ALGEBRAIC TOPOLOGY III SPRING 2023 CHROMATIC HOMOTOPY THEORY

CHAPTER 8: SPECTRAL SEQUENCES

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Given a map $f\colon X \to Y$ of spectra, we can use the long exact sequence of homotopy groups

$$\dots \to \pi_{*+1}(Cf) \xrightarrow{\partial} \pi_*(X) \xrightarrow{f_*} \pi_*(Y) \longrightarrow \pi_*(Cf) \xrightarrow{\partial} \pi_{*-1}(X) \to \dots$$

to attempt to calculate $\pi_*(Y)$ from $\pi_*(X)$ and $\pi_*(Cf)$. By exactness at $\pi_*(Y)$, these two graded abelian groups give an upper bound for $\pi_*(X)$. By also taking into account exactness at $\pi_*(X)$ and at $\pi_*(Cf)$ we can replace $\pi_*(X)$ by $\operatorname{cok}(\partial\colon \pi_{*+1}(Cf) \to \pi_*(X))$, and replace $\pi_*(Cf)$ by $\ker(\partial\colon \pi_*(Cf) \to \pi_{*-1}(X))$, and still have an exact sequence

$$0 \to \operatorname{cok}(\partial) \longrightarrow \pi_*(Y) \longrightarrow \ker(\partial) \to 0\,.$$

This then gives a precise upper bound for $\pi_*(Y)$, determining this graded abelian group up to extension. We now aim to extend this discussion from the case of $f\colon X \to Y$ to longer sequences of maps, possibly continuing without bound to the left, to the right, or in both directions.

1. Sequences of spectra and exact couples

Let

$$\dots \to Y_{s+2} \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} Y_{s-1} \to \dots$$

be a sequence of spectra. We call $s \in \mathbb{Z}$ the filtration index.

Let the mapping telescope, or sequential homotopy colimit $Y_{-\infty} = \operatorname{hocolim}_s Y_s$ be the homotopy coequalizer of the two maps

$$\begin{tikzcd} \bigvee_s Y_s \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\alpha^\vee"'] & \bigvee_s Y_s \end{tikzcd}$$

where

$$\begin{tikzcd} Y_s \arrow[r, "\alpha"] \arrow[d, "in_s"'] & Y_{s-1} \arrow[d, "in_{s-1}"] \\ \bigvee_s Y_s \arrow[r, "\alpha^\vee"] & \bigvee_s Y_s \end{tikzcd}$$

commutes for each s. We get a homotopy cofiber sequence

$$\bigvee_s Y_s \xrightarrow{\mathrm{id}-\alpha^\vee} \bigvee_s Y_s \xrightarrow{\iota} Y_{-\infty}\,,$$

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where

$$\bigoplus_s \pi_*(Y_s) \xrightarrow{\mathrm{id} - \alpha_*^\vee} \bigoplus_s \pi_*(Y_s)$$

is injective with cokernel $\mathrm{colim}_s\, \pi_*(Y_s)$. Hence the long exact sequence in homotopy breaks up into short exact sequences, and

$$\iota\colon \operatorname*{colim}_s \pi_*(Y_s) \cong \pi_*(Y_{-\infty}) \,.$$

Let the mapping microscope, or sequential homotopy limit $Y_\infty = \mathrm{holim}_s\, Y_s$ be the homotopy equalizer of the two maps

$$\prod_s Y_s \underset{\alpha^\Pi}{\overset{\mathrm{id}}{\rightrightarrows}} \prod_s Y_s$$

where

$$\begin{tikzcd} \prod_s Y_s \arrow[r, "\alpha^\Pi"] \arrow[d, "pr_{s+1}"'] & \prod_s Y_s \arrow[d, "pr_s"] \\ Y_{s+1} \arrow[r, "\alpha"] & Y_s \end{tikzcd}$$

commutes for each s. We get a homotopy (co-)fiber sequence

$$Y_\infty \xrightarrow{\pi} \prod_s Y_s \xrightarrow{\mathrm{id} - \alpha^\Pi} \prod_s Y_s$$

where

$$\prod_s \pi_*(Y_s) \xrightarrow{\mathrm{id} - \alpha_*^\Pi} \prod_s \pi_*(Y_s)$$

has kernel $\lim_s \pi_*(Y_s)$ and cokernel $\mathrm{Rlim}_s\, \pi_*(Y_s)$. Here $\mathrm{Rlim}_s = \lim^1_s$ is the (first) right derived functor of the sequential limit. The long exact sequence in homotopy yields short exact sequences

$$0 \to \operatorname*{Rlim}_s \pi_{*+1}(Y_s) \xrightarrow{\partial} \pi_*(Y_\infty) \xrightarrow{\pi} \lim_s \pi_*(Y_s) \to 0 \,.$$

For $r \geq 1$ define $Y_{s,r}$ be the homotopy cofiber sequence

$$Y_{s+r} \xrightarrow{\alpha^r} Y_s \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r} \,.$$

In particular, for $r = 1$ we have the homotopy cofiber sequence

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1}$$

which we can draw as a distinguished triangle

$$\begin{tikzcd} Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[d, "\beta"] \\ & Y_{s,1} \arrow[ul, dashed, "\gamma"] \end{tikzcd}$$

for each s. The dashed arrow means a morphism to the suspension of the indicated target. We get one long exact sequence in homotopy for each s, which fit together as in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y_{s+2}) \arrow[r, "\alpha_*"] & \pi_*(Y_{s+1}) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_s) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_{s-1}) \arrow[r] \arrow[d, "\beta_*"] & \dots \\
& & \pi_*(Y_{s+1,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{s,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{s-1,1}) \arrow[ul, dashed, "\gamma_*"] &
\end{tikzcd}$$

This is called an (unrolled) exact couple [Mas52], [Boa99]. We aim to determine $\pi_*(Y_{-\infty})$ from information about the $\pi_*(Y_{s,1})$ for all s, concentrating on cases when $\pi_*(Y_\infty) = 0$.

Example 1.1. Let X be a CW complex, with skeleton filtration

$$\dots \subset X^{(s-1)} \subset X^{(s)} \subset \dots ,$$

and E any spectrum. The sequence of spectra

$$\begin{tikzcd}
\dots \arrow[r] & F(X/X^{(s)}, E) \arrow[r, "\alpha"] & F(X/X^{(s-1)}, E) \arrow[r] \arrow[d, "\beta"] & \dots \\
& & F(X^{(s)}/X^{(s-1)}, E) \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

with

$$Y_s = \begin{cases} F(X/X^{(s-1)}, E) & \text{for } s \geq 0, \\ F(X_+, E) & \text{for } s \leq 0 \end{cases}$$

has homotopy colimit $Y_{-\infty} \simeq F(X_+, E)$ and homotopy limit $Y_\infty \simeq F(X/X, E) \simeq *$. We have

$$Y_{s,1} \simeq F(X^{(s)}/X^{(s-1)}, E) \simeq \prod \Omega^s E$$

for each $s \geq 0$, where the product ranges over the set of s-cells in X. Hence the starting data in this case are the graded abelian groups

$$\pi_*(Y_{s,1}) \cong E^{-*}(X^{(s)}, X^{(s-1)}) \cong C^s_{CW}(X; E_{s+*})$$

given by the cellular cochains of X with coefficients in E_*. The aim is to calculate $\pi_* F(X_+, E) = E^{-*}(X)$.

Example 1.2. Let X be any space, and let

$$\dots \to \tau_{\geq s+1} E \to \tau_{\geq s} E \to \dots$$

be the Whitehead tower of E, with $\mathrm{hocolim}_s\, \tau_{\geq s} E \simeq E$ and $\mathrm{holim}_s\, \tau_{\geq s} E \simeq *$. We have Puppe cofiber sequences

$$\tau_{\geq s+1} E \longrightarrow \tau_{\geq s} E \longrightarrow \Sigma^s H\pi_s(E) \longrightarrow \Sigma \tau_{\geq s+1} E\,.$$

The sequence of spectra

$$\begin{tikzcd}
\dots \arrow[r] & F(X_+, \tau_{\geq s+1} E) \arrow[r, "\alpha"] & F(X_+, \tau_{\geq s} E) \arrow[r] \arrow[d, "\beta"] & \dots \\
& & F(X_+, \Sigma^s H\pi_s(E)) \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

with

$$Y_s = F(X_+, \tau_{\geq s} E)$$

for all $s \in \mathbb{Z}$ has homotopy colimit $Y_{-\infty} \simeq F(X_+, E)$ ((ETC: this uses that each $\Sigma^k X$ is bounded below)) and homotopy limit $Y_\infty \simeq F(X_+, *) = *$. Hence the starting data in this case are the graded abelian groups

$$\pi_*(Y_{s,1}) = \pi_* F(X_+, \Sigma^s H\pi_s(E)) \cong H^{s-*}(X; \pi_s(E))$$

and the aim is to calculate $\pi_* F(X_+, E) = E^{-*}(X)$.

Example 1.3. Let Y be any spectrum, let (E, η, ϕ) be a ring spectrum up to homotopy, define I by the homotopy cofiber sequence

$$I \longrightarrow S \xrightarrow{\eta} E \longrightarrow \Sigma I\,,$$

and let $I^{\wedge s} = I \wedge \cdots \wedge I$ be the s-fold smash power. Consider the sequence of spectra

$$\begin{tikzcd}
\dots \arrow[r] & I^{\wedge s+1} \wedge Y \arrow[r, "\alpha"] & I^{\wedge s} \wedge Y \arrow[r] \arrow[d, "\beta"] & \dots \\
& & E \wedge I^{\wedge s} \wedge Y \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

with

$$Y_s = \begin{cases} I^{\wedge s} \wedge Y & \text{for } s \ge 0, \\ Y & \text{for } s \le 0. \end{cases}$$

Additional hypotheses are needed to ensure that $Y_\infty = \operatorname{holim}_s Y_s$ will be trivial, but clearly $Y \simeq Y_{-\infty} = \operatorname{hocolim}_s Y_s$. Suppose now that E is flat, so that

$$\dots \to E_*(Y_{s+1}) \xrightarrow{\alpha_*} E_*(Y_s) \xrightarrow{\beta_*} E_*(Y_{s,1}) \xrightarrow{\gamma_*} E_{*-1}(Y_{s+1}) \to \dots$$

is an exact sequence of E_*E-comodules. Here β_* is split injective as an E_*-module homomorphism, with left inverse

$$\pi_*(\phi \wedge \mathrm{id})\colon E_*(E \wedge Y_s) = E_*(Y_{s,1}) \longrightarrow E_*(Y_s)$$

induced by the ring spectrum multiplication, so $\alpha_* = 0$ and the long exact sequence breaks up into short exact sequences. Letting s vary, these can be spliced into a resolution

$$0 \to E_*(Y) \xrightarrow{\beta_*} E_*(Y_{0,1}) \xrightarrow{\beta_*\gamma_*} E_{*-1}(Y_{1,1}) \xrightarrow{\beta_*\gamma_*} E_{*-2}(Y_{2,1}) \to \dots$$

of $E_*(Y)$ in the category of E_*E-comodules. Moreover,

$$\begin{aligned} \pi_*(Y_{s,1}) &\xrightarrow{\cong} \operatorname{Hom}_{E_*E}(E_*, E_*(Y_{s,1})) \\ [f] &\longmapsto f_* = E_*(f) \end{aligned}$$

is an isomorphism for each s. Hence the starting data in this case are the graded abelian groups $\operatorname{Hom}_{E_*E}(E_*, E_*(Y_{s,1}))$, where $E_*(Y_{s,1})$ is part of an E_*E-comodule resolution of $E_*(Y)$, and the aim is to calculate $\pi_*(Y)$, at least when $\pi_*(Y_\infty) = 0$.

The first two examples both lead to the Atiyah–Hirzebruch spectral sequence from [AH61], while the third example leads to the E-based Adams spectral sequence. In the case $H = H\mathbb{F}_p$ this is the classical mod p Adams spectral sequence [Ada58], while for $E = MU$ it is the Adams–Novikov spectral sequence from [Nov67].

2. The spectral sequence associated to an exact couple

Definition 2.1. A spectral sequence is a sequence $(\mathcal{E}_r, d_r)_{r\ge 1}$ of bigraded abelian groups $\mathcal{E}_r = (\mathcal{E}_r^{s,*})_s$ and differentials

$$d_r \colon \mathcal{E}_r^{s,*} \longrightarrow \mathcal{E}_r^{s+r,*}$$

increasing the filtration degree s by r (and reducing the homotopical/homological degree by 1), together with isomorphisms

$$\mathcal{E}_{r+1}^{s,*} \cong H^s(\mathcal{E}_r^{*,*}, d_r) = \frac{\ker(d_r \colon \mathcal{E}_r^{s,*} \to \mathcal{E}_r^{s+r,*})}{\operatorname{im}(d_r \colon \mathcal{E}_r^{s-r,*} \to \mathcal{E}_r^{s,*})} .$$

(The usual notation is (E_r, d_r), but we write $\mathcal{E}$ here to distinguish spectral sequence $\mathcal{E}_r$-terms from E-(co-)homology for a spectrum E.)

For each $r' \ge r \ge 1$ the $\mathcal{E}_{r'}$-term is a subquotient of the $\mathcal{E}_r$-term, so we can view the $\mathcal{E}_1$-term as an initial upper bound for the target of a computation, which is gradually improved by the $\mathcal{E}_r$-terms as r grows.

Consider any exact couple

$$\begin{tikzcd}
\dots \arrow[r] & A^{s+2} \arrow[r, "\alpha"] & A^{s+1} \arrow[r, "\alpha"] \arrow[d, "\beta"] & A^s \arrow[r, "\alpha"] \arrow[d, "\beta"] & A^{s-1} \arrow[r] \arrow[d, "\beta"] & \dots \\
 & & \mathcal{E}_1^{s+1} \arrow[ul, dashed, "\gamma"] & \mathcal{E}_1^{s} \arrow[ul, dashed, "\gamma"] & \mathcal{E}_1^{s-1} \arrow[ul, dashed, "\gamma"] &
\end{tikzcd} \tag{2.1}$$

where each A^s and each $\mathcal{E}_1^s$ is a graded abelian group, α and β have degree 0, γ has (homotopical/homological) degree -1, and each triangle is exact. We shall associate a spectral sequence $(\mathcal{E}_r, d_r)$ to this exact couple.

For each s, we find one decreasing and one increasing family of subgroups within $\mathcal{E}_1^s$:

$$0 = B_1^s \subset B_2^s \subset \dots \subset B_r^s \subset \dots \subset Z_r^s \subset \dots \subset Z_2^s \subset Z_1^s = \mathcal{E}_1^s .$$

To define these, let $r \ge 1$ and consider the following subdiagram.

$$\begin{tikzcd}
A^{s+r+1} \arrow[r, "\alpha"] & A^{s+r} \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & A^{s+1} \arrow[r, "\alpha"] & A^s \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & A^{s-r+1} \arrow[r, "\alpha"] & A^{s-r} \arrow[d, "\beta"] \\
 & \mathcal{E}_1^{s+r} \arrow[ul, dashed, "\gamma"] & & \mathcal{E}_1^{s} \arrow[ul, dashed, "\gamma"] & & \mathcal{E}_1^{s-r} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

Let

$$Z_r^s = \gamma^{-1}(\operatorname{im}(\alpha^{r-1})) \qquad \text{and} \qquad B_r^s = \beta(\ker(\alpha^{r-1}))$$

be the r-th (co-)cycles and (co-)boundaries in filtration degree s. These are then nested as claimed. We let

$$\mathcal{E}_r^s = Z_r^s / B_r^s$$

be the filtration degree s part of the $\mathcal{E}_r$-term. Let

$$\begin{aligned} d_r \colon \mathcal{E}_r^s &\longrightarrow \mathcal{E}_r^{s+r} \\ [x] &\longmapsto [\beta(y)] \end{aligned}$$

map the coset of $x \in Z_r^s$ to the coset of $\beta(y) \in Z_r^{s+r}$, where $\alpha^{r-1}(y) = \gamma(x)$. In particular, $d_1 = \beta\gamma$.

Lemma 2.2. $\ker(d_r) = Z_{r+1}^s / B_r^s$ *and* $\operatorname{im}(d_r) = B_{r+1}^s / B_r^s$*, so* $H^s(\mathcal{E}_r^*, d_r) \cong \mathcal{E}_{r+1}^s$*.*

Hence we have the terms and differentials of a spectral sequence $(\mathcal{E}_r, d_r)$, for $1 \le r < \infty$. We use the following notation for its limiting term as $r \to \infty$.

Definition 2.3. Let the graded abelian groups

$$Z^s_\infty = \bigcap_r Z^s_r \qquad \text{and} \qquad B^s_\infty = \bigcup_r B^s_r$$

be the infinite (co-)cycles and (co-)boundaries in filtration degree s, so that

$$0 = B^s_1 \subset \cdots \subset B^s_r \subset \cdots \subset B^s_\infty \subset Z^s_\infty \subset \cdots \subset Z^s_r \subset \cdots \subset Z^s_1 = \mathcal{E}^s_1 .$$

Let

$$\mathcal{E}^s_\infty = Z^s_\infty / B^s_\infty$$

be the filtration degree s component of the $\mathcal{E}_\infty$-term of the spectral sequence.

Let $A^{-\infty} = \operatorname{colim}_s A^s$, $A^\infty = \lim_s A^s$ and $RA^\infty = \operatorname{Rlim}_s A^s$. We aim to calculate the graded abelian group $G = A^{-\infty}$, under the assumption that $A^\infty = 0$ and $RA^\infty = 0$. More realistically, we aim to identify the associated graded for a good filtration of G with the spectral sequence $\mathcal{E}_\infty$-term.

Definition 2.4. Let

$$F^s G = \operatorname{im}(A^s \longrightarrow A^{-\infty})$$

for each $s \in \mathbb{Z}$, so that

$$\cdots \subset F^{s+1} G \subset F^s G \subset \cdots \subset G$$

is a decreasing filtration of $G = A^{-\infty} = \operatorname{colim}_s A^s$. We say that the filtration is exhaustive if $\operatorname{colim}_s F^s G = G$, it is Hausdorff if $\lim_s F^s G = 0$, and it is complete if $\operatorname{Rlim}_s F^s G = 0$. The filtration subquotients $(F^s G / F^{s+1} G)_s$ form a bigraded abelian group, called the associated graded of the filtration.

The group G is often called the abutment of the spectral sequence, and we write

$$\mathcal{E}^s_1 \Longrightarrow_s G \qquad \text{or} \qquad \mathcal{E}^s_2 \Longrightarrow_s G$$

to present information about the $\mathcal{E}_1$- or $\mathcal{E}_2$-term and the abutment, and to indicate that s is the filtration index.

Lemma 2.5. *There is a natural injective homomorphism*

$$\zeta^s \colon \frac{F^s G}{F^{s+1} G} \longrightarrow \mathcal{E}^s_\infty$$
$$[\xi] \longmapsto [\beta(\eta)]$$

for each $s \in \mathbb{Z}$, where $\eta \in A^s$ maps to $\xi \in F^s G$ under $A^s \to A^\infty$.

Definition 2.6. If $\xi \in F^s G \setminus F^{s+1} G$ then its coset $[\xi] \in F^s G / F^{s+1} G$ is nonzero, hence corresponds to a nonzero class $x = \zeta^s([\xi]) \in \mathcal{E}^s_\infty$. We say that x detects ξ, and that ξ is detected by (or represents) x. (This terminology is not standardized.) Note that any other class $\xi' \in \xi + F^{s+1} G$ in the same coset as ξ will be detected by the same class x.

Definition 2.7. The spectral sequence $(\mathcal{E}_r, d_r)$ converges strongly to the filtered group G if

(1) $\zeta = (\zeta^s)_s$ is an isomorphism of bigraded abelian groups, and
(2) $\{F^s G\}_s$ is an exhaustive complete Hausdorff filtration of G.

Lemma 2.8. *If $\{F^sG\}_s$ is an exhaustive complete Hausdorff filtration of G then*

$$\operatorname*{colim}_a \lim_b \frac{F^aG}{F^bG} \cong G \cong \lim_b \operatorname*{colim}_a \frac{F^aG}{F^bG},$$

so that G can be algebraically recovered from the finite filtration quotients F^aG/F^bG for $-\infty < a < b < \infty$.

Hence strong convergence lets us recover G from $\mathcal{E}_\infty$, assuming that we can inductively resolve the extension problem of determining $F^aG/F^{s+1}G$ from F^aG/F^sG and $F^sG/F^{s+1}G \cong \mathcal{E}^s_\infty$, using the short exact sequence

$$0 \to \frac{F^sG}{F^{s+1}G} \longrightarrow \frac{F^aG}{F^{s+1}G} \longrightarrow \frac{F^aG}{F^sG} \to 0\,.$$

A convenient criterion for strong convergence was given by Boardman in a preprint circulating from ca. 1981 [Boa99].

Definition 2.9. The exact couple (2.1) (and its associated spectral sequence) is conditionally convergent if $A^\infty = 0$ and $RA^\infty = 0$.

Note that for $A^s = \pi_*(Y_s)$ we have conditional convergence if and only if $\pi_*(Y_\infty) = 0$, where $Y_\infty = \operatorname{holim}_s Y_s$.

Definition 2.10. Let $R\mathcal{E}^s_\infty = \operatorname{Rlim}_r Z^s_r$ for each s.

If there is a finite r' such that $d_r = 0$ for all $r \ge r'$ then $\mathcal{E}_{r'} = \mathcal{E}_\infty$ and we say that the spectral sequence collapses at the $\mathcal{E}_{r'}$-term. This is certainly sufficient to ensure that $R\mathcal{E}_\infty = 0$. A little more generally, the derived limit vanishes in bidegree (s,t) if only finitely many of the d_r-differentials from $\mathcal{E}^{s,t}_r$ are nonzero.

See [Boa99, (8.7)] or [HR19] for the definition

$$W = \operatorname*{colim}_s \operatorname*{Rlim}_r K_\infty \operatorname{im}^r A^s$$

of Boardman's whole-plane obstruction group W.

Theorem 2.11 ([Boa99, §6, §7, §8])**.** *(a) (Exiting differentials) Suppose that $A^s = 0$ for all $s > 0$, so that the spectral sequence is concentrated in the half-plane $s \le 0$. Then the spectral sequence is strongly convergent to the colimit G.*

(b) (Entering differentials) Suppose that $\mathcal{E}^s_1 = 0$ for all $s < 0$, and that the spectral sequence is conditionally convergent. Then the spectral sequence is strongly convergent to G if (and only if) $R\mathcal{E}_\infty = 0$.

(c) (Whole-plane spectral sequence) Suppose that the spectral sequence is conditionally convergent. Then the spectral sequence is strongly convergent to G if $R\mathcal{E}_\infty = 0$ and $W = 0$.

3. The additive Atiyah–Hirzebruch spectral sequence

The unrolled exact couple associated to the sequence of spectra from Example 1.1 has the form

$$\begin{tikzcd}
E^*(X, X^{(s+1)}) \arrow[r, "\alpha"] & E^*(X, X^{(s)}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & E^*(X, X^{(s-1)}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & E^*(X, X^{(s-2)}) \arrow[d, "\beta"] \\
& E^*(X^{(s+1)}, X^{(s)}) \arrow[ul, dashed, "\gamma"] & E^*(X^{(s)}, X^{(s-1)}) \arrow[ul, dashed, "\gamma"] & E^*(X^{(s-1)}, X^{(s-2)}) \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

(continuing to the left and the right), so the associated (cohomologically graded) Atiyah–Hirzebruch spectral sequence has $\mathcal{E}_1$-term

$$\mathcal{E}_1^{s,*} = E^*(X^{(s)}, X^{(s-1)}) = C^s_{CW}(X; E^*)$$

given by the cellular cochains with coefficients in the graded abelian group $E^* = \pi_{-*}(E)$. Moreover, the d_1-differential is the composite

$$d_1 = \beta\gamma \colon E^*(X^{(s)}, X^{(s-1)}) \longrightarrow E^*(X^{(s+1)}, X^{(s)}) ,$$

which is equal to the cellular coboundary

$$\delta \colon C^s_{CW}(X; E^*) \longrightarrow C^{s+1}_{CW}(X; E^*) .$$

Hence the $\mathcal{E}_2$-term is

$$\mathcal{E}_2^{s,*} = H^s(\mathcal{E}_1^{*,*}, d_1) = H^s(X; E^*) ,$$

i.e., the (cellular = singular) cohomology groups of X with coefficients in E^*. Note that $\operatorname{hocolim}_s F(X/X^{(s-1)}, E) \simeq F(X_+, E)$ and $\operatorname{holim}_s F(X/X^{(s-1)}, E) \simeq *$, so the limiting terms of the exact couple are $G = A^{-\infty} = E^*(X)$, $A^\infty = 0$ and $RA^\infty = 0$. We therefore have a conditionally convergent spectral sequence (with entering differentials)

$$\mathcal{E}_2^{s,*} = H^s(X; E^*) \Longrightarrow_s E^*(X) .$$

By Boardman's theorem, this spectral sequence is strongly convergent if (and only if) $R\mathcal{E}_\infty = 0$.

We now make the bigrading more explicit. In addition to the (decreasing) filtration degree s we let t denote the complementary (= internal) degree, so that $s+t$ is the total cohomological degree preserved by α and β and incremented by 1 by γ. The $\mathcal{E}_1$-term is then

$$\mathcal{E}_1^{s,t} = E^{s+t}(X^{(s)}, X^{(s-1)}) = C^s_{CW}(X; E^t)$$

in view of the suspension isomorphism $E^{s+t}(D^s, \partial D^s) \cong \tilde{E}^{s+t}(S^s) \cong \tilde{E}^t(S^0) = E^t$. The d_r-differential $d_r \colon \mathcal{E}_r^{s,*} \to \mathcal{E}_r^{s+r,*}$ is derived from

$$\begin{tikzcd}
E^{s+t+1}(X, X^{(s+r-1)}) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & E^{s+t+1}(X, X^{(s)}) & \\
E^{s+t+1}(X^{(s+r)}, X^{(s+r-1)}) & & E^{s+t}(X^{(s)}, X^{(s-1)}) \arrow[ul, "\gamma"']
\end{tikzcd}$$

hence has components

$$d_r \colon \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t-r+1}$$

of cohomological bidegree $(r, 1-r)$, for all s and t. In particular, $d_1 \colon \mathcal{E}_1^{s,t} \to \mathcal{E}_1^{s+1,t}$, as indicated for δ above.

The abutment $G^n = E^n(X)$ in total degree n is exhaustively filtered by

$$F^s G^n = \operatorname{im}(E^n(X, X^{(s-1)}) \to E^n(X))$$

with $F^0G^n = G^n$, and the comparison homomorphism ζ^s has components derived from

$$\begin{tikzcd}
E^n(X, X^{(s-1)}) \arrow[r] \arrow[d, "\beta"] & E^n(X) \\
E^n(X^{(s)}, X^{(s-1)}) &
\end{tikzcd}$$

that can be written

$$\frac{F^sG^n}{F^{s+1}G^n} \longrightarrow \mathcal{E}_\infty^{s,n-s} \qquad \text{or} \qquad \frac{F^sG^{s+t}}{F^{s+1}G^{s+t}} \longrightarrow \mathcal{E}_\infty^{s,t}\,.$$

The latter is more common, and we usually express the bigrading of the spectral sequence and its abutment as follows:

$$\mathcal{E}_2^{s,t} = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X)\,.$$

Here is part of the $\mathcal{E}_2$-term and the d_2-differentials, drawn in the left half of the $(-s, -t)$-plane:

$$\begin{tikzcd}
H^4(X;E^{-2}) & H^3(X;E^{-2}) & H^2(X;E^{-2}) & H^1(X;E^{-2}) & H^0(X;E^{-2}) \\
H^4(X;E^{-1}) & H^3(X;E^{-1}) & H^2(X;E^{-1}) \arrow[ull] & H^1(X;E^{-1}) \arrow[ull] & H^0(X;E^{-1}) \arrow[ull] \\
H^4(X;E^{0}) & H^3(X;E^{0}) & H^2(X;E^{0}) \arrow[ull] & H^1(X;E^{0}) \arrow[ull] & H^0(X;E^{0}) \arrow[ull] \\
H^4(X;E^{1}) & H^3(X;E^{1}) & H^2(X;E^{1}) \arrow[ull] & H^1(X;E^{1}) \arrow[ull] & H^0(X;E^{1}) \arrow[ull] \\
H^4(X;E^{2}) & H^3(X;E^{2}) & H^2(X;E^{2}) \arrow[ull] & H^1(X;E^{2}) \arrow[ull] & H^0(X;E^{2}) \arrow[ull]
\end{tikzcd}$$

Replacing each $\mathcal{E}_2^{s,t} = H^s(X; E^t)$ with $\mathcal{E}_3^{s,t} = H^{s,t}(\mathcal{E}_2^{*,*}, d_2) = \ker(d_2)^{s,t}/\operatorname{im}(d_2)^{s,t}$ we obtain the $\mathcal{E}_3$-term, here shown with the d_3-differentials.

$$\begin{tikzcd}
\mathcal{E}_3^{4,-2} & \mathcal{E}_3^{3,-2} & \mathcal{E}_3^{2,-2} & \mathcal{E}_3^{1,-2} & \mathcal{E}_3^{0,-2} \\
\mathcal{E}_3^{4,-1} & \mathcal{E}_3^{3,-1} & \mathcal{E}_3^{2,-1} & \mathcal{E}_3^{1,-1} & \mathcal{E}_3^{0,-1} \\
\mathcal{E}_3^{4,0} & \mathcal{E}_3^{3,0} & \mathcal{E}_3^{2,0} & \mathcal{E}_3^{1,0} \arrow[uulll] & \mathcal{E}_3^{0,0} \arrow[uulll] \\
\mathcal{E}_3^{4,1} & \mathcal{E}_3^{3,1} & \mathcal{E}_3^{2,1} & \mathcal{E}_3^{1,1} \arrow[uulll] & \mathcal{E}_3^{0,1} \arrow[uulll] \\
\mathcal{E}_3^{4,2} & \mathcal{E}_3^{3,2} & \mathcal{E}_3^{2,2} & \mathcal{E}_3^{1,2} \arrow[uulll] & \mathcal{E}_3^{0,2} \arrow[uulll]
\end{tikzcd}$$

In the end we are left with the $\mathcal{E}_\infty$-term.

$$
\begin{array}{ccccc}
\mathcal{E}_\infty^{4,-2} & \mathcal{E}_\infty^{3,-2} & \mathcal{E}_\infty^{2,-2} & \mathcal{E}_\infty^{1,-2} & \mathcal{E}_\infty^{0,-2} \\
\mathcal{E}_\infty^{4,-1} & \mathcal{E}_\infty^{3,-1} & \mathcal{E}_\infty^{2,-1} & \mathcal{E}_\infty^{1,-1} & \mathcal{E}_\infty^{0,-1} \\
\mathcal{E}_\infty^{4,0} & \mathcal{E}_\infty^{3,0} & \mathcal{E}_\infty^{2,0} & \mathcal{E}_\infty^{1,0} & \mathcal{E}_\infty^{0,0} \\
\mathcal{E}_\infty^{4,1} & \mathcal{E}_\infty^{3,1} & \mathcal{E}_\infty^{2,1} & \mathcal{E}_\infty^{1,1} & \mathcal{E}_\infty^{0,1} \\
\mathcal{E}_\infty^{4,2} & \mathcal{E}_\infty^{3,2} & \mathcal{E}_\infty^{2,2} & \mathcal{E}_\infty^{1,2} & \mathcal{E}_\infty^{0,2}
\end{array}
$$

In total degree n, the associated graded groups $F^sE^n(X)/F^{s+1}E^n(X)$ of the filtration of $E^n(X)$

$$
\begin{tikzcd}
\dots \arrow[r, tail] & F^4E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^3E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^2E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^1E^n(X) \arrow[r, tail] \arrow[d, two heads] & E^n(X) \arrow[d, two heads] \\
 & \dfrac{F^4E^n(X)}{F^5E^n(X)} & \dfrac{F^3E^n(X)}{F^4E^n(X)} & \dfrac{F^2E^n(X)}{F^3E^n(X)} & \dfrac{F^1E^n(X)}{F^2E^n(X)} & \dfrac{E^n(X)}{F^1E^n(X)}
\end{tikzcd}
$$

map to the groups $\mathcal{E}_\infty^{s,n-s}$ in the $\mathcal{E}_\infty$-term, which lie on the dashed line of slope -1 in total degree $s+t=n$. When the spectral sequence is (strongly) convergent, these maps are isomorphisms, so that we can think of the group $\mathcal{E}_\infty^{s,t}$ as the filtration quotient $F^sE^{s+t}(X)/F^{s+1}E^{s+t}(X)$ for each $s \ge 0$ and $t \in \mathbb{Z}$.

Example 3.1. If $\pi_0(E) = A$ and $\pi_*(E) = 0$ for $* \ne 0$ then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = \begin{cases} H^s(X;A) & \text{for } t = 0, \\ 0 & \text{otherwise} \end{cases}$$

is concentrated on the line $t = 0$. Each differential $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t-r+1}$ for $r \ge 2$ maps from or to a trivial group (or both), so the spectral sequence collapses at the $\mathcal{E}_2$-term, hence is strongly convergent to $E^{s+t}(X)$. In total degree n the groups $\mathcal{E}_\infty^{s,n-s}$ are trivial, except in the one case $n - s = 0$, so there are no extension problems and $E^n(X) \cong \mathcal{E}_\infty^{n,0} = \mathcal{E}_2^{n,0} = H^n(X;A)$. Hence E represents ordinary cohomology with coefficients in A and $E \simeq HA$.

Example 3.2. Suppose that $H_*(X) = H_*(X;\mathbb{Z})$ is free in each even degree, and trivial in each odd degree. This is the case, for instance, when $X = \mathbb{C}P^m$, $\mathbb{C}P^\infty = BU(1)$, $(\mathbb{C}P^\infty)^n = BU(1)^n$, $BU(n)$ or BU. Suppose also that E is even, in the sense that E^* is trivial in odd degrees. This is the case, for instance, when $E = KU$ or MU. The Atiyah–Hirzebruch $\mathcal{E}_2$-term

$$\mathcal{E}_2^{s,t} = H^s(X;E^t) \cong \mathrm{Hom}(H_s(X), E^t)$$

is then concentrated in bidegrees (s,t) with s and t even. In particular, $\mathcal{E}_2^{s,t}$ is zero if $s+t$ is odd. Since $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t-r+1}$ maps total degree $s+t$ to total degree

$(s+r)+(t-r+1) = s+t+1$, its source or target is trivial for each $r \ge 2$, so the spectral sequence collapses at the $\mathcal{E}_2$-term. It is therefore strongly convergent, so $E^n(X) = 0$ for n odd, and for n even there is a complete Hausdorff filtration

$$\cdots \subset F^4E^n(X) \subset F^2E^n(X) \subset F^0E^n(X) = E^n(X)$$

with filtration quotients

$$F^{2m}E^n(X)/F^{2m+2}E^n(X) \cong H^{2m}(X; E^{n-2m}) .$$

For example, when $X = \mathbb{C}P^\infty$ and $E = KU$ we have a complete Hausdorff filtration

$$\cdots \subset F^4KU^n(\mathbb{C}P^\infty) \subset F^2KU^n(\mathbb{C}P^\infty) \subset KU^n(\mathbb{C}P^\infty)$$

for each even n, with filtration quotients

$$F^{2m}KU^n(\mathbb{C}P^\infty)/F^{2m+2}KU^n(\mathbb{C}P^\infty) \cong H^{2m}(\mathbb{C}P^\infty; KU^n) \cong \mathbb{Z} .$$

Since $\mathbb{Z}$ is free, it follows by induction on m that

$$KU^n(\mathbb{C}P^\infty)/F^{2m+2}KU^n(\mathbb{C}P^\infty) \cong \bigoplus_{i=0}^{m} \mathbb{Z} \cong \prod_{i=0}^{m} \mathbb{Z} ,$$

and, by passage to the limit over m,

$$KU^n(\mathbb{C}P^\infty) \cong \prod_{i=0}^{\infty} \mathbb{Z} .$$

On the other hand, $KU^n(\mathbb{C}P^\infty) = 0$ for n odd.

4. The additive Whitehead tower spectral sequence

The unrolled exact couple associated to the sequence of spectra from Example 1.2 has the form

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*F(X_+, \tau_{\geq s+1}E) \arrow[r, "\alpha"] & \pi_*F(X_+, \tau_{\geq s}E) \arrow[r] \arrow[d, "\beta"] & \dots \\
 & & \pi_*F(X_+, \Sigma^s HE_s) \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

where $E_s = \pi_s(E)$, so the associated spectral sequence has $\mathcal{E}_1$-term

$$\mathcal{E}_1^{s,*} = H^*(X; E_s) .$$

The limiting terms of the exact couple are $G = A^{-\infty} = \operatorname{colim}_s(\tau_{\geq s}E)^*(X) \cong E^*(X)$, $A^\infty = 0$ and $RA^\infty = 0$. We therefore have a conditionally convergent spectral sequence (with entering differentials). By Boardman's theorem it is strongly convergent to $E^*(X)$ if (and only if) $R\mathcal{E}_\infty = 0$.

The abutment $G^n = E^n(X)$ in total degree n is exhaustively filtered by

$$F^sG^n = \operatorname{im}(\pi_{-n}F(X_+, \tau_{\geq s}E) \to \pi_{-n}F(X_+, E))$$

so in order to have $n = s+t$, with complementary degree t, we must have

$$\mathcal{E}_1^{s,t} = \pi_{-s-t}F(X_+, \Sigma^s HE_s) = H^{2s+t}(X; E_s) .$$

The d_r-differential is then derived from

$$\begin{tikzcd}
\pi_{-s-t-1}F(X_+,\tau_{\geq s+r}E) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & \pi_{-s-t-1}F(X_+,\tau_{\geq s+1}E) \\
\pi_{-s-t-1}F(X_+,\Sigma^{s+r}HE_{s+r}) & \pi_{-s-t}F(X_+,\Sigma^{s}HE_{s})\,, \arrow[u, "\gamma"']
\end{tikzcd}$$

hence has components $d_r\colon \mathcal{E}_r^{s,t}\to\mathcal{E}_r^{s+r,t-r+1}$ of cohomological bidegree $(r,1-r)$. In other words, we have a cohomologically (bi-)graded spectral sequence

$$\mathcal{E}_1^{s,t}=H^{2s+t}(X;E_s)\Longrightarrow_s E^{s+t}(X)\,.$$

Here is part of the $\mathcal{E}_1$-term and the d_1-differentials, drawn in the $(-s,-t)$-plane:

$$\begin{tikzcd}
H^4(X;E_3) & H^2(X;E_2) \arrow[l] & H^0(X;E_1) \arrow[l] & 0 & 0 \\
H^5(X;E_3) & H^3(X;E_2) \arrow[l] & H^1(X;E_1) \arrow[l] & 0 & 0 \\
H^6(X;E_3) & H^4(X;E_2) \arrow[l] & H^2(X;E_1) \arrow[l] & H^0(X;E_0) \arrow[l] & 0 \\
H^7(X;E_3) & H^5(X;E_2) \arrow[l] & H^3(X;E_1) \arrow[l] & H^1(X;E_0) \arrow[l] & 0 \\
H^8(X;E_3) & H^6(X;E_2) \arrow[l] & H^4(X;E_1) \arrow[l] & H^2(X;E_0) \arrow[l] & H^0(X;E_{-1}) \arrow[l]
\end{tikzcd}$$

This Whitehead tower spectral sequence is isomorphic to the Atiyah–Hirzebruch spectral sequence, up to a reindexing of the terms, taking the $\mathcal{E}_r^{s,t}$-term and d_r-differential of the former to the $\mathcal{E}_{r+1}^{2s+t,-s}$-term and d_{r+1}-differential of the latter. This was first proved by Maunder [Mau63], who showed that the Whitehead tower exact couple is isomorphic to the derived Atiyah–Hirzebruch exact couple, in the sense of [Mas52]. By reference to a later construction due to Deligne (in the context of filtered chain complexes), it is now common to call the Whitehead tower spectral sequence the décalage of the Atiyah–Hirzebruch spectral sequence.

5. Pairings of sequences and Cartan–Eilenberg systems

If $Y=Y_{-\infty}$ is a ring spectrum, we may hope to use the homotopy spectral sequence

$$\mathcal{E}_1^s=\pi_*(Y_{s,1})\Longrightarrow_s \pi_*(Y)$$

to access the ring structure on $\pi_*(Y)$. If $Y=F(X_+,E)$ with E a ring spectrum, this is the same as the cup product structure on $\pi_*(Y)=E^{-*}(X)$, induced by the diagonal $\Delta\colon X\to X\times X$ and the product $\phi\colon E\wedge E\to E$. More generally, we may consider pairings $\mu\colon Y\wedge Y'\to Y''$ and study $\mu_*\colon \pi_*(Y)\otimes\pi_*(Y')\to\pi_*(Y'')$.

Definition 5.1. Let

$$\dots \to Y_{s+2} \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} Y_{s-1} \to \dots$$
$$\dots \to Y'_{s'+2} \xrightarrow{\alpha} Y'_{s'+1} \xrightarrow{\alpha} Y'_{s'} \xrightarrow{\alpha} Y'_{s'-1} \to \dots$$
$$\dots \to Y''_{s''+2} \xrightarrow{\alpha} Y''_{s''+1} \xrightarrow{\alpha} Y''_{s''} \xrightarrow{\alpha} Y''_{s''-1} \to \dots$$

be three sequences of orthogonal spectra, briefly denoted $Y_\star$, $Y'_\star$ and $Y''_\star$. A pairing $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ of sequences of orthogonal spectra is a collection of maps

$$\mu_{s,s'}\colon Y_s \wedge Y'_{s'} \longrightarrow Y''_{s+s'}$$

in $\mathcal{S}p^{\mathbb{O}}$, such that the squares

$$\begin{tikzcd}
Y_{s+1}\wedge Y'_{s'} \arrow[r, "\alpha\wedge\mathrm{id}"] \arrow[d, "\mu_{s+1,s'}"'] & Y_s\wedge Y'_{s'} \arrow[d, "\mu_{s,s'}"] & Y_s \wedge Y'_{s'+1} \arrow[l, "\mathrm{id}\wedge\alpha"'] \arrow[d, "\mu_{s,s'+1}"] \\
Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'} & Y''_{s+s'+1} \arrow[l, "\alpha"']
\end{tikzcd}$$

commute for all $s, s' \in \mathbb{Z}$.

Given a pairing $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ as above, the following 3-dimensional diagram commutes in $\mathcal{S}p^{\mathbb{O}}$.
(5.1)

$$\begin{tikzcd}
& Y_{s+1}\wedge Y'_{s'} \arrow[rrr, "\alpha\wedge\mathrm{id}"] \arrow[ddr, "\mu_{s+1,s'}"] & & & Y_s\wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"] \\
Y_{s+1}\wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"] \arrow[rrr, "\alpha\wedge\mathrm{id}"] \arrow[d, "\mu_{s+1,s'+1}"] & & & Y_s\wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"] \arrow[dl, "\mu_{s,s'+1}"] & \\
Y''_{s+s'+2} \arrow[rr, "\alpha"] & & Y''_{s+s'+1} \arrow[rr, "\alpha"] & & Y''_{s+s'}
\end{tikzcd}$$

Recall the notation $Y_{s,1} = Y_s \cup_\alpha CY_{s+1}$. A homotopy class $x \in \pi_n(Y_{s,1})$ can be represented by a map of pairs

$$f\colon (D^n, S^{n-1}) \longrightarrow (Y_s, Y_{s+1})$$

where $D^n = CS^{n-1}$. Given maps f and f' representing $x \in \pi_n(Y_{s,1})$ and $x' \in \pi_{n'}(Y'_{s',1})$ we obtain a map

$$\begin{aligned} f \wedge f'\colon (D^n \wedge D^{n'}, S^{n-1}\wedge D^{n'} \cup D^n \wedge S^{n'-1}) \\ \longrightarrow (Y_s \wedge Y'_{s'}, Y_{s+1}\wedge Y'_{s'} \cup Y_s \wedge Y'_{s'+1}), \end{aligned}$$

where the source is isomorphic to $(D^{n+n'}, S^{n+n'-1})$. Composing with μ we obtain a map

$$\mu(f\wedge f')\colon (D^{n+n'}, S^{n+n'-1}) \longrightarrow (Y''_{s+s'}, Y''_{s+s'+1})$$

representing a class $\mu_*(x \otimes x')$ in $\pi_{n+n'}(Y''_{s+s',1})$. This defines a pairing

$$\mu_*\colon \pi_*(Y_{s,1}) \otimes \pi_*(Y'_{s',1}) \longrightarrow \pi_*(Y''_{s+s',1}).$$

Definition 5.2. Let $(\mathcal{E}_r, d_r)$, $('\mathcal{E}_r, 'd_r)$ and $(''\mathcal{E}_r, ''d_r)$ be three spectral sequences. A pairing $\mu\colon (\mathcal{E}_r, '\mathcal{E}_r) \to ''\mathcal{E}_r$ of spectral sequences is a collection of chain maps

$$\mu_r\colon \mathcal{E}_r \otimes '\mathcal{E}_r \longrightarrow ''\mathcal{E}_r\,,$$

where the source has the boundary operator $d_r \otimes 1 + 1 \otimes 'd_r$ and the target has the boundary operator $''d_r$, such that the diagram

$$\begin{tikzcd}
\mathcal{E}_{r+1} \otimes '\mathcal{E}_{r+1} \arrow[rr, "\mu_{r+1}"] \arrow[d, "\cong"'] & & ''\mathcal{E}_{r+1} \arrow[dd, "\cong"] \\
H^*(\mathcal{E}_r, d_r) \otimes H^*('\mathcal{E}_r, 'd_r) \arrow[d, "\otimes"'] & & \\
H^*(\mathcal{E}_r \otimes '\mathcal{E}_r, d_r \otimes 1 + 1 \otimes 'd_r) \arrow[rr, "H^*(\mu_r)"] & & H^*(''\mathcal{E}_r, ''d_r)
\end{tikzcd}$$

commutes.

The condition that μ_r is a chain map is a form of the Leibniz rule:

$$''d_r(\mu_r(x \otimes x')) = \mu_r(d_r(x) \otimes x' + (-1)^{|x|} x \otimes 'd_r(x'))\,.$$

Note that a pairing of spectral sequences is determined by its initial component μ_1, but not every bilinear pairing of $\mathcal{E}_1$-terms will induce chain complex pairings of $(\mathcal{E}_r, d_r)$-terms for all $r \geq 1$.

Definition 5.3. Let $(F^sG)_s$, $(F^{s'}G')_{s'}$ and $(F^{s''}G'')_{s''}$, be filtered graded abelian groups. A pairing $\mu\colon G \otimes G' \to G''$ is filtration-preserving if

$$\mu(F^sG \otimes F^{s'}G') \subset F^{s+s'}G''$$

for all $s, s' \in \mathbb{Z}$. It then induces pairings

$$\bar\mu\colon \frac{F^sG}{F^{s+1}G} \otimes \frac{F^{s'}G'}{F^{s'+1}G'} \longrightarrow \frac{F^{s+s'}G''}{F^{s+s'+1}G''}\,.$$

A pairing $\mu\colon (\mathcal{E}_r, '\mathcal{E}_r) \to ''\mathcal{E}_r$ of spectral sequences, with abutments G, G' and G'', is compatible with the filtration-preserving pairing μ if the diagram

$$\begin{tikzcd}
\frac{F^sG}{F^{s+1}G} \otimes \frac{F^{s'}G'}{F^{s'+1}G'} \arrow[r, "\bar\mu"] \arrow[d, "\zeta\otimes\zeta"'] & \frac{F^{s+s'}G''}{F^{s+s'+1}G''} \arrow[d, "\zeta"] \\
\mathcal{E}_\infty \otimes '\mathcal{E}_\infty \arrow[r, "\mu_\infty"] & ''\mathcal{E}_\infty
\end{tikzcd}$$

commutes.

Since ζ is injective, the pairing μ_∞ determines $\bar\mu$, which in turn determines $\mu\colon G \otimes G' \to G''$ modulo the given filtrations.

Theorem 5.4. *Let $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ be a pairing of sequences of orthogonal spectra, and let*

$$(\mathcal{E}_r, d_r) = (\mathcal{E}_r(Y), d_r)\ ,\ \ ('\mathcal{E}_r, 'd_r) = (\mathcal{E}_r(Y'), d_r) \text{ and } (''\mathcal{E}_r, ''d_r) = (\mathcal{E}_r(Y''), d_r)$$

be the spectral sequences associated to $Y_\star$, $Y'_\star$ and $Y''_\star$, respectively. Then there is a (unique) pairing of spectral sequences $\mu\colon (\mathcal{E}_r, '\mathcal{E}_r) \to ''\mathcal{E}_r$ with $\mu_1 = \mu_*$. It is compatible with the filtration-preserving pairing $\mu\colon \pi_*(Y_{-\infty}) \otimes \pi_*(Y'_{-\infty}) \to \pi_*(Y''_{-\infty})$.

Sketch proof. See e.g. [Hedenlund–Rognes, arXiv:2008.09095, Thm. 4.27]. The proof uses Cartan–Eilenberg systems [CE56, §XV.7] in an essential way, which are intermediate between sequences of spectra and exact couples. There is a useful notion of pairings of Cartan–Eilenberg systems, which induce pairings of spectral sequences. (The definition in [Mas54] of pairings of exact couples is too close to tautological to be useful.) □

Some authors only assume that the two squares in Definition 5.1 commute up to homotopy, i.e., they work in the 1-category $\mathrm{Ho}(\mathcal{S}p)$, in which case the 3-dimensional diagram (5.1) also commutes in $\mathrm{Ho}(\mathcal{S}p)$. However, this will not be sufficient to obtain a pairing of spectral sequences, since (at least) a 2-categorical compatibility between given choices of commuting homotopies for the front faces

$$\begin{tikzcd}
Y_{s+1} \wedge Y'_{s'+1} \arrow[rr, "\alpha\wedge\alpha"] \arrow[dr, "\alpha\wedge\mathrm{id}"] \arrow[dd, "\mu_{s+1,s'+1}"'] & & Y_s \wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"] \\
& Y_s \wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"] \arrow[d, "\mu_{s,s'+1}"] & \\
Y''_{s+s'+2} \arrow[r, "\alpha"] & Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'}
\end{tikzcd}$$

and the back faces

$$\begin{tikzcd}
Y_{s+1} \wedge Y'_{s'+1} \arrow[rr, "\alpha\wedge\alpha"] \arrow[dr, "\mathrm{id}\wedge\alpha"] \arrow[dd, "\mu_{s+1,s'+1}"'] & & Y_s \wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"] \\
& Y_{s+1} \wedge Y'_{s'} \arrow[ur, "\alpha\wedge\mathrm{id}"] \arrow[d, "\mu_{s+1,s'}"] & \\
Y''_{s+s'+2} \arrow[r, "\alpha"] & Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'}\,,
\end{tikzcd}$$

is required to prove the Leibniz rule, i.e., that μ_r takes $d_r \otimes 1 + 1 \otimes {}'d_r$ to ${}''d_r$. One should therefore assume that the 3-dimensional diagram (5.1) commutes in a k-category of spectra, for $2 \le k \le \infty$. (Any discussion internal to the stable homotopy category will contain a gap.) Our assumption that it commutes strictly in the topological category of orthogonal spectra is certainly sufficient.

We often apply the theorem in the case where the three sequences are the same, so that we have an internal pairing. If this is unital and associative, then we say that we have an algebra spectral sequence.

Corollary 5.5. *Let $Y_\star$ be a multiplicative sequence of orthogonal spectra, i.e., a sequence with a pairing $\mu\colon Y_\star \wedge Y_\star \to Y_\star$, and let*

$$\mathcal{E}_1^s = \pi_*(Y_{s,1}) \Longrightarrow_s \pi_*(Y_{-\infty})$$

be the associated spectral sequence. Then there is a (unique) pairing of spectral sequences $\mu\colon (\mathcal{E}_r, \mathcal{E}_r) \to \mathcal{E}_r$ with $\mu_1 = \mu_*$. It is compatible with the filtration-preserving pairing $\mu\colon \pi_*(Y_{-\infty}) \otimes \pi_*(Y_{-\infty}) \to \pi_*(Y_{-\infty})$.

6. The multiplicative Atiyah–Hirzebruch spectral sequence

Let X be a CW complex and E a spectrum with a pairing $\phi\colon E \wedge E \to E$, e.g., a ring spectrum up to homotopy or an orthogonal ring spectrum. The diagonal map

$$\Delta\colon X \longrightarrow X \times X$$

rarely preserves the skeleton filtration, but by cellular approximation it is homotopic to a cellular map

$$D\colon X \longrightarrow X \times X .$$

In particular,

$$D(X^{(s+s'-1)}) \subset (X \times X)^{(s+s'-1)} \subset (X^{(s-1)} \times X) \cup (X \times X^{(s'-1)}) .$$

so that D induces a map

$$\bar{D}\colon \frac{X}{X^{(s+s'-1)}} \longrightarrow \frac{X}{X^{(s-1)}} \wedge \frac{X}{X^{(s'-1)}} .$$

Let $Y_s = F(X/X^{(s-1)}, E)$ as before. The composite maps

$$\mu\colon F(X/X^{(s-1)}, E) \wedge F(X/X^{(s'-1)}, E)$$
$$\xrightarrow{\wedge} F(X/X^{(s-1)} \wedge X/X^{(s'-1)}, E \wedge E) \overset{F(\bar{D},\mu)}{\longrightarrow} F(X/X^{(s+s'-1)}, E)$$

then define a pairing of sequences of orthogonal spectra.

Hence the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_1^{s,t} = C^s_{CW}(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

admits a pairing $\mu\colon (\mathcal{E}_r, \mathcal{E}_r) \to \mathcal{E}_r$ that is given at the $\mathcal{E}_1$-term by

$$C^s_{CW}(X; E^t) \otimes C^{s'}_{CW}(X; E^{t'}) \xrightarrow{D^*} C^{s+s'}_{CW}(X; E^t \otimes E^{t'}) \xrightarrow{\phi_*} C^{s+s'}_{CW}(X; E^{t+t'})$$

and at the $\mathcal{E}_2$-term by the E-cohomology cup product

$$\mathcal{E}_2^{s,t} \otimes \mathcal{E}_2^{s',t'} = H^s(X; E^t) \otimes H^{s'}(X; E^{t'}) \xrightarrow{\cup} H^{s+s'}(X; E^{t+t'}) = \mathcal{E}_2^{s+s',t+t'} ,$$

converging to the cup product

$$E^n(X) \otimes E^{n'}(X) \xrightarrow{\cup} E^{n+n'}(X) .$$

Note that the $\mathcal{E}_1$-term and the pairing μ_1 depend on the CW structure on X and the cellular approximation D to Δ, while for $r \geq 2$ the $\mathcal{E}_r$-term and the pairing μ_r are homotopy invariants. If E is a ring spectrum up to homotopy, then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

is an algebra spectral sequence. If E is homotopy commutative, then the $\mathcal{E}_r$-terms for $r \geq 2$ are graded commutative, and we have an E^*-algebra spectral sequence.

Example 6.1. Consider the case $X = \mathbb{C}P^\infty$ with E a homotopy commutative ring spectrum. Let $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$ with $|y| = 2$. The Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{*,*} = H^*(\mathbb{C}P^\infty; E^*) \Longrightarrow E^*(\mathbb{C}P^\infty)$$

then has $\mathcal{E}_2$-term

$$\mathcal{E}_2^{*,*} = \mathbb{Z}[y] \otimes E^* = E^*[y]$$

with $y \in \mathcal{E}_2^{2,0}$ and $\mathcal{E}_2^{0,t} = E^t$ for all t. We now suppose that E is even, so that $\mathcal{E}_2 = \mathcal{E}_\infty$, the spectral sequence is strongly convergent, and

$$\frac{F^s E^n(\mathbb{C}P^\infty)}{F^{s+1}E^n(\mathbb{C}P^\infty)} \cong \mathcal{E}_\infty^{s,n-s}$$

for all s and n. Choose a class $\eta \in F^2E^2(\mathbb{C}P^\infty) \setminus F^3E^2(\mathbb{C}P^\infty)$ whose coset $[\eta]$ corresponds to y under the isomorphism above. Then $\eta^m \in F^{2m}E^{2m}(\mathbb{C}P^\infty)$, so there is an E^*-algebra homomorphism

$$E^*[\eta]/(\eta^m) = \mathbb{Z}[\eta]/(\eta^m) \otimes E^* \longrightarrow E^*(\mathbb{C}P^\infty)/F^{2m}E^*(\mathbb{C}P^\infty)$$

for each $m \geq 0$. In fact each of these is an isomorphism, which we can prove by induction on m using the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathbb{Z}\{\eta^m\} \otimes E^* \arrow[r, "\cong"] \arrow[d] & \dfrac{F^{2m}E^*(\mathbb{C}P^\infty)}{F^{2m+2}E^*(\mathbb{C}P^\infty)} \arrow[d] & \mathcal{E}_\infty^{2m,*} \arrow[l, "\cong"'] \\
\mathbb{Z}[\eta]/(\eta^{m+1}) \otimes E^* \arrow[r] \arrow[d] & \dfrac{E^*(\mathbb{C}P^\infty)}{F^{2m+2}E^*(\mathbb{C}P^\infty)} \arrow[d] & \\
\mathbb{Z}[\eta]/(\eta^m) \otimes E^* \arrow[r] \arrow[d] & \dfrac{E^*(\mathbb{C}P^\infty)}{F^{2m}E^*(\mathbb{C}P^\infty)} \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

Passing to limits over m, we obtain an E_*-algebra isomorphism

$$E^*(\mathbb{C}P^\infty) \cong E^*[[\eta]]\,,$$

where

$$E^*[[\eta]] = \lim_m E^*[\eta]/(\eta^m)$$

denotes the E^*-algebra of formal power series in η. In cohomological degree n it has elements of the form

$$\sum_{m=0}^{\infty} e_m \eta^m$$

with $e_m \in E^{n-2m}$. If the spectrum E is bounded above, i.e., t-truncated for some finite t, then $e_m = 0$ for $2m - n > t$, in which case each such formal sum is finite.

Hereafter we shall generally simply write y in place of η for a choice of class in $F^2E^2(\mathbb{C}P^\infty) = \tilde{E}^2(\mathbb{C}P^\infty)$ that is detected by $y \in \mathcal{E}_\infty^{2,0} = H^2(\mathbb{C}P^\infty; E^0)$.

Similar arguments show:

Proposition 6.2. *If E is a commutative ring spectrum up to homotopy, with E^* concentrated in even degrees, then there are E^*-algebra isomorphisms*

$$\begin{aligned} E^*(\mathbb{C}P^m) &\cong E^*[y]/(y^{m+1}) \\ E^*(\mathbb{C}P^\infty) &\cong E^*[[y]] \\ E^*((\mathbb{C}P^\infty)^n) &\cong E^*[[y_1,\dots,y_n]] \\ E^*(BU(n)) &\cong E^*[[c_1,\dots,c_n]] \\ E^*(BU) &\cong E^*[[c_k \mid k \ge 1]]\,. \end{aligned}$$

Remark 6.3. These calculations show that the E^*-algebra structure of $E^*(\mathbb{C}P^\infty)$ (or any of the other algebras listed) does not carry any more information about E than the coefficients ring E^*. However, we shall see that the E^*-algebra homomorphism

$$\begin{aligned} E^*[[y]] \cong E^*(\mathbb{C}P^\infty) &\xrightarrow{m^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1,y_2]] \\ y &\longmapsto F_E(y_1,y_2) \end{aligned}$$

(induced by the map $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of complex line bundles) often carries significantly more information about E. Here

$$F_E(y_1,y_2) = y_1 + y_2 + \sum_{i,j\ge 1} a_{i,j} y_1^i y_2^j$$

is a formal group law.

((ETC: Also homological Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}^2_{s,t} = H_s(X;E_t) \Longrightarrow_s E_{s+t}(X)\,,$$

and evaluation pairing.))

7. The multiplicative Whitehead tower spectral sequence

The Whitehead tower approach to the Atiyah–Hirzebruch spectral sequence also gives a multiplicative spectral sequence, but this requires 2-categorical flexibility.

Let X be a CW complex and E a ring spectrum, with product $\phi\colon E \wedge E \to E$. For each pair (s,s') consider the diagram

$$\begin{tikzcd} & \tau_{\ge s}E \wedge \tau_{\ge s'}E \arrow[r] \arrow[d, dashed, "\phi_{s,s'}"'] & E \wedge E \arrow[d, "\phi"] \\ \Sigma^{-1}\tau_{<s+s'}E \arrow[r] & \tau_{\ge s+s'}E \arrow[r] \arrow[ur, phantom, "C_{s,s'}"] & E \arrow[r] & \tau_{<s+s'}E\,. \end{tikzcd}$$

Here $\tau_{\ge s}E \wedge \tau_{\ge s'}E$ is $(s+s')$-connective and $\tau_{<s+s'}E$ is $(s+s'-1)$-coconnective, so the mapping space $\mathrm{Map}(\tau_{\ge s}E \wedge \tau_{\ge s'}E, \tau_{<s+s'}E)$ is contractible. Hence the space of pairs $(\phi_{s,s'}, C_{s,s'})$, where $\phi_{s,s'}$ is a map and $C_{s,s'}$ is a commuting homotopy, is (nonempty and) contractible. For simplicity, let us assume that each map in the Whitehead tower is a fibration, so that we may take $C_{s,s'}$ to be the constant homotopy, i.e., so that $\phi_{s,s'}$ makes the square commute "on the nose".

It follows that the two composite maps around the square

$$\begin{tikzcd} \tau_{\ge s+1}E \wedge \tau_{\ge s'}E \arrow[r] \arrow[d, "\phi_{s+1,s'}"'] & \tau_{\ge s}E \wedge \tau_{\ge s'}E \arrow[d, "\phi_{s,s'}"] \\ \tau_{\ge s+1+s'}E \arrow[r] \arrow[ur, phantom, "H_{s,s'}"] & \tau_{\ge s+s'}E \end{tikzcd}$$

both make the square

$$\begin{tikzcd}
& \tau_{\geq s+1}E \wedge \tau_{\geq s'}E \arrow[r] \arrow[d, dashed] & E \wedge E \arrow[d, "\phi"] & \\
\Sigma^{-1}\tau_{<s+s'}E \arrow[r] & \tau_{\geq s+s'}E \arrow[r] & E \arrow[r] & \tau_{<s+s'}E
\end{tikzcd}$$

commute, which implies that these two maps are homotopic, since also the mapping space $\mathrm{Map}(\tau_{\geq s+1}E \wedge \tau_{\geq s'}E, \tau_{<s+s'}E)$ is contractible. Let $H_{s,s'}$ be such a "horizontal" homotopy, which we may assume projects to the constant homotopy of maps $\tau_{\geq s+1}E \wedge \tau_{\geq s'}E \to E$. A similar argument applies for the two composite maps around the square

$$\begin{tikzcd}
\tau_{\geq s}E \wedge \tau_{\geq s'+1}E \arrow[r] \arrow[d, "\phi_{s,s'+1}"'] & \tau_{\geq s}E \wedge \tau_{\geq s'}E \arrow[d, "\phi_{s,s'}"] \\
\tau_{\geq s+s'+1}E \arrow[r] \arrow[ur, phantom, "V_{s,s'}"] & \tau_{\geq s+s'}E\,.
\end{tikzcd}$$

Let $V_{s,s'}$ be a "vertical" homotopy connecting them, projecting to the constant homotopy of maps $\tau_{\geq s}E \wedge \tau_{\geq s'+1}E \to E$. We now need a 2-homotopy connecting the front composite homotopy $H_{s,s'+1} * V_{s,s'}$ to the back composite homotopy $V_{s+1,s'} * H_{s,s'}$, both of which connect

$$\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \xrightarrow{\phi_{s+1,s'+1}} \tau_{\geq s+s'+2}E \longrightarrow \tau_{\geq s+s'}E$$

to

$$\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \longrightarrow \tau_{\geq s}E \wedge \tau_{\geq s'}E \xrightarrow{\phi_{s,s'}} \tau_{\geq s+s}E\,,$$

and which project to the constant homotopy of maps $\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \to E$. The existence of this 2-homotopy now follows from the fact that $\mathrm{Map}(\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E, \tau_{<s+s'}E)$ is contractible.

The diagonal $\Delta\colon X \to X \times X$ makes $F(X_+, -)$ a lax monoidal functor. Applying it to all of these spectra, maps, homotopies and 2-homotopies, we obtain a map

$$\mu_{s,s'}\colon Y_s \wedge Y_{s'} = F(X_+, \tau_{\geq s}E) \wedge F(X_+, \tau_{\geq s'}E) \xrightarrow{F(\Delta, \phi_{s,s'})} F(X_+, \tau_{\geq s+s'}E) = Y_{s+s'}$$

for each $s, s' \in \mathbb{Z}$, making each square in (5.1) commute up to homotopy, so that the combined homotopies are connected by a 2-homotopy.

Hence the Whitehead tower spectral sequence

$$\mathcal{E}_1^{s,t} = H^{2s+t}(X; E_s) \Longrightarrow_s E^{s+t}(X)$$

is an algebra spectral sequence, with product on the $\mathcal{E}_1$-term given by the cup product with coefficients in E^*, converging to the E-cohomology cup product.

References

[Ada58] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214, DOI 10.1007/BF02564578. MR96219

[AH61] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38. MR0139181

[Boa99] J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84, DOI 10.1090/conm/239/03597. MR1718076

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR0077480

[HR19] Gard Olav Helle and John Rognes, *Boardman's whole-plane obstruction group for Cartan-Eilenberg systems*, Doc. Math. **24** (2019), 1855–1878. MR4369361

[Mac71] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin, 1971. MR0354798

[Mau63] C. R. F. Maunder, *The spectral sequence of an extraordinary cohomology theory*, Proc. Cambridge Philos. Soc. **59** (1963), 567–574, DOI 10.1017/s0305004100037245. MR150765

[Mas52] W. S. Massey, *Exact couples in algebraic topology. I, II*, Ann. of Math. (2) **56** (1952), 363–396, DOI 10.2307/1969805. MR52770

[Mas54] ——, *Products in exact couples*, Ann. of Math. (2) **59** (1954), 558–569, DOI 10.2307/1969719. MR60829

[Nov67] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951 (Russian). MR0221509

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