ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 13: TELESCOPIC LOCALIZATION

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1. The thick subcategory theorem

Implicitly, suppose that all spectra are p-local, for a fixed prime p.

The stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$ is a triangulated category, with Puppe cofiber sequences as its distinguished triangles. The analogues of Serre classes and hereditary torsion theories for triangulated categories are called thick and localizing subcategories, respectively. The full subcategory $\mathrm{Ho}(\mathcal{S}p^\omega)$ of finite spectra is also triangulated, but does not admit infinite coproducts.

Definition 1.1 ([HS99, Def. 1.3])**.** A thick subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ is a full subcategory that closed under cofiber sequences and retracts, meaning that

- if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle and two of X, Y and Z are in $\mathcal{T}$, then so is the third, and
- if X is a retract of Y and Y is in $\mathcal{T}$, then X is in $\mathcal{T}$.

A property of objects of $\mathcal{C}$ is said to be generic if the class of objects with that property is closed under cofiber sequences and retracts, i.e., spans a thick subcategory.

A localizing subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ (with all coproducts) is a thick subcategory that is closed under coproducts, meaning that

- if $\{X_\alpha\}_{\alpha\in J}$ is a set of objects in $\mathcal{T}$, then $\coprod_{\alpha\in J} X_\alpha$ is an object in $\mathcal{T}$.

The n-th term $L_nX = L_{E(n)}X$ in the chromatic tower

$$X \longrightarrow \dots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_0X$$

of localization functors is the left adjoint in an adjunction

$$L_n \colon \mathrm{Ho}(\mathcal{S}p) \rightleftarrows \mathrm{Ho}(L_n\mathcal{S}p) \colon U\,.$$

It annihilates the localizing subcategory

$$\mathrm{Ho}(\mathcal{S}p_{\ge n+1}) := \{Z \mid L_nZ \simeq *\} = \{Z \mid E(n)_*(Z) = 0\} \subset \mathrm{Ho}(\mathcal{S}p)$$

of (p-local) $E(n)$-acyclic spectra. When restricted to (p-local) finite spectra F, it annihilates the thick subcategory

$$\mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1}) := \{F \mid L_nF \simeq *\} = \{F \mid E(n)_*(F) = 0\} \subset \mathrm{Ho}(\mathcal{S}p^\omega)$$

of finite $E(n)$-acyclic spectra. These full subcategories are the preimages under $MU_*(-)\colon \mathrm{Ho}(\mathcal{S}p) \to LB-\mathrm{coMod}$ of the hereditary torsion theory $\mathcal{T}_n$.

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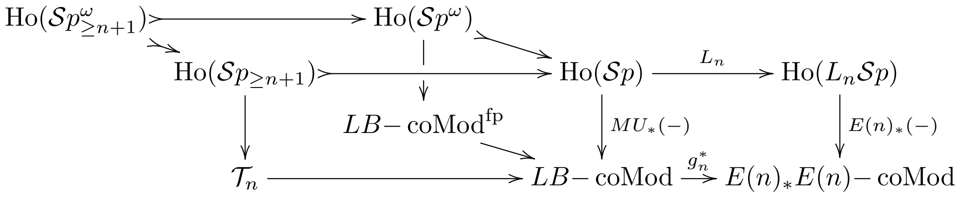


Definition 1.2. A finite (p-local) spectrum F has type $\geq n$ if $E(n-1)_*(F) = 0$, i.e., if $F \in \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n})$. It has type $= n$ if $E(n-1)_*(F) = 0$ and $E(n)_*(F) \neq 0$.

Example 1.3. A Smith–Toda complex $V(n-1)$ has type $= n$, when it exists.

Stephen Mitchell proved that there are finite spectra of each (chromatic) type.

Let $A(n) \subset \mathscr{A}$ denote the finite subalgebra generated by $Sq^1, \dots, Sq^{2^n}$ for $p = 2$, or by $\beta, P^1, \dots, P^{p^{n-1}}$ for p odd. It contains the exterior algebra $\Lambda(Q_0, \dots, Q_n)$ on the first Milnor primitives.

Theorem 1.4 (Mitchell [Mit85, Thm. B])**.** *For each prime p and integer $n \geq 0$ there exists a finite spectrum $F(n)$ such that*

- *$H^*(F(n); \mathbb{F}_p)$ is a (finitely generated) free module over $A(n-1)$,*
- *$K(m)_*(F(n)) = 0$ for $0 \leq m < n$, and*
- *$K(n)_*(F(n)) \neq 0$,*

so that $F(n)$ has type $= n$.

The proof uses the Steinberg idempotent from representation theory to split $F(n)$ off as a summand of the suspension spectrum of a homogeneous space $SO(2^n)/(\mathbb{Z}/2)^n$ for $p = 2$ or $U(p^n)/(\mathbb{Z}/p)^n$ for p odd.

Lemma 1.5. *Let F be a finite p-local spectrum. If F is not contractible, then F has type $= n$ for some finite $0 \leq n < \infty$. Otherwise, F has type $\geq n$ for all n.*

Proof. The homology $H_*(F; \mathbb{F}_p) = 0$ is concentrated in a finite range $0 \leq * \leq d$. Choose n so large that $|v_n| = 2p^n - 2 \geq d$. Then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}^2_{s,t} = H_s(F; K(n)_t) \Longrightarrow_s K(n)_{s+t}(F)$$

collapses at the $\mathcal{E}^2$-term for bidegree reasons. Hence $K(n)_*(F) = 0$ if and only if $H_*(F; \mathbb{F}_p) = 0$. For finite p-local F this happens if and only if F is contractible. Hence, for non-contractible F there exist n such that $K(n)_*(F) \neq 0$. The minimal such n is then the exact type of F, which is finite. □

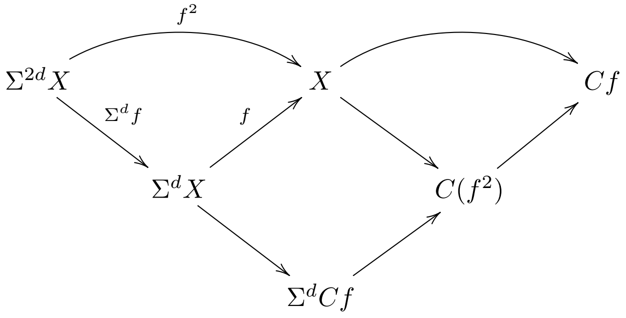
Let $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq 0})$ be the category of all p-local finite spectra, and let $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq \infty})$ be the category of all contractible finite spectra, so that there are proper inclusions

$$\mathrm{Ho}(\mathcal{S}p^{\omega}_{(p)}) = \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq 0}) \supsetneq \cdots \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n}) \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n+1}) \supsetneq \cdots \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq \infty}).$$

The Hopkins–Smith thick subcategory theorem asserts that these account for all the thick subcategories of the category of finite spectra.

Theorem 1.6 (Hopkins–Smith [HS98, Thm. 7])**.** *If $\mathcal{T} \subset \mathrm{Ho}(\mathcal{S}p^{\omega})$ is a thick subcategory of the triangulated category of p-local finite spectra, then $\mathcal{T} = \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n})$ for some $0 \leq n \leq \infty$.*

This is proved as a consequence of the Devinatz–Hopkins–Smith nilpotence theorem (Chapter 11, Theorem 4.3 or 4.4). See also [Rav92a, Ch. 5]. As a hint of how thick subcategories/generic properties are related to nilpotence, note that if $f\colon \Sigma^d X \to X$ is a self-map and X lies in a thick subcategory $\mathcal{T}$, then Cf also lies in $\mathcal{T}$. Conversely, if Cf lies in $\mathcal{T}$, then the braid diagram



shows that $C(f^2)$ lies in $\mathcal{T}$. By induction, $C(f^{2^i})$ lies in $\mathcal{T}$ for all $i \geq 0$. If we now assume that f is nilpotent, so that $f^{2^i} \simeq *$ for some i, then $C(f^{2^i}) \simeq X \vee \Sigma^{2^i d+1} X$ contains X as a retract, which implies that X also lies in $\mathcal{T}$.

Remark 1.7. An algebraic analogue of the thick subcategory theorem, classifying the Serre subcategories of $LB\!-\mathrm{coMod}^{\mathrm{fp}}$, is stated as [Rav92a, Thm. 3.4.2]. Working p-locally, these are the full subcategories $LB\!-\mathrm{coMod}^{\mathrm{fp}}_{\geq n}$ of v_{n-1}-power torsion comodules, for $0 \leq n \leq \infty$. The proof is corrected in [JLR96, Thm. 1.6], and is an application of the Landweber filtration theorem (Chapter 11, Theorems 7.1 and 7.2.).

Remark 1.8. The Hopkins–Ravenel smash product theorem (Chapter 11, Theorem 5.14) is proved [Rav92a, §8] using the thick subcategory theorem. One needs to prove that the $E(n)$-local sphere $L_n S$ is $E(n)$-nilpotent, i.e., lies in the thick ideal of $\mathrm{Ho}(\mathcal{S}p)$ generated by $E(n)$. The full category of finite spectra F for which $L_n F$ is $E(n)$-nilpotent is a thick subcategory, so to prove that it contains S it suffices to show that it contains some rationally nontrivial spectrum F with this property. This is then carried out.

The coherent sheaves $MU_*(F)^\sim$ associated to finite spectra F have "closed" support that is invariant under specialization (to greater heights), in the following sense.

Theorem 1.9 (Ravenel [Rav84, Thm. 2.11])**.** *Let F be a finite spectrum. Then*

$$\dim_{K(n)_*} K(n)_*(F) \leq \dim_{K(n+1)_*} K(n+1)_*(F)$$

for all $n \geq 0$. In particular, $K(n)_*(F) \neq 0$ implies $K(n+1)_*(F) \neq 0$, while $K(n+1)_*(F) = 0$ implies $K(n)_*(F) = 0$. Hence $K(n)_*(F) = 0$ if and only if $E(n)_*(F) = 0$.

Proof. Consider the MU-module spectrum $E = E(n+1)/I_n = E/(p, \dots, v_{n-1})$, with coefficient ring $E_* = \mathbb{F}_p[v_n, v_{n+1}^{\pm 1}]$. (For $n = 0$, this is to be read as $E_* = E(1)_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$.) Since E_* is a graded PID (= principal ideal domain) and F is finite, $E_*(F)$ is a finite direct sum of cyclic E_*-modules, i.e., of a free summands E_* and b torsion summands E_*/v_n^k for $k \geq 1$, up to suspensions.

The number a of free summands is the same as the dimension of $v_n^{-1}E_*(F)$ over $v_n^{-1}E_* = \mathbb{F}_p[v_n^{\pm 1}, v_{n+1}^{\pm 1}]$, which by Johnson–Wilson [JW75, Thm. 3.1], is the same as $\dim_{K(n)_*} K(n)_*(F)$. ((ETC: This uses that $B(n)_*(F)$ is a free $B(n)_*$-module, which follows since there are no invariant ideals in $B(n)_*$ other than (0) and (1).))

The cofiber sequence $\Sigma^{|v_n|}E \xrightarrow{v_n} E \longrightarrow K(n+1)$ induces a universal coefficient short exact sequence

$$0 \to K(n+1)_* \otimes_{E_*} E_*(F) \to K(n+1)_*(F) \to \mathrm{Tor}_1^{E_*}(K(n+1)_*, E_{*-1}(F)) \to 0\,.$$

Each free summand E_* contributes a copy of $K(n+1)_*$ to the left hand term. Each v_n-power torsion summand E_*/v_n^k contributes one copy of $K(n+1)_*$ at the left hand side and one copy at the right hand side. Hence $\dim_{K(n+1)_*} K(n+1)_*(F) = a + 2b \ge a = \dim_{K(n)_*} K(n)_*(F)$. ((ETC: If F were not finite, then $E_*(F)$ could contain uniquely v_n-divisible summands such as $v_n^{-1}E_*$, which would contribute to $K(n)_*(F)$ but not to $K(n+1)_*(F)$.))

The final claim follows from $\langle E(n) \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle$. □

Corollary 1.10. *A finite p-local spectrum has type $\ge n$ if and only if $K(n-1)_*(F) = 0$. It has type $= n$ if and only if $K(n-1)_*(F) = 0$ and $K(n)_*(F) \ne 0$.*

This does not explicitly refer to Johnson–Wilson $E(n)$-theory, and is the more usual way of defining (chromatic) type $\ge n$, but relies on Theorem 1.9 to make good sense.

Example 1.11. A finite p-local spectrum F has type 0 if and only if $H_*(F;\mathbb{Q}) \cong \pi_*(F) \otimes \mathbb{Q}$ is nonzero. It has type ≥ 1 if and only if $H_*(F;\mathbb{Q}) \cong \pi_*(F) \otimes \mathbb{Q} = 0$. In that case it has type $= 1$ if and only if $K(1)_*(F) \ne 0$, which is equivalent to $KU_*(F) \ne 0$. It has type ≥ 2 if and only if $K(0)_*(F) = 0$ and $K(1)_*(F) = 0$, which is equivalent to $KU_*(F) = 0$. The Moore spectrum $F = V(0) = S/p = S \cup_p e^1$ has type 1, while (for p odd) the cofiber $V(1) = S/(p, v_1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ of the Adams self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$ has type ≥ 2, since $KU_*(S/p) = KU_*/p \ne 0$ while $KU_*(S/(p, v_1)) = 0$.

Example 1.12. In Chapter 11, Remark 2.5 we saw that $\dim_{K(n)_*} K(n)_*(BC_p)$ is finite, and grows with n, even if BC_p is not a finite spectrum. In the $K(n)$-local category the spectra $\hat{L}_n \Sigma^\infty BG_+$ are in fact dualizable, for all finite groups G, hence are somewhat close to being finite in that category [Rav82], [HS99, Cor. 8.7].

In contrast to Ravenel's result for finite spectra F, Jeremy Hahn proved that $\mathbb{H}_\infty$ ring spectra R (and even less ring structure is needed) have "open" support that is invariant under generalization (to lower heights).

Theorem 1.13 (Hahn (arXiv:1612.04386)). *Let R be an $\mathbb{H}_\infty$ ring spectrum. If $K(n)_*(R) = 0$ for some $n \ge 0$, then $K(n+1)_*(R) = 0$. Hence $K(n+1)_*(R) \ne 0$ implies $K(n)_*(R) \ne 0$.*

The orthogonality result $K(n)_*(K(m)) = 0$ for $n \ne m$ (Chapter 12, Proposition 6.10) shows that for general p-local spectra X the support

$$\{n \ge 0 \mid K(n)_*(X) \ne 0\}$$

can be arbitrary, often being invariant neither under specialization nor under generalization.

2. The periodicity theorem

Definition 2.1. Let F be a finite p-local spectrum and $n \geq 0$. A map $v\colon \Sigma^d F \to F$ is said to be a v_n self-map if

$$K(m)_*(v)\colon K(m)_*(\Sigma^d F) \longrightarrow K(m)_*(F)$$

is multiplication by a power of v_n for $m = n$, and zero otherwise.

Multiplication by p defines a v_0 self-map $p\colon F \to F$ for any F in $\mathrm{Ho}(\mathcal{S}p^{\omega}_{(p)})$. The Adams self-maps $v_1 : \Sigma^{2p-2} S/p \to S/p$ (for p odd) and $v_1^4\colon \Sigma^8 S/2 \to S/2$ (for $p = 2$) are v_1 self-maps. Sometimes one refines the terminology, and calls v a v_n^k self-map if $K(n)_*(v)$ is multiplication by v_n^k, and says v_n-power self-map if the exponent k is not specified. If a v_n-power self-map exists, one may always find one where the exponent $k = p^N$ is a power of p.

Hopkins–Smith [HS98, §3] show that the property of admitting a v_n self-map, for a fixed $n \geq 0$, is generic. In other words, the collection of such F generates a thick subcategory of $\mathrm{Ho}(\mathcal{S}p^{\omega}_{(p)})$. By the thick subcategory theorem it must therefore be $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq m})$ for some $0 \leq m \leq \infty$. In fact, $m = n$.

Theorem 2.2 (Hopkins–Smith [HS98, Thm. 9])**.** *Let p be a prime and $n \geq 0$ an integer. A finite p-local spectrum admits a v_n self-map if and only if it has (chromatic) type $\geq n$.*

Outline of proof. One implication is easy: Let $v\colon \Sigma^d F \to F$ be a v_n self-map, with homotopy cofiber Cv. The case $m = n$ of the long exact sequence

$$\cdots \to K(m)_*(\Sigma^d F) \overset{K(m)_*(v)}{\longrightarrow} K(m)_*(F) \longrightarrow K(m)_*(Cv) \to \dots \tag{2.1}$$

shows that $K(n)_*(Cv) = 0$, since $K(n)_*(v)$ is an isomorphism. If F had type $m < n$ then the sequence would also show that $K(m)_*(Cv) \cong K(m)_*(F) \oplus K(m)_*(\Sigma^{d+1} F) \neq 0$, since $K(m)_*(v) = 0$ and $K(m)_*(F) \neq 0$. This contradicts Theorem 1.9 for the finite spectrum Cv.

It follows that the thick subcategory of spectra admitting v_n self-maps is contained in $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n})$. To prove equality, it suffices to exhibit a single finite spectrum of type n admitting a v_n self-map. This is done in [HS98, §4] and [Rav92a, App. C]. Jeff Smith used idempotents in the group rings of symmetric groups to construct a spectrum with particular cohomology as a module over the Steenrod algebra ((ETC: and more)), and the Adams spectral sequence is then used to construct the v_n self-map.

Once this one type n spectrum with a v_n self-map has been constructed, it follows from the thick subcategory theorem that every spectrum if type $\geq n$ admits such maps. This is a powerful existence result. □

Note that $E(m)_*(F) = 0$ if and only if $v_m^{-1} MU_*(F) = 0$, since $E(m)_*$ and $v_m^{-1} MU_*$ are both Landweber exact of height m, so a finite spectrum F has type $\geq n$ if and only if the LB-comodule $MU_*(F)$ satisfies $v_{n-1}^{-1} MU_*(F) = 0$.

The periodicity theorem has the following algebraic precursor.

Proposition 2.3 ([Rav92a, Cor. 3.3.9])**.** *Let M be a finitely presented LB-comodule. Then $v_n^k\colon \Sigma^{k|v_n|} M \to M$ is an LB-comodule homomorphism for some $k > 0$ if and only if $v_{n-1}^{-1} M = 0$.*

Proof. The proof uses the Landweber filtration theorem (Chapter 11, Theorems 7.1 and 7.3), giving a filtration

$$0 = M(0) \subset \cdots \subset M(s-1) \subset M(s) \subset \cdots \subset M(\ell) = M$$

by finitely presented LB-comodules, where $M(s)/M(s-1) = \Sigma^{d_s} L/I_{n_s}$.

If $v_n^k \colon \Sigma^{k|v_n|} M \to M$ commutes with the LB-coaction, then so does its restriction to $M(s)$ for each s, hence also its corestriction to $M(s)/M(s-1)$. But multiplication by v_n^k acts as an LB-comodule homomorphism on L/I_m only for $m \geq n$, by the calculation $P(L/I_m) = \mathbb{F}_p[v_m]$ of LB-comodule primitives (Chapter 11, Theorem 7.2). Hence $n_s \geq n$ for each $1 \leq s \leq \ell$, which implies $v_{n-1}^{-1} L/I_{n_s} = 0$, $v_{n-1}^{-1} M(s) = 0$ and $v_{n-1}^{-1} M = 0$.

Conversely, if $v_{n-1}^{-1} M = 0$ then $n_s \geq n$ for each $1 \leq s \leq \ell$. It follows that M is annihilated by I_n^ℓ. By the invariance of v_n under strict isomorphisms (Chapter 10, Lemma 4.10)

$$\eta_L(v_n) \equiv \eta_R(v_n) \mod LB \cdot I_n \,,$$

which implies that

$$\eta_L(v_n^{p^{\ell-1}}) \equiv \eta_R(v_n^{p^{\ell-1}}) \mod LB \cdot I_n^\ell \,.$$

It follows that $v_n^k = v_n^{p^{\ell-1}}$ is LB-comodule primitive in LB/I_n^ℓ, and acts on M as an LB-comodule homomorphism. □

Lemma 2.4. *If F has type $= n$ and $v \colon \Sigma^d F \to F$ is a v_n self-map then Cf has type $= n+1$.*

Proof. We have $K(m)_*(F) = 0$ for $m < n$ and $K(m)_*(F) \neq 0$ for $m \geq n$. Moreover, $K(m)_*(v)$ is an isomorphism for $m = n$ and zero for $m > n$. By (2.1) it follows that $K(m)_*(Cv) = 0$ for $m \leq n$ and $K(m)_*(Cv) \neq 0$ for $m > n$. □

Example 2.5. The periodicity theorem provides an alternative approach to the existence Theorem 1.4 (but Smith's construction is no easier than Mitchell's). To start an induction, let $F(0) = S$. For $n \geq 0$, suppose we have constructed a type n finite spectrum $F(n) = S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$, with $i_s \geq 1$ for $0 \leq s < n$ and

$$MU_*(S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})) \cong L/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$$

as an L-module. (It will also be an LB-comodule, so $(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \subset L$ will be an invariant ideal.) These are sometimes called generalized Moore spectra. By the periodicity theorem, there exists a v_n self-map $v \colon \Sigma^d F(n) \to F(n)$ inducing multiplication by v_n^k in $K(n)$-homology. Since $p, \dots, v_{n-1}$ are nilpotent in $MU_*(F(n))$ we may arrange that v induces multiplication by $v_n^{i_n}$ in MU-homology, for some $i_n > 0$. Let

$$F(n+1) = S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}, v_n^{i_n}) = Cv$$

be the homotopy cofiber of this v_n self-map.

The degree p map $p \colon S \to S$ is a v_0 map for each prime p, so we may take $i_0 = 1$ and $F(1) = V(0) = S/p$. For odd p the Adams self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$ corresponds to $i_1 = 1$, so we can form the type 2 Smith–Toda complex $F(2) = V(1) = S/(p, v_1)$. For $p = 2$, the Adams self-map $v_1^4 \colon \Sigma^8 S/2 \to S/2$ realizes the smallest possible value $i_1 = 4$, so we can form $F(2) = S/(2, v_1^4)$. ((ETC: Also survey v_1 self-maps of S/p^{i_0} for $i_0 \geq 2$.))

For $p \geq 5$ the Smith–Toda [Smi71], [Tod71] self-map $v_2\colon \Sigma^{2p^2-2}S/(p,v_1) \to S/(p,v_1)$ realizes $i_2 = 1$, with homotopy cofiber $F(3) = V(2) = S/(p,v_1,v_2)$. For $p = 3$, Behrens–Pemmaraju [BP04] proved the existence of a v_2^9 self-map $v_2^9\colon \Sigma^{144}S/(3,v_1) \to S/(3,v_1)$, with homotopy cofiber $F(3) = S/(p,v_1,v_2^9)$. Belmont–Shimomura (arXiv: 2109.01059) recently obtained a v_2^9 self-map of $S/(3,v_1^8)$, which is useful for propagating 3-torsion classes that are v_1^8-torsion but not (strict) v_1-torsion. For $p = 2$, Behrens–Hill–Hopkins–Mahowald [BHHM08] established the existence of a v_2^{32} self-map $v_2^{32}\colon \Sigma^{192}S/(2,v_1^4) \to S/(2,v_1^4)$ with type 3 homotopy cofiber $F(3) = S/(2,v_1^4,v_2^{32})$. Behrens–Mahowald–Quigley (arXiv:2011.08956) also obtained a v_2^{32} self-map $v_2^{32}\colon \Sigma^{192}S/(8,v_1^8) \to S/(8,v_1^8)$, with homotopy cofiber $S/(8,v_1^8,v_2^{32})$. This is useful for propagating 8-torsion and v_1^8-torsion classes. The proofs for $p \in \{2,3\}$ use topological modular forms, and suffice to determine the image of the homomorphism $\pi_*(S) \to \pi_*(\mathrm{tmf})$.

For $p \geq 7$, Toda [Tod71] constructed the type 4 spectrum $F(4) = V(3) = S/(p,v_1,v_2,v_3)$ as the homotopy cofiber of a v_3 self-map $v_3\colon \Sigma^{2p^3-2}S/(p,v_1,v_2) \to S/(p,v_1,v_2)$. On the other hand, Lee Nave [Nav10] proved that $V((p+1)/2)$ does not exist, so $V(2) = S/(5,v_1,v_2)$ at $p = 5$ does not admit a strict v_3 self-map. It is not known whether $V(3)$ admits a strict v_4 self-map for any prime p.

The existence statement of the periodicity theorem is supplemented with the following weak uniqueness statement.

Proposition 2.6 ([HS98, Cors. 3.7, 3.8])**.** *Let $v\colon \Sigma^d F \to F$ and $v'\colon \Sigma^{d'}F' \to F'$ be v_n self-maps. There are $i, i' > 0$ (with $id = i'd'$) such that for every map $g\colon F \to F'$ the diagram*

$$\begin{array}{ccc} \Sigma^{id}F & \xrightarrow{\Sigma^{id}g} & \Sigma^{i'd'}F' \\ \downarrow{\scriptstyle v^i} & & \downarrow{\scriptstyle (v')^{i'}} \\ F & \xrightarrow{g} & F' \end{array}$$

commutes up to homotopy. In particular, if $F = F'$ and $g = \mathrm{id}_F$ then $v^i \simeq (v')^{i'}$.

This has the following consequence.

Definition 2.7. Let $F(n)$ be a (finite, p-local) type n spectrum, with v_n self-map $v\colon \Sigma^d F(n) \to F(n)$. The telescope

$$T(n) = v_n^{-1}F(n) = \mathrm{hocolim}(F \xrightarrow{v} \Sigma^{-d}F \xrightarrow{v} \Sigma^{-2d}F \longrightarrow \dots)$$

is, up to homotopy equivalence under $F(n)$, independent of the choice of v_n self-map. Each map v is an $E(n)$-equivalence, so there is a factorization

$$F(n) \xrightarrow{\beta} v_n^{-1}F(n) = T(n) \xrightarrow{\tau} L_nF(n)$$

of the $E(n)$-localization map $\eta\colon F(n) \to L_nF(n) = \hat{L}_nF(n)$.

For small n we usually take $T(0) = p^{-1}S_{(p)} = S\mathbb{Q} = H\mathbb{Q}$ for all p, $T(1) = v_1^{-1}S/p$ for p odd and $T(1) = v_1^{-4}S/2$ for $p = 2$. The v_1-periodic homotopy in $\pi_*(S/p)$ is fully understood, by the following theorems of Mark Mahowald and of Haynes Miller.

Theorem 2.8 (Mahowald [Mah70], [Mah81], [Mah84])**.**

$$\tau\colon v_1^{-1}\pi_*(S/2) \xrightarrow{\cong} \pi_*(L_1S/2) \cong \pi_*(J/2)$$

is an isomorphism. Hence $T(1) = v_1^{-1}F(1) \simeq L_1F(1) \simeq \hat{L}_1F(1)$ *for any type* 1 *finite 2-local spectrum* $F(1)$.

See Chapter 5, Section 8, Figure 2 for a picture of a fundamental domain for $\cong \pi_*(J/2)$, which repeats v_1^4-periodically. For any homotopy class $x \in \pi_*(S/2)$, the product $v_1^{4N}x$ lies in the summand $\pi_*(J/2)$ for all sufficiently large N.

Sketch proof. The original argument works with $F(1) = S/2$, but working with $F(1) = Y = S/2 \wedge S/\eta = \Sigma^{-3}\mathbb{R}P^2 \wedge \mathbb{C}P^2$ is a little less difficult. Here $H^*(Y; \mathbb{F}_2) \cong A(1)//\Lambda(Q_1)$. The proof amounts to a careful analysis of the ko-based Adams spectral sequence for $F(1)$, using a splitting of $ko \wedge ko$ in terms of integral Brown–Gitler spectra, and determining differentials in a range by a comparison along a map $Th(\xi \downarrow \Omega S^5) \to ko$ from a Thom spectrum over ΩS^5. □

Theorem 2.9 (Miller [Mil81, Thm. 4.11])**.**

$$\tau \colon v_1^{-1}\pi_*(S/p) \xrightarrow{\cong} \pi_*(L_1S/p) \cong \pi_*(J/p)$$

is an isomorphism for odd primes p. Hence $T(1) = v_1^{-1}F(1) \simeq L_1F(1) \simeq \hat{L}_1F(1)$ *for any type* 1 *finite p-local spectrum* $F(1)$.

Let $g \in \mathbb{Z}_p^\times$ be a topological generator. The fiber sequence

$$J/p \longrightarrow KU/p \xrightarrow{\psi^g - 1} KU/p$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_*(J/p) \xrightarrow{\pi} \mathbb{F}_p[u^{\pm 1}] \xrightarrow{\psi^g - 1} \mathbb{F}_p[u^{\pm 1}] \longrightarrow \dots$$

in homotopy, where $(\psi^g - 1)(u^n) = (g^n - 1)u^n$. Here $g^n - 1 \equiv 0 \mod p$ if and only if $n \equiv 0 \mod p - 1$, so we have a short exact sequence

$$0 \to \Sigma^{-1}\mathbb{F}_p[u^{\pm(p-1)}] \xrightarrow{\partial} \pi_*(J/p) \xrightarrow{\pi} \mathbb{F}_p[u^{\pm(p-1)}] \to 0$$

and an algebra isomorphism

$$\pi_*(J/p) \cong \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1^{\pm 1}],$$

where $\alpha_1 = \partial(u^{p-1})$ and $\pi(v_1) = u^{p-1}$ have degree $2p-3$ and $2p-2$, respectively. See also Chapter 11, Section 11, Figure 1 for the v_1-periodic Adams–Novikov chart for J/p.

Sketch proof. The proof compares the (strongly convergent) Adams spectral sequence

$$\mathcal{E}_2^{*,*}(S/p) = \mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_p, H_*(S/p; \mathbb{F}_p)) \Longrightarrow \pi_*(S/p)$$

with a (potentially non-convergent) localized Adams spectral sequence

$$v_1^{-1}\mathcal{E}_2^{*,*}(S/p) = v_1^{-1}\,\mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_p, H_*(S/p; \mathbb{F}_p)) \Longrightarrow v_1^{-1}\pi_*(S/p)\,.$$

A comparison via the Adams–Novikov spectral sequence is used to transfer known d_2-differentials from an algebraic May spectral sequence to the localized Adams spectral sequence. This shows that $\mathcal{E}_\infty^{*,*}(S/p)$ above a line of slope $1/(p^2-p-1)$, in the usual Adams $(t-s, s)$-bigrading, consists only of classes detecting $\Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]$, on a line of slope $1/(2p-2)$. Since v_1-multiplication acts parallel to the Adams vanishing line for S/p, this suffices to deduce that there are no other v_1-periodic classes than those mentioned. □

3. Finite localizations

We follow Miller's article [Mil92], which responds to [Rav93] and [MS95].

Recall from Chapter 12, Section 4 that for any Landweber exact L-module E_* of height n, such as $E(n)_*$ or $v_n^{-1}L$, the full abelian subcategory

$$\mathcal{T}_E = \{M \mid E_* \otimes_L M = 0\} \subset LB-\mathrm{coMod}$$

only depends on n, and is equal to the hereditary torsion theory (= Serre subcategory closed under coproducts) generated by $L/I_{n+1} = L/(p, \dots, v_n)$.

Let X and E be spectra. The (co-)fiber sequence

$$C_E X \to X \to L_E X$$

is characterized by $C_E X$ being E-acyclic and $[Z, L_E X] = 0$ for any E-acyclic Z. Hence the Bousfield E-localization and E-colocalization functors L_E and C_E are fully determined by the full triangulated subcategory

$$\mathrm{Ho}(C_E\mathcal{S}p) := \{Z \mid E_*(Z) = 0\} \subset \mathrm{Ho}(\mathcal{S}p)$$

of E-acyclic spectra. This is a localizing subcategory, i.e., a thick subcategory closed under coproducts.

For any Landweber exact spectrum E of height n, such as $E(n)$ or $v_n^{-1}MU$, the finite p-local E-acyclic spectra

$$\mathrm{Ho}(\mathcal{S}p^\omega_{(p)}) \cap \mathrm{Ho}(C_E\mathcal{S}p) = \mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1})$$

span the thick subcategory of finite p-local spectra of type $\ge n+1$. By the Hopkins–Smith thick subcategory theorem, it is generated as a thick subcategory by any one type $= n+1$ spectrum $F(n+1)$. For example, if p and n are such that the Smith–Toda spectrum $V(n)$ exists, then it has type $n+1$ and $MU_*(V(n)) = L/I_{n+1}$ is the LB-comodule generating $\mathcal{T}_n$.

Let us write

$$\mathrm{Ho}(C^f_E\mathcal{S}p) = \mathrm{Loc}(F(n+1)) \subset \mathrm{Ho}(\mathcal{S}p)$$

for the localizing subcategory generated by $\mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1})$, which is equal to the localizing subcategory generated by any one $F(n+1)$. Clearly

$$\mathrm{Ho}(C^f_E\mathcal{S}p) \subset \mathrm{Ho}(C_E\mathcal{S}p)\,. \tag{3.1}$$

Miller shows that for any spectrum X there is a (co-)fiber sequence

$$C^f_n X \longrightarrow X \longrightarrow L^f_n X$$

with $C^f_n X$ in $\mathrm{Ho}(C^f_E\mathcal{S}p)$ and $[Z, L^f_n X] = 0$ for each $Z \in \mathrm{Ho}(C^f_E\mathcal{S}p)$. We call $L^f_n X$ and $C^f_n X$ the finite E-localization and finite E-colocalization of X. The inclusion (3.1) implies that there is a natural, unique, factorization

$$X \xrightarrow{\eta^f} L^f_n X \xrightarrow{\tau} L_n X$$

of the E-localization map $\eta\colon X \to L_n X$.

Definition 3.1. Let $\mathcal{A}$ be a set of homotopy types of finite spectra.

- A spectrum N is finitely $\mathcal{A}$-local if $[Z, N]_* = 0$ for each $Z \in \mathcal{A}$.
- A spectrum Z is finitely $\mathcal{A}$-acyclic if $[Z, N]_* = 0$ for each finitely $\mathcal{A}$-local spectrum N.
- A map $f\colon X \to Y$ is a finite $\mathcal{A}$-equivalence if its mapping cone Cf is finitely $\mathcal{A}$-acyclic.

Clearly $f\colon X \to Y$ is a finite $\mathcal{A}$-equivalence if and only if $f^*\colon [Y,N] \to [X,N]$ is a bijection for each finitely $\mathcal{A}$-local N. The finitely $\mathcal{A}$-acyclic spectra form a localizing subcategory of $\mathrm{Ho}(\mathcal{S}p)$, containing each element of $\mathcal{A}$. In particular, it is closed under sequential homotopy colimits (= mapping telescopes).

Theorem 3.2 (Miller [Mil92, Thm. 4]). *For any set $\mathcal{A}$ of (homotopy types of) finite spectra and any spectrum X there is a finite $\mathcal{A}$-equivalence $X \to L^f_{\mathcal{A}}X$ to a finitely $\mathcal{A}$-local spectrum.*

Proof. We may assume $\mathcal{A}$ is closed under (de-)suspensions. Miller constructs $L^f_{\mathcal{A}}X$ as the homotopy colimit of a sequence

$$X = X_0 \xrightarrow{i_0} X_1 \longrightarrow \dots \longrightarrow X_m \xrightarrow{i_m} X_{m+1} \longrightarrow \dots \longrightarrow L^f_{\mathcal{A}}X = \operatorname*{hocolim}_m X_m \,.$$

Let $X_0 = X$ and suppose that X_m has been defined. Let

$$W_m = \bigvee_{f\colon A \to X_m} A$$

be a wedge sum of spectra, where A ranges over all elements in $\mathcal{A}$ and $f\colon A \to X_m$ ranges over all homotopy classes of maps $f\colon A \to X_m$. The maps f combine to a map $f_m\colon W_m \to X_m$, and we let $X_{m+1} = Cf_m$ be its homotopy cofiber:

$$W_m \xrightarrow{f_m} X_m \xrightarrow{i_m} X_{m+1} \,.$$

Each W_m is finitely $\mathcal{A}$-acyclic, since $[W_m, N]_* \cong \prod_{f\colon A\to X_m} [A, N]_*$ vanishes if N is finitely $\mathcal{A}$-local. The homotopy cofiber of each $X_0 \to X_m$ is finitely $\mathcal{A}$-acyclic, by induction on m, so the homotopy cofiber of $X \to L^f_{\mathcal{A}}X$ is finitely $\mathcal{A}$-acyclic, by passage to the sequential homotopy colimit. Thus this map is a finite $\mathcal{A}$-equivalence.

If $Z \in \mathcal{A}$ and $g\colon Z \to L^f_{\mathcal{A}}X$ is any map, then g factors

$$g\colon Z \xrightarrow{\tilde g} X_m \longrightarrow L^f_{\mathcal{A}}X$$

through some X_m, since Z is finite. Here $\tilde g$ is one of the components of f_m, so $i_m \tilde g$ is null-homotopic. Hence g is null-homotopic and $[Z, L^f_{\mathcal{A}}X] = 0$, so that $L^f_{\mathcal{A}}X$ is finitely $\mathcal{A}$-local. □

In the resulting homotopy cofiber sequence

$$C^f_{\mathcal{A}}X \longrightarrow X \xrightarrow{\eta^f} L^f_{\mathcal{A}}X$$

we call $L^f_{\mathcal{A}}X$ the finite $\mathcal{A}$-localization of X, and $C^f_{\mathcal{A}}X$ the finite $\mathcal{A}$-colocalization of X. When $\mathcal{A}$ is the set of homotopy types of E-acyclic finite spectra, for a given spectrum E, we say finitely E-local, finitely E-acyclic and finite E-equivalence for finitely $\mathcal{A}$-local, finitely $\mathcal{A}$-acyclic and finite $\mathcal{A}$-equivalence, respectively. We set $L^f_E X = L^f_{\mathcal{A}}X$ and $C^f_E X = C^f_{\mathcal{A}}X$.

When $E = E(n)$ we write $L^f_n X = L^f_{E(n)}X$ and $C^f_n X = C^f_{E(n)}X$ for the finite $E(n)$-localization and finite $E(n)$-colocalization of X. Since a finite p-local spectrum is $E(n)$-acyclic if and only if it is $K(n)$-acyclic, these are the same as the finite $K(n)$-localization and finite $K(n)$-colocalization of X, respectively.

Proposition 3.3 ([Mil92, Prop. 5, Cor. 6]). *A spectrum is finitely $\mathcal{A}$-acyclic if and only if it is the homotopy colimit of a sequence of maps with homotopy cofibers* *that are wedge sums of integer suspensions of elements in $\mathcal{A}$. Hence the finitely $\mathcal{A}$-acyclic spectra span the localizing subcategory of* $\mathrm{Ho}(\mathcal{S}p)$ *generated by the elements of $\mathcal{A}$.*

This follows from Miller's proof, since X is finitely $\mathcal{A}$-acyclic if and only if $L^f_{\mathcal{A}}X \simeq *$. In particular, the full subcategory of finitely $E(n)$-acyclic spectra is equal to the localizing subcategory $\mathrm{Ho}(C^f_n\mathcal{S}p)$ generated by the finite p-local spectra of type $\geq n+1$.

Proposition 3.4 ([Mil92, Prop. 9, Cor. 11])**.** *Finite $\mathcal{A}$-localization is smashing, so that*

$$L^f_{\mathcal{A}}X \simeq X \wedge L^f_{\mathcal{A}}S$$

for all spectra X. Hence $L^f_{\mathcal{A}}$ is Bousfield localization with respect to the ring spectrum $L^f_{\mathcal{A}}S$.

The proof that $X \wedge L^f_{\mathcal{A}}S$ is finitely $\mathcal{A}$-local uses Spanier–Whitehead duality.

Proposition 3.5 ([Mil92, Prop. 14])**.** *If F is a type $\geq n$ finite p-local spectrum, with v_n self-map $v\colon \Sigma^d F \to F$, then the map*

$$F \longrightarrow v^{-1}F = T \simeq L^f_n F$$

inverting v is the finite $E(n)$-localization.

Proof. The mapping cone Cv is finite and $E(n)$-acyclic, which implies that the homotopy cofiber of $F \to v^{-1}F = T$ is finitely $E(n)$-acyclic. Hence this map is a finite $E(n)$-equivalence.

Let Z be any finite $E(n)$-acyclic spectrum, and consider any map $g\colon Z \to T$. It factors through $\Sigma^{-md}F \to T$ for some m, since Z is finite. Write $\tilde{g}\colon Z \to \Sigma^{-md}F$ for one such lift. The trivial map $0\colon \Sigma^d Z \to Z$ is a v_n self-map, so (by the weak uniqueness result Proposition 2.6) the square

$$\begin{array}{ccc}
 & & F \\
 & & \downarrow{\scriptstyle v^m} \\
Z & \xrightarrow{\tilde{g}} & \Sigma^{-md}F \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle v^i} \\
\Sigma^{-id}Z & \xrightarrow{\Sigma^{-id}\tilde{g}} & \Sigma^{-(i+m)d}F \\
 & & \downarrow \\
 & & T
\end{array}$$

commutes up to homotopy for some $i > 0$. This proves that $g \simeq 0$, so $T = v^{-1}F$ is finitely $E(n)$-local. ☐

We now follow Bousfield and Mahowald–Sadofsky, to show that the finite localization L^f_n can be rewritten as the Bousfield localization at $T(0) \vee \cdots \vee T(n)$.

Lemma 3.6. $\langle T(n)\rangle \geq \langle K(n)\rangle$ *for each $n \geq 0$. Hence*

$$\langle T(0) \vee \cdots \vee T(n)\rangle \geq \langle K(0) \vee \cdots \vee K(n)\rangle = \langle E(n)\rangle$$

and there are natural transformations

$$L_{T(n)}X \xrightarrow{\tau} L_{K(n)}X = \hat{L}_n X$$

and

$$L_{T(0)\vee\cdots\vee T(n)}X \xrightarrow{\tau} L_{E(n)}X = L_nX\,.$$

Proof. We have $K(n)_*F(n) \neq 0$ since $F(n)$ has type $= n$. Any choice of v_n self-map induces an isomorphism in $K(n)$-homology, so $K(n)_*F(n) \cong K(n)_*T(n)$ is also nonzero. Hence $K(n) \wedge T(n)$ is a wedge sum of one or more suspensions of $K(n)$, and contains a suspension of $K(n)$ as a retract. If $T(n)_*(Z) = 0$, then $K(n) \wedge T(n) \wedge Z \simeq *$, and this implies $K(n)_*(Z) = 0$. □

Definition 3.7. If $\langle D\rangle \vee \langle E\rangle = \langle S\rangle$ and $\langle D\rangle \wedge \langle E\rangle = \langle *\rangle$, then we say that $\langle D\rangle = \langle E\rangle^c$ is a (Bousfield) complement of $\langle E\rangle$.

Not every Bousfield class admits a complement, but for those that do it is unique.

Lemma 3.8. *If $\langle C\rangle$ and $\langle D\rangle$ are complements of $\langle E\rangle$, then $\langle C\rangle = \langle D\rangle$.*

Proof. If $C_*(X) = 0$ then $\langle X\rangle = \langle C \wedge X\rangle \vee \langle E \wedge X\rangle = \langle E \wedge X\rangle$ so $\langle D\rangle \wedge \langle X\rangle = \langle D\rangle \wedge \langle E \wedge X\rangle = \langle D \wedge E \wedge X\rangle = \langle *\rangle$ and $D_*(X) = 0$. Hence $\langle C\rangle \geq \langle D\rangle$. The same argument applies with C and D switched. □

Lemma 3.9 (Ravenel [Rav84, Lem. 1.34])**.** *For any self-map $f\colon \Sigma^d X \to X$ with homotopy cofiber $Cf = X/f$ and telescope $f^{-1}X$, we have*

$$\langle X\rangle = \langle f^{-1}X\rangle \vee \langle X/f\rangle\,.$$

Hence

$$\langle S\rangle = \langle T(0) \vee \cdots \vee T(n)\rangle \vee \langle F(n+1)\rangle\,.$$

Proof. If $X_*Z = 0$ then $(X/f)_*Z = 0$ by the long exact sequence, and $f^{-1}X_*Z = 0$ by algebraic localization.

Conversely, if $(X/f)_*Z = 0$ then $f_*\colon X_*Z \to X_{*+d}Z$ is an isomorphism by the long exact sequence, so $X_*Z \cong f^{-1}X_*Z$ since inverting an isomorphism has no effect. If $f^{-1}X_*Z = 0$ it then follows that $X_*Z = 0$. □

Lemma 3.10. *$T(m) \wedge F(n+1) \simeq *$ for each $m \leq n$. Hence*

$$\langle T(0) \vee \cdots \vee T(n)\rangle \wedge \langle F(n+1)\rangle = \langle *\rangle\,,$$

so that $\langle F(n+1)\rangle^c = \langle T(0) \vee \cdots \vee T(n)\rangle$ is a Bousfield complement.

Proof. Let $v\colon \Sigma^d F(m) \to F(m)$ be a v_m self-map. The smash product $F(m) \wedge F(n+1)$ has type $= n+1$, so both $v_m \wedge \mathrm{id}$ and the zero map are v_m self-maps. Hence $v_m \wedge \mathrm{id}$ is nilpotent, by Proposition 2.6, and its telescope $T(m) \wedge F(n+1)$ must be contractible. □

Let X be any spectrum, and consider the case $\mathcal{A} = \{F(n+1)\}$ of Miller's homotopy cofiber sequence

$$C^f_{\mathcal{A}}X \longrightarrow X \longrightarrow L^f_{\mathcal{A}}X\,.$$

By the construction

$$X = X_0 \to \cdots \to X_m \to X_{m+1} \to \cdots \to X_\infty = L^f_{\mathcal{A}}X$$

with $W_m \to X_m \to X_{m+1}$, where W_m is a wedge sum of suspensions of $F(n+1)$, the finite $\mathcal{A}$-colocalization $C^f_{\mathcal{A}}X$ is a sequential homotopy colimit along maps with homotopy cofibers given by wedge sums of suspensions of $F(n+1)$. Hence it is $[F(n+1),\ ]_*$-colocal in the sense of [Bou79a, p. 369], and is a sequential homotopy colimit of finite $T(0) \vee \cdots \vee T(n)$-acyclic spectra. In particular, $C^f_{\mathcal{A}} X$ is $T(0) \vee \cdots \vee T(n)$-acyclic.

Moreover, $[F(n{+}1), L^f_{\mathcal{A}} X]_* = 0$, so the finite $\mathcal{A}$-localization $L^f_{\mathcal{A}} X$ is $[F(n{+}1), \ ]_*$-trivial, and is equal to the $[F(n+1), \ ]_*$-trivialization $X^{F(n+1)}$ of X in the sense of [Bou79a, p. 371].

Proposition 3.11 (Bousfield [Bou79a, Prop. 2.9])**.** *If F is a finite spectrum, then $\langle F \rangle$ has the complement $\langle F \rangle^c = \langle S^F \rangle$, where $S^F = L^f_{\{F\}} S$ is the $[F, \ ]_*$-trivialization of S.*

Proposition 3.12 (Bousfield [Bou79b, Prop. 3.5])**.** *If F is a finite spectrum, then a spectrum X is $(S^F)_*$-local if and only if $[F, X]_* = 0$.*

Proposition 3.13 (Mahowald–Sadofsky [MS95, Prop. 3.3])**.** *(a) A spectrum is $T(0) \vee \cdots \vee T(n)$-local if and only if it is finitely $\{F(n+1)\}$-local.*

(b) Finite $E(n)$-localization, finite $\{F(n+1)\}$-localization and Bousfield $T(0) \vee \cdots \vee T(n)$-localization all agree:

$$L^f_n X \simeq L_{\{F(n+1)\}} X \simeq L_{T(0)\vee\cdots\vee T(n)} X \,.$$

(c) Every $T(0) \vee \cdots \vee T(n)$-acyclic is a sequential homotopy colimit of finite $T(0) \vee \cdots \vee T(n)$-acyclics.

Proof. (a) By Lemmas 3.8, 3.10 and Proposition 3.11 we know that

$$\langle T(0) \vee \cdots \vee T(n) \rangle = \langle F(n+1) \rangle^c = \langle S^{F(n+1)} \rangle \,,$$

so by Proposition 3.12 any spectrum X is $T(0) \vee \cdots \vee T(n)$-local if and only if $[F(n+1), X]_* = 0$, i.e., if and only if it is finitely $\mathcal{A}$-local for $\mathcal{A} = \{F(n+1)\}$.

(b) The finite $E(n)$-acyclics are generated as a thick subcategory by $F(n+1)$, so they generate the same localizing subcategory of $\mathrm{Ho}(\mathcal{S}p)$, which implies that $L^f_n X = L^f_{E(n)} X$ agrees with $L^f_{\{F(n+1)\}} X$. The equivalence with $L_{T(0)\vee\cdots\vee T(n)} X$ follows from (a).

(c) Suppose that Z is $T(0) \vee \cdots \vee T(n)$-acyclic. Since $C^f_{\mathcal{A}} Z$ is $T(0) \vee \cdots \vee T(n)$-acyclic, it follows that $L^f_{\mathcal{A}} Z$ is $T(0) \vee \cdots \vee T(n)$-acyclic. By (a), $L^f_{\mathcal{A}} Z$ is also $T(0) \vee \cdots \vee T(n)$-local, so it must be contractible. Hence $Z \simeq C^f_{\mathcal{A}} Z$ is a sequential homotopy colimit of finite $T(0) \vee \cdots \vee T(n)$-acyclic spectra. □

((ETC: Is $L^f_n F \simeq L_{T(n)} F$ for F finite of type n?))

By analogy with the chromatic tower from Chapter 12, (1.1), (1.2) and (1.3), there is a telescopic tower

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \ldots \longrightarrow \mathrm{Ho}(L^f_n \mathcal{S}p) \longrightarrow \mathrm{Ho}(L^f_{n-1} \mathcal{S}p) \longrightarrow \ldots \longrightarrow \mathrm{Ho}(L^f_0 \mathcal{S}p)$$

of localization functors between the full subcategories

$$\mathrm{Ho}(\mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L^f_n \mathcal{S}p) \supset \mathrm{Ho}(L^f_{n-1} \mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L^f_0 \mathcal{S}p)$$

that defines the telescopic filtration of (p-local) stable homotopy theory. Applied to a spectrum X, this gives the telescopic tower

$$X \longrightarrow \ldots \longrightarrow L^f_n X \longrightarrow L^f_{n-1} X \longrightarrow \ldots \longrightarrow L^f_0 X$$

in $\mathrm{Ho}(\mathcal{S}p)$.

It appears to be an open problem whether telescopic convergence holds, i.e., whether

$$X \longrightarrow \operatorname*{holim}_n L_n^f X$$

is an equivalence for finite p-local X. As was noted in [MS95, p. 114] it is a split injection, since the composite with

$$\tau \colon \operatorname*{holim}_n L_n^f X \longrightarrow \operatorname*{holim}_n L_n X$$

is an equivalence by the chromatic convergence theorem (Chapter 12, Theorem 7.3).

4. The telescope conjecture

Based on the results of Mahowald and Miller (Theorems 2.8 and 2.9), a hope to calculate the v_n-periodic homotopy groups $v_n^{-1}\pi_* F(n) = \pi_* L_n^f F(n)$ for $n \geq 2$, and the ability to calculate the chromatically localized homotopy groups $\pi_* L_n F(n)$ in some nontrivial cases (starting with $n = 2$ and $p \geq 5$, see Chapter 12, Proposition 11.9), Ravenel made the following conjecture around 1977:

Conjecture 4.1 ([Rav84, Conj. 10.5])**.** $\langle T(n)\rangle = \langle K(n)\rangle$.

We already know that $\langle T(n)\rangle \geq \langle K(n)\rangle$ for all n. If $\langle T(m)\rangle = \langle K(m)\rangle$ for all $0 \leq m \leq n$ then

$$\langle T(0) \vee \cdots \vee T(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle = \langle E(n)\rangle$$

so that the natural map

$$L_n^f X \simeq L_{T(0)\vee\cdots\vee T(n)} X \xrightarrow{\tau} L_{K(0)\vee\cdots\vee K(n)} X \simeq L_n X$$

is an equivalence. This is the usual formulation of the height n Telescope Conjecture for X. It is equivalent to the assertion that a spectrum X is finitely $E(n)$-local if and only if it is $E(n)$-local. It is also equivalent to the assertion that in $\mathrm{Ho}(\mathcal{S}p)$ the subcategory $\mathrm{Ho}(\mathcal{S}p_{\geq n+1})$ of $E(n)$-acyclic spectra is generated, as a localizing subcategory, by the (thick) subcategory $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n+1})$ of finite $E(n)$-acyclic spectra.

Since both L_n^f and L_n are smashing localizations, they commute with homotopy colimits, so if the height n telescope conjecture holds for all finite (p-local) spectra F then it holds for all (p-local) spectra X. In particular, if a counterexample exists, then there also exists a finite (p-local) counterexample.

If the height n telescope conjecture holds for a finite spectrum F, then it also holds for all spectra in the thick subcategory generated by F. It is trivially true for finite F of type $\geq n + 1$, The main case to consider is thus that when F has type $= n$.

In the case $T(2) = v_2^{-1} S/(p, v_1)$ for $p \geq 5$, Ravenel [Rav92b], [Rav93], [Rav95] made calculations with a localized Adams spectral sequence (similar to Miller's proof strategy for $n = 1$), that strongly suggest that $\pi_* T(2) = v_2^{-1}\pi_*(S/(p, v_1))$ is different from $\pi_* L_2 S/(p, v_1)$. The latter is a subquotient of an exterior algebra over $K(2)_*$ on $n^2 = 4$ generators, while the former appears to be a subquotient of an exterior algebra on only $\binom{n+1}{2} = 3$ generators, tensored with $\binom{n}{2} = 1$ factor(s) of the form $K(2)_*[\mathbb{Q}/\mathbb{Z}_{(2)}] = K(2)_*[\mathbb{Z}/2^\infty]$. The expectation is therefore that the telescope conjecture is false for $n = 2$ and $p \geq 5$, and most likely for all $n \geq 2$ and all p.

Calculations for $n = 2$ and $p = 2$, with a similar conclusion, were made by Mahowald–Ravenel–Shick [MRS01], but these efforts did also not reach a definite conclusion.

More recently, Beaudry–Behrens–Bhattacharya–Culver–Xu [BBB+21] made calculations with the tmf-based Adams spectral sequence at $n = 2$ and $p = 2$ (similar to Mahowald's proof strategy for $n = 1$). For a specific type 2 spectrum Z with $H^*(Z; \mathbb{F}_2) \cong A(2)//\Lambda(Q_2)$ they obtain specific conjectures about the v_2-localized Adams spectral sequence with abutment $v_2^{-1}\pi_*(Z)$, which would contradict the telescope conjecture.

In contrast to these partial calculations for finite spectra, complete computations of v_n-periodic homotopy have been for some infinite spectra, including algebraic K-theory and topological cyclic homology spectra. Bökstedt–Madsen [BM94], [BM95] calculated

$$T(1)_*K(\mathbb{Z}_p) = v_1^{-1}V(0)_*K(\mathbb{Z}_p)$$

at primes $p \geq 3$ to be a (finitely generated and free) $K(1)_*$-module of rank $p+3$. The result agrees with $L_1V(0) \wedge K(\mathbb{Z}_p) \simeq V(0) \wedge K^{\mathrm{et}}(\mathbb{Q}_p)$, confirming the Lichtenbaum–Quillen conjecture for $\mathbb{Q}_p$ at these primes. Ausoni–Rognes [AR02] calculated

$$T(2)_*K(BP\langle 1\rangle) = v_2^{-1}V(1)_*K(BP\langle 1\rangle)$$

at primes $p \geq 5$ to be a (finitely generated and free) $K(2)_*$-module of rank $4p + 4$, and Angelini-Knoll–Ausoni–Culver–Höning–Rognes (arXiv:2204.05890) calculated

$$T(3)_*K(BP\langle 2\rangle) = v_3^{-1}V(2)_*K(BP\langle 2\rangle)$$

at primes $p \geq 7$ to be a (finitely generated and free) $K(3)_*$-module of rank $12p + 4$. In the latter two cases the chromatic localizations $L_2V(1) \wedge K(BP\langle 1\rangle)$ and $L_3V(2) \wedge K(BP\langle 2\rangle)$ are not currently known, so at the time of writing (May 2023) the telescope conjecture remains open.

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