n$-bundle over $O(n)/O(n-1) \cong S^{n-1}$, and the map is a $(2n-1)$-connected retraction. Hence

$$\pi_k(\lambda) = \operatorname*{colim}_n \pi_{k+n}(\lambda_n)$$

is an isomorphism for each $k \in \mathbb{Z}$. □

Remark 3.8. For symmetric spectra, λ should be a (stable, weak) equivalence, but is not a π_*-isomorphism. Hence more maps than the π_*-isomorphisms need to be inverted to pass from $\mathcal{S}p^\Sigma$ to $\mathrm{Ho}(\mathcal{S}p^\Sigma) \simeq \mathcal{B}$.

Definition 3.9. Given a map $\phi\colon S^{m-1} \to X$, we say that $C\phi = X \cup CS^{m-1}$ is obtained from X by attaching an m-cell along ϕ. A spectrum that can be obtained from $*$ by attaching (transfinitely) many cells is called a cell spectrum. ((ETC: Also allow $\Sigma^i S^{j-1}$ as source of ϕ?))

Definition 3.10. A sequential or orthogonal spectrum X is called an Ω-spectrum if the adjoint structure map

$$\tilde{\sigma}\colon X_n \longrightarrow \Omega X_{n+1}$$

is a weak homotopy equivalence, for each $n \geq 0$.

If X is an Ω-spectrum, then each space X_m is an infinite loop space, in the sense that there is an infinite sequence of weak equivalences

$$X_m \simeq \Omega X_{m+1} \simeq \cdots \simeq \Omega^n X_{m+n} \simeq \dots .$$

Theorem 3.11 ([BF78, Thm. 2.3], [MMSS01, Thm. 9.2])**.** *There is a model structure on the category of sequential, resp. orthogonal, spectra, with weak equivalences given by the π_*-isomorphisms, such that cell spectra are cofibrant and Ω-spectra are fibrant.*

Hence the homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}}) = \mathcal{S}p^{\mathbb{N}}[\mathcal{W}_{\mathbb{N}}^{-1}]$, *resp.* $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) = \mathcal{S}p^{\mathbb{O}}[\mathcal{W}_{\mathbb{O}}^{-1}]$, *exists, and*

$$[X, Y] = \{X^c \to Y^f\}/{\simeq}$$

where $X^c \to X$ is a π_*-equivalence from a cell spectrum, $Y \to Y^f$ is a π_*-equivalence to an Ω-spectrum, and $\simeq$ denotes homotopy classes of spectrum maps $X^c \to Y^f$.

Proposition 3.12. *There is a natural isomorphism*

$$\pi_k(Y) \cong [S^k, Y]$$

for each sequential, resp. orthogonal, spectrum Y.

Proof. Note that

$$S^k = \begin{cases} \Sigma^\infty S^k & \text{for } k \geq 0, \\ \Sigma^\infty_{-k} S^0 & \text{for } k \leq 0 \end{cases}$$

is a cell spectrum, hence its own cofibrant replacement. Let $Y \to Y^f$ be a fibrant replacement, i.e., a π_*-isomorphism to an Ω-spectrum. It suffices to prove that $\pi_k(Y^f) \cong [S^k, Y^f]$. Here

$$[S^k, Y^f] = \{S^k \to Y^f\}/\simeq \; \cong \begin{cases} \pi_k(Y^f_0) & \text{for } k \geq 0, \\ \pi_0(Y^f_{-k}) & \text{for } k \leq 0, \end{cases}$$

which indeed is isomorphic to $\pi_k(Y^f)$. □

Theorem 3.13 ([MMSS01, Thm. 10.4])**.** *The model categories of sequential and orthogonal spectra are Quillen equivalent, so that*

$$R\mathbb{U}\colon \operatorname{Ho}(\mathcal{S}p^{\mathbb{O}}) \xrightarrow{\simeq} \operatorname{Ho}(\mathcal{S}p^{\mathbb{N}})$$

is an equivalence of categories.

This uses that the underlying sequential spectra $\mathbb{U}S^m$ of orthogonal sphere spectra are π_*-isomorphic to the corresponding sequential sphere spectra. Hereafter we write $\operatorname{Ho}(\mathcal{S}p)$ for either one of these equivalent categories.

4. Stability and triangulated structure

The model structures on $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ are stable, which implies that the homotopy category $\operatorname{Ho}(\mathcal{S}p)$ is triangulated.

Theorem 4.1. *The suspension and loop functors induce inverse equivalences*

$$\Sigma\colon \operatorname{Ho}(\mathcal{S}p) \overset{\cong}{\rightleftarrows} \operatorname{Ho}(\mathcal{S}p)\colon \Omega$$

In particular, the adjunction unit $\eta\colon X \to \Omega\Sigma X$ and counit $\epsilon\colon \Sigma\Omega Y \to Y$ are both π_*-isomorphisms.

For one proof, using that the cyclic permutation of $S^1 \wedge S^1 \wedge S^1$ is homotopic to the identity, see [Rognes, MAT9580/2021, Spectral Sequences, §9.3].

Lemma 4.2. *Loop composition gives each morphism set*

$$[X, Y] \cong [\Sigma^2 X, \Sigma^2 Y] \cong [X, \Omega^2\Sigma^2 Y]$$

the structure of an abelian group, and composition of morphisms is bilinear.

We say that $\operatorname{Ho}(\mathcal{S}p)$ is an $\mathcal{A}b$-category. An additive category is an $\mathcal{A}b$-category with all finite sums (= coproducts). It follows that it has all finite products, and that the canonical map from any finite sum to the corresponding finite product is an isomorphism. We now give May's version [May01] of Verdier's axioms.

Definition 4.3. A triangulated category is an additive category $\mathcal{C}$ with an additive equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and a collection Δ of diagrams

$$X \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X\,, \tag{4.1}$$

called distinguished triangles. We assume that:

(1) (a) For each object X in $\mathcal{C}$ the triangle

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

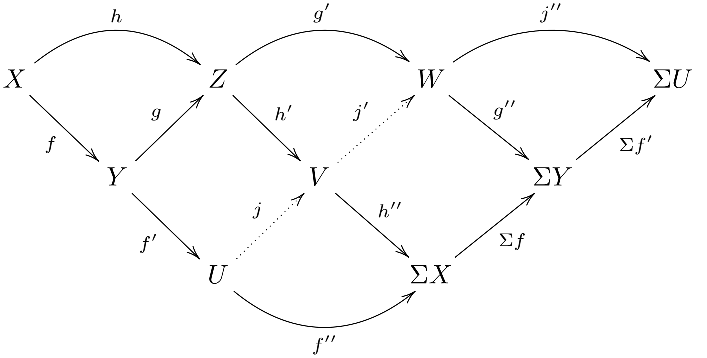
is distinguished. (b) For each morphism $f\colon X \to Y$ in $\mathcal{C}$ there exists a distinguished triangle (4.1). (c) Any diagram isomorphic to a distinguished triangle is also a distinguished triangle.

(2) For each distinguished triangle (4.1) its rotation

$$Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is a distinguished triangle.

(3) Consider the following braid diagram.



Assume that $h = gf$ and $j'' = (\Sigma f')g''$, and that

$$X \xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} \Sigma Y$$

$$X \xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} \Sigma X$$

are distinguished. Then there exist maps j and j' such that the diagram commutes and

$$U \xrightarrow{j} V \xrightarrow{j'} W \xrightarrow{j''} \Sigma U$$

is distinguished.

The braid axiom is usually known as the octahedral axiom, since the four distinguished triangles and the four commuting triangles can be viewed as the eight faces of an octahedron. The two commuting squares then appear in the interior of the octahedron.

The following fill-in lemma was taken as an axiom by Puppe and (unnecessarily so) by Verdier.

Lemma 4.4. *If the rows are distinguished and the left hand square commutes in the following diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

then there exists a map k making the remaining two squares commute.

It is a consequence of the following 3×3-lemma, which is proved by comparing the braid diagrams for the compositions jf and for $f'i$.

Lemma 4.5. *Assume that $jf = f'i$ and the two top rows and two left columns are distinguished in the following diagram.*

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle j'} & & \vdots{\scriptstyle k'} & & \downarrow{\scriptstyle \Sigma i'} \\
X'' & \overset{f''}{\cdots\to} & Y'' & \overset{g''}{\cdots\to} & Z'' & \overset{h''}{\cdots\to} & \Sigma X'' \\
{\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle j''} & & \vdots{\scriptstyle k''} & (-) & \downarrow{\scriptstyle \Sigma i''} \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma g} & \Sigma Z & \xrightarrow{\Sigma h} & \Sigma^2 X
\end{array}
$$

Then there is an object Z'' and maps f'', g'', h'', k, k' and k'' such that the diagram is commutative, except for its bottom right hand square, which commutes up to the sign -1, and all four rows and columns are distinguished.

In all cases, no uniqueness is assumed for these existence statements. This makes it difficult to glue together triangulated categories. This issue can be resolved by working with richer structures, i.e., stable ∞-categories.

The fill-in lemma implies that distinguished triangles are exact and coexact.

Proposition 4.6. *For any distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Y \xrightarrow{h} \Sigma X$$

and object T, in a triangulated category $\mathcal{C}$, the sequences

$$\mathcal{C}(T, X) \xrightarrow{f_*} \mathcal{C}(T, Y) \xrightarrow{g_*} \mathcal{C}(T, Z) \xrightarrow{h_*} \mathcal{C}(T, \Sigma X)$$

and

$$\mathcal{C}(\Sigma X, T) \xrightarrow{h^*} \mathcal{C}(Z, T) \xrightarrow{g^*} \mathcal{C}(Y, T) \xrightarrow{f^*} \mathcal{C}(X, T)$$

are exact.

In view of stability and rotation invariance, these extend in both directions to long exact sequences. Recall the mapping cone $Cf = Y \cup CX$.

Theorem 4.7. *The stable homotopy category* $\mathrm{Ho}(\mathcal{S}p)$ *is triangulated, with distinguished triangles the diagrams that are isomorphic to the Puppe cofiber sequences*

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X\,.$$

Sketch proof. The braid axiom may be unfamiliar. We may assume that $U = Cf$, $W = Cg$ and $V = C(gf)$. There is then a commuting diagram

$$
\begin{array}{ccccc}
Y & \xrightarrow{g} & Z & \longrightarrow & Cg \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \simeq} \\
Cf & \xrightarrow{j} & C(gf) & \longrightarrow & Cj \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma X & \xrightarrow{\mathrm{id}} & \Sigma X & \longrightarrow & C\Sigma X
\end{array}
$$

of vertical cofiber sequences and horizontal homotopy cofiber sequences, formed in $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$. The map $Cg \to Cj$ is an equivalence, since $C\Sigma X \simeq *$. □

Corollary 4.8. *For any map $f\colon X \to Y$ of (sequential or orthogonal) spectra there is a long exact sequence*

$$\dots \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \longrightarrow \pi_k(Cf) \xrightarrow{\partial} \pi_{k-1}(X) \to \dots .$$

This could also be proved directly from Theorem 4.1. Following G. Whitehead [Whi60], each spectrum defines a (generalized) homology and cohomology theory on spaces.

Theorem 4.9. *(a) Let E be a (sequential or orthogonal) spectrum. The functors*

$$T \mapsto \tilde{E}_k(T) = \pi_k(E \wedge T)$$

and the suspension isomorphisms

$$\tilde{E}_k(T) = \pi_k(E \wedge T) \xrightarrow{\cong} \pi_{k+1}(E \wedge \Sigma T) = \tilde{E}_{k+1}(\Sigma T)$$

define a reduced homology theory on all based spaces T.

(b) The functors

$$X \mapsto E^k(X) = [X, \Sigma^k E]$$

and the suspension isomorphisms

$$E^k(X) = [X, \Sigma^k E] \xrightarrow{\cong} [\Sigma X, \Sigma^{k+1} E] = E^{k+1}(X)\,,$$

for $k \in \mathbb{Z}$, define a cohomology theory on all spectra X, which restricts to a reduced cohomology theory on all based spaces T via

$$\tilde{E}^k(T) = E^k(\Sigma^\infty T) = [\Sigma^\infty T, \Sigma^k E]\,.$$

We will extend the homology theory $E_*(-)$ to all spectra after defining the smash product of orthogonal spectra.

Definition 4.10. Let $\mathscr{A}_E^* = E^*E = E^*(E)$ be the E-based Steenrod algebra.

Proposition 4.11. *The composition pairing*

$$E^iE \otimes E^j(X) = [E, \Sigma^i E] \otimes [X, \Sigma^j E] \xrightarrow{\circ} [X, \Sigma^{i+j} E] = E^{i+j}(X)$$

gives $E^*(X)$ a natural left E^*E-module structure. The multiplication in E^*E corresponds to the case $X = E$.

In the case $E = H\mathbb{F}_p$ we recover the mod p Steenrod algebra, and its natural left action on $H^*(X;\mathbb{F}_p)$. The structure of $\mathscr{A}_{MU}^* = MU^*(MU)$ was determined by Novikov [Nov67] (announced at the 1966 ICM) and Landweber [Lan67], cf. [Ada74, Part I]. Its action on $MU^*(X)$ naturally is that of a topological ring acting continuously on a topological module. Following Adams [Ada69, Lec. III] we shall instead view $\mathscr{A}_*^{MU} = MU_*(MU)$ as a generalized coalgebra, called a Hopf algebroid, with a natural coaction on $MU_*(X)$. This avoids the technical issues about topological actions.

5. Truncation structure

The method of killing homotopy groups shows that for each spectrum X there exists a Postnikov tower

$$X \to \dots \to \tau_{\le t}X \to \tau_{\le t-1}X \to \dots$$

where $\pi_i(X) \to \pi_i(\tau_{\le t}X)$ is an isomorphism for each $i \le t$, while $\pi_i(\tau_{\le t}X) = 0$ for all $i > t$. We say that $\tau_{\le t}X$ is t-coconnective (omitting t when $t = 0$), or t-truncated. It follows that

$$X \simeq \operatorname{holim}_t \tau_{\le t}X$$

(the mapping microscope). We may write $\tau_{<t}X$ for $\tau_{\le t-1}X$.

There is a homotopy cofiber sequence

$$\tau_{>t}X \longrightarrow X \longrightarrow \tau_{\le t}X \longrightarrow \Sigma\tau_{>t}X$$

for each $t \in \mathbb{Z}$. Writing $\tau_{\ge t+1}X$ for $\tau_{>t}X$, we obtain a Whitehead tower

$$\dots \to \tau_{\ge t+1}X \to \tau_{\ge t}X \to \dots \to X$$

where $\pi_i(\tau_{\ge t}X) \to \pi_i(X)$ is an isomorphism for each $i \ge t$, while $\pi_i(\tau_{\ge t}X) = 0$ for all $i < t$. We say that $\tau_{\ge t}X$ is t-connective, omit t when $t = 0$, and say that $\tau_{\ge 0}X \to X$ is the connective cover of X. It follows that

$$\operatorname{hocolim}_t X \simeq X$$

(the mapping telescope).

Example 5.1. The spectra S, MO, MSO, MU, HA are connective, for any abelian group A. The connective covers of KO and KU are denoted ko and ku, respectively, with

$$\pi_*(ku) = \mathbb{Z}[u]$$

and

$$\pi_*(ko) = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)\,.$$

The formal properties of Postnikov towers were axiomatized by Beilinson–Bernstein–Deligne.

Definition 5.2 ([BBD82, §1.3])**.** A t-structure (= truncation structure, I presume) on a triangulated category $\mathcal{C}$ is a pair of full subcategories $\mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le 0}$. With the notations $\mathcal{C}_{\ge t} = \Sigma^t\mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le t} = \Sigma^t\mathcal{C}_{\le 0}$ we assume that:

(1)
$$\dots \subset \mathcal{C}_{\ge 1} \subset \mathcal{C}_{\ge 0} \subset \dots \qquad \text{and} \qquad \dots \subset \mathcal{C}_{\le 0} \subset \mathcal{C}_{\le 1} \subset \dots\,.$$

(2) For each object Y in $\mathcal{C}$ there exists a distinguished triangle
$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$
with $X \in \mathcal{C}_{\ge 1}$ and $Z \in \mathcal{C}_{\le 0}$.

(3) If $X \in \mathcal{C}_{\ge 1}$ and $Z \in \mathcal{C}_{\le 0}$ then $\mathcal{C}(X, Z) = 0$.

Definition 5.3. An abelian category is an additive category such that

(1) each morphism has a kernel and a cokernel,
(2) each monomorphism is a kernel, and each epimorphism is a cokernel.

For each morphism $f\colon A \to B$ in an abelian category, there is an exact sequence

$$0 \to \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{cok}(f) \to 0\,,$$

and $A/\ker(f) = \operatorname{coim}(f) \cong \operatorname{im}(f)$. Abelian categories are convenient settings for homological algebra.

Theorem 5.4. *The heart $\mathcal{C}^\heartsuit = \tau_{\ge 0}\mathcal{C} \cap \tau_{\le 0}\mathcal{C}$ of a t-structure is an abelian category.*

Proposition 5.5. *The categories $\mathrm{Ho}(\mathcal{S}p)_{\ge 0}$ of connective spectra and $\mathrm{Ho}(\mathcal{S}p)_{\le 0}$ of coconnective spectra define a t-structure on the stable homotopy category, with heart the abelian category of abelian groups.*

Sketch proof. If X is 1-connective and Z is 0-coconnective, then $[X, Z] = 0$ by induction over a CW structure on X.

The heart $\mathrm{Ho}(\mathcal{S}p)^\heartsuit = \mathrm{Ho}(\mathcal{S}p)_{\ge 0} \cap \mathrm{Ho}(\mathcal{S}p)_{\le 0}$ consists of the spectra with $\pi_*(X)$ concentrated in degree 0, i.e., the Eilenberg–MacLane spectra HA for all abelian groups A. □

The derived category $\mathcal{D}(\mathbb{Z})$ of chain complexes of abelian groups, up to quasi-isomorphism, is another triangulated category with t-structure, having the same heart as $\mathrm{Ho}(\mathcal{S}p)$. ((ETC: Realize $\mathcal{D}(\mathbb{Z})$ as $\mathrm{Ho}(\mathrm{Mod}_{H\mathbb{Z}})$, with base change along $S \to H\mathbb{Z}$ defining a functor $\mathrm{Ho}(\mathcal{S}p) = \mathrm{Ho}(\mathrm{Mod}_S) \to \mathrm{Ho}(\mathrm{Mod}_{H\mathbb{Z}})$.))

6. Smash products and function spectra

We now make use of the fact that S is a commutative monoid in $\mathbb{O}$-spaces to define a smash product

$$X \wedge Y = X \otimes_S Y$$

and a function object

$$F(Y, Z) = \mathrm{Hom}_S(Y, Z)$$

for orthogonal spectra, i.e., right S-modules, X, Y and Z.

Definition 6.1. Given right S-modules X, Y and Z let $X \wedge Y = X \otimes_S Y$ be the coequalizer

$$X \otimes S \otimes Y \underset{\mathrm{id}\otimes\sigma'}{\overset{\sigma\otimes\mathrm{id}}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} X \otimes_S Y$$

in $\mathcal{T}^{\mathbb{O}}$, where $\sigma' = \sigma\gamma\colon S \otimes Y \to Y$ defines a left S-action on Y. Let $F(Y, Z) = \mathrm{Hom}_S(Y, Z)$ be the equalizer

$$\mathrm{Hom}_S(Y, Z) \xrightarrow{\iota} \mathrm{Hom}(Y, Z) \underset{\sigma^\vee}{\overset{\sigma^*}{\rightrightarrows}} \mathrm{Hom}(Y \otimes S, Z)$$

in $\mathcal{T}^{\mathbb{O}}$, where $\sigma^\vee$ has left adjoint

$$\mathrm{Hom}(Y, Z) \otimes Y \otimes S \xrightarrow{\epsilon\otimes\mathrm{id}} Z \otimes S \xrightarrow{\sigma} Z\,.$$

Then $X \wedge Y$ has a right S-module structure making the square

$$\begin{array}{ccc}
X \otimes Y \otimes S & \xrightarrow{\mathrm{id}\otimes\sigma} & X \otimes Y \\
\downarrow{\scriptstyle \pi\otimes\mathrm{id}} & & \downarrow{\scriptstyle \pi} \\
(X \wedge Y) \otimes S & \longrightarrow & X \wedge Y
\end{array}$$

commute, while $F(Y,Z)$ has a right S-module structure making the rectangle

$$\begin{array}{ccccc} F(Y,Z)\otimes S & & \longrightarrow & & F(Y,Z) \\ \downarrow{\scriptstyle \iota\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle \iota} \\ \mathrm{Hom}(Y,Z)\otimes S & \xrightarrow{\kappa} & \mathrm{Hom}(Y,Z\otimes S) & \xrightarrow{\sigma_*} & \mathrm{Hom}(Y,Z) \end{array}$$

commute. Here κ has left adjoint

$$\mathrm{Hom}(Y,Z)\otimes S\otimes Y \xrightarrow{\mathrm{id}\otimes\gamma} \mathrm{Hom}(Y,Z)\otimes Y\otimes S \xrightarrow{\epsilon\otimes\mathrm{id}} Z\otimes S\,.$$

Remark 6.2. More explicitly, the smash product $X\wedge Y$ is given at level n by the coequalizer of two maps

$$\begin{array}{c} \bigvee_{a+b+c=n} O(n)_+ \wedge_{O(a)\times O(b)\otimes O(c)} X_a\wedge S^b\wedge Y_c \\ \downarrow\downarrow \\ \bigvee_{i+j=n} O(n)_+ \times_{O(i)\times O(j)} X_i\wedge Y_j\,. \end{array}$$

A map of orthogonal spectra $\mu\colon X\wedge Y\to Z$ is equivalent to a collection of $O(i)\times O(j)$-equivariant maps

$$\mu_{i,j}\colon X_i\wedge Y_j \longrightarrow Z_{i+j}$$

for $i,j\ge 0$, making the bilinearity diagram

$$\begin{array}{ccccc} X_a\wedge S^1\wedge Y_c & \xrightarrow[\cong]{\mathrm{id}\wedge\gamma} & X_a\wedge Y_c\wedge S^1 & & \\ \downarrow{\scriptstyle \sigma\wedge\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\wedge\sigma} & \searrow{\scriptstyle \mu_{a,c}\wedge\mathrm{id}} & \\ X_{a+1}\wedge Y_c & & X_a\wedge Y_{c+1} & & Z_{a+c}\wedge S^1 \\ \downarrow{\scriptstyle \mu_{a+1,c}} & & \downarrow{\scriptstyle \mu_{a,c+1}} & \swarrow{\scriptstyle \sigma} & \\ Z_{a+1+c} & \xrightarrow[\cong]{(I_a\oplus\chi_{1,c})\cdot} & Z_{a+c+1} & & \end{array}$$

commute, for all $a,c\ge 0$. Note the appearance of the action of $I_a\oplus\chi_{1,c}\in O(a+1+c)$ on Z_{a+1+c}, which is not available for sequential spectra. See [Schwede, Symmetric Spectra, diagram (5.1)].

Theorem 6.3. *The category $\mathcal{S}p^{\mathbb{O}}$ of orthogonal spectra is closed symmetric monoidal, with unit object S, monoidal pairing $X,Y\mapsto X\wedge Y$, symmetry isomorphism*

$$\gamma\colon X\wedge Y\cong Y\wedge X$$

and internal function object $F(Y,Z)$.

Sketch proof. The diagram

$$X\otimes S\otimes S \underset{\mathrm{id}\otimes\phi}{\overset{\sigma\otimes\mathrm{id}}{\rightrightarrows}} X\otimes S \xrightarrow{\sigma} X$$

is a split coequalizer, which shows that $X\wedge S\cong X$. Left unitality and associativity admits similar proofs. The symmetry isomorphism is induced by $\gamma\colon X\otimes Y\cong Y\otimes X$. The natural adjunction homeomorphism

$$\mathcal{T}^{\mathbb{O}}(X\wedge Y,Z)\cong\mathcal{T}^{\mathbb{O}}(X,F(Y,Z))$$

lifts to a natural isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)) \,.$$

□

This smash product of orthogonal spectra extends that of based spaces.

Lemma 6.4. *There are natural isomorphisms*

$$\Sigma^\infty T \wedge \Sigma^\infty T' \cong \Sigma^\infty(T \wedge T')$$

in $\mathcal{S}p^{\mathbb{O}}$*, for* $T, T' \in \mathcal{T}$*.*

Proof. $\Sigma^\infty T \wedge \Sigma^\infty T' = T \wedge S \wedge T' \wedge S \cong T \wedge T' \wedge S \wedge S \cong T \wedge T' \wedge S = \Sigma^\infty(T \wedge T')$. □

We give the category $gr\mathcal{A}b$ of graded abelian groups the usual symmetric monoidal structure, with symmetry

$$\gamma \colon A \otimes B \cong B \otimes A$$

taking $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$. A lax monoidal functor $\Phi \colon \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories comes with a natural transformation $\cdot \colon \Phi(X) \otimes \Phi(Y) \to \Phi(X \otimes Y)$ and a morphism $U \to \Phi(U)$, and takes monoids to monoids and modules to modules. It is symmetric if

$$\begin{array}{ccc} \Phi(X) \otimes \Phi(Y) & \xrightarrow{\gamma} & \Phi(Y) \otimes \Phi(X) \\ \downarrow{\scriptstyle \cdot} & & \downarrow{\scriptstyle \cdot} \\ \Phi(X \otimes Y) & \xrightarrow{\Phi(\gamma)} & \Phi(Y \otimes X) \end{array}$$

commutes, in which case it takes commutative monoids to commutative monoids.

((ETC: Properly define lax (symmetric) monoidal and closed functors?))

Theorem 6.5. *There is natural pairing*

$$\begin{aligned} \cdot \colon \pi_*(X) \otimes \pi_*(Y) &\longrightarrow \pi_*(X \wedge Y) \\ \alpha \otimes \beta &\longmapsto \alpha \cdot \beta \end{aligned}$$

and a homomorphism $\mathbb{Z} \to \pi_*(S)$ *that make* π_* *a closed and lax symmetric monoidal functor from* $(\mathcal{S}p^{\mathbb{O}}, S, \wedge)$ *to* $(gr\mathcal{A}b, \mathbb{Z}, \otimes)$*.*

Proof. See [Rognes, MAT9580/2017, Stable Homotopy Theory, Thm. 6.8]. Let X and Y be orthogonal spectra, and let $\iota_{n,m} \colon X_n \wedge Y_m \to (X \wedge Y)_{n+m}$ be the $O(n) \times O(m)$-equivariant components of the identity map of $X \wedge Y$. Given $\alpha \in \pi_k(X)$ and $\beta \in \pi_\ell(Y)$, represented by $f \colon S^{k+n} \to X_n$ and $g \colon S^{\ell+m} \to Y_m$, respectively, we can form the composite

$$f * g \colon S^{k+n} \wedge S^{\ell+m} \overset{f \wedge g}{\longrightarrow} X_n \wedge Y_m \overset{\iota_{n,m}}{\longrightarrow} (X \wedge Y)_{n+m} \,.$$

Its homotopy class in $\pi_{k+n+\ell+m}((X \wedge Y)_{n+m})$ only depends on $[f]$ and $[g]$, so we can let $[f] * [g] = [f * g]$. Let

$$\begin{aligned} f' &= \sigma(f \wedge \mathrm{id}) \colon S^{k+n+1} \to X_{n+1} \\ g' &= \sigma(g \wedge \mathrm{id}) \colon S^{\ell+m+1} \to Y_{m+1} \\ (f * g)' &= \sigma(f * g \wedge \mathrm{id}) \colon S^{k+n+\ell+m+1} \to (X \wedge Y)_{n+m+1} \end{aligned}$$

denote the stabilized maps. The bilinearity diagram shows that

$$f * g' = (f * g)' \qquad \text{and} \qquad (f * g)'(\mathrm{id} \wedge \gamma) = (I_n \oplus \chi_{1,m})(f' * g) \,.$$

Here $\gamma\colon S^1 \wedge S^{\ell+m} \to S^{\ell+m} \wedge S^1$ has degree $(-1)^{\ell+m}$, and multiplication by $I_n \oplus \chi_{1,m}$ has degree $(-1)^m$, so it follows that

$$[f * g'] = [(f * g)'] = (-1)^\ell [f' * g] .$$

To compensate for the sign $(-1)^\ell$ that appears when n is incremented, we let

$$[f] \cdot [g] = (-1)^{\ell n} [f * g]$$

in $\pi_{k+\ell+n+m}((X \wedge Y)_{m+n})$. We define $\alpha \cdot \beta$ to be its stable class in $\pi_{k+\ell}(X \wedge Y)$, which only depends on the stable classes α and β. □

If $\ell \geq 0$, then the sign $(-1)^{\ell n}$ is realized by $\gamma\colon S^\ell \wedge S^n \cong S^n \wedge S^\ell$, so $[f] \cdot [g]$ is the homotopy class of the composite

$$f \cdot g\colon S^{k+\ell+n+m} = S^k \wedge S^\ell \wedge S^n \wedge S^m \overset{\mathrm{id} \wedge\gamma\wedge \mathrm{id}}{\longrightarrow} S^k \wedge S^n \wedge S^\ell \wedge S^m$$
$$\overset{f \wedge g}{\longrightarrow} X_n \wedge Y_m \overset{\iota_{n,m}}{\longrightarrow} (X \wedge Y)_{n+m} .$$

This suffices, e.g., to present the right $\pi_*(S)$-action on $\pi_*(X)$.

7. Orthogonal ring and module spectra

Definition 7.1. An orthogonal ring spectrum, also called an S-algebra, is a monoid in $(\mathcal{S}p^{\mathbb{O}}, S, \wedge)$, i.e., an orthogonal spectrum R with a unit map $\eta\colon S \to R$ and product map $\phi\colon R \wedge R \to R$, satisfying unitality and associativity.

A commutative orthogonal ring spectrum, or commutative S-algebra, is a commutative monoid in $\mathcal{S}p^{\mathbb{O}}$, meaning that $\phi = \phi\gamma\colon R \wedge R \to R$.

A right R-module spectrum is a right R-module in $\mathcal{S}p^{\mathbb{O}}$, i.e., an orthogonal spectrum M with a right action map $\rho\colon M \wedge R \to M$ satisfying unitality and associativity. A left R-module spectrum is a left R-module in $\mathcal{S}p^{\mathbb{O}}$, i.e., an orthogonal spectrum N with a left action map $\lambda\colon R \wedge N \to N$ satisfying unitality and associativity.

With M and N as above, the relative smash product $M \wedge_R N$ is the coequalizer

$$M \wedge R \wedge N \underset{\mathrm{id} \wedge \lambda}{\overset{\rho \wedge \mathrm{id}}{\rightrightarrows}} M \wedge N \overset{\pi}{\longrightarrow} M \wedge_R N$$

in $\mathcal{S}p^{\mathbb{O}}$. If R is commutative, then left and right R-actions are interchangeable, and $M \wedge_R N$ is again an R-module.

((ETC: Can also discuss $F_R(M, N)$.))

Lemma 7.2. *If R is an orthogonal ring spectrum, then $\pi_*(R)$ is a graded ring. If R is commutative, then $\pi_*(R)$ is graded commutative. If M is a right R-module, then $\pi_*(M)$ is a right $\pi_*(R)$-module. If N is a left R-module, then $\pi_*(N)$ is a left $\pi_*(R)$-module. There is a natural homomorphism*

$$\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N) \overset{\cdot}{\longrightarrow} \pi_*(M \wedge_R N) .$$

((ETC: Also $\pi_* F_R(M, N) \to \mathrm{Hom}_{\pi_*(R)}(\pi_*(M), \pi_*(N))$.))

Proof. The lax monoidal pairing $\pi_*(M) \otimes \pi_*(N) \to \pi_*(M \wedge N) \to \pi_*(M \wedge_R N)$ equalizes the two homomorphisms from $\pi_*(M) \otimes \pi_*(R) \otimes \pi_*(N)$, hence factors through $\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N)$. □

Example 7.3. The spectra S, MO, MSO, MU, KO, KU and HR for any commutative ring R admit models as commutative orthogonal ring spectra. For example, the multiplication $\mu\colon MO \wedge MO \to MO$ is given by the maps

$$\mu_{i,j}\colon MO(i) \wedge MO(j) \longrightarrow MO(i+j)$$

obtained by Thomification from the Whitney sum map $BO(i) \times BO(j) \to BO(i+j)$. Each

$$MO(n) = EO(n)_+ \wedge_{O(n)} S^n = B(O(n), S^n)$$

(using the bar construction from Chapter 3, Definition 10.8) comes with a left $O(n)$-action, given by conjugation on the group $O(n)$ and the standard action on S^n, and $\mu_{i,j}$ becomes $O(i) \times O(j)$-equivariant. The spectrum MU is most naturally a unitary spectrum, but is π_*-isomorphic to an orthogonal spectrum with n-th space $\Omega^n MU(n)$, equipped with the multiplication

$$\Omega^i MU(i) \wedge \Omega^j MU(j) \longrightarrow \Omega^{i+j}(MU(i) \wedge MU(j)) \xrightarrow{\Omega^{i+j}\mu^{\mathbb{C}}_{i,j}} \Omega^{i+j} MU(i+j)\,.$$

See [Schwede, Symmetric Spectra, Example 1.18].

((ETC: Discuss (orthogonal) ring spectrum maps $S \to MU \to KU$ later.))

8. The smash product in the stable homotopy category

The model structure on $\mathcal{S}p^{\mathbb{O}}$ is monoidal, satisfying a so-called pushout-product axiom. This implies that for any cofibrant replacements $X^c \to X$, $Y^c \to Y$ and fibrant replacement $Z \to Z^f$ the induced maps

$$X^c \wedge Y \xleftarrow{\simeq} X^c \wedge Y^c \xrightarrow{\simeq} X \wedge Y^c$$

and

$$F(Y^c, Z) \xrightarrow{\simeq} F(Y^c, Z^f) \xleftarrow{\simeq} F(Y, Z^f)$$

are π_*-isomorphisms. Hence the closed symmetric monoidal structure on $\mathcal{S}p^{\mathbb{O}}$ descends to $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$, giving a (derived) smash product

$$\begin{aligned} \wedge\colon \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \times \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) &\longrightarrow \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \\ X, Y &\longmapsto X^c \wedge Y^c \end{aligned}$$

and (derived) function spectrum

$$\begin{aligned} F\colon \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})^{op} \times \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) &\longrightarrow \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \\ Y, Z &\longmapsto F(Y^c, Z^f) \end{aligned}$$

making $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$ closed symmetric monoidal. In particular, there are compatible isomorphisms

$$\begin{gathered} S \wedge Y \cong Y \cong Y \wedge S \\ (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z) \\ \gamma\colon X \wedge Y \cong Y \wedge X \\ F(X \wedge Y, Z) \cong F(X, F(Y, Z)) \end{gathered}$$

in $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$. The symmetric monoidal part of this structure was developed "by hand" on pages 158–190 of [Ada74].

The closed symmetric monoidal and triangulated structures on $\mathrm{Ho}(\mathcal{S}p)$ are compatible.

Lemma 8.1. *(a) For each distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and spectrum W the triangles

$$W \wedge X \xrightarrow{\mathrm{id} \wedge f} W \wedge Y \xrightarrow{\mathrm{id} \wedge g} W \wedge Z \xrightarrow{\mathrm{id} \wedge h} \Sigma(W \wedge X)$$

$$X \wedge W \xrightarrow{f \wedge \mathrm{id}} Y \wedge W \xrightarrow{g \wedge \mathrm{id}} Z \wedge W \xrightarrow{(\mathrm{id} \wedge \gamma)(h \wedge \mathrm{id})} \Sigma(X \wedge W)$$

$$F(W, X) \xrightarrow{F(\mathrm{id}, f)} F(W, Y) \xrightarrow{F(\mathrm{id}, g)} F(W, Z) \xrightarrow{\kappa^{-1} F(\mathrm{id}, h)} \Sigma F(W, X)$$

$$\Sigma^{-1} F(X, W) \xrightarrow{-F(h, \mathrm{id})} F(Z, W) \xrightarrow{F(g, \mathrm{id})} F(Y, W) \xrightarrow{F(f, \mathrm{id})} F(X, W)$$

are distinguished.

(b) The composite

$$\Sigma S^1 = S^1 \wedge S^1 \xrightarrow{\gamma} S^1 \wedge S^1 = \Sigma S^1$$

is multiplication by -1.

Note the minus sign in $-F(h, \mathrm{id})$. Mapping out of a cofiber sequence defines a fiber sequence, which stably differs by this sign from a cofiber sequence. May [May01] gives more compatibility conditions satisfied in $\mathrm{Ho}(\mathcal{S}p)$. The full compatibility story is perhaps best accounted for by presentably symmetric monoidal stable ∞-categories.

The symmetric monoidal and truncation structures on $\mathrm{Ho}(\mathcal{S}p)$ are also compatible.

Lemma 8.2. *(a) S is connective, with $\mathbb{Z} \cong \pi_0(S)$.*

(b) If X and Y are connective, then so is $X \wedge Y$, with $\pi_0(X) \otimes \pi_0(Y) \cong \pi_0(X \wedge Y)$.

Proof. (a) This is a consequence of the Hurewicz theorem.

(b) There are cofibrant replacements $X^c \to X$ and $Y^c \to Y$ where X^c and Y^c are CW spectra with cellular complexes ending with the exact sequences

$$C_1(X^c) \xrightarrow{\partial} C_0(X^c) \longrightarrow \pi_0(X) \longrightarrow 0$$

$$C_1(Y^c) \xrightarrow{\partial} C_0(Y^c) \longrightarrow \pi_0(Y) \longrightarrow 0 .$$

Then $X^c \wedge Y^c$ is a CW spectrum with cellular complex ending with an exact sequence

$$C_1(X^c) \otimes C_0(Y^c) \oplus C_0(X^c) \otimes C_1(Y^c) \xrightarrow{\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial} C_0(X^c) \otimes C_0(Y^c) \longrightarrow \pi_0(X \wedge Y) \longrightarrow 0 .$$

This implies that $\pi_0(X) \otimes \pi_0(Y) \cong \pi_0(X \wedge Y)$. □

((ETC: Can also note that $\pi_0(X) \cong H_0(X)$ for connective X, and appeal to the Künneth theorem in homology.))

Example 8.3. For abelian groups A and B the 0-truncation

$$HA \wedge HB \longrightarrow \tau_{\geq 0}(HA \wedge HB) \simeq H(A \otimes B)$$

of the smash product of two Eilenberg–MacLane spectra is the Eilenberg–MacLane spectrum of the tensor product. In general, this map is not an equivalence. For instance, $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = \mathscr{A}_*$ is the mod p Steenrod algebra.

Lemma 8.4. *Let R be an orthogonal ring spectrum, M a right R-module and N a left R-module.*

(a) If $\pi_*(M) \cong \pi_*(R)\{g_\alpha\}_\alpha$ is free as a right $\pi_*(R)$-module then

$$M \simeq \bigvee_\alpha \Sigma^{|g_\alpha|} R$$

as right R-modules, and $\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N) \cong \pi_*(M \wedge_R N)$.

(b) More generally, there is a natural strongly convergent Tor*-spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(R)}_{*,*}(\pi_*(M), \pi_*(N)) \Longrightarrow \pi_*(M \wedge_R N)\,.$$

Proof. (a) We represent the module generator g_α by maps

$$g_\alpha \colon S^{|g_\alpha|} \longrightarrow M$$

and extend these using the R-action to obtain maps

$$\Sigma^{|g_\alpha|} R \cong S^{|g_\alpha|} \wedge R \xrightarrow{g_\alpha \wedge \mathrm{id}} M \wedge R \xrightarrow{\rho} M\,.$$

Their direct sum g over α induces the assumed isomorphism

$$\bigoplus_\alpha \Sigma^{|g_\alpha|} \pi_*(R) \cong \pi_*(\bigvee_\alpha \Sigma^{|g_\alpha|} R) \xrightarrow{\cong} \pi_*(M)\,,$$

hence is an equivalence. It follows that

$$\bigvee_\alpha \Sigma^{|g_\alpha|} N \cong \bigvee_\alpha \Sigma^{|g_\alpha|} R \wedge_R N \xrightarrow{g \wedge_R \mathrm{id}} M \wedge_R N$$

also is an equivalence, and here

$$\pi_*(\bigvee_\alpha \Sigma^{|g_\alpha|} N) \cong \bigoplus_\alpha \Sigma^{|g_\alpha|} \pi_*(R) \otimes_{\pi_*(R)} \pi_*(N) \cong \pi_*(M) \otimes_{\pi_*(R)} \pi_*(N)\,.$$

(b) Any free $\pi_*(R)$-module resolution

$$\dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi_*(M) \longrightarrow 0$$

can be spectrally realized by the associated graded of a filtered R-module spectrum

$$* \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_\infty$$

with $M_\infty \simeq M$. Apply $- \wedge_R N$ to this filtration, and consider the associated spectral sequence. See [EKMM97, §IV.5] for the details. □

Corollary 8.5. *Let R be a ring, M a right R-module and N a left R-module, so that HR is an orthogonal ring spectrum, HM a right HR-module and HN a left HR-module. Then*

$$\pi_*(HM \wedge_{HR} HN) \cong \mathrm{Tor}^R_*(M, N)\,.$$

((ETC: Can also discuss $F_R(M, N)$ and the Ext spectral sequence.))

Corollary 8.6. *Let R be a ring, and M and N right R-modules, so that HR is an orthogonal ring spectrum, and HM and HN are right HR-modules. Then*

$$\pi_{-*}(F_{HR}(HM, HN)) \cong \mathrm{Ext}^*_R(M, N)\,.$$

Remark 8.7. Before the invention of symmetric and orthogonal spectra, the term "ring spectrum" meant a monoid in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p) \simeq \mathcal{B}$, i.e., a spectrum R with morphisms $\eta\colon S \to R$ and $\phi\colon R \wedge R \to R$ such that the unitality and associativity diagrams commute in $\mathrm{Ho}(\mathcal{S}p)$, i.e., up to homotopy. Similarly, a "module spectrum" meant a module in $\mathrm{Ho}(\mathcal{S}p)$, with a morphism $\rho\colon M \wedge R \to M$ such that the unitality and associativity diagrams commute up to homotopy. This makes $\pi_*(R)$ a graded ring and $\pi_*(M)$ a right $\pi_*(R)$-module, but does not suffice to define $M \wedge_R N$. Nonetheless, if $\pi_*(M)$ is free as a right $\pi_*(R)$-module, then M is equivalent to a wedge sum of suspensions of R, as in the first part of Lemma 8.4(a).

Definition 8.8. We refer to monoids and modules in $\mathrm{Ho}(\mathcal{S}p)$ as ring spectra up to homotopy, and module spectra up to homotopy, respectively.

Example 8.9. Let p be a prime, and let the mod p Moore spectrum $S/p = Cp$ be the mapping cone of the multiplication-by-p map $p\colon S \to S$. The smash product of S/p with the homotopy cofiber sequence

$$S \xrightarrow{p} S \xrightarrow{i} S/p \xrightarrow{j} \Sigma S$$

is a homotopy cofiber sequence

$$S/p \wedge S \xrightarrow{\mathrm{id} \wedge p} S/p \wedge S \xrightarrow{\mathrm{id} \wedge i} S/p \wedge S/p \xrightarrow{\mathrm{id} \wedge j} \Sigma(S/p \wedge S)$$

which is isomorphic to

$$S/p \xrightarrow{p} S/p \xrightarrow{i'} S/p \wedge S/p' \xrightarrow{j'} \Sigma S/p\,.$$

If p is odd then $[S/p, S/p] \cong \mathbb{Z}/p$ and the map $p\colon S/p \to S/p$ is null-homotopic. Hence there exists a retraction $S/p \wedge S/p \to S/p$ in the stable homotopy category. This is left and right unital up to homotopy, and turns out to be associative up to homotopy if $p \neq 3$. Hence S/p is a ring spectrum up to homotopy for $p \geq 5$, while $S/3$ is a "non-associative" ring spectrum up to homotopy.

If $p = 2$ then $[S/2, S/2] \cong \mathbb{Z}/4$ and the map $2\colon S/2 \to S/2$ is essential. Hence there is no (left or right) unital pairing $S/2 \wedge S/2 \to S/2$, and $S/2$ is not a ring spectrum. One way to see this, due to Barratt, is to use that $H^*(S/2; \mathbb{F}_2) \cong \mathbb{F}_2\{1, Sq^1\}$ and

$$H^*(S/2 \wedge S/2; \mathbb{F}_2) \cong \mathbb{F}_2\{1, Sq^1\} \otimes \mathbb{F}_2\{1, Sq^1\}$$

with $Sq^2(1 \otimes 1) = Sq^1 \otimes Sq^1 \neq 0$ by the Cartan formula. This would have to be zero if $S/2$ were a retract up to homotopy of $S/2 \wedge S/2$.

The following recent result was contrary to every expectation.

Theorem 8.10 (Burklund (arXiv:2203.14787))**.** *The Moore spectra $S/8$ and S/p^2, for any odd prime p, can be realized as (strictly unital and associative) orthogonal ring spectra.*

9. Spectral homology and cohomology

We now extend G. Whitehead's Theorem 4.9.

Theorem 9.1. *Let E be a spectrum. The functors*

$$X \mapsto E_k(X) = \pi_k(E \wedge X)$$

and the suspension isomorphisms

$$E_k(X) = \pi_k(E \wedge X) \xrightarrow{\cong} \pi_{k+1}(E \wedge \Sigma X) = E_{k+1}(\Sigma X)$$

define a homology theory on all spectra X.

Theorem 9.2. *Let E be a ring spectrum in the homotopy category. There are natural pairings*

$$E_i(X) \wedge E_j(Y) \longrightarrow E_{i+j}(X \wedge Y)$$

and

$$E^i(X) \wedge E^j(Y) \longrightarrow E^{i+j}(X \wedge Y)$$

making $E_*(-)$ a multiplicative homology theory and $E^*(-)$ a multiplicative cohomology theory. In particular, $E_*(Y)$ is naturally a left $E_*(S) = \pi_*(E)$-module and $E^*(Y)$ is naturally a left $E^*(S) = \pi_{-*}(E)$-module.

Sketch proof. The composition

$$\pi_i(E \wedge X) \otimes \pi_j(E \wedge Y) \xrightarrow{\cdot} \pi_{i+j}(E \wedge X \wedge E \wedge Y) \xrightarrow{\cong} \pi_{i+j}(E \wedge E \wedge X \wedge Y) \xrightarrow{\phi_*} \pi_{i+j}(E \wedge X \wedge Y)$$

defines the homology pairing. The composition

$$[X, \Sigma^i E] \otimes [Y, \Sigma^j E] \xrightarrow{\wedge} [X \wedge Y, \Sigma^i E \wedge \Sigma^j E] \cong [X \wedge Y, \Sigma^{i+j}(E \wedge E)] \xrightarrow{\phi_*} [X \wedge Y, \Sigma^{i+j} E]$$

defines the cohomology pairing. The left module actions correspond to the case $X = S$. □

Next we follows Adams [Ada69, Lec. III] and interpret $E_*(X)$ as an E_*E-comodule, subject to a flatness condition on E.

Definition 9.3. Let (E, η, ϕ) be a ring spectrum in the homotopy category. We briefly write

$$E_* = \pi_*(E) \qquad \text{and} \qquad E_*E = E_*(E) = \pi_*(E \wedge E)\,.$$

Then E_* is a graded ring, and E_*E is an E_*-E_*-bimodule, with left E_*-action induced by

$$E \wedge E \wedge E \xrightarrow{\phi \wedge \mathrm{id}} E \wedge E$$

and right E_*-action induced by

$$E \wedge E \wedge E \xrightarrow{\mathrm{id} \wedge \phi} E \wedge E\,.$$

Moreover, E_*E is a graded ring, with multiplication induced by

$$E \wedge E \wedge E \wedge E \xrightarrow{\mathrm{id} \wedge \gamma \wedge \mathrm{id}} E \wedge E \wedge E \wedge E \xrightarrow{\phi \wedge \phi} E \wedge E\,.$$

The left E_*-action on E_*E is then given by $\lambda(a \otimes b) = \eta_L(a) \cdot b$, where $\eta_L \colon E_* \to E_*E$ is the left unit homomorphism induced by

$$E \cong E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E\,,$$

and the right E_*-action on E_*E is given by $\rho(b \otimes c) = b \cdot \eta_R(c)$, where $\eta_R \colon E_* \to E_*E$ is the right unit homomorphism induced by

$$E \cong S \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E\,.$$

The ring spectrum multiplication $\phi\colon E\wedge E \to E$ induces an augmentation $\epsilon\colon E_*E \to E_*$, with $\epsilon \circ \eta_L = \mathrm{id} = \epsilon \circ \eta_R$.

In the case $E = H\mathbb{F}_p$ we have $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the mod p dual Steenrod algebra. The left and right units are both the degree zero inclusion $\mathbb{F}_p \to \mathscr{A}_*$. In general, the left and right units $\eta_L, \eta_R\colon E_* \to E_*E$ will be different homomorphisms. If E is homotopy commutative, i.e., a commutative ring spectrum in the homotopy category, then E_* and E_*E are graded commutative, and the conjugation (= antipode/involution) isomorphism

$$\chi\colon E_*E \xrightarrow{\cong} E_*E$$

induced by the symmetry $\gamma\colon E \wedge E \cong E \wedge E$ satisfies $\chi^2 = \mathrm{id}$ and $\chi \circ \eta_L = \eta_R$. Hence the left E_*-module E_*E is isomorphic via χ to the right E_*-module E_*E.

Definition 9.4. Let E be a commutative ring spectrum in the homotopy category. We say that E is flat if E_*E is flat as a left (or, equivalently, right) E_*-module.

The map

$$E \wedge E \wedge E \wedge X \xrightarrow{\mathrm{id}\wedge\phi\wedge\mathrm{id}} E \wedge E \wedge X$$

induces a pairing

$$E_*E \otimes E_*(X) \longrightarrow \pi_*(E \wedge E \wedge X)$$

which equalizes the two usual homomorphisms from $E_*E\otimes E_*\otimes E_*(X)$ and therefore factors uniquely through $E_*E \otimes_{E_*} E_*(X)$.

Lemma 9.5. *If E is flat, then*

$$E_*E \otimes_{E_*} E_*(X) \stackrel{\cdot}{\longrightarrow} \pi_*(E \wedge E \wedge X)$$

is an isomorphism, for each spectrum X.

Proof. Since E_*E is flat as a right E_*-module, this is a morphism of homology theories that is an isomorphism for $X = S$. It follows that it is an isomorphism for all X. (If E_*E is free as a right E_*-module, then one can also prove this using a splitting of $E \wedge E$ as a wedge sum of suspensions of E.) □

Definition 9.6. If E is flat, let

$$\nu\colon E_*(X) \longrightarrow E_*E \otimes_{E_*} E_*(X)$$

be the composite homomorphism

$$\pi_*(E \wedge X) = \pi_*(E \wedge S \wedge X) \xrightarrow{(\mathrm{id}\wedge\eta\wedge\mathrm{id})_*} \pi_*(E \wedge E \wedge X) \cong E_*E \otimes_{E_*} E_*(X)\,.$$

In the case $X = E$, we write

$$\psi\colon E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$$

for this homomorphism. Note that in the target the tensor product is formed with respect to the right E_*-action on the left hand copy of E_*E and with respect to the left E_*-action on the right hand copy of E_*E.

Lemma 9.7. *If E is flat, then the left E_*-module $E_*(X)$ is naturally a left E_*E-comodule, in the sense that the diagrams*

$$\begin{array}{ccc} E_*(X) & \xrightarrow{\nu} & E_*E \otimes_{E_*} E_*(X) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \epsilon\otimes \mathrm{id}} \\ & & E_* \otimes_{E_*} E_*(X) \end{array}$$

and

$$\begin{array}{ccc} E_*(X) & \xrightarrow{\nu} & E_*E \otimes_{E_*} E_*(X) \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\nu} \\ E_*E \otimes_{E_*} E_*(X) & \xrightarrow{\psi\otimes\mathrm{id}} & E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_*(X) \end{array}$$

commute.

Let $E_*E-\mathrm{coMod} = \mathrm{coMod}_{E_*E}$ denote the category of E_*E-comodules. The E_*E-coaction ν defines a lift

$$\begin{array}{ccc} & & E_*E-\mathrm{coMod} \\ & \nearrow^{(E_*(-),\nu)} & \downarrow{\scriptstyle \mathbb{U}} \\ \mathrm{Ho}(\mathcal{S}p) & \xrightarrow[E_*(-)]{} & E_*-\mathrm{Mod} \end{array}$$

of the E-homology functor $X \mapsto E_*(X)$, also keeping track of the E_*E-coaction, or cooperations.

Example 9.8. When $E = H\mathbb{F}_p$, so that $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the left $\mathscr{A}_*$-coaction

$$\nu\colon H_*(X;\mathbb{F}_p) \longrightarrow \mathscr{A}_* \otimes H_*(X;\mathbb{F}_p)$$

is now naturally defined for arbitrary spectra X, and agrees with that obtained earlier, under suitable finiteness hypotheses, by dualization from the left $\mathscr{A}$-module action λ on $H^*(X;\mathbb{F}_p)$.

10. Hopf algebroids

Definition 10.1. Let $\mathscr{A}_*^E = E_*E = E_*(E)$ be the E-based dual Steenrod algebra.

So far we have only discussed comodules over coalgebras (and bialgebras), but in general E_*E is not a coalgebra in the classical sense. We shall now pin down its precise bialgebraic structure. This will involve structure on the pair (E_*, E_*E).

Theorem 10.2 ([Ada69, Lec. III])**.** *If E is flat, then (E_*, E_*E) is a Hopf algebroid.*

This means that E_* and E_*E are graded commutative rings, there are ring homomorphisms

$$\eta_L\colon E_* \longrightarrow E_*E$$
$$\eta_R\colon E_* \longrightarrow E_*E$$
$$\epsilon\colon E_*E \longrightarrow E_*$$
$$\psi\colon E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$$
$$\chi\colon E_*E \longrightarrow E_*E\,,$$

these satisfy the relations

$$\epsilon\eta_L = \mathrm{id} = \epsilon\eta_R$$

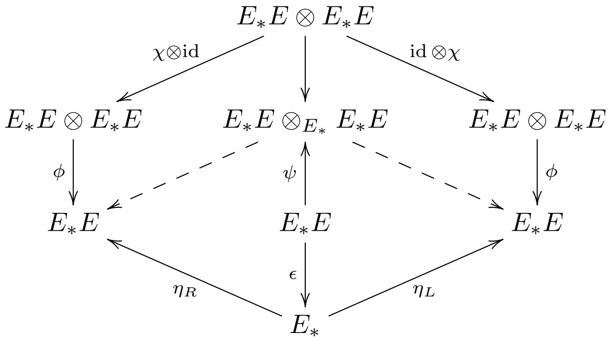
$$\psi\eta_L = (\mathrm{id}\otimes\eta_L)\eta_L \qquad \text{and} \qquad \psi\eta_R = (\eta_R\otimes\mathrm{id})\eta_R$$

$$(\epsilon\otimes\mathrm{id})\psi = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\psi$$

$$(\psi\otimes\mathrm{id})\psi = (\mathrm{id}\otimes\psi)\psi$$

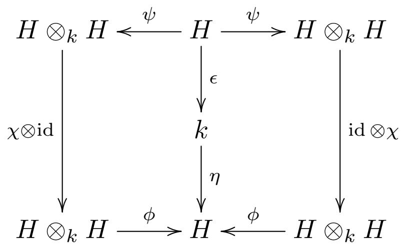
$$\chi^2 = \mathrm{id} \qquad \text{and} \qquad \chi\eta_L = \eta_R\,,$$

and there are dashed arrows making the diagram



commute. See [Rav86, Def. A1.1.1]. The terminology "Hopf algebroid" is due to Haynes Miller, and can be motivated by Grothendieck's functor of points perspective, as we now discuss. The appendix [Rav86, A1] is a standard reference for Hopf algebroids and their homological algebra.

Definition 10.3. Let k be a (graded) commutative ring. A k-Hopf algebra is a k-bialgebra (H, ϵ, ψ) with a k-linear homomorphism $\chi\colon H \to H$, called the conjugation (or antipode) such that



commutes.

A bialgebra admits at most one conjugation χ, so being a Hopf algebra is a property of bialgebras. If H is commutative, then χ is a k-algebra isomorphism with $\chi^2 = \mathrm{id}$.

Proposition 10.4. *Let $\mathcal{C}Alg_k$ be the category of commutative k-algebras.*

(a) A commutative k-algebra A corepresents a functor

$$\mathrm{Spec}(A)\colon \mathcal{C}Alg_k \longrightarrow \mathcal{S}\mathrm{et}$$
$$R \longmapsto \mathcal{C}Alg_k(A, R)$$

to the category of sets.

(b) If (B, ϵ, ψ) is a commutative k-bialgebra, then $\mathrm{Spec}(B)$ *lifts to a functor*

$$\mathrm{Spec}(B) \colon \mathcal{C}Alg_k \longrightarrow \mathcal{M}on$$

to the category of monoids, with unit $e \in \mathrm{Spec}(B)$ corresponding to ϵ and multiplication $\mathrm{Spec}(B) \times \mathrm{Spec}(B) \to \mathrm{Spec}(B)$ corresponding to ψ.

(c) For a commutative k-bialgebra (H, ϵ, ψ) the functor $\mathrm{Spec}(H)$ lifts to a functor

$$\mathrm{Spec}(H) \colon \mathcal{C}Alg_k \longrightarrow \mathcal{G}p$$

to the category of groups if and only if H is a Hopf k-algebra. In this case the conjugation χ corepresents the group inverse $\mathrm{Spec}(H) \to \mathrm{Spec}(H)$.

Proof. (a) Clear. (b) We have a natural bijection

$$\mathcal{C}Alg_k(B \otimes_k B, R) \cong \mathcal{C}Alg_k(B, R) \times \mathcal{C}Alg_k(B, R)$$

so the k-algebra $B \otimes_k B$ corepresents $\mathrm{Spec}(B) \times \mathrm{Spec}(B)$, while k itself corepresents $*$. Hence $\psi \colon B \to B \otimes_k B$ and $\epsilon \colon B \to k$ induce a natural pairing on $\mathrm{Spec}(B)(R)$ and a preferred element. The counitality and coassociativity axioms for a bialgebra show that these define a natural monoid structure on $\mathrm{Spec}(B)(R)$, so that $\mathrm{Spec}(B)$ lifts through the forgetful functor $\mathcal{M}on \to \mathcal{S}\mathrm{et}$.

(c) The identities $\phi(\chi \otimes \mathrm{id})\psi = \eta\epsilon = \phi(\mathrm{id} \otimes \chi)\psi$ show that for each k-algebra homomorphism $g \colon H \to R$ in the monoid $\mathrm{Spec}(H)(R)$ the composite $g\chi$ represents a group inverse. □

Remark 10.5. We can view $\mathrm{Spec}(A)$ as a representable contravariant functor from $\mathcal{C}Alg_k^{op}$ to $\mathcal{S}\mathrm{et}$, i.e., as an affine presheaf on $\mathcal{C}Alg_k^{op}$. It satisfies faithfully flat descent (meaning that there are equalizer diagrams

$$\mathrm{Spec}(A)(R) \xrightarrow{\iota} \mathrm{Spec}(A)(T) \rightrightarrows \mathrm{Spec}(A)(T \otimes_R T)$$

for $R \to T$ faithfully flat), hence is a flat, étale, Nisnevich and Zariski sheaf defined over $\mathrm{Spec}(k)$. We may refer to it as an affine (étale) sheaf. In the situation of the proposition, $\mathrm{Spec}(B)$ is then an affine monoid sheaf and $\mathrm{Spec}(H)$ is an affine group sheaf.

Recall that a (small) groupoid is a (small) category in which each morphism is invertible, i.e., an isomorphism. Given any morphism $f \colon X \to Y$ we refer to $X = s(f)$ and $Y = t(f)$ as the source and target of f.

Proposition 10.6. *Let E be a flat homotopy commutative ring spectrum. The (graded) commutative rings E_* and E_*E corepresent functors*

$$\begin{aligned} \mathcal{O} = \mathrm{Spec}(E_*) \colon \mathcal{C}Ring &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathcal{C}Ring(E_*, R) \\ \mathcal{M} = \mathrm{Spec}(E_*E) \colon \mathcal{C}Ring &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathcal{C}Ring(E_*E, R) \end{aligned}$$

that constitute the object and morphism components of a functor

$$\begin{aligned} \mathcal{G} \colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{G}(R) \end{aligned}$$

to the category of (small) groupoids. In other words, $\mathcal{G}(R)$ is a groupoid with

$$\operatorname{obj}\mathcal{G}(R) = \mathcal{O}(R) = \mathcal{CR}ing(E_*, R)$$
$$\operatorname{mor}\mathcal{G}(R) = \mathcal{M}(R) = \mathcal{CR}ing(E_*E, R)$$

for all (graded) commutative rings R. The left unit $\eta_L \colon E_* \to E_*E$ corepresents the target $t \colon \mathcal{M}(R) \to \mathcal{O}(R)$, the right unit $\eta_R \colon E_* \to E_*E$ corepresents the source $s \colon \mathcal{M}(R) \to \mathcal{O}(R)$, the augmentation $\epsilon \colon E_*E \to E_*$ corepresents the identity morphism $\mathrm{id} \colon \mathcal{O}(R) \to \mathcal{M}(R)$, the coproduct $\psi \colon E_*E \to E_*E \otimes_{E_*} E_*E$ corepresents the composition law

$$\circ \colon \mathcal{M}(R) \times_{\mathcal{O}(R)} \mathcal{M}(R) \longrightarrow \mathcal{M}(R)\,,$$

and the conjugation $\chi \colon E_*E \to E_*E$ corepresents the passage to inverse $\mathcal{M}(R) \to \mathcal{M}(R)$. The relations and commuting diagram from Theorem 10.2 express the axioms for composition and existence of inverses in a groupoid.

Remark 10.7. The point of Miller's terminology is thus that the Hopf algebroid (E_*, E_*E) corepresents the affine groupoid presheaf

$$R \longmapsto \mathcal{G}(R) = \begin{cases} \mathcal{O}(R) = \mathcal{CR}ing(E_*, R) \\ \mathcal{M}(R) = \mathcal{CR}ing(E_*E, R) \\ \text{plus structure maps,} \end{cases}$$

and this is in fact an affine groupoid sheaf, i.e., a contravariant functor $\mathcal{CR}ing^{op} \to \mathcal{G}pd$ satisfying suitable descent properties. Since a groupoid is more than a set, these descent properties are better described by applying the nerve functor to simplicial sets, and ask that the simplicial presheaf $R \mapsto N\mathcal{G}(R)$ satisfies descent. (This means that the coaugmentation from $N\mathcal{G}(R)$ to the homotopy limit (= totalization) of the (pre-)cosimplicial diagram

$$N\mathcal{G}(T) \rightrightarrows N\mathcal{G}(T \otimes_R T) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} N\mathcal{G}(T \otimes_R T \otimes_R T) \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \dots$$

is a homotopy equivalence, for all covers $R \to T$ in the relevant topology.)

Example 10.8. When $E = H\mathbb{F}_p$, so that $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the conjugation

$$\chi \colon \mathscr{A}_* \longrightarrow \mathscr{A}_*$$

is characterized by the relation $\phi(\mathrm{id} \otimes \chi)\psi = \eta\epsilon$, meaning that

$$\sum_{i+j=k} \zeta_i^{2^j} \chi(\zeta_j) = 0$$

for $k \ge 1$ when $p = 2$, and

$$\tau_k + \sum_{i+j=k} \xi_i^{p^j} \chi(\tau_j) = 0$$

for $k \ge 0$ and

$$\sum_{i+j=k} \xi_i^{p^j} \chi(\xi_j) = 0$$

for $k \ge 1$ when p odd. This uses Milnor's Theorems 8.7 and 8.8 from Chapter 2. These formulas recursively determine χ on the algebra generators, and $\chi^2 = \mathrm{id}$.

Remark 10.9. The groupoid $\mathcal{G}(R)$ has a single object $\mathcal{O}(R) = \mathcal{C}Ring(\mathbb{F}_p, R)$ for each graded commutative $\mathbb{F}_p$-algebra R (and is otherwise empty), and a group $\mathcal{M}(R) = \mathcal{C}Ring(\mathscr{A}_*, R)$ of automorphisms of this object. When $p = 2$, so that $\mathscr{A}_* = \mathbb{F}_2[\zeta_i \mid i \geq 1]$, a homomorphism $\theta\colon \mathscr{A}_* \to R$ corresponds to a sequence of elements $b_i = \theta(\zeta_i)$ in R, for $i \geq 1$. These sequences in turn correspond to formal power series

$$f(x) = \sum_{i\geq 0} b_i x^{2^i} \in x + x^2 R[[x]]$$

with $b_0 = 1$. The composition law in $\mathcal{M}(R)$ takes (θ', θ'') corresponding to $((b'_i)_i, (b''_j)_j)$ and (f', f'') to the homomorphism

$$\theta\colon \mathscr{A}_* \xrightarrow{\psi} \mathscr{A}_* \otimes \mathscr{A}_* \xrightarrow{\theta'\otimes\theta''} R \otimes R \xrightarrow{\phi} R$$

corresponding to the sequence

$$b_k = \sum_{i+j=k} (b'_i)^{2^j} b''_j$$

for $k \geq 1$ and the formal power series

$$f(x) = \sum_{k\geq 0} b_k x^{2^k} = \sum_{i,j\geq 0} (b'_i)^{2^j} b''_j x^{2^{i+j}} ,$$

which is also equal to the formal composition

$$f''(f'(x)) = f''(\sum_{i\geq 0} b'_i x^{2^i}) = \sum_{j\geq 0} b''_j (\sum_{i\geq 0} b'_i x^{2^i})^{2^j} .$$

Hence $\mathcal{G}(R) = \mathcal{B}(\mathcal{M}(R))$ is the one-object groupoid associated to

$$\mathcal{M}(R) \cong \{f(x) = \sum_{i\geq 0} b_i x^{2^i}\} \subset x + x^2 R[[x]]$$

with the group structure $(f', f'') \mapsto f'' \circ f'$ given by composition of certain formal power series. These power series $f(x) = x + \sum_{i\geq 1} b_i x^{2^i}$ are precisely those satisfying the functional equation

$$f(x) + f(y) = f(x + y) .$$

In other words, these $f(x)$ are the strict automorphisms $f\colon F_a \to F_a$ of the additive formal group law $F_a(x, y) = x + y$ over $\mathbb{F}_2$. The groupoid sheaf for $E = H\mathbb{F}_2$ is thus isomorphic

$$\mathcal{G}_{H\mathbb{F}_2} \cong \mathcal{B}\,\mathrm{Aut}_s(F_a/\mathbb{F}_2)$$

to the classifying sheaf for the strict automorphism group sheaf of F_a over $\mathbb{F}_2$. The corresponding result for $E = MU$ is central to chromatic homotopy theory.

((ETC: Harder to say this for odd p?))
((ETC: Can add grading, or interpret that in terms of $\mathbb{G}_m$-bundles.))

11. Spanier–Whitehead duality

((ETC: For finite cell spectra Y let $DY = F(Y, S)$. Then $\kappa\colon DY \wedge Z \to F(Y, Z)$ is an equivalence, so $[X \wedge Y, Z] \cong [X, DY \wedge Z]$ and $Y \simeq DDY$. In particular, there are natural isomorphisms $E^{-k}(Y) \cong E_k(DY)$ and $E_k(Y) \cong E^{-k}(DY)$. Lift to account for E-based Steenrod operations?))

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Department of Mathematics, University of Oslo, Norway
Email address: rognes@math.uio.no