

**Complex Cobordism and  
Stable Homotopy Groups of Spheres**

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To my wife, Michelle



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## Preface to the first edition

My initial inclination was to call this book *The Music of the Spheres*, but I was dissuaded from doing so by my diligent publisher, who is ever mindful of the sensibilities of librarians. The purpose of this book is threefold: (i) to make  $BP$ -theory and the Adams–Novikov spectral sequence more accessible to nonexperts, (ii) to provide a convenient reference for workers in the field, and (iii) to demonstrate the computational potential of the indicated machinery for determining stable homotopy groups of spheres. The reader is presumed to have a working knowledge of algebraic topology and to be familiar with the basic concepts of homotopy theory. With this assumption the book is almost entirely self-contained, the major exceptions (e.g., Sections 5.4, 5.4, A1.4, and A1.5) being cases in which the proofs are long, technical, and adequately presented elsewhere.

The subject matter is a difficult one and this book will not change that fact. We hope that it will make it possible to learn the subject other than by the only practical method heretofore available, i.e., by numerous exhausting conversations with one of a handful of experts. Much of the material here has been previously published in journal articles too numerous to keep track of. However, a lot of the foundations of the subject, e.g., Chapter 2 and Appendix 1, have not been previously worked out in sufficient generality and the author found it surprisingly difficult to do so.

The reader (especially if she is a graduate student) should be warned that many portions of this volume contain more than he is likely to want or need to know. In view of (ii), results are given (e.g., in Sections 4.3, 6.3, and A1.4) in greater strength than needed at present. We hope the newcomer to the field will not be discouraged by abundance of material.

The homotopy groups of spheres is a highly computational topic. The serious reader is strongly encouraged to reproduce and extend as many of the computations presented here as possible. There is no substitute for the insight gained by carrying out such calculations oneself.

Despite the large amount of information and techniques currently available, stable homotopy is still very mysterious. Each new computational breakthrough heightens our appreciation of the difficulty of the problem. The subject has a highly experimental character. One computes as many homotopy groups as possible with existing machinery, and the resulting data form the basis for new conjectures and new theorems, which may lead to better methods of computation. In contrast with physics, in this case the experimentalists who gather data and the theoreticians who interpret them are the same individuals.

The core of this volume is Chapters 2–6 while Chapter 1 is a casual nontechnical introduction to this material. Chapter 7 is a more technical description of actual computations of the Adams–Novikov spectral sequence for the stable homotopy

groups of spheres through a large range of dimensions. Although it is likely to be read closely by only a few specialists, it is in some sense the justification for the rest of the book, the computational payoff. The results obtained there, along with some similar calculations of Tangora, are tabulated in Appendix 3.

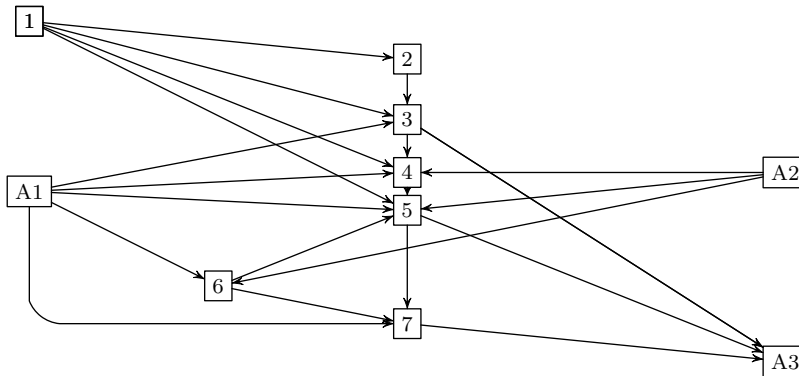
Appendices 1 and 2 are utilitarian in nature and describe technical tools used throughout the book. Appendix 1 develops the theory of Hopf algebroids (of which Hopf algebras are a special case) and useful homological tools such as relative injective resolutions, spectral sequences, Massey products, and algebraic Steenrod operations. It is not entertaining reading; we urge the reader to refer to it only when necessary.

Appendix 2 is a more enjoyable self-contained account of all that is needed from the theory of formal group laws. This material supports a bridge between stable homotopy theory and algebraic number theory. Certain results (e.g., the cohomology of some groups arising in number theory) are carried across this bridge in Chapter 6. The house they inhabit in homotopy theory, the chromatic spectral sequence, is built in Chapter 5.

The logical interdependence of the seven chapters and three appendixes is displayed in the accompanying diagram.

It is a pleasure to acknowledge help received from many sources in preparing this book. The author received invaluable editorial advice from Frank Adams, Peter May, David Pengelley, and Haynes Miller. Steven Mitchell, Austin Pearlman, and Bruce McQuistan made helpful comments on various stages of the manuscript, which owes its very existence to the patient work of innumerable typists at the University of Washington.

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## Preface to the second edition

The subject of  $BP$ -theory has grown dramatically since the appearance of the first edition 17 years ago. One major development was the proof by Devinatz, Hopkins and Smith (see Devinatz, Hopkins and Smith [2] and Hopkins and Smith [3]) of nearly all the conjectures made in Ravenel [8]. An account of this work can be found in our book Ravenel [13]. The only conjecture of Ravenel [8] that remains is Telescope Conjecture. An account of our unsuccessful attempt to disprove it is given in Mahowald, Ravenel, and Shick [1].

Another big development is the emergence of elliptic cohomology and the theory of topological modular forms. There is still no comprehensive introduction to this topic. Some good papers to start with are Ando, Hopkins and Strickland [1], Hopkins and Mahowald [2], Landweber, Ravenel and Stong [8], and Rezk [1], which is an account of the still unpublished Hopkins-Miller theorem.

The seventh and final chapter of the book has been completely rewritten and is nearly twice as long as the original. We did this with an eye to carrying out future research in this area.

I am grateful to the many would be readers who urged me to republish this book and to the AMS for its assistance in getting the original manuscript retypeset. Peter Landweber was kind enough to provide me with a copious list of misprints he found in the first edition. Nori Minami and Igor Kriz helped in correcting some errors in § 4.3. Mike Hill and his fellow MIT students provided me with a timely list of typos in the online version of this edition. Hirofumi Nakai was very helpful in motivating me to make the revisions of Chapter 7.



## Commonly Used Notations

$\mathbf{Z}$	Integers
$\mathbf{Z}_p$	$p$ -adic integers
$\mathbf{Z}_{(p)}$	Integers localized at $p$
$\mathbf{Z}/(p)$	Integers mod $p$
$\mathbf{Q}$	Rationals
$\mathbf{Q}_p$	$p$ -adic numbers
$P(x)$	Polynomial algebra on generators $x$
$E(x)$	Exterior algebra on generators $x$
$\square$	Cotensor product (Section A1.1)

Given suitable objects  $A$ ,  $B$ , and  $C$  and a map  $f: A \rightarrow B$ , the evident map  $A \otimes C \rightarrow B \otimes C$  is denoted by  $f \otimes C$ .



## CHAPTER 1

# An Introduction to the Homotopy Groups of Spheres

This chapter is intended to be an expository introduction to the rest of the book. We will informally describe the spectral sequences of Adams and Novikov, which are the subject of the remaining chapters. Our aim here is to give a conceptual picture, suppressing as many technical details as possible.

In Section 1 we list some theorems which are classical in the sense that they do *not* require any of the machinery described in this book. These include the Hurewicz theorem 1.1.2, the Freudenthal suspension theorem 1.1.4, the Serre finiteness theorem 1.1.8, the Nishida nilpotence theorem 1.1.9, and the Cohen–Moore–Neisendorfer exponent theorem 1.1.10. They all pertain directly to the homotopy groups of spheres and are not treated elsewhere here. The homotopy groups of the stable orthogonal group  $SO$  are given by the Bott periodicity theorem 1.1.11. In 1.1.12 we define the  $J$ -homomorphism from  $\pi_i(SO(n))$  to  $\pi_{n+i}(S^n)$ . Its image is given in 1.1.13, and in 1.1.14 we give its cokernel in low dimensions. Most of the former is proved in Section 5.3.

In Section 2 we describe Serre’s method of computing homotopy groups using cohomological techniques. In particular, we show how to find the first element of order  $p$  in  $\pi_*(S^3)$  1.2.4. Then we explain how these methods were streamlined by Adams to give his celebrated spectral sequence 1.2.10. The next four theorems describe the Hopf invariant one problem. A table showing the Adams spectral sequence at the prime 2 through dimension 45 is given in 1.2.15. In Chapter 2 we give a more detailed account of how the spectral sequence is set up, including a convergence theorem. In Chapter 3 we make many calculations with it at the prime 2.

In 1.2.16 we summarize Adams’ method for purposes of comparing it with that of Novikov. The basic idea is to use complex cobordism (1.2.17) in place of ordinary mod  $(p)$  cohomology. FIG. 1.2.19 is a table of the Adams–Novikov spectral sequence for comparison with FIG. 1.2.15.

In the next two sections we describe the algebra surrounding the  $E_2$ -term of the Adams–Novikov spectral sequence. To this end formal group laws are defined in 1.3.1 and a complete account of the relevant theory is given in Appendix 2. Their connection with complex cobordism is the subject of Quillen’s theorem (1.3.4) and is described more fully in Section 4.1. The Adams–Novikov  $E_2$ -term is described in terms of formal group law theory (1.3.5) and as an Ext group over a certain Hopf algebra (1.3.6).

The rest of Section 3 is concerned with the Greek letter construction, a method of producing infinite periodic families of elements in the  $E_2$ -term and (in favorable cases) in the stable homotopy groups of spheres. The basic definitions are given in

1.3.17 and 1.3.19 and the main algebraic fact required is the Morava–Landweber theorem (1.3.16). Applications to homotopy are given in 1.3.11, 1.3.15, and 1.3.18. The section ends with a discussion of the proofs and possible extensions of these results. This material is discussed more fully in Chapter 5.

In Section 4 we describe the deeper algebraic properties of the  $E_2$ -term. We start by introducing  $BP$  and defining a Hopf algebroid. The former is a minimal wedge summand of  $MU$  localized at a prime. A Hopf algebroid is a generalized Hopf algebra needed to describe the Adams–Novikov  $E_2$ -term more conveniently in terms of  $BP$  (1.4.2). The algebraic and homological properties of such objects are the subject of Appendix 1.

Next we give the Lazard classification theorem for formal group laws (1.4.3) over an algebraically closed field of characteristic  $p$ , which is proved in Section A2.2. Then we come to Morava’s point of view. Theorem 1.3.5 describes the Adams–Novikov  $E_2$ -term as the cohomology of a certain group  $G$  with coefficients in a certain polynomial ring  $L$ .  $\text{Spec}(L)$  (in the sense of abstract algebraic geometry) is an infinite dimensional affine space on which  $G$  acts. The points in  $\text{Spec}(L)$  can be thought of as formal group laws and the  $G$ -orbits as isomorphism classes, as described in 1.4.3. This orbit structure is described in 1.4.4. For each orbit there is a stabilizer or isotropy subgroup of  $G$  called  $S_n$ . Its cohomology is related to that of  $G$  (1.4.5), and its structure is known. The theory of Morava stabilizer algebras is the algebraic machinery needed to exploit this fact and is the subject of Chapter 6. Our next topic, the chromatic spectral sequence (1.4.8, the subject of Chapter 5), connects the theory above to the Adams–Novikov  $E_2$ -term. The Greek letter construction fits into this apparatus very neatly.

Section 5 is about unstable homotopy groups of spheres and is not needed for the rest of the book. Its introduction is self-explanatory.

## 1. Classical Theorems Old and New

We begin by recalling some definitions. The  $n$ th homotopy group of a connected space  $X$ ,  $\pi_n(X)$ , is the set of homotopy classes of maps from the  $n$ -sphere  $S^n$  to  $X$ . This set has a natural group structure which is abelian for  $n \geq 2$ .

We now state three classical theorems about homotopy groups of spheres. Proofs can be found, for example, in Spanier [1].

1.1.1. THEOREM.  $\pi_1(S^1) = \mathbf{Z}$  and  $\pi_m(S^1) = 0$  for  $m > 1$ . □

1.1.2. HUREWICZ’S THEOREM.  $\pi_n(S^n) = \mathbf{Z}$  and  $\pi_m(S^n) = 0$  for  $m < n$ . A generator of  $\pi_n(S^n)$  is the class of the identity map. □

For the next theorem we need to define the suspension homomorphism  $\sigma: \pi_m(S^n) \rightarrow \pi_{m+1}(S^{n+1})$ .

1.1.3. DEFINITION. The  $k$ th suspension  $\Sigma^k X$  of a space  $X$  is the quotient of  $I^k \times X$  obtained by collapsing  $\partial I^k \times X$  onto  $\partial I^k$ ,  $\partial I^k$  being the boundary of  $I^k$ , the  $k$ -dimensional cube. Note that  $\Sigma^i \Sigma^j X = \Sigma^{i+j} X$  and  $\Sigma^k f: \Sigma^k X \rightarrow \Sigma^k Y$  is the quotient of  $1 \times f: I^k \times X \rightarrow I^k \times Y$ . In particular, given  $f: S^m \rightarrow S^n$  we have  $\Sigma f: S^{m+1} \rightarrow S^{n+1}$ , which induces a homomorphism  $\pi_m(S^n) \rightarrow \pi_{m+1}(S^{n+1})$ . □

1.1.4. FREUDENTHAL SUSPENSION THEOREM. The suspension homomorphism  $\sigma: \pi_{n+k}(S^n) \rightarrow \pi_{n+k+1}(S^{n+1})$  defined above is an isomorphism for  $k < n - 1$  and a surjection for  $k = n - 1$ . □

1.1.5. COROLLARY. *The group  $\pi_{n+k}(S^n)$  depends only on  $k$  if  $n > k + 1$ .  $\square$*

1.1.6. DEFINITION. *The stable  $k$ -stem or  $k$ th stable homotopy group of spheres  $\pi_k^S$  is  $\pi_{n+k}(S^n)$  for  $n > k + 1$ . The groups  $\pi_{n+k}(S^n)$  are called stable if  $n > k + 1$  and unstable if  $n \leq k + 1$ . When discussing stable groups we will not make any notational distinction between a map and its suspensions.  $\square$*

The subsequent chapters of this book will be concerned with machinery for computing the stable homotopy groups of spheres. Most of the time we will not be concerned with unstable groups. The groups  $\pi_k^S$  are known at least for  $k \leq 45$ . See the tables in Appendix 3, along with Theorem 1.1.13. Here is a table of  $\pi_k^S$  for  $k \leq 15$ :

$k$	0	1	2	3	4	5	6	7	8
$\pi_k^S$	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(24)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(240)$	$(\mathbf{Z}/(2))^2$

$k$	9	10	11	12	13	14	15
$\pi_k^S$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/(504)$	0	$\mathbf{Z}/(3)$	$(\mathbf{Z}/(2))^2$	$\mathbf{Z}/(480) \oplus \mathbf{Z}/(2)$

This should convince the reader that the groups do not fall into any obvious pattern. Later in the book, however, we will present evidence of some deep patterns not apparent in such a small amount of data. The nature of these patterns will be discussed later in this chapter.

When homotopy groups were first defined by Hurewicz in 1935 it was hoped that  $\pi_{n+k}(S^n) = 0$  for  $k > 0$ , since this was already known to be the case for  $n = 1$  (1.1.1). The first counterexample is worth examining in some detail.

1.1.7. EXAMPLE.  $\pi_3(S^2) = \mathbf{Z}$  generated by the class of the Hopf map  $\eta: S^3 \rightarrow S^2$  defined as follows. Regard  $S^2$  (as Riemann did) as the complex numbers  $\mathbf{C}$  with a point at infinity.  $S^3$  is by definition the set of unit vectors in  $\mathbf{R}^4 = \mathbf{C}^2$ . Hence a point in  $S^3$  is specified by two complex coordinates  $(z_1, z_2)$ . Define  $\eta$  by

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0. \end{cases}$$

It is easy to verify that  $\eta$  is continuous. The inverse image under  $\eta$  of any point in  $S^2$  is a circle, specifically the set of unit vectors in a complex line through the origin in  $\mathbf{C}^2$ , the set of all such lines being parameterized by  $S^2$ . Closer examination will show that any two of these circles in  $S^3$  are linked. One can use quaternions and Cayley numbers in similar ways to obtain maps  $\nu: S^7 \rightarrow S^4$  and  $\sigma: S^{15} \rightarrow S^8$ , respectively. Both of these represent generators of infinite cyclic summands. These three maps ( $\eta$ ,  $\nu$ , and  $\sigma$ ) were all discovered by Hopf [1] and are therefore known as the Hopf maps.

We will now state some other general theorems of more recent vintage.

1.1.8. FINITENESS THEOREM (Serre [3]).  *$\pi_{n+k}(S^n)$  is finite for  $k > 0$  except when  $n = 2m$ ,  $k = 2m - 1$ , and  $\pi_{4m-1}(S^{2m}) = \mathbf{Z} \oplus F_m$ , where  $F_m$  is finite.  $\square$*

The next theorem concerns the ring structure of  $\pi_*^S = \bigoplus_{k \geq 0} \pi_k^S$  which is induced by composition as follows. Let  $\alpha \in \pi_i^S$  and  $\beta \in \pi_j^S$  be represented by  $f: S^{n+i} \rightarrow S^n$  and  $g: S^{n+i+j} \rightarrow S^{n+i}$ , respectively, where  $n$  is large. Then

$\alpha\beta \in \pi_{i+j}^S$  is defined to be the class represented by  $f \cdot g: S^{n+i+j} \rightarrow S^n$ . It can be shown that  $\beta\alpha = (-1)^{ij}\alpha\beta$ , so  $\pi_*^S$  is an anticommutative graded ring.

1.1.9. NILPOTENCE THEOREM (Nishida [1]). *Each element  $\alpha \in \pi_k^S$  for  $k > 0$  is nilpotent, i.e.,  $\alpha^t = 0$  for some finite  $t$ .*  $\square$

For the next result recall that 1.1.8 says  $\pi_{2i+1+j}(S^{2i+1})$  is a finite abelian group for all  $j > 0$ .

1.1.10. EXPONENT THEOREM (Cohen, Moore, and Neisendorfer [1]). *For  $p \geq 5$  the  $p$ -component of  $\pi_{2i+1+j}(S^{2i+1})$  has exponent  $p^i$ , i.e., each element in it has order  $\leq p^i$ .*  $\square$

This result is also true for  $p = 3$  (Neisendorfer [1]) as well, but is known to be false for  $p = 2$ . For example, the 2-component of 3-stem is cyclic of order 4 (see Fig. 3.3.18) on  $S^3$  and of order 8 on  $S^8$  (see Fig. 3.3.10). It is also known (Gray [1]) to be the best possible, i.e.,  $\pi_{2i+1+j}(S^{2i+1})$  is known to contain elements of order  $p^i$  for certain  $j$ .

We now describe an interesting subgroup of  $\pi_*^S$ , the image of the Hopf–Whitehead  $J$ -homomorphism, to be defined below. Let  $SO(n)$  be the space of  $n \times n$  special orthogonal matrices over  $\mathbf{R}$  with the standard topology.  $SO(n)$  is a subspace of  $SO(n+1)$  and we denote  $\bigcup_{n>0} SO(n)$  by  $SO$ , known as the stable orthogonal group. It can be shown that  $\pi_i(SO) = \pi_i(SO(n))$  if  $n > i+1$ . The following result of Bott is one of the most remarkable in all of topology.

1.1.11. BOTT PERIODICITY THEOREM (Bott [1]; see also Milnor [1]).

$$\pi_i(SO) = \begin{cases} \mathbf{Z} & \text{if } i \equiv -1 \pmod{4} \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \text{ or } 1 \pmod{8} \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

We will now define a homomorphism  $J: \pi_i(SO(n)) \rightarrow \pi_{n+i}(S^n)$ . Let  $\alpha \in \pi_i(SO(n))$  be the class of  $f: S^i \rightarrow SO(n)$ . Let  $D^n$  be the  $n$ -dimensional disc, i.e., the unit ball in  $\mathbf{R}^n$ . A matrix in  $SO(n)$  defines a linear homeomorphism of  $D^n$  to itself. We define  $\hat{f}: S^i \times D^n \rightarrow D^n$  by  $\hat{f}(x, y) = f(x)(y)$ , where  $x \in S^i$ ,  $y \in D^n$ , and  $f(x) \in SO(n)$ . Next observe that  $S^n$  is the quotient of  $D^n$  obtained by collapsing its boundary  $S^{n-1}$  to a single point, so there is a map  $p: D^n \rightarrow S^n$ , which sends the boundary to the base point. Also observe that  $S^{n+i}$ , being homeomorphic to the boundary of  $D^{i+1} \times D^n$ , is the union of  $S^i \times D^n$  and  $D^{i+1} \times S^{n-1}$  along their common boundary  $S^i \times S^{n-1}$ . We define  $\tilde{f}: S^{n+i} \rightarrow S^n$  to be the extension of  $p\hat{f}: S^i \times D^n \rightarrow S^n$  to  $S^{n+i}$  which sends the rest of  $S^{n+i}$  to the base point in  $S^n$ .

1.1.12. DEFINITION. *The Hopf–Whitehead  $J$ -homomorphism  $J: \pi_i(SO(n)) \rightarrow \pi_{n+i}(S^n)$  sends the class of  $f: S^i \rightarrow SO(n)$  to the class of  $\tilde{f}: S^{n+i} \rightarrow S^n$  as described above.*  $\square$

We leave it to the skeptical reader to verify that the above construction actually gives us a homomorphism.

Note that both  $\pi_i(SO(n))$  and  $\pi_{n+i}(S^n)$  are stable, i.e., independent of  $n$ , if  $n > i+1$ . Hence we have  $J: \pi_k(SO) \rightarrow \pi_k^S$ . We will now describe its image.

1.1.13. THEOREM (Adams [1] and Quillen [1]).  *$J: \pi_k(SO) \rightarrow \pi_k^S$  is a monomorphism for  $k \equiv 0 \text{ or } 1 \pmod{8}$  and  $J(\pi_{4k-1}(SO))$  is a cyclic group whose 2-component*



is  $\mathbf{Z}_{(2)}/(8k)$  and whose  $p$ -component for  $p \geq 3$  is  $\mathbf{Z}_{(p)}/(pk)$  if  $(p-1) \mid 2k$  and 0 if  $(p-1) \nmid 2k$ , where  $\mathbf{Z}_{(p)}$  denotes the integers localized at  $p$ . In dimensions 1, 3, and 7,  $\text{im } J$  is generated by the Hopf maps (1.1.7)  $\eta$ ,  $\nu$ , and  $\sigma$ , respectively. If we denote by  $x_k$  the generator in dimension  $4k-1$ , then  $\eta x_{2k}$  and  $\eta^2 x_{2k}$  are the generators of  $\text{im } J$  in dimensions  $8k$  and  $8k+1$ , respectively.  $\square$

The image of  $J$  is also known to a direct summand; a proof can be found for example at the end of Chapter 19 of Switzer [1]. The order of  $J(\pi_{4k-1}(SO))$  was determined by Adams up to a factor of two, and he showed that the remaining ambiguity could be resolved by proving the celebrated Adams conjecture, which Quillen and others did. Denote this number by  $a_k$ . Its first few values are tabulated here.

$k$	1	2	3	4	5	6	7	8	9	10
$a_k$	24	240	504	480	264	65,520	24	16,320	28,728	13,200

The number  $a_k$  has interesting number theoretic properties. It is the denominator of  $B_k/4k$ , where  $B_k$  is the  $k$ th Bernoulli number, and it is the greatest common divisor of numbers  $n^{t(n)}(n^{2k}-1)$  for  $n \in \mathbf{Z}$  and  $t(n)$  sufficiently large. See Adams [1] and Milnor and Stasheff [5] for details.

Having determined  $\text{im } J$ , one would like to know something systematic about coker  $J$ , i.e., something more than its structure through a finite range of dimensions. For the reader's amusement we record some of that structure now.

1.1.14. THEOREM. *In dimensions  $\leq 15$ , the 2-component of coker  $J$  has the following generators, each with order 2:*

$$\begin{aligned} \eta^2 \in \pi_2^S, \quad \nu^2 \in \pi_6^S, \quad \bar{\nu} \in \pi_8^S, \quad \eta\bar{\nu} = \nu^3 \in \pi_9^S, \quad \mu \in \pi_9^S, \\ \eta\mu \in \pi_{10}^S, \quad \sigma^2 \in \pi_{14}^S, \quad \kappa \in \pi_{14}^S \quad \text{and} \quad \eta\kappa \in \pi_{15}^S. \end{aligned}$$

(There are relations  $\eta^3 = 4\nu$  and  $\eta^2\mu = 4x_3$ ). For  $p \geq 3$  the  $p$ -component of coker  $J$  has the following generators in dimensions  $\leq 3pq-6$  (where  $q = 2p-2$ ), each with order  $p$ :

$$\beta_1 \in \pi_{pq-2}^S, \quad \alpha_1\beta_1 \in \pi_{(p+1)q-3}^S$$

where  $\alpha_1 = x_{(p-1)/2} \in \pi_{q-1}^S$  is the first generator of the  $p$ -component of  $\text{im } J$ ,

$$\begin{aligned} \beta_1^2 \in \pi_{2pq-4}^S, \quad \alpha_1\beta_1^2 \in \pi_{(2p+1)q-5}^S, \quad \beta_2 \in \pi_{(2p+1)q-2}^S, \\ \alpha_1\beta_2 \in \pi_{(2p+2)q-3}^S, \quad \text{and} \quad \beta_1^3 \in \pi_{3pq-6}^S. \end{aligned} \quad \square$$

The proof and the definitions of new elements listed above will be given later in the book, e.g., in Section 4.4.

## 2. Methods of Computing $\pi_*(S^n)$

In this section we will informally discuss three methods of computing homotopy groups of spheres, the spectral sequences of Serre, Adams, and Novikov. A fourth method, the EHP sequence, will be discussed in Section 5. We will not give any proofs and in some cases we will sacrifice precision for conceptual clarity, e.g., in our identification of the  $E_2$ -term of the Adams–Novikov spectral sequence.

The Serre spectral sequence (circa 1951) (Serre [2]) is included here mainly for historical interest. It was the first systematic method of computing homotopy groups and was a major computational breakthrough. It has been used as late as the 1970s by various authors (Toda [1], Oka [1, 2, 3]), but computations made

with it were greatly clarified by the introduction of the Adams spectral sequence in 1958 in Adams [3]. In the Adams spectral sequence the basic mechanism of the Serre spectral sequence information is organized by homological algebra.

For the 2-component of  $\pi_*(S^n)$  the Adams spectral sequence is indispensable to this day, but the odd primary calculations were streamlined by the introduction of the Adams–Novikov spectral sequence (Adams–Novikov spectral sequence) in 1967 by Novikov [1]. It is the main subject in this book. Its  $E_2$ -term contains more information than that of the Adams spectral sequence; i.e., it is a more accurate approximation of stable homotopy and there are fewer differentials in the spectral sequence. Moreover, it has a very rich algebraic structure, as we shall see, largely due to the theorem of Quillen [2], which establishes a deep (and still not satisfactorily explained) connection between complex cobordism (the cohomology theory used to define the Adams–Novikov spectral sequence; see below) and the theory of formal group laws. Every major advance in the subject since 1969, especially the work of Jack Morava, has exploited this connection.

We will now describe these three methods in more detail. The starting point for Serre’s method is the following classical result.

1.2.1. THEOREM. *Let  $X$  be a simply connected space with  $H_i(X) = 0$  for  $i < n$  for some positive integer  $n \geq 2$ . Then*

(a) (Hurewicz [1]).  $\pi_n(X) = H_n(X)$ .

(b) (Eilenberg and Mac Lane [2]). *There is a space  $K(\pi, n)$ , characterized up to homotopy equivalence by*

$$\pi_i(K(\pi, n)) = \begin{cases} \pi & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

*If  $X$  is above and  $\pi = \pi_n(X)$  then there is a map  $f: X \rightarrow K(\pi, n)$  such that  $H_n(f)$  and  $\pi_n(f)$  are isomorphisms.  $\square$*

1.2.2. COROLLARY. *Let  $F$  be the fiber of the map  $f$  above. Then*

$$\pi_i(F) = \begin{cases} \pi_i(X) & \text{for } i \geq n + 1 \\ 0 & \text{for } i \leq n. \end{cases} \quad \square$$

In other words,  $F$  has the same homotopy groups as  $X$  in dimensions above  $n$ , so computing  $\pi_*(F)$  is as good as computing  $\pi_*(X)$ . Moreover,  $H_*(K(\pi, n))$  is known, so  $H_*(F)$  can be computed with the Serre spectral sequence applied to the fibration  $F \rightarrow X \rightarrow K(\pi, n)$ .

Once this has been done the entire process can be repeated: let  $n' > n$  be the dimension of the first nontrivial homology group of  $F$  and let  $H_{n'}(F) = \pi'$ . Then  $\pi_{n'}(F) = \pi_{n'}(X) = \pi'$  is the next nontrivial homotopy group of  $X$ . Theorem 1.2.1 applied to  $F$  gives a map  $f': F \rightarrow K(\pi', n')$  with fiber  $F'$ , and 1.2.2 says

$$\pi_i(F') = \begin{cases} \pi_i(X) & \text{for } i > n' \\ 0 & \text{for } i \leq n'. \end{cases}$$

Then one computes  $H_*(F')$  using the Serre spectral sequence and repeats the process.

As long as one can compute the homology of the fiber at each stage, one can compute the next homotopy group of  $X$ . In Serre [3] a theory was developed which allows one to ignore torsion of order prime to a fixed prime  $p$  throughout the

calculation if one is only interested in the  $p$ -component of  $\pi_*(X)$ . For example, if  $X = S^3$ , one uses 1.2.1 to get a map to  $K(\mathbf{Z}, 3)$ . Then  $H_*(F)$  is described by:

1.2.3. LEMMA. *If  $F$  is the fibre of the map  $f: S^3 \rightarrow K(\mathbf{Z}, 3)$  given by 1.2.1, then*

$$H_i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m \text{ and } m > 1 \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

1.2.4. COROLLARY. *The first  $p$ -torsion in  $\pi_*(S^3)$  is  $\mathbf{Z}/(p)$  in  $\pi_{2p}(S^3)$  for any prime  $p$ .*  $\square$

PROOF OF 1.2.3. (It is so easy we cannot resist giving it.) We have a fibration

$$\Omega K(\mathbf{Z}, 3) = K(\mathbf{Z}, 2) \rightarrow F \rightarrow S^3$$

and  $H^*(K(\mathbf{Z}, 2)) = H^*(\mathbf{C}P^\infty) = \mathbf{Z}[x]$ , where  $x \in H^2(\mathbf{C}P^\infty)$  and  $\mathbf{C}P^\infty$  is an infinite-dimensional complex projective space. We will look at the Serre spectral sequence for  $H^*(F)$  and use the universal coefficient theorem to translate this to the desired description of  $H_*(F)$ . Let  $u$  be the generator of  $H^3(S^3)$ . Then in the Serre spectral sequence we must have  $d_3(x) = \pm u$ ; otherwise  $F$  would not be 3-connected, contradicting 1.1.2. Since  $d_3$  is a derivation we have  $d_3(x^n) = \pm nu x^{n-1}$ . It is easily seen that there can be no more differentials and we get

$$H^i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m + 1, m > 1 \\ 0 & \text{otherwise} \end{cases}$$

which leads to the desired result.  $\square$

If we start with  $X = S^n$  the Serre spectral sequence calculations will be much easier for  $\pi_{k+n}(S^n)$  for  $k < n - 1$ . Then all of the computations are in the stable range, i.e., in dimensions less than twice the connectivity of the spaces involved.

This means that for a fibration  $F \xrightarrow{i} X \xrightarrow{f} K$ , the Serre spectral sequence gives a long exact sequence

$$(1.2.5) \quad \cdots \rightarrow H_j(F) \xrightarrow{i_*} H_j(X) \xrightarrow{f_*} H_j(K) \xrightarrow{d} H_{j-1}(F) \rightarrow \cdots,$$

where  $d$  corresponds to Serre spectral sequence differentials. Even if we know  $H_*(X)$ ,  $H_*(K)$ , and  $f_*$ , we still have to deal with the short exact sequence

$$(1.2.6) \quad 0 \rightarrow \operatorname{coker} f_* \rightarrow H_*(F) \rightarrow \ker f_* \rightarrow 0.$$

It may lead to some ambiguity in  $H_*(F)$ , which must be resolved by some other means. For example, when computing  $\pi_*(S^n)$  for large  $n$  one encounters this problem in the 3-component of  $\pi_{n+10}(S^n)$  and the 2-component of  $\pi_{n+14}(S^n)$ . This difficulty is also present in the Adams spectral sequence, where one has the possibility of a nontrivial differential in these dimensions. These differentials were first calculated by Adams [12], Liulevicius [2], and Shimada and Yamanoshita [3] by methods involving secondary cohomology operations and later by Adams and Atiyah [13] by methods involving  $K$ -theory

The Adams spectral sequence of Adams [3] begins with a variation of Serre's method. One works only in the stable range and only on the  $p$ -component. Instead of mapping  $X$  to  $K(\pi, n)$  as in 1.2.1, one maps to  $K = \prod_{j>0} K(H^j(X; \mathbf{Z}/(p)), j)$  by a certain map  $g$  which induces a surjection in mod  $(p)$  cohomology. Let  $X_1$  be the fiber of  $g$ . Define spaces  $X_i$  and  $K_i$  inductively by  $K_i = \prod_{j>0} K(H^j(X_i; \mathbf{Z}/(p)), j)$

and  $X_{i+1}$  is the fiber of  $g: X_i \rightarrow K_i$  (this map is defined in Section 2.1, where the Adams spectral sequence is discussed in more detail). Since  $H^*(g_i)$  is onto, the analog of 1.2.5 is an short exact sequence in the stable range

$$(1.2.7) \quad 0 \leftarrow H^*(X_i) \leftarrow H^*(K_i) \leftarrow H^*(\Sigma X_{i+1}) \leftarrow 0,$$

where all cohomology groups are understood to have coefficients  $\mathbf{Z}/(p)$ . Moreover,  $H^*(K_i)$  is a free module over the mod  $(p)$  Steenrod algebra  $A$ , so if we splice together the short exact sequences of 1.2.7 we get a free  $A$ -resolution of  $H^*(X)$

$$(1.2.8) \quad 0 \leftarrow H^*(X) \leftarrow H^*(K) \leftarrow H^*(\Sigma^1 K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \dots$$

Each of the fibration  $X_{i+1} \rightarrow X_i \rightarrow K_i$  gives a long exact sequence of homotopy groups. Together these long exact sequences form an exact couple and the associated spectral sequence is the Adams spectral sequence for the  $p$ -component of  $\pi_*(X)$ . If  $X$  has finite type, the diagram

$$(1.2.9) \quad K \rightarrow \Sigma^{-1} K_1 \rightarrow \Sigma^{-2} K_2 \rightarrow \dots$$

(which gives 1.2.8 in cohomology) gives a cochain complex of homotopy groups whose cohomology is  $\text{Ext}_A(H^*(X); \mathbf{Z}/(p))$ . Hence one gets

1.2.10. THEOREM (Adams [3]). *There is a spectral sequence converging to the  $p$ -component of  $\pi_{n+k}(S^n)$  for  $k < n - 1$  with*

$$E_2^{s,t} = \text{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) =: H^{s,t}(A)$$

and  $d_r: E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}$ . Here the groups  $E_\infty^{s,t}$  for  $t - s = k$  form the associated graded group to a filtration of the  $p$ -component of  $\pi_{n+k}(S^n)$ .  $\square$

Computing this  $E_2$ -term is hard work, but it is much easier than making similar computations with Serre spectral sequence. The most widely used method today is the spectral sequence of May [1, 2] (see Section 3.2). This is a trigraded spectral sequence converging to  $H^{**}(A)$ , whose  $E_2$ -term is the cohomology of a filtered form of the Steenrod algebra. This method was used by Tangora [1] to compute  $E_2^{s,t}$  for  $p = 2$  and  $t - s \leq 70$ . Most of his table is reproduced here in FIG. A3.1a–C. Computations for odd primes can be found in Nakamura [2].

As noted above, the Adams  $E_2$ -term is the cohomology of the Steenrod algebra. Hence  $E_2^{1,*} = H^1(A)$  is the indecomposables in  $A$ . For  $p = 2$  one knows that  $A$  is generated by  $Sq^{2^i}$  for  $i \geq 0$ ; the corresponding elements in  $E_2^{1,*}$  are denoted by  $h_i \in E_2^{1,2^i}$ . For  $p > 2$  the generators are the Bockstein  $\beta$  and  $\mathcal{P}^{p^i}$  for  $i \geq 0$  and the corresponding elements are  $a_0 \in E_2^{1,1}$  and  $h_i \in E_2^{1,qp^i}$ , where  $q = 2p - 2$ .

For  $p = 2$  these elements figure in the famous Hopf invariant one problem.

1.2.11. THEOREM (Adams [12]). *The following statements are equivalent.*

(a)  $S^{2^i-1}$  is parallelizable, i.e., it has  $2^i - 1$  globally linearly independent tangent vector fields.

(b) There is a division algebra (not necessarily associative) over  $\mathbf{R}$  of dimension  $2^i$ .

(c) There is a map  $S^{2 \cdot 2^i - 1} \rightarrow S^{2^i}$  of Hopf invariant one (see 1.5.2).

(d) There is a 2-cell complex  $X = S^{2^i} \cup e^{2^{i+1}}$  [the cofiber of the map in (c)] in which the generator of  $H^{2^{i+1}}(X)$  is the square of the generator of  $H^{2^i}(X)$ .

(e) The element  $h_i \in E_2^{1,2^i}$  is a permanent cycle in the Adams spectral sequence.  $\square$

Condition (b) is clearly true for  $i = 0, 1, 2$  and  $3$ , the division algebras being the reals  $\mathbf{R}$ , the complexes  $\mathbf{C}$ , the quaternions  $\mathbf{H}$  and the Cayley numbers, which are nonassociative. The problem for  $i \geq 4$  is solved by

1.2.12. THEOREM (Adams [12]). *The conditions of 1.2.11 are false for  $i \geq 4$  and in the Adams spectral sequence one has  $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$  for  $i \geq 4$ .  $\square$*

For  $i = 4$  the above gives the first nontrivial differential in the Adams spectral sequence. Its target has dimension 14 and is related to the difficulty in Serre's method referred to above.

The analogous results for  $p > 2$  are

1.2.13. THEOREM (Liulevicius [2] and Shimada and Yamanoshita [3]). *The following are equivalent.*

- (a) *There is a map  $S^{2p^{i+1}-1} \rightarrow \widehat{S}^{2p^i}$  with Hopf invariant one (see 1.5.3 for the definition of the Hopf invariant and the space  $\widehat{S}^{2m}$ ).*
- (b) *There is a  $p$ -cell complex  $X = S^{2p^i} \cup e^{4p^i} \cup e^{6p^i} \cup \dots \cup e^{2p^{i+1}}$  [the cofiber of the map in (a)] whose mod  $(p)$  cohomology is a truncated polynomial algebra on one generator.*
- (c) *The element  $h_i \in E_2^{1,qp^i}$  is a permanent cycle in the Adams spectral sequence.  $\square$*

The element  $h_0$  is the first element in the Adams spectral sequence above dimension zero so it is a permanent cycle. The corresponding map in (a) suspends to the element of  $\pi_{2p}(S^3)$  given by 1.2.4. For  $i \geq 1$  we have

1.2.14. THEOREM (Liulevicius [2] and Shimada and Yamanoshita [3]). *The conditions of 1.2.13 are false for  $i \geq 1$  and  $d_2(h_i) = a_0 b_{i-1}$ , where  $b_{i-1}$  is a generator of  $E_2^{2,qp^i}$  (see Section 5.2).  $\square$*

For  $i = 1$  the above gives the first nontrivial differential in the Adams spectral sequence for  $p > 2$ . For  $p = 3$  its target is in dimension 10 and was referred to above in our discussion of Serre's method.

FIG. 1.2.15 shows the Adams spectral sequence for  $p = 3$  through dimension 45. We present it here mainly for comparison with a similar figure (1.2.19) for the Adams–Novikov spectral sequence.  $E_2^{s,t}$  is a  $\mathbf{Z}/(p)$  vector space in which each basis element is indicated by a small circle. Fortunately in this range there are just two bigradings [(5,28) and (8,43)] in which there is more than one basis element. The vertical coordinate is  $s$ , the cohomological degree, and the horizontal coordinate is  $t - s$ , the topological dimension. These extra elements appear in the chart to the right of where they should be, and the lines meeting them should be vertical. A  $d_r$  is indicated by a line which goes up by  $r$  and to the left by 1. The vertical lines represent multiplication by  $a_0 \in E_2^{1,1}$  and the vertical arrow in dimension zero indicates that all powers of  $a_0$  are nonzero. This multiplication corresponds to multiplication by  $p$  in the corresponding homotopy group. Thus from the figure one can read off  $\pi_0 = \mathbf{Z}$ ,  $\pi_{11} = \pi_{45} = \mathbf{Z}/(9)$ ,  $\pi_{23} = \mathbf{Z}/(9) \oplus \mathbf{Z}/(3)$ , and  $\pi_{35} = \mathbf{Z}/(27)$ . Lines that go up 1 and to the right by 3 indicate multiplication by  $h_0 \in E_2^{1,4}$ , while those that go to the right by 7 indicate the Massey product  $\langle h_0, h_0, - \rangle$  (see A1.4.1). The elements  $a_0$  and  $h_i$  for  $i = 0, 1, 2$  were defined above and the elements  $b_0 \in E_2^{2,12}$ ,  $k_0 \in E_2^{2,28}$ , and  $b_1 \in E_2^{2,36}$  are up to the sign the Massey products  $\langle h_0, h_0, h_0 \rangle$ ,  $\langle h_0, h_1, h_1 \rangle$ , and  $\langle h_1, h_1, h_1 \rangle$ , respectively. The unlabeled elements in

$E_2^{i,5i-1}$  for  $i \geq 2$  (and  $h_0 \in E_2^{1,4}$ ) are related to each other by the Massey product  $\langle h_0, a_0, - \rangle$ . This accounts for all of the generators except those in  $E_2^{3,26}$ ,  $E_2^{7,45}$  and  $E_2^{8,50}$ , which are too complicated to describe here.

We suggest that the reader take a colored pencil and mark all of the elements which survive to  $E_\infty$ , i.e., those which are not the source or target of a differential. There are in this range 31 differentials which eliminate about two-thirds of the elements shown.

Now we consider the spectral sequence of Adams and Novikov, which is the main object of interest in this book. Before describing its construction we review the main ideas behind the Adams spectral sequence. They are the following.

1.2.16. PROCEDURE. (i) Use mod  $(p)$ -cohomology as a tool to study the  $p$ -component of  $\pi_*(X)$ . (ii) Map  $X$  to an appropriate Eilenberg–Mac Lane space  $K$ , whose homotopy groups are known. (iii) Use knowledge of  $H^*(K)$ , i.e., of the Steenrod algebra, to get at the fiber of the map in (ii). (iv) Iterate the above and codify all information in a spectral sequence as in 1.2.10.  $\square$

An analogous set of ideas lies behind the Adams–Novikov spectral sequence, with mod  $p$  cohomology being replaced by complex cobordism theory. To elaborate, we first remark that “cohomology” in 1.2.16(i) can be replaced by “homology” and 1.2.10 can be reformulated accordingly; the details of this reformulation need not be discussed here. Recall that singular homology is based on the singular chain complex, which is generated by maps of simplices into the space  $X$ . Cycles in the chain complex are linear combinations of such maps that fit together in an appropriate way. Hence  $H_*(X)$  can be thought of as the group of equivalence classes of maps of certain kinds of simplicial complexes, sometimes called “geometric cycles,” into  $X$ .

Our point of departure is to replace these geometric cycles by closed complex manifolds. Here we mean “complex” in a very weak sense; the manifold  $M$  must be smooth and come equipped with a complex linear structure on its stable normal bundle, i.e., the normal bundle of some embedding of  $M$  into a Euclidean space of even codimension. The manifold  $M$  need not be analytic or have a complex structure on its tangent bundle, and it may be odd-dimensional.

The appropriate equivalence relation among maps of such manifolds into  $X$  is the following.

1.2.17. DEFINITION. *Maps  $f_i: M \rightarrow X$  ( $i = 1, 2$ ) of  $n$ -dimensional complex (in the above sense) manifolds into  $X$  are bordant if there is a map  $g: W \rightarrow X$  where  $W$  is a complex manifold with boundary  $\partial W = M_1 \cup M_2$  such that  $g|M_i = f_i$ . (To be correct we should require the restriction to  $M_2$  to respect the complex structure on  $M_2$  opposite to the given one, but we can ignore such details here.)  $\square$*

One can then define a graded group  $MU_*(X)$ , the *complex bordism* of  $X$ , analogous to  $H_*(X)$ . It satisfies all of the Eilenberg–Steenrod axioms except the dimension axiom, i.e.,  $MU_*(\text{pt})$ , is not concentrated in dimension zero. It is by definition the set of equivalence classes of closed complex manifolds under the relation of 1.2.17 with  $X = \text{pt}$ , i.e., without any condition on the maps. This set is a ring under disjoint union and Cartesian product and is called the *complex bordism ring*, as are the analogous rings for several other types of manifolds; see Stong [1].

1.2.18. THEOREM (Thom [1], Milnor [4], Novikov [2]). *The complex bordism ring,  $MU_*(\text{pt})$ , is  $\mathbf{Z}[x_1, x_2, \dots]$  where  $\dim x_i = 2i$ .  $\square$*

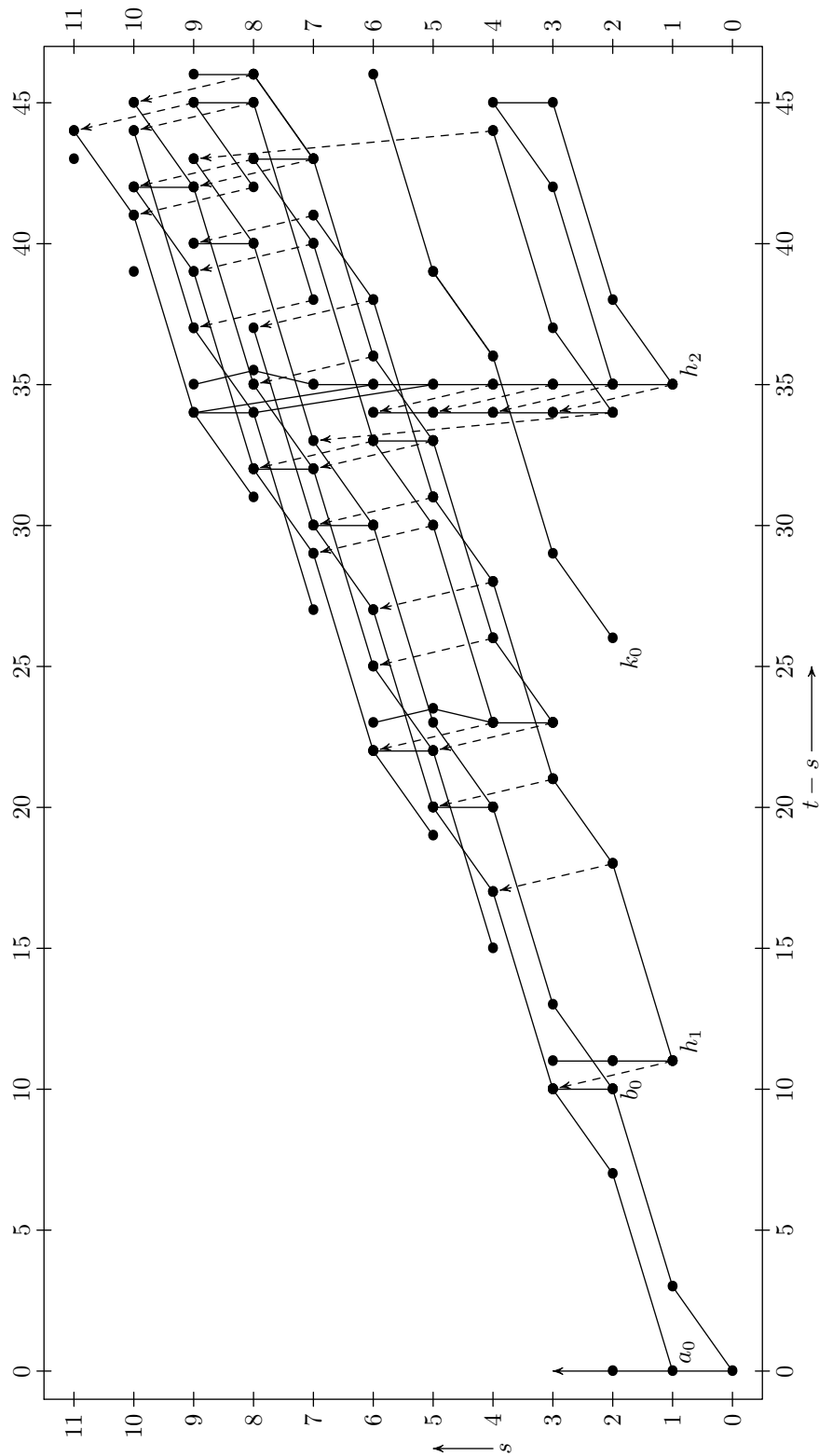


FIGURE 1.2.15. The Adams spectral sequence for  $p = 3, t - s \leq 45$ .

Now recall 1.2.16. We have described an analog of (i), i.e., a functor  $MU_*(-)$  replacing  $H_*(-)$ . Now we need to modify (ii) accordingly, e.g., to define analogs of the Eilenberg–Mac Lane spaces. These spaces (or rather the corresponding spectrum  $MU$ ) are described in Section 4.1. Here we merely remark that Thom’s contribution to 1.2.18 was to equate  $MU_i(\text{pt})$  with the homotopy groups of certain spaces and that these spaces are the ones we need.

To carry out the analog of 1.2.16(iii) we need to know the complex bordism of these spaces, which is also described (stably) in Section 4.1. The resulting spectral sequence is formally introduced in Section 4.4, using constructions given in Section 2.2. We will not state the analog of 1.2.10 here as it would be too much trouble to develop the necessary notation. However we will give a figure analogous to 1.2.15.

The notation of FIG. 1.2.19 is similar to that of FIG. 1.2.15 with some minor differences. The  $E_2$ -term here is not a  $\mathbf{Z}/(3)$ -vector space. Elements of order  $> 3$  occur in  $E_2^{0,0}$  (an infinite cyclic group indicated by a square), and in  $E_2^{1,12t}$  and  $E_2^{3,48}$ , in which a generator of order  $3^{k+1}$  is indicated by a small circle with  $k$  parentheses to the right. The names  $\alpha_t$ ,  $\beta_t$ , and  $\beta_{s/t}$  will be explained in the next section. The names  $\alpha_{3t}$  refer to elements of order 3 in, rather than generators of,  $E_2^{1,12t}$ . In  $E_2^{3,48}$  the product  $\alpha_1\beta_3$  is divisible by 3.

One sees from these two figures that the Adams–Novikov spectral sequence has far fewer differentials than the Adams spectral sequence. The first nontrivial Adams–Novikov differential originates in dimension 34 and leads to the relation  $\alpha_1\beta_1^3$  in  $\pi_*(S^0)$ . It was first established by Toda [2, 3].

### 3. The Adams–Novikov $E_2$ -term, Formal Group Laws, and the Greek Letter Construction

In this section we will describe the  $E_2$ -term of the Adams–Novikov spectral sequence introduced at the end of the previous section. We begin by defining formal group laws (1.3.1) and describing their connection with complex cobordism (1.3.4). Then we characterize the  $E_2$ -term in terms of them (1.3.5 and 1.3.6). Next we describe the Greek letter construction, an algebraic method for producing periodic families of elements in the  $E_2$ -term. We conclude by commenting on the problem of representing these elements in  $\pi_*^S$ .

Suppose  $T$  is a one-dimensional commutative analytic Lie group and we have a local coordinate system in which the identity element is the origin. Then the group operation  $T \times T \rightarrow T$  can be described locally as a real-valued analytic function of two variables. Let  $F(x, y) \in \mathbf{R}[[x, y]]$  be the power series expansion of this function about the origin. Since 0 is the identity element we have  $F(x, 0) = F(0, x) = x$ . Commutativity and associativity give  $F(x, y) = F(y, x)$  and  $F(F(x, y), z) = F(x, F(y, z))$ , respectively.

1.3.1. DEFINITION. A formal group law over a commutative ring with unit  $R$  is a power series  $F(x, y) \in R[[x, y]]$  satisfying the three conditions above.  $\square$

Several remarks are in order. First, the power series in the Lie group will have a positive radius of convergence, but there is no convergence condition in the definition above. Second, there is no need to require the existence of an inverse because it exists automatically. It is a power series  $i(x) \in R[[x]]$  satisfying  $F(x, i(x)) = 0$ ;



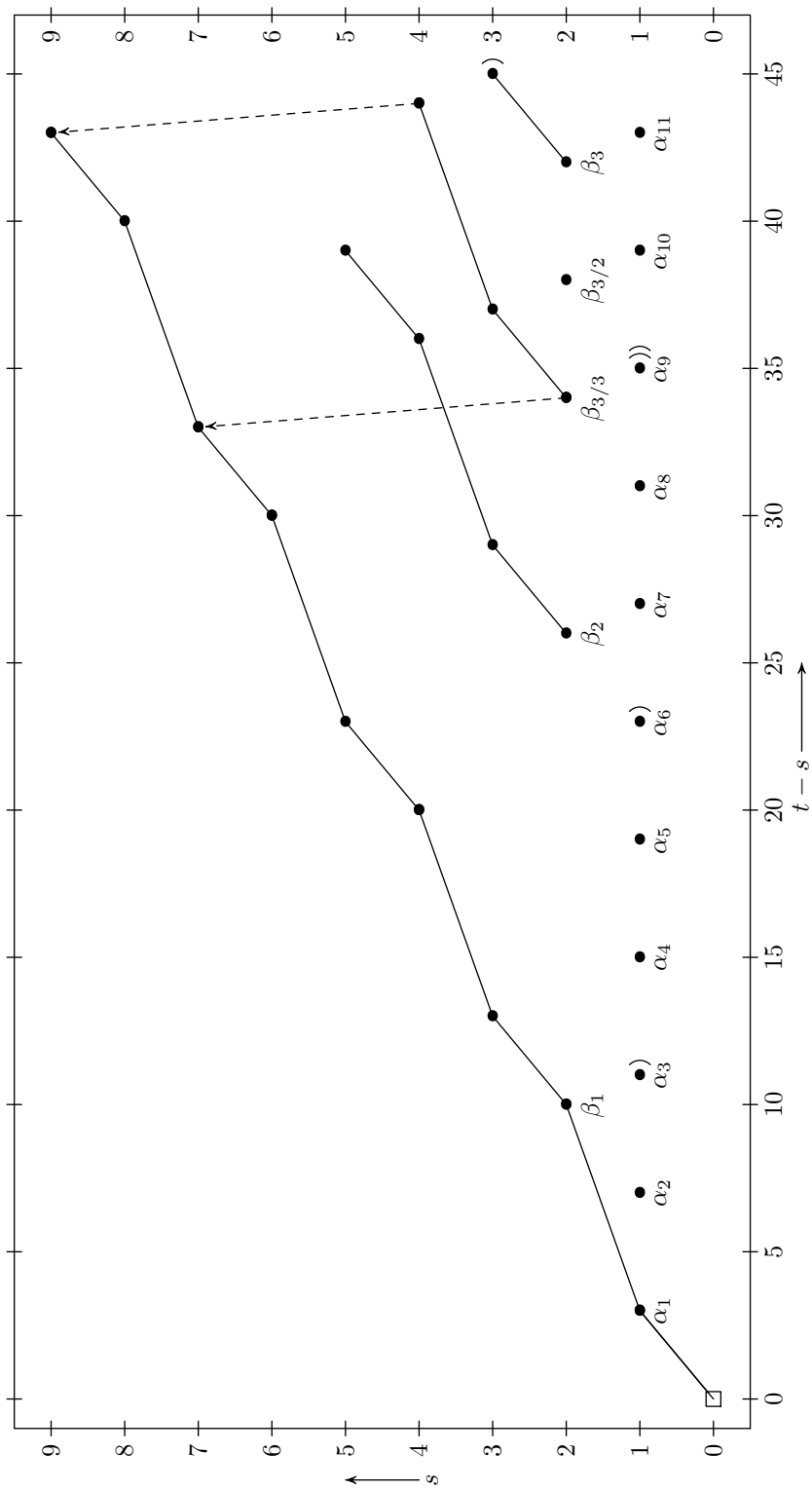


FIGURE 1.2.19. The Adams–Novikov spectral sequence for  $p = 3$ ,  $t - s \leq 45$

it is an easy exercise to solve this equation for  $i(x)$  given  $F$ . Third, a rigorous self-contained treatment of the theory of formal group laws is given in Appendix 2.

Note that  $F(x, 0) = F(0, x) = x$  implies that  $F \equiv x + y \pmod{(x, y)^2}$  and that  $x + y$  is therefore the simplest example of a formal group law; it is called the *additive* formal group law and is denoted by  $F_a$ . Another easy example is the *multiplicative* formal group law,  $F_m = x + y + rxy$  for  $r \in \mathbf{R}$ . These two are known to be the only formal group laws which are polynomials. Other examples are given in A2.1.4.

To see what formal group laws have to do with complex cobordism and the Adams–Novikov spectral sequence, consider  $MU^*(\mathbf{C}P^\infty)$ , the complex cobordism of infinite-dimensional complex projective space. Here  $MU^*(-)$  is the cohomology theory dual to the homology theory  $MU_*(-)$  (complex bordism) described in Section 2. Like ordinary cohomology it has a cup product and we have

1.3.2. THEOREM. *There is an element  $x \in MU^2(\mathbf{C}P^\infty)$  such that*

$$MU^*(\mathbf{C}P^\infty) = MU^*(\text{pt})[[x]]$$

and

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = MU^*(\text{pt})[[x \otimes 1, 1 \otimes x]].$$

□

Here  $MU^*(\text{pt})$  is the complex cobordism of a point; it differs from  $MU_*(\text{pt})$  (described in 1.2.18) only in that its generators are negatively graded. The generator  $x$  is closely related to the usual generator of  $H^2(\mathbf{C}P^\infty)$ , which we also denote by  $x$ . The alert reader may have expected  $MU^*(\mathbf{C}P^\infty)$  to be a polynomial rather than a power series ring since  $H^*(\mathbf{C}P^\infty)$  is traditionally described as  $\mathbf{Z}[x]$ . However, the latter is really  $\mathbf{Z}[[x]]$  since the cohomology of an infinite complex maps onto the inverse limit of the cohomologies of its finite skeleta. [ $MU^*(\mathbf{C}P^n)$ , like  $H^*(\mathbf{C}P^n)$ , is a truncated polynomial ring.] Since one usually considers only homogeneous elements in  $H^*(\mathbf{C}P^\infty)$ , the distinction between  $\mathbf{Z}[x]$  and  $\mathbf{Z}[[x]]$  is meaningless. However, one can have homogeneous infinite sums in  $MU^*(\mathbf{C}P^\infty)$  since the coefficient ring is negatively graded.

Now  $\mathbf{C}P^\infty$  is the classifying space for complex line bundles and there is a map  $\mu: \mathbf{C}P^\infty \times \mathbf{C}P^\infty \rightarrow \mathbf{C}P^\infty$  corresponding to the tensor product; in fact,  $\mathbf{C}P^\infty$  is known to be a topological abelian group. By 1.3.2 the induced map  $\mu^*$  in complex cobordism is determined by its behavior on the generator  $x \in MU^2(\mathbf{C}P^\infty)$  and one easily proves, using elementary facts about line bundles,

1.3.3. PROPOSITION. *For the tensor product map  $\mu: \mathbf{C}P^\infty \times \mathbf{C}P^\infty \rightarrow \mathbf{C}P^\infty$ ,  $\mu^*(x) = F_U(x \otimes 1, 1 \otimes x) \in MU^*(\text{pt})[[x \otimes 1, 1 \otimes x]]$  is an formal group law over  $MU^*(\text{pt})$ .* □

A similar statement is true of ordinary cohomology and the formal group law one gets is the additive one; this is a restatement of the fact that the first Chern class of a tensor product of complex line bundles is the sum of the first Chern classes of the factors. One can play the same game with complex  $K$ -theory and get a multiplicative formal group law.

$\mathbf{C}P^\infty$  is a good test space for both complex cobordism and  $K$ -theory. One can analyze the algebra of operations in both theories by studying their behavior in  $\mathbf{C}P^\infty$  (see Adams [5]) in the same way that Milnor [2] analyzed the mod (2) Steenrod algebra by studying its action on  $H^*(\mathbf{R}P^\infty; \mathbf{Z}/(2))$ . (See also Steenrod and Epstein [1].)

The formal group law of 1.3.3 is not as simple as the ones for ordinary cohomology or  $K$ -theory; it is complicated enough to have the following universal property.

1.3.4. THEOREM (Quillen [2]). *For any formal group law  $F$  over any commutative ring with unit  $R$  there is a unique ring homomorphism  $\theta: MU^*(\text{pt}) \rightarrow R$  such that  $F(x, y) = \theta F_U(x, y)$ .  $\square$*

We remark that the existence of such a universal formal group law is a triviality. Simply write  $F(x, y) = \sum a_{i,j} x^i y^j$  and let  $L = \mathbf{Z}[a_{i,j}]/I$ , where  $I$  is the ideal generated by the relations among the  $a_{i,j}$  imposed by the definition 1.3.1 of an formal group law. Then there is an obvious formal group law over  $L$  having the universal property. Determining the explicit structure of  $L$  is much harder and was first done by Lazard [1]. Quillen's proof of 1.3.4 consisted of showing that Lazard's universal formal group law is isomorphic to the one given by 1.3.3.

Once Quillen's Theorem 1.3.4 is proved, the manifolds used to define complex bordism theory become irrelevant, however pleasant they may be. All of the applications we will consider follow from purely algebraic properties of formal group laws. This leads one to suspect that the spectrum  $MU$  can be constructed somehow using formal group law theory and without using complex manifolds or vector bundles. Perhaps the corresponding infinite loop space is the classifying space for some category defined in terms of formal group laws. Infinite loop space theorists, where are you?

We are now just one step away from a description of the Adams–Novikov spectral sequence  $E_2$ -term. Let  $G = \{f(x) \in \mathbf{Z}[[x]] \mid f(x) \equiv x \pmod{(x)^2}\}$ . Here  $G$  is a group under composition and acts on the Lazard/complex cobordism ring  $L = MU_*(\text{pt})$  as follows. For  $g \in G$  define a formal group law  $F_g$  over  $L$  by  $F_g(x, y) = g^{-1} F_U(g(x), g(y))$ . By 1.3.4  $F_g$  is induced by a homomorphism  $\theta_g: L \rightarrow L$ . Since  $g$  is invertible under composition,  $\theta_g$  is an automorphism and we have a  $G$ -action on  $L$ .

Note that  $g(x)$  defines an isomorphism between  $F$  and  $F_g$ . In general, isomorphisms between formal group laws are induced by power series  $g(x)$  with leading term a unit multiple (not necessarily one) of  $x$ . An isomorphism induced by a  $g$  in  $G$  is said to be *strict*.

1.3.5. THEOREM. *The  $E_2$ -term of the Adams–Novikov spectral sequence converging to  $\pi_*^S$  is isomorphic to  $H^{**}(G; L)$ .  $\square$*

There is a difficulty with this statement: since  $G$  does not preserve the grading on  $L$ , there is no obvious bigrading on  $H^*(G; L)$ . We need to reformulate in terms of  $L$  as a comodule over a certain Hopf algebra  $B$  defined as follows.

Let  $g \in G$  be written as  $g(x) = \sum_{i \geq 0} b_i x^{i+1}$  with  $b_0 = 1$ . Each  $b_i$  for  $i > 0$  can be thought of as a  $\mathbf{Z}$ -valued function on  $G$  and they generate a graded algebra of such functions

$$B = \mathbf{Z}[b_1, b_2, \dots] \quad \text{with } \dim b_i = 2i.$$

(Do not confuse this ring with  $L$ , to which it happens to be isomorphic.) The group structure on  $G$  corresponds to a coproduct  $\Delta: B \rightarrow B \otimes B$  on  $B$  given by  $\Delta(b) = \sum_{i \geq 0} b^{i+1} \otimes b_i$ , where  $b = \sum_{i \geq 0} b_i$  and  $b_0 = 1$  as before. To see this suppose  $g(x) = g^{(1)}(g^{(2)}(x))$  with  $g^{(k)}(x) = \sum b_i^{(k)} x^{i+1}$ . Then we have

$$\sum b_i x^{i+1} = \sum b_i^{(1)} \left( \sum b_j^{(2)} x^{j+1} \right)^{i+1}$$

from which the formula for  $\Delta$  follows. This coproduct makes  $B$  into a graded connected Hopf algebra over which  $L$  is a graded comodule. We can restate 1.3.5 as

1.3.6. THEOREM. *The  $E_2$ -term of the Adams–Novikov spectral sequence converging to  $\pi_*^S$  is given by  $E_2^{s,t} = \text{Ext}_B^{s,t}(\mathbf{Z}, L)$ .  $\square$*

The definition of this Ext is given in A1.2.3; all of the relevant homological algebra is discussed in Appendix 1.

Do not be alarmed if the explicit action of  $G$  (or coaction of  $B$ ) on  $L$  is not obvious to you. It is hard to get at directly and computing its cohomology is a very devious business.

Next we will describe the *Greek letter construction*, which is a method for producing lots (but by no means all) of elements in the  $E_2$ -term, including the  $\alpha_t$ 's and  $\beta_t$ 's seen in 1.2.19. We will use the language suggested by 1.3.5; the interested reader can translate our statements into that of 1.3.6. Our philosophy here is that group cohomology in positive degrees is too hard to comprehend, but  $H^0(G; M)$  (the  $G$ -module  $M$  will vary in the discussion), the submodule of  $M$  fixed by  $G$ , is relatively straightforward. Hence our starting point is

1.3.7. THEOREM.  $H^0(G; L) = \mathbf{Z}$  concentrated in dimension 0.  $\square$

This corresponds to the 0-stem in stable homotopy. Not a very promising beginning you say? It does give us a toehold on the problem. It tells us that the only principal ideals in  $L$  which are  $G$ -invariant are those generated by integers and suggests the following. Fix a prime number  $p$  and consider the short exact sequence of  $G$ -modules

$$(1.3.8) \quad 0 \rightarrow L \xrightarrow{p} L \rightarrow L/(p) \rightarrow 0.$$

We have a connecting homomorphism

$$\delta_0: H^i(G; L/(p)) \rightarrow H^{i+1}(G; L).$$

1.3.9. THEOREM.  $H^0(G; L/(p)) = \mathbf{Z}/(p)[v_1]$ , where  $v_1 \in L$  has dimension  $q = 2(p-1)$ .  $\square$

1.3.10. DEFINITION. For  $t > 0$  let  $\alpha_t = \delta_0(v_1^t) \in E_2^{1,qt}$ .  $\square$

It is clear from the long exact sequence in cohomology associated with 1.3.8 that  $\alpha_t \neq 0$  for all  $t > 0$ , so we have a collection of nontrivial elements in the Adams–Novikov  $E_2$ -term. We will comment below on the problems of constructing corresponding elements in  $\pi_*^S$ ; for now we will simply state the result.

1.3.11. THEOREM. (a) (Toda [4, IV]) For  $p > 2$  each  $\alpha_t$  is represented by an element of order  $p$  in  $\pi_{qt-1}^S$  which is in the image of the  $J$ -homomorphism (1.1.12).

(b) For  $p = 2$   $\alpha_t$  is so represented provided  $t \not\equiv 3 \pmod{4}$ . If  $t \equiv 2 \pmod{4}$  then the element has order 4; otherwise it has order 2. It is in  $\text{im } J$  if  $t$  is even.  $\square$

Theorem 1.3.9 tells us that

$$(1.3.12) \quad 0 \rightarrow \Sigma^q L/(p) \xrightarrow{v_1} L/(p) \rightarrow L/(p, v_1) \rightarrow 0$$

is an short exact sequence of  $G$ -modules and there is a connecting homomorphism

$$\delta_1: H^i(G; L/(p, v_1)) \rightarrow H^{i+1}(G; L/(p)).$$

The analogs of 1.3.9 and 1.3.10 are

1.3.13. THEOREM.  $H^0(G; L/(p, v_1)) = \mathbf{Z}/(p)[v_2]$  where  $v_2 \in L$  has dimension  $2(p^2 - 1)$ .  $\square$

1.3.14. DEFINITION. For  $t > 0$  let  $\beta_t = \delta_0 \delta_1(v_2^t) \in E_2^{2, t(p+1)q-q}$ .  $\square$

More work is required to show that these elements are nontrivial for  $p > 2$ , and  $\beta_1 = 0$  for  $p = 2$ . The situation in homotopy is

1.3.15. THEOREM (Smith [1]). For  $p \geq 5$   $\beta_t$  is represented by a nontrivial element of order  $p$  in  $\pi_{(p+1)tq-q-2}(S^0)$ .  $\square$

You are probably wondering if we can continue in this way and construct  $\gamma_t$ ,  $\delta_t$ , etc. The following results allow us to do so.

1.3.16. THEOREM (Morava [3], Landweber [4]). (a) There are elements  $v_n \in L$  of dimension  $2(p^n - 1)$  such that  $I_n = (p, v_1, v_2, \dots, v_{n-1}) \subset L$  is a  $G$ -invariant prime ideal for all  $n > 0$ .

(b)  $0 \rightarrow \Sigma^{2(p^n - 1)} L/I_n \xrightarrow{v_n} L/I_n \rightarrow L/I_{n+1} \rightarrow 0$  is an short exact sequence of modules with connecting homomorphism

$$\delta: H^i(G; L/I_{n+1}) \rightarrow H^{i+1}(G; L/I_n).$$

(c)  $H^0(G; L/I_n) = \mathbf{Z}/(p)[v_n]$ .

(d) The only  $G$ -invariant prime ideals in  $L$  are the  $I_n$  for  $0 < n \leq \infty$  for all primes  $p$ .  $\square$

Part (d) above shows how rigid the  $G$ -action on  $L$  is; there are frightfully many prime ideals in  $L$ , but only the  $I_n$  for various primes are  $G$ -invariant. Using (b) and (c) we can make

1.3.17. DEFINITION. For  $t, n > 0$  let  $\alpha_t^{(n)} = \delta_0 \delta_1 \dots \delta_{n-1}(v_n^t) \in E_2^{n,*}$ .  $\square$

Here  $\alpha^{(n)}$  stands for the  $n$ th letter of the Greek alphabet, the length of which is more than adequate given our current state of knowledge. The only other known result comparable to 1.3.11 or 1.3.15 is

1.3.18. THEOREM. (a) (Miller, Ravenel, and Wilson [1]) The element  $\gamma_t \in E_2^{3, tq(p^2+p+1)-q(p+2)}$  is nontrivial for all  $t > 0$  and  $p > 2$ .

(b) (Toda [1]) For  $p \geq 7$  each  $\gamma_t$  is represented by a nontrivial element of order  $p$  in  $\pi_{tq(p^2+p+1)-q(p+2)-3}(S^0)$ .  $\square$

It is known that not all  $\gamma_t$  exist in homotopy for  $p = 5$  (see 7.6.1). Part (b) above was proved several years before part (a). In the intervening time there was a controversy over the nontriviality of  $\gamma_1$  which was unresolved for over a year, ending in 1974 (see Thomas and Zahler [1]). This unusual state of affairs attracted the attention of the editors of *Science* [1] and the *New York Times* [1], who erroneously cited it as evidence of the decline of mathematics.

We conclude our discussion of the Greek letter construction by commenting briefly on *generalized* Greek letter elements. Examples are  $\beta_{3/3}$  and  $\beta_{3/2}$  (and the elements in  $E_2^{1,*}$  of order  $> 3$ ) in 1.2.19. The elements come via connecting homomorphisms from  $H^0(G; L/J)$ , where  $J$  is a  $G$ -invariant *regular* (instead of prime) ideal. Recall that a regular ideal  $(x_0, x_1, \dots, x_{n-1}) \subset L$  is one in which each  $x_i$  is not a zero divisor modulo  $(x_0, \dots, x_{i-1})$ . Hence  $G$ -invariant prime ideals are regular as are ideals of the form  $(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ . Many but not all  $G$ -invariant regular ideals have this form.

1.3.19. DEFINITION.  $\beta_{s/t}$  (for appropriate  $s$  and  $t$ ) is the image of  $v_2^s \in H^0(G; L/(p, v_1^t))$  and  $\alpha_{s/t}$  is the image of  $v_1^s \in H^0(G; L/(p^t))$ .  $\square$

Hence  $p\alpha_{s/t} = \alpha_{s/t-1}$ ,  $\alpha_{s/1} = \alpha_s$ , and  $\beta_{t/1} = \beta_t$  by definition.

Now we will comment on the problem of representing these elements in the  $E_2$ -term by elements in stable homotopy, e.g., on the proofs of 1.3.11, 1.3.15, and 1.3.18(b). The first thing we must do is show that the elements produced are actually nontrivial in the  $E_2$ -term. This has been done only for  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's. For  $p = 2$ ,  $\beta_1$  and  $\gamma_1$  are zero but for  $t > 1$   $\beta_t$  and  $\gamma_t$  are nontrivial; these results are part of the recent computation of  $E_2^{2,*}$  at  $p = 2$  by Shimomura [1], which also tells us which generalized  $\beta$ 's are defined and are nontrivial. The corresponding calculation at odd primes was done in Miller, Ravenel, and Wilson [1], as was that of  $E_2^{1,*}$  for all primes.

The general strategy for representing Greek letter elements geometrically is to realize the relevant short exact sequences [e.g., 1.3.8, 1.3.12, and 1.3.16(b)] by cofiber sequences of finite spectra. For any connective spectrum  $X$  there is an Adams–Novikov spectral sequence converging to  $\pi_*(X)$ . Its  $E_2$ -term [denoted by  $E_2(X)$ ] can be described as in 1.3.5 with  $L = MU_*(S^0)$  replaced by  $MU_*(X)$ , which is a  $G$ -module. For 1.3.8 we have a cofiber sequence

$$S^0 \xrightarrow{p} S^0 \rightarrow V(0),$$

where  $V(0)$  is the mod  $(p)$  Moore spectrum. It is known (2.3.4) that the long exact sequence of homotopy groups is compatible with the long exact sequence of  $E_2$ -terms. Hence the elements  $v_1^t$  of 1.3.9 live in  $E_2^{0,qt}(V(0))$  and for 1.3.11(a) [which says  $\alpha_t$  is represented by an element of order  $p$  in  $\pi_{qt-1}(S^0)$  for  $p > 2$  and  $t > 0$ ] it would suffice to show that these elements are permanent cycles in the Adams–Novikov spectral sequence for  $\pi_*(V(0))$  with  $p > 0$ . For  $t = 1$  (even if  $p = 2$ ) one can show this by brute force; one computes  $E_2(V(0))$  through dimension  $q$  and sees that there is no possible target for a differential coming from  $v_1 \in E_2^{0,q}$ . Hence  $v_1$  is realized by a map

$$S^q \rightarrow V(0)$$

If we can extend it to  $\Sigma^q V(0)$ , we can iterate and represent all powers of  $v_1$ . We can try to do this either directly, using obstruction theory, or by showing that  $V(0)$  is a ring spectrum. In the latter case our extension  $\alpha$  would be the composite

$$S^q \wedge V(0) \rightarrow V(0) \wedge V(0) \rightarrow V(0),$$

where the first map is the original map smashed with the identity on  $V(0)$  and the second is the multiplication on  $V(0)$ . The second method is generally (in similar situation of this sort) easier because it involves obstruction theory in a lower range of dimensions.

In the problem at hand both methods work for  $p > 2$  but both fail for  $p = 2$ . In that case  $V(0)$  is not a ring spectrum and our element in  $\pi_2(V(0))$  has order 4, so it does not extend to  $\Sigma^2 V(0)$ . Further calculations show that  $v_1^2$  and  $v_1^3$  both support nontrivial differentials (see 5.3.13) but  $v_1^4$  is a permanent cycle represented by map  $S^8 \rightarrow V(0)$ , which *does* extend to  $\Sigma^8 V(0)$ . Hence iterates of this map produce the homotopy elements listed in 1.3.11(b) once certain calculation have been made in dimensions  $\leq 8$ .

For  $p > 2$  the map  $\alpha: \Sigma^q V(0) \rightarrow V(0)$  gives us a cofiber sequence

$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \rightarrow V(1),$$

realizing the short exact sequence 1.3.12. Hence to arrive at 1.3.15 (which describes the  $\beta$ 's in homotopy) we need to show that  $v_2 \in E_2^{0,(p+1)q}(V(1))$  is a permanent cycle represented by a map which extends to  $\beta: \Sigma^{(p+1)q}V(1) \rightarrow V(1)$ . We can do this for  $p \geq 5$  but not for  $p = 3$ . Some partial results for  $\beta$ 's at  $p = 3$  and  $p = 2$  are described in Section 5.5.

The cofiber of the map  $\beta$  (corresponding to  $v_2$ ) for  $p \geq 5$  is called  $V(2)$  by Toda [1]. In order to construct the  $\gamma$ 's [1.3.18(b)] one needs a map

$$\gamma: \Sigma^{2(p^3-1)}V(2) \rightarrow V(2)$$

corresponding to  $v_3$ . Toda [1] produces such a map for  $p \geq 7$  but it is known not to exist for  $p = 5$  (see 7.6.1).

Toda [1] first considered the problem of constructing the spectra  $V(n)$  above, and hence of the representation of Greek letter elements in  $\pi_*^S$ , although that terminology (and 1.3.16) was not available at the time. While the results obtained there have not been surpassed, the methods used leave something to be desired. Each positive result is proved by brute force; the relevant obstruction groups are shown to be trivial. This approach can be pushed no further; the obstruction to realizing  $v_4$  lies in a nontrivial group for all primes (5.6.13). Homotopy theorists have yet to learn how to compute obstructions in such situations.

The negative results of Toda [1] are proved by ingenious but *ad hoc* methods. The nonexistence of  $V(1)$  for  $p = 2$  follows easily from the structure of the Steenrod algebra; if it existed its cohomology would contradict the Adem relation  $Sq^2Sq^2 = Sq^1Sq^2Sq^1$ . For the nonexistence of  $V(2)$  at  $p = 3$  Toda uses a delicate argument involving the nonassociativity of the mod (3) Moore spectrum, which we will not reproduce here. We will give another proof (5.5.1) which uses the multiplicative structure of the Adams–Novikov  $E_2$ -term to show that the nonrealizability of  $\beta_4 \in E_2^{2,60}$ , and hence of  $V(2)$ , is a formal consequence of that of  $\beta_{3/3} \in E_2^{2,36}$ . This was shown by Toda [2, 3] using an extended power construction, which will also not be reproduced here. Indeed, all of the differentials in the Adams–Novikov spectral sequence for  $p = 3$  in the range we consider are formal consequences of that one in dimension 34. A variant of the second method used for  $V(2)$  at  $p = 3$  works for  $V(3)$  (the cofiber of  $\gamma$ ) at  $p = 5$ .

#### 4. More Formal Group Law Theory, Morava's Point of View, and the Chromatic Spectral Sequence

We begin this section by introducing  $BP$ -theory, which is essentially a  $p$ -local form of  $MU$ -theory. With it many of the explicit calculations behind our results become a lot easier. Most of the current literature on the subject is written in terms of  $BP$  rather than  $MU$ . On the other hand,  $BP$  is not essential for the overall picture of the  $E_2$ -term we will give later, so it could be regarded as a technicality to be passed over by the casual reader. Next we will describe the classification of formal group laws over an algebraically closed field of characteristic  $p$ . This is needed for Morava's point of view, which is a useful way of understanding the action of  $G$  on  $L$  (1.3.5). The insights that come out of this approach are made computationally precise in the chromatic spectral sequence, which is the pivotal idea in this book. Technically the chromatic spectral sequence is a trigraded spectral sequence converging to the Adams–Novikov  $E_2$ -term; heuristically it is like a spectrum in the astronomical sense in that it resolves the  $E_2$ -term into various

components each having a different type of periodicity. In particular, it incorporates the Greek letter elements of the previous section into a broader scheme which embraces the entire  $E_2$ -term.

$BP$ -theory began with Brown and Peterson [1] (after whom it is named), who showed that after localization at any prime  $p$ , the  $MU$  spectrum splits into an infinite wedge suspension of identical smaller spectra subsequently called  $BP$ . One has

$$(1.4.1) \quad \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots],$$

where  $\mathbf{Z}_{(p)}$  denotes the integers localized at  $p$  and the  $v_n$ 's are the same as the generators appearing in the Morava–Landweber theorem 1.3.16. Since  $\dim v_n = 2(p^n - 1)$ , this coefficient ring, which we will denote by  $BP_*$ , is much smaller than  $L = \pi_*(MU)$ , which has a polynomial generator in every even dimension.

Next Quillen [2] observed that there is a good formal group law theoretic reason for this splitting. A theorem of Cartier [1] (A2.1.18) says that every formal group law over a  $\mathbf{Z}_{(p)}$ -algebra is canonically isomorphic to one in a particularly convenient form called a  $p$ -typical formal group law (see A2.1.17 and A2.1.22 for the definition, the details of which need not concern us now). This canonical isomorphism is reflected topologically in the above splitting of the localization of  $MU$ . This fact is more evidence in support of our belief that  $MU$  can somehow be constructed in purely formal group law theoretic terms.

There is a  $p$ -typical analog of Quillen's theorem 1.3.4; i.e.,  $BP^*(\mathbf{C}P^\infty)$  gives us a  $p$ -typical formal group law with a similar universal property. Also, there is a  $BP$  analog of the Adams–Novikov spectral sequence, which is simply the latter tensored with  $\mathbf{Z}_{(p)}$ ; i.e., its  $E_2$ -term is the  $p$ -component of  $H^*(G; L)$  and it converges to the  $p$ -component of  $\pi_*^S$ . However, we encounter problems in trying to write an analog of our metaphor 1.3.5 because *there is no  $p$ -typical analog of the group  $G$* .

In other words there is no suitable group of power series over  $\mathbf{Z}_{(p)}$  which will send any  $p$ -typical formal group law into another. Given a  $p$ -typical formal group law  $F$  over  $\mathbf{Z}_{(p)}$  there is a set of power series  $g \in \mathbf{Z}_{(p)}[[x]]$  such that  $g^{-1}F(g(x), g(y))$  is also  $p$ -typical, but this set depends on  $F$ . Hence  $\text{Hom}(BP_*, K)$  the set of  $p$ -typical formal group laws over a  $\mathbf{Z}_{(p)}$ -algebra  $K$ , is acted on not by a group analogous to  $G$ , but by a *groupoid*.

Recall that a groupoid is a small category in which every morphism is an equivalence, i.e., it is invertible. A groupoid with a single object is a group. In our case the objects are  $p$ -typical formal group laws over  $K$  and the morphisms are isomorphisms induced by power series  $g(x)$  with leading term  $x$ .

Now a Hopf algebra, such as  $B$  in 1.3.6, is a *cogroup object* in the category of commutative rings  $R$ , which is to say that  $\text{Hom}(B, R) = G_R$  is a group-valued functor. In fact  $G_R$  is the group (under composition) of power series  $f(x)$  over  $R$  with leading term  $x$ . For a  $p$ -typical analog of 1.3.6 we need to replace  $b$  by *cogroupoid object* in the category of commutative  $\mathbf{Z}_{(p)}$ -algebras  $K$ . Such an object is called a *Hopf algebroid* (A1.1.1) and consists of a pair  $(A, \Gamma)$  of commutative rings with appropriate structure maps so that  $\text{Hom}(A, K)$  and  $\text{Hom}(\Gamma, K)$  are the sets of objects and morphisms, respectively, of a groupoid. The groupoid we have in mind, of course, is that of  $p$ -typical formal group laws and isomorphisms as above. Hence  $BP_*$  is the appropriate choice for  $A$ ; the choice for  $\Gamma$  turns out to be  $BP_*(BP)$ , the  $BP$ -homology of the spectrum  $BP$ . Hence the  $p$ -typical analog of 1.3.6 is



1.4.2. THEOREM. *The  $p$ -component of the  $E_2$ -term of the Adams–Novikov spectral sequence converging to  $\pi_*^S$  is*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*). \quad \square$$

Again this Ext is defined in A1.2.3 and the relevant homological algebra is discussed in Appendix 1.

We will now describe the classification of formal group laws over an algebraically closed field of characteristic  $p$ . First we define power series  $[m]_F(x)$  associated with a formal group law  $F$  and natural numbers  $m$ . We have  $[0]_F(x) = 0$ ,  $[1]_F(x) = x$ , and  $[m]_F(x) = F(x, [m-1]_F(x))$ . An easy lemma (A2.1.6) says that if  $F$  is defined over a field of characteristic  $p$ , then  $[p]_F(x)$  is in fact a power series over  $x^{p^n}$  with leading term  $ax^{p^n}$ ,  $a \neq 0$ , for some  $n > 0$ , provided  $F$  is not isomorphic to the additive formal group law, in which case  $[p]_F(x) = 0$ . This integer  $n$  is called the *height* of  $F$ , and the height of the additive formal group law is defined to be  $\infty$ . Then we have

1.4.3. CLASSIFICATION THEOREM (Lazard [2]).

(a) *Two formal group laws defined over the algebraic closure of  $\mathbf{F}_p$  are isomorphic iff they have the same height.*

(b) *If  $F$  is nonadditive, its height is the smallest  $n$  such that  $\theta(v_n) \neq 0$ , where  $\theta: L \rightarrow K$  is the homomorphism of 1.3.4 and  $v_n \in L$  is as in 1.3.16, where  $K$  is finite field.*  $\square$

Now we come to Morava’s point of view. Let  $K = \overline{\mathbf{F}}_p$ , the algebraic closure of the field with  $p$  elements, and let  $G_K \subset K[[x]]$  be the group (under composition) of power series with leading term  $x$ . We have seen that  $G_K$  acts on  $\mathrm{Hom}(L, K)$ , the set formal group laws defined over  $K$ . Since  $L$  is a polynomial ring, we can think of  $\mathrm{Hom}(L, K)$  as an infinite-dimensional vector space  $V$  over  $K$ ; a set of polynomial generators of  $L$  gives a topological basis of  $V$ . For a vector  $v \in V$ , let  $F_v$  be the corresponding formal group law.

Two vectors in  $V$  are in the same orbit iff the corresponding formal group laws are strictly isomorphic (strict isomorphism was defined just prior to 1.3.5), and the stabilizer group of  $v \in V$  (i.e., the subgroup of  $G_K$  leaving  $V$  fixed) is the strict automorphism group of  $F_v$ . This group  $S_n$  (where  $n$  is the height) can be described explicitly (A2.2.18); it is a profinite group of units in a certain  $p$ -adic division algebra, but the details need not concern us here. Theorem 1.4.3 enables us to describe the orbits explicitly.

1.4.4. THEOREM. *There is one  $G_K$ -orbit of  $V$  for each height as in 1.4.3. The height  $n$  orbit  $V_n$  is the subset defined by  $v_i = 0$  for  $i < n$  and  $v_n \neq 0$ .*  $\square$

Now observe that  $V$  is the set of closed points in  $\mathrm{Spec}(L_n \otimes K)$ , and  $V_n$  is the set of closed points in  $\mathrm{Spec}(L_n \otimes K)$ , where  $L_n = v_n^{-1}L/I_n$ . Here  $V_n$  is a homogeneous  $G_K$ -space and a standard change-of-rings argument gives

1.4.5. CHANGE-OF-RINGS THEOREM.  $H^*(G_K; L_n \otimes K) = H^*(S_n; K)$ .  $\square$

We will see in Chapter 6 that a form of this isomorphism holds over  $\mathbf{F}_p$  as well as over  $K$ . In it the right-hand term is the cohomology of a certain Hopf algebra [called the  $n$ th Morava stabilizer algebra  $\Sigma(n)$ ] defined over  $\mathbf{F}_p$ , which, when tensored with  $\mathbf{F}_{p^n}$ , becomes isomorphic to the dual of  $\mathbf{F}_{p^n}[S_n]$ , the  $\mathbf{F}_{p^n}$ -group algebra of  $S_n$ .

Now we are ready to describe the central construction of this book, the chromatic spectral sequence, which enables us to use the results above to get more explicit information about the Adams–Novikov  $E_2$ -term. We start with a long exact sequence of  $G$ -modules, called the chromatic resolution

$$(1.4.6) \quad 0 \rightarrow L \otimes \mathbf{Z}_{(p)} \rightarrow M^0 \rightarrow M^1 \rightarrow \dots$$

defined as follows.  $M^0 = L \otimes \mathbf{Q}$ , and  $N^1$  is the cokernel in the short exact sequence

$$0 \rightarrow L \otimes \mathbf{Z}_{(p)} \rightarrow M^0 \rightarrow N^1 \rightarrow 0.$$

$M^n$  and  $N^n$  are defined inductively for  $n > 0$  by short exact sequences

$$(1.4.7) \quad 0 \rightarrow N^n \rightarrow M^n \rightarrow N^{n+1} \rightarrow 0,$$

where  $M^n = v_n^{-1}N^n$ . Hence we have

$$N^1 = L \otimes \mathbf{Q}/\mathbf{Z}_{(p)} = \varinjlim L/(p^i) = L/(p^\infty)$$

and

$$N^{n+1} = \varinjlim L/(p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n}) = L/(p^\infty, v_1^\infty, \dots, v_n^\infty).$$

The fact that these are short exact sequences of  $G$ -modules is nontrivial. The long exact sequence 1.4.6 is obtained by splicing together the short exact sequences 1.4.7. In Chapter 5, where the chromatic spectral sequence is described in detail,  $M^n$  and  $N^n$  denote the corresponding objects defined in terms of  $BP_*$ . In what follows here  $\text{Ext}_B(\mathbf{Z}, M)$  will be abbreviated by  $\text{Ext}(M)$  for a  $B$ -module (e.g.,  $G$ -module)  $M$ . Standard homological algebra (A1.3.2) gives

1.4.8. PROPOSITION. *There is a spectral sequence converging to  $\text{Ext}(L \otimes \mathbf{Z}_{(p)})$  with  $E_1^{n,s} = \text{Ext}^s(M^n)$ ,  $d_r: E_r^{n,s} \rightarrow E_r^{n+r,s-r+1}$ , and  $d_1: \text{Ext}(M^n) \rightarrow \text{Ext}(M^{n+1})$  being induced by the maps  $M^n \rightarrow M^{n+1}$  in 1.4.6. [ $E_\infty^{n,s}$  is a subquotient of  $\text{Ext}^{n+s}(L \otimes \mathbf{Z}_{(p)})$ .]  $\square$*

This is the *chromatic spectral sequence*. We can use 1.4.5 to get at its  $E_1$  term as follows. Define  $G$ -modules  $M_i^n$  for  $0 \leq i \leq n$  by  $M_0^n = M^n$ , and  $M_i^n$  is the kernel in the short exact sequence

$$(1.4.9) \quad 0 \rightarrow M_i^n \rightarrow M_{i-1}^n \xrightarrow{v_{i-1}} M_{i-1}^n \rightarrow 0,$$

where  $v_0 = p$ . This gives  $M_n^n = L_n = v_n^{-1}L/I_n$ , so the  $\mathbf{F}_p$ -analog of 1.4.5 describes  $\text{Ext}(M_n^n)$  in terms of the cohomology of the stabilizer group  $S_n$ . Equation 1.4.9 gives a long exact sequence of Ext groups of a Bockstein spectral sequence computing  $\text{Ext}(M_{i-1}^n)$  in terms of  $\text{Ext}(M_i^n)$ . Hence in principle we can get from  $H^*(S_n)$  to  $\text{Ext}(M^n)$ , although the Bockstein spectral sequences are difficult to handle in practice.

Certain general facts about  $H^*(S_n)$  are worth mentioning here. If  $(p-1)$  divides  $n$  then this cohomology is periodic (6.2.10); i.e., there is an element  $c \in H^*(S_n; \mathbf{F}_p)$  such that  $H^*(S_n; \mathbf{F}_p)$  is a finitely generated free module over  $\mathbf{F}_p[c]$ . In this case  $S_n$  has a cyclic subgroup of order  $p$  to whose cohomology  $c$  restricts nontrivially. This cohomology can be used to detect elements in the Adams–Novikov  $E_2$ -term of high cohomological degree, e.g., to prove

1.4.10. THEOREM. *For  $p > 2$ , all monomials in the  $\beta_{p^i/p^i}$  (1.3.19) are nontrivial.  $\square$*

If  $n$  is not divisible by  $p - 1$  then  $S_n$  has cohomological dimension  $n^2$ ; i.e.,  $H^i(S_n) = 0$  if  $i > n^2$ , and  $H^*(S_n)$  has a certain type of Poincaré duality (6.2.10). It is essentially the cohomology of a certain  $n$ -stage nilpotent Lie algebra (6.3.5), at least for  $n < p - 1$ . The cohomological dimension implies

1.4.11. MORAVA VANISHING THEOREM. *If  $(p - 1) \nmid n$ , then in the chromatic spectral sequence (1.4.8)  $E_1^{n,s} = 0$  for  $s > n^2$ .*  $\square$

It is also known (6.3.6) that every sufficiently small open subgroup of  $S_n$  has the same cohomology as a free abelian group of rank  $n^2$ . This fact can be used to get information about the Adams–Novikov spectral sequence  $E_2$ -term for certain Thom spectra (6.5.6).

Now we will explain how the Greek letter elements of 1.3.17 and 1.3.19 appear in the chromatic spectral sequence. If  $J$  is a  $G$ -invariant regular ideal with  $n$  generators [e.g., the invariant prime ideal  $I_n = (p, v_1, \dots, v_{n-1})$ ], then  $L/J$  is a submodule of  $N^n$  and  $M^n$ , so  $\text{Ext}^0(L/J) \subset \text{Ext}^0(N^n) \subset \text{Ext}^0(M^n) = E_1^{n,0}$ . Recall that the Greek letter elements are images of elements in  $\text{Ext}^0(J)$  under the appropriate composition of connecting homomorphisms. This composition corresponds to the edge homomorphism  $E_2^{n,0} \rightarrow E_\infty^{n,0}$  in the chromatic spectral sequence. [Note that every element in the chromatic  $E_2^{n,0}$  is a permanent cycle; i.e., it supports no nontrivial differential although it may be the target of one. Elements in  $E_1^{n,0}$  coming from  $\text{Ext}(L/J)$  lift to  $\text{Ext}(N^n)$  are therefore in  $\ker d_1$  and live in  $E_2^{n,0}$ .] The module  $N^n$  is the union of the  $L/J$  over all possible invariant regular ideals  $J$  with  $n$  generators, so  $\text{Ext}^0(N^n)$  contains all possible  $n$ th Greek letter elements.

To be more specific about the particular elements discussed in Section 3 we must introduce *chromatic notation* for elements in  $N^n$  and  $M^n$ . Such elements will be written as fractions  $\frac{x}{y}$  with  $x \in L$  and  $y = p^{i_0} v^{i_1} \dots v_{n-1}^{i_{n-1}}$  with all exponent positive, which stands for the image of  $y$  in  $L/J \subseteq N^n$  where  $J = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ . Hence  $x/y$  is annihilated by  $J$  and depends only on the mod  $J$  reduction of  $x$ . The usual rules of addition, subtraction, and cancellation of fractions apply here.

1.4.12. PROPOSITION. *Up to sign the elements  $\alpha_t^{(n)}$  (1.3.17),  $\alpha_{s/t}$  and  $\beta_{s/t}$  (1.3.19) are represented in the chromatic spectral sequence by  $v_n^t/pv_1 \dots v_{n-1} \in E_2^{n,0}$ ,  $v_1^s/p^t \in E_2^{1,0}$ , and  $v_2^s/pv_1^t \in E_2^{2,0}$ , respectively.*  $\square$

The signs here are a little tricky and come from the double complex used to prove 1.4.8 (see 5.1.18). The result suggests elements of a more complicated nature; e.g.,  $\beta_{s/i_2, i_1}$  stands for  $v_2^s/p^{i_1} v_1^{i_2}$ , with the convention that if  $i_1 = 1$  it is omitted from the notation. The first such element with  $i_1 > 1$  is  $\beta_{p^2/p, 2}$ . We also remark that some of these elements require correcting terms in their numerators; e.g.,  $(v_1^4 + 8v_1v_2)/2^4$  (but not  $v_1^4/2^4$ ) is in  $\text{Ext}^0(N^1)$  and represents  $\alpha_{4/4}$ , which corresponds to the generator  $\sigma \in \pi_*(S^0)$ .

We will describe  $E_1^{n,*}$  for  $n \leq 1$  at  $p > 2$ . For all primes  $E_1^{0,0} = \mathbf{Q}$  (concentrated in dimension 0) and  $E_1^{0,s} = 0$  for  $s > 0$ . For  $p > 2$ ,  $E_1^{1,s} = 0$  for  $s > 1$  and  $E_1^{1,1} = \mathbf{Q}/\mathbf{Z}_{(p)}$  concentrated in dimension 0, and  $E_1^{1,0}$  is trivial in dimensions not divisible by  $q = 2(p - 1) = \dim v_1$  and is generated by all elements of the form  $v_1^t/pt$  for  $t \in \mathbf{Z}$ . Hence if  $p^i$  is the largest power of  $p$  dividing  $t$ , then  $E_1^{1,0} \approx \mathbf{Z}/(p^{i+1})$  in dimension  $qt$ , and in dimension 0,  $E_1^{1,0} = \mathbf{Q}/\mathbf{Z}_{(p)}$ .

The differential  $d_1: E_1^{0,0} \rightarrow E_1^{1,0}$  is the usual map  $\mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z}_{(p)}$ . Its kernel  $\mathbf{Z}_{(p)}$  is  $\text{Ext}^0(L \otimes \mathbf{Z}_{(p)})$ . On  $E_1^{1,1} = \mathbf{Q}/\mathbf{Z}_{(p)}$  the kernel of  $d_1$  is trivial, so  $E_2^{1,1} = E_2^{0,2} = 0$  and  $\text{Ext}^2(L \otimes \mathbf{Z}_{(p)}) = E_2^{2,0}$ . On  $E_1^{1,0}$ , the kernel of  $d_1$  consists of all elements in nonnegative dimensions. Since the  $\mathbf{Q}/\mathbf{Z}_{(p)}$  in dimension 0 is hit by  $d_1$ ,  $E_2^{1,0}$  consists of the positive dimensional elements in  $E_1^{1,0}$  and this group is  $\text{Ext}^1(L \otimes \mathbf{Z}_{(p)})$ . In  $\pi_*(S^0)$  it is represented by the  $p$ -component of  $\text{im } J$ .

Now the chromatic  $E_1$ -term is periodic in the following sense. By definition,

$$M^n = \varinjlim v_n^{-1}L/J,$$

where the direct limit is over all invariant regular ideals  $J$  with  $n$  generators. For each  $J$ ,  $\text{Ext}^0(v_n^{-1}L/J)$  contains some power of  $v_n$ , say  $v_n^k$ . Then  $\text{Ext}(v_n^{-1}L/J)$  is a module over  $\mathbf{Z}_{(p)}[v_n^k, v_n^{-k}]$ , i.e., multiplication by  $v_n^k$  is an isomorphism, so we say that this  $\text{Ext}$  is  $v_n$ -periodic. Hence  $E_1^{n,*} = \text{Ext}(M^n)$  is a direct limit of such groups. We may say that an element in the Adams–Novikov spectral sequence  $E_2$ -term is  $v_n$ -periodic if it represents an element in  $E_\infty^{n,*}$  of the chromatic spectral sequence.

Hence the chromatic spectral sequence  $E_\infty$ -term is the trigraded group associated with the filtration of  $\text{Ext}(L \otimes \mathbf{Z}_{(p)})$  by  $v_n$ -periodicity. This filtration is decreasing and has an infinite number of stages in each cohomological degree. One sees this from the diagram

$$\text{Ext}^s(N^0) \leftarrow \text{Ext}^{s-1}(N^1) \leftarrow \dots \leftarrow \text{Ext}^0(N^s)$$

where  $N^0 = L \otimes \mathbf{Z}_{(p)}$ ; the filtration of  $\text{Ext}(N^0)$  is by images of the groups  $\text{Ext}(N^n)$ . This local finiteness allows us to define an increasing filtration on  $\text{Ext}(N^0)$  by  $F_i \text{Ext}^s(N^0) = \text{im } \text{Ext}^i(N^{s-i})$  for  $0 \leq i \leq s$ , and  $F_0 \text{Ext}(N)$  is the subgroup of Greek letter elements in the most general possible sense.

## 5. Unstable Homotopy Groups and the EHP Spectral Sequence

In this section we will describe the EHP sequence, which is an inductive method for computing  $\pi_{n+k}(S^n)$  beginning with our knowledge of  $\pi_*(S^1)$  (1.1.7). We will explain how the Adams vector field theorem, the Kervaire invariant problem, and the Segal conjecture are related to the unstable homotopy groups of spheres. We will not present proofs here or elsewhere in the book, nor will we pursue the topic further except in Section 3.3. We are including this survey here because no comparable exposition exists in the literature and we believe these results should be understood by more than a handful of experts. In particular, this section could serve as an introduction to Mahowald [4]. For computations at the prime 3, see Toda [8], which extends the known range for unstable 3-primary homotopy groups from 55 to 80.

The EHP sequences are the long exact sequences of homotopy groups associated with certain fibration constructed by James [1] and Toda [6]. There is a different set of fibrations for each prime  $p$ . All spaces and groups are assumed localized at the prime in question. We start with  $p = 2$ . There we have a fibration

$$(1.5.1) \quad S^n \rightarrow \Omega S^{n+1} \rightarrow \Omega S^{2n+1},$$

which gives the long exact sequence

$$(1.5.2) \quad \dots \rightarrow \pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+1}(S^{2n+1}) \xrightarrow{P} \pi_{n+k-1}(S^n) \rightarrow \dots$$

Here  $E$  stands for Einhangung (suspension),  $H$  for Hopf invariant, and  $P$  for Whitehead product. If  $n$  is odd the fibration is valid for all primes and it splits at odd primes, so for  $p > 2$  we have

$$\pi_{2m+k}(S^{2m}) = \pi_{2m+k-1}(S^{2m-1}) \oplus \pi_{2m+k}(S^{4m-1}).$$

This means that even-dimensional spheres at odd primes are uninteresting. Instead one considers the fibration

$$(1.5.3) \quad \widehat{S}^{2m} \rightarrow \Omega S^{2m+1} \rightarrow \Omega S^{2pm+1},$$

where the second map is surjective in  $H_*(\ ; \mathbf{Z}_{(p)})$ , and  $\widehat{S}^{2m}$  is the  $(2mp-1)$ -skeleton of  $\Omega S^{2m+1}$ , which is a  $CW$ -complex with  $p-1$  cells of the form  $S^{2m} \cup e^{4m} \cup \dots \cup e^{2(p-1)m}$ . The corresponding long exact sequence is

$$(1.5.4) \quad \dots \rightarrow \pi_i(\widehat{S}^{2m}) \xrightarrow{E} \pi_{i+1}(S^{2m+1}) \xrightarrow{H} \pi_{i+1}(S^{2pm+1}) \xrightarrow{P} \pi_{i-1}(\widehat{S}^{2m}) \rightarrow \dots.$$

There is also a fibration

$$(1.5.5) \quad S^{2m-1} \rightarrow \Omega \widehat{S}^{2m} \rightarrow \Omega S^{2pm-1},$$

which gives

$$(1.5.6) \quad \dots \rightarrow \pi_{i-1}(S^{2m-1}) \xrightarrow{E} \pi_i(\widehat{S}^{2m}) \xrightarrow{H} \pi_i(S^{2pm-1}) \xrightarrow{P} \pi_{i-2}(S^{2m-1}) \rightarrow \dots.$$

1.5.4 and 1.5.6 are the EHP sequences for odd primes. Note that for  $p = 2$ ,  $\widehat{S}^{2m} = S^{2m}$  and both sequences coincide with (1.5.2).

For each prime these long exact sequences fit together into an exact couple (2.1.6) and we can study the associated spectral sequence, namely

1.5.7. PROPOSITION.

(a) For  $p = 2$  there is a spectral sequence converging to  $\pi_*^S$  (stable homotopy) with

$$E_1^{k,n} = \pi_{k+n}(S^{2n-1}) \quad \text{and} \quad d_r: E_r^{k,n} \rightarrow E_r^{k-1,n-r}.$$

$E_\infty^{n,k}$  is the subquotient  $\text{im } \pi_{n+k}(S^n) / \text{im } \pi_{n+k-1}(S^{n-1})$  of  $\pi_k^S$ . There is a similar spectral sequence converging to  $\pi_*(S^j)$  with  $E_1^{k,n}$  as above for  $n \leq j$  and  $E_1^{k,n} = 0$  for  $n > j$ .

(b) For  $p > 2$  there are similar spectral sequences with

$$E_1^{k,2m+1} = \pi_{k+2m+1}(S^{2pm+1}) \quad \text{and} \quad E_1^{k,2m} = \pi_{k+2m}(S^{2pm-1}).$$

The analogous spectral sequence with  $E_1^{k,n} = 0$  for  $n > j$  converges to  $\pi_*(S^j)$  if  $j$  is odd and to  $\pi_*(\widehat{S}^j)$  if  $j$  is even.  $\square$

This is the *EHP spectral sequence*. We will explain below how it can be used to compute  $\pi_{n+k}(S^n)$  [or  $\pi_{n+k}(\widehat{S}^n)$  if  $n$  is even and  $p$  is odd] by double induction on  $n$  and  $k$ . First we make some easy general observations.

1.5.8. PROPOSITION.

(a) For all primes  $E_1^{k,1} = \pi_{1+k}(S^1)$ , which is  $\mathbf{Z}_{(p)}$  for  $k = 0$  and 0 for  $k > 0$ .

(b) For  $p = 2$ ,  $E_1^{k,n} = 0$  for  $k < n - 1$ .

(c) For  $p = 2$ ,  $E_1^{k,n} = \pi_{k-n+1}^S$  for  $k < 3n - 3$ .

(d) For  $p > 2$ ,  $E_1^{k,2m+1} = 0$  for  $k < qm$  and  $E_1^{k,2m} = 0$  for  $k < qm - 1$ , where  $q = 2(p-1)$ .

(e) For  $p > 2$ ,  $E_1^{k,2m+1} = \pi_{k-qm}^S$  for  $k < q(pm + m + 1) - 2$ , and  $E_1^{k,2m} = \pi_{k+1-qm}^S$  for  $k < q(pm + m) - 3$ .  $\square$

Part (b) follows from the connectivity of the  $(2n - 1)$ -sphere and similarly for (d); these give us a vanishing line for the spectral sequence. (c) and (e) follow from the fact that  $\pi_{2m-1+k}(S^{2m-1}) = \pi_k^S$  for  $k < 2m - 2$ , which is in turn a consequence of 1.5.7. We will refer to the region where  $n - 1 \leq k$  and  $E_1^{k,n}$  is a stable stem as the *stable zone*.

Now we will describe the inductive aspect of the EHP spectral sequence. Assume for the moment that we know how to compute differentials and solve the group extension problems. Also assume inductively that we have computed  $E_1^{i,j}$  for all  $(i, j)$  with  $i < k$  and all  $(k, j)$  for  $j > n$ . For  $p = 2$  we have  $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$ . This group is in the  $(k - n + 1)$ -stem. If  $n = 1$ , this group is  $\pi_{1+k}(S^1)$ , which is known, so assume  $n > 1$ . If  $n = 2$  this group is  $\pi_{2+k}(S^3)$ , which is 0 for  $k = 0$ ,  $\mathbf{Z}$  for  $k = 1$ , and for  $k > 1$  is the middle term in the short exact sequence

$$0 \rightarrow E_2^{k-1,2} \rightarrow \pi_{k+2}(S^3) \rightarrow \ker d_1 \subset E_2^{k-1,3} \rightarrow 0.$$

Note that  $E_2^{k-1,2}$  is the cokernel of the  $d_1$  coming from  $E_1^{k,3}$  and is therefore known by induction. Finally, if  $n > 2$ ,  $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$  can be read off from the already computed portion of the EHP spectral sequence as follows. As in 1.5.7 one obtains a spectral sequence for  $\pi_*(S^{2n-1})$  by truncating the EHP spectral sequence, i.e., by setting all  $E_1^{j,m} = 0$  for  $m > 2n - 1$ . The group  $\pi_{n+k}(S^{2n-1})$  lies in a stem which is already known, so we have  $E_1^{k,n}$ . Similar remarks apply to odd primes.

We will illustrate the method in detail for  $p = 2$  by describing what happens for  $0 \leq k \leq 7$  in FIG. 1.5.9. By 1.5.8(c) we have  $E_1^{k,k+1} = \pi_0^S = \mathbf{Z}$ . Let  $x_k$  denote the standard generator of this group. We will see below (1.5.13) that  $d_1(x_k) = 2x_{k-1}$  for even positive  $k$  and  $d_1(x_k) = 0$  otherwise. Hence  $E_2^{1,2} = E_\infty^{1,2} = \pi_1^S = \mathbf{Z}/(2)$ , so  $E_1^{k,k} = \mathbf{Z}/(2)$  for all  $k \geq 2$ . We denote the generator of each of these groups by 1 to indicate that, if the generator is a permanent cycle, it corresponds to an element whose Hopf invariant suspends to the element corresponding to  $x_1$ . Now the first such generator, that of  $E_1^{2,2}$ , is not hit by a differential, so we have  $E_1^{k,k-1} = \pi_{2k-1}(S^{2k-3}) = \mathbf{Z}/(2)$  for all  $k \geq 3$ . We denote these generators by 11, to indicate that their Hopf invariants each desuspend to elements with Hopf invariant  $x_1$ .

In general we can specify an element  $\alpha \in \pi_{n+k}(S^n)$  by a sequence of integers adding up to  $k$  as follows. Desuspend  $\alpha$  as far as possible, say to  $S^{m+1}$ . The first integer is then  $m$  (necessarily  $\leq k$ ) and the desuspended  $\alpha$  has a Hopf invariant  $\beta \in \pi_{m+1+k}(S^{2m+1})$ . To get the second integer we desuspend  $\beta$ , and so forth. After a finite number of steps we get an element with Hopf invariant in the zero stem and stop the process. Of course there is some indeterminacy in desuspending but we can ignore it for now. We call this sequence of integers the *serial number* of  $\alpha$ . In FIG. 1.5.9 we indicate each element of  $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$  by its serial number. In almost all cases if  $p\alpha \neq 0$ , its serial number differs from that of  $\alpha$  itself.

To get back to FIG. 1.5.9, we now have to determine the groups  $E_1^{k,k-t} = \pi_{2k-2}(S^{2k-5})$  for  $k \geq 4$ , which means examining the 3-stem in detail. The groups  $E_1^{3,2}$  and  $E_1^{3,3}$  are not touched by differentials, so there is an short exact sequence

$$0 \rightarrow E_1^{3,2} \rightarrow \pi_6(S^3) \rightarrow E_1^{3,3} \rightarrow 0.$$

The two end terms are  $\mathbf{Z}/(2)$  and the group extension can be shown to be nontrivial, so  $E_1^{4,2} = \pi_6(S^3) = \mathbf{Z}/(4)$ . Using the serial number notation, we denote the generator by 21 and the element of order 2 by 111. Similarly one sees  $\pi_5(S^2) =$

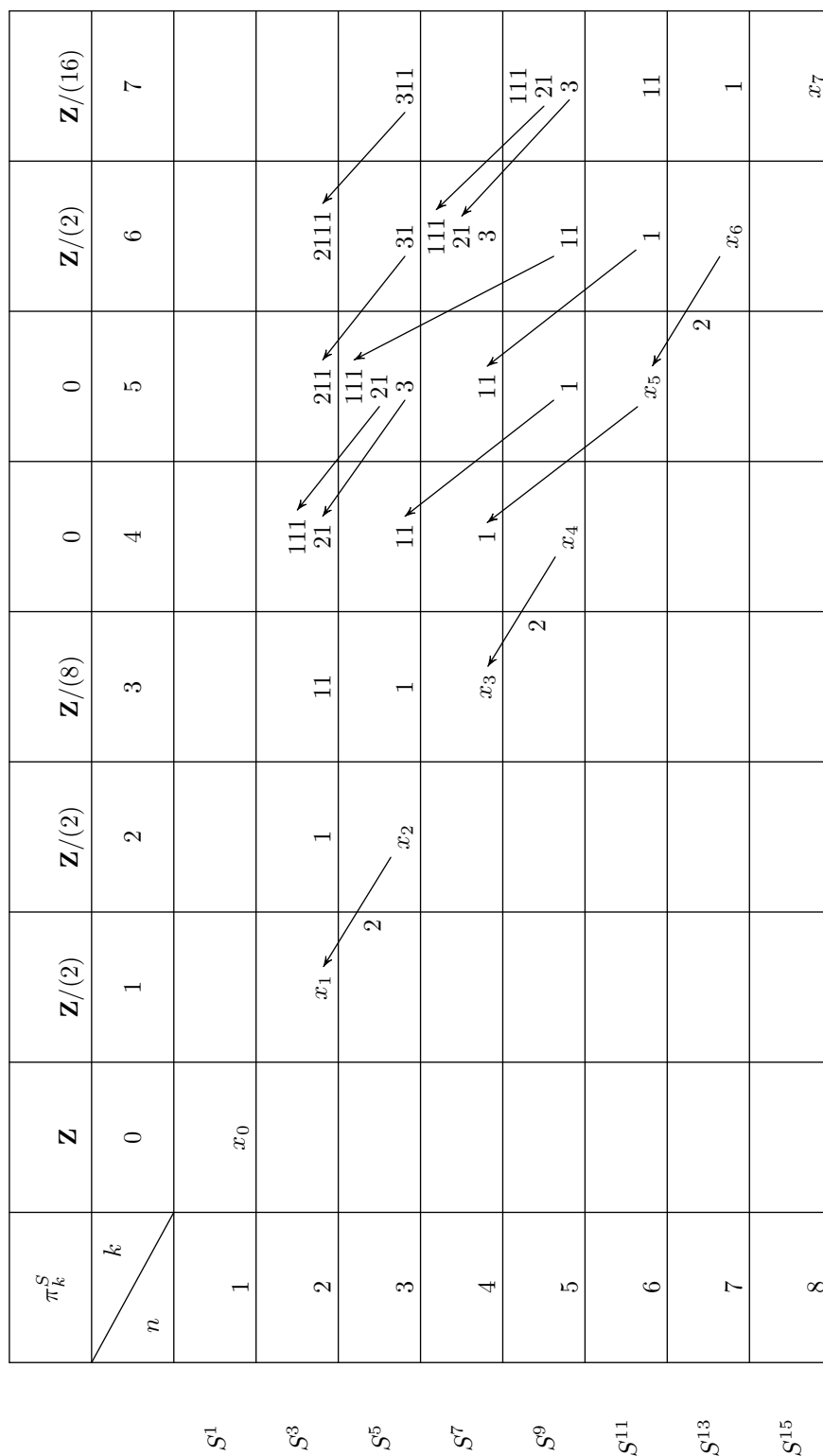


FIGURE 1.5.9. The EHP spectral sequence for  $p = 2$  and  $k \leq 7$ .

$\mathbf{Z}/(2)$ ,  $\pi_7(S^4) = \mathbf{Z} \oplus \mathbf{Z}/(4)$  and there is an short exact sequence

$$0 \rightarrow \pi_6(S^3) \rightarrow \pi_8(S^5) \rightarrow E_2^{3,4} \rightarrow 0.$$

Here the subgroup and cokernel are  $\mathbf{Z}/(4)$  and  $\mathbf{Z}/(2)$ , respectively, and the group extension is again nontrivial, so  $\pi_8(S^5) = E_1^{k,k-2} = \mathbf{Z}/(8)$  for  $k \geq 5$ . The generator of this group is the suspension of the Hopf map  $\nu: S^7 \rightarrow S^4$  and is denoted by 3.

To determine  $E_1^{k,k-3} = \pi_{2k-3}(S^{2k-7})$  for  $k \geq 5$  we need to look at the 4-stem, i.e., at the column  $E_r^{4,*}$ . The differentials affecting those groups are indicated on the chart. Hence we have  $E_2^{4,2} = 0$  so  $\pi_7(S^3) = E_1^{5,2} = \mathbf{Z}/(2)$ ; the  $d_2$  hitting  $E_1^{4,3}$  means that the corresponding element *dies* (i.e., becomes null homotopic) when suspended to  $\pi_9(S^5)$ ; since it first appears on  $S^3$  we say it is *born* there. Similarly, the generator of  $E_1^{4,4}$  corresponds to an element that is born on  $S^4$  and dies on  $S^6$  and hence shows up in  $E_1^{6,3} = \pi_9(S^5)$ . We leave it to the reader to determine the remaining groups shown in the chart, assuming the differentials are as shown.

We now turn to the problem of computing differentials and group extensions in the EHP spectral sequence. For the moment we will concentrate on the prime 2. The fibration 1.5.1 can be looped  $n$  times to give

$$\Omega^n S^n \rightarrow \Omega^{n+1} S^{n+1} \rightarrow \Omega^{n+1} S^{2n+1}.$$

In Snaith [1] a map is constructed from  $\Omega^n S^n$  to  $Q\mathbf{R}P^{n-1}$  which is compatible with the suspension map  $\Omega^n S^n \rightarrow \Omega^{n+1} S^{n+1}$ . (Here  $QX$  denotes  $\varinjlim \Omega^k \Sigma^k X$ .) Hence we get a commutative diagram

$$(1.5.10) \quad \begin{array}{ccccc} \Omega^n S^n & \longrightarrow & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathbf{R}P^{n-1} & \longrightarrow & Q\mathbf{R}P^n & \longrightarrow & QS^n \end{array}$$

where both rows are fibre sequences and the right-hand vertical map is the standard inclusion. The long exact sequence in homotopy for the bottom row leads to an exact couple and a spectral sequence as in 1.5.7. We call it the *stable EHP spectral sequence*.

There is an odd primary analog of 1.5.10 in which  $\mathbf{R}P^n$  is replaced by an appropriate skeleton of  $B\Sigma_p$ , the classifying space for the symmetric group on  $p$  letters. Recall that its mod  $(p)$  homology is given by

$$(1.5.11) \quad H_i(B\Sigma_p; \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } i \equiv 0 \text{ or } -1 \pmod{q} \\ 0 & \text{otherwise.} \end{cases}$$

1.5.12. PROPOSITION. (a) For  $p = 2$  there is a spectral sequence converging to  $\pi_*^S(\mathbf{R}P^\infty)$  (stable homotopy of  $\mathbf{R}P^\infty$ ) with  $E_1^{k,n} = \pi_{k-n+1}^S$  for  $n \geq 2$  and  $d_r: E_r^{k,n} \rightarrow E_r^{k-1, n-r}$ . Here  $E_\infty^{k,n}$  is the subquotient  $\text{im } \pi_k^S(\mathbf{R}P^{n-1}) / \text{im } \pi_k^S(\mathbf{R}P^{n-2})$  of  $\pi_k^S(\mathbf{R}P^\infty)$ . There is a similar spectral sequence converging to  $\pi_*^S(\mathbf{R}P^{j-1})$  with  $E_1^{k,n}$  as above for  $n \leq j$  and  $E_1^{k,n} = 0$  for  $n > j$ .

(b) For  $p > 2$  there is a similar spectral sequence converging to  $\pi_*^S(B\Sigma_p)$  with  $E_1^{k,2m+1} = \pi_k^S$  and  $E_1^{k,2m} = \pi_{k+1-mq}$ . There is a similar spectral sequence with  $E_1^{k,n} = 0$  for  $n > j$  converging to  $\pi_*^S(B\Sigma_p^{(q)j-1})$  if  $j$  is even and to  $\pi_*(B\Sigma_p^{(q)(j-1)})$  if  $j$  is odd.



(c) There are homomorphisms to these from the corresponding EHP spectral sequences of 1.5.7 induced by suspension on the  $E_1$  level, e.g., at  $p = 2$  by the suspension map  $\pi_{k+n}(S^{2n-1}) \rightarrow \pi_{k-n+1}^S$ . Hence the  $E_1$ -terms are isomorphic in the stable zone.

We remark that this stable EHP spectral sequence is nothing but a reindexed form of the Atiyah–Hirzebruch spectral sequence (see Adams [4], Section 7) for  $\pi_*^S(B\Sigma_p)$ . In the latter one has  $E_2^{s,t} = H_s(B\Sigma_p; \pi_t^S)$  and this group is easily seen to be  $E_2^{s+t, f(s)}$  in the EHP spectral sequence where

$$f(s) = \begin{cases} s/(p-1) + 1 & \text{if } s \equiv 0 \pmod{2p-2} \\ (s+1)/(p-1) & \text{if } s \equiv -1 \pmod{2p-2}. \end{cases}$$

Since everything in 1.5.12 is stable one can use stable homotopy theoretic methods, such as the Adams spectral sequence and  $K$ -theory, to compute differentials and group extensions. This is a major theme in Mahowald [1]. Differentials originating  $E_r^{k,k+1}$  for  $p = 2$  correspond to attaching maps in the cellular structure of  $\mathbf{R}P^\infty$ , and similarly for  $p > 2$ . For example, we have

1.5.13. PROPOSITION. In the stable EHP spectral sequence (1.5.12), the differential  $d_1: E_1^{k,n} \rightarrow E_1^{k-1,n-1}$  is multiplication by  $p$  if  $k$  is even and trivial if  $k$  is odd.  $\square$

Another useful feature of this spectral sequence is *James periodicity*: for each  $r$  there is a finite  $i$  and an isomorphism  $E_r^{k,n} \approx E_r^{k+qp^i, n+2p^i}$  which commutes with differentials (note that  $q = 2$  when  $p = 2$ ). This fact is a consequence of the vector field theorem and will be explained more fully below (1.5.18).

For  $p = 2$ , the diagram 1.5.10 can be enlarged as follows. An element in the orthogonal group  $O(n)$  gives a homeomorphism  $S^{n-1} \rightarrow S^{n-1}$ . Suspension gives a basepoint-preserving map  $S^n \rightarrow S^n$  and therefore an element in  $\Omega^n S^n$ . Hence we have a map  $J: O(n) \rightarrow \Omega^n S^n$  (compare 1.1.12). We also have the reflection map  $r: \mathbf{R}P^{n-1} \rightarrow O(n)$  sending a line through the origin in  $\mathbf{R}^n$  to the orthogonal matrix corresponding to reflection through the orthogonal hyperplane. Combining these we get

$$(1.5.14) \quad \begin{array}{ccccc} \mathbf{R}P^{n-1} & \longrightarrow & \mathbf{R}P^n & \longrightarrow & S^n \\ \downarrow & & \downarrow & & \parallel \\ O(n) & \longrightarrow & O(n+1) & \longrightarrow & S^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^n S^n & \longrightarrow & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathbf{R}P^{n-1} & \longrightarrow & Q\mathbf{R}P^n & \longrightarrow & QS^n. \end{array}$$

Here the top row is a cofiber sequence while the others are fiber sequences. The right-hand vertical maps are all suspensions, as is the composite  $\mathbf{R}P^n \rightarrow Q\mathbf{R}P^n$ . The second row leads to a spectral sequence (which we call the *orthogonal spectral sequence*) converging to  $\pi_*(O)$  which maps to the EHP spectral sequence. The map on

$E_1^{k,n} = \pi_k(S^{n-1})$  is an isomorphism for  $k < 2n - 3$  by the Freudenthal suspension theorem 1.1.10. The middle right square of this diagram only commutes after a single looping. This blemish does not affect calculations of homotopy groups.

Hence we have three spectral sequences corresponding to the three lower rows of 1.5.14 and converging to  $\pi_*(O)$ , the 2-component of  $\pi_*^S$ , and  $\pi_*^S(\mathbf{R}P^\infty)$ . In all three we have generators  $x_k \in E_1^{k,k+1} = \mathbf{Z}$  and we need to determine the first nontrivial differential (if any exists) on it for  $k$  odd. We will see that this differential always lands in the zone where all three spectral sequences are isomorphic. In the orthogonal spectral sequence  $x_k$  survives to  $E_r$  iff the projection  $O(k+1)/O(k+1-r) \rightarrow S^k$  admits a cross section. It is well known (and easy to prove) that such a cross section exists iff  $S^k$  admits  $r-1$  linearly independent tangent vector fields. The question of how many such vector fields exist is the *vector field problem*, which was solved by Adams [16] (see 1.5.16). We can give equivalent formulations of the problem in terms of the other two spectral sequences.

1.5.15. THEOREM (James [2, 3]). *The following three statements are equivalent:*

- (a)  $S^{k-1}$  admits  $r-1$  linearly independent tangent vector fields.
- (b) Let  $\iota$  be the generator of  $\pi_{2k-1}(S^{2k-1}) = \mathbf{Z}$ . Then  $P(\iota) \in \pi_{2k-3}(S^{k-1})$  (see 1.5.2) desuspend to  $\pi_{2k-r-2}(S^{k-r})$ .
- (c) The stable map  $\mathbf{R}P^{k-1}/\mathbf{R}P^{k-r} \rightarrow S^{k-1}$  admits a cross section.  $\square$

The largest possible  $r$  above depends on the largest powers of 2 dividing  $k+1$ . Let  $k = 2^j(2s+1)$ ,

$$\phi(j) = \begin{cases} 2j & \text{if } j \equiv 1 \text{ or } 2 \pmod{4} \\ 2j+1 & \text{if } j \equiv 0 \pmod{4} \\ 2j+2 & \text{if } j \equiv 3 \pmod{4} \end{cases}$$

and  $\rho(k) = \phi(j)$ .

1.5.16. THEOREM (Adams [16]).

(a) *With notation as above,  $S^{k-1}$  admits  $\rho(k) - 1$  linearly independent tangent vector fields and no more.*

(b) *Let  $\bar{\alpha}_0 = 2 \in \pi_0^S$  and for  $j > 0$  let  $\bar{\alpha}_j$  denote the generator of  $\text{im } J$  in  $\pi_{\rho(j)-1}^S$  (see 1.5.15 (c)). Then in the 2-primary EHP spectral sequence (1.5.7)  $d_{\phi(j)}(x_{k-1})$  is the (nontrivial) image of  $\bar{\alpha}_j$  in  $E_{\phi(j)}^{k-2,k-j}$ .  $\square$*

We remark that the  $\rho(k) - 1$  vector fields on  $S^k$  were constructed long ago by Hurwitz and Radon (see Eckmann [1]). Adams [16] showed that no more exist by using real  $K$ -theory to solve the problem as formulated in 1.5.15(c).

Now we turn to the odd primary analog of this problem, i.e., finding differentials on the generators  $x_{qk-1}$  of  $E_1^{qk-1,2k} = \mathbf{Z}$ . We know of no odd primary analog of the enlarged diagram 1.5.14, so we have no analogs of 1.5.15(a) or 1.5.16(a), but we still call this the *odd primary vector field problem*. The solution is

1.5.17. THEOREM (Kambe, Matsunaga and Toda [1]). *Let  $\bar{\alpha}_j$  generate  $\text{im } J \subset \pi_{qj-1}^S$  (1.1.12), let  $x_{qk-1}$  generate  $E_1^{qk-1,2k}$  in the EHP spectral sequence (1.5.7) for an odd prime  $p$  (here  $q = 2p - 2$ ), and let  $k = p^j s$  with  $s$  not divisible by  $p$ . Then  $x_{qk-1}$  lives to  $E_{2j+2}$  and  $d_{2j+2}(x_{qk-1})$  is the (nontrivial) image of  $\bar{\alpha}_{j+1}$  in  $E_{2j}^{qk-2,2k-2j-2}$ .  $\square$*

Now we will explain the James periodicity referred to above. For  $p = 2$  let  $\mathbf{R}P_m^n = \mathbf{R}P^n / \mathbf{R}P^{m-1}$  for  $m \leq n$ . There is an  $i$  depending only on  $n - m$  such that  $\mathbf{R}P_{m+2^{i+1}}^{n+2^{i+1}} \simeq \Sigma^{2^{i+1}} \mathbf{R}P_m^n$ , a fact first proved by James [3]. To prove this, let  $\lambda$  be the canonical real line bundle over  $\mathbf{R}P^{n-m}$ . Then  $\mathbf{R}P_m^n$  is the Thom space for  $m\lambda$ . The reduced bundle  $\lambda - 1$  is an element of finite order  $2i + 1$  in  $KO^*(\mathbf{R}P^{n-m})$ , so  $(2^{i+1} + m)\lambda = m\lambda + 2^{i+1}$  and the respective Thom spaces  $\mathbf{R}P_{m+2^{i+1}}^{n+2^{i+1}}$  and  $\Sigma^{2^{i+1}} \mathbf{R}P_m^n$  are equivalent. The relevant computations in  $KO^*(\mathbf{R}P^{n-m})$  are also central to the proof of the vector field theorem 1.5.16. Similar statements can be made about the odd primary case. Here one replaces  $\lambda$  by the  $\mathbf{C}^{p-1}$  bundle obtained by letting  $\Sigma_p$  act via permutation matrices on  $\mathbf{C}^p$  and splitting off the diagonal subspace on which  $\Sigma_p$  acts trivially.

For  $p = 2$  one can modify the stable EHP spectral sequence to get a spectral sequence converging to  $\pi_*(\mathbf{R}P_m^n)$  by setting  $E_1^{k,j} = 0$  for  $j < m - 1$  and  $j > n - 1$ . Clearly the  $d_r: E_r^{k,n} \rightarrow E_r^{k-1, n-r}$  in the stable EHP spectral sequence is the same as that in the spectral sequence for  $\pi_*(\mathbf{R}P_{n-r-1}^{n-1})$  and similar statements can be made for  $p > 2$ , giving us

1.5.18. JAMES PERIODICITY THEOREM. *In the stable EHP spectral sequence (1.5.12) there is an isomorphism  $E_r^{k,n} \rightarrow E_r^{k+qp^i, n+2p^i}$  commuting with  $d_r$ , where  $i = \lceil r/2 \rceil$ .*  $\square$

Note that 1.5.17 is simpler than its 2-primary analog 1.5.16(b). The same is true of the next question we shall consider, that of the general behavior of elements in  $\text{im } J$  in the EHP spectral sequence. It is ironic that most of the published work in this area, e.g., Mahowald [2, 4], is concerned exclusively with the prime 2, where the problem appears to be more difficult.

Theorem 1.5.17 describes the behavior of the elements  $x_{qk-1}$  in the odd primary EHP spectral sequence and indicates the need to consider the behavior of  $\text{im } J$ . The elements  $\alpha_j$  and their multiples occur in the stable EHP spectral sequence in the groups  $E_1^{qk-2, 2m}$  and  $E_1^{qk-1, 2m+1}$  for all  $k > m$ . To get at this question we use the spectrum  $J$ , which is the fibre of a certain map  $bu \rightarrow \Sigma^2 bu$ , where  $bu$  is the spectrum representing connective complex  $K$ -theory, i.e., the spectrum obtained by delooping the space  $\mathbf{Z} \times BU$ . There is a stable map  $S^0 \rightarrow J$  which maps  $\text{im } J \subset \pi_*^S$  isomorphically onto  $\pi_*(J)$ . The stable EHP spectral sequence, which converges to  $\pi_*^S(B\Sigma_p)$ , maps to a similar spectral sequence converging to  $J_*(B\Sigma_p) = \pi_*(J \wedge B\Sigma_p)$ . This latter spectral sequence is completely understood and gives information about the former and about the EHP spectral sequence itself.

1.5.19. THEOREM.

(a) *For each odd prime  $p$  there is a connective spectrum  $J$  and a map  $S^0 \rightarrow J$  sending the  $p$ -component of  $\text{im } J$  (1.1.12) isomorphically onto  $\pi_*(J)$ , i.e.,*

$$\pi_i(J) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } i = 0 \\ \mathbf{Z}/(p^{j+1}) & \text{if } i = qk - 1, k > 0, k = sp^j \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases}$$

(b) *There is a spectral sequence converging to  $J_*(B\Sigma_p)$  with*

$$E_1^{k, 2m+1} = \pi_{k-mq}(J) \quad \text{and} \quad E_1^{k, 2m} = \pi_{k+1-mq}(J);$$

the map  $S^0 \rightarrow J$  induces a map to this spectral sequence from the stable EHP spectral sequence of 1.5.12.

(c) The  $d_1$  in this spectral sequence is determined by 1.5.13. The resulting  $E_2$ -term has the following nontrivial groups and no other:

$$\begin{aligned} E_2^{qk-1,2k} &= \mathbf{Z}/(p) \quad \text{generated by } x_{qk-1} \text{ for } k > 0, \\ E_2^{q(k+j)-2,2k} &= \mathbf{Z}/(p) \quad \text{generated by } \bar{\alpha}_j \text{ for } k, j > 0, \end{aligned}$$

and

$$E_2^{q(k+j)-1,2k+1} = \mathbf{Z}/(p) \quad \text{generated by } \alpha_j \text{ for } k, j > 0,$$

where  $\alpha_j$  is an element of order  $p$  in  $\pi_{qj-1}(J)$ .

(d) The higher differentials are determined by 1.5.17 and the fact that all group extensions in sight are nontrivial, i.e., with  $k$  and  $j$  as in 1.5.17,

$$d_{2j+2}(x_{qk-1}) = \bar{\alpha}_{j+1} \in E_{w_{j+2}}^{qk-2,2(k-j-1)}$$

and  $d_{2j+3}$  is nontrivial on  $E_{2j+3}^{qk-1,2m+1}$  for  $j+2 < m < k$ .

(e) The resulting  $E_\infty$ -term has the following nontrivial groups and no others:  $E_\infty^{qk-2,2m}$  for  $k > m \geq k-j$  and  $E_\infty^{qk-1,2m+1}$  for  $1 \leq m \leq j+1$ . The group extensions are all nontrivial and we have for  $i > 0$

$$J_i(B\Sigma_p) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(p^j) & \text{for } i = qsp^j - 2 \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

We will sketch the proof of this theorem. We have the fibration  $J \rightarrow bu \rightarrow \Sigma^2 bu$  for which the long exact sequence of homotopy groups is known; actually  $bu$  (when localized at the odd prime  $p$ ) splits into  $p-1$  summands each equivalent to an even suspension of  $BP\langle 1 \rangle$ , where  $\pi_*(BP\langle 1 \rangle) = \mathbf{Z}_{(p)}[v_1]$  with  $\dim v_1 = q$ . It is convenient to replace the above fibration by  $J \rightarrow BP\langle 1 \rangle \rightarrow \Sigma^q BP\langle 1 \rangle$ . We also have a transfer map  $B\Sigma_p \rightarrow S_{(p)}^0$ , which is the map which Kahn and Priddy [2] show induces a surjection of homotopy groups in positive dimensions (see also Adams [15]); the same holds for  $J$ -homology groups. Let  $R$  be the cofiber of this map. One can show that  $S_{(p)}^0 \rightarrow R$  induces a monomorphism in  $BP\langle 1 \rangle$ -homology (or equivalently in  $bu$ -homology) and that  $BP\langle 1 \rangle \wedge R \simeq \bigvee_{j \geq 0} \Sigma^{qj} H\mathbf{Z}_{(p)}$ , i.e., a wedge of suspensions of integral Eilenberg–Mac Lane spectra localized at  $p$ . Smashing these two fibrations together gives us a diagram

$$(1.5.20) \quad \begin{array}{ccccc} J \wedge R & \longrightarrow & BP\langle 1 \rangle \wedge R & \longrightarrow & \Sigma^q BP\langle 1 \rangle \wedge R \\ \uparrow & & \uparrow & & \uparrow \\ J & \longrightarrow & BP\langle 1 \rangle & \xrightarrow{f} & \Sigma^q BP\langle 1 \rangle \\ \uparrow & & \uparrow & & \uparrow \\ J \wedge B\Sigma_p & \longrightarrow & BP\langle 1 \rangle \wedge B\Sigma_p & \longrightarrow & \Sigma^q BP\langle 1 \rangle \wedge B\Sigma_p \end{array}$$

in which each row and column is a cofiber sequence. The known behavior of  $\pi_*(f)$  determines that of  $\pi_*(f \wedge R)$  and enables one to compute  $\pi_*(J \wedge B\Sigma_p) = J_*(B\Sigma_p)$ . The answer, described in 1.5.19(c), essentially forces the spectral sequence of 1.5.19

to behave in the way it does. The  $E_2$ -term [1.5.19(c)] is a filtered form of  $\pi_*(BP\langle 1 \rangle \wedge B\Sigma_p) \oplus \pi_*(\Sigma^{q-1}BP\langle 1 \rangle \wedge B\Sigma_p)$ .

Corresponding statements about the EHP spectral sequence are not yet known but can most likely be proven by using methods of Mahowald [4]. We surmise they can be derived from the following.

1.5.21. CONJECTURE.

(a) *The composite  $\pi_k(\Omega^{2n+1}S^{2n+1}) \rightarrow \pi_k(QB\Sigma_p^{qn}) \rightarrow J_k(B\Sigma_p^{qn})$  is onto unless  $k = qsp^j - 2$  (with  $j > 0$ ,  $sp^j > p$  and  $p \nmid s$ ) and  $n = sp^j - i$  for  $1 \leq i \leq j$ .*

(b) *The groups  $E_\infty^{qk-1, 2m+1}$  of 1.5.19(e) pull back to the  $E_\infty$ -term of the EHP spectral sequence and correspond to the element  $\alpha_{k/m}$  (1.3.19) of order  $p^m$  in  $\text{im } J \in \pi_{qk-1}^S$ . Hence  $\alpha_{k/m}$  is born in  $S^{2m+1}$  and has Hopf invariant  $\alpha_{k-m}$  except for  $\alpha_1$ , which is born on  $\widehat{S}^2$  with Hopf invariant one. (This was not suspected when the notation was invented!)  $\square$*

We will give an example of an exception to 1.5.21(a) for  $p = 3$ . One has age  $\alpha_8 \in E_3^{39,5}$ , which should support a  $d_3$  hitting  $\bar{\alpha}_9 \in E_3^{38,2}$ , but  $E_1^{38,2} = \pi_{40}(S^5)$  and  $\alpha_9$  is only born on  $S^7$ , so the proposed  $d_3$  cannot exist (this problem does not occur in the stable EHP spectral sequence). In fact,  $\alpha_1\alpha_8 \neq 0 \in \pi_{41}(S^7) = E_1^{38,3}$  and this element is hit by a  $d_2$  supported by the  $\alpha_8 \in E_2^{39,5}$ .

The other groups in 1.5.19(e),  $J_{pq_i-2}(B\Sigma_p)$ , are harder to analyze.  $E_\infty^{pq-2, q}$  pulls back to the EHP spectral sequence and corresponds to  $\beta_1 \in \pi_{pq-2}^S$  (1.3.14), the first stable element in  $\text{coker } J$  (1.1.12), so  $\beta_1$  is born on  $\widehat{S}^q$  and has Hopf invariant  $\alpha_1$ . Presumably the corresponding generators of  $E_r^{piq-2, 2pi-2}$  for  $i > 1$  each supports a nontrivial  $d_q$  hitting a  $\beta_1$  in the appropriate group. The behavior of the remaining elements of this sort is probably determined by that of the generators of  $E_2^{p^j q-2, wp^j-2j}$  for  $j \geq 2$ , which we now denote by  $\tilde{\theta}_j$ . These appear to be closely related to the Arf invariant elements  $\theta_j = \beta_{p^{j-1}/p^{j-1}}$  (1.4.10) in  $E_2^{2, p^j q}$  of the Adams–Novikov spectral sequence. The latter are known not to survive (6.4.1), so presumably the  $\tilde{\theta}_j$  do not survive either. In particular we know  $d_{2p^2-6}(\tilde{\theta}_2) = \beta_1^p$  in the appropriate group. There are similar elements at  $p = 1$  as we shall see below. In that case the  $\theta_j$  are presumed but certainly not known (for  $j > 5$ ) to exist in  $\pi_{2^{j+1}-2}^S$ . Hence any program to prove their existence at  $p = 2$  is doomed to fail if it would also lead to a proof for  $p > 2$ .

We now consider the 2-primary analog of 1.5.19 and 1.5.21. The situation is more complicated for four reasons.

- (1)  $\text{im } J$  (1.5.15) is more complicated at  $p = 2$  than at odd primes.
- (2) The homotopy of  $J$  (which is the fiber of a certain map  $bo \rightarrow \Sigma^4 bsp$ , where  $bo$  and  $bsp$  are the spectra representing connective real and symplectic  $K$ -theory, respectively) contains more than just  $\text{im } J$ .
- (3) Certain additional exceptions have to be made in the analog 1.5.21.
- (4) The groups corresponding to the  $J_{piq-2}(B\Sigma_p)$  are more complicated and lead us to the elements  $\eta_j \in \pi_{2^j}^S$  of Mahowald [6] in addition to the hypothetical  $\theta_j \in \pi_{2^{j+1}-2}^S$ .

Our first job then is to describe  $\pi_*(J)$  and how it differs from  $\text{im } J$  as described in 1.1.12. We have  $\pi_i(bo) = \pi_{i+7}(O)$  and  $\pi_i(bsp) = \pi_{i+3}(O)$  for  $i \geq 0$  and  $\pi_*(O)$  is

described in 1.1.11, i.e.,

$$\pi_i(O) = \begin{cases} \mathbf{Z} & \text{if } i \equiv 3 \pmod{4} \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \text{ or } 1 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

The map  $bo \rightarrow \Sigma^4 bsp$  used to define  $J$  is trivial on the torsion in  $\pi_*(bo)$ , so these groups pull back to  $\pi_*(J)$ . Hence  $\pi_{8i+1}(J)$  and  $\pi_{8i+2}(J)$  for  $i \geq 1$  contain summands of order 2 not coming from  $\text{im } J$ .

1.5.22. PROPOSITION. *At  $p = 2$*

$$\pi_i(J) = \begin{cases} \mathbf{Z}/(2) & \text{if } i = 0 \\ \mathbf{Z}/(2) & \text{if } i = 1 \text{ or } 2 \\ \mathbf{Z}/(8) & \text{if } i \equiv 3 \pmod{8} \text{ and } i > 0 \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \text{ or } 2 \pmod{8} \text{ and } i \geq 8 \\ \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) & \text{if } i \equiv 1 \pmod{8} \text{ and } i \geq 9 \\ \mathbf{Z}/(2^{j+1}) & \text{if } i = 8m - 1, m \geq 1 \text{ and } 8m = 2^j(2s + 1). \end{cases}$$

Here,  $\text{im } J \subset \pi_*(J)$  consists of cyclic summands in  $\pi_i(J)$  for  $i > 0$  and  $i \equiv 7, 0, 1$  or  $3 \pmod{8}$ .  $\square$

Now we need to name certain elements in  $\pi_*(J)$ . As in 1.5.16 let  $\bar{\alpha}_j$  denote the generator of  $\text{im } J$  in dimension  $\phi(j) - 1$ , where

$$\phi(j) - 1 = \begin{cases} 2j - 1 & \text{if } j \equiv 1 \text{ or } 2 \pmod{4} \\ 2j & \text{if } j \equiv 0 \pmod{4} \\ 2j + 1 & \text{if } j \equiv 3 \pmod{4}. \end{cases}$$

We also define elements  $\alpha_j$  in  $\pi_*(J)$  of order 2 as follows.  $\alpha_1 = \eta \in \pi_1(J)$  and  $\alpha_{4k+1} \in \pi_{8k+1}(J)$  is a certain element not in  $\text{im } J$  for  $k \geq 1$ .  $\alpha_{4k+2} = \eta\alpha_{4k+1}$ ,  $\alpha_{4k+3} = \eta^2\alpha_{4k+1} = 4\bar{\alpha}_{4k+2}$ , and  $\alpha_{4k} \in \pi_{8k-1}(J)$  is an element of order 2 in that cyclic group.

1.5.23. THEOREM (Mahowald [4]). (a) *There is a spectral sequence converging to  $J_*(\mathbf{R}P^\infty)$  with  $E_1^{k,n} = \pi_{k-n+1}(J)$ ; the map  $S^0 \rightarrow J$  induces a homomorphism to this spectral sequence from the stable EHP spectral sequence of 1.5.12. (We will denote the generator of  $E_1^{k,k+1}$  by  $x_k$  and the generator of  $E_1^{k,k+1+m}$  for  $m > 0$  by the name of the corresponding element in  $\pi_m(J)$ .)*

(b) *The  $d_1$  in this spectral sequence is determined by 1.5.13. The following is a complete list of nontrivial  $d_2$ 's and  $d_3$ 's.*

For  $k \geq 1$  and  $t \geq 0$ ,  $d_2$  sends

$$\begin{aligned} x_{4k+1} &\in E_2^{4k+1,4k+2} && \text{to } \alpha_1 \\ \bar{\alpha}_{4t+3+i} &\in E_2^{4k+8+i+8t,4k+2} && \text{to } \bar{\alpha}_{4t+i} \text{ for } i = 0, 1 \\ \alpha_{4t+1} &\in E_2^{4k+2+8t,4k+2} && \text{to } \alpha_{4t+2} \\ \bar{\alpha}_{4t+4} &\in E_2^{4k+1+8t+7,4k+1} && \text{to } \bar{\alpha}_{4t+5} \end{aligned}$$

and

$$\alpha_{4t+i} \in E_2^{4k+i+8t,4k+1} \quad \text{to } \alpha_{4t+i+1} \quad \text{for } i = 1, 2.$$

For  $k \geq 1$  and  $t \geq 1$ ,  $d_3$  sends

$$\alpha_{4t} \in E_2^{4k+1+8t, 4k+3} \quad \text{to} \quad \alpha_{4t+1}$$

and

$$\bar{\alpha}_{4t+1} \in E_2^{4k+8t+1, 4k+1} \quad \text{to} \quad \bar{\alpha}_{4t+2}.$$

See FIG. 1.5.24.

(c) The resulting  $E_4$ -term is a  $\mathbf{Z}/(2)$ -vector space on the following generators for  $k \geq 1$ ,  $t \geq 0$ .

$$\begin{aligned} x_1 &\in E_4^{1,2}; \quad \bar{\alpha} \in E_4^{4,2}; \quad \alpha_{4t+i} \in E_4^{8t+i+1,2} \quad \text{for } i = 1, 2; \\ \bar{\alpha}_{4t+i} &\in E_4^{8t+i+5,2} \quad \text{for } i = 3, 4, 5; \quad \alpha_{4t+1} \in E_4^{8t+3,3}; \quad \alpha_{4t+4} \in E_4^{8t+9,3}; \\ \bar{\alpha}_{4t+4} &\in E_4^{8t+10,3}; \quad x_{4k-1} \in E_4^{4k-1,4k}; \quad \bar{\alpha}_{4t+2} \in E_4^{4k+8t+2,4k}; \\ \bar{\alpha}_{4t+3} &\in E_4^{4k+8t+6,4k}; \quad \alpha_{4t+3} \in E_4^{4k+8t+3,4k+1}; \quad \alpha_{4t+4} \in E_4^{4k+8t+7,4k+1}; \\ \alpha_{4t+2} &\in E_4^{4k+8t+3,4k+2}; \quad \bar{\alpha}_2 \in E_4^{4k+4,4k+2}; \quad \bar{\alpha}_{4t+5} \in E_4^{4k+8t+10,4k+2}; \\ \alpha_{4t+1} &\in E_4^{4k+8t+3,4k+3}; \quad \text{and} \quad \bar{\alpha}_{4t+4} \in E_4^{4k+8t+10,4k+3}. \end{aligned}$$

(d) The higher differentials are determined by 1.5.15 and the fact that most group extensions in sight are nontrivial. The resulting  $E_\infty$ -term has the following additive generators and no others for  $t \geq 0$ .

$$\begin{aligned} x_1 &\in E_\infty^{1,2}; \quad \alpha_{4t+4} \in E_\infty^{8t+9,3}; \quad \alpha_{4t+i} \in E_\infty^{8t+i+1,2} \quad \text{for } i = 1, 2; \\ \alpha_{4t+1} &\in E_\infty^{8t+3,3}; \quad x_3 \in E_\infty^{3,4}; \quad \alpha_{4t+4} \in E_\infty^{8t+11,5}; \\ \bar{\alpha}_{4t+i} &\in E_\infty^{8t+i+5,2} \quad \text{for } i = 3, 4; \quad x_7 \in E_\infty^{7,8}; \\ \alpha_{4t+4} &\in E_\infty^{8t+15,9}; \quad \alpha_{4t+i} \in E_\infty^{8t+7,8-i} \quad \text{for } i = 1, 2, 3; \\ \alpha_{2^j t + 2^j - j - 2} &\in E_\infty^{2^j + 1(1+t) - 1, *}; \quad \text{for } j \geq 3; \\ \bar{\alpha}_2 &\in E_\infty^{4t+4, 4t+2}; \quad \text{and} \quad \bar{\alpha}_j \in E_\infty^{2^j + 1(t+1) - 2, *}; \quad \text{for } j \geq 2. \end{aligned}$$

(e) For  $i > 0$

$$J_i(\mathbf{R}P^\infty) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(2) & \text{if } i \equiv 0 \pmod{4} \\ \mathbf{Z}/(2^j) & \text{if } i = 2^{j+2}s - 2 \text{ for } s \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Note that the portion of the  $E_\infty$ -term corresponding to the summand  $\pi_*(J)$  in 1.5.23(e) [i.e., all but the last two families of elements listed in 1.5.23(d)] is near the line  $n = 0$ , while that corresponding to the second summand is near the line  $n = k$ .

The proof of 1.5.23 is similar to that of 1.5.19 although the details are messier. One has fibrations  $J \rightarrow bo \rightarrow \Sigma^4 bsp$  and  $\mathbf{R}P^\infty \rightarrow S_{(2)}^0 \rightarrow R$ . We have  $R \wedge bo \simeq \bigvee_{j \geq 0} \Sigma^{4j} H\mathbf{Z}_{(2)}$  and we can get a description of  $R \wedge bsp$  from the fibration  $\Sigma^4 bo \rightarrow bsp \rightarrow H\mathbf{Z}_{(2)}$ . The  $E_4$ -term in 1.5.22 is a filtered form of  $\pi_*(\Sigma^3 bsp \wedge \mathbf{R}P^\infty) \oplus \pi_*(bo \wedge \mathbf{R}P^\infty)$ ; elements with Hopf invariants of the form  $\bar{\alpha}_j$  are in the first summand while the other generators make up the second summand. By studying the analog of 1.5.20 we can compute  $J_*(\mathbf{R}P^\infty)$  and again the answer [1.5.23(e)] forces the spectral sequence to behave the way it does.

Now we come to the analog of 1.5.21.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	$x_1$	$\alpha_1$		$\bar{\alpha}_2$				$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ $\alpha_5$	$\bar{\alpha}_6$			
3			$\alpha_1$	$\alpha_2$	$\alpha_3$			$\alpha_4$			$\bar{\alpha}_5$ $\alpha_5$	$\alpha_6$		
4			$x_3$	$\alpha_1$	$\alpha_2$					$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\alpha_5$	$\alpha_7$	$\bar{\alpha}_6$
5				$\alpha_1$	$\alpha_2$							$\bar{\alpha}_5$ $\bar{\alpha}_4$	$\alpha_6$	
6					$x_5$	$\alpha_1$	$\alpha_3$				$\alpha_4$	$\bar{\alpha}_3$	$\alpha_5$	$\bar{\alpha}_5$ $\alpha_6$
7						$\alpha_2$	$\alpha_1$					$\bar{\alpha}_4$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ $\alpha_5$
							$\alpha_1$	$\alpha_2$	$\alpha_3$				$\alpha_4$	$\bar{\alpha}_4$

FIGURE 1.5.24. A portion of the  $E_2$ -term of the spectral sequence of Theorem 1.5.23 converging to  $J_*(\mathbf{R}P^\infty)$  and showing the  $d_2$ 's and  $d_3$ 's listed in Theorem 1.5.23, part (c).



1.5.25. THEOREM (Mahowald [4]). (a) *The composite*

$$\pi_k(\Omega^{2n+k}S^{2n+1}) \rightarrow \pi_k(Q\mathbf{R}P^{2n}) \rightarrow J_k(\mathbf{R}P^{2n})$$

is onto unless  $k \equiv 0 \pmod{4}$  and  $k \leq 2n$ , or  $k \equiv 6 \pmod{8}$ . It is also onto if  $k = 2^j$  for  $j \geq 3$  or if  $k \equiv 2^j - 2 \pmod{2^{j+1}}$  and  $k \geq 2n + 8 + 2j$ . When  $k \leq 2n$  is a multiple of 4 and not a power of 2 at least 8, then the cokernel is  $\mathbf{Z}/(2)$ ; when  $k \leq 2n$  is 2 less than a multiple of 8 but not 2 less than a power of 2, then the cokernel is  $J_k(\mathbf{R}P^{2n}) = J_k(\mathbf{R}P^\infty)$ .

(b) All elements in the  $E_\infty$ -term corresponding to elements in  $\pi_*(J)$  pull back to the EHP spectral sequence except some of the  $\bar{\alpha}_{4t+i} \in E_\infty^{8t+i+5,2}$  for  $i = 3, 4$  and  $t \geq 0$ . Hence  $H(\alpha_1) = H(\bar{\alpha}_2) = H(\bar{\alpha}_3) = 1$ ,  $H(\alpha_{t+1}) = \alpha_t$ , and if  $2^i x = \alpha_{t+1}$  for  $x \in \text{im } J$  then  $H(x) = \alpha_{t-i}$ .  $\square$

Theorem 1.5.23 leads one to believe that  $H(\bar{\alpha}_{4t+i}) = \bar{\alpha}_{4t+i-1}$  for  $i = 4, 5$  and  $t \geq 0$ , and that these elements are born on  $S^2$ , but this cannot be true in all cases. If  $\bar{\alpha}_4$  were born on  $S^2$ , its Hopf invariant would be in  $\pi_{10}(S^3)$ , but this group does not contain  $\bar{\alpha}_3$ , which is born on  $S^4$ . In fact we find  $H(\bar{\alpha}_4) = \bar{\alpha}_2$ ,  $H(\bar{\alpha}_5) = \bar{\alpha}_2^2$ , and  $H(\bar{\alpha}_8)$  is an unstable element.

1.5.26. REMARK. Theorem 1.5.25(b) shows that the portion of  $\text{im } J$  generated by  $\bar{\alpha}_{4t+2}$  and  $\bar{\alpha}_{4t+3}$ , i.e., the cyclic summands of order  $\geq 8$  in dimensions  $4k - 1$ , are born on low-dimensional spheres, e.g.,  $\bar{\alpha}_{4t+2}$  is born on  $S^5$ . However, simple calculations with 1.5.14 show that the generator of  $\pi_{4k-1}(O)$  pulls back to  $\pi_{4k-1}(O(2k+1))$  and no further. Hence  $\bar{\alpha}_{4t+2} \in \pi_{qt+8}(S^5)$  is not actually in the image of the unstable  $J$ -homomorphism until it is suspended to  $S^{4t+3}$ .

Now we consider the second summand of  $J_*(\mathbf{R}P^\infty)$  of 1.5.23(e). The elements  $\bar{\alpha}_2 \in E_\infty^{4k,4k-2}$  for  $k \geq 1$  have no odd primary analog and we treat them first. The main result of Mahowald [6] says there are elements  $\eta_j \in \pi_{2j}(S^0)$  for  $j \geq 3$  with Hopf invariant  $v = \bar{\alpha}_2$ . This takes care of the case  $k = 2^{j-2}$  above.

1.5.27. THEOREM. *In the EHP spectral sequence the element  $\nu = \bar{\alpha}_2 \in E_1^{4k,4k-2}$  for  $k \geq 2$  behaves as follows (there is no such element for  $k = 1$ ).*

(a) *If  $k = 2^{j-2}$ ,  $j \geq 3$  then the element is a permanent cycle corresponding to  $\eta_j$ ; this is proved by Mahowald [6].*

(b) *If  $k = 2s + 1$  then  $d_4(\nu) = \nu^2$ .*  $\square$

1.5.28. CONJECTURE. *If  $k = (2s + 1)2^{j-2}$  with  $s > 0$  then  $d_{2j-2}(\nu) = \eta_j$ .*  $\square$

The remaining elements in 1.5.23(e) appear to be related to the famous Kervaire invariant problem (Mahowald [7], Browder [1]).

1.5.29. CONJECTURE. *In the EHP spectral sequence the elements*

$$\bar{\alpha}_j \in E_2^{2^{j+1}(t+1)-2,*} \quad \text{for } j \geq 2, t \geq 0$$

behave as follows:

(a) *If there is a framed  $(2^{j+1} - 2)$ -manifold with Kervaire invariant one then  $\bar{\alpha}_j \in E_2^{2^{j+1}-2,*}$  is a nontrivial permanent cycle corresponding to an element  $\theta_j \in \pi_{2^{j+1}-2}(S^0)$  (These elements are known (Barratt, Jones, and Mahowald [2]) to exist for  $j \geq 0$ .)*

(b) *If (a) is true then the element  $\bar{\alpha}_j \in E_2^{2^{j+1}(2s+1)-2,*}$  satisfies  $d_r(\bar{\alpha}_j) = \theta_j$  where  $r = 2^{j+1} - 1 - \dim(\bar{\alpha}_j)$ .*  $\square$

The converse of 1.5.29(a) is proved by Mahowald [4] 7.11.

Now we will describe the connection of the EHP spectral sequence with the Segal conjecture. For simplicity we will limit our remarks to the 2-primary case, although everything we say has an odd primary analog. As remarked above, the stable EHP spectral sequence (1.5.12) can be modified so as to converge to the stable homotopy of a stunted projective space. Let  $\mathbf{R}P_j = \mathbf{R}P^\infty/\mathbf{R}P_{j-1}$  for  $j > 0$ ; i.e.,  $\mathbf{R}P^j$  is the infinite-dimensional stunted projective space whose first cell is in dimension  $j$ . It is easily seen to be the Thom spectrum of the  $j$ -fold Whitney sum of the canonical line bundle over  $\mathbf{R}P^\infty$ . This bundle can be defined stably for  $j \leq 0$ , so we get Thom spectra  $\mathbf{R}P_j$  having one cell in each dimension  $\geq j$  for any integer  $j$ .

1.5.30. PROPOSITION. *For each  $j \in \mathbf{Z}$  there is a spectral sequence converging to  $\pi_*(\mathbf{R}P_j)$  with*

$$E_1^{k,n} = \begin{cases} \pi_{k-n+1}(S^0) & \text{if } n-1 \geq j \\ 0 & \text{if } n-1 < j \end{cases}$$

and  $d_r: E_r^{k,n} \rightarrow E_r^{k-1, n-r}$ . For  $j = 1$  this is the stable EHP spectral sequence of 1.5.12. If  $j < 1$  this spectral sequence maps to the stable EHP spectral sequence, the map being an isomorphism on  $E_1^{k,n}$  for  $n \geq 2$ .  $\square$

The Segal conjecture for  $\mathbf{Z}/(2)$ , first proved by Lin [1], has the following consequence.

1.5.31. THEOREM. *For each  $j < 0$  there is a map  $S^{-1} \rightarrow \mathbf{R}P_j$  such that the map  $S^{-1} \rightarrow \mathbf{R}P_{-\infty} = \varprojlim \mathbf{R}P_j$  is a homotopy equivalent after 2-adic completion of the source (the target is already 2-adically complete since  $\mathbf{R}P_j$  is for  $j$  odd). Consequently the inverse limit over  $j$  of the spectral sequences of 1.5.30 converges to the 2-component of  $\pi_*(S^{-1})$ . We will call this limit spectral sequence the superstable EHP spectral sequence.  $\square$*

Nothing like this is stated in Lin [1] even though it is an easy consequence of his results. A proof and some generalizations are given in Ravenel [4]. Notice that  $H_*(\mathbf{R}P_{-\infty}) \neq \varprojlim H_*(\mathbf{R}P_j)$ ; this is a spectacular example of the failure of homology to commute with inverse limits. Theorem 1.5.31 was first conjectured by Mahowald and was discussed by Adams [14].

Now consider the spectrum  $\mathbf{R}P_0$ . It is the Thom spectrum of the trivial bundle and is therefore  $S^0 \vee \mathbf{R}P_1$ . Hence for each  $j < 0$  there is a map  $\mathbf{R}P_j \rightarrow S^0$  which is nontrivial in mod (2) homology. The cofiber of this map for  $j = -1$  can be shown to be  $R$ , the cofiber of the map  $\mathbf{R}P_1 \rightarrow S^0$  of Kahn and Priddy [2]. The Kahn–Priddy theorem says this map is surjective in homotopy in positive dimensions. Using these facts we get

1.5.32. THEOREM. *In the spectral sequence of 1.5.30 for  $j < 0$ ,*

- (a) *no element in  $E_r^{0,k}$  supports a nontrivial differential;*
- (b) *no element in  $E_r^{1,k}$  is the target of a nontrivial differential;*
- (c) *every element of  $E_1^{0,k} = \pi_{k+1}(S^0)$  that is divisible by 2 is the target of a nontrivial  $d_1$  and every element of  $E_2^{0,k}$  for  $k > -1$  is the target of some  $d_r$  for  $r \geq 2$ ; and*
- (d) *every element in  $E_1^{1,k} = \pi_k(S^0)$  not of order 2 supports a nontrivial  $d_1$  and every element of  $E_2^{1,k}$  supports a nontrivial  $d_r$  for some  $r \geq 2$ .  $\square$*

PROOF. Parts (a) and (b) follow from the existence of maps  $S^{-1} \rightarrow \mathbf{R}P_j \rightarrow S^0$ , (c) follows from the Kahn–Priddy theorem, and (d) follows from the fact that the map  $\varprojlim \mathbf{R}P_j \rightarrow S^0$  is trivial.  $\square$

Now the spectral sequence converges to  $\pi_*(S^{-1})$ , yet 1.5.32(c) indicates that the map  $S^{-1} \rightarrow \mathbf{R}P_{-\infty}$  induces a trivial map of  $E_\infty$ -terms, except for  $E_\infty^{-1,0}$ , where it is the projection of  $\mathbf{Z}$  onto  $\mathbf{Z}/(2)$ . [Here we are using a suitably indexed, collapsing AHSS for  $\pi_*(S^{-1})$ .] This raises the following question: what element in  $E_\infty^{k,-n}$  (for some  $n > 0$ ) corresponds to a given element  $x \in \pi_k(S^{-1})$ ? The determination of  $n$  is equivalent to finding the smallest  $n$  such that the composite  $S^k \xrightarrow{x} S^{-1} \rightarrow \mathbf{R}P_{-n-1}$  is nontrivial. The Kahn–Priddy theorem tells us this composite is trivial for  $n = 0$  if  $k \geq 0$  or  $k = -1$  and  $x$  is divisible by 2; and the Segal conjecture (via 1.5.31) says the map is nontrivial for some  $n > 0$ . Now consider the cofiber sequence  $S^{-n-1} \rightarrow \mathbf{R}P_{-n-1} \rightarrow \mathbf{R}P_{-n}$ . The map from  $S^k$  to  $\mathbf{R}P_{-n}$  is trivial by assumption so we get a map from  $S^k$  to  $S^{-1-n}$ , defined modulo some indeterminacy. Hence  $x \in \pi_{k+1}(S^0)$  gives us a coset  $M(x) \subset \pi_{k+1+n}(S^0)$  which does not contain zero. We call  $M(x)$  the Mahowald invariant of  $x$ , and note that  $n$ , as well as the coset, depends on  $x$ . The invariant can be computed in some cases and appears to be very interesting. For example, we have

1.5.33. THEOREM. *Let  $\iota$  be a generator of  $\pi_0(S^0)$ . Then for each  $j > 0$ ,  $M(2^j \iota)$  contains  $\alpha_j$ , a preimage in  $\pi_*(S^0)$  of the  $\alpha_j \in \pi_*(J)$  of 1.5.23.*  $\square$

A similar result holds for odd primes. In 1.5.31 we replace the  $\mathbf{R}P_j$  by Thom spectra of certain bundles over  $B\Sigma_p$ , and  $M(p^j \iota) \ni \alpha_j$  for  $\alpha_j$ , as in 1.5.19. We also have

1.5.34. CONJECTURE.  *$M(\theta_j)$  contains  $\theta_{j+1}$  for  $\theta_j$  as in 1.5.29.*  $\square$

1.5.35. CONJECTURE. *Whenever the Greek letter elements (1.3.17)  $\alpha_j^{(n)}$  and  $\alpha_j^{(n+1)}$  exist in homotopy,  $\alpha_j^{(n+1)} \in M(\alpha_j^{(n)})$ .*  $\square$

One can mimic the definition of the Mahowald invariant in terms of the Adams spectral sequence or Adams–Novikov  $E_2$ -terms and in the latter case prove an analog of these conjectures. At  $p = 2$  one can show (in homotopy) that  $M(\alpha_1) \ni \bar{\alpha}_2$ ,  $M(\bar{\alpha}_2) \ni \bar{\alpha}_3$ , and  $M(\bar{\alpha}_3) \ni \bar{\alpha}_3^2 = \theta_3$ . This suggests using the iterated Mahowald invariant to *define* (up to indeterminacy) Greek letter elements in homotopy, and that  $\theta_j$  is a special case (namely  $\alpha_1^{(j+1)}$ ) of this definition.



## CHAPTER 2

# Setting up the Adams Spectral Sequence

In this chapter we introduce the spectral sequence that will be our main object of study. We do not intend to give a definitive account of the underlying theory, but merely to make the rest of the book intelligible. Nearly all of this material is due to Adams. The classical Adams spectral sequence [i.e., the one based on ordinary mod  $(p)$  cohomology] was first introduced in Adams [3] and a most enjoyable exposition of it can be found in Adams [7]. In Section 1 we give a fairly self-contained account of it, referring to Adams [4] only for standard facts about Moore spectra and inverse limits. We include a detailed discussion of how one extracts differentials from an exact couple and a proof of convergence.

In Section 2 we describe the Adams spectral sequence based on a generalized homology theory  $E_*$  satisfying certain assumptions (2.2.5). We rely heavily on Adams [4], referring to it for the more difficult proofs. The  $E_*$ -Adams resolutions (2.2.1) and spectral sequences (2.2.4) are defined, the  $E_2$ -term is identified, and the convergence question is settled (2.2.3). We do not give the spectral sequence in its full generality; we are only concerned with computing  $\pi_*(Y)$ , not  $[X, Y]$  for spectra  $X$  and  $Y$ . Most of the relevant algebraic theory, i.e., the study of Hopf algebroids, is developed in Appendix 1.

In Section 3 we study the pairing of Adams spectral sequences induced by a map  $\alpha: X' \wedge X'' \rightarrow X$  and the connecting homomorphism associated with a cofibration realizing a short exact sequence in  $E$ -homology. Our smash product result implies that for a ring spectrum the Adams spectral sequence is one of differential algebras. To our knowledge these are the first published proofs of these results in such generality.

Throughout this chapter and the rest of the book we assume a working knowledge of spectra and the stable homotopy category as described, for example, in the first few sections of Adams [4].

### 1. The Classical Adams Spectral Sequence

In this section we will set up the Adams spectral sequence based on ordinary mod  $(p)$  cohomology for the homotopy groups of a spectrum  $X$ . Unless otherwise stated all homology and cohomology groups will have coefficients in  $\mathbf{Z}/(p)$  for a prime number  $p$ , and  $X$  will be a connective spectrum such that  $H^*(X)$  (but not necessarily  $X$  itself) has finite type.

Recall that  $H^*(X)$  is a module over the mod  $(p)$  Steenrod algebra  $A$ , to be described explicitly in the next chapter. Our object is to prove

2.1.1. THEOREM (Adams [3]). *Let  $X$  be a spectrum as above. There is a spectral sequence*

$$E_*^{**}(X) \quad \text{with} \quad d_r: E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$$

such that

- (a)  $E_2^{s,t} = \text{Ext}_A^{s,t}(H^*(X), \mathbf{Z}/(p))$ .
- (b) if  $X$  is of finite type,  $E_\infty^{**}$  is the bigraded group associated with a certain filtration of  $\pi_*(X) \otimes \mathbf{Z}_p$ , where  $\mathbf{Z}_p$  denotes the ring of  $p$ -adic integers.  $\square$

Let  $E = H\mathbf{Z}/(p)$ , the mod  $(p)$  Eilenberg–Mac Lane spectrum. We recall some of its elementary properties.

2.1.2. PROPOSITION.

- (a)  $H_*(X) = \pi_*(E \wedge X)$ .
- (b)  $H^*(X) = [X, E]$ .
- (c)  $H^*(E) = A$ .
- (d) If  $K$  is a locally finite wedge of suspensions of  $E$ , i.e., a generalized mod  $(p)$  Eilenberg–Mac Lane spectrum, then  $\pi_*(K)$  is a graded  $\mathbf{Z}/(p)$ -vector space with one generator for each wedge summand of  $K$ . More precisely,  $\pi_*(K) = \text{Hom}_A(H^*(K), \mathbf{Z}/(p))$ .
- (e) A map from  $X$  to  $K$  is equivalent to a locally finite collection of elements in  $H^*(X)$  in the appropriate dimensions. Conversely, any locally finite collection of elements in  $H^*(X)$  determines a map to such a  $K$ .
- (f) If a locally finite collection of elements in  $H^*(X)$  generate it as an  $A$ -module, then the corresponding map  $f: X \rightarrow K$  induces a surjection in cohomology.
- (g)  $E \wedge X$  is a wedge of suspensions of  $E$  with one wedge summand for each  $\mathbf{Z}/(p)$  generator of  $H^*(X)$ .  $H^*(E \wedge X) = A \otimes H^*(X)$  and the map  $f: X \rightarrow E \wedge X$  (obtained by smashing  $X$  with the map  $S^0 \rightarrow E$ ) induces the  $A$ -module structure map  $A \otimes H^*(X) \rightarrow H^*(X)$  in cohomology. In particular  $H^*(F)$  is a surjection.  $\square$

The idea behind the Adams spectral sequence is to use maps such as those of (f) or (g) and our knowledge of  $\pi_*(K)$  or  $\pi_*(E \wedge X)$  to get information about  $\pi_*(X)$ . We enlist the aid of homological algebra to make the necessary calculations.

More specifically, we have

2.1.3. DEFINITION. A mod  $(p)$  Adams resolution  $(X_s, g_s)$  for  $X$  is a diagram

$$\begin{array}{ccccccc} X & = & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \dots \\ & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ & & K_0 & & K_1 & & K_2 & & \end{array}$$

where each  $K_s$  is a wedge of suspensions of  $E$ ,  $H^*(f_s)$  is onto and  $X_{s+1}$  is the fiber of  $f_s$ .  $\square$

Proposition 2.1.2(f) and (g) enable us to construct such resolutions for any  $X$ , e.g., by setting  $K_s = E \wedge X_s$ . Since  $H^*(f_s)$  is onto we have short exact sequences

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(\Sigma X_{s+1}) \leftarrow 0.$$

We can splice these together to obtain a long exact sequence

$$(2.1.4) \quad 0 \leftarrow H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(\Sigma K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \dots$$

Since the maps are  $A$ -module homomorphisms and each  $H^*(K_s)$  is free over  $A_p$ , 2.1.4 is a free  $A$ -resolution of  $H^*(X)$ .

Unfortunately, the relation of  $\pi_*(K_s)$  to  $\pi_*(X)$  is not as simple as that between the corresponding cohomology groups. Life would be very simple if we knew  $\pi_*(f_s)$

was onto, but in general it is not. We have instead long exact sequences

$$(2.1.5) \quad \begin{array}{ccccc} \pi_*(X_{s+1}) & \xrightarrow{\pi_*(g_s)} & \pi_*(X_s) & \xrightarrow{\pi_*(f_s)} & \pi_*(K_s) \\ & & \uparrow & & \downarrow \\ & & \xrightarrow{\partial_{s,*}} & & \end{array}$$

arising from the fibrations

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s.$$

If we regard  $\pi_*(X_s)$  and  $\pi_*(K_s)$  for all  $s$  as bigraded abelian groups  $D_1$  and  $E_1$ , respectively [i.e.,  $D_1^{s,t} = \pi_{t-s}(X_s)$  and  $E_1^{s,t} = \pi_{t-s}(K_s)$ ] then 2.1.5 becomes

$$(2.1.6) \quad \begin{array}{ccc} D_1 & \xrightarrow{i_1} & D_1 \\ & \swarrow k_1 & \searrow j_1 \\ & E_1 & \end{array}$$

where

$$\begin{aligned} i_1 &= \pi_{t-s}(g_s): D_1^{s+1,t+1} \rightarrow D_1^{s,t}, \\ j_1 &= \pi_{t-s}(f_s): D_1^{s,t} \rightarrow E_1^{s,t}, \end{aligned}$$

and

$$k_1 = \partial_{s,t-s}: E_1^{s,t} \rightarrow D_1^{s+1,t}.$$

The exactness of 2.1.5 translates to  $\ker i_1 = \text{im } k_1$ ,  $\ker j_1 = \text{im } i_1$ , and  $\ker k_1 = \text{im } j_1$ . A diagram such as 2.1.6 is known as an *exact couple*. It is standard homological algebra that an exact couple leads one to a spectral sequence; accounts of this theory can be found in Cartan and Eilenberg [1, Section XV.7], Mac Lane [1, Section XI.5], and Hilton and Stammbach [1, Chapter 8] as well as Massey [2].

Briefly,  $d_1 = j_1 k_1: E_1^{s,t} \rightarrow E_1^{s+1,t}$  has  $(d_1)^2 = j_1 k_1 j_1 k_1 = 0$  so  $(E_1, d_1)$  is a complex and we define  $E_2 = H(E_1, d_1)$ . We get another exact couple, called the derived couple,

$$(2.1.7) \quad \begin{array}{ccc} D_2 & \xrightarrow{i_2} & D_2 \\ & \swarrow k_2 & \searrow j_2 \\ & E_2 & \end{array}$$

where  $D_2^{s,t} = i_1 D_1^{s,t}$ ,  $i_2$  is induced by  $i_1$ ,  $j_2(i_1 d) = j_1 d$  for  $d \in D_1$ , and  $k_2(e) = k_1(e)$  for  $e \in \ker d, \subset E_1$ . Since 2.1.7 is also an exact couple (this is provable by a diagram chase), we can take its derived couple, and iterating the procedure gives a sequence of exact couples

$$\begin{array}{ccc} D_r & \xrightarrow{i_r} & D_r \\ & \swarrow k_r & \searrow j_r \\ & E_r & \end{array}$$

where  $D_{r+1} = i_r D_r$ ,  $d_r = j_r k_r$ , and  $E_{r+1} = H(E_r, d_r)$ . The sequences of complexes  $\{(E_r, d_r)\}$  constitutes a spectral sequence. A close examination of the indices will





$\varprojlim X_i$ , is the fiber of the map

$$g: \prod X_i \rightarrow \prod X_i$$

whose  $i$ th component is the difference between the projection  $p_i: \prod X_j \rightarrow X_i$  and the composite

$$\prod X_j \xrightarrow{p_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} X_i. \quad \square$$

For the existence of products in the stable category see 3.13 of Adams [4]. This  $\varprojlim$  is not a categorical inverse limit (Mac Lane [1, Section III.4] because a compatible collection of maps to the  $X_i$ , does not give a unique map to  $\varprojlim X_i$ . For this reason some authors (e.g., Bousfield and Kan [1]) denote it instead by  $\text{holim}$ . The same can be said of the direct limit, which can be defined as the cofiber of the appropriate self-map of the coproduct of the spectra in question. However this  $\varprojlim$  has most of the properties one would like, such as the following.

2.1.11. LEMMA. *Given spectra  $X_{i,j}$  for  $i, j \geq 0$  and maps  $f: X_{i,j} \rightarrow X_{i-1,j}$  and  $g: X_{i,j} \rightarrow X_{i,j-1}$  such that  $fg$  is homotopic to  $gf$ ,*

$$\varprojlim_i \varprojlim_j X_{i,j} = \varprojlim_j \varprojlim_i X_{i,j}.$$

PROOF. We have for each  $i$  a cofiber sequence

$$\varprojlim_j X_{i,j} \rightarrow \prod_j X_{i,j} \rightarrow \prod_j X_{i,j}.$$

Next we need to know that products preserve cofiber sequences. For this fact, recall that the product of spectra  $\prod Y_i$ , is defined via Brown's representability theorem (Adams [4], Theorem 3.12) as the spectrum representing the functor  $\prod[-, Y_i]$ . Hence the statement follows from the fact that a product (although not the inverse limit) of exact sequences is again exact.

Hence we get the following homotopy commutative diagram in which both rows and columns are cofiber sequences.

$$\begin{array}{ccccc} \varprojlim_i \varprojlim_j X_{i,j} & \longrightarrow & \varprojlim_i \prod_j X_{i,j} & \longrightarrow & \varprojlim_i \prod_j X_{i,j} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i \varprojlim_j X_{i,j} & \longrightarrow & \prod_i \prod_j X_{i,j} & \longrightarrow & \prod_i \prod_j X_{i,j} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i \varprojlim_j X_{i,j} & \longrightarrow & \prod_i \prod_j X_{i,j} & \longrightarrow & \prod_i \prod_j X_{i,j} \end{array}$$

Everything in sight is determined by the two self-maps of  $\prod_i \prod_j X_{i,j}$  and the homotopy that makes them commute. Since the product is categorical we have  $\prod_i \prod_j X_{i,j} = \prod_j \prod_i X_{i,j}$ . It follows that  $\prod_i \varprojlim_j X_{i,j} = \varprojlim_j \prod_i X_{i,j}$  because they are each the fiber of the same map.

Similarly

$$\prod_j \varprojlim_i X_{i,j} = \varprojlim_i \prod_j X_{i,j}$$

so one gets an equivalent diagram with  $\varprojlim_j \varprojlim_i X_{i,j}$  in the upper left corner.  $\square$

Now we will show that for suitable  $X$ ,  $E_\infty^{s,t}$  is a certain subquotient of  $\pi_u(X)$ .

2.1.12. LEMMA. *Let  $X$  be a spectrum with an Adams resolution  $(X_s, g_s)$  such that  $\varprojlim X_s = \text{pt}$ . Then  $E_\infty^{s,t}$  is the subquotient  $\text{im } \pi_u(X_s) / \text{im } \pi_u(X_{s+1})$  of  $\pi_u(X)$  and  $\bigcap \text{im } \pi_*(X_s) = 0$ .*

PROOF. For the triviality of the intersection we have  $\varprojlim \pi_*(X_s) = 0$  since  $\varprojlim X_s = \text{pt}$ . Let  $G_s = \pi_*(X_s)$  and

$$G_s^t = \begin{cases} G_s & \text{if } s \geq t \\ \text{im } G_t < G_s & \text{if } t \geq s. \end{cases}$$

We have injections  $G_s^t \rightarrow G_s^{t-1}$  and surjections  $G_s^t \rightarrow G_{s-1}^t$ , so  $\varprojlim_t G_s^t = \bigcap_t G_s^t$  and  $\varprojlim_s G_s^t = G_t$ . We are trying to show  $\varprojlim_t G_0^t = 0$ .  $\varprojlim_t G_s^t$  maps onto  $\varprojlim_t G_{s-1}^t$ , so  $\varprojlim_s \varprojlim_t G_s^t$  maps onto  $\varprojlim_t G_0^t$ . But  $\varprojlim_s \varprojlim_t G_s^t = \varprojlim_t \varprojlim_s G_s^t = \varprojlim_t G_t = 0$ .

For the identification of  $E_\infty^{s,t}$ , let  $0 \neq [x] \in E_\infty^{s,t}$ .

First we show  $\partial_{s,u}(x) = 0$ . Since  $d_r[x] = 0$ ,  $\partial_{s,u}(x)$  can be lifted to  $\pi_{u-1}(X_{s+r+1})$  for each  $r$ . It follows that  $\partial_{s,u}(x) \in \text{im } \varprojlim \pi_{u-1}(X_{s+r}) = 0$ , so  $\partial_{s,u}(x) = 0$ .

Hence we have  $x = \pi_u(f_s)(y)$  for  $y \in \pi_u(X_s)$ . It suffices to show that  $y$  has a nontrivial image in  $\pi_u(X)$ . If not, let  $r$  be the largest integer such that  $y$  has a nontrivial image  $z \in \pi_u(X_{s-r+1})$ . Then  $z = \partial_{s-r,u}(w)$  for  $w \in \pi_u(X_{s-r})$  and  $\partial_r[w] = [x]$ , contradicting the nontriviality of  $[x]$ .  $\square$

Now we prove 2.1.1(a), the identification of the  $E_2$ -term.

By 2.1.2(d),  $E_1^{s,t} = \text{Hom}_A(H^{t-s}(K_s), \mathbf{Z}/(p))$ . Hence applying  $\text{Hom}_A(-, \mathbf{Z}/(p))$  to 2.1.4 gives a complex

$$E_1^{0,t} \xrightarrow{\delta} E_1^{1,t} \xrightarrow{\delta} E_1^{2,t} \rightarrow \dots$$

The cohomology of this complex is by definition the indicated Ext group. It is straightforward to identify the coboundary  $\delta$  with the  $d_1$  in the spectral sequence and 2.1.1(a) follows.

2.1.13. COROLLARY. *If  $f: X \rightarrow Y$  induces an isomorphism in  $\text{mod } (p)$  homology then it induces an isomorphism (from  $E_2$  onward) in the  $\text{mod } (p)$  Adams spectral sequence.*  $\square$

2.1.14. DEFINITION. *Let  $G$  be an abelian group and  $X$  a spectrum. Then  $XG = X \wedge SG$ , where  $SG$  is the Moore spectrum associated with  $G$  (Adams [4], p. 200]. Let  $\widehat{X} = X\mathbf{Z}_p$  (the  $p$ -adic completion of  $X$ ), where  $\mathbf{Z}_p$  is the  $p$ -adic integers, and  $X^m = X\mathbf{Z}/(p^m)$ .*  $\square$

2.1.15. LEMMA. (a) *The map  $X \rightarrow \widehat{X}$  induces an isomorphism of  $\text{mod } (p)$  Adams spectral sequences.*

(b)  $\pi_*(\widehat{X}) = \pi_*(X) \otimes \mathbf{Z}_p$ .

(c)  $\widehat{X} = \varprojlim X^m$ , if  $x$  has finite type.

PROOF. For (a) it suffices by 2.1.11 to show that the map induces an isomorphism in  $\text{mod } (p)$  homology. For this see Adams [4], proposition 6.7, which also shows (b).

Part (c) does not follow immediately from the fact that  $S\mathbf{Z}_p = \varprojlim SZ/(p^m)$  because inverse limits do not in general commute with smash products. Indeed our assertion would be false for  $X = S\mathbf{Q}$ , but we are assuming that  $X$  has finite type.

By 2.1.10 there is a cofibration

$$S\mathbf{Z}_p \rightarrow \prod_m SZ/(p^m) \rightarrow \prod_m SZ/(p^m),$$

so it suffices to show that

$$X \wedge \prod SZ/(p^m) \simeq \prod XZ/(p^m).$$

This is a special case (with  $X = E$  and  $R = \mathbf{Z}$ ) of Theorem 15.2 of Adams [4].  $\square$

2.1.16. LEMMA. *If  $X$  is a connective spectrum with each  $\pi_i(X)$  a finite  $p$ -group, then for any mod  $(p)$  Adams resolution  $(X_s, g_s)$  of  $X$ ,  $\varprojlim X_s = \text{pt}$ .*

PROOF. Construct a diagram

$$X = X'_0 \leftarrow X'_1 \leftarrow X'_2 \leftarrow \cdots$$

(not an Adams resolution) by letting  $X'_{s+1}$  be the fiber in

$$X'_{s+1} \rightarrow X'_s \rightarrow K_s,$$

where the right-hand map corresponds [2.1.2(e)] to a basis for the bottom cohomology group of  $X_s$ . Then the finiteness of  $\pi_i(X)$  implies that for each  $i$ ,  $\pi_i(X'_s) = 0$  for large  $s$ . Moreover,  $\pi_*(X'_{s+1}) \rightarrow \pi_*(X'_s)$  is monomorphic so  $\varprojlim X'_s = \text{pt}$ .

Now if  $(X_s, g_s)$  is an Adams resolution, the triviality of  $g_s$  in cohomology enables us to construct compatible maps  $X_s \rightarrow X'_s$ . It follows that the map  $\varprojlim \pi_*(X_s) \rightarrow \pi_*(X)$  is trivial. Each  $X_s$  also satisfies the hypotheses of the lemma, so we conclude that  $\varprojlim \pi_*(X_s)$  has trivial image in each  $\pi_*(X_s)$  and is therefore trivial. Since  $\pi_i(X_s)$  is finite for all  $i$  and  $s$ ,  $\varprojlim^1 \pi_*(X_s) = 0$  so  $\varprojlim X_s = \text{pt}$ .  $\square$

We are now ready to prove 2.1.1(b), i.e., to identify the  $E_\infty$ -term. By 2.1.15(a) it suffices to replace  $X$  by  $\widehat{X}$ . Note that since  $S\mathbf{Z}_p \wedge SZ/(p^m) = SZ/(p^m)$ ,  $X^m = \widehat{X}^m$ . It follows that given a mod  $(p)$  Adams resolution  $(X_s, g_s)$  for  $X$ , smashing with  $S\mathbf{Z}_p$  and  $SZ/(p^m)$  gives resolutions  $(\widehat{X}_s, \widehat{g}_s)$  and  $(X_s^m, g_s^m)$  for  $\widehat{X}$  and  $X^m$ , respectively. Moreover,  $X^m$  satisfies 2.1.16 so  $\varprojlim_s X_s^m = \text{pt}$ . Applying 2.1.15(c) to each  $X_s$ , we get  $\widehat{X}_s = \varprojlim_m X_s^m$ , so

$$\begin{aligned} \varprojlim_s X_s &= \varprojlim_s \varprojlim_m X_s^m \\ &= \varprojlim_m \varprojlim_s X_s^m \quad \text{by 2.1.11} \\ &= \text{pt}. \end{aligned}$$

Hence the result follows from 2.1.12.  $\square$

2.1.17. REMARK. The  $E_\infty$  term only gives us a series of subquotients of  $\pi_*(X) \otimes \mathbf{Z}_p$ , not the group itself. After computing  $E_\infty$  one may have to use other methods to solve the extension problem and recover the group.

We close this section with some examples.

2.1.18. EXAMPLE. Let  $X = H\mathbf{Z}$ , the integral Eilenberg–Mac Lane spectrum. The fundamental cohomology class gives a map  $f: X \rightarrow E$  with  $H^*(f)$  surjective. The fiber of  $f$  is also  $X$ , the inclusion map  $g: X \rightarrow X$  having degree  $p$ . Hence we get an Adams resolution (2.1.3) with  $X_s = X$  and  $K_s = E$  for all  $s$ , the map  $X = X_s \rightarrow X_0 = X$ , having degree  $p^s$ . We have then

$$E_1^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

There is no room for nontrivial differentials so the spectral sequence collapses, i.e.,  $E_\infty = E_1$ . We have  $E_\infty^{s,s} = \mathbf{Z}/(p) = \text{im } \pi_0(X_s) / \text{im } \pi_0(X_{s+1})$ . In this case  $\widehat{X} = H\mathbf{Z}_p$ , the Eilenberg–Mac Lane spectrum for  $\mathbf{Z}_p$ .

2.1.19. EXAMPLE. Let  $X = H\mathbf{Z}/(p^i)$  with  $i > 1$ . It is known that  $H^*(X) = H^*(Y) \oplus \Sigma H^*(Y)$  as  $A$ -modules, where  $Y = H\mathbf{Z}$ . This splitting arises from the two right-hand maps in the cofiber sequence

$$Y \rightarrow Y \rightarrow X \rightarrow \Sigma Y,$$

where the left-hand map has degree  $p^i$ . Since the  $E_2$ -term of the Adams spectral sequence depends only on  $H^*(X)$  as an  $A$ -module, the former will enjoy a similar splitting. In the previous example we effectively showed that

$$\text{Ext}_A^{s,t}(H^*(Y), \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

It follows that in the spectral sequence for  $X$  we have

$$E_2^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t - s = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

In order to give the correct answer we must have  $E_\infty^{s,t} = 0$  if  $t - s = 1$  and  $E_\infty^{s,t} = 0$  if  $t = s$  for all but  $i$  values of  $s$ . Multiplicative properties of the spectral sequence to be discussed in Section 3 imply that the only way we can arrive at a suitable  $E_\infty$  term is to have  $d_i: E_i^{s,s+1} \rightarrow E_i^{s+i,s+i}$  nontrivial for all  $s \geq 0$ . A similar conclusion can be drawn by chasing the relevant diagrams.

2.1.20. EXAMPLE. Let  $X$  be the fiber in  $X \rightarrow \widehat{S}^0 \rightarrow H\mathbf{Z}_p$  where the right-hand map is the fundamental integral cohomology class on  $S^0$ . Smashing the above fibration with  $X$  we get

$$X \wedge X \xrightarrow{g_0} X \xrightarrow{f_0} X \wedge H\mathbf{Z}$$

It is known that the integral homology of  $X$  has exponent  $p$ , so  $X \wedge H\mathbf{Z}$  is a wedge of  $E$  and  $H^*(f_0)$  is surjective. Similar statements hold after smashing with  $X$  any number of times, so we get an Adams resolution (2.1.3) with  $K_s = X_s \wedge H\mathbf{Z}$  and  $X_s = X^{(s+1)}$ , the  $(s+1)$ -fold smash product of  $X$  with itself, i.e., one of the form

$$\begin{array}{ccccccc} X & \longleftarrow & X \wedge X & \longleftarrow & X \wedge X \wedge X & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X \wedge H\mathbf{Z} & & X \wedge X \wedge H\mathbf{Z} & & X \wedge X \wedge X \wedge H\mathbf{Z} & & \end{array}$$

Since  $X$  is  $(2p-4)$ -connected  $X_s$  is  $((s+1)(2p-3)-1)$ -connected, so  $\varprojlim X_s$  is contractible.

## 2. The Adams Spectral Sequence Based on a Generalized Homology Theory

In this section we will define a spectral sequence similar to that of 2.1.1 (the classical Adams spectral sequence) in which the mod  $(p)$  Eilenberg–Mac Lane spectrum is replaced by some more general spectrum  $E$ . The main example we have in mind is of course  $E = BP$ , the Brown–Peterson spectrum, to be defined in 4.1.12. The basic reference for this material is Adams [4] (especially Section 15, which includes the requisite preliminaries on the stable homotopy category).

Our spectral sequence should have the two essential properties of the classical one: it converges to  $\pi_*(X)$  localized or completed at  $p$  and its  $E_2$ -term is a functor of  $E^*(X)$  (the generalized cohomology of  $X$ ) as a module over the algebra of cohomology operations  $E^*(E)$ ; i.e., the  $E_2$ -term should be computable in some homological way, as in 2.1.1. Experience has shown that with regard to the second property we should dualize and consider instead  $E_*(X)$  (the generalized homology of  $X$ ) as a comodule over  $E_*(E)$  (sometimes referred to as the coalgebra of cooperations). In the classical case, i.e., when  $E = H\mathbf{Z}/(p)$ ,  $E_*(E)$  is the dual Steenrod algebra  $A_*$ .

Theorem 2.1.1(a) can be reformulated as  $E_2 = \text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X))$  using the definition of  $\text{Ext}$  in the category of comodules given in A1.2.3. In the case  $E = BP$  substantial technical problems can be avoided by using homology instead of cohomology. Further discussion of this point can be found in Adams [6, pp. 51–55].

Let us assume for the moment that we have known enough about  $E$  and  $E_*(E)$  to say that  $E_*(X)$  is a comodule over  $E_*(E)$  and we have a suitable definition of  $\text{Ext}_{E_*(E)}(E_*(S^0), E_*(X))$ , which we abbreviate as  $\text{Ext}(E_*(X))$ . Then we might proceed as follows.

2.2.1. DEFINITION. *An  $E_*$ -Adams resolution for  $X$  is a diagram*

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{\quad} & \cdots \\ & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 \\ & & K_0 & & K_1 & & K_2 \end{array}$$

such that for all  $s \geq 0$  the following conditions hold.

- (a)  $X_{s+1}$  is the fiber of  $f_s$ .
- (b)  $E \wedge X_s$  is a retract of  $E \wedge K_s$ , i.e., there is a map  $h_s: E \wedge K_s \rightarrow E \wedge X_s$  such that  $h_s(E \wedge f_s)$  is an identity map of  $E \wedge X_s$ . particular  $E_*(f_s)$  is a monomorphism.
- (c)  $K_s$  is a retract of  $E \wedge K_s$ .
- (d)

$$\text{Ext}^{t,u}(E_*(K_s)) = \begin{cases} \pi_u(K_s) & \text{if } t = 0 \\ 0 & \text{if } t > 0. \end{cases} \quad \square$$

As we will see below, conditions (b) and (c) are necessary to insure that the spectral sequence is natural, while (d) is needed to give the desired  $E_2$ -term. As before it is convenient to consider a spectrum with the following properties.

2.2.2. DEFINITION. *An  $E$ -completion  $\widehat{X}$  of  $X$  is a spectrum such that*

- (a) There is a map  $X \rightarrow \widehat{X}$  inducing an isomorphism in  $E_*$ -homology.
- (b)  $\widehat{X}$  has an  $E_*$ -Adams resolution  $\{\widehat{X}_s\}$  with  $\varprojlim \widehat{X}_s = \text{pt}$ . □

This is not necessarily the same as the  $\widehat{X}$  of 2.1.14, which will be denoted in this section by  $X_p$  (2.2.12). Of course, the existence of such a spectrum (2.2.13) is not obvious and we will not give a proof here. Assuming it, we can state the main result of this section.

2.2.3. THEOREM (Adams [4]). *An  $E_*$ -Adams resolution for  $X$  (2.2.1) leads to a natural spectral sequence  $E_*^{**}(X)$  with  $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$  such that*

(a)  $E_2^{s,t} = \text{Ext}(E_*(X))$ .

(b)  $E_\infty^{**}$  is the bigraded group associated with a certain filtration of  $\pi_*(\widehat{X})$ , in other words, the spectral sequence converges to the latter. (This filtration will be described in 2.2.14.)

2.2.4. DEFINITION. *The spectral sequence of 2.2.3 is the Adams spectral sequence for  $X$  based on  $E$ -homology.*  $\square$

2.2.5. ASSUMPTION. We now list the assumptions on  $E$  which will enable us to define  $\text{Ext}$  and  $\widehat{X}$ .

(a)  $E$  is a commutative associative ring spectrum.

(b)  $E$  is *connective*, i.e.,  $\pi_r(E) = 0$  for  $r < 0$ .

(c) The map  $\mu_*: \pi_0(E) \otimes \pi_0(E) \rightarrow \pi_0(E)$  induced by the multiplication  $\mu: E \wedge E \rightarrow E$  is an isomorphism.

(d)  $E$  is *flat*, i.e.,  $E_*(E)$  is flat as a left module over  $\pi_*(E)$ .

(e) Let  $\theta: \mathbf{Z} \rightarrow \pi_0(E)$  be the unique ring homomorphism, and let  $R \subset \mathbf{Q}$  be the largest subring to which  $\theta$  extends. Then  $H_r(E; R)$  is finitely generated over  $R$  for all  $r$ .

2.2.6. PROPOSITION.  *$H\mathbf{Z}/(p)$  and  $BP$  satisfy 2.2.5(a)–(e)*  $\square$

The flatness condition 2.2.5(d) is only necessary for identifying  $E_2^{**}$  as an  $\text{Ext}$ . Without it one still has a spectral sequence with the specified convergence properties. Some well-known spectra which satisfy the remaining conditions are  $H\mathbf{Z}$ ,  $bo$ ,  $bu$ , and  $MSU$ . In these cases  $E \wedge E$  is not a wedge of suspensions of  $E$  as it is when  $E = H\mathbf{Z}/(p)$ ,  $BP$ , or  $MU$ .  $H\mathbf{Z} \wedge H\mathbf{Z}$  is known to be a certain wedge of suspensions of  $H\mathbf{Z}/(p)$  and  $H\mathbf{Z}$ ,  $bo \wedge bo$  is described by Milgram [1],  $bu \wedge bu$  by Adams [4], Section 17, and  $MSU \wedge MSU$  by Pengelley [1].

We now turn to the definition of  $\text{Ext}$ . It follows from our assumptions 2.2.5 that  $E_*(E)$  is a ring which is flat as a left  $\pi_*(E)$  module. Moreover,  $E_*(E)$  is a  $\pi_*(E)$  bimodule, the right and left module structures being induced by the maps

$$E = S^0 \wedge E \rightarrow E \wedge E \quad \text{and} \quad E = E \wedge S^0 \rightarrow E \wedge E,$$

respectively. In the case  $E = H\mathbf{Z}/(p)$  these two module structures are identical, but not when  $E = BP$ . Following Adams [4], Section 12, let  $\mu: E \wedge E$  be the multiplication on  $E$  and consider the map

$$(E \wedge E) \wedge (E \wedge X) \xrightarrow{1 \wedge \mu \wedge 1} E \wedge E \wedge X.$$

2.2.7. LEMMA. *The above map induces an isomorphism*

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow \pi_*(E \wedge E \wedge X).$$

PROOF. The result is trivial for  $X = S^n$ . It follows for  $X$  finite by induction on the number of cells using the 5-lemma, and for arbitrary  $X$  by passing to direct limits.  $\square$

Now the map

$$E \wedge X = E \wedge S^0 \wedge X \rightarrow E \wedge E \wedge X$$

induces

$$\psi: E_*(X) \rightarrow \pi_*(E \wedge E \wedge X) = E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

In particular, if  $X = E$  we get

$$\Delta: E_*(E) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(E).$$

Thus  $E_*(E)$  is a coalgebra over  $\pi_*(E)$  as well as an algebra, and  $E_*(X)$  is a comodule over  $E_*(E)$ . One would like to say that  $E_*(E)$ , like the dual Steenrod algebra, is a commutative Hopf algebra, but that would be incorrect since one uses the bimodule structure in the tensor product  $E_*(E) \otimes_{\pi_*(E)} E_*(E)$  (i.e., the product is with respect to the right module structure on the first factor and the left module structure on the second). In addition to the coproduct  $\Delta$  and algebra structure, it has a right and left unit  $\eta_R, \eta_L: \pi_*(E) \rightarrow E_*(E)$  corresponding to the two module structures, a counit  $\varepsilon: E_*(E) \rightarrow \pi_*(E)$  induced by  $\mu: E \wedge E \rightarrow E$ , and a conjugation  $c: E_*(E) \rightarrow E_*(E)$  induced by interchange the factors in  $E \wedge E$ .

**2.2.8. PROPOSITION.** *With the above structure maps  $(\pi_*(E), E_*(E))$  is a Hopf algebroid (A1.1.1), and  $E$ -homology is a functor to the category of left  $E_*(E)$ -comodules (A1.1.2), which is abelian (A1.1.3).  $\square$*

The problem of computing the relevant Ext groups is discussed in Appendix 1, where an explicit complex (the cobar complex A1.2.11) for doing so is given. This complex can be realized geometrically by the canonical  $E_*$ -Adams resolution defined below.

**2.2.9. LEMMA.** *Let  $K_s = E \wedge X_s$ , and let  $X_{s+1}$  be the fiber of  $f_s: X_s \rightarrow K_s$ . Then the resulting diagram (2.2.1) is an  $E_*$ -Adams resolution for  $X$ .*

**PROOF.** Since  $E$  is a ring spectrum it is a retract of  $E \wedge E$ , so  $E \wedge X_s$  is a retract of  $E \wedge K_s = E \wedge E \wedge X_s$  and 2.2.1(b) is satisfied.  $E \wedge X_s$  is an  $E$ -module spectrum so 2.2.1(c) is satisfied. For 2.2.1(d) we have  $E_*(K_s) = E_*(E) \otimes_{\pi_*(E)} E_*(X_s)$  by 2.2.7 and  $\text{Ext}(E_*(K_s))$  has the desired properties by A1.2.1 and A1.2.4.  $\square$

**2.2.10. DEFINITION.** *The canonical  $E_*$ -Adams resolution for  $X$  is the one given by 2.2.9.*

Note that if  $E$  is not a ring spectrum then the above  $f_s$  need not induce a monomorphism in  $E$ -homology, in which case the above would not be an Adams resolution.

Note also that the canonical resolution for  $X$  can be obtained by smashing  $X$  with the canonical resolution for  $S^0$ .

**2.2.11. PROPOSITION.** *The  $E_1$ -term of the Adams spectral sequence associated with the resolution of 2.2.9 is the cobar complex  $C^*(E_*(X))$  (A1.2.11).  $\square$*

Next we describe an  $E$ -completion  $\widehat{X}$  (2.2.2). First we need some more terminology.

**2.2.12. DEFINITION.**  $X_{(p)} = X\mathbf{Z}_{(p)}$ , where  $\mathbf{Z}_{(p)}$  denotes the integers localized at  $p$ , and  $X_p = X\mathbf{Z}_p$  (see 2.1.14).

2.2.13. THEOREM. *If  $X$  is connective and  $E$  satisfies 2.2.5(a)–(e) then an  $E$ -completion (2.2.2) of  $X$  is given by*

$$\widehat{X} = \begin{cases} X\mathbf{Q} & \text{if } \pi_0(E) = \mathbf{Q} \\ X_{(p)} & \text{if } \pi_0(E) = \mathbf{Z}_{(p)} \\ X & \text{if } \pi_0(E) = \mathbf{Z} \\ X_p & \text{if } \pi_0(E) = \mathbf{Z}/(p) \text{ and } \pi_n(X) \\ & \text{is finitely generated for all } n. \end{cases} \quad \square$$

These are not the only possible values of  $\pi_0(E)$ , but the others will not concern us. A proof is given by Adams [4], Theorem 14.6 and Section 15. We will sketch a proof using the additional hypothesis that  $\pi_1(E) = 0$ , which is true in all of the cases we will consider in this book.

For simplicity assume that  $\pi_0(X)$  is the first nonzero homotopy group. Then in the cases where  $\pi_0(E)$  is a subring of  $\mathbf{Q}$  we have  $\pi_i(\widehat{X} \wedge \overline{E}^{(s)}) = 0$  for  $i < s$ , so by setting  $\widehat{X}_s = \widehat{X} \wedge \overline{E}^{(s)}$  we get  $\varprojlim \widehat{X}_s = \text{pt}$ .

The remaining case,  $\pi_0(E) = \mathbf{Z}/(p)$  can be handled by an argument similar to that of the classical case. We show  $X\mathbf{Z}/(p^m)$  is its own  $E$ -completion by modifying the proof of 2.1.16 appropriately. Then  $X_p$  can be shown to be  $E$ -complete just as in the proof of 2.1.1(b) (following 2.1.16).

Now we are ready to prove 2.2.3(a). As in Section 1 the diagram 2.2.1 leads to an exact couple which gives the desired spectral sequence. To identify the  $E_2$ -term, observe that 2.2.1(a) implies that each fibration in the resolution gives a short (as opposed to long) exact sequence in  $E$ -homology. These splice together to give a long exact sequence replacing 2.1.3,

$$0 \rightarrow E_*(X) \rightarrow E_*(K_0) \rightarrow E_*(\Sigma K_1) \rightarrow \cdots$$

Condition 2.2.1(c) implies that the  $E_2$ -term of the spectral sequence is the cohomology of the complex

$$\text{Ext}^0(E_*(K_0)) \rightarrow \text{Ext}^0(E_*(\Sigma K_1)) \rightarrow \cdots$$

By A1.2.4 this is  $\text{Ext}(E_*(X))$ .

For 2.2.3(b) we know that the map  $X \rightarrow \widehat{X}$  induces a spectral sequence isomorphism since it induces an  $E$ -homology isomorphism. We also know that  $\varprojlim \widehat{X}_s = \text{pt}$ , so we can identify  $E_\infty^{**}$  as in 2.1.12.

We still need to show that the spectral sequence is natural and independent (from  $E_2$  onward) of the choice of resolution. The former implies the latter as the identity map on  $X$  induces a map between any two resolutions and standard homological arguments show that such a map induces an isomorphism in  $E_2$  and hence in  $E_r$  for  $r \geq 2$ . The canonical resolution is clearly natural so it suffices to show that any other resolution admits maps to and from the canonical one.

We do this in stages as follows. Let  $\{f_s: X_s \rightarrow K_s\}$  be an arbitrary resolution and let  $R^0$  be the canonical one. Let  $R^n = \{f_s^n: X_s^n \rightarrow K_s^n\}$  be defined by  $X_s^n = X_s$ , and  $K_s^n = K_n$  for  $s < n$  and  $K_s^n = E \wedge X_s^n$ ; for  $s \geq n$ . Then  $R^\infty$  is the arbitrary resolution and we construct maps  $R^0 \leftrightarrow R^\infty$  by constructing maps  $R^n \leftrightarrow R^{n+1}$ , for which it suffices to construct maps between  $K_s$  and  $E \wedge X_s$  compatible with the map from  $X_s$ . By 2.2.1(b) and (c),  $K_s$  and  $E \wedge X_s$  are both retracts of  $E \wedge K_s$ , so



we have a commutative diagram

$$\begin{array}{ccccc}
 X_s & \longrightarrow & K_s & & \\
 \downarrow & & \downarrow & \searrow & \\
 E \wedge X_s & \longrightarrow & E \wedge K_s & \longrightarrow & E \wedge X_s \\
 & \searrow & \downarrow & & \\
 & & K_s & & 
 \end{array}$$

in which the horizontal and vertical composite maps are identities. It follows that the diagonal maps are the ones we want.

The Adams spectral sequence of 2.2.3 is useful for computing  $\pi_*(X)$ , i.e.,  $[S^0, X]$ . With additional assumptions on  $E$  one can generalize to a spectral sequence for computing  $[W, X]$ . This is done in Adams [4] for the case when  $E_*(W)$  is projective over  $\pi_*(E)$ . We omit this material as we have no need for it.

Now we describe the filtration of 2.2.3(b), which will be referred to as the  $E_*$ -Adams filtration on  $\pi_*(\widehat{X})$ .

2.2.14. FILTRATION THEOREM. *The filtration on  $\pi_*(\widehat{X})$  of 2.2.3(b) is as follows. A map  $f: S^n \rightarrow X$  has filtration  $\geq s$  if  $f$  can be factored into  $s$  maps each of which becomes trivial after smashing the target with  $E$ .*

PROOF. We have seen above that  $F^s \pi_*(\widehat{X}) = \text{im } \pi_*(X_s)$ . We will use the canonical resolution (2.2.10). Let  $\overline{E}$  be the fiber of the unit map  $S^0 \rightarrow E$ . Then  $X_2 = \overline{E}^{(s)} \wedge X$ , where  $\overline{E}^{(s)}$  is the  $s$ -fold smash product of  $\overline{E}$ .  $X_{i+1} \rightarrow X_i \rightarrow X_i \wedge E$  is a fiber sequence so each such composition is trivial and a map  $S^n \rightarrow X$  which lifts to  $X_s$  clearly satisfies the stated condition. It remains to show the converse, i.e., that if a map  $f: S^n \rightarrow X$  factors as

$$S^n \rightarrow Y_s \xrightarrow{g_s} Y_{s-1} \xrightarrow{g_{s-1}} \cdots \rightarrow Y_0 = X,$$

where each composite  $Y_i \xrightarrow{g_i} Y_{i-1} \rightarrow Y_{i-1} \wedge E$  is trivial, then it lifts to  $X_s$ . We argue by induction on  $i$ . Suppose  $Y_{i-1} \rightarrow X$  lifts to  $X_{i-1}$  (a trivial statement for  $i = 1$ ). Since  $Y_i$  maps trivially to  $Y_{i-1} \wedge E$ , it does so to  $X_{i-1} \wedge E$  and therefore lifts to  $X_i$ .  $\square$

### 3. The Smash Product Pairing and the Generalized Connecting Homomorphism

In this section we derive two properties of the Adams spectral sequence which will prove useful in the sequel. The first concerns the structure induced by a map

$$(2.3.1) \quad \alpha: X' \wedge X'' \rightarrow X,$$

e.g., the multiplication on a ring spectrum. The second concerns a generalized connecting homomorphism arising from a cofiber sequence

$$(2.3.2) \quad W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$$

when  $E_*(h) = 0$ . Both of these results are folk theorems long known to experts in the field but to our knowledge never before published in full generality. The first property in the classical case was proved in Adams [3], while a weaker form of the second property was proved by Johnson, Miller, Wilson, and Zahler [1].

Throughout this section the assumptions 2.2.5 on  $E$  will apply. However, the flatness condition [2.2.5(d)] is only necessary for statements explicitly involving  $\text{Ext}$ , i.e., 2.3.3(e) and 2.3.4(a). For each spectrum  $X$  let  $E_*^{**}(X)$  be the Adams spectral sequence for  $X$  based on  $E$ -homology (2.2.3). Our first result is

2.3.3. THEOREM. *Let  $2 \leq r \leq \infty$ . Then the map  $a$  above induces a natural pairing*

$$E_r^{**}(X') \otimes E_r^{**}(X'') \rightarrow E_r^{**}(X)$$

such that

$$(a) \text{ for } a' \in E_r^{s',t'}(X'), a'' \in E_r^{s'',t''}(X''),$$

$$d_r(a', a'') = d_r(a')a'' + (-1)^{t'-s'} a' d_r(a'');$$

(b) the pairing on  $E_{r+1}$ , is induced by that on  $E_r$ ;

(c) the pairing on  $E_\infty$ , corresponds to  $a_*: \pi_*(X') \otimes \pi_*(X'') \rightarrow \pi_*(X)$ ;

(d) if  $X' = X'' = X$  and  $E_*(\alpha): E_*(X) \otimes E_*(X) \rightarrow E_*(X)$  is commutative or associative, then so is the pairing [modulo the usual sign conventions, i.e.,  $a'a'' = (-1)^{(t'-s')(t''-s'')} a''a'$ ];

(e) for  $r = 2$  the pairing is the external cup product (A1.2.13)

$$\text{Ext}(E_*(X')) \otimes \text{Ext}(E_*(X'')) \rightarrow \text{Ext}(E_*(X') \otimes_{\pi_*(E)} E_*(X''))$$

composed with the map in  $\text{Ext}$  induced by the composition of canonical maps

$$E_*(X') \otimes_{\pi_*(E)} E_*(X'') \rightarrow E_*(X' \wedge X'') \xrightarrow{\alpha_*} E_*(X).$$

In particular, by setting  $X' = S^0$  and  $X'' = X$  we find that the spectral sequence for  $X$  is a module (in the appropriate sense) over that for the sphere  $S^0$ .  $\square$

The second result is

2.3.4. THEOREM. *Let  $E_*(h) = 0$  in 2.3.2. Then for  $2 \leq r \leq \infty$  there are maps  $\delta_r: E_r^{s,*}(Y) \rightarrow E_r^{s+1,*}(W)$  such that*

(a)  $\delta_2$  is the connecting homomorphism associated with the short exact sequence

$$0 \rightarrow E_*(W) \rightarrow E_*(X) \rightarrow E_*(Y) \rightarrow 0,$$

(b)  $\delta_r d_r = d_r \delta_r$  and  $\delta_{r+1}$  induced by  $\delta_r$ ,

(c)  $\delta_\infty$  is a filtered form of the map  $\pi_*(h)$ .

The connecting homomorphism in  $\text{Ext}$  can be described as the Yoneda product (Hilton and Stammach [1, p. 155] with the element of  $\text{Ext}_{E_*(E)}^1(E_*(Y), E_*(W))$ ) corresponding to the short exact sequence

$$0 \rightarrow E_*(W) \rightarrow E_*(X) \rightarrow E_*(Y) \rightarrow 0.$$

Similarly, given a sequence of maps

$$X_0 \xrightarrow{f_0} \Sigma X_1 \xrightarrow{f_1} \Sigma^2 X_2 \rightarrow \cdots \rightarrow \Sigma^n X_n$$

with  $E_*(f_i) = 0$  one gets maps

$$\delta_r: E_r^{s,*}(X_0) \rightarrow E_r^{s+n,*}(X_n)$$

commuting with differentials where  $\delta_2$  can be identified as the Yoneda product with the appropriate element in

$$\text{Ext}_{E(E_*)}^n(E_*(X_0), E_*(X_n)). \quad \square$$

If one generalizes the spectral sequence to source spectra other than the sphere one is led to a pairing induced by composition of maps. This has been studied by Moss [1], where it is assumed that one has Adams resolutions satisfying much stronger conditions than 2.2.1. In the spectral sequence for the sphere it can be shown that the composition and smash product pairings coincide, but we will not need this fact.

To prove 2.3.3 we will use the canonical resolutions (2.2.9) for  $X'$ ,  $X''$  and  $X$ . Recall that these can be obtained by smashing the respective spectra with the canonical resolution for  $S^0$ . Let  $K_{s,s+r}$  be the cofiber in

$$(2.3.5) \quad \overline{E}^{(s+r)} \rightarrow \overline{E}^{(s)} \rightarrow K_{s,s+r},$$

where  $E$  is the fiber of  $S^0 \rightarrow E$ .

These spectra have the following properties.

2.3.6. LEMMA.

(a) *There are canonical fibrations*

$$K_{s+i,s+i+j} \rightarrow K_{s,s+i+j} \rightarrow K_{s,s+i}.$$

(b)  $E_1^{s,*}(X) = \pi_*(X \wedge K_{s,s+1})$ .

Let  $Z_r^{s,*}(X)$ ,  $B_r^{s,*}(X) \subset E_1^{s,*}(X)$  be the images of  $\pi_*(X \wedge K_{s,s+r})$  and  $\pi_*(X \wedge \Sigma^{-1}K_{s-r+1,s})$ , respectively. Then  $E_r^{s,*}(X) = Z_r^{s,*}(X)/B_r^{s,*}(X)$  and  $d_r$  is induced by the map

$$X \wedge K_{s,s+r} \rightarrow X \wedge \Sigma K_{s+r,s+2r}.$$

(c)  $\alpha$  induces map  $X'_s \wedge X''_t \rightarrow X_{s+1}$  (where these are the spectra in the canonical resolutions) compatible with the maps  $g'_s$ ,  $g''_t$ , and  $g_{s+t}$  of 2.2.1.

(d) *The map*

$$K_{s,s+1} \wedge K_{t,t+1} \rightarrow K_{s+t,s+t+1},$$

given by the equivalence

$$K_{n,n+1} = E \wedge \overline{E}^{(n)}$$

and the multiplication on  $E$ , lifts to maps

$$K_{s,s+r} \wedge K_{t,t+r} \rightarrow K_{s+t,s+t+r}$$

for  $r > 1$  such that the following diagram commutes

$$\begin{array}{ccc} K_{s,s+r+1} \wedge K_{t,t+r+1} & \longrightarrow & K_{s+t,s+t+r+1} \\ \downarrow & & \downarrow \\ K_{s,s+r} \wedge K_{t,t+r} & \longrightarrow & K_{s+t,s+t+r} \end{array}$$

where the vertical maps come from (a).

(e) *The following diagram commutes*

$$\begin{array}{ccc} K_{s,s+r} \wedge K_{t,t+r} & \longrightarrow & (\Sigma K_{s+r,s+2r} \wedge K_{t,t+r}) \vee (K_{s,s+r} \wedge \Sigma K_{t+r,t+2r}) \\ \downarrow & & \downarrow \\ K_{s+t,s+t+r} & \longrightarrow & \Sigma K_{s+t+r,s+t+2r} \end{array}$$

where the vertical maps are those of (d) and the horizontal maps come from (a), the maps to and from the wedge being the sums of the maps to and from the summands.

PROOF. Part (a) is elementary. For (b) we refer the reader to Cartan and Eilenberg [1], Section XV.7, where a spectral sequence is derived from a set of abelian groups  $H(p, q)$  satisfying certain axioms. Their  $H(p, q)$  in this case is our  $\pi_*(K_{p,q})$ , and (a) guarantees that these groups have the appropriate properties. For (c) we use the fact that  $X'_s = X' \wedge \overline{E}^{(s)}$ ,  $X''_t = X'' \wedge \overline{E}^{(t)}$ , and  $X_{s+t} = X \wedge \overline{E}^{(s+t)}$ .

For (d) we can assume the maps  $\overline{E}^{(s+1)} \rightarrow \overline{E}^{(s)}$  are all inclusions with  $K_{s,s+r} = \overline{E}^{(s)}/\overline{E}^{(s+r)}$ . Hence we have

$$K_{s,s+r} \wedge K_{t,t+r} = E^{(s)} \wedge \overline{E}^{(t)} / (\overline{E}^{(s+r)} \wedge \overline{E}^{(t)} \cup \overline{E}^{(s)} \wedge \overline{E}^{(t+r)})$$

and this maps naturally to

$$\overline{E}^{(s+t)} / \overline{E}^{(s+t+r)} = K_{s+t,s+t+r}.$$

For (e) if  $\overline{E}^{(s+2r)} \rightarrow \overline{E}^{(s+r)} \rightarrow \overline{E}^{(s)}$  are inclusions then so is  $K_{s+r,s+2r} \rightarrow K_{s,s+2r}$  so we have  $K_{s,s+r} = K_{s,s+2r}/K_{s+r,s+2r}$  and  $K_{t,t+r} = K_{t,t+2r}/K_{t+r,t+2r}$ . With this in mind we get a commutative diagram

$$\begin{array}{ccc} K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r} & \longrightarrow & K_{s+t+r,s+t+2r} \\ \downarrow & & \downarrow \\ K_{s,s+2r} \wedge K_{t,t+2r} & \longrightarrow & K_{s+t,s+t+2r} \\ \downarrow & & \downarrow \\ K_{s,s+r} \wedge K_{t,t+r} & \longrightarrow & K_{s+t,s+t+r} \\ \downarrow & & \downarrow \\ \Sigma(K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r}) & \longrightarrow & \Sigma K_{s+t+r,s+t+2r} \end{array}$$

where the horizontal maps come from (d) and the upper vertical maps are inclusions. The lower left-hand map factors through the wedge giving the desired diagram.  $\square$

We are now ready to prove 2.3.3. In light of 2.3.6(b), the pairing is induced by the maps of 2.3.6(d). Part 2.3.3(a) then follows from 2.3.6(e) as the differential on  $E_r^{**}(X') \otimes E_r^{**}(X'')$  is induced by the top map of 2.3.6(e). Part 2.3.3(b) follows from the commutative diagram in 2.3.6(d). Part 2.3.3(c) follows from the compatibility of the maps in 2.3.6(c) and (d).

Assuming 2.3.3(e), (d) is proved as follows. The pairing on Ext is functorial, so if  $E_*(X)$  has a product which is associative or commutative, so will  $E_2^{**}(X)$ . Now suppose inductively that the product on  $E_r^{**}(X)$  has the desired property. Since the product on  $E_{r+1}$  is induced by that on  $E_r$  the inductive step follows.

It remains then to prove 2.3.3(e). We have  $E_*(X' \wedge K_{s,s+1}) = D^s(E_*(X'))$  (A1.2.11) and similarly for  $X''$ , so our pairing is induced by a map

$$E_*(X' \wedge K_{s,s+1}) \otimes_{\pi_*(E)} E_*(X'' \wedge K_{t,t+1}) \rightarrow E_*(X \wedge K_{s+t,s+t+1}),$$

i.e., by a pairing of resolutions. Hence the pairing on  $E_2$  coincides with the specified algebraic pairing by the uniqueness of the latter (A1.2.14).

We prove 2.3.4 by reducing it to the following special case.

2.3.7. LEMMA. *Theorem 2.3.4 holds when  $X$  is such that  $\text{Ext}^s(E_*(X)) = 0$  for  $s > 0$  and  $\pi_*(X) = \text{Ext}^0(E_*(X))$ .*  $\square$

PROOF OF 2.3.4. Let  $W'$  be the fiber of the composite

$$W \xrightarrow{f} X \rightarrow X \wedge E.$$

Since  $\Sigma fh$  is trivial,  $h$  lifts to a map  $h': Y \rightarrow \Sigma W'$ . Now consider the cofiber sequence

$$W \rightarrow X \wedge E \rightarrow \Sigma W' \rightarrow \Sigma W.$$

Lemma 2.3.7 applies here and gives maps

$$\delta_r: E_r^{s,*}(\Sigma W') \rightarrow E_r^{s+1,*}(\Sigma W).$$

Composing this with the maps induced by  $h'$  gives the desired result.  $\square$

PROOF OF 2.3.7. Disregarding the notation used in the above proof, let  $W' = \Sigma^{-1}Y$ ,  $X' = \Sigma^{-1}Y \wedge E$ , and  $Y' = Y \wedge \bar{E}$ . Then we have a commutative diagram in which both rows and columns are cofiber sequences

$$\begin{array}{ccccc} X & \longleftarrow & W & \longleftarrow & W' \\ \downarrow & & \downarrow & & \downarrow \\ X \vee (Y \wedge E) & \longleftarrow & X & \longleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y \wedge E & \longleftarrow & Y & \longleftarrow & Y' \end{array}$$

Each row is the beginning of an Adams resolution (possibly noncanonical for  $W$  and  $X$ ) which we continue using the canonical resolutions (2.2.9) for  $W'$ ,  $X'$ , and  $Y'$ . Thus we get a commutative diagram

$$(2.3.8) \quad \begin{array}{ccccccc} W & \longleftarrow & W' & \longleftarrow & W' \wedge \bar{E} & \longleftarrow & W' \wedge \bar{E}^{(2)} \longleftarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X' & \longleftarrow & X' \wedge \bar{E} & \longleftarrow & X' \wedge \bar{E}^{(2)} \longleftarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y' & \longleftarrow & Y' \wedge \bar{E} & \longleftarrow & Y' \wedge \bar{E}^{(2)} \longleftarrow \dots \end{array}$$

in which each column is a cofiber sequence. The map  $Y \xrightarrow{\cong} \Sigma W'$  induces maps  $\delta_r: E_r^{s,*}(Y) \rightarrow E_r^{s+1,*}(W)$  which clearly satisfy 2.3.4(a) and (b), so we need only to verify that  $\delta_2$  is the connecting homomorphism. The resolutions displayed in 2.3.8 make this verification easy because they yield a short exact sequence of  $E_1$ -terms which is additively (though not differentially) split. For  $s = 0$  we have

$$\begin{aligned} E_1^{0,*}(W) &= \pi_*(X), & E_1^{0,*}(X) &= \pi_*(X \vee (Y \wedge E)), \\ E_1^{0,*}(Y) &= \pi_*(Y \wedge E), & E_1^{1,*}(W) &= \pi_*(Y \wedge E), \\ E_1^{1,*}(X) &= \pi_*(Y \wedge E \wedge E) & \text{and} & E_1^{1,*}(Y) = \pi_*(\Sigma Y \wedge \bar{E} \wedge E), \end{aligned}$$

so the relevant diagram for the connecting homomorphism is

$$\begin{array}{ccccc} X & \longrightarrow & X \vee (Y \wedge E) & \xleftarrow{b} & Y \wedge E \\ \downarrow & & \downarrow d & & \downarrow \\ Y \wedge E & \xleftarrow{a} & Y \wedge E \wedge E & \longrightarrow & \Sigma Y \wedge \bar{E} \wedge \bar{E} \end{array}$$

where  $a$  and  $b$  are splitting maps. The connecting homomorphism is induced by  $adb$ , which is the identity on  $Y \wedge E$ , which also induces  $\delta_2$ .

For  $s > 0$  we have

$$\begin{aligned} E_1^{s,*}(W) &= \pi_*(\Sigma^{s-1}Y \wedge E \wedge \bar{E}^{(s-1)}), \\ E_1^{s,*}(X) &= \pi_*(\Sigma^{s-1}Y \wedge E^{(2)} \wedge \bar{E}^{(s-1)}), \end{aligned}$$

and

$$E_1^{s,*}(Y) = \pi_*(\Sigma^s Y \wedge E \wedge \bar{E}^{(s)}),$$

so the relevant diagram is

$$\begin{array}{ccccc} E & \longrightarrow & E \wedge E & \xleftarrow{\quad} & E \wedge \bar{E} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma E \wedge \bar{E} & \xleftarrow{\quad} & \Sigma E \wedge E \wedge \bar{E} & \longrightarrow & \Sigma^2 E \wedge \bar{E}^2 \end{array}$$

and again the connecting homomorphism is induced by the identity on  $\Sigma^s Y \wedge E \wedge \bar{E}^s$ .  $\square$

## The Classical Adams Spectral Sequence

In Section 1 we make some simple calculations with the Adams spectral sequence which will be useful later. In particular, we use it to compute  $\pi_*(MU)$  (3.1.5), which will be needed in the next chapter. The computations are described in some detail in order to acquaint the reader with the methods involved.

In Sections 2 and 3 we describe the two best methods of computing the Adams spectral sequence for the sphere, i.e., the May spectral sequence and the lambda algebra. In both cases a table is given showing the result in low dimensions (3.2.9 and 3.3.10). Far more extensive charts are given in Tangora [1, 4]. The main table in the former is reproduced in Appendix 3.

In Section 4 we survey some general properties of the Adams spectral sequence. We give  $E_2^{s,*}$  for  $s \leq 3$  (3.4.1 and 3.4.2) and then say what is known about differentials on these elements (3.4.3 and 3.4.4). Then we outline the proof of the Adams vanishing and periodicity theorems (3.4.5 and 3.4.6). For  $p = 2$  they say that  $E_s^{s,t}$  vanishes roughly for  $0 < t - s < 2s$  and has a very regular structure for  $t - s < 5s$ . The  $E_\infty$ -term in this region is given in 3.4.16. An elementary proof of the nontriviality of most of these elements is given in 3.4.21.

In Section 5 we survey some other past and current research and suggest further reading.

### 1. The Steenrod Algebra and Some Easy Calculations

In this section we begin calculating with the classical mod  $(p)$  Adams spectral sequence of 2.1.1. We start by describing the dual Steenrod algebra  $A_*$ , referring the reader to Milnor [2] or Steenrod and Epstein [1] for the proof. Throughout this book,  $P(x)$  will denote a polynomial algebra (over a field which will be clear from the context) on one or more generators  $x$ , and  $E(x)$  will denote the exterior algebra on same.

3.1.1. THEOREM (Milnor [2]).  $A_*$  is a graded commutative, noncocommutative Hopf algebra.

(a) For  $p = 2$ ,  $A_* = P(\xi_1, \xi_2, \dots)$  as an algebra where  $|\xi_n| = 2^n - 1$ . The coproduct  $\Delta: A_* \rightarrow A_* \otimes A_*$  is given by  $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i$ , where  $\xi_0 = 1$ .

(b) For  $p > 2$ ,  $A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$  as an algebra, where  $|\xi_n| = 2(p^n - 1)$ , and  $|\tau_n| = 2p^n - 1$ . The coproduct  $\Delta: A_* \rightarrow A_* \otimes A_*$  is given by  $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} \otimes \xi_i$ , where  $\xi_0 = 1$  and  $\Delta\tau_n = \tau_n \otimes 1 + \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} \otimes \tau_i$ .

(c) For each prime  $p$ , there is a unit  $\eta: \mathbf{Z}/(p) \rightarrow A_*$ , a counit  $\varepsilon: A_* \rightarrow \mathbf{Z}/(p)$  (both of which are isomorphisms in dimension 0), and a conjugation (canonical antiautomorphism)  $c: A_* \rightarrow A_*$  which is an algebra map given recursively by  $c(\xi_0) = 1$ ,  $\sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} c(\xi_i) = 0$  for  $n > 0$  and  $\tau_n + \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} c(\tau_i) = 0$  for  $n \geq 0$ .  $\bar{A}_*$

will denote  $\ker \varepsilon$ ; i.e.,  $\bar{A}_*$  is isomorphic to  $A_*$  in positive dimensions, and is trivial in dimension 0.  $\square$

$A_*$  is a commutative Hopf algebra and hence a Hopf algebroid. The homological properties of such objects are discussed in Appendix 1.

We will consider the classical Adams spectral sequence formulated in terms of homology (2.2.3) rather than cohomology (2.1.1). The most obvious way of computing the  $E_2$ -term is to use the cobar complex. The following description of it is a special case of 2.2.10 and A1.2.11.

3.1.2. PROPOSITION. *The  $E_2$ -term for the classical Adams spectral sequence for  $\pi_*(X)$  is the cohomology of the cobar complex  $C_{A_*}^s(H_*(X))$  defined by*

$$C_{A_*}^s(H_*(X)) = \bar{A}_* \otimes \cdots \otimes \bar{A}_* \otimes H_*(X)$$

(with  $s$  tensor factors of  $\bar{A}_*$ ). For  $a_i \in A_*$  and  $x \in H_*(X)$ , the element  $a_1 \otimes \cdots \otimes a_s \otimes x$  will be denoted by  $[a_1|a_2|\cdots|a_s]x$ . The coboundary operator  $d_s: C_{A_*}^s(H_*(X)) \rightarrow C_{A_*}^{s+1}(H_*(X))$  is given by

$$\begin{aligned} d_s[a_1|\cdots|a_s]x &= [1|a_1|\cdots|a_s]x + \sum_{i=1}^s (-1)^i [a_1|\cdots|a_{i-1}|a'_i|a''_i|a_{i+1}|\cdots|a_s]x \\ &\quad + (-1)^{s+1} [a_1|\cdots|a_s|x']x'', \end{aligned}$$

where  $\Delta a_i = a'_i \otimes a''_i$  and  $\psi(x) = x' \otimes x'' \in A_* \otimes H_*(X)$ . [A priori this expression lies in  $A_*^{\otimes s+1} \otimes H_*(X)$ . The diligent reader can verify that it actually lies in  $\bar{A}_*^{\otimes s+1} \otimes H_*(X)$ .]  $\square$

This  $E_2$ -term will be abbreviated by  $\text{Ext}(H_*(X))$ .

Whenever possible we will omit the subscript  $A_*$ .

The following result will be helpful in solving group extension problems in the Adams spectral sequence. For  $p > 2$  let  $a_0 \in \text{Ext}_{A_*}^{1,1}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  be the class represented by  $[\tau_0] \in C(\mathbf{Z}/(p))$ . The analogous element for  $p = 2$  is represented by  $[\xi_1]$  and is denoted by  $a_0$ ,  $h_{1,0}$ , or  $h_0$ .

3.1.3. LEMMA.

(a) For  $s \geq 0$ ,  $\text{Ext}^{s,s}(H_*(S^0))$  is generated by  $a_0^s$ .

(b) If  $x \in \text{Ext}(H_*(X))$  is a permanent cycle in the Adams spectral sequence represented by  $\alpha \in \pi_*(X)$ , then  $a_0x$  is a permanent cycle represented by  $\rho\alpha$ . [The pairing  $\text{Ext}(H_*(S^0)) \otimes \text{Ext}(H_*(X)) \rightarrow \text{Ext}(H_*(X))$  is given by 2.3.3.]  $\square$

PROOF. Part (a) follows from inspection of  $C^*(\mathbf{Z}/(p))$ ; there are no other elements in the indicated bidegrees. For (b) the naturality of the smash product pairing (2.3.3) reduces the problem to the case  $x = 1 \in \text{Ext}(H_*(S^0))$ , where it follows from the fact that  $\pi_0(S^0) = \mathbf{Z}$ .  $\square$

The cobar complex is so large that one wants to avoid using it directly at all costs. In this section we will consider four spectra ( $MO$ ,  $MU$ ,  $bo$ , and  $bu$ ) in which the change-of-rings isomorphism of A1.1.18 can be used to great advantage. The most important of these for our purposes is  $MU$ , so we treat it first. The others are not used in the sequel. Much of this material is covered in chapter 20 of Switzer [1].

The computation of  $\pi_*(MU)$  is due independently to Milnor [4] and Novikov [2, 3]. For the definition and basic properties of  $MU$ , including the following lemma, we refer the reader to Milnor [4] or Stong [1] or to Section 4.1.



3.1.4. LEMMA.

(a)  $H_*(MU; \mathbf{Z}) = \mathbf{Z}[b_1, b_2, \dots]$ , where  $b_i \in H_{2i}$ .

(b) Let  $H/(p)$  denote the mod  $(p)$  Eilenberg–Mac Lane spectrum for a prime  $p$  and let  $u: MU \rightarrow H/(p)$  be the Thom class, i.e., the generator of  $H^0(MU; \mathbf{Z}/(p))$ . Then  $H_*(u)$  is an algebra map and its image in  $H_*(H/(p)) = A_*$  is  $P(\xi_1^2, \xi_2^2, \dots)$  for  $p = 2$  and  $P(\xi_1, \xi_2, \dots)$  for  $p > 2$ .  $\square$

The main result concerning  $MU$  is the following.

3.1.5. THEOREM (Milnor [4], Novikov [2, 3]).

(a)  $\pi_*(MU) = \mathbf{Z}[x_1, x_2, \dots]$  with  $x_i \in \pi_{2i}(MU)$ .

(b) Let  $h: \pi_*(MU) \rightarrow H_*(MU; \mathbf{Z})$  be the Hurewicz map. Then modulo decomposables in  $H_*(MU; \mathbf{Z})$ ,

$$h(x_i) = \begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ -b_i & \text{otherwise.} \end{cases} \quad \square$$

We will prove this in essentially the same way that Milnor and Novikov did. After some preliminaries on the Steenrod algebra we will use the change-of-rings isomorphisms A1.1.18 and A1.3.13 to compute the  $E_2$ -term (3.1.10). It will follow easily that the spectral sequence collapses; i.e., it has no nontrivial differentials.

To compute the  $E_2$ -term we need to know  $H_*(MU; \mathbf{Z}/(p))$  as an  $A_*$ -comodule algebra. Since it is concentrated in even dimensions, the following result is useful.

3.1.6. LEMMA. Let  $M$  be a left  $A_*$ -comodule which is concentrated in even dimensions. Then  $M$  is a comodule over  $P_* \subset A_*$  defined as follows. For  $p > 2$ ,  $P_* = P(\xi_1, \xi_2, \dots)$  and for  $p = 2$ ,  $P_* = P(\xi_1^2, \xi_2^2, \dots)$ .

PROOF. For  $m \in M$ , let  $\psi(m) = \Sigma m' \otimes m''$ . Then each  $m' \in A_*$  must be even-dimensional and by coassociativity its coproduct expansion must consist entirely of even-dimensional factors, which means it must lie in  $P_*$ .  $\square$

3.1.7. LEMMA. As a left  $A_*$ -comodule,  $H_*(MU) = P_* \otimes C$ , where  $C = P(u_1, u_2, \dots)$  with  $\dim u_i = 2i$  and  $i$  is any positive integer not of the form  $p^k - 1$ .

PROOF.  $H_*(MU; \mathbf{Z}/(p))$  is a  $P_*$ -comodule algebra by 3.1.4 and 3.1.6. It maps onto  $P_*$  by 3.1.4(b), so by A1.1.18 it is  $P_* \otimes C$ , where  $C = \mathbf{Z}/(p) \square_{P_*} H_*(MU)$ . An easy counting argument shows that  $C$  must have the indicated form.  $\square$

3.1.8. LEMMA. Let  $M$  be a comodule algebra over  $A_*$  having the form  $P_* \otimes N$  for some  $A_*$ -comodule algebra  $N$ . Then

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), M) = \text{Ext}_E(\mathbf{Z}/(p), N)$$

where

$$E = A_* \otimes_{P_*} \mathbf{Z}/(p) = \begin{cases} E(\xi_1, \xi_2, \dots) & \text{for } p = 2 \\ E(\tau_0, \tau_1, \dots) & \text{for } p > 2. \end{cases}$$

In particular,

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = \text{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p)) \otimes C.$$

PROOF. The statement about  $H_*(MU)$  follows from the general one by 3.1.7. For the latter we claim that  $M = A_* \square_E N$ . We have  $A_* = P_* \otimes E$  as vector spaces and hence as  $E$ -comodules by A1.1.20, so

$$A_* \square_E N = P_* \otimes E \square_E N = P_* \otimes N = M,$$

and the result follows from A1.3.13.  $\square$

Hence we have reduced the problem of computing the Adams  $E_2$ -term for  $MU$  to that of computing  $\text{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p))$ . This is quite easy since  $E$  is dual to an exterior algebra of finite type.

3.1.9. LEMMA. *Let  $\Gamma$  be a commutative, graded connected Hopf algebra of finite type over a field  $K$  which is an exterior algebra on primitive generators  $x_1, x_2, \dots$ , each having odd degree if  $K$  has characteristic other than 2 (e.g., let  $\Gamma = E$ ). Then*

$$\text{Ext}_\Gamma(K, K) = P(y_1, y_2, \dots),$$

where  $y_i \in \text{Ext}^{1, |x_i|}$  is represented by  $[x_i]$  in  $C_\Gamma(K)$  (the cobar complex of A1.2.11).

PROOF. Let  $\Gamma_i \subset \Gamma$  be the exterior algebra on  $x_i$ . Then an injective  $\Gamma_i$ -resolution of  $K$  is given by

$$0 \rightarrow K \rightarrow \Gamma_i \xrightarrow{d} \Gamma_i \rightarrow \Gamma_i \rightarrow \dots$$

where  $d(x_i) = 1$  and  $d(1) = 0$  applying  $\text{Hom}_{\Gamma_i}(K, \_)$  gives a complex with trivial boundary operator and shows  $\text{Ext}_{\Gamma_i}(K, K) = P(Y_i)$ . Tensoring all the  $R_i$  together gives an injective  $\Gamma$ -resolution of  $K$  and the result follows from the Kunnet theorem.  $\square$

Combining the last three lemmas gives

3.1.10. COROLLARY.

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = C \otimes P(a_0, a_1, \dots),$$

where  $C$  is as in 3.1.7 and  $a_i \in \text{Ext}^{1, 2p^i - 1}$  is represented by  $[\tau_i]$  for  $p > 2$  and  $[\xi_i]$  for  $p = 2$  in  $C_{A_*}(H_*(MU))$ .  $\square$

Thus we have computed the  $E_2$ -term of the classical Adams spectral sequence for  $\pi_*(MU)$ . Since it is generated by even-dimensional classes, i.e., elements in  $E_2^{s, t}$  with  $t - s$  even, there can be no nontrivial differentials, i.e.,  $E_2 = E_\infty$ .

The group extension problems are solved by 3.1.3; i.e., all multiples of  $a_0^s$  are represented in  $\pi_*(MU)$  by multiples of  $p^s$ . It follows that  $\pi_*(MU) \otimes \mathbf{Z}/(p)$  is as claimed for each  $p$ ; i.e., 3.1.5(a) is true locally. Since  $\pi_i(MU)$  is finitely generated for each  $i$ , we can conclude that it is a free abelian group of the appropriate rank.

To get at the global ring structure note that the mod  $(p)$  indecomposable quotient in dimension  $2i$ ,  $Q_{2i}\pi_*(MU) \otimes \mathbf{Z}/(p)$  is  $\mathbf{Z}/(p)$  for each  $i > 0$ , so  $Q_{2i}\pi_*(MU) = \mathbf{Z}$ . Pick a generator  $x_i$  in each even dimension and let  $R = \mathbf{Z}[x_1, x_2, \dots]$ . The map  $R \rightarrow \pi_*(MU)$  gives an isomorphism after tensoring with  $\mathbf{Z}/(p)$  for each prime  $p$ , so it is isomorphism globally.

To study the Hurewicz map

$$h: \pi_*(MU) \rightarrow H_*(MU; \mathbf{Z}),$$

recall  $H_*(X; \mathbf{Z}) = \pi_*(X \wedge H)$ , where  $H$  is the integral Eilenberg–Mac Lane spectrum. We will prove 3.1.5(b) by determining the map of Adams spectral sequences

induced by  $i: MU \rightarrow MU \wedge H$ . We will assume  $p > 2$ , leaving the obvious changes for  $p = 2$  to the reader. The following result on  $H_*(H)$  is standard.

3.1.11. LEMMA. *The mod  $(p)$  homology of the integer Eilenberg–Mac Lane spectrum*

$$H_*(H) = P_* \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots)$$

as an  $A_*$  comodule, where  $\bar{\tau}_i$  denotes the conjugate  $\tau_i$ , i.e., its image under the conjugation  $c$ .  $\square$

Hence we have

$$H_*(H) = A_* \square_{E(\tau_0)} \mathbf{Z}/(p)$$

and an argument similar to that of 3.1.8 shows

$$(3.1.12) \quad \text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X \wedge H)) = \text{Ext}_{E(\tau_0)}(\mathbf{Z}/(p), H_*(X)).$$

In the case  $X = MU$  the comodule structure is trivial, so by 3.1.11,

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H)) = H_*(MU) \otimes P(a_0).$$

To determine the map of Ext groups induced by  $i$ , we consider three cobar complexes,  $C_{A_*}(H_*(MU))$ ,  $C_E(C)$ , and  $C_{E(\tau_0)}(H_*(MU))$ . The cohomologies of the first two are both  $\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU))$ , by 3.1.2 and 3.1.8, respectively, while that of the third is  $\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H))$  by 3.1.12. There are maps from  $C_{A_*}(H_*(MU))$  to each of the other two.

The class  $A_n \in \text{Ext}_{A_*}^{1, 2p^n - 1}(\mathbf{Z}/(p), H_*(MU))$  is represented by  $[\tau_n] \in C_E(C)$ . The element  $-\sum_i [\bar{\tau}_i] \bar{\xi}_{n-i}^{p^i} \in C_{A_*}(H_*(MU))$  [using the decomposition of  $H_*(MU)$  given by 3.1.7] is a cycle which maps to  $[\tau_n]$  and therefore it also represents  $a_n$ . Its image in  $C_{E(\tau_0)}(H_*(MU))$  is  $[\tau_0] \bar{\xi}_n$ , so we have  $i_*(a_n) = a_0 \bar{\xi}_n$ . Since  $\bar{\xi}_n \in H_*(MU)$  is a generator it is congruent modulo decomposables to a nonzero scalar multiple of  $b_{p^n - 1}$ , while  $u_i$  (3.1.9) can be chosen to be congruent to  $b_i$ . It follows that the  $x_i \in \pi_{2i}(MU)$  can be chosen to satisfy 3.1.5(b).

We now turn to the other spectra in our list,  $MO$ ,  $bu$ , and  $bo$ . The Adams spectral sequence was not used originally to compute the homotopy of these spectra, but we feel these calculations are instructive examples. In each case we will quote without proof a standard theorem on the spectrum's homology as an  $A_*$ -comodule and proceed from there.

For similar treatments of  $MSO$ ,  $MSU$ , and  $MSp$  see, respectively, Pengelley [2], Pengelley [1], and Kochman [1].

The following result on  $MO$  was first proved by Thom [1]. Proofs can also be found in Liulevicius [1] and Stong [1, p. 95].

3.1.13. THEOREM. *For  $p = 2$ ,  $H_*(MO) = A_* \otimes N$ , where  $N$  is a polynomial algebra with one generator in each degree not of the form  $2^k - 1$ . For  $p > 2$ ,  $H_*(MO) = 0$ .  $\square$*

It follows immediately that

$$(3.1.14) \quad \text{Ext}_{A_*}^s(\mathbf{Z}/(2), H_*(MO)) = \begin{cases} N & \text{if } s = 0 \\ 0 & \text{if } s > 0, \end{cases}$$

the spectral sequence collapses and  $\pi_*(MO) = N$ .

For  $bu$  we have

3.1.15. THEOREM (Adams [8]).

$$H_*(bu) = \bigoplus_{0 \leq i < p-1} \Sigma^{2i} M$$

where

$$\begin{aligned} M &= P_* \otimes E(\bar{\tau}_2, \bar{\tau}_3, \dots) && \text{for } p > 2 \\ M &= P_* \otimes E(\bar{\xi}_3, \bar{\xi}_4, \dots) && \text{for } p = 2 \end{aligned}$$

where  $\bar{\alpha}$  for  $\alpha \in A_*$  is the conjugate  $c(\alpha)$ . □

Using 3.1.8 we get

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), M) = \text{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots))$$

(again we assume for convenience that  $p > 2$ ) and by an easy calculation A1.3.13 gives

$$\text{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots)) = \text{Ext}_{E(\tau_0, \tau_1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1)$$

by 3.1.11, so we have

3.1.16. THEOREM.

$$\text{Ext}_{A_*}(\mathbf{Z}/(p), H_*(bu)) = \bigoplus_{i=0}^{p-2} \Sigma^{2i} P(a_0, a_1)$$

where  $a_0 \in \text{Ext}^{1,1}$  and  $a_1 \in \text{Ext}^{1,2p-1}$ . □

As in the  $MU$  case the spectral sequence collapses because the  $E_2$ -term is concentrated in even dimensions. The extensions can be handled in the same way, so we recover the fact that

$$\pi_i(bu) = \begin{cases} \mathbf{Z} & \text{if } i \geq 0 \text{ and } i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The  $bo$  spectrum is of interest only at the prime 2 because at odd primes it is a summand of  $bu$  (see Adams [8]). For  $p = 2$  we have

3.1.17. THEOREM (Stong [2]). For  $p = 2$ ,  $H_*(bo) = P(\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_4, \dots)$  where  $\bar{\xi}_i = c(\xi_i)$ . □

Let  $A(1)_* = A_*/(\xi_1^4, \xi_2^2, \xi_3, \xi_4, \dots)$ . We leave it as an exercise for the reader to show that  $A(1)_*$  is dual to the subalgebra  $A(1)$  of  $A$  generated by  $Sq^1$  and  $Sq^2$ , and that

$$H_*(bo) = A_* \square_{A(1)_*} \mathbf{Z}/(2),$$

so by A1.3.13,

$$(3.1.18) \quad \text{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \text{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)).$$

$A(1)$  is not an exterior algebra, so 3.1.9 does not apply. We have to use the Cartan–Eilenberg spectral sequence A1.3.15. The reader can verify that the following is an extension (A1.1.15)

$$(3.1.19) \quad \Phi \rightarrow A(1)_* \rightarrow E(\bar{\xi}_2),$$

where  $\Phi = P(\xi_1)/(\xi_1^4)$ .  $\Phi$  is isomorphic as a coalgebra to an exterior algebra on elements corresponding to  $\xi_1$  and  $\xi_1^2$ , so by 3.1.9

$$\text{Ext}_\Phi(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{10}, h_{11})$$

and

$$\text{Ext}_{E(\bar{\xi}_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{20}),$$

where  $h_{i,j}$  is represented by  $[\bar{\xi}_i^{2^j}]$  in the appropriate cobar complex. Since  $P(h_{20})$  has only one basis element in each degree, the coaction of  $\Phi$  on it is trivial, so by [A1.3.15](#) we have a Cartan–Eilenberg spectral sequence converging to  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$  with

$$(3.1.20) \quad E_2 = P(h_{10}, h_{11}, h_{20})$$

where  $h_{1i} \in E_2^{1,0}$  and  $h_{20} \in E_2^{0,1}$ . We claim

$$(3.1.21) \quad d_2(h_{20}) = h_{10}h_{11}.$$

This follows from the fact that

$$d(\xi_2) = \xi_1 \otimes \xi_1^2$$

in  $C_{A(1)_*}(\mathbf{Z}/(2))$ . It follows that

$$(3.1.22) \quad E_3 = P(u, h_{10}, h_{11})/(h_{10}h_{11})$$

where  $u \in E_3^{0,2}$  corresponds to  $h_{20}^2$ . Next we claim

$$(3.1.23) \quad d_3(u) = h_{11}^3.$$

We have in  $C_{A(1)_*}(\mathbf{Z}/(2))$ ,

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2) = \bar{\xi}_2 \otimes \xi_1 \otimes \xi_1^2 + \xi_1 \otimes \xi_1^2 \otimes \bar{\xi}_2.$$

In this  $E_2$  this gives

$$d_2 h_{20}^2 = h_{10}h_{11}h_{20} + h_{20}h_{10}h_{11} = 0$$

since  $E_2$  is commutative. However, the cobar complex is not commutative and when we add correcting terms to  $\bar{\xi}_2 \otimes \bar{\xi}_2$  in the hope of getting a cycle, we get instead

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2 + \xi_1 \otimes \xi_1^2 \bar{\xi}_2 + \xi_1 \bar{\xi}_2 \otimes \xi_1^2) = \xi_1^2 \otimes \xi_1^2 \otimes \xi_1^2,$$

which implies [3.1.23](#). It follows that

$$(3.1.24) \quad E_4 = P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}),$$

where  $v \in E_4^{1,2}$  and  $w \in E_4^{0,4}$  correspond to  $h_{10}h_{20}^2$  and  $h_{20}^4$ , respectively.

Finally, we claim that  $E_4 = E_\infty$ ; inspection of  $E_4$  shows that there cannot be any higher differentials because there is no  $E_r^{s,t}$  for  $r \geq 4$  which is nontrivial and for which  $E_r^{s+r,t-r+1}$  is also nontrivial. There is also no room for any nontrivial extensions in the multiplicative structure. Thus we have proved

**3.1.25. THEOREM.** *The  $E_2$ -term for the mod (2) Adams spectral sequence for  $\pi_*(bo)$ ,*

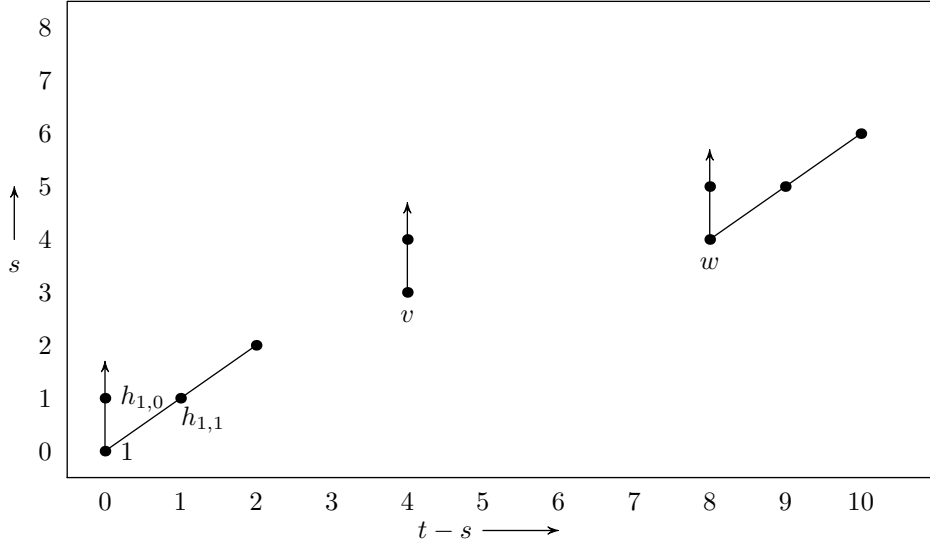
$$\text{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \text{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

is

$$P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}),$$

where

$$h_{10} \in \text{Ext}^{1,1}, \quad h_{11} \in \text{Ext}^{1,2}, \quad v \in \text{Ext}^{3,7}, \quad \text{and} \quad w \in \text{Ext}^{4,12}. \quad \square$$



This  $E_2$ -term is displayed in the accompanying figure. A vertical arrow over an element indicates that  $h_{10}^s x$  is also present and nontrivial for all  $s > 0$ .

Now we claim that this Adams spectral sequence also collapses, i.e.,  $E_2 = E_\infty$ . Inspection shows that the only possible nontrivial differential is  $d_r(w^n h_{11}) = w^n h_{10}^{n+r}$ . However,  $bo$  is a ring spectrum so by 2.3.3 the differentials are derivations and we cannot have  $d_r(h_{11}) = h_{10}^{r+1}$  because it contradicts the relation  $h_{10} h_{11} = 0$ . The extension problem is solved by 3.1.3, giving

3.1.26. THEOREM (Bott [1]).

$$\pi_*(bo) = \mathbf{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with  $\eta \in \pi_1$ ,  $\alpha \in \pi_4$ ,  $\beta \in \pi_8$ , i.e., for  $i \geq 0$

$$\pi_i(bo) = \begin{cases} \mathbf{Z} & \text{if } i \equiv 0 \pmod{4} \\ \mathbf{Z}/2 & \text{if } i \equiv 1 \text{ or } 2 \pmod{8} \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

For future reference we will compute  $\text{Ext}_{A(1)}(\mathbf{Z}/(2), M)$  for  $M = A(0)_* \equiv E(\xi_1)$  and  $M = Y \equiv P(\xi_1)/(\xi_1^4)$ . Topologically these are the Adams  $E_2$ -terms for the mod (2)-Moore spectrum smashed with  $bo$  and  $bu$ , respectively. We use the Cartan–Eilenberg spectral sequence as above and our  $E_2$ -term is

$$\text{Ext}_{\Phi}(\mathbf{Z}/(2), \text{Ext}_{E(\xi_2)}(\mathbf{Z}/(2), M)).$$

An easy calculation shows that

$$E_2 = P(h_{11}, h_{20}) \quad \text{for } M = A(0)_*$$

and

$$E_2 = P(h_{20}) \quad \text{for } M = Y.$$

In the latter case the Cartan–Eilenberg spectral sequence collapses. In the former case the differentials are not derivations since  $A(0)_*$  is not a comodule algebra.

From 3.1.23 we get  $d_3(h_{20}^2) = h_{11}^3$ , so

$$E_\infty = E_4 = P(w) \otimes \{1, h_{11}, h_{11}^2, h_{20}, h_{20}h_{11}, h_{20}h_{11}^2\}.$$

This Ext is not an algebra but it is a module over  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ . We will show that there is a nontrivial extension in this structure, namely  $h_{10}h_{20} = h_{11}^2$ . We do this by computing in the cobar complex  $C_{A(1)_*}(A(0)_*)$ . There the class  $h_{20}$  is represented by  $[\xi_2] + [\xi_1^2]\xi_1$ , so  $h_{10}h_{20}$  is represented by  $[\xi_1|\xi_2] + [\xi_1|\xi_1^2]\xi_1$ . The sum of this and  $[\xi_1^2|\xi_1^2]$  (which represents  $h_{1,1}^2$ ) is the coboundary of  $[\xi_1\xi_2] + [\xi_1^3 + \xi_2]\xi_1$ .

From these considerations we get

3.1.27. THEOREM. *As a module over  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$  (3.1.25) we have*

(a)  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), A(0)_*)$  is generated by  $1 \in \text{Ext}^{0,0}$  and  $h_{20} \in \text{Ext}^{1,3}$  with  $h_{10} \cdot 1 = 0$ ,  $h_{10}h_{20} = h_{11}^2 \cdot 1$ ,  $v \cdot 1 = 0$ , and  $vh_{20} = 0$ .

(b)  $\text{Ext}_{A(1)_*}(\mathbf{Z}(2), Y)$  is generated by  $\{h_{20}^i : 0 \leq i \leq 3\}$  with  $h_{10}h_{20}^i = h_{11}h_{20}^i = vh_{20}^i = 0$ .  $\square$

We will also need an odd primary analog of 3.1.27(a).  $A(1) = E(\tau_0, \tau_1) \otimes P(\xi_1)/(\xi_1^p)$  is the dual to the subalgebra of  $A$  generated by the Bockstein  $\beta$  and the Steenrod reduced power  $P^1$ . Instead of generalizing the extension 3.1.19 we use

$$P(0)_* \rightarrow A(0)_* \rightarrow E(1)_*,$$

where  $P(0)_* = P(\xi_1)/(\xi_1^p)$  and  $E(1)_* = E(\tau_0, \tau_1)$ . The Cartan–Eilenberg spectral sequence  $E_2$ -term is therefore

$$\text{Ext}_{p(0)_*}(\mathbf{Z}/(p), \text{Ext}_{E(1)_*}(\mathbf{Z}/(p), A(0)_*)),$$

where  $A(0)_* = E(\tau_0)$ . An easy calculation gives

3.1.28. THEOREM. *For  $p > 2$*

$$\text{Ext}_{A(1)_*}(\mathbf{Z}/(p), A(0)_*) = E(h_0) \otimes P(a_1, b_0),$$

where  $h_0 \in \text{Ext}^{1,q}$ ,  $a_1 \in \text{Ext}^{1,q+1}$ , and  $b_0 \in \text{Ext}^{2,pq}$  are represented by  $[\xi_1]$ ,  $[\xi_1]\tau_0 + [\tau_1]$ , and  $\sum_{0 < i < p} p^{-1} \binom{p}{i} [\xi_1^i|\xi_1^{p-i}]$ , respectively.  $\square$

## 2. The May Spectral Sequence

In this section we discuss a method for computing the classical Adams  $E_2$ -term,  $\text{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ , which we will refer to simply as Ext. For the reader hoping to understand the classical Adams spectral sequence we offer two pieces of advice. First, do as many explicit calculations as possible yourself. Seeing someone else do it is no substitute for the insight gained by firsthand experience. The computations sketched below should be reproduced in detail and, if possible, extended by the reader. Second, the  $E_2$ -term and the various patterns within it should be examined and analyzed from as many viewpoints as possible. For this reason we will describe several methods for computing Ext. For reasons to be given in Section 4.4, we will limit our attention here to the prime 2.

The most successful method for computing Ext through a range of dimensions is the spectral sequence of May [1]. Unfortunately, crucial parts of this material have never been published. The general method for computing Ext over a Hopf algebra is described in May [2], and the computation of the differentials in the May spectral sequence for the Steenrod algebra through dimension 70 is described by Tangora [1]. A revised account of the May  $E_2$ -term is given in May [4].

In our language May's approach is to filter  $A_*$  by copowers of the unit coideal (A1.3.10) and to study the resulting spectral sequence. Its  $E_2$ -term is the Ext over the associated graded Hopf algebra  $E^0A_*$ . The structure of this Hopf algebra is as follows.

3.2.1. THEOREM (May [1]). (a) For  $p = 2$ ,

$$E^0A_* = E(\xi_{i,j} : i > 0, j \geq 0)$$

with coproduct

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j},$$

where  $\xi_{0,j} = 1$  and  $\xi_{i,j} \in E_i^0A_*$  is the projection of  $\xi_i^{2^j}$ .

(b) For  $p > 2$ ,

$$E^0A_* = E(\tau_i : i \geq 0) \otimes T(\xi_{i,j} : i > 0, j \geq 0)$$

with coproduct given by

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j}$$

and

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{0 \leq k \leq i} \xi_{i-k,k} \otimes \tau_k,$$

where  $T(\ )$  denotes the truncated polynomial algebra of height  $p$  on the indicated generators,  $\tau_i \in E_{i+1}^0A_*$  is the projection of  $\tau_i \in A_*$ , and  $\xi_{i,j} \in E_i^0A_p^*$  is the projection of  $\xi_i^{p^j}$ .  $\square$

May actually filters the Steenrod algebra  $A$  rather than its dual, and proves that the associated bigraded Hopf algebra  $E_0A$  is primitively generated, which is dual to the statement that each primitive in  $E^0A_p^*$  is a generator. A theorem of Milnor and Moore [3] says that every graded primitively generated Hopf algebra is isomorphic to the universal enveloping algebra of a restricted Lie algebra. For  $p = 2$  let  $x_{i,j} \in E_0A$  be the primitive dual to  $\xi_{i,j}$ . These form the basis of a Lie algebra under commutation, i.e.,

$$[x_{i,j}, x_{k,m}] \equiv x_{i,j}x_{k,m} - x_{k,m}x_{i,j} = \delta_k^i x_{i,m} - \delta_i^m x_{k,j}$$

where  $\delta_j^i$  is the Kronecker  $\delta$ . A *restriction* in a graded Lie algebra  $L$  is an endomorphism  $\xi$  which increases the grading by a factor of  $p$ . In the case at hand this restriction is trivial. The universal enveloping algebra  $V(L)$  of a restricted Lie algebra  $L$  (often referred to as the restricted enveloping algebra) is the associative algebra generated by the elements of  $L$  subject to the relations  $xy - yx = [x, y]$  and  $x^p = \xi(x)$  for  $x, y \in L$ .

May [1] constructs an efficient complex (i.e., one which is much smaller than the cobar complex) for computing Ext over such Hopf algebras. In particular, he proves

3.2.2. THEOREM (May [1]). For  $p = 2$ ,  $\text{Ext}_{E^0A_*}^{***}(\mathbf{Z}/(2), \mathbf{Z}/(2))$  (the third grading being the May filtration) is the cohomology of the complex

$$V^{***} = P(h_{i,j} : i > 0, j \geq 0)$$

with  $d(h_{i,j}) = \sum_{0 < k < i} h_{k,j} h_{i-k,k+j}$ , where  $h_{i,j} \in V^{1,2^j(2^i-1),i}$  corresponds to  $\xi_{i,j} \in A_2^*$ .  $\square$



Our  $h_{i,j}$  is written  $R_i^j$  by May [1] and  $R_{ji}$  by Tangora [1], but as  $h_{i,j}$  (in a slightly different context) by Adams [3]. Notice that in  $C^*(\mathbf{Z}/(2))$  one has  $d[\xi_i^{2^j}] = \sum_{0 < k < i} [\xi_{k-i}^{2^{i+j}} | \xi_k^{2^j}]$ , which corresponds to the formula for  $d(h_{i,j})$  above. The theorem asserts that  $E^0 C^*(\mathbf{Z}/(2))$  is chain homotopy equivalent to the polynomial algebra on the  $[\xi_{i,j}]$ . We will see below (3.2.7) that  $C^*(\mathbf{Z}/(2))$  itself does not enjoy the analogous property and that the May differentials are a measure of its failure to do so.

From 3.2.2 May derives a spectral sequence of the following form.

3.2.3. THEOREM (May [1]). *There is a spectral sequence converging to*

$$\mathrm{Ext}_{A_*}^{***}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

with  $E_1^{***} = V^{***}$  and  $d_r: E_r^{s,t,u} \rightarrow E_r^{s+1,t,u+1-r}$ .

PROOF OF 3.2.2 AND 3.2.3. The spectral sequence is a reindexed form of that of A1.3.9, so 3.2.3 follows from 3.2.2. We will show that the same spectral sequence can be obtained more easily by using a different increasing filtration of  $A_*$ . An increasing filtration is defined by setting  $|\xi_i^{2^j}| = 2i - 1$ . Then it follows easily that this  $E^0 A_*$  has the same algebra structure as in 3.2.1 but with each  $\xi_{i,j}$  primitive. Hence  $E^0 A_*$  is dual to an exterior algebra and its Ext is  $V^{***}$  (suitably reindexed) by 3.1.9. A1.3.9 gives us a spectral sequence associated to this filtration. In particular, it will have  $d_1(h_{i,j}) = \sum h_{k,j} h_{i-k,j+k}$  as in 3.2.2. Since all of the  $h_{i,j}$  have odd filtration degree, all of the nontrivial differentials must have odd index. It follows that this spectral sequence can be reindexed in such a way that each  $d_{2r-1}$  gets converted to a  $d_r$  and the resulting spectral sequence is that of 3.2.3.  $\square$

For  $p > 2$  the spectral sequence obtained by this method is not equivalent to May's but is perhaps more convenient as the latter has an  $E_1$ -term which is nonassociative. In the May filtration one has  $|\tau_{i-1}| = |\xi_i^{p^j}| = i$ . If we instead set  $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i - 1$ , then the resulting  $E^0 A_*$  has the same algebra structure (up to indexing) as that of 3.2.1(b), but all of the generators are primitive. Hence it is dual to a product of exterior algebras and truncated polynomial algebras of height  $p$ . To compute its Ext we need, in addition to 3.1.11, the following result.

3.2.4. LEMMA. *Let  $\Gamma = T(x)$  with  $\dim x = 2n$  and  $x$  primitive. Then*

$$\mathrm{Ext}_{\Gamma}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = E(h) \otimes P(b),$$

where

$$h \in \mathrm{Ext}^1 \quad \text{is represented in } C_{\Gamma}(\mathbf{Z}/(p)) \text{ by } [x]$$

and

$$b \in \mathrm{Ext}^2 \quad \text{by} \quad \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} [x^i | x^{p-i}]. \quad \square$$

The proof is a routine calculation and is left to the reader.

To describe the resulting spectral sequence we have

3.2.5. THEOREM. *For  $p > 2$  the dual Steenrod algebra (3.1.1)  $A_*$  can be given an increasing filtration with  $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i - 1$  for  $i - 1, j \geq 0$ . The associated bigraded Hopf algebra  $E^0 A_*$  is primitively generated with the algebra structure of 3.2.1(b). In the associated spectral sequence (A1.3.9)*

$$E_1^{***} = E(h_{i,j}: i > 0, j \geq 0) \otimes P(b_{i,j}: i > 0, j \geq 0) \otimes P(a_i: i \geq 0),$$

where

$$\begin{aligned} h_{i,j} &\in E_1^{1,2(p^i-1)p^j,2i-1}, \\ b_{i,j} &\in E_1^{2,2(p^i-1)p^{1+j},p(2i-1)}, \end{aligned}$$

and

$$a_i \in E_1^{1,2p^i-1,2i+1}$$

( $h_{i,j}$  and  $a_i$  correspond respectively to  $\xi_i^{p^j}$  and  $\tau_i$ ). One has  $d_r: E_r^{s,t,u} \rightarrow E_r^{s-1,t,u-r}$ , and if  $x \in E_r^{s,t,u}$  then  $d_r(xy) = d_r(x)y + (-1)^{t-s}xd_r(y)$ .  $d_1$  is given by

$$\begin{aligned} d_1(h_{i,j}) &= - \sum_{0 < k < i} h_{k,j}h_{i-k,k+j}, \\ d_1(a_i) &= - \sum_{0 \leq k < i} a_k h_{i-k,k}, \\ d_1(b_{i,j}) &= 0. \end{aligned} \quad \square$$

In May's spectral sequence for  $p > 2$ , indexed as in 3.2.3, the  $E_1$ -term has the same additive structure (up to indexing) as 3.2.5 and  $d_1$  is the same on the generators, but it is a derivation with respect to a different multiplication, which is unfortunately nonassociative.

We will illustrate this nonassociativity with a simple example for  $p = 3$ .

3.2.6. EXAMPLE. In the spectral sequence of 3.2.5 the class  $h_{10}h_{20}$  corresponds to a nontrivial permanent cycle which we call  $g_0$ . Clearly  $h_{10}g_0=0$  in  $E_\infty$ , but for  $p = 3$  it could be a nonzero multiple of  $h_{11}b_{10}$  in Ext. The filtration of  $h_{10}g_0$  and  $h_{11}b_{10}$  are 5 and 4, respectively. Using Massey products (A1.4), one can show that this extension in the multiplicative structure actually occurs in the following way. Up to nonzero scalar multiplication we have  $b_{10} = \langle h_{10}, h_{10}, h_{10} \rangle$  and  $g_0 = \langle h_{10}, h_{10}, h_{11} \rangle$  (there is no indeterminacy), so

$$\begin{aligned} h_{10}g_0 &= h_{10}\langle h_{10}, h_{10}, h_{11} \rangle \\ &= \langle h_{10}, h_{10}, h_{10} \rangle h_{11} \\ &= b_{10}h_{11}. \end{aligned}$$

Now in the May filtration, both  $h_{10}g_0$  and  $b_{10}h_{11}$  have weight 4, so this relation must occur in  $E_1$ , i.e., we must have

$$0 \neq h_{10}g_0 = h_{10}(h_{10}g_0) \neq (h_{10}h_{10})g_0 = 0,$$

so the multiplication is nonassociative.

To see a case where this nonassociativity affects the behavior of May's  $d_1$ , consider the element  $h_{10}h_{20}h_{30}$ . It is a  $d_1$  cycle in 3.2.5. In  $E_2$  the Massey product  $\langle h_{10}, h_{11}, h_{12} \rangle$  is defined and represented by  $\pm(h_{10}h_{21} + h_{20}h_{12}) = \pm d_1(h_{30})$ . Hence in Ext we have

$$\begin{aligned} 0 &= g_0\langle h_{10}, h_{11}, h_{12} \rangle \\ &= \langle g_0h_{10}, h_{11}, h_{12} \rangle \\ &= \pm\langle h_{11}b_{10}, h_{11}, h_{12} \rangle \\ &= \pm b_{10}\langle h_{11}, h_{11}, h_{12} \rangle. \end{aligned}$$

The last bracket is represented by  $\pm h_{11}h_{21}$ , which is a permanent cycle  $g_1$ . This implies (A1.4.12)  $d_2(h_{10}h_{20}h_{30}) = \pm b_{11}g_1$ . In May's grading this differential is a  $d_1$ .

Now we return to the prime 2.

3.2.7. EXAMPLE. The computation leading to 3.1.25, the Adams  $E_2$ -term for  $bo$ , can be done with the May spectral sequence. One filters  $A(1)_*$  (see 3.1.18) and gets the sub-Hopf algebra of  $E^0A_*$  generated by  $\xi_{10}$ ,  $\xi_{11}$ , and  $\xi_{20}$ . The complex analogous to 3.2.2 is  $P(h_{10}, h_{11}, h_{20})$  with  $d(h_{20}) = h_{10}h_{11}$ . Hence the May  $E_2$ -term is the Cartan–Eilenberg  $E_3$ -term (3.1.22) suitably reindexed, and the  $d_3$  of 3.1.23 corresponds to a May  $d_2$ .

We will illustrate the May spectral sequence for the mod (2) Steenrod algebra through the range  $t - s \leq 13$ . This range is small enough to be manageable, large enough to display some nontrivial phenomena, and is convenient because no May differentials originate at  $t - s = 14$ . May [1, 4] was able to describe his  $E_2$ -term (including  $d_2$ ) through a very large range,  $t - s \leq 164$  (for  $t - s \leq 80$  this description can be found in Tangora [1]). In our small range the  $E_2$ -term is as follows.

3.2.8. LEMMA. *In the range  $t - s \leq 13$  the  $E_2$ -term for the May spectral sequence (3.2.3) has generators*

$$\begin{aligned} h_j &= h_{1,j} \in E_2^{1,2^j,1}, \\ b_{i,j} &= h_{i,j}^2 \in E_2^{2,2^{j+1}(2^i-1),2i}, \end{aligned}$$

and

$$x_7 = h_{20}h_{21} + h_{11}h_{30} \in E_2^{2,9,4}$$

with relations

$$\begin{aligned} h_j h_{j+1} &= 0, \\ h_2 b_{20} &= h_0 x_7, \end{aligned}$$

and

$$h_2 x_7 = h_0 b_{21}. \quad \square$$

This list of generators is complete through dimension 37 if one adds  $x_{16}$  and  $x_{34}$ , obtained from  $x_7$  by adding 1 and 2 to the second component of each index. However, there are many more relations in this larger range.

The  $E_2$ -term in this range is illustrated in FIG. 3.2.9. Each dot represents an additive generator. If two dots are joined by a vertical line then the top element is  $h_0$  times the lower element; if they are joined by a line of slope  $\frac{1}{3}$  then the right-hand element is  $h_2$  times the left-hand element. Vertical and diagonal arrows mean that the element has linearly independent products with all powers of  $h_0$  and  $h_1$ , respectively.

3.2.10. LEMMA. *The differentials in 3.2.3 in this range are given by*

- (a)  $d_r(h_j) = 0$  for all  $r$ ,
- (b)  $d_2(b_{2,j}) = h_j^2 h_{j+2} + h_{j+1}^3$ ,
- (c)  $d_2(x_7) = h_0 h_2^2$ ,
- (d)  $d_2(b_{30}) = h_1 b_{21} + h_3 b_{20}$ , and
- (e)  $d_4(b_{20}^2) = h_0^4 h_3$ .

PROOF. In each case we make the relevant calculation in the cobar complex  $C_{A_*}(\mathbf{Z}/(2))$  of 3.1.2. For (a),  $[\xi_i^{2^j}]$  is a cycle. For (b) we have

$$d([\xi_2|\xi_2] + [\xi_1^2|\xi_1\xi_2] + [\xi_2\xi_1^2|\xi_1]) = [\xi_1^2|\xi_1^2|\xi_1^2] + [\xi_1^4|\xi_1|\xi_1].$$

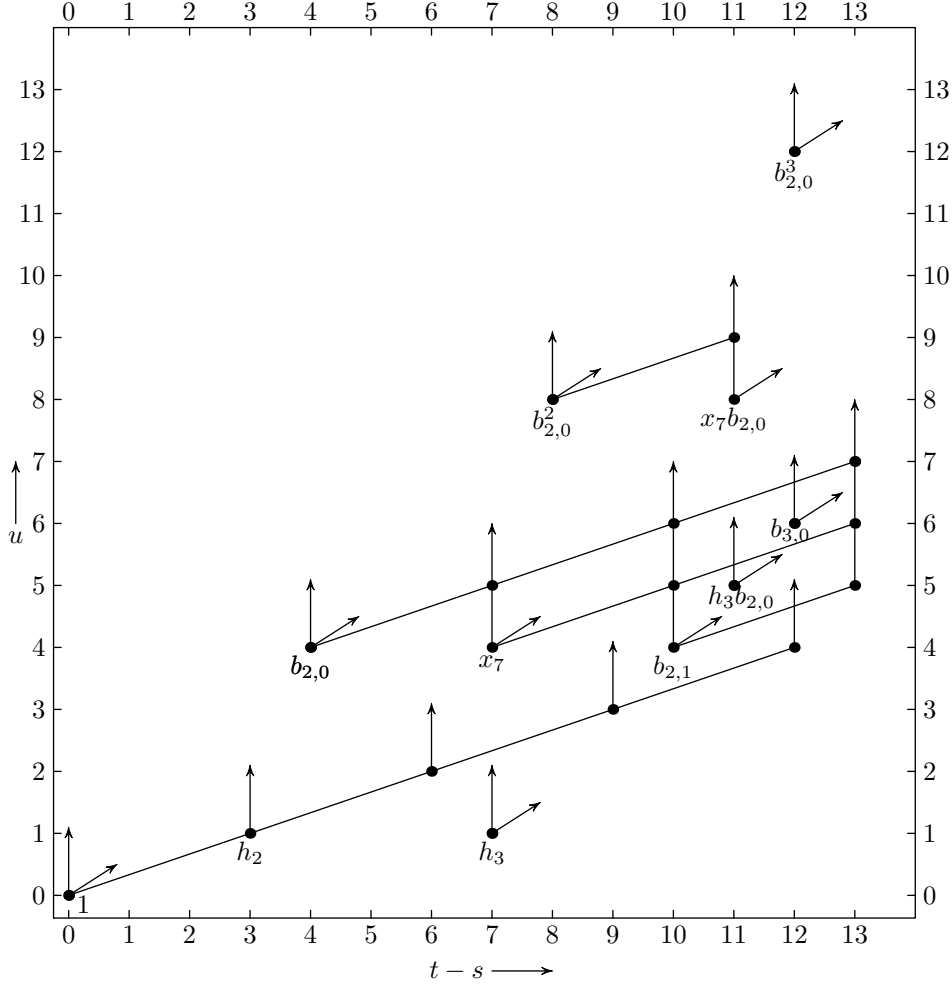


FIGURE 3.2.9. The May  $E_2$ -term for  $p = 2$  and  $t - s \leq 13$

For (c) we have

$$d([\xi_1^3 + \xi_2]|\xi_2^2] + [(\xi_3 + \xi_1^4\xi_2 + \xi_1\xi_2^2 + \xi_1^7)|\xi_1^2] + [\xi_1|\xi_1^2\xi_2^2]) = [\xi_1|\xi_1^4|\xi_1^4].$$

For (d) we use the relation  $x_7^2 = h_1^2b_{30} + b_{20}b_{21}$  (which follows from the definition of the elements in question); the right-hand term must be a cycle in  $E_2$  and we can use this fact along with (b) to calculate  $d_2(b_{30})$ .

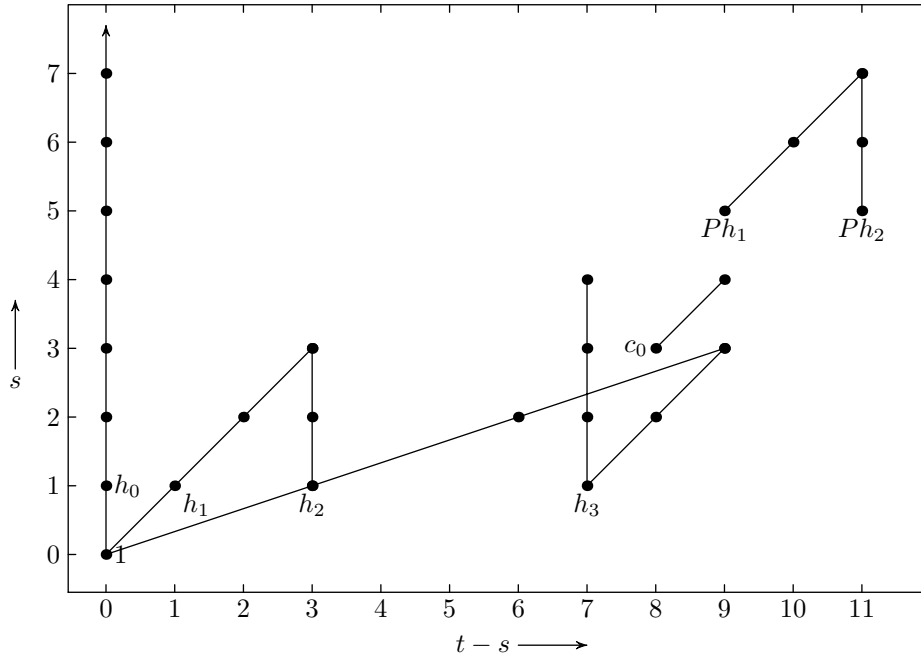
Part (e) follows from the fact that  $h_0^4h_3 = 0$  in Ext, for which three different proofs will be given below. These are by direct calculation in the  $\Lambda$ -algebra (Section 3.3), by application of a Steenrod squaring operation to the relation  $h_0h_1 = 0$ , and by the Adams vanishing theorem (3.4.5).  $\square$

It follows by inspection that no other differentials can occur in this range. Since no May differentials originate in dimension 14 we get

3.2.11. THEOREM.  $\text{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}(2))$  for  $t - s \leq 13$  and  $s \leq 7$  is generated as a vector space by the elements listed in the accompanying table. (There are no

generators for  $t - s = 12$  and  $13$ , and the only generators in this range with  $s > 7$  are powers of  $h_0$ .)

In the table  $c_0$  corresponds to  $h_1x_7$ , while  $Px$  corresponds to  $b_{2,0}^2x$ . There are relations  $h_1^3 = h_0^2h_2$ ,  $h_2^3 = h_1^2h_3$ , and  $Ph_1^3 = Ph_0^2h^2 = h_0^2Ph_2$ .  $\square$



Inspecting this table one sees that there are no differentials in the Adams spectral sequence in this range, and all of the group extensions are solved by 3.1.3 and we get

3.2.12. COROLLARY. *For  $n \leq 13$  the 2-component of  $\pi_n(S^0)$  are given by the following table.*

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_n(S^0)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(16)$	$(\mathbf{Z}/(2))^2$	$(\mathbf{Z}/(2))^3$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0

In general the computation of higher May differentials is greatly simplified by the use of algebraic Steenrod operations (see Section A1.5). For details see Nakamura [1].

Now we will use the May spectral sequence to compute  $\text{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$ , where  $A(n)_* = P(\xi_1, \xi_2, \dots, \xi_{n+1})/(\xi_i^{2^{n+2} - 1})$  is dual to the subalgebra  $A(n) \subset A$  generated by  $Sq^1, Sq^2, \dots, Sq^{2^n}$ . We filter  $A(2)_*$  just as we filter  $A_*$ . The resulting May  $E_1$ -term is  $P(h_{11}, h_{12}, h_{20}, h_{21}, h_{30})$  with  $d_1(h_{1,i}) = 0 = d_1(h_{20})$ ,  $d_1(h_{21}) = h_{11}h_{12}$ , and  $d_1(h_{30}) = h_{20}h_{12}$ . This gives

$$(3.2.13) \quad E_2 = P(b_{21}, b_{30}) \otimes ((P(h_{11}, h_{20}) \otimes E(x_7)) \oplus \{h_{12}^i : i > 0\}),$$

where  $b_{21} = h_{21}^2$ ,  $b_{30} = h_{30}^2$ , and  $x_7 = h_{11}h_{30} + h_{20}h_{21}$ . The  $d_2$ 's are trivial except for

$$(3.2.14) \quad d_2(h_{20}^2) = h_{11}^3, \quad d_2(b_{21}) = h_{12}^3, \quad \text{and} \quad d_2(b_{30}) = h_{11}b_{21}.$$

Since  $A(0)_*$  is not a comodule algebra, this is not a spectral sequence of algebras, but there is a suitable pairing with the May spectral sequence of 3.2.3.

Finding the resulting  $E_3$ -term requires a little more ingenuity. In the first place we can factor out  $P(b_{30}^2)$ , i.e.,  $E_2 = E_2/(b_{30}^2) \otimes P(b_{30}^2)$  as complexes. We denote  $E_2/(b_{30}^2)$  by  $\overline{E}_2$  and give it an increasing filtration as a differential algebra by letting  $F_0 = P(h_{11}, h_{20}) \otimes E(x_7) \oplus \{h_{12}^i : i > 0\}$  and letting  $b_{21}, b_{30} \in F_1$ . The cohomology of the subcomplex  $F_0$  is essentially determined by 3.1.27(a), which gives  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), A(0)_*)$ . Let  $B$  denote this object suitably regraded for the present purpose. Then we have

$$(3.2.15) \quad H^*(F_0) = B \otimes E(x_7) \oplus \{h_{12}^i : i > 0\}.$$

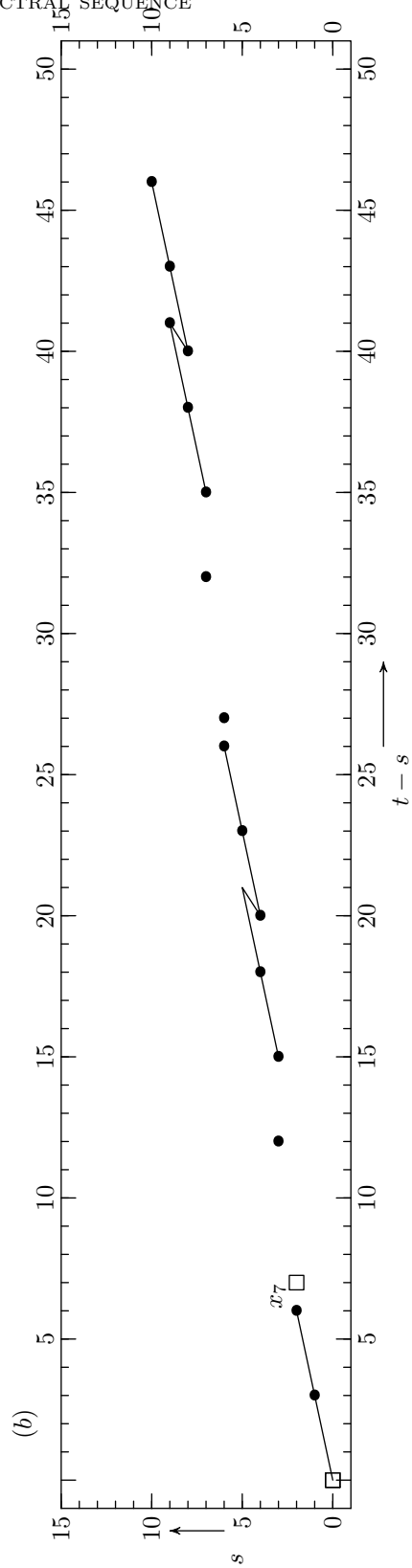
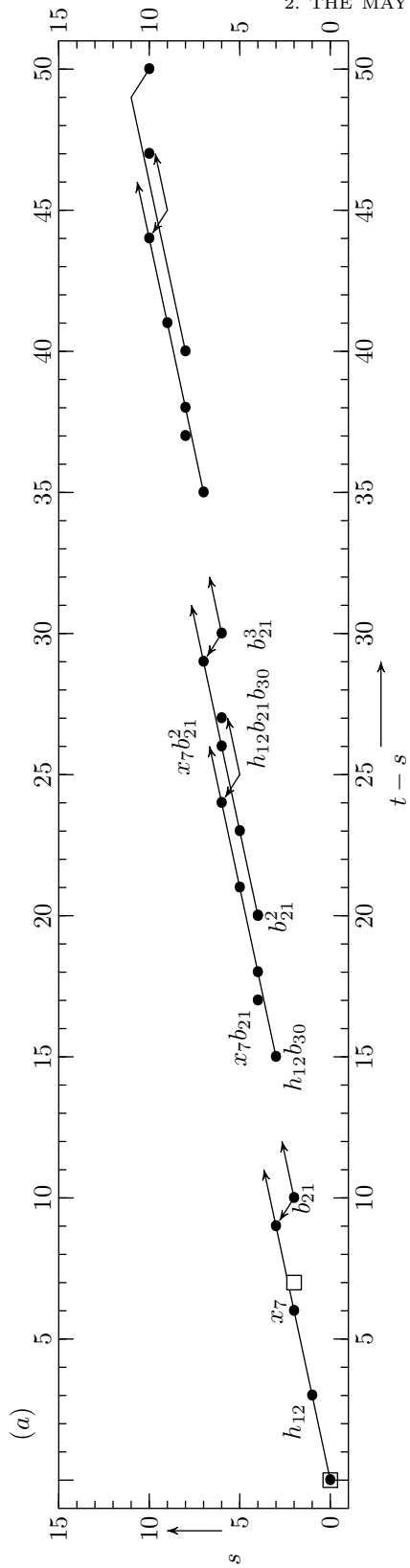
For  $k > 0$  we have  $F_k/F_{k-1} = \{b_{21}^k, b_{21}^{k-1}b_{30}\} \otimes F_0$  with  $d_2(b_{21}^{k-1}b_{30}) = b_{21}^k h_{11}$ . Its cohomology is essentially determined by 3.1.27(b), which describes  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(2), Y)$ . Let  $C$  denote this object suitably regraded, i.e.,  $C = P(h_{20})$ . Then we have for  $k > 0$

$$(3.2.16) \quad H^*(F_k/F_{k-1}) = C\{b_{21}^k\} \otimes E(X_7) \oplus \{b_{21}^k h_{12}^i, b_{30} b_{21}^{k-1} h_{12}^i : i > 0\}.$$

This filtration leads to a spectral sequence converging to  $\overline{E}_3$  in which the only nontrivial differential sends

$$b_{21}^k b_{30}^\varepsilon h_{12}^i \quad \text{to} \quad k b_{21}^{k-1} b_{30}^\varepsilon h_{12}^{i+3}$$

for  $\varepsilon = 0, 1$ ,  $k > 0$  and  $i \geq 1$ . This is illustrated in FIG. 3.2.17(a), where a square indicates a copy of  $B$  and a large circle indicates a copy of  $C$ . Arrows pointing to the left indicate further multiplication by  $h_{12}$ , and diagonal lines indicate differentials. Now  $b_{21}$  supports a copy of  $C$  and a differential. This leads to a copy of  $C$  in  $\overline{E}_3$  supported by  $h_{20}b_{21}$  shown in 3.2.17(b). There is a nontrivial multiplicative extension  $h_{20}h_{12}b_{30} = x_7b_{21}$  which we indicate by a copy of  $C$  in place of  $h_{12}b_{30}$  in (b). Fig. 3.2.17(b) also shows the relation  $h_{11}b_{21}^2 = h_{12}^3b_{30}$ .



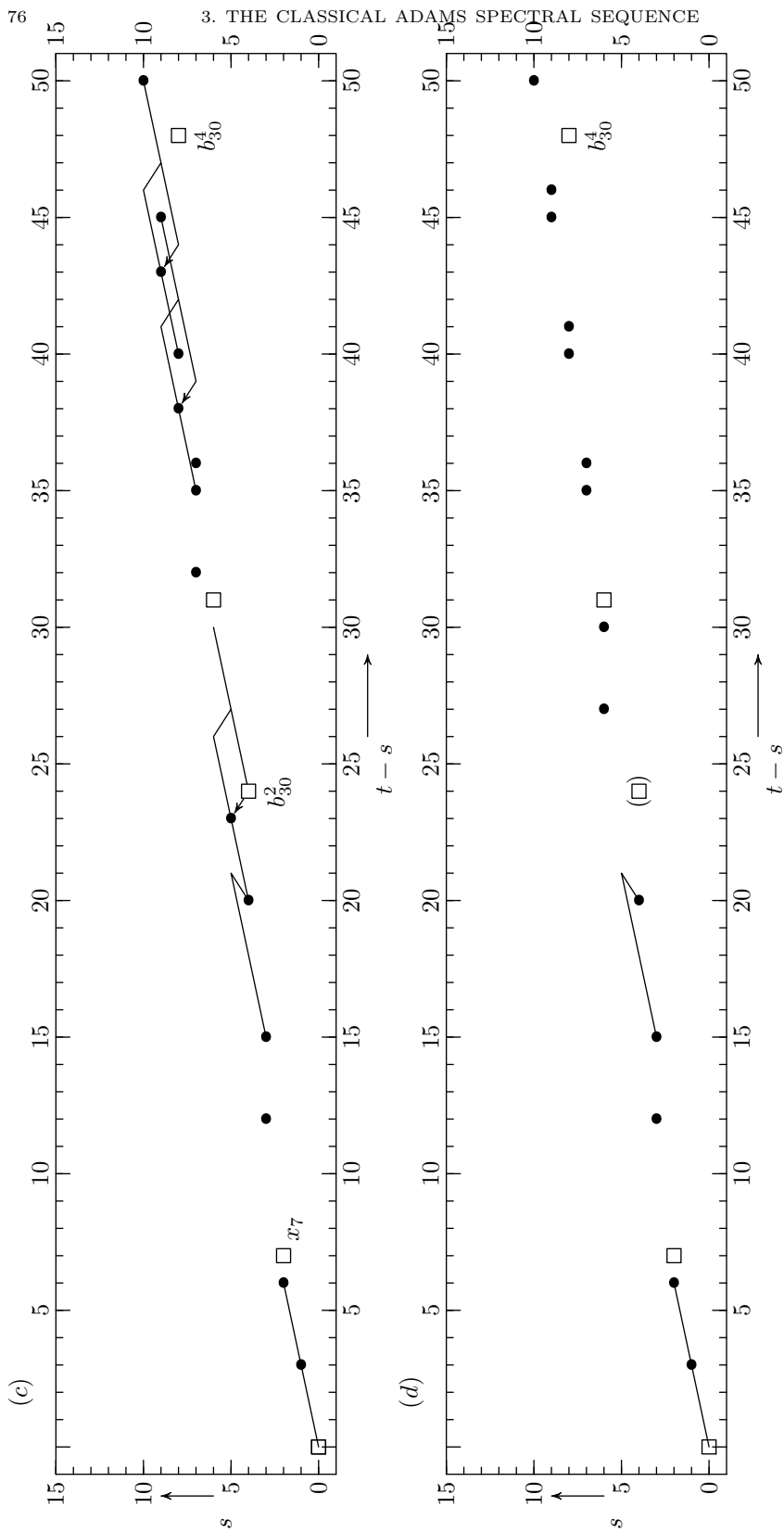


FIGURE 3.2.17. The May spectral sequence for  $\text{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$ . (a) The spectral sequence for  $\bar{E}_3$ ; (b) the  $\bar{E}_3$ -term; (c) differentials in  $E_3$ ; (d)  $E_\infty$



The differentials in  $E_3$  are generated by  $d_3(b_{30}^2) = h_{12}b_{21}^2$  and are shown in 3.2.17(c). The resulting  $E_4 = E_\infty$  is shown in 3.2.17(d), where the symbol in place of  $b_{30}^2$  indicates a copy of  $B$  with the first element missing.

### 3. The Lambda Algebra

In this section we describe the lambda algebra of Bousfield et al. [2] at the prime 2 and the algorithm suggested by it for computing Ext. For more details, including references, see Tangora [2, 3] and Richter [1]. For most of this material we are indebted to private conversations with E. B. Curtis. It is closely related to that of Section 1.5.

The lambda algebra  $\Lambda$  is an associative differential bigraded algebra whose cohomology, like that of the cobar complex, is Ext. It is much smaller than the cobar complex; it is probably the smallest such algebra generated by elements of cohomological degree one with cohomology isomorphic to Ext. Its greatest attraction, which will not be exploited here, is that it contains for each  $n > 0$  a subcomplex  $\Lambda(n)$  whose cohomology is the  $E_2$ -term of a spectral sequence converging to the 2-component of the *unstable* homotopy groups of  $S^n$ . In other words  $\Lambda(n)$  is the  $E_1$ -term of an unstable Adams spectral sequence.

More precisely,  $\Lambda$  is a bigraded  $\mathbf{Z}/(2)$ -algebra with generators  $\lambda_n \in \Lambda^{1,n+1}$  ( $n \geq 0$ ) and relations

$$(3.3.1) \quad \lambda_i \lambda_{2i+1+n} = \sum_{j \geq 0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j} \quad \text{for } i, n \geq 0$$

with differential

$$(3.3.2) \quad d(\lambda_n) = \sum_{j \geq 1} \binom{n-j}{j} \lambda_{n-j} \lambda_{j-1}.$$

Note that  $d$  behaves formally like left multiplication by  $\lambda_{-1}$ .

**3.3.3. DEFINITION.** *A monomial  $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_s} \in \Lambda$  is admissible if  $2i_r \geq i_{r+1}$  for  $1 \leq r < s$ .  $\Lambda(n) \subset \Lambda$  is the subcomplex spanned by the admissible monomials with  $i_1 < n$ .*

The following is an easy consequence of 3.3.1 and 3.3.2.

**3.3.4. PROPOSITION.**

- (a) *The admissible monomials constitute an additive basis for  $\Lambda$ .*
- (b) *There are short exact sequences of complexes*

$$0 \rightarrow \Lambda(n) \rightarrow \Lambda(n+1) \rightarrow \Sigma^n \Lambda(2n+1) \rightarrow 0. \quad \square$$

The significant property of  $\Lambda$  is the following.

**3.3.5. THEOREM** (Bousfield et al. [2]). (a)  $H(\Lambda) = \text{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ , the classical Adams  $E_2$ -term for the sphere.

(b)  $H(\Lambda(n))$  is the  $E_2$ -term of a spectral sequence converging to  $\pi_*(S^n)$ .

(c) The long exact sequence in cohomology (3.3.6) given by 3.3.4(b) corresponds to the EHP sequence, i.e., to the long exact sequence of homotopy groups of the fiber sequence (at the prime 2)

$$S^n \rightarrow \Omega S^{n+1} \rightarrow \Omega S^{2n+1} \quad (\text{see 1.5.1}). \quad \square$$

The spectral sequence of (b) is the *unstable Adams spectral sequence*. The long exact sequence in (c) above is

$$(3.3.6) \quad \begin{array}{c} \rightarrow H^{s,t}(\Lambda(n)) \xrightarrow{E} H^{s,t}(\Lambda(n+1)) \xrightarrow{H} H^{s-1,t-n-1}(\Lambda(2n+1)) \\ \xrightarrow{P} H^{s+1,t}(\Lambda(n)) \rightarrow . \end{array}$$

The letters  $E$ ,  $H$ , and  $P$  stand respectively for suspension (*Einhangung* in German), Hopf invariant, and Whitehead product. The map  $H$  is obtained by dropping the first factor of each monomial. This sequence leads to an inductive method for calculating  $H^{s,t}(\Lambda(n))$  which we will refer to as the *Curtis algorithm*.

Calculations with this algorithm up to  $t = 51$  (which means up to  $t - s = 33$ ) are recorded in an unpublished table prepared by G. W. Whitehead. Recently, Tangora [4] has programmed a computer to find  $H^{s,t}(\Lambda)$  at  $p = 2$  for  $t \leq 48$  and  $p = 3$  for  $t \leq 99$ . Some related machine calculations are described by Wellington [1].

For the Curtis algorithm, note that the long exact sequences of 3.3.6 for all  $n$  constitute an exact couple (see Section 2.1) which leads to the following spectral sequence, similar to that of 1.5.7.

3.3.7. PROPOSITION (Algebraic EHP spectral sequence).

(a) *There is a trigraded spectral sequence converging to  $H^{s,t}(\Lambda)$  with*

$$E_1^{s,t,n} = H^{s-1,t-n}(\Lambda(2n-1)) \quad \text{for } s > 0$$

and

$$E_1^{0,t,n} = \begin{cases} \mathbf{Z}/(2) & \text{for } t = n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and  $d_r: E_r^{s,t,n} \rightarrow E_r^{s+1,t,n-r}$ .

(b) *For each  $m > 0$  there is a similar spectral sequence converging to  $H^{s,t}(\Lambda(m))$  with*

$$E_1^{s,t,n} = \begin{cases} \text{as above} & \text{for } n \leq m \\ 0 & \text{for } n > m. \end{cases} \quad \square$$

The EHP sequence in homotopy leads to a similar spectral sequence converging to stable homotopy filtered by sphere of origin which is described in Section 1.5.

At first glance the spectral sequence of 3.3.7 appears to be circular in that the  $E_1$ -term consists of the same groups one is trying to compute. However, for  $n > 1$  the groups in  $E_1^{s,t,n}$  are from the  $(t - s - n + 1)$ -stem, which is known by induction on  $t - s$ . Hence 3.3.7(b) for odd values of  $m$  can be used to compute the  $E_1$ -terms. For  $n = 1$ , we need to know  $H^*(\Lambda(1))$  at the outset, but it is easy to compute.  $\Lambda(1)$  is generated simply by the powers of  $\lambda_0$  and it has trivial differential. This corresponds to the homotopy of  $S^1$ .

Hence the EHP spectral sequence has the following properties,

3.3.8. LEMMA. *In the spectral sequence of 3.3.7(a),*

(a)  $E_1^{s,t,n} = 0$  for  $t - s < n - 1$  (vanishing line);

(b)  $E_1^{s,t,n} = \mathbf{Z}/(2)$  for  $t - s = n - 1$  and all  $s \geq 0$  and if in addition  $n - 1$  is even and positive,  $d_1: E_1^{s,t,n} \rightarrow E_1^{s+1,t,n-1}$  is nontrivial for all  $s \geq 0$  (diagonal groups);

(c)  $E_1^{s,t} = H^{s-1,t-n}(\Lambda)$  for  $t - s < 3n$  (stable zone); and

(d)  $E_1^{s,t,1} = 0$  for  $t > s$ .

PROOF. The groups in (a) vanish because they come from negative stems in  $\Lambda(2n-1)$ . The groups in (b) are in the 0-stem of  $\Lambda(2n-1)$  and correspond to  $\lambda_{n-1}\lambda_0^{s-1} \in \Lambda$ . If  $n-1$  is even and positive, 3.3.2 gives

$$d(\lambda_{n-1}\lambda_0^{s-1}) \equiv \lambda_{n-2}\lambda_0^s \pmod{\Lambda(n-2)},$$

which means  $d_1$  behaves as claimed. The groups in (c) are independent of  $n$  by 3.3.6. The groups in (d) are in  $\Lambda(1)$  in positive stems.  $\square$

The above result leaves undecided the fate of the generators of  $E_1^{0,n-1,n}$  for  $n-1$  odd, which correspond to the  $\lambda_{n-1}$ . We use 3.3.2 to compute the differentials on these elements. (See Tangora [2] for some helpful advice on dealing with these binomial coefficients.) We find that if  $n$  is a power of 2,  $\lambda_{n-1}$  is a cycle, and if  $n = k \cdot 2^j$  for odd  $k > 1$  then

$$d(\lambda_{n-1}) \equiv \lambda_{n-1-2^j}\lambda_{2^j-1} \pmod{\Lambda(n-1-2^j)}.$$

This equation remains valid after multiplying on the right by any cycle in  $\Lambda$ , so we get

3.3.9. PROPOSITION. *In the spectral sequence of 3.3.7(a) every element in  $E_1^{s,t,2^j}$  is a permanent cycle. For  $n = k2^j$  for  $k > 1$  odd, then every element in  $E_r^{s,t,k2^j}$  is a  $d_r$ -cycle for  $r < 2^j$  and*

$$d_{2^j} : E_{2^j}^{0,k \cdot 2^j - 1, k2^j} \rightarrow E_2^{1,k \cdot 2^j - 1, (k-1)2^j}$$

*is nontrivial, the target corresponding to  $\lambda_{2^j-1}$  under the isomorphism of 3.3.7. The cycle  $\lambda_{2^j-1}$  corresponds to  $h_j \in \text{Ext}^{1,2^j}$ .*  $\square$

Before proceeding any further it is convenient to streamline the notation. Instead of  $\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_s}$  we simply write  $i_1i_2 \dots i_s$ , e.g., we write 411 instead of  $\lambda_4\lambda_1\lambda_1$ . If an integer  $\geq 10$  occurs we underline all of it but the first digit, thereby removing the ambiguity; e.g.,  $\lambda_{15}\lambda_3\lambda_{15}$  is written as  $\underline{15}3\underline{15}$ . Sums of monomials are written as sums of integers, e.g.,  $d(9) = 71 + 53$  means  $d(\lambda_9) = \lambda_7\lambda_1 + \lambda_5\lambda_3$ ; and we write  $\phi$  for zero, e.g.,  $d(15) = \phi$  means  $d(\lambda_{15}) = 0$ .

We now study the EHP spectral sequence [3.3.7(a)] for  $t-s \leq 14$ . It is known that no differentials or unexpected extensions occur in this range in any of the unstable Adams spectral sequences, so we are effectively computing the 2-component of  $\pi_{n+k}(S^n)$  for  $k \leq 13$  and all  $n$ .

For  $t-s = 0$  we have  $E_1^{s,s,1} = \mathbf{Z}/(2)$  for all  $s \geq 0$  and  $E_1^{s,s,n} = 0$  for  $n > 1$ . For  $t-s = 1$  we have  $E_2^{1,2,2} = \mathbf{Z}/(2)$ , corresponding to  $\lambda_1$  or  $h_1$ , while  $E_2^{s,1+s,n} = 0$  for all other  $s$  and  $n$ . From this and 3.3.8(c) we get  $E_1^{2,n+2,n} = \mathbf{Z}/(2)$  generated by  $\lambda_{n-1}\lambda_1$  for all  $n \geq 2$ , while  $E_1^{s,t,t-s} = 0$  for all other  $s, t$ . The element 11 cannot be hit by a differential because 3 is a cycle, so it survives to a generator of the 2-stem, and it gives generators of  $E_1^{3,n+4,n}$  (corresponding to elements with Hopf invariant 11) for  $n \geq 2$ , while  $E_1^{s,t,t-s-1} = 0$  for all other  $s$  and  $t$ .

This brings us to  $t-s = 3$ . In addition to the diagonal groups  $E_1^{s,s+3,4}$  given by 3.3.8(b) we have  $E_1^{2,5,3}$  generated by 21 and  $E_1^{3,6,2}$  generated by 111, with no other generators in this stem. These two elements are easily seen to be nontrivial permanent cycles, so  $H^{s,s+3}(\Lambda)$  has three generators; 3, 21, and 111. Using 3.3.1 one sees that they are connected by left multiplication by 0 (i.e., by  $\lambda_0$ ).

Thus for  $t - s \leq 3$  we have produced the same value of Ext as given by the May spectral sequence in 3.2.11. The relation  $h_0^2 h_2 = h_1^3$  corresponds to the relation  $003 = 111$  in  $\Lambda$ , the latter being easier to derive. It is also true that 300 is cohomologous in  $\Lambda$  to 111, the difference being the coboundary of  $40 + 22$ . So far no differentials have occurred other than those of 3.3.8(b).

These and subsequent calculations are indicated in FIG. 3.3.10, which we now describe. The gradings  $t - s$  and  $n$  are displayed; we find this more illuminating than the usual practice of displaying  $t - s$  and  $s$ . All elements in the spectral sequence in the indicated range are displayed except the infinite towers along the diagonal described in 3.3.8(b). Each element (except the diagonal generators) is referred to by listing the leading term of its Hopf invariant with respect to the left lexicographic ordering; e.g., the cycle  $4111 + 221 + 1123$  is listed in the fifth row as 111. An important feature of the Curtis algorithm is that it *suffices to record the leading term of each element*. We will illustrate this principle with some examples. For more discussion see Tangora [3]. The arrows in the figure indicate differentials in the spectral sequence. Nontrivial cycles in  $\Lambda$  for  $0 < t - s < 14$  are listed at the bottom. We do not list them for  $t - s = 14$  because the table does not indicate which cycles in the 14th column are hit by differentials coming from the 15th column.

3.3.11. EXAMPLE. Suppose we are given the leading term 4111 of the cycle above. We can find the other terms as follows. Using 3.3.1 and 3.3.2 we find  $d(4111) = 21111$ . Referring to Fig. 3.3.10 we find 1111 is hit by the differential from 221, so we add 2221 to 4111 and find that  $d(4111 + 2221) = 11121$ . The figure shows that 121 is killed by 23, so we add 1123 to our expression and find that  $d(4111 + 2221 + 1123) = \phi$  i.e., we have found all of the terms in the cycle.

Now suppose the figure has been completed for  $t - s < k$ . We wish to fill in the column  $t - s = k$ . The box for  $n = 1$  is trivial by 3.3.8(d) and the boxes for  $n \geq 3$  can be filled in on the basis of previous calculations. (See 3.3.12.) The elements in the box for  $n = 2$  will come from the *cycles* in the box for  $n = 3$ ,  $t - s = k - 1$ , and the elements in the box for  $n = 2$ ,  $t - s = k - 1$  which are *not hit* by  $d_1$ 's. Hence before we can fill in the box for  $b = 2$ ,  $t - s = k$ , we must find the  $d_1$ 's originating in the box for  $n = 3$ . The procedure for computing differentials will be described below. Once the column  $t - s = k$  has been filled in, one computes the differentials for successively larger values of  $n$ .

The above method is adequate for the limited range we will consider, but for more extensive calculations it has a drawback. One could work very hard to show that some element is a cycle only to find at the next stage that it is hit by an easily computed differential. In order to avoid such redundant work one should work by induction on  $t$ , then on  $s$  and then on  $n$ ; i.e., one should compute differentials originating in  $E_r^{s,t,n}$  only after one has done so for all  $E_r^{s',t',n'}$  with  $t' < t$ , with  $t' = t$  and  $s' < s$ , and with  $s' = s$ ,  $t' = t$ , and  $n' < n$ . This triple induction is awkward to display on a sheet of paper but easy to write into a computer program. On the other hand Tangora [4, last paragraph starting on page 48] used downward rather than upward induction on  $s$  because given knowledge of what happens at all lower values of  $t$ , the last group needed for the  $(t - s)$ -stem is the one with the largest value of  $s$  possible under the vanishing line, the unstable analog of 3.4.5. There are advantages to both approaches.

The procedure for finding differentials in the EHP spectral sequence (3.3.7) is the following. We start with some sequence  $\alpha$  in the  $(n + 1)$ th row. Suppose



inductively that some correcting terms have already been added to  $\lambda_n\alpha$ , in the manner about to be described, to give an expression  $x$ . We use 3.3.1 and 3.3.2 to find the leading term  $i_1i_2\dots i_{s+1}$  of  $d(x)$ . If  $d(x) = 0$ , then our  $\alpha$  is a permanent cycle in the spectral sequence. If not, then beginning with  $u = 0$  we look in the table for the sequence  $i_{s-u+1}i_{s-u+2}\dots i_{s+1}$  in the  $(i_{s-u} + 1)$ th row until we find one that is hit by a differential from some sequence  $\beta$  in the  $(m + 1)$ th row or until  $u = s - 1$ . In the former event we add  $\lambda_{i_1}\dots\lambda_{i_{s-u-1}}\lambda_m\beta$  to  $x$  and repeat the process. The coboundary of the new expression will have a smaller leading term since we have added a correcting term to cancel out the original leading coboundary term.

If we get up to  $u = s - 1$  without finding a target of a differential, then it follows that our original  $\alpha$  supports a  $d_{n-i_1}$  whose target is  $i_2\dots i_{s+1}$ .

It is not necessary to add all of the correcting terms to  $x$  to show that our  $\alpha$  is a permanent cycle. The figure will provide a finite list of possible targets for the differential in question. As soon as the leading term of  $d(x)$  is smaller (in the left lexicographic ordering) than any of these candidates then we are done.

In practice it may happen that one of the sequences  $i_{s-u+1}\dots i_{s+1}$  in the  $(i_{s-u} + 1)$ th row supports a nontrivial differential. This would be a contradiction indicating the presence of an error, which should be found and corrected before proceeding further. Inductive calculations of this sort have the advantage that mistakes usually reveal themselves by producing contradictions a few stems later. Thus one can be fairly certain that a calculation through some range that is free of contradictions is also correct through most of that range. In publishing such computations it is prudent to compute a little beyond the stated range to ensure the accuracy of one's results.

We now describe some sample calculations in 3.2.11.

3.3.12. EXAMPLE. FILLING IN THE TABLE. Consider the boxes with

$$t - s - (n - 1) = 8.$$

To fill them in we need to know the 8-stem of  $H(\Lambda(2n - 1))$ . For convenience the values of  $2n - 1$  are listed at the extreme left. The first element in the 8-stem is 233, which originates on  $S^3$  and hence appears in all boxes for  $n \geq 2$ . Next we have the elements 53, 521, and 5111 originating on  $S^6$ . The latter two are trivial on  $S^7$  and so do not appear in any of our boxes, while 53 appears in all boxes with  $n \geq 4$ . The element 611 is born on  $S^7$  and dies on  $S^9$  and hence appears only in the box for  $n = 4$ . Similarly, 71 appears only in the box for  $n = 5$ .

3.3.13. EXAMPLE. COMPUTING DIFFERENTIALS We will compute the differentials originating in the box for  $t - s = 11$ ,  $n = 11$ . To begin we have  $d(101) = (90 + 72 + 63 + 54)1 = 721 + 631 + 541$ . The table shows that 721 is hit by 83 and we find

$$d(83) = (70 + 61 + 43)3 = 721 + 433.$$

Hence

$$d(101 + 83) = 631 + 541 + 433.$$

The figure shows that 31 is hit by 5 so we compute

$$d(65) = 631 + (50 + 32)5 = 631 + 541,$$

so

$$d(101 + 83 + 65) = 433,$$

which is the desired result.

Even in this limited range one can see the beginnings of several systematic phenomena worth commenting on.

3.3.14. REMARK. JAMES PERIODICITY. (Compare 1.5.18.) In a neighborhood of the diagonal one sees a certain in the differentials in addition to that of 3.3.9. For example, the leading term of  $d(\lambda_n \lambda_1)$  is  $\lambda_{n-2} \lambda_1 \lambda_1$  if  $n \equiv 0$  or  $1 \pmod{4}$  and  $n \geq 4$ , giving a periodic family of  $d_2$ 's in the spectral sequence. The differential computed in 3.3.13 can be shown to recur every 8 stems; add any positive multiple of 8 to the first integer in each sequence appearing in the calculation and the equation remains valid modulo terms which will not affect the outcome.

More generally, one can show that  $\Lambda(n)$  is isomorphic to

$$\Sigma^{-2^m} \Lambda(n + 2^m) / \Lambda(2^m)$$

through some range depending on  $n$  and  $m$ , and a general result on the periodicity of differentials follows. It can be shown that  $H^*(\Lambda(n+k)/\Lambda(n))$  is isomorphic in the stable zone [3.3.8(c)] to the Ext for  $H^*(RP^{n+k-1}/RP^{n-1})$  and that this periodicity of differentials corresponds to James periodicity. The latter is the fact that the stable homotopy type of  $RP^{n+k}/RP^n$  depends (up to suspension) only on the congruence class of  $n$  modulo a suitable power of 2. For more on this subject see Mahowald [1, 2, 3, 4].

3.3.15. REMARK. THE ADAMS VANISHING LINE. Define a collection of admissible sequences (3.3.3)  $a_i$  for  $i > 0$  as follows.

$$\begin{aligned} a_1 &= 1, & a_2 &= 11, & a_3 &= 111, & a_4 &= 4111, \\ a_5 &= 24111, & a_6 &= 124111, & a_7 &= 1124111, & a_8 &= 41124111, \text{ etc.} \end{aligned}$$

That is, for  $i > 1$

$$a_i = \begin{cases} (1, a_{i-1}) & \text{for } i \equiv 2, 3 \pmod{4} \\ (2, a_{i-1}) & \text{for } i \equiv 1 \pmod{4} \\ (4, a_{i-1}) & \text{for } i \equiv 0 \pmod{4} \end{cases}$$

It can be shown that all of these are nontrivial permanent cycles in the EHP spectral sequence and that they correspond to the elements on the Adams vanishing line (3.4.5). Note that  $H(a_{i+1}) = a_i$ . All of these elements have order 2 (i.e., are killed by  $\lambda_0$  multiplication) and half of them, the  $a_i$  for  $i \equiv 3$  and  $0 \pmod{4}$ , are divisible by 2. The  $a_{4i+3}$  are divisible by 4 but not by 8; the sequences obtained are  $(2, a_{4i+2})$  and  $(4, a_{4i+1})$  except for  $i = 1$ , when the latter sequence is 3. These little towers correspond to cyclic summands of order 8 in  $\pi_{8i+3}^S$  (see 5.3.7). The  $a_{4i}$  are the tops of longer towers whose length depends on  $i$ . The sequences in the tower are obtained in a similar manner; i.e., sequences are contracted by adding the first two integers; e.g., in the 7-stem we have 4111, 511, 61, and 7. Whenever  $i$  is a power of 2 the tower goes all the way down to filtration 1; i.e., it has  $4i$  elements, of which the bottom one is  $8i - 1$ . The table of Tangora [1] shows that the towers in the 23-, 29-, and 55-stems have length 6, while that in the 47-stem has length 12. Presumably this result generalizes in a straightforward manner. These towers are also discussed in 3.4.21 and following 4.4.47.

3.3.16. REMARK.  $d_1$ 's. It follows from 3.3.9 that all  $d_1$ 's originate in rows with  $n$  odd and that they can be computed by left multiplication by  $\lambda_0$ . In particular, the towers discussed in the above remark will appear repeatedly in the  $E_1$ -term and be almost completely cancelled by  $d_1$ 's, as one can see in Fig. 3.3.10. The elements cancelled by  $d_1$ 's do not appear in any  $H^*(\Lambda(2n-1))$ , so if one is not interested in  $H^*(\Lambda(2n))$  they can be ignored. This indicates that a lot of repetition could be avoided if one had an algorithm for computing the spectral sequence starting from  $E_2$  instead of  $E_1$ .

3.3.17. REMARK.  $S^3$ . As indicated in 3.3.5,  $\Lambda$  gives unstable as well as stable Ext groups. From a figure such as 3.3.11 one can extract unstable Adams  $E_2$ -terms for each sphere. For the reader's amusement we do this for  $S^3$  for  $t-s \leq 28$  in FIG. 3.3.18. One can show that if we remove the infinite tower in the 0-stem, what remains is isomorphic above a certain line of slope  $\frac{1}{5}$  to the stable Ext for the mod (2) Moore spectrum. This is no accident but part of a general phenomenon described by Mahowald [3].

It is only necessary to label a few of the elements in FIG. 3.3.18 because most of them are part of certain patterns which we now describe. There are clusters of six elements known as *lightning flashes*, the first of which consists of 1, 11, 111, 21, 211, 2111. Vertical and diagonal lines as usual represent right multiplication by  $\lambda_0$  and  $\lambda_1$ , i.e., by  $h_0$  and  $h_0$  respectively. This point is somewhat delicate. For example the element with in the 9-stem with filtration 4 has leading term (according to 3.3.10) 1233, not 2331. However these elements are cohomologous, their difference being the coboundary of 235.

If the first element of a lightning flash is  $x$ , the others are  $1x$ ,  $11x$ ,  $2x$ ,  $21x$ , and  $211x$ . In the clusters containing 23577 and 233577, the first elements are missing, but the others behave as if the first ones were 4577 and 43577, respectively. For example, the generator of  $E_2^{5,30}$  is 24577. In these two cases the sequences  $1x$  and  $11x$  are not admissible, but since  $14 = 23$  by 3.3.1, we get the indicated values for  $1x$ .

If  $x \in E_2^{s,t}$  is the first element of a lightning flash, there is another one beginning with  $Px \in E_2^{s+4,t+12}$ . The sequence for  $Px$  is obtained from that for  $x$  by adding 1 to the last integer and then adjoining 4111 on the right, e.g.,  $P(233) = 2344111$ . This operator  $P$  can be iterated any number of times, is related to Bott periodicity, and will be discussed more in the next section.

There are other configurations which we will call *rays* beginning with 245333 and 235733. Successive elements in a ray are obtained by left multiplication by  $\lambda_2$ . This operation is related to complex Bott periodicity.

In the range of this figure the only elements in positive stems not part of a ray or lightning flash are 23333 and 2335733. This indicates that the Curtis algorithm would be much faster if it could be modified in some way to incorporate this structure.

Finally, the figure includes Tangora's labels for the stable images of certain elements. This unstable Adams spectral sequence for  $\pi_*(S^3)$  is known to have nontrivial  $d_2$ 's originating on 245333, 222245333, and 2222245333, and  $d_3$ 's on 2235733 and 22235733. Related to these are some exotic additive and multiplicative extensions: the homotopy element corresponding to  $Ph_1d_0 = 243344111$  is twice any representative of  $h_0h_2g = 235733$  and  $\eta$  (the generator of the 1-stem) times a representative of 2245333. Hence the permanent cycles 2245333, 24334111, 235733,



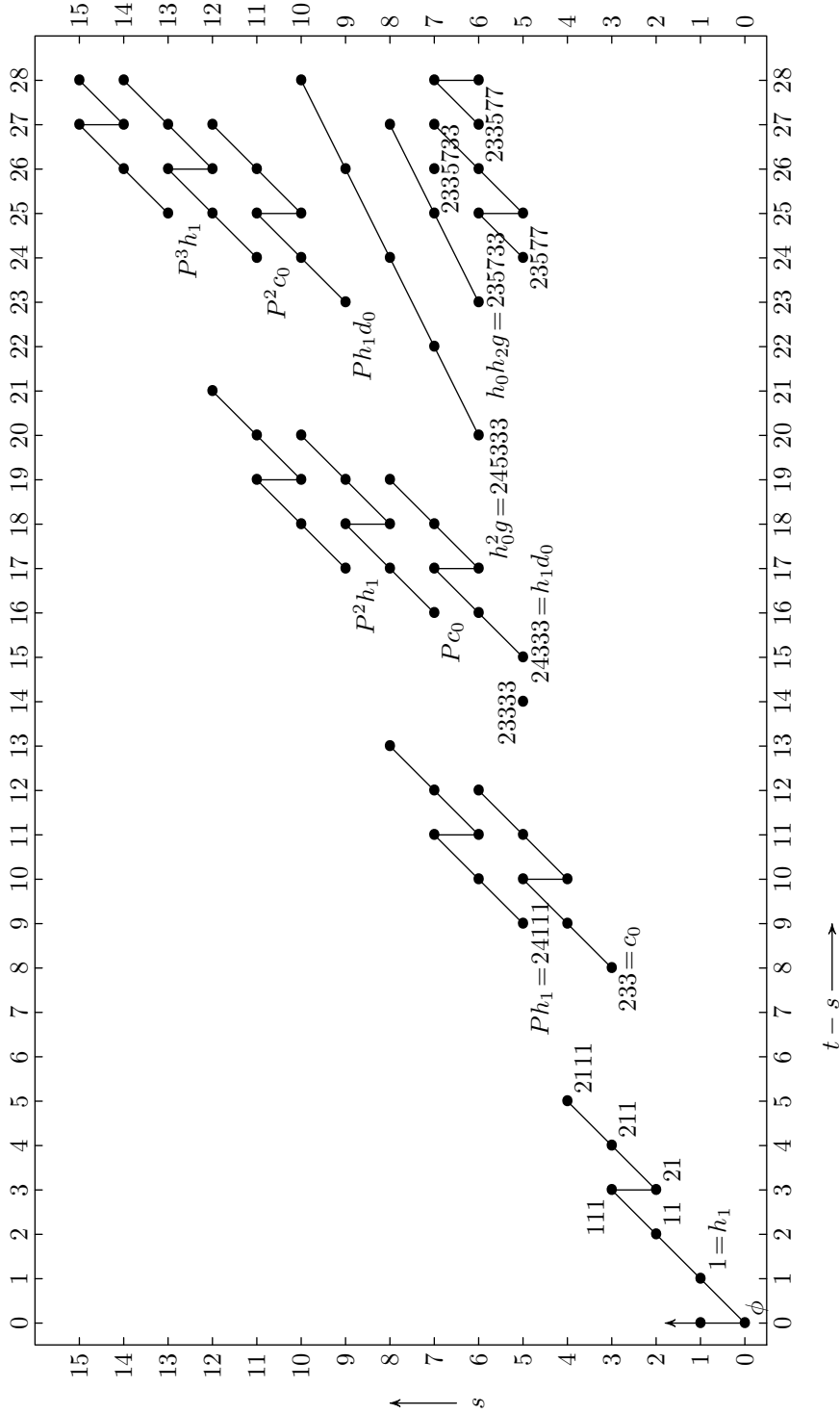


FIGURE 3.3.18. The unstable Adams  $E_2$ -term for  $S^3$ .

22245333, 224334111, and the missing element 35733 in some sense constitute an exotic lightning flash.

#### 4. Some General Properties of Ext

In this section we abbreviate  $\text{Ext}_{A^*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  by  $\text{Ext}$ . First we describe  $\text{Ext}^s$  for small values of  $s$ . Then we comment on the status of its generators in homotopy. Next we give a vanishing line, i.e., a function  $f(s)$  such that  $\text{Ext}^{s,t} = 0$  for  $0 < t - s < f(s)$ . Then we give some results describing  $\text{Ext}^{s,t}$  for  $t$  near  $f(s)$ .

3.4.1. THEOREM. For  $p = 2$

- (a)  $\text{Ext}^0 = \mathbf{Z}/(2)$  generated by  $1 \in \text{Ext}^{0,0}$ .
- (b)  $\text{Ext}^1$  is spanned by  $\{h_i : i \geq 0\}$  with  $h_i \in \text{Ext}^{1,2^i}$  represented by  $[\xi_1^{2^i}]$ .
- (c) (Adams [12])  $\text{Ext}^2$  is spanned by  $\{h_i h_j : 0 \leq i \leq j, j \neq i + 1\}$ .
- (d) (Wang [1])  $\text{Ext}^3$  is spanned by  $h_i h_j h_k$ , subject to the relations

$$h_i h_j = h_j h_i, \quad h_i h_{i+1} = 0 \quad h_1 h_{i+2}^2 = 0 \quad h_i^2 h_{i+2} = h_{i+1}^3,$$

along with the elements

$$c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle \in \text{Ext}^{3,11 \cdot 2^i}. \quad \square$$

3.4.2. THEOREM. For  $p > 2$

- (a)  $\text{Ext}^0 = \mathbf{Z}/(p)$  generated by  $1 \in \text{Ext}^{0,0}$ .
- (b)  $\text{Ext}^1$  is spanned by  $a_0$  and  $\{h_i : i \geq 0\}$  where  $a_0 \in \text{Ext}^{1,1}$  is represented by  $[\tau_0]$  and  $h_i \in \text{Ext}^{1,qp^i}$  is represented by  $[\xi_i^{p^i}]$ .
- (c) (Liulevicius [2])  $\text{Ext}^2$  is spanned by  $\{h_i h_j : 0 \leq i < j - 1\}$ ,  $a_0^2$ ,  $\{a_0 h_i : i > 0\}$ ,  $\{g_i : i \geq 0\}$ ,  $\{k_i : i \geq 0\}$ ,  $\{b_i : i \geq 0\}$ , and  $\Pi_0 h_0$ , where

$$g_i = \langle h_i, h_i, h_{i+1} \rangle \in \text{Ext}^{2,(2+p)p^i q}, \quad k_i = \langle h_i, h_{i+1}, h_{i+1} \rangle \in \text{Ext}^{2,(2p+1)p^i q},$$

$$b_i = \langle h_i, h_i, \dots, h_i \rangle \in \text{Ext}^{2,qp^{i+1}} \quad (\text{with } p \text{ factors } h_i),$$

and

$$\Pi_0 h_0 = \langle h_0, h_0, a_0 \rangle \in \text{Ext}^{2,1+2q}. \quad \square$$

$\text{Ext}^3$  for  $p > 2$  has recently been computed by Aikawa [1].

The behavior of the elements in  $\text{Ext}^1$  in the Adams spectral sequence is described in Theorems 1.2.11–1.2.14.

We know that most of the elements in  $\text{Ext}^2$  cannot be permanent cycles, i.e.,

3.4.3. THEOREM. (a) (Mahowald and Tangora [8]). *With the exceptions  $h_0 h_2$ ,  $h_0 h_3$ , and  $h_2 h_4$  the only elements in  $\text{Ext}^2$  for  $p = 2$  which can possibly be permanent cycles are  $h_j^2$  and  $h_1 h_j$ .*

(b) (Miller, Ravenel, and Wilson [1]). *For  $p > 2$  the only elements in  $\text{Ext}^2$  which can be permanent cycles are  $a_0^2$ ,  $\Pi_0 h_0$ ,  $k_0$ ,  $h_0 h_i$ , and  $b_i$ .*  $\square$

Part (b) was proved by showing that the elements in question are the only ones with preimages in the Adams–Novikov  $E_2$ -term. A similar proof for  $p = 2$  is possible using the computation of Shimomura [1]. The list in Mahowald and Tangora [8] includes  $h_2 h_5$  and  $h_3 h_6$ ; the latter is known not to come from the Adams–Novikov spectral sequence and the former is known to support a differential.

The cases  $h_0 h_i$  and  $b_i$ , for  $p > 3$  and  $h_1 h_i$  for  $p = 2$  are now settled.

3.4.4. THEOREM. (a) (Browder [1]). For  $p = 2$   $h_j^2$  is a permanent cycle iff there is a framed manifold of dimension  $2^{j+1} - 2$  with Kervaire invariant one. Such are known to exist for  $j \leq 5$ . For more discussion see 1.5.29 and 1.5.35.

(b) (Mahowald [6]). For  $p = 2$   $h_1 h_j$  is a permanent cycle for all  $j \geq 3$ .

(c) (Ravenel [7]). For  $p > 3$  and  $i \geq 1$ ,  $b_i$  is not a permanent cycle. (At  $p = 3$   $b_1$  is not permanent but  $b_2$  is;  $b_0$  is permanent for all odd primes.)

(d) (R. L. Cohen [3]). For  $p > 2$   $h_0 b_i$  is a permanent cycle corresponding to an element of order  $p$  for all  $i \geq 0$ .  $\square$

The proof of (c) will be given in Section 6.4.

Now we describe a vanishing line. The main result is

3.4.5. VANISHING THEOREM (Adams [17]). (a) For  $p = 2$   $\text{Ext}^{s,t} = 0$  for  $0 < t - s < f(s)$ , where  $f(s) = 2s - \varepsilon$  and  $\varepsilon = 1$  for  $s \equiv 0, 1 \pmod{4}$ ,  $\varepsilon = 2$  for  $s \equiv 2$  and  $\varepsilon = 3$  for  $s \equiv 3$ .

(b) (May [6]). For  $p > 2$   $\text{Ext}^{s,t} = 0$  for  $0 < t - s < sq - \varepsilon$ , where  $\varepsilon = 1$  if  $s \not\equiv 0 \pmod{p}$  and  $\varepsilon = 2$  if  $s \equiv 0$ .  $\square$

Hence in the usual picture of the Adams spectral sequence, where the  $x$  and  $y$  coordinates are  $t - s$  and  $s$ , the  $E_2$ -term vanishes above a certain line of slope  $1/q$  (e.g.,  $\frac{1}{2}$  for  $p = 2$ ). Below this line there are certain periodicity operators  $\Pi_n$  which raise the bigrading so as to move elements in a direction parallel to the vanishing line. In a certain region these operators induce isomorphisms.

3.4.6. PERIODICITY THEOREM (Adams [17], May [6]).

(a) For  $p = 2$  and  $n \geq 1$   $\text{Ext}^{s,t} \simeq \text{Ext}^{s+2^{n+1}, t+3 \cdot 2^{n+1}}$  for

$$0 < t - s < \min(g(s) + 2^{n+2}, h(s)),$$

where  $g(s) = 2s - 4 - \tau$  with  $\tau = 2$  if  $s \equiv 0, 1 \pmod{4}$ ,  $\tau = 1$  if  $s \equiv 3$ , and  $\tau = 0$  if  $s \equiv 2$ , and  $h(s)$  is defined by the following table:

$s$	1	2	3	4	5	6	7	8	$\geq 9$
$h(s)$	1	1	7	10	17	22	25	32	$5s - 7$

(b) For  $p > 2$  and  $n \geq 0$   $\text{Ext}^{s,t} \simeq \text{Ext}^{s+p^n, s+(q+1)/p^n}$  for

$$0 < t - s < \min(g(s) + p^n q, h(s)),$$

where  $g(s) = qs - 2p - 1$  and  $h(s) = 0$  for  $s = 1$  and  $h(s) = (p^2 - p - 1)s - \tau$  with  $\tau = 2p^2 - 2p + 1$  for even  $s > 1$  and  $\tau = p^2 + p - 2$  for odd  $s > 1$ .  $\square$

These two theorems are also discussed in Adams [7].

For  $p = 2$  these isomorphisms are induced by Massey products (A1.4) sending  $x$  to  $\langle x, h_0^{2^{n+1}}, h_{n+2} \rangle$ . For  $n = 1$  this operator is denoted in Tangora [1] and elsewhere in this book by  $P$ . The elements  $x$  are such that  $h_0^{2^{n+1}} x$  is above the vanishing line of 3.4.5, so the Massey product is always defined. The indeterminacy of the product has the form  $xy + h_{n+2}z$  with  $y \in \text{Ext}^{2^{n+1}, 3 \cdot 2^{n+1}}$  and  $z \in \text{Ext}^{s-1+2^{n+1}, t+2^{n+1}}$ . The group containing  $y$  is just below the vanishing line and we will see below that it is always trivial. The group containing  $z$  is above the vanishing line so the indeterminacy is zero.

Hence the theorem says that any group close enough to the vanishing line [i.e., satisfying  $t - s < 2^{n+2} + g(s)$ ] and above a certain line with slope  $\frac{1}{5}[t - s < h(s)]$  is acted on isomorphically by the periodicity operator. In Adams [17] this line

had slope  $\frac{1}{3}$ . It is known that  $\frac{1}{5}$  is the best possible slope, but the intercept could probably be improved by pushing the same methods further. The odd primary case is due entirely to May [6]. We are grateful to him for permission to include this unpublished material here.

Hence for  $p = 2$   $\text{Ext}^{s,t}$  has a fairly regular structure in the wedge-shaped region described roughly by  $2s < t - s < 5s$ . Some of this (partially below the line of slope  $\frac{1}{5}$  given above) is described by Mahowald and Tangora [14] and an attempt to describe the entire structure for  $p = 2$  is made by Mahowald [13].

However, this structure is of limited interest because we know that almost all of it is wiped out by differentials. All that is left in the  $E_\infty$ -term are certain few elements near the vanishing line related to the  $J$ -homomorphism (1.1.12). We will not formulate a precise statement or proof of this fact, but offer the following explanation. In the language of Section 1.4, the periodicity operators  $\Pi_n$  in the Adams spectral sequence correspond to  $v_1$ -periodicity in the Adams–Novikov spectral sequence. More precisely,  $\Pi_n$  corresponds to multiplication by  $v_1^{p^n}$ . The behavior of the  $v_1$ -periodic part of the Adams–Novikov spectral sequence is analyzed completely in Section 5.3. The  $v_1$ -periodic part of the Adams–Novikov  $E_\infty$ -term must correspond to the portion of the Adams spectral sequence  $E_\infty$ -term lying above (for  $p = 2$ ) a suitable line of slope  $\frac{1}{5}$ . Once the Adams–Novikov spectral sequence calculation has been made it is not difficult to identify the corresponding elements in the Adams spectral sequence. The elements in the Adams–Novikov spectral sequence all have low filtrations, so it is easy to establish that they cannot be hit by differentials. The elements in the Adams spectral sequence are up near the vanishing line so it is easy to show that they cannot support a nontrivial differential. We list these elements in 3.4.16 and in 3.4.21 give an easy direct proof (i.e., one that does not use  $BP$ -theory or  $K$ -theory) that most (all for  $p > 2$ ) of them cannot be hit by differentials.

The proof of 3.4.5 involves the comodule  $M$  given by the short exact sequence

$$(3.4.7) \quad 0 \rightarrow \mathbf{Z}/(p) \rightarrow A_* \square_{A(0)_*} \mathbf{Z}/(p) \rightarrow M \rightarrow 0,$$

where  $A(0)_* = E(\tau_0)$  for  $p > 2$  and  $E(\xi_1)$  for  $p = 2$ .  $M$  is the homology of the cofiber of the map from  $S^0$  to  $H$ , the integral Eilenberg–Mac Lane spectrum. The  $E_2$ -term for  $H$  was computed in 2.1.18 and it gives us the tower in the 0-stem. Hence the connecting homomorphism of 3.4.7 gives an isomorphism

$$(3.4.8) \quad \text{Ext}_{A_*}^{s-1,t}(\mathbf{Z}/(p), M) \simeq \text{Ext}^{s,t}$$

for  $t - s > 0$ .

We will consider the subalgebras  $A(n) \subset A$  generated by  $\{Sq^1, Sq^2, \dots, Sq^{2^n}\}$  for  $p = 2$  and  $\{\beta, P^1, P^p, \dots, P^{p^{n-1}}\}$  for  $p > 2$ . Their duals  $A(n)_*$  are  $P(\xi_1, \xi_2, \dots, \xi_{n+1})/(\xi_i^{2^{n+2-i}})$  for  $p = 2$  and

$$E(\tau_0, \dots, \tau_n) \otimes P(\xi_1, \dots, \xi_n)/(\xi_i^{p^{n+1-i}})$$

for  $p > 2$ .

We will be considering  $A_*$ -comodules  $N$  which are free over  $A(0)_*$  and  $(-1)$ -connected.  $\Sigma^{-1}M$  is an example. Unless stated otherwise  $N$  will be assumed to have these properties for the rest of the section.

Closely related to the questions of vanishing and periodicity is that of approximation. For what  $(s, t)$  does  $\text{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p), N) = \text{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p), N)$ ? This relation is illustrated by

3.4.9. APPROXIMATION LEMMA. *Suppose that there is a nondecreasing function  $f_n(s)$  defined such that for any  $N$  as above,  $\text{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p), N) = 0$  for  $t - s < f_n(s)$ . Then for  $r \geq n$  this group is isomorphic to  $\text{Ext}_{A(r)_*}^{s,t}(\mathbf{Z}/(p), N)$  for  $t - s < p^n q + f_n(s - 1)$ , and the map from the former to the latter is onto for  $t - s = p^n q + f_n(s)$ .  $\square$*

Hence if  $f_n(s)$  describes a vanishing line for  $A(n)$ -cohomology then there is a parallel line below it, above which it is isomorphic to  $A$ -cohomology. For  $n = 1$  such a vanishing line follows easily from 3.1.27(a) and 3.1.28, and it has the same slope as that of 3.4.5.

PROOF OF 3.4.9. The comodule structure map  $N \rightarrow A(r)_* \otimes N$  gives a monomorphism  $N \rightarrow A(r)_* \square_{A(n)_*} N$  with cokernel  $C$ . Then  $C$  is  $A(0)_*$ -free and  $(p^n q - 1)$ -connected. Then we have

$$\begin{array}{ccc} \text{Ext}_{A(r)_*}^{s-1}(C) \rightarrow \text{Ext}_{A(r)_*}^s(N) \rightarrow \text{Ext}_{A(r)_*}^s(A(r)_* \square_{A(n)_*} N) \rightarrow \text{Ext}_{A(r)_*}^s(C) & & \\ & \searrow & \downarrow \simeq \\ & & \text{Ext}_{A(n)_*}^s(N) \end{array}$$

where  $\text{Ext}_{A(r)_*}(-)$  is an abbreviation for  $\text{Ext}_{A(r)_*}(\mathbf{Z}/(p), -)$ . The isomorphism is given by A1.1.18 and the diagonal map is the one we are considering. The high connectivity of  $C$  and the exactness of the top row give the desired result.  $\square$

PROOF OF 3.4.5. We use 3.4.9 with  $N = M$  as in 3.4.7. An appropriate vanishing line for  $M$  will give 3.4.5 by 3.4.8. By 3.4.9 it suffices to get a vanishing line for  $\text{Ext}_{A(1)_*}(\mathbf{Z}/(p), M)$ . We calculate this by filtering  $M$  skeletally as an  $A(0)_*$ -comodule. Then  $E^0 M$  is an extended  $A(0)_*$ -determined by 3.1.27(a) or 3.1.28 and the additive structure of  $M$ . Considering the first two (three for  $p = 2$ ) subquotients is enough to get the vanishing line. We leave the details to the reader.  $\square$

The periodicity operators in 3.4.6 which raise  $s$  by  $p^n$  correspond in  $A(n)$ -cohomology to multiplication by an element  $\omega_n \in \text{Ext}^{p^n, (q+1)p^n}$ . In view of 3.4.9, 3.4.6 can be proved by showing that this multiplication induces an isomorphism in the appropriate range. For  $p = 2$  our calculation of  $\text{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$  (3.2.17) is necessary to establish periodicity above a line of slope  $\frac{1}{5}$ . To get these  $\omega_n$  we need

3.4.10. LEMMA. *There exist cochains  $c_n \in C_{A_*}$  satisfying the following.*

(a) *For  $p = 2$   $c_n \equiv [\xi_2 | \cdots | \xi_2]$  with  $2^n$  factors modulo terms involving  $\xi_1$ , and for  $p > 2$   $c_n \equiv [\tau_1 | \cdots | \tau_1]$  with  $p^n$  factors.*

(b) *For  $p = 2$   $d(c_1) = [\xi_1 | \xi_1 | \xi_1^4] + [\xi_1^2 | \xi_1^2 | \xi_1^2]$  and for  $n > 1$   $d(c_n) = [\xi_1 | \cdots | \xi_1 | \xi_1^{2^{n+1}}]$  factors  $\xi_1$ ; and for  $p > 2$   $d(c_n) = -[\tau_0 | \cdots | \tau_0 | \xi_1^{p^n}]$ .*

(c)  *$c_n$  is uniquely determined up to a coboundary by (a) and (b).*

(d) *For  $n \geq 1$  ( $p > 2$ ) or  $n \geq 2$  ( $p = 2$ )  $c_n$  projects to a cocycle in  $C_{A(n)_*}$  representing a nontrivial element  $\omega_n \in \text{Ext}_{A(n)_*}^{p^n, (q+1)p^n}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ .*

(e) *For  $p = 2$ ,  $\omega_2$  maps to  $\omega$  as in 3.1.27, and in general  $\omega_{n+1}$  maps to  $\omega_n^p$ .*

PROOF. We will rely on the algebraic Steenrod operations in Ext described in Section A1.5. We treat only the case  $p = 2$ . By A1.5.2 there are operations

$Sq^i: \text{Ext}^{s,t} \rightarrow \text{Ext}^{s+i,2t}$  satisfying a Cartan formula with  $Sq^0(h_i) = h_{i+1}$  (A1.5.3) and  $Sq^1(h_i) = h_1^2$ . Applying  $Sq^1$  to the relation  $h_0h_1 = 0$  we have

$$\begin{aligned} 0 &= Sq^1(h_0h_1) = Sq^0(h_0)Sq^1(h_1) + Sq^1(h_0)Sq^0(h_1) \\ &= h_1^3 + h_0^2h_2. \end{aligned}$$

Applying  $Sq^2$  to this gives  $h_1^4h_2 + h_0^4h_3 = 0$ . Since  $h_1h_2 = 0$  this implies  $h_0^4h_3 = 0$ . Applying  $Sq^4$  to this gives  $h_0^8h_4 = 0$ . Similarly, we get  $h_0^{2^i}h_{i+1} = 0$  for all  $i \geq 2$ . Hence there must be cochains  $c_n$  satisfying (b) above.

To show that these cochains can be chosen to satisfy (a) we will use the Kudo transgression theorem A1.5.7. Consider the cocentral extension of Hopf algebras (A1.1.15)

$$P(\xi_1) \rightarrow P(\xi_1, \xi_2) \rightarrow P(\xi_2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.17) for

$$\text{Ext}_{P(\xi_1, \xi_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

one has  $E_2 = P(h_{1j}, h_{2j}: y \geq 0)$  with  $h_{1j} \in E_2^{1,0}$  and  $h_{2j} \in E_2^{0,1}$ . By direct calculation one has  $d_2(h_{20}) = h_{10}h_{11}$ . Applying  $Sq^2Sq^1$  one gets  $d_5(h_{20}^4) = h_{10}^4h_{13} + h_{11}^4h_{12}$ . The second term was killed by  $d_2(h_{11}^3h_{21})$  so we have  $d_5(h_{20}^4) = h_{10}h_{13}^4$ . Applying appropriate Steenrod operations gives  $d_{2^n+1}(h_{20}^{2^n}) = h_{10}^{2^n}h_{1n+1}$ . Hence our cochain  $c_n$  can be chosen in  $C_{P(\xi_1, \xi_2)}$  so that its image in  $C_{P(\xi_2)}$  is  $[\xi_2 | \cdots | \xi_2]$  representing  $h_{20}^{2^n}$ , so (a) is verified.

For (c), note that (b) determines  $c_n$  up to a cocycle, so it suffices to show that each cocycle in that bidegree is a coboundary, i.e., that  $\text{Ext}^{2^n, 3 \cdot 2^n} = 0$ . This group is very close to the vanishing line and can be computed directly by what we already know.

For (d), (a) implies that  $c_n$  projects to a cocycle in  $C_{A(n)_*}$  which is nontrivial by (b); (e) follows easily from the above considerations.  $\square$

For  $p = 2$  suppose  $x \in \text{Ext}$  satisfies  $h_0^{2^n}x = 0$ . Let  $\hat{x} \in C_{A_*}$  be a cocycle representing  $x$  and let  $y$  be a cochain with  $d(y) = \hat{x}[\xi_1 | \cdots | \xi_1]$  with  $2^n$  factors. Then  $\hat{x}c_n + y[\xi_1^{2^n+1}]$  is a cocycle representing the Massey product  $\langle x, h_0^{2^n}, h_{n+1} \rangle$ , which we define to be the  $n$ th periodicity operator  $\Pi_n$ . This cocycle maps to  $\hat{x}c_n$  in  $c_{A(n)_*}$ , so  $\Pi_n$  corresponds to multiplication by  $\omega_n$  as claimed. The argument for  $p > 2$  is similar.

Now we need to examine  $\omega_1$  multiplication in  $\text{Ext}(A(1)_*)(\mathbf{Z}/(p), A(0)_*)$  for  $p > 2$  using 3.1.28 and  $\omega_2$  multiplication in  $\text{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$  using 3.2.17. The result is

#### 3.4.11. LEMMA.

(a) For  $p = 2$ , multiplication by  $\omega_2$  in  $\text{Ext}_{A(2)_*}^{s,t}(\mathbf{Z}/(2), A(0)_*)$  is an isomorphism for  $t - s < v(s)$  and an epimorphism for  $t - s < w(s)$ , where  $v(s)$  and  $w(s)$  are given in the following table.

$s$	0	1	2	3	4	5	$\geq 6$
$v(s)$	1	8	6	18	18	21	$5s + 3$
$w(s)$	1	8	10	18	23	25	$5s + 3$

(b) For  $p > 2$  multiplication by  $\omega_1$  in  $\text{Ext}_{A(1)_*}^{s,t}(\mathbf{Z}/(p), A(0)_*)$  is a monomorphism for all  $s \geq 0$  and an epimorphism for  $t - s < w(s)$  where

$$w(s) = \begin{cases} (p^2 - p - 1)s - 1 & \text{for } s \text{ even} \\ (p^2 - p - 1)s + p^2 - 3p & \text{for } s \text{ odd} \end{cases} \quad \square$$

Next we need an analogous result where  $A(0)_*$  is replaced by a  $(-1)$ -connected comodule  $N$  free over  $A(0)_*$ . Let  $N^0 \subset N$  be the smallest free  $A(0)_*$ -subcomodule such that  $N/N^0$  is 1-connected. Then

$$0 \rightarrow N^0 \rightarrow N \rightarrow N/N^0 \rightarrow 0$$

is a short exact sequence of  $A(0)_*$ -free comodules inducing a long exact sequence of  $A(n)$ -Ext groups on which  $\omega_n$  acts. Hence one can use induction and the 5-lemma to get

3.4.12. LEMMA. *Let  $N$  be a connective  $A(n)_*$ -comodule free over  $A(0)_*$ .*

(a) For  $p = 2$  multiplication by  $\omega_2$  in  $\text{Ext}_{A(2)_*}^{s,t}(\mathbf{Z}/(2), M)$  is an isomorphism for  $t - s < \tilde{v}(s)$  and an epimorphism for  $t - s < \tilde{w}(s)$ , where these functions are given by the following table

$s$	0	1	2	3	4	5	6	$\geq 7$
$\tilde{v}(s)$	-4	1	6	10	18	21	25	$5s - 2$
$\tilde{w}(s)$	1	7	10	18	22	25	33	$5s + 3$

(b) For  $p > 2$  a similar result holds for  $\omega_1$ -multiplication where

$$\tilde{v}(s) = \begin{cases} (p^2 - p - 1)s - 2p + 1 & \text{for } s \text{ even} \\ (p^2 - p - 1)s - p^2 + p & \text{for } s \text{ odd} \end{cases}$$

and

$$\tilde{w}(s) = \begin{cases} (p^2 - p - 1)s - 1 & \text{for } s \text{ even} \\ (p^2 - p - 1)s - p^2 + 2p - 1 & \text{for } s \text{ odd.} \end{cases} \quad \square$$

3.4.13. REMARK. If  $N/N^0$  is  $(q-1)$ -connected, as it is when  $N = \Sigma^{-q}M$  (3.4.7), then the function  $\tilde{v}(s)$  can be improved slightly. This is reflected in 3.4.6 and we leave the details to the reader.

The next step is to prove an analogous result for  $\omega_n$ -multiplication. We sketch the proof for  $p = 2$ . Let  $N$  be as above and define  $\bar{N} = A(n)_* \square_{A(2)_*} N$ , and let  $C = \bar{N}/N$ . Then  $C$  is 7-connected if  $N$  is  $(-1)$ -connected, and  $\text{Ext}_{A(n)_*}(\mathbf{Z}/(2), \bar{N}) = \text{Ext}_{A(2)_*}(\mathbf{Z}/(2), N)$ . Hence in this group  $\omega_n = \omega_2^{2^{n-2}}$  and we know its behavior by 3.4.12. We know the behavior of  $\omega_n$  on  $C$  by induction, since  $C$  is highly connected, so we can argue in the usual way by the 5-lemma on the long exact sequence of Ext groups. If  $N$  satisfies the condition of 3.4.13, so will  $\bar{N}$  and  $\bar{C}$ , so we can use the improved form of 3.4.12 to start the induction. The result is

3.4.14. LEMMA. *Let  $N$  be as above and satisfy the condition of 3.4.13. Then multiplication by  $\omega_n$  (3.4.10) in  $\text{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p), N)$  is an isomorphism for  $t - s < h(s+1) - 1$  and an epimorphism for  $t - s < h(s) - 1$ , where  $h(s)$  is as in 3.4.6.  $\square$*

Now the periodicity operators  $\Pi_n$ , defined above as Massey products, can be described in terms of the cochains  $c_n$  of 3.4.10 as follows. Let  $x$  represent a class in  $\text{Ext}$  (also denoted by  $x$ ) which is annihilated by  $h_0^{2^n}$  and let  $y$  be a cochain whose coboundary is  $x[\xi_1|\xi_1|\cdots|\xi_1]$  with  $2^n$  factors  $\xi_1$ . Then  $y[\xi_1^{2^{n+1}}] + xc_n$  is a cochain representing  $\Pi_n(x)$ .

Hence it is evident that the action of  $\Pi_n$  in  $\text{Ext}$  corresponds to multiplication by  $\omega_n$  in  $A(n)_*$ -cohomology. Hence 3.4.14 gives a result about the behavior of  $\Pi_n$  in  $\text{Ext}_{A_*}(\mathbf{Z}/(p), M)$  with  $M$  as in 3.4.7, so 3.4.6 follows from the isomorphism 3.4.8.

Having proved 3.4.6 we will list the periodic elements in  $\text{Ext}$  which survive to  $E_\infty$  and correspond to nontrivial homotopy elements. First we have

3.4.15. LEMMA. *For  $p = 2$  and  $n \geq 2$ ,  $\Pi_n(h_0^{2^n-1}h_{n+1}) = h_0^{2^{n+1}-1}h_{n+2}$ . For  $p > 2$  and  $n \geq 1$ ,  $\Pi_n(a_0^{p^n-1}h_n) = a_0^{p^{n+1}-1}h_{n+1}$  up to a nonzero scalar. [It is not true that  $\Pi_0(h_0) = a_0^{p-1}h_1$ .]*

PROOF. We do not know how to make this computation directly. However, 3.4.6 says the indicated operators act isomorphically on the indicated elements, and 3.4.21 below shows that the indicated image elements are nontrivial. Since the groups in question all have rank one the result follows. (3.4.6 does not apply to  $\Pi_0$  acting on  $h_0$  for  $p > 2$ .)  $\square$

### 3.4.16. THEOREM.

(a) *For  $p > 2$  the set of elements in the Adams  $E_\infty$ -term on which all iterates of some periodicity operator  $\Pi_n$  are nontrivial is spanned by  $\Pi_n^i(a_0^{p^n-j}h_n)$  with  $n \geq 0$ ,  $0 < j \leq n+1$  and  $i \not\equiv -1 \pmod{p}$ . (For  $i \equiv -1$  these elements vanish for  $n = 0$  and are determined by 3.4.15 for  $n > 0$ .) The corresponding subgroup of  $\pi_*(S^0)$  is the image of the  $J$ -homomorphism (1.1.12). (Compare 1.5.19.)*

(b) *For  $p = 2$  the set is generated by all iterates of  $\Pi_2$  on  $h_1$ ,  $h_1^2$ ,  $h_1^3 = h_0^2h_2$ ,  $h_0h_2$ ,  $h_2$ ,  $c_0$ , and  $h_1c_0$  (where  $c_0 = \langle h_1, h_0, h_2 \rangle \in \text{Ext}^{3,11}$ ) and by  $\Pi_n^i h_n h_0^{2^n-1-j}$  with  $n \geq 3$ ,  $i$  odd, and  $0 < j \leq n+1$ . (For even  $i$  these elements are determined by 3.4.15.) The corresponding subgroup of  $\pi_*(S^0)$  is  $\pi_*(J)$  (1.5.22). In particular,  $\text{im } J$  corresponds to the subgroup of  $E_\infty$  spanned by all of the above except  $\Pi_2^i h_1$  for  $i > 0$  and  $\Pi_2^i h_1^2$  for  $i \geq 0$ .  $\square$*

This can be proved in several ways. The cited results in Section 1.5 are very similar and their proofs are sketched there; use is made of  $K$ -theory. The first proof of an essentially equivalent theorem is the one of Adams [1], which also uses  $K$ -theory. For  $p = 2$  see also Mahowald [15] and Davis and Mahowald [1]. The computations of Section 5.3 can be adapted to give a  $BP$ -theoretic proof.

The following result is included because it shows that most (all if  $p > 2$ ) of the elements listed above are not hit by differentials, and the proof makes no use of any extraordinary homology theory. We will sketch the construction for  $p = 2$ . It is a strengthened version of a result of Maunder [1]. Recall (3.1.9) the spectrum  $bo$  (representing real connective  $K$ -theory) with  $H_*(bo) = A_* \square_{A(1)_*} \mathbf{Z}/(2) = P(\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \dots)$ . For each  $i \geq 0$  there is a map to  $\Sigma^{4i}H$  (where  $H$  is the integral Eilenberg–Mac Lane spectrum) under which  $\xi_1^{4i}$  has a nontrivial image. Together these define a map  $f$  from  $bo$  to  $W = \bigvee_{i \geq 0} \Sigma^{4i}H$ . We denote its cofiber



by  $\overline{W}$ . There is a map of cofiber sequences

$$(3.4.17) \quad \begin{array}{ccccc} S^0 & \longrightarrow & H & \longrightarrow & \overline{H} \\ \downarrow & & \downarrow & & \downarrow \\ bo & \xrightarrow{f} & W & \longrightarrow & \overline{W} \end{array}$$

in which each row induces an short exact sequence in homology and therefore an long exact sequence of Ext groups. Recall (3.1.26) that the Ext group for  $bo$  has a tower in every fourth dimension, as does the Ext group for  $W$ . One can show that the former map injectively to the latter. Then it is easy to work out the Adams  $E_2$ -term for  $\overline{W}$ , namely

$$(3.4.18) \quad \text{Ext}^{s,t}(H_*(\overline{W})) = \begin{cases} \text{Ext}^{s+1,t}(H_*(bo)) & \text{if } t-s \not\equiv 0 \pmod{4} \\ \mathbf{Z}/(2) & \text{if } t-s \equiv 0 \text{ and } \text{Ext}^{s,t}(H_*(bo)) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\text{Ext}(M)$  is an abbreviation for  $\text{Ext}_{A_*}(\mathbf{Z}/(2), M)$ . See FIG. 3.4.20. Combin- ing 3.4.17 and 3.4.8 gives us a map

$$(3.4.19) \quad \text{Ext}^{s,t}(\mathbf{Z}/(2)) \rightarrow \text{Ext}^{s-1,t}(H_*(\overline{W})) \quad \text{for } t-s > 0$$

Since this map is topologically induced it commutes with Adams differentials. Hence any element in Ext with a nontrivial image in 3.4.19 cannot be the target of a differential.

One can show that each  $h_n$  for  $n > 0$  is mapped monomorphically in 3.4.19, so each  $h_n$  supports a tower going all the way up to the vanishing line as is required in the proof of 3.4.15. Note that the vanishing here coincides with that for Ext given in 3.4.5.

A similar construction at odd primes detects a tower going up to the vanishing line in every dimension  $\equiv -1 \pmod{2p-2}$ .

To summarize

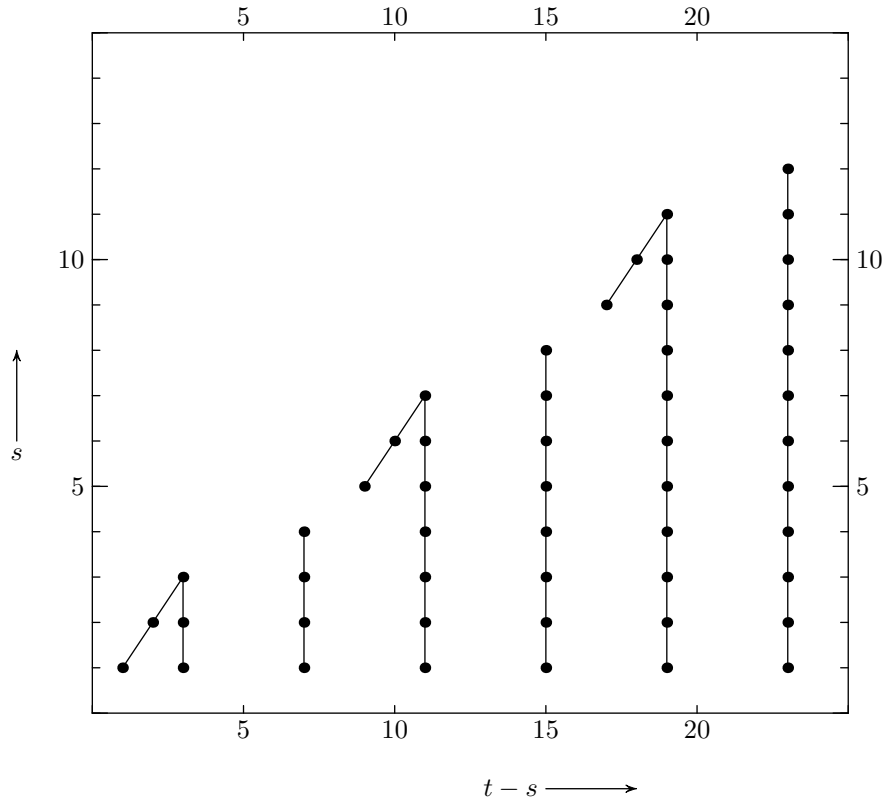
#### 3.4.21. THEOREM.

(a) For  $p = 2$  there is a spectrum  $\overline{W}$  with Adams  $E_2$ -term described in 3.4.18 and 3.4.20. The resulting map 3.4.19 commutes with Adams differentials and is nontrivial on  $h_n$  for all  $n > 0$  and all  $\Pi_2$  iterates of  $h_1$ ,  $h_1^2$ ,  $h_1^3 = h_0^2 h_2$ ,  $h_2$ , and  $h_0^3 h_3$ . Hence none of these elements is hit by Adams differentials.

(b) A similar construction for  $p > 2$  gives a map as above which is nontrivial on  $h_n$  for all  $n \geq 0$  and on all the elements listed in 3.4.16(a).  $\square$

The argument above does not show that the elements in question are permanent cycles. For example, all but a few elements at the top of the towers built on  $h_n$  for large  $n$  support nontrivial differentials, but map to permanent cycles in the Adams spectral sequence for  $\overline{W}$ .

We do not know the image of the map in 3.4.19. For  $p = 2$  it is clearly onto for  $t-s = 2^n - 1$ . For  $t-s+1 = (2k+1)2^n$  with  $k > 0$  the image is at least as big as it is for  $k = 0$ , because the appropriate periodicity operator acts on  $h_n$ . However, the actual image appears to be about  $\frac{3}{2}$  as large. For example, the towers in Ext in dimensions 23 and 39 have 6 elements instead of the 4 in dimension 7, while the one in dimension 47 has 12. We leave this as a research question for the interested reader.

FIGURE 3.4.20.  $\text{Ext}^{s-1,t} H_*(\overline{W})$ .

### 5. Survey and Further Reading

In this section we survey some other research having to do with the classical Adams spectral sequence, published and unpublished. We will describe in sequence results related to the previous four sections and then indicate some theorems not readily classified by this scheme.

In Section 1 we made some easy Ext calculations and thereby computed the homotopy groups of such spectra as  $MU$  and  $bo$ . The latter involved the cohomology of  $A(1)$ , the subalgebra of the mod (2) Steenrod algebra generated by  $Sq^1$  and  $Sq^2$ . A pleasant partial classification of  $A(1)$ -modules is given in section 3 of Adams and Priddy [10]. They compute the Ext groups of all of these modules and show that many of them can be realized as  $bo$ -module spectra. For example, they use this result to analyze the homotopy type of  $bo \wedge bo$ .

The cohomology of the subalgebra  $A(2)$  was computed by Shimada and Iwai [2]. Recently, Davis and Mahowald [4] have shown that  $A//A(2)$  is *not* the cohomology of any connective spectrum. In Davis and Mahowald [5] they compute  $A(2)$ -Ext groups for the cohomology of stunted real projective spaces.

More general results on subalgebras of  $A$  can be found in Adams and Margolis [11] and Moore and Peterson [1].

The use of the Adams spectral sequence in computing cobordism rings is becoming more popular. The spectra  $MO$ ,  $MSO$ ,  $MSU$ , and  $MSpin$  were originally

analyzed by other methods (see Stong [1] for references) but in theory could be analyzed with the Adams spectral sequence; see Pengelley [1, 2] and Giambalvo and Pengelley [1].

The spectrum  $MO\langle 8 \rangle$  (the Thom spectrum associated with the 7-connected cover of  $BO$ ) has been investigated by Adams spectral sequence methods in Giambalvo [2], Bahri [1], Davis [3, 6], and Bahri and Mahowald [1].

In Johnson and Wilson [5] the Adams spectral sequence is used to compute the bordism ring of manifolds with free  $G$ -action for an elementary abelian  $p$ -group  $G$ .

The most prodigious Adams spectral sequence calculation to date is that for the symplectic cobordism ring by Kochman [1, 2, 3]. He uses a change-of-rings isomorphism to reduce the computation of the  $E_2$ -term to finding Ext over the coalgebra

$$(3.5.1) \quad B = P(\xi_1, \xi_2, \dots)/(\xi_i^4)$$

for which he uses the May spectral sequence. The  $E_2$ -term for  $MSp$  is a direct sum of many copies of this Ext and these summands are connected to each other by higher Adams differentials. He shows that  $MSp$  is indecomposable as a ring spectrum and that the Adams spectral sequence has nontrivial  $d_r$ 's for arbitrarily large  $r$ .

In Section 2 we described the May spectral sequence. The work of Nakamura [1] enables one to use algebraic Steenrod operations (A1.5) to compute May differentials.

The May spectral sequence is obtained from an increasing filtration of the dual Steenrod algebra  $A_*$ . We will describe some decreasing filtrations of  $A_*$  for  $p = 2$  and the spectral sequences they lead to. The method of calculation these results suggest is conceptually more complicated than May's but it may have some practical advantages. The  $E_2$ -term (3.5.2) can be computed by another spectral sequence (3.5.4) whose  $E_2$ -term is the  $A(n)$  cohomology (for some fixed  $n$ ) of a certain tri-graded comodule  $T$ . The structure of  $T$  is given by a third spectral sequence (3.5.10) whose input is essentially the cohomology of the Steenrod algebra through a range of dimensions equal to  $2^{-n-1}$  times the range one wishes to compute.

This method is in practice very similar to Mahowald's unpublished work on "Koszul resolutions".

**3.5.2. PROPOSITION.** *For each  $n \geq 0$ ,  $A_*$  has a decreasing filtration (A1.3.5)  $\{F^s A_*\}$  where  $F^s$  is the smallest possible subgroup satisfying  $\bar{\xi}_i^{2^j} \in F^{2^{i+j-n-1}-1}$  for  $j \leq n+1$ .*

□

In particular,  $F^0/F^1 = A(n)_*$ , so  $A(n)_* \subset E_0 A_*$  where

$$A(n)_* = A_*/(\bar{\xi}_1^{2^{n+1}}, \bar{\xi}_2^{2^n}, \dots, \bar{\xi}_n^2, \bar{\xi}_{n+1}, \bar{\xi}_{n+2}, \dots).$$

We also have  $\bar{\xi}_i^{2^j} \in F^{2^{j-n-1}(2^i-1)}$  for  $j \geq n+1$ . Hence there is a spectral sequence (A1.3.9) converging to  $\text{Ext}_{A_*}(\mathbf{Z}/c(2), M)$  with  $E_1^{s,t,u} = \text{Ext}_{E_0 A_*}^{s,t}(\mathbf{Z}/(2), E_0 M)$  and  $d_r: E_r^{s,t,u} \rightarrow E_r^{s+1,t,u+r}$ , where the third grading is that given by the filtration,  $M$  is any  $A_*$ -comodule, and  $E_0 M$  is the associated  $E_0 A_*$ -comodule (A1.3.7).

Now let  $G(n)_* = E_0 A_* \square_{A(n)_*} \mathbf{Z}/(2)$ . It inherits a Hopf algebra structure from  $E_0 A_*$  and

$$(3.5.3) \quad A(n)_* \rightarrow E_0 A_* \rightarrow G(n)_*$$

is an extension of Hopf algebras (A1.1.15). Hence we have a Cartan–Eilenberg spectral sequence (A1.3.14), i.e.,

3.5.4. LEMMA. *Associated with the extension 3.5.3 there is a spectral sequence with*

$$E_2^{s_1, s_2, t, u} = \text{Ext}_{A(n)_*}^{s_1}(\mathbf{Z}/(2), \text{Ext}_{G(n)_*}^{s_2, t, u}(\mathbf{Z}/(2), M))$$

with  $d_r: E_r^{s_1, s_2, t, u} \rightarrow E_r^{s_1+r, s_2-r+1, t, u}$  converging to  $\text{Ext}_{E_0 A_*}^{s_1+s_2, t, u}$  for any  $E_0 A_*$  comodule  $M$ . [ $\text{Ext}_{G(n)_*}(\mathbf{Z}/(2), M)$  is the  $T$  referred to above.]  $\square$

3.5.5. REMARK. According to A1.3.11(a) the cochain complex  $W$  used to compute  $\text{Ext}$  over  $G(n)_*$  can be taken to be one of  $A(n)_*$ -comodules. The  $E_2$ -term of the spectral sequence is the  $A(n)_*$   $\text{Ext}$  of the cohomology of  $W$ , and the  $E_\infty$ -term is the cohomology of the double complex obtained by applying  $C_{A(n)_*}^*(\ )$  (A1.2.11) to  $W$ . This  $W$  is the direct sum [as a complex of  $A(n)_*$ -comodules] of its components for various  $u$  (the filtration grading). The differentials are computed by analyzing this  $W$ .

Next observe that  $E_0 A_*$  and  $G(n)_*$  contain a sub-Hopf algebra  $A_*^{(n+1)}$  isomorphic up to regrading to  $A_*$ ; i.e.,  $A_*^{(n+1)} \subset E_0 A_*$  is the image of  $P(\bar{\xi}_i^{2^{n+1}}) \subset A_*$ . The isomorphism follows from the fact that the filtration degree  $2^i - 1$  of  $\bar{\xi}_i^{2^{n+1}}$  coincides with the topological degree of  $\bar{\xi}_i$ . Hence we have

$$(3.5.6) \quad \text{Ext}_{A_*}^{s, t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \text{Ext}_{A_*^{(n+1)}}^{s, 2^{n+1}t, t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

and we can take these groups as known inductively.

Let  $L(n)_* = G(n)_* \otimes_{A_*^{(n+1)}} \mathbf{Z}/(2)$  and get an extension

$$(3.5.7) \quad A_*^{(n+1)} \rightarrow G(n)_* \rightarrow L(n)_*.$$

$L(n)_*$  is easily seen to be cocommutative with

$$(3.5.8) \quad \text{Ext}_{L(n)_*}^{s, t, u}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{i, j}: 0 \leq j \leq n, i \geq n+2-j),$$

where  $h_{i, j} \in \text{Ext}^{1, 2^j(2^i-1), 2^{i+j-n-1}-1}$  corresponds as usual to  $\bar{\xi}_i^{2^j}$ . This  $\text{Ext}$  is a comodule algebra over  $A_*^{(n+1)}$  (A1.3.14) with coaction given by

$$(3.5.9) \quad \psi(h_{i, j}) = \sum_{k>0} \xi_k^{2^{i+j-k}} \otimes h_{i-k, j}$$

Hence by A1.3.14 we have

3.5.10. LEMMA. *The extension 3.5.7 leads to a spectral sequence as in 3.5.4 with*

$$E_2^{s_1, s_2, t, u} = \text{Ext}_{A_*^{(n+1)}}^{s_1}(\mathbf{Z}/(2), \text{Ext}_{L(n)_*}^{s_2, t, u}(\mathbf{Z}/(2), M))$$

converging to  $\text{Ext}_{G(n)_*}^{s_1+s_2, t, u}(\mathbf{Z}/(2), M)$  for any  $G(n)_*$ -comodule  $M$ . For  $M = \mathbf{Z}/(2)$ , the  $\text{Ext}$  over  $L(n)_*$  and its comodule algebra structure are given by 3.5.8 and 3.5.9. Moreover, this spectral sequence collapses from  $E_2$ .

PROOF. All is clear but the last statement, which we prove by showing that  $G(n)_*$  possesses an extra grading which corresponds to  $s_2$  in the spectral sequence.

It will follow that differentials must respect this grading so  $d_r = 0$  for  $r \geq 2$ . Let  $\bar{\xi}_{i,j} \in G(n)_*$  be the element corresponding to  $\bar{\xi}_i^{2^j}$ . The extra grading is defined by

$$|\bar{\xi}_{i,j}| = \begin{cases} 1 & \text{if } j \leq n \\ 0 & \text{if } j > n. \end{cases}$$

Since the  $\bar{\xi}_{i,j}$  for  $j \leq n$  are all exterior generators, the multiplication in  $G(n)_*$  respects this grading. The coproduct is given by

$$\Delta(\bar{\xi}_{i,j}) = \sum_k \bar{\xi}_{k,j} \otimes \bar{\xi}_{i=k,k+j}.$$

If  $j \geq n + 1$  then all terms have degree 0, and if  $j \leq n$ , we have  $k + j \geq n + 2$  so all terms have degree 1, so  $\Delta$  also respects the extra grading.  $\square$

We now describe how to use these results to compute Ext. If one wants to compute through a fixed range of dimensions, the isomorphism 3.5.6 reduces the calculation of the spectral sequence of 3.5.10 to a much smaller range, so we assume inductively that this has been done. The next step is to compute in the spectral sequence of 3.5.4. The input here is the trigraded  $A(n)_*$ -comodule  $\text{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ . We began this discussion by assuming we could compute Ext over  $A(n)_*$ , but in practice we cannot do this directly if  $n > 1$ . However, for  $0 \leq m < n$  we can reduce an  $A(n)_*$  calculation to an  $A(m)_*$  calculation by proceeding as above, starting with the  $m$ th filtration of  $A(n)_*$  instead of  $A_*$ . We leave the precise formulation to the reader. Thus we can compute the  $A(n)_*$  Ext of  $\text{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$  separately for each  $u$ ; the slogan here is divide and conquer.

This method can be used to compute the cohomology of the Hopf algebra  $B$  (3.5.1) relevant to  $MSp$ . Filtering with  $n = 1$ , the SS analogous to 3.5.4 has

$$E_2 = \text{Ext}_{A(1)_*}(\mathbf{Z}/(2), P(h_{21}, h_{30}, h_{31}, h_{40}, \dots))$$

with  $\psi(h_{i+1,0}) = \xi_1 \otimes h_{i,1} + 1 \otimes h_{i+1,0}$  and  $\psi(h_{i,1}) = 1 \otimes h_{i,1}$  for  $i \geq 2$ . This Ext is easy to compute. Both this spectral sequence and the analog of the one in 3.5.2 collapse from  $E_2$ . Hence we get a description of the cohomology of  $B$  which is more concise though less explicit than that of Kochman [1].

In Section 3 we described  $\Lambda$  and hinted at an unstable Adams spectral sequence. For more on this theory see Bousfield and Kan [3], Bousfield and Curtis [4], Bendersky, Curtis, and Miller [1], Curtis [1], and Singer [3, 4, 5]. A particularly interesting point of view is developed by Singer [2].

In Mahowald [3] the double suspension homomorphism

$$\Lambda(2n - 1) \rightarrow \Lambda(2n + 1)$$

is studied. He shows that the cohomology of its cokernel  $W(n)$  is isomorphic to  $\text{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \Sigma^{2n-1}A(0)_*)$  for  $t - s < 5s + k$  for some constant  $k$ , i.e., above a line with slope  $\frac{1}{5}$ . This leads to a similar isomorphism between  $H^*(\Lambda(2n+1)/\Lambda(1))$  and  $\text{Ext}_{A_*}(\mathbf{Z}/(2), \tilde{H}_*(RP^{2n}))$ . In Mahowald [4] he proves a geometric analog, showing that a certain subquotient of  $\pi_*(S^{2n+1})$  is isomorphic to that of  $\pi_*(RP^{2n})$ . The odd primary analog of the algebraic result has been demonstrated by Harper and Miller [1]. The geometric result is very likely to be true but is still an open question. This point was also discussed in Section 1.5.

Now we will describe some unpublished work of Mahowald concerning generalizations of  $\Lambda$ . In 3.3.3 we defined subcomplexes  $\Lambda(n) \subset \Lambda$  by saying that an admissible monomial  $\lambda_{i_1} \cdots \lambda_{i_k}$  is in  $\Lambda(n)$  if  $i_1 < n$ . The short exact sequence

$$\Lambda(n-1) \rightarrow \Lambda(n) \rightarrow \Sigma^n \Lambda(2n-1)$$

led to the algebraic EHP spectral sequence of 3.3.7. Now we define quotient complexes  $\Lambda(n)$  by  $\Lambda\langle n \rangle = \Lambda/\Lambda(\lambda_0, \dots, \lambda_{n-1})$ , so  $\Lambda(0) = \Lambda$  and  $\varinjlim M\Lambda\langle n \rangle = \mathbf{Z}/(2)$ . Then there are short exact sequences

$$(3.5.11) \quad 0 \rightarrow \Sigma^n \Lambda\langle (n+1)/2 \rangle \rightarrow \Lambda\langle n \rangle \rightarrow \Lambda\langle n+1 \rangle \rightarrow 0$$

where the fraction  $(n+1)/2$  is taken to be the integer part thereof. This leads to a spectral sequence similar to that of 3.3.7 and an inductive procedure for computing  $H_*(\Lambda)$ .

Next we define  $A_*$ -comodules  $B_n$  as follows. Define an increasing filtration on  $A_*$  (different from those of 3.5.2) by  $\xi_i \in F_{2^i}$  and let  $B_n = F_n$ . The  $B_n$  is realized by the spectra of Brown and Gitler [3]. They figure critically in the construction of the  $\eta_j$ 's in Mahowald [6] and in the Brown–Peterson–Cohen program to prove that every closed smooth  $n$ -manifold immerses in  $\mathbf{R}^{2n-\alpha(n)}$ , where  $\alpha(n)$  is the number of ones in the dyadic expansion of  $n$ . Brown and Gitler [3] show that  $\text{Ext}_{A_*}(\mathbf{Z}/(2), B_n) = H^*(\Lambda\langle n \rangle)$  and that the short exact sequence 3.5.11 is realized by a cofibration. It is remarkable that the Brown–Gitler spectra and the unstable spheres both lead in this way to  $\Lambda$ .

Now let  $N = (n_1, n_2, \dots)$  be a nonincreasing sequence of nonnegative integers. Let  $A(N) = A_*/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$ . This is a Hopf algebra. Let  $M(N) = A_* \square_{A(N)} \mathbf{Z}/(2)$ , so  $M(N) = P(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$ . The filtration of  $A_*$  defined above induces one on  $M(N)$  and we have

$$(3.5.12) \quad F_i M(N)/F_{i-1} M(N) = \begin{cases} \Sigma^i F_{[i/2]} M(N^1) & \text{if } 2^{n_1} \mid i \\ 0 & \text{otherwise} \end{cases}$$

where  $N^k$  is the sequence  $(n_{k+1}, n_{k+2}, \dots)$ . For  $N = (0, 0, \dots)$   $A(N) = A_*$  and this is equivalent to 3.5.11.

3.5.13. PROPOSITION. *The short exact sequence*

$$0 \rightarrow F_{i-1} M(N) \rightarrow F_i M(N) \rightarrow F_i/F_{i-1} \rightarrow 0$$

*is split over  $A(N)$ .* □

This result can be used to construct an long exact sequence of  $A_*$ -comodules

$$(3.5.14) \quad 0 \rightarrow \mathbf{Z}/(2) \rightarrow C_N^0 \rightarrow C_N^1 \rightarrow C_N^2 \rightarrow \cdots$$

such that  $C_N^k$  is a direct sum of suspensions of  $M(N^k)$  indexed by sequences  $(i_1, i_2, \dots, i_k)$  satisfying  $1 + i_j \equiv 0 \pmod{2^{n_1+k-j}}$  and  $i_j \leq 2i_{j-1}$ . Equation 3.5.14 leads to a spectral sequence (A1.3.2) converging to Ext with

$$(3.5.15) \quad E_1^{k,s} = \text{Ext}_{A_*}^s(\mathbf{Z}/(2), C_N^k).$$

The splitting of  $C_N^k$  and the change-of-rings isomorphism A1.3.13 show that  $E_1^{k,*}$  is a direct sum of suspensions  $\text{Ext}_{A(N^k)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ .

The  $E_1$ -term of this spectral sequence is a “generalized  $\Lambda$ ” in that it consists of copies of  $A(N^k)$  Ext groups indexed by certain monomials in  $\Lambda$ . The  $d_1$  is closely related to the differential in  $\Lambda$ .

We will describe the construction of 3.5.14 in more detail and then discuss some examples. Let  $\overline{M}(N)$  be the quotient in

$$0 \rightarrow \mathbf{Z}/(2) \rightarrow M(N) \rightarrow \overline{M}(N) \rightarrow 0.$$

In 3.5.14 we want  $C_N^0 = M(N)$  and  $C_N^1 = \bigoplus_{i>0} \Sigma^{i2^{n1}} M(N^1)$ , so we need to embed  $\overline{M}(N)$  in this putative  $C_N^1$ . The filtration on  $M(N)$  induces ones on  $M(N)$  and  $C_N^s$ ; in the latter  $F_i$  should be a direct sum of suspensions of  $M(N^1)$ . Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{i-1}\overline{M}(N) & \longrightarrow & F_i\overline{M}(N) & \longrightarrow & F_i/F_{i-1}\overline{M}(N) & \longrightarrow & 0 \\ & & \downarrow & \swarrow \text{---} & \downarrow & & \parallel & & \\ 0 & \longrightarrow & F_{i-1}C_N^1 & \longrightarrow & X_i & \longrightarrow & F_i/F_{i-1}\overline{M}(N) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{i-1}C_N^1 & \longrightarrow & F_iC_N^i & \longrightarrow & \Sigma^? N(N^1) & \longrightarrow & 0 \end{array}$$

with exact rows. The upper short exact sequence splits over  $A(N)$  (3.5.13) and hence over  $A(N^1)$ . Since  $F_{i-1}C_N^1$  splits as above, the change-of-rings isomorphism A1.3.13 implies that the map

$$\mathrm{Hom}_{A_*}(F_i\overline{M}(N), F_{i-1}C_N^1) \rightarrow \mathrm{Hom}_{A_*}(F_{i-1}\overline{M}(N), F_{i-1}C_N^1)$$

is onto, so the diagonal map exists. It can be used to split the middle short exact sequence, so the lower short exact sequence can be taken to be split and  $C_N^1$  is as claimed.

The rest of 3.5.14 can be similarly constructed.

Now we consider some examples. If  $N = (0, 0, \dots)$  the spectral sequence collapses and we have the  $\Lambda$ -algebra. If  $N = (1, 1, \dots)$  we have  $\mathrm{Ext}_{A(N)} = P(a_0, a_1, \dots)$  as computed in 3.1.9, and the  $E_1$ -term is this ring tensored with the subalgebra of  $\Lambda$  generated by  $\lambda_i$  with  $i$  odd, which is isomorphic up to regrading with  $\Lambda$  itself. This is also the  $E_1$ -term of a spectral sequence converging to the Adams–Novikov  $E_2$ -term to be discussed in Section 4.4. The SS of 3.5.15 in this case can be identified with the one obtained by filtering  $\Lambda$  by the number of  $\lambda_i$ , with  $i$  odd occurring in each monomial.

For  $N = (2, 2, \dots)$  we have  $A(N) = B$  as in 3.5.1, so the  $E_1$ -term is  $\mathrm{Ext}_B$  tensored with a regraded  $\Lambda$ .

Finally, consider the case  $N = (2, 1, 0, 0, \dots)$ . We have  $E_1^{0,s} = \mathrm{Ext}_{A(1)_*}^s$  and  $E_1^{1,s} = \bigoplus_{i>0} \Sigma^{4i} \mathrm{Ext}_{A(0)_*}^s$ . One can study the quotient spectral sequence obtained by setting  $E_1^{k,s} = 0$  for  $k > 1$ . The resulting  $E_2 = E_\infty$  is the target of a map from  $\mathrm{Ext}$ , and this map is essentially the one given in 3.4.19. More generally, the first few columns of the spectral sequence of 3.5.15 can be used to detect elements in  $\mathrm{Ext}$ .

In Section 4 we gave some results concerning vanishing and periodicity. In particular we got a vanishing line of slope  $\frac{1}{2}$  (for  $p = 2$ ) for any connective comodule free over  $A(0)_*$ . This result can be improved if the comodule is free over  $A(n)_*$  for some  $n > 0$ ; e.g., one gets a vanishing line of slope  $\frac{1}{5}$  for  $n = 1, p = 2$ . See Anderson and Davis [1] and Miller and Wilkerson [8].

The periodicity in Section 4 is based on multiplication by powers of  $h_{20}$  ( $p = 2$ ) or  $a_1$  ( $p > 2$ ) and these operators act on classes annihilated by some power of  $h_{10}$  or  $a_0$ . As remarked above, this corresponds to  $v_1$ -periodicity in the Adams–Novikov spectral sequence (see Section 1.4). Therefore one would expect to find other operators based on multiplication by powers of  $h_{n+1,0}$  or  $a_n$  corresponding to  $v_n$ -periodicity for  $n > 1$ . A  $v_n$ -periodicity operator should be a Massey product defined on elements annihilated by some  $v_{n-1}$ -periodicity operator. For  $n = 2$ ,  $p = 2$  this phenomenon is investigated by Davis and Mahowald [1] and Mahowald [10, 11, 12].

More generally one can ask if there is an Adams spectral sequence version of the chromatic SS (1.4.8). For this one would need an analog of the chromatic resolution (1.4.6), which means inverting periodicity operators. This problem is addressed by Miller [4, 7].

A  $v_n$ -periodicity operator in the Adams spectral sequence for  $p = 2$  moves an element along a line of slope  $1/(2^{n+1} - 2)$ . Thus  $v_n$ -periodic families of stable homotopy elements would correspond to families of elements in the Adams spectral sequence lying near the line through the origin with this slope. We expect that elements in the  $E_\infty$ -term cluster around such lines.

Now we will survey some other research with the Adams spectral sequence not directly related to the previous four sections. For  $p = 2$  and  $t - s \leq 45$ , differentials and extensions are analyzed by Mahowald and Tangora [9], Barratt, Mahowald, and Tangora [1], Tangora [5], and Bruner [2]. Some systematic phenomena in the  $E_2$ -term are described in Davis [2], Mahowald and Tangora [14], and Margolis, Priddy, and Tangora [1]. Some machinery useful for computing Adams spectral sequence differentials involving Massey products is developed by Kochman [4] and Section 12 of Kochman [2]. See also Milgram [2] and Kahn [2] and Bruner *et al* [1], and Makinen [1].

The Adams spectral sequence was used in the proof of the Segal conjecture for  $\mathbf{Z}/(2)$  by Lin [1] and Lin *et al.* [2]. Computationally, the heart of the proof is the startling isomorphism

$$\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), M) = \mathrm{Ext}_{A_*}^{s,t+1}(\mathbf{Z}/(2), \mathbf{Z}/(2)),$$

where  $M$  is dual to the  $A$ -module  $\mathbf{Z}/(2)[x, x^{-1}]$  with  $\dim x = 1$  and  $Sq^k x^i = \binom{i}{k} x^{i+k}$  (this binomial coefficient makes sense for any integer  $i$ ). This isomorphism was originally conjectured by Mahowald (see Adams [14]). The analogous odd primary result was proved by Gunawardena [1]. The calculation is streamlined and generalized to elementary abelian  $p$ -groups by Adams, Gunawardena, and Miller [18]. This work makes essential use of ideas due to Singer [1] and Li and Singer [1].

In Ravenel [4] we proved the Segal conjecture for cyclic groups by means of a modified form of the Adams spectral sequence in which the filtration is altered. This method was used by Miller and Wilkerson [9] to prove the Segal conjecture for periodic groups.

The general Segal conjecture, which is a statement about the stable homotopy type of the classifying space of a finite group, has been proved by Gunnar Carlsson [1]. A related result is the Sullivan conjecture, which concerns says among other things that there are no nontrivial maps to a finite complex from such a classifying space. It was proved by Haynes Miller in [10]. New insight into both proofs was provided by work of Jean Lannes on unstable modules over the Steenrod algebra, in particular his  $T$ -functor, which is an adjoint to a certain tensor product. See



Lannes [1], Lannes [2] and Lannes and Schwartz [3]. An account of this theory is given in the book by Lionel Schwartz [1].

Recent work of Palmieri (Palmieri [1] and Palmieri [2]) gives a global description of Ext over the Steenrod algebra modulo nilpotent elements.

Finally, we must mention the Whitehead conjecture. The  $n$ -fold symmetric product  $Sp^n(X)$  of a space  $X$  is the quotient of the  $n$ -fold Cartesian product by the action of the symmetric group  $\Sigma_n$ . Dold and Thom [1] showed that  $Sp^\infty(X) = \varprojlim Sp^n(X)$  is a product of Eilenberg–Mac Lane spaces whose homotopy is the homotopy of  $X$ . Symmetric products can be defined on spectra and we have  $Sp^\infty(S^0) = HJ$ , the integer Eilenberg–Mac Lane spectrum. After localizing at the prime  $p$  one considers

$$S^0 \rightarrow Sp^p(S^0) \rightarrow Sp^{p^2}(S^0) \rightarrow \dots$$

and

$$(3.5.16) \quad H \leftarrow S^0 \leftarrow \Sigma^{-1}Sp^p(S^0)/S^0 \leftarrow \Sigma^{-2}Sp^{p^2}(S^0)/Sp^p(S^0) \leftarrow \dots$$

Whitehead conjectured that this diagram induces an long exact sequence of homotopy groups. In particular, the map  $\Sigma^{-1}Sp^p(S^0)/S^0 \rightarrow S^0$  should induce a surjection in homotopy in positive dimensions; this is the famous theorem of Kahn and Priddy [2]. The analogous statement about Ext groups was proved by Lin [3]. Miller [6] generalized this to show that 3.5.16 induces an long exact sequence of Ext groups. The long exact sequence of homotopy groups was established by Kuhn [1]. The spectra in 3.5.16 were studied by Welcher [1, 2]. He showed that  $H_*(Sp^{p^{n+1}}(S^0)/Sp^{p^n}(S^0))$  is free over  $A(n)_*$ , so its Ext groups has a vanishing line given by Anderson and Davis [1] and Miller and Wilkerson [8] and the long exact sequence of 3.5.16 is finite in each bigrading.



## *BP*-Theory and the Adams–Novikov Spectral Sequence

In this chapter we turn to the main topic of this book, the Adams–Novikov spectral sequence. In Section 1 we develop the basic properties of  $MU$  and the Brown–Peterson spectrum  $BP$ , using the calculation of  $\pi_*(MU)$  (3.1.5) and the algebraic theory of formal group laws as given in Appendix 2. The main result is 4.1.19, which describes  $BP_*(BP)$ , the  $BP$ -theoretic analog of the dual Steenrod algebra.

Section 2 is a survey of other aspects of  $BP$ -theory not directly related to this book.

In Section 3 we study  $BP_*(BP)$  more closely and obtain some formulas, notably 4.3.13, 4.3.18, 4.3.22, and 4.3.33, which will be useful in subsequent calculations.

In Section 4 we set up the Adams–Novikov spectral sequence and use it to compute the stable homotopy groups of spheres through a middling range of dimensions, namely  $\leq 24$  for  $p = 2$  and  $\leq 2p^3 - 2p - 1$  for  $p > 2$ .

### 1. Quillen’s Theorem and the Structure of $BP_*(BP)$

In this section we will construct the Brown–Peterson spectrum  $BP$  and determine the structure of its Hopf algebroid of cooperations,  $BP_*(BP)$ , i.e., the analog of the dual Steenrod algebra. This will enable us to begin computing with the Adams–Novikov spectral sequence (ANSS) in Section 4. The main results are Quillen’s theorem 4.1.6, which identifies  $\pi_*(MU)$  with the Lazard ring  $L$  (A2.1.8); the Landweber–Novikov theorem 4.1.11, which describes  $MU_*(MU)$ ; the Brown–Peterson theorem 4.1.12, which gives the spectrum  $BP$ ; and the Quillen–Adams theorem 4.1.19, which describes  $BP_*(BP)$ .

We begin by informally defining the spectrum  $MU$ . For more details see Milnor and Stasheff [5]. Recall that for each  $n \geq 0$  the group of complex unitary  $n \times n$  matrices  $U(n)$  has a classifying space  $BU(n)$ . It has a complex  $n$ -plane bundle  $\gamma_n$  over it which is universal in the sense that any such bundle  $\xi$  over a paracompact space  $X$  is the pullback of  $\gamma_n$ , induced by a map  $f: X \rightarrow BU(n)$ . Isomorphism classes of such bundles  $\xi$  are in one-to-one correspondence with homotopy classes of maps from  $X$  to  $BU(n)$ . Any  $\mathbf{C}^n$ -bundle  $\xi$  has an associated disc bundle  $D(\xi)$  and sphere bundle  $S(\xi)$ . The *Thom space*  $T(\xi)$  is the quotient  $D(\xi)/S(\xi)$ . Alternatively, for compact  $X$ ,  $T(\xi)$  is the one-point compactification of the total space of  $\xi$ .

$MU(n)$  is  $T(\gamma_n)$ , the Thom space of the universal  $n$ -plane bundle  $\gamma_n$  over  $BU(n)$ . The inclusion  $U(n) \rightarrow U(n+1)$  induces a map  $BU(n) \rightarrow BU(n+1)$ . The pullback of  $\gamma_{n+1}$  under this map has Thom space  $\Sigma^2 MU(n)$ . Thom spaces are functorial so we have a map  $\Sigma^2 MU(n) \rightarrow MU(n+1)$ . Together these maps give the spectrum  $MU$ .

It follows from the celebrated theorem of Thom [1] that  $\pi_*(MU)$  is isomorphic to the complex cobordism ring (see Milnor [4]) which is defined as follows. A *stably complex manifold* is one with a complex structure on its stable normal bundle. (This notion of a complex manifold is weaker than others, e.g., algebraic, analytic, and almost complex.) All such manifolds are oriented. Two closed stably complex manifolds  $M_1$  and  $M_2$  are *cobordant* if there is a stably complex manifold  $W$  whose boundary is the disjoint union of  $M_1$  (with the opposite of the given orientation) and  $M_2$ . Cobordism, i.e., being cobordant, is an equivalence relation and the set of equivalence classes forms a ring (the complex cobordism ring) under disjoint union and Cartesian product. Milnor and Novikov’s calculation of  $\pi_*(MU)$  (3.1.5) implies that two such manifolds are cobordant if they have the same Chern numbers. For the definition of these and other details of the above we refer the reader to Milnor and Stasheff [5] or Stong [1].

This connection between  $MU$  and complex manifolds is, however, not relevant to most of the applications we will discuss, nor is the connection between  $MU$  and complex vector bundles. On the other hand, the connection with formal group laws (A2.1.1) discovered by Quillen [2] (see 4.1.6) is essential to all that follows. This leads one to suspect that there is some unknown formal group theoretic construction of  $MU$  or its associated infinite loop space. For example, many well-known infinite loop spaces have been constructed as classifying spaces of certain types of categories (see Adams [9], section 2.6), but to our knowledge no such description exists for  $MU$ . This infinite loop space has been studied in Ravenel and Wilson [2].

In order to construct  $BP$  and compute  $BP_*(BP)$  we need first to analyze  $MU$ . Our starting points are 3.1.4, which describes its homology, and the Milnor–Novikov theorem 3.1.5, which describes its homotopy and the behavior of the Hurewicz map. The relevant algebraic information is provided by A2.1, which describes universal formal group laws and related concepts and which should be read before this section. The results of this section are also derived in Adams [5].

Before we can state Quillen’s theorem (4.1.6), which establishes the connection between formal group laws and complex cobordism, we need some preliminary discussion.

4.1.1. DEFINITION. *Let  $E$  be an associative commutative ring spectrum. A complex orientation for  $E$  is a class  $x_E \in \tilde{E}^2(\mathbf{C}P^\infty)$  whose restriction to*

$$\tilde{E}(\mathbf{C}P^1) \simeq \tilde{E}^2(S^2) \cong \pi_0(E)$$

*is 1, where  $\mathbf{C}P^n$  denotes  $n$ -dimensional complex projective space.  $\square$*

This definition is more restrictive than that given in Adams [5] (2.1), but it is adequate for our purposes.

Of course, not all ring spectra (e.g., *bo*) are orientable in this sense. Two relevant examples of oriented spectra are the following.

4.1.2. EXAMPLE. Let  $E = H$ , the integral Eilenberg–Mac Lane spectrum. Then the usual generator of  $H^2(\mathbf{C}P^\infty)$  is a complex orientation  $x_H$ .

4.1.3. EXAMPLE. Let  $E = MU$ . Recall that  $MU$  is built up out of Thom spaces  $MU(n)$  of complex vector bundles over  $BU(n)$  and that the map  $BU(n) \rightarrow MU(n)$  is an equivalence when  $n = 1$ . The composition

$$\mathbf{C}P^\infty = BU(1) \xrightarrow{\cong} MU(1) \rightarrow MU$$

gives a complex orientation  $x_{MU} \in MU^2(\mathbf{C}P^\infty)$ . Alternatively,  $x_{MU}$  could be defined to be the first Conner–Floyd Chern class of the canonical complex line bundle over  $\mathbf{C}P^\infty$  (see Conner and Floyd [1]).

4.1.4. LEMMA. *Let  $E$  be a complex oriented ring spectrum.*

(a)  $E^*(\mathbf{C}P^\infty) = E^*(\text{pt})[[x_E]]$ .

(b)  $E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = E^*(\text{pt})[[x_E \otimes 1, 1 \otimes x_E]]$ .

(c) *Let  $t: \mathbf{C}P^\infty \times \mathbf{C}P^\infty \rightarrow \mathbf{C}P^\infty$  be the  $H$ -space structure map, i.e., the map corresponding to the tensor product of complex line bundles, and let  $F_E(x, y) \in E^*(\text{pt})[[x, y]]$  be defined by  $t^*(x_E) = F_E(x_E \otimes 1, 1 \otimes x_E)$ . Then  $F_E$  is a formal group law (A2.1.1) over  $E^*(\text{pt})$ .*

PROOF. For (c), the relevant properties of  $F_E$  follow from the fact that  $\mathbf{C}P^\infty$  is an associative, commutative  $H$ -space with unit.

For (a) and (b) one has the Atiyah–Hirzebruch spectral sequence (AHSS)  $H^*(X; E^*(\text{pt})) \Rightarrow E^*(X)$  (see section 7 of Adams [4]). For  $X = \mathbf{C}P^\infty$  the class  $x_E$  represents a unit multiple of  $x_H \in H^2(\mathbf{C}P^\infty)$ . Hence  $x_H$  and all of its powers are permanent cycles so the spectral sequence collapses and (a) follows. The argument for (b) is similar.  $\square$

Hence a complex orientation  $x_E$  leads to a formal group law  $F_E$  over  $E^*(\text{pt})$ . Lazard’s theorem A2.1.8 asserts that  $F_E$  is induced by a homomorphism  $\theta_E: L \rightarrow E^*(\text{pt})$ , where  $L$  is a certain ring over which a universal formal group law is defined. Recall that  $L = \mathbf{Z}[x_1, x_2, \dots]$ , where  $x_i$  has degree  $2i$ . There is a power series over  $L \otimes \mathbf{Q}$

$$\log(x) = \sum_{i \geq 0} m_i x^{i+1}$$

where  $m_0 = 1$  such that

$$L \otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2, \dots]$$

and

$$\log(F(x, y)) = \log(x) + \log(y)$$

This formula determines the formal group law  $F(x, y)$ .

The following geometric description of  $\theta_{MU}$ , while interesting, is not relevant to our purposes, so we refer the reader to Adams [5, Theorem 9.2] for a proof.

4.1.5. THEOREM (Mischenko [1]). *The element  $(n + 1)\theta_{MU}(m_n) \in \pi_*(MU)$  is represented by the complex manifold  $\mathbf{C}P^n$ .*  $\square$

4.1.6. THEOREM (Quillen [2]).  *$\theta_{MU}$  is an isomorphism.*  $\square$

We will prove this with the help of the diagram

$$\begin{array}{ccc} L & \xrightarrow{j} & M \\ \theta_{MU} \downarrow & & \downarrow \phi \\ \pi_*(MU) & \xrightarrow{h} & H_*(MU) \end{array}$$

where  $M = \mathbf{Z}[m_1, m_2, \dots]$  is defined in A2.1.9(b) and contains  $L$ . The map  $\phi$  will be constructed below. Recall [A2.1.10(b)] that modulo decomposables in  $M$ , the

image of  $j$  is generated by

$$\begin{cases} pm_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ m_i & \text{otherwise.} \end{cases}$$

Recall also that  $H_*(MU) = \mathbf{Z}[b_1, b_2, \dots]$  [3.1.4(a)] and that modulo decomposables in  $H_*(MU)$ , the image of  $h$  is generated by

$$\begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ -b_i & \text{otherwise} \end{cases}$$

Hence it suffices to construct  $\phi$  and show that it is an isomorphism.

Before doing this we need two lemmas.

First we must compute  $E_*(MU)$ . It follows easily from 4.1.4(a) that  $E_*(\mathbf{C}P^\infty)$  is a free  $\pi_*(E)$  module on elements  $\beta_i^E$  dual to  $x_E^i$ . We have a stable map  $\mathbf{C}P^\infty \rightarrow \Sigma^2 MU$  and we denote by  $b_i^E$  the image of  $\beta_{i+1}^E$ .

4.1.7. LEMMA. *If  $E$  is a complex oriented ring spectrum then*

$$E_*(MU) = \pi_*(E)[b_1^E, b_2^E, \dots].$$

PROOF. We use the Atiyah–Hirzebruch spectral sequence  $H_*(MU, \pi_*(E)) \rightarrow E_*(MU)$ . The  $b_i^E$  represent unit multiples of  $b_i \in H_{2i}(MU)$  [3.1.4(a)], so the  $b_i$  are permanent cycles and the Atiyah–Hirzebruch spectral sequence collapses.  $\square$

If  $E$  is complex oriented so is  $E \wedge MU$ . The orientations  $x_E$  and  $x_{MU}$  both map to orientations for  $E \wedge MU$  which we denote by  $\hat{x}_E$  and  $\hat{x}_{MU}$ , respectively. We also know by 4.1.7 that

$$\pi_*(E \wedge MU) = E_*(MU) = \pi_*(E)[b_i^E]$$

4.1.8. LEMMA. *Let  $E$  be a complex oriented ring spectrum. Then in  $(E \wedge MU)^2(\mathbf{C}P^\infty)$ .*

$$\hat{x}_{MU} = \sum_{i \geq 0} b_i^E \hat{x}_E^{i+1},$$

where  $b_0 = 1$ . This power series will be denoted by  $g_E(\hat{x}_E)$ .

PROOF. We will show by induction on  $n$  that after restricting to  $\mathbf{C}P^n$  we get

$$\hat{x}_{MU} = \sum_{0 \leq i < n} b_i^E \hat{x}_E^{i+1}.$$

For  $n = 1$  this is clear since  $x_E$  and  $x_{MU}$  restrict to the canonical generators of  $\tilde{E}^*(\mathbf{C}P^1)$  and  $MU^*(\mathbf{C}P^1)$ . Now notice that  $x_E^n$  is the composite

$$\mathbf{C}P^n \rightarrow S^{2n} \rightarrow \Sigma^{2n} E$$

where the first map is collapsing to the top cell and the second map is the unit. Also  $b_{n-1}^E$  is by definition the composite

$$S^{2n} \xrightarrow{\beta_n^E} \mathbf{C}P^n \wedge E \xrightarrow{x_{MU} \wedge E} \Sigma^2 MU \wedge E.$$

Hence we have a diagram

$$\begin{array}{ccccc}
\mathbf{C}P^{n-1} & \longrightarrow & \mathbf{C}P^{n-1} \wedge E & & \\
\downarrow & & \uparrow g & \searrow x_{MU \wedge E} & \\
\mathbf{C}P^n & \longrightarrow & \mathbf{C}P^n \wedge E & \xrightarrow{x_{MU \wedge E}} & \Sigma^2 MU \wedge E \\
\downarrow & \searrow x_E^n & \uparrow \mathbf{C}P^n \wedge m & & \uparrow \Sigma^2 MU \wedge m \\
& & \mathbf{C}P^n \wedge E \wedge E & \xrightarrow{x_{MU \wedge E \wedge E}} & \Sigma^2 MU \wedge E \wedge E \\
& & \uparrow \beta_n^E \wedge E & \nearrow b_{n-1}^E \wedge E & \\
S^{2n} & \longrightarrow & S^{2n} \wedge E & & 
\end{array}$$

where  $m: E \wedge E \rightarrow E$  is the multiplication and  $g$  is the cofiber projection of  $(\mathbf{C}P^n \wedge m)$   $(\beta_n^E \wedge E)$ . is now split as  $(\mathbf{C}P^{n-1} \wedge E) \vee (S^{2n} \wedge E)$  and  $x_{MU \wedge E}: \mathbf{C}P \wedge E \rightarrow \Sigma^2 MU \wedge E$  is the sum of  $(x_{MU \wedge E})g$  and the map from  $S^{2n} \wedge E$ . Since  $x_{MU}$  is the composition

$$\mathbf{C}P^n \rightarrow \mathbf{C}P^n \wedge E \xrightarrow{x_{MU \wedge E}} \Sigma^2 MU \wedge E$$

and the lower composite map from  $\mathbf{C}P^n$  to  $\Sigma^2 MU \wedge E$  is  $b_{n-1}^E x_E^n$ , the inductive step and the result follow.  $\square$

4.1.9. COROLLARY. In  $\pi_*(E \wedge MU)[[x, y]]$ ,

$$F_{MU}(x, y) = g_E(F_E(g_E^{-1}(x), g_E^{-1}(y))).$$

PROOF. In  $(E \wedge MU)^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$ ,

$$\begin{aligned}
F_{MU}(\hat{x}_{MU} \otimes 1, 1 \otimes \hat{x}_{MU}) &= t^*(x_{MU}) \\
&= g_E(t^*(\hat{x}_E)) \\
&= g_E(F_E(\hat{x}_E \otimes 1, 1 \otimes \hat{x}_E)) \\
&= g_E(F_E(g_E^{-1}(x_{MU}) \otimes 1, 1 \otimes g_E^{-1}(\hat{x}_{MU}))).
\end{aligned}$$

$\square$

Now we are ready to prove 4.1.6. The map  $\phi$  in 4.1.6 exists if the logarithm of the formal group law defined over  $H_*(MU)$  by  $h\theta_{MU}$  is integral, i.e., if the formal group law is isomorphic to the additive one. For  $E = H$ ,  $F_E(x, y) = x + y$ , so the formal group law over  $H_*(MU) = \pi_*(H \wedge MU)$  is indeed isomorphic to the additive one, so  $\phi$  exists. Moreover,  $\log_E(\hat{x}_E) = \hat{x}_E$ , so

$$\hat{x}_E = \sum \phi(m_i) \hat{x}_{MU}^{i-1} = g_H^{-1}(\hat{x}_{MU})$$

by 4.1.9. It follows that  $\sum \phi(m_i) x^{i+1}$  is the functional inverse of  $\sum b_i x^{i+1}$ , i.e.,

$$(4.1.10) \quad h\theta_{MU} \exp(x) = \sum_{i \geq 0} b_i x^{i+1},$$

where  $\exp$  is the functional inverse of the logarithm (A2.1.5), so  $\phi(m_i) \equiv -b_i$ , modulo decomposables in  $H^*(MU)$  and 4.1.6 follows.

Now we will determine the structure of  $MU_*(MU)$ . We know it as an algebra by 4.1.7. In particular, it is a free  $\pi_*(MU)$  module, so  $MU$  is a flat ring spectrum. Hence by 2.2.8  $(\pi_*(MU), MU_*(MU))$  is a Hopf algebroid (A1.1.1). We will show that it is isomorphic to  $(L, LB)$  of A2.1.16. We now recall its structure. As an

algebra,  $LB = L[b_1, b_2 \dots]$  with  $\deg b_i = 2i$ . There are structure maps  $\varepsilon: LB \rightarrow L$  (augmentation),  $\eta_L, \eta_R: L \rightarrow LB$  (left and right units),  $\Delta: LB \rightarrow LB \otimes_L LB$  (coproduct), and  $c: LB \rightarrow LB$  (conjugation) satisfying certain identities listed in [A1.1.1](#).

$\varepsilon: LB \rightarrow L$  is defined by  $\varepsilon(b_i) = 0$ ;  $\eta_L: L \rightarrow LB$  is the standard inclusion, while  $\eta_R: L \otimes \mathbf{Q} \rightarrow LB \otimes \mathbf{Q}$  is given by

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left( \sum_{j \geq 0} c(b_j) \right)^{i+1},$$

where  $m_0 = b_0 = 1$ ;

$$\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} \left( \sum_{i \geq 0} b_i \right)^{j+1} \otimes b_j;$$

and  $c: LB \rightarrow LB$  is determined by  $c(m_i) = \eta_R(m_i)$  and

$$\sum_{i \geq 0} c(b_i) \left( \sum_{j \geq 0} b_j \right)^{i+1} = 1.$$

Note that the maps  $\eta_L$  and  $\eta_R$ , along with the identities of [A1.1.1](#), determine the remaining structure maps  $\varepsilon$ ,  $\Delta$ , and  $c$ .

The map  $\theta_{MU}$  of [4.1.6](#) is an isomorphism which can be extended to  $LB$  by defining  $\theta_{MU}(b_i)$  to be  $b_i^{MU} \in MU_{2i}(MU)$  ([4.1.8](#)).

**4.1.11. THEOREM** (Novikov [\[1\]](#), Landweber [\[2\]](#)). *The map  $\theta_{MU}: LB \rightarrow MU_*(MU)$  defined above gives a Hopf algebroid isomorphism  $(L, LB) \rightarrow (\pi_*(MU), MU_*(MU))$ .*

**PROOF.** Recall that the Hopf algebroid structure of  $(L, LB)$  is determined by the right unit  $\eta_R: L \rightarrow LB$ . Hence it suffices to show that  $\theta_{MU}$  respects  $\eta_R$ . Now the left and right units in  $MU_*(MU)$  are induced by  $MU \wedge S^0 \rightarrow MU \wedge MU$  and  $S^0 \wedge MU \rightarrow MU \wedge MU$ , respectively. These give complex orientations  $x_L$  and  $x_R$  for  $MU \wedge MU$  and hence formal group laws ([4.1.4](#))  $F_R$  and  $F_L$  over  $MU_*(MU)$ . The  $b_i$  in  $LB$  are the coefficients of the power series of the universal isomorphism between two universal formal group laws. Hence it suffices to show that  $x_R = \sum_{i \geq 0} b_i^{MU} x_R^{i+1}$ , but this is the special case of [4.1.9](#) where  $E = MU$ .  $\square$

Our next objective is

**4.1.12. THEOREM.** [*Brown and Peterson* [\[1\]](#), *Quillen* [\[1\]](#)] *For each prime  $p$  there is a unique associative commutative ring spectrum  $BP$  which is a retract of  $MU_{(p)}$  ([2.1.12](#)) such that the map  $g: MU_{(p)} \rightarrow BP$  is multiplicative,*

(a)  $\pi_*(BP) \otimes \mathbf{Q} = \mathbf{Q}[g_*(m_{p^k-1}): k > 0]$  with  $g_*(m_n) = 0$  for  $n \neq p^k - 1$ ;

(b)  $H_*(BP; \mathbf{Z}/(p)) = P_*$  ([3.1.6](#)) as comodule algebras over the dual Steenrod algebra  $A_*$  ([3.1.1](#)); and

(c)  $\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2 \dots]$  with  $v_n \in \pi_{2(p^n-1)}$  and the composition  $\pi_*(g)\theta_{MU_{(p)}}$  factors through the map  $L \times \mathbf{Z}_{(p)} \rightarrow V$  of [A2.1.25](#), giving an isomorphism from  $V$  to  $\pi_*(BP)$ .  $\square$

The spectrum  $BP$  is named after Brown and Peterson, who first constructed it via its Postnikov tower. Recall ([3.1.9](#)) that  $H_*(MU; \mathbf{Z}/(p))$  splits as an  $A_*$ -comodule into many copies of  $P_*$ . Theorem [4.1.12](#) implies that there is a corresponding splitting of  $MU_{(p)}$ . Since  $P_*$  is dual to a cyclic  $A$ -module, it is clear that



$BP$  cannot be split any further. Brown and Peterson [1] also showed that  $BP$  can be constructed from  $H$  (the integral Eilenberg–Mac Lane spectrum) by killing all of the torsion in its integral homology with Postnikov fibrations. More recently, Priddy [1] has shown that  $BP$  can be constructed from  $S_{(p)}^0$  by adding local cells to kill off all of the torsion in its homotopy.

The generators  $v_n$  of  $\pi_*(BP)$  will be defined explicitly below.

Quillen [2] constructed  $BP$  in a more canonical way which enabled him to determine the structure of  $BP_*(BP)$ .  $BP$  bears the same relation to  $p$ -typical formal group laws (A2.1.17) that  $MU$  bears to formal group laws as seen in 4.1.6. The algebraic basis of Quillen's proof of 4.1.12 is Cartier's theorem A2.1.18, which states that any formal group law over a  $\mathbf{Z}_{(p)}$ -algebra is canonically isomorphic to  $p$ -typical one. Accounts of Quillen's work are given in Adams [5] and Araki [1].

Following Quillen [2], we will construct a multiplicative map  $g: MU_{(p)} \rightarrow MU_{(p)}$  which is idempotent, i.e.,  $g^2 = g$ . This map will induce an idempotent natural transformation or cohomology operation on  $MU_{(p)}^*(-)$ . The image of this map will be a functor satisfying the conditions of Brown's representability theorem (see Brown [2] or, in terms of spectra, 3.12 of Adams [4]) and will therefore be represented by a spectrum  $BP$ . The multiplicativity of  $BP$  and its other properties will follow from the corresponding properties of  $g$ .

To construct  $g$  we need two lemmas.

4.1.13. LEMMA. *Let  $E$  be an oriented ring spectrum. Then orientations of  $E$  are in one-to-one correspondence with multiplicative maps from  $MU$  to  $E$ ; i.e., given an orientation  $y_E \in E^2(\mathbf{C}P^\infty)$ , there is a unique multiplicative map  $g: MU \rightarrow E$  such that  $g^*(x_{MU}) = y_E$  and vice versa.*

PROOF. By 4.1.4,  $E^*(\mathbf{C}P^\infty) = \pi_*(E)[[x_E]]$  so we have

$$y_E = f(x_E) = \sum_{i \geq 0} f_i x^{i+1}$$

with  $f_0 = 1$  and  $f_i \in \pi_{2i}(E)$ . Using arguments similar to that of 4.1.8 and 4.1.6 one shows

$$(4.1.14) \quad E^*(MU) \cong \text{Hom}_{\pi_*(E)}(E_*(MU), \pi_*(E))$$

and

$$E^*(\mathbf{C}P^\infty) \cong \text{Hom}_{\pi_*(E)}(E_*(\mathbf{C}P^\infty), \pi_*(E)).$$

A diagram chase shows that a map  $MU \rightarrow E$  is multiplicative if the corresponding map  $E_*(MU) \rightarrow \pi_*(E)$  is a  $\pi_*(E)$ -algebra map. The map  $y_E$  corresponds to the map which sends  $\beta_{i+1}^E$  to  $f_i$  and  $\beta_{i+1}^E$  by definition maps to  $b_i^E \in E_{2i}(MU)$ , so we let  $g$  be the map which sends  $b_i^E$  to  $f_i$ .  $\square$

4.1.15. LEMMA. *A map  $g: MU_{(p)} \rightarrow MU_{(p)}$  (or  $MU \rightarrow MU$ ) is determined up to homotopy by its behavior on  $\pi_*$ .*

PROOF. We do the  $MU$  case first. By 4.1.14,

$$MU^*(MU) = \text{Hom}_{\pi_*(MU)}(MU_*(MU), \pi_*(MU)).$$

This object is torsion-free so we lose no information by tensoring with  $\mathbf{Q}$ . It follows from 4.1.11 that  $MU_*(MU) \otimes \mathbf{Q}$  is generated over  $\pi_*(MU) \otimes \mathbf{Q}$  by the image of  $\eta_R$ , which is the Hurewicz map. Therefore the map

$$MU^*(MU) \otimes \mathbf{Q} \rightarrow \text{Hom}_{\mathbf{Q}}(\pi_*(MU) \otimes \mathbf{Q}, \pi_*(MU) \otimes \mathbf{Q})$$

is an isomorphism, so the result follows for  $MU$ .

For the  $MU_{(p)}$  case we need to show

$$(4.1.16) \quad MU_{(p)}^*(MU_{(p)}) = MU^*(MU) \otimes \mathbf{Z}_{(p)}.$$

This will follow from 4.1.13 if we can show that the map

$$(4.1.17) \quad MU_{(p)}^*(MU) \rightarrow MU_{(p)}^*(MU_{(p)})$$

is an isomorphism, i.e., that  $MU_{(p)}^*(C) = 0$ , where  $C$  is the cofiber of  $MU \rightarrow MU_{(p)}$ . Now  $C$  is trivial when localized at  $p$ , so any  $p$ -local cohomology theory vanishes on it. Thus 4.1.15 and the  $MU_{(p)}$  case follow.  $\square$

We are now ready to prove 4.1.12. By 4.1.13 and 4.1.15 a multiplicative map  $g: MU_{(p)} \rightarrow MU_{(p)}$  is determined by a power series  $f(x)$  over  $\pi_*(MU_{(p)})$ . We take  $f(x)$  to be as defined by A2.1.23. By 4.1.15 the corresponding map  $g$  is idempotent if  $\pi_*(g) \otimes \mathbf{Q}$  is. To compute the latter we need to see the effect of  $g^*$  on

$$\log(x_{MU}) = \sum m_i x_{MU}^{i+1} \in MU^2(\mathbf{C}P^\infty) \otimes \mathbf{Q}.$$

Let  $F'_{MU_{(p)}}$  be the formal group law associated with the orientation  $f(x_{MU})$ , and let  $\text{mog}(x)$  be its logarithm (A2.1.6). The map  $g^*$  preserves formal group laws and hence their logarithms, so we have  $g^*(\log(x_{MU})) = \text{mog}(f(x_{MU}))$ . By A2.1.24  $\text{mog}(x) = \sum_{k \geq 0} m_{p^k-1} x^{p^k}$  and it follows that  $\pi_*(g)$  has the indicated behavior; i.e., we have proved 4.1.12(a).

For (b), we have  $H_*(BP; \mathbf{Q}) = \pi_*(BP) \otimes \mathbf{Q}$ , and  $H_*(BP; \mathbf{Z}_{(p)})$  is torsionfree, so  $H_*(BP; \mathbf{Z}/(p)) = P_*$  as algebras. Since  $BP$  is a retract of  $MU_{(p)}$  its homology is a direct summand over  $A_*$  and (b) follows.

For (c) the structure of  $\pi_*(BP)$  follows from (a) and the fact that  $BP$  is a retract of  $MU_{(p)}$ . For the isomorphism from  $V$  we need to complete the diagram

$$\begin{array}{ccc} L \otimes \mathbf{Z}_{(p)} & \longrightarrow & V \\ \theta_{MU_{(p)}} \downarrow & & \downarrow \\ \pi_*(MU_{(p)}) & \xrightarrow{g^*} & \pi_*(BP) \end{array}$$

The horizontal maps are both onto and the left-hand vertical map is an isomorphism so it suffices to complete the diagram tensored with  $\mathbf{Q}$ . In this case the result follows from (a) and A2.1.25. This completes the proof of 4.1.12.

Our last objective in this section is the determination of the Hopf algebroid (A1.1.1)  $(\pi_*(BP), BP_*(BP))$ . (Proposition 2.2.8 says that this object is a Hopf algebroid if  $BP$  is flat. It is since  $MU_{(p)}$  is flat.) We will show that it is isomorphic to  $(V, VT)$  of A2.1.27, which bears the same relation to  $p$ -typical formal group laws that  $(L, LB)$  (A2.1.16 and 4.1.11) bears to ordinary formal group laws. The ring  $V$  (A2.1.25), over which the universal  $p$ -typical formal group law is defined, is isomorphic to  $\pi_*(BP)$  by 4.1.12(c).  $V \otimes \mathbf{Q}$  is generated by  $m_{p^i-1}$  for  $i \geq 0$ , and we denote this element by  $\lambda_i$ . Then from A2.1.27 we have

4.1.18. THEOREM. *In the Hopf algebroid  $(V, VT)$  (see A1.1.1)*

(a)  $V = Z_{(p)}[v_1, v_2, \dots]$  with  $|v_n| = 2(p^n - 1)$ ,

(b)  $VT = V[t_1, t_2, \dots]$  with  $|t_n| = 2(p^n - 1)$ , and

(c)  $\eta_L: V \rightarrow VT$  is the standard inclusion and  $\varepsilon: VT \rightarrow V$  is defined by  $\varepsilon(t_i) = 0$ ,  $\varepsilon(v_i) = v_i$ .

- (d)  $\eta_R: V \rightarrow VT$  is determined by  $\eta_R(\lambda_n) = \sum_{0 \leq i \leq n} \lambda_i t_{n-i}^{p^i}$ , where  $\lambda_0 = t_0 = 1$ ,  
 (e)  $\Delta$  is determined by

$$\sum_{i,j \geq 0} \lambda_i \Delta(t_j)^{p^i} = \sum_{i,j,k \geq 0} \lambda_i t_j^{p^i} \otimes t_k^{p^{i+j}},$$

and

- (f)  $c$  is determined by

$$\sum_{i,j,k \geq 0} \lambda_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i \geq 0} \lambda_i.$$

(g) The forgetful functor from  $p$ -typical formal group laws to formal group laws induces a surjection of Hopf algebroids (A1.1.19)

$$(L \otimes \mathbf{Z}_{(p)}, LB \otimes \mathbf{Z}_{(p)}) \rightarrow (V, VT). \quad \square$$

4.1.19. THEOREM (Quillen [2], Adams [5]). *The Hopf algebroid  $(\pi_*(BP), BP_*(BP))$  is isomorphic to  $(V, VT)$  described above.*

PROOF. Consider the diagram

$$\begin{array}{ccc} (L, LB) \otimes \mathbf{Z}_{(p)} & \longrightarrow & (V, VT) \\ \downarrow \theta_{MU} & & \downarrow \gamma \\ MU_*(MU) \otimes \mathbf{Z}_{(p)} & \xrightarrow{g_*} & (\pi_*(BP), BP_*(BP)). \end{array}$$

The left-hand map is an isomorphism by 4.1.11 and the horizontal maps are both onto by (g) above and by 4.1.12. Therefore it suffices to complete the diagram with an isomorphism over  $\mathbf{Q}$ . One sees easily that  $VT \otimes \mathbf{Q}$  and  $BP_*(BP) \otimes \mathbf{Q}$  are both isomorphic to  $V \otimes V \otimes \mathbf{Q}$ .  $\square$

## 2. A Survey of BP-Theory

In this section we will give an informal survey of some aspects of complex cobordism theory not directly related to the Adams–Novikov spectral sequence. (We use the terms complex cobordism and  $BP$  interchangeably in light of 4.1.12.) Little or no use of this material will be made in the rest of the book. This survey is by no means exhaustive.

The history of the subject shows a movement from geometry to algebra. The early work was concerned mainly with applications to manifold theory, while more recent work has dealt with the internal algebraic structure of various cohomology theories and their applications to homotopy theory. The present volume is, of course, an example of the latter. The turning point in this trend was Quillen's theorem 4.1.6, which established a link with the theory of formal groups treated in Appendix 2. The influential but mostly unpublished work of Jack Morava in the early 1970s was concerned with the implications of this link.

Most geometric results in the theory, besides the classification of closed manifolds up to cobordism, rest on the notion of the *bordism groups*  $\Omega_*(X)$  of a space  $X$ , first defined by Conner and Floyd [2].  $\Omega_n(X)$  is the group (under disjoint union) of equivalence classes of maps from closed  $n$ -dimensional manifolds (possibly with some additional structure such as an orientation or a stable complex structure) to  $X$ . Two such maps  $f_i: M_i \rightarrow X$  ( $i = 1, 2$ ) are equivalent if there is a map

$f: W \rightarrow X$  from a manifold whose boundary is  $M_1 \cup M_2$  with  $f$  extending  $f_1$  and  $f_2$ . It can be shown (Conner and Floyd [2]) that the functor  $\Omega_*(-)$  is a generalized homology theory and that the spectrum representing it is the appropriate Thom spectrum for the manifolds in question. For example, if the manifolds are stably complex (see the beginning of Section 1) the bordism theory, denoted by  $\Omega_*^U(-)$ , coincides with  $MU_*(-)$ , the generalized homology theory represented by the spectrum  $MU$ , i.e.,  $\Omega_n^U(X) = \pi_n(MU \wedge X)$ . The notation  $\Omega_*(-)$  with no superscript usually denotes the oriented bordism group, while the unoriented bordism group is usually denoted by  $N_*(-)$ .

These bordism groups are usually computed by algebraic methods that use properties of the Thom spectra. Thom [1] showed that  $MO$ , the spectrum representing unoriented bordism, is a wedge of mod (2) Eilenberg–Mac Lane spectra, so  $N_*(X)$  is determined by  $H_*(X; \mathbf{Z}/(2))$ .  $MSO$  (which represents oriented bordism) when localized at the prime 2 is known (Stong [1, p. 209]) to be a wedge of integral and mod (2) Eilenberg–Mac Lane spectra, so  $\Omega_*(X)_{(2)}$  is also determined by ordinary homology. Brown and Peterson [1] showed that when localized at any odd prime the spectra  $MSO$ ,  $MSU$ , and  $MSP$  as well as  $MU$  are wedges of various suspensions of  $BP$ , so the corresponding bordism groups are all determined by  $BP_*(X)$ . Conner and Floyd [2] showed effectively that  $BP_*(X)$  is determined by  $H_*(X; \mathbf{Z}_{(p)})$  when the latter is torsion-free.

For certain spaces the bordism groups have interesting geometric interpretations. For example,  $\Omega_*(BO)$  is the cobordism group of vector bundles over oriented manifolds. Since  $H_*(BSO)$  has no odd torsion, it determines this group. If  $X_n$  is the  $n$ th space in the  $\Omega$ -spectrum for  $MSO$ , then  $\Omega_*(X_n)$  is the cobordism group of maps of codimension  $n$  between oriented manifolds. The unoriented analog was treated by Stong [3] and the complex analog by Ravenel and Wilson [2].

For a finite group  $G$ ,  $\Omega_*(BG)$  is the cobordism group of oriented manifolds with free  $G$ -actions, the manifolds mapped to  $BG$  being the orbit spaces. These groups were studied by Conner and Floyd [2] and Conner [4]. Among other things they computed  $\Omega_*(BG)$  for cyclic  $G$ . In Landweber [6] it was shown that the map  $MU_*(BG) \rightarrow H_*(BG)$  is onto iff  $G$  has periodic cohomology. In Floyd [1] and tom Dieck [1] it is shown that the ideal of  $\pi_*(MU)$  represented by manifolds on which an abelian  $p$ -group with  $n$  cyclic summands can act without stationary points is the prime ideal  $I_n$  defined below. The groups  $BP_*(BG)$  for  $G = (\mathbf{Z}/(p))^n$  have been computed by Johnson and Wilson [5].

We now turn to certain other spectra related to  $MU$  and  $BP$ . These are constructed by means of either the Landweber exact functor theorem (Landweber [3]) or the *Sullivan–Baas construction* (Baas [1]), which we now describe. Dennis Sullivan (unpublished, circa 1969) wanted to construct “manifolds with singularities” (admittedly a contradiction in terms) with which any ordinary homology class could be represented; i.e., any element in  $H_*(X; \mathbf{Z})$  could be realized as the image of the fundamental homology class of such a “manifold”  $M$  under some map  $M \rightarrow X$ . It was long known that not all homology classes were representable in this sense by ordinary manifolds, the question having been originally posed by Steenrod. (I heard Sullivan begin a lecture on the subject by saying that homology was like the weather; everybody talks about it but nobody does anything about it.)

In terms of spectra this nonrepresentability is due to the fact that  $MU$  (if we want our manifolds to be stably complex) is *not* a wedge of Eilenberg–Mac Lane

spectra. The Sullivan–Baas construction can be regarded as a way to get from  $MU$  to  $H$ .

Let  $y \in \pi_k(MU)$  be represented by a manifold  $X$ . A closed  $n$ -dimensional manifold with *singularity of type*  $(y)$  ( $n > k$ ) is a space  $W$  of the form  $A \cup (B \times CM)$ , where  $CM$  denotes the cone on a manifold  $M$  representing  $y$ ,  $B$  is a closed  $(n-k-1)$ -dimensional manifold,  $A$  is an  $n$ -dimensional manifold with boundary  $B \times M$ , and  $A$  and  $B \times CM$  are glued together along  $B \times M$ . It can be shown that the bordism group defined using such objects is a homology theory represented by a spectrum  $C(y)$  which is the cofiber of

$$\Sigma^k MU \xrightarrow{y} MU, \quad \text{so} \quad \pi_* C(Y) = \pi_*(MU)/y.$$

This construction can be iterated any number of times. Given a sequence  $y_1, y_2, \dots$  of elements in  $\pi_*(MU)$  we get spectra  $C(y_1, y_2, \dots, y_n)$  and cofibrations

$$\Sigma^{|y_n|} C(y_1, \dots, y_{n-1}) \rightarrow C(y_1, \dots, y_{n-1}) \rightarrow C(y_1, \dots, y_n).$$

If the sequence is regular, i.e., if  $y_n$  is not a zero divisor in  $\pi_*(MU)/(y_1, \dots, y_{n-1})$ , then each of the cofibrations will give a short exact sequence in homotopy, so we get

$$\pi_*(C(y_1, \dots, y_n)) = \pi(MU)/(y_1, \dots, y_n).$$

In this way one can kill off any regular ideal in  $\pi_*(MU)$ . In particular, one can get  $H$  by killing  $(x_1, x_2, \dots)$ . Sullivan’s idea was to use this to show that any homology class could be represented by the corresponding type of manifold with singularity. One could also get  $BP$  by killing the kernel of the map  $\pi_*(MU) \rightarrow \pi_*(BP)$  and then localizing at  $p$ . This approach to  $BP$  does not reflect the splitting of  $MU_{(p)}$ .

Much more delicate arguments are needed to show that the resulting spectra are multiplicative (Shimada and Yagita [1], Morava [1], Mironov [1]), and the proof only works at odd primes. Once they are multiplicative, it is immediate that they are orientable in the sense of 4.1.1.

The two most important cases of this construction are the *Johnson–Wilson spectra*  $BP\langle n \rangle$  (Johnson and Wilson [2]) and the Morava  $K$ -theories  $K(n)$  (Morava’s account remains unpublished; see Johnson and Wilson [3]).

$BP\langle n \rangle$  is the spectrum obtained from  $BP$  (one can start there instead of  $MU$  since  $BP$  itself is a product of the Sullivan–Baas construction) by killing  $(v_{n+1}, v_{n+2}, \dots) \subset \pi_*(BP)$ . One gets

$$\pi_*(BP\langle n \rangle) = \mathbf{Z}_{(p)}[v_1, \dots, v_n]$$

and

$$H_*(BP\langle n \rangle, \mathbf{Z}/(p)) = P_* \otimes E(\tau_{n+1}, \tau_{n+2}, \dots).$$

(It is an easy exercise using the methods of Section 3.1 to show that a connective spectrum with that homology must have the indicated homotopy.) One has fibrations

$$\Sigma^{2(p^n-1)} BP\langle n \rangle \xrightarrow{v_n} BP\langle n \rangle \rightarrow BP\langle n-1 \rangle.$$

$BP\langle 0 \rangle$  is  $H_{(p)}$  and  $BP\langle 1 \rangle$  is a summand of  $bu_{(p)}$ , the localization at  $p$  of the spectrum representing connective complex  $K$ -theory. One can iterate the map

$$v_n: \Sigma^{2(p^n-1)} BP\langle n \rangle \rightarrow BP\langle n \rangle$$

and form the direct limit

$$E(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)} BP\langle n \rangle.$$

$E(1)$  is a summand of periodic complex  $K$ -theory localized at  $p$ . Johnson and Wilson [2] showed that

$$E(n)_*(X) = BP_*(X) \otimes_{BP_*} E(n)_*.$$

$E(n)$  can also be obtained by using the Landweber exact functor theorem below.

The  $BP\langle n \rangle$  are interesting for two reasons. First, the fibrations mentioned above split unstably; i.e., if  $BP\langle n \rangle_k$  is the  $k$ th space in the  $Q$ -spectrum for  $BP\langle n \rangle$  (i.e., the space whose homotopy starts in dimension  $k$ ) then

$$BP\langle n \rangle_k \simeq BP\langle n-1 \rangle_k \times BP\langle n \rangle_{k+2(p^n-1)}$$

for  $k \leq 2(p^n-1)/(p-1)$  (Wilson [2]). This means that if  $X$  is a finite complex then  $BP_*(X)$  is determined by  $BP\langle n \rangle_*(X)$  for an appropriate  $n$  depending on the dimension of  $X$ .

The second application of  $BP\langle n \rangle$  concerns  $\text{Hom dim } BP_*(X)$ , the projective dimension of  $BP_*(X)$  as a module over  $\pi_*(BP)$ , known in some circles as the ugliness number. Johnson and Wilson [2] show that the map  $BP_*(X) \rightarrow BP\langle n \rangle_*(X)$  is onto iff  $\text{Hom dim } BP_*(X) \leq n+1$ . The cases  $n=0$  and  $n=1$  of this were obtained earlier by Conner and Smith [3].

We now turn to the *Morava  $K$ -theories*  $K(n)$ . These spectra are periodic, i.e.,  $\Sigma^{2(p^n-1)}K(n) = K(n)$ . Their connective analogs  $k(n)$  are obtained from  $BP$  by killing  $(p, v_1, \dots, v_{n-1}, v_{n+1}, v_{n+2}, \dots)$ . Thus one has  $\pi_*(k(n)) = \mathbf{Z}/(p)[v_n]$  and  $H_*(k(n), \mathbf{Z}/(p)) = A/(Q_n)_*$ . One has fibrations

$$\Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \rightarrow H\mathbf{Z}/(p),$$

and one defines

$$K(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)}k(n).$$

$K(1)$  is a summand of mod  $p$  complex  $K$ -theory and it is consistent to define  $K(0)$  to be  $HQ$ , rational homology.

The coefficient ring  $\pi_*(K(n)) = \mathbf{F}_p[v_n, v_n^{-1}]$  is a graded field in the sense that every graded module over it is free. One has a Künneth isomorphism

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes_{\pi_*(K(n))} K(n)_*(Y).$$

This makes  $K(n)_*(-)$  much easier to compute with than any of the other theories mentioned here. In Ravenel and Wilson [3] we compute the Morava  $K$ -theories of all the Eilenberg–Mac Lane spaces, the case  $n=1$  having been done by Anderson and Hodgkin [2]. We show that for a finite abelian group  $G$ ,  $K(n)_*(K(G, m))$  is finite-dimensional over  $\pi_*(K(n))$  for all  $m$  and  $n$ , and is isomorphic to  $K(n)_*(\text{pt})$  if  $m > n$ .  $K(n+1)^*K(\mathbf{Z}, m+2)$  for  $m, n \geq 0$  is a power series ring on  $\binom{n}{m}$  variables. In all cases the  $K(n)$ -theory is concentrated in even dimensions. These calculations enabled us to prove the conjecture of Conner and Floyd [2] which concerns  $BP_*(B(\mathbf{Z}/p)^n)$ .

To illustrate the relation between the  $K(n)$ 's and  $BP$  we must introduce some more theories. Let  $I_n = (p, v_1, \dots, v_{n-1}) \subset \pi_*(BP)$  (see 4.3.2) and let  $P(n)$  be the spectrum obtained from  $BP$  by killing  $I_n$ . Then one has fibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \rightarrow P(n+1)$$

and we define

$$B(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)} P(n).$$

$P(n)_*(X)$  is a module over  $\mathbf{F}_p[v_n]$  and its torsion-free quotient maps monomorphically to  $B(n)_*(X)$ . In Johnson and Wilson [3] it is shown that  $B(n)_*(X)$  is determined by  $K(n)_*(X)$ . In Würgler [2] it is shown that a certain completion of  $B(n)$  splits into a wedge of suspensions of  $K(n)$ .

This splitting has the following algebraic antecedent. The formal group law associated with  $K(n)$  (4.1.4) is essentially the standard height  $n$  formal group law  $F_n$  of A2.2.11, while  $\pi_*(B(n)) = \mathbf{F}_p[v_n, v_n^{-1}, v_{n+1}, \dots]$  is the universal ring for all  $p$ -typical formal group laws of height  $n$  (A2.2.8). In A2.2.12 it is shown that over the algebraic closure of  $\mathbf{F}_p$  any height  $n$  formal group law is isomorphic to the standard one. Heuristically this is why  $B(n)_*(X)$  is determined by  $K(n)_*(X)$ .

This connection between  $K(n)$  and height  $n$  formal group laws also leads to a close relation between  $K(n)_*(K(n))$  and the endomorphism ring of  $F_n$  (A2.2.18). An account of  $K(n)_*(K(n))$  is given in Yagita [1]. The reader should be warned that  $K(n)_*(K(n))$  is not the Hopf algebroid  $K(n)_*K(n)$  of Ravenel [5, 6], which is denoted herein by  $\Sigma(n)$ ; in fact,  $K(n)_*(K(n)) = \Sigma(n) \otimes E(\tau_0, \tau_1, \dots, \tau_{n-1})$ , where the  $\tau_i$  are analogous to the  $\tau_i$  in  $A_*$ .

Most of the above results on  $K(n)$  (excluding the results about Eilenberg–Mac Lane spaces) were known to Morava and communicated by him to the author in 1973.

The invariance of the  $I_n$  (4.3.2) under the  $BP$ -operations makes it possible to construct the spectra  $P(n)$ ,  $B(n)$ , and  $K(n)$  and to show that they are ring spectra for  $p > 2$  by more algebraic means, i.e., without using the Sullivan–Baas construction. This is done in Würgler [1], where the structure of  $P(n)_*(P(n))$  is also obtained.  $k(n)_*(k(n))$  is described in Yagita [2].

We now turn to the important work of Peter Landweber on the internal algebraic structure of  $MU$ - and  $BP$ -theories. The starting point is the invariant prime ideal theorem 4.3.2, which first appeared in Landweber [4], although it was probably first proved by Morava. It states that the only prime ideals in  $\pi_*(BP)$  which are invariant (A1.1.21), or, equivalently, which are subcomodules over  $BP_*(BP)$ , are the  $I_n = (p, v_1, v_2, \dots, v_{n-1})$  for  $0 \leq n \leq \infty$ . In Conner and Smith [3] it is shown that for a finite complex  $X$ ,  $BP_*(X)$  is finitely presented as a module over  $\pi_*(BP)$ . [The result there is stated in terms of  $MU_*(X)$ , but the two statements are equivalent.] From commutative algebra one knows that such a module over such a ring has a finite filtration in which each of the successive subquotients is isomorphic to the quotient of the ring by some prime ideal. Of course, as anyone who has contemplated the prospect of algebraic geometry knows, a ring such as  $\pi_*(BP)$  has a very large number of prime ideals. However, Landweber [3] shows that the coaction of  $BP_*(BP)$  implies that the filtration of  $BP_*(X)$  [or of any  $BP_*(BP)$ -comodule which is finitely presented as a module over  $\pi_*(BP)$ ] can be chosen so that each successive subquotient has the form  $\pi_*(BP)/I_n$  for some finite  $n$ . {The corresponding statement about  $MU_*(X)$  appeared earlier in Landweber [5].} The submodules in the filtration can be taken to be subcomodules and  $n$  (the number of generators of the prime ideal) never exceeds the projective dimension of the module. This useful result is known as the *Landweber filtration theorem*.



It leads to the *Landweber exact functor theorem*, which addresses the following question. For which  $\pi_*(BP)$ -modules  $M$  is the functor  $BP_*(-) \otimes_{\pi_*(BP)} M$  a generalized homology theory? Such a functor must be exact in the sense that it converts cofiber sequences into long exact sequences of modules. This will be the case if  $M$  is flat, i.e., if  $\mathrm{Tor}_1^{\pi_*(BP)}(M, N) = 0$  for all modules  $N$ . However, in view of the filtration theorem it suffices for this Tor group to vanish only for  $N = \pi_*(BP)/I_n$  for all  $n$ . This weaker (than flatness) condition on  $M$  can be made more explicit as follows. For each  $n$ , multiplication by  $v_n$  in  $M \otimes_{\pi_*(BP)} \pi_*(BP)/I_n$  is monic. Thus Landweber [3] shows that any  $M$  satisfying this condition gives a homology theory.

For example, the spectrum  $E(n)$  mentioned above (in connection with Johnson–Wilson spectra) can be so obtained since

$$\pi_*(E(n)) = \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$$

satisfies Landweber’s condition. [Multiplication by  $v_i$  is monic in  $\pi_*(E(n))$  itself for  $i \leq n$ , while for  $i > n$ ,  $\pi_*(E(n)) \otimes_{\pi_*(BP)} \pi_*(BP)/I_i = 0$  so the condition is vacuous.]

As remarked earlier,  $E(1)$  is a summand of complex  $K$ -theory localized at  $p$ . The exact functor theorem can be formulated globally in terms of  $MU$ -theory and  $\pi_*(K)$  [viewed as a  $\pi_*(MU)$ -module via the Todd genus  $td: \pi_*(MU) \rightarrow \mathbf{Z}$ ] satisfies the hypotheses. Thereby one recovers the *Conner–Floyd isomorphism*

$$K_*(X) = MU_*(X) \otimes_{\pi_*(MU)} \pi_*(K)$$

and similarly for cohomology. In other words, complex  $K$ -theory is determined by complex cobordism. This result was first obtained by Conner and Floyd [1], whose proof relied on an explicit  $K$ -theoretic orientation of a complex vector bundle. Using similar methods they were able to show that real  $K$ -theory is determined by symplectic cobordism.

Landweber’s results have been generalized as follows. Let  $J \subset \pi_*(BP)$  be an invariant regular ideal (see Landweber [7]), and let  $BPJ$  be the spectrum obtained by killing  $J$ ; e.g.,  $P(n)$  above is  $BPI_n$ . Most of the algebra of  $BP$ -theory carries over to these spectra, which are studied systematically in a nice paper by Johnson and Yosimura [4]. The case  $J = I_n$  was treated earlier by Yagita [3] and Yosimura [1]. The mod  $I_n$  version of the exact functor enables one to get  $K(n)$  from  $P(n)$ .

Johnson and Yosimura [4] also prove some important facts about  $\pi_*(BP)$  modules  $M$  which are comodules over  $BP_*(BP)$ . They show that if an element  $m \in M$  is  $v_n$ -torsion (i.e., it is annihilated by some power of  $v_n$ ) then it is  $v_{n-1}$ -torsion. If all of the primitive elements in  $M$  [i.e., those with  $\psi(m) = 1 \otimes m$ ] are  $v_n$ -torsion, then so is every element, and, if none is, then  $M$  is  $v_n$ -torsion free. If  $M$  is a  $v_{n-1}$ -torsion module, then  $v_n^{-1}M$  is still a comodule over  $BP_*(BP)$ . Finally, they show that  $v_n^{-1}BP_*(X) = 0$  if  $E(n)_*(X) = 0$ .

This last result may have been prompted by an erroneous claim by the author that the spectrum  $v_n^{-1}BP$  splits as a wedge of suspensions of  $E(n)$ . It is clear from the methods of Würzler [2] that one must complete the spectra in some way before such a splitting can occur. Certain completions of  $MU$  are studied in Morava [2].

We now turn to the last topic of this section, the applications of  $BP$ -theory to unstable homotopy theory. This subject began with Steve Wilson’s thesis (Wilson [1, 2]) in which he studied the spaces in the  $\Omega$ -spectra for  $MU$  and  $BP$ . He obtained the splitting mentioned above (in connection with the Johnson–Wilson spectra) and showed that all of the spaces in question have torsion-free homology.



Both the homology and cohomology of each space are either an exterior algebra on odd-dimensional generators or a polynomial algebra on even-dimensional generators.

These spaces were studied more systematically in Ravenel and Wilson [2]. There we found it convenient to consider all of them simultaneously as a graded space. The mod  $(p)$  homology of such an object is a bigraded coalgebra. The fact that this graded space represents a multiplicative homology theory implies that its homology is a ring object in the category of bigraded coalgebras; we call such an object a *Hopf ring*. We show that the one in question has a simple set of generators and relations which are determined by the structure of  $MU^*(\mathbf{C}P^\infty)$ , i.e., by  $\pi_*(MU)$  and the associated formal group law. We obtain similar results for the value on this graded space of any complex oriented (4.1.1) generalized homology theory.

As mentioned earlier, the complex bordism of the graded space associated with  $MU$  is the cobordism group of maps between stably complex manifolds. We show that it is a Hopf ring generated by maps from a manifold to a point and the linear embeddings of  $\mathbf{C}P^n$  in  $\mathbf{C}P^{n+1}$ .

The Hopf ring point of view is also essential in Ravenel and Wilson [3], where we calculate  $K(n)_*(K(G, m))$ . We show that the Hopf ring  $K(n)_*(K(\mathbf{Z}/(p^j), *))$  is a certain type of free object on  $K(n)_*(K(\mathbf{Z}/(p^j), 1))$ . The ordinary homology of  $K(\mathbf{Z}/(p^j), *)$  can be described in similar terms and the methods of our paper may lead to simpler proofs of the classical theorems about it (see Wilson [3], section II.8).

Knowing the  $BP$  homology of the spaces in the  $BP$  spectrum is analogous to knowing the mod  $(p)$  homology of the mod  $(p)$  Eilenberg–Mac Lane spaces. This information, along with some ingenious formal machinery, is needed to construct the unstable Adams spectral sequence, i.e., a spectral sequence for computing the homotopy groups of a space  $X$  rather than a spectrum. This was done in the  $BP$  case by Bendersky, Curtis, and Miller [1]. Their spectral sequence is especially convenient for  $X = S^{2n+1}$ . In that case they get an  $E_1$ -term which is a subcomplex of the usual  $E_1$ -term for the sphere spectrum, i.e., of the cobar complex of A1.2.11. Their  $E_2$ -term is  $\text{Ext}$  in an appropriate category. For  $S^{2n+1}$  they compute  $\text{Ext}^1$ , which is a subgroup of the stable  $\text{Ext}^1$ , and get some corresponding information about  $\pi_*(S^{2n+1})$ .

In Bendersky [2] the spectral sequence is applied to the special unitary groups  $SU(n)$ . In Bendersky, Curtis, and Ravenel [3] the  $E_2$ -terms for various spheres are related by an analog of the EHP sequence.

### 3. Some Calculations in $BP_*(BP)$

In this section we will prove the Morava–Landweber theorem (4.3.2), which classifies invariant prime ideals in  $\pi_*(BP)$ . Then we will derive several formulas in  $BP_*(BP)$  (4.1.18 and 4.1.19). These results are rather technical. Some of them are more detailed than any of the applications in this book require and they are included here only for possible future reference. The reader is advised to refer to this material only when necessary.

Theorem 4.3.3 is a list of invariant regular ideals that will be needed in Chapter 5. Lemma 4.3.8 gives some generalizations of the Witt polynomials. They are

used to give more explicit formulas for the formal group law (4.3.9), the coproduct (4.3.13), and the right unit (4.3.18). We define certain elements,  $b_{i,J}$  (4.3.14) and  $c_{i,J}$  (4.3.19), which are used to give approximations (modulo certain prime ideals) of the coproduct (4.3.15) and right unit (4.3.20). Explicit examples of the right unit are given in 4.3.21. The coboundaries of  $b_{i,J}$  and  $c_{i,J}$  in the cobar complex are given in 4.3.22.

In 4.3.23 we define a filtration of  $BP_*(BP)/I_n$  which leads to a May spectral sequence which will be used in Section 6.3. The structure of the resulting bigraded Hopf algebroid is given in 4.3.32–34.

From now on  $\pi_*(BP)$  will be abbreviated by  $BP_*$ . Recall (A2.2.3) that we have two sets of generators for the ring  $BP_*$  given by Hazewinkel [2] (A2.2.1) and Araki [1] (A2.2.2). The behavior of the right unit  $\eta_R: BP_* \rightarrow BP_*(BP)$  on the Araki generators is given by A2.2.6, i.e.,

$$(4.3.1) \quad \sum_{i,j \geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{i,j \geq 0}^F v_i t_j^{p^i}$$

For the Hazewinkel generators this formula is true only mod  $(p)$ .

This formula will enable us to define some invariant ideals in  $BP_*$ . In each case it will be easy to show that the ideal in question is independent of the choice of generators used. The most important result of this sort is the following.

4.3.2. THEOREM (Morava [3], Landweber [4]). *Let  $I_n = (p, v_1, \dots, v_{n-1}) \subset BP_*$ .*

- (a)  $I_n$  is invariant.
- (b) For  $n > 0$ ,

$$\mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*/I_n) = \mathbf{Z}/(p)[v_n]$$

and

$$\mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*) = \mathbf{Z}_{(p)}.$$

- (c)  $0 \rightarrow \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \rightarrow BP_*/I_{n+1} \rightarrow 0$  is a short exact sequence of comodules.

- (d) The only invariant prime ideals in  $BP_*$  are the  $I_n$  for  $0 \leq n \leq \infty$ .

PROOF. Part (a) follows by induction on  $n$ , using (c) for the inductive step. Part (c) is equivalent to the statement that

$$v_n \in \mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*/I_n)$$

and is therefore a consequence of (b). For (d) suppose  $J$  is an invariant prime ideal which properly contains some  $I_n$ . Then the smallest dimensional element of  $J$  not in  $I_n$  must be invariant modulo  $I_n$ , i.e., it must be in  $\mathrm{Ext}_{BP_*(BP)}^0(BP_*(BP)/I_n)$ , so by (b) it must be a power of  $v_n$  (where  $v_0 = p$ ). Since  $J$  is prime this element must be  $v_n$  itself, so  $J \supset I_{n+1}$ . If this containment is proper the argument can be repeated. Hence, if  $J$  is finitely generated, it is  $I_n$  for some  $n < \infty$ . If  $J$  is infinitely generated we have  $J \supset I_\infty$ , which is maximal, so (d) follows.

Hence it remains only to prove (b). It is clear from 4.3.1 that  $\eta_R(v_n) \equiv v_n \pmod{I_n}$ , so it suffices to show that  $\mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*/I_n)$  is no bigger than indicated. From 4.3.1 we see that in  $BP_*(BP)/I_n$ ,

$$\eta_R(v_{n+j}) \equiv v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j \pmod{(t_1, t_2, \dots, t_{j-1})},$$

so the set  $\{v_{n+j}, \eta_R(v_{n+j}) \mid j > 0\} \cup \{v_n\}$  is algebraically independent. It follows that if  $\eta_R(v) = v$  then  $v$  must be a polynomial in  $v_n$ .  $\square$

Now we will construct some invariant regular ideals in  $BP_*$ . Recall that an ideal  $(x_0, x_1, \dots, x_{n-1})$  is regular if  $x_i$  is not a zero divisor in  $BP_*/(x_0, \dots, x_{i-1})$  for  $0 \leq i < n$ . This means that the sequence

$$0 \rightarrow BP_*/(x_0, \dots, x_{i-1}) \xrightarrow{x_i} BP_*/(x_0, \dots, x_{i-1}) \rightarrow BP_*/(x_0, \dots, x_i) \rightarrow 0$$

is exact. The regular sequence  $(x_0, x_1, \dots)$  is invariant if the above is a short exact sequence of comodules. Invariant regular ideals have been studied systematically by Landweber [7]. He shows that an invariant regular ideal with  $n$  generators is primary with radical  $I_n$ , and that any invariant ideal with  $n$  generators and radical  $I_n$  is regular. Invariant ideals in general need not be regular, e.g.,  $I_n^k$  for  $k > 1$ .

**4.3.3. THEOREM.** *Let  $i_1, i_2, \dots$  be a sequence of positive integers such that for each  $n > 0$ ,  $i_{n+i}$  is divisible by the smallest power of  $p$  not less than  $i_n$ , and let  $k \geq 0$ . Then for each  $n > 0$ , the regular ideal  $(p^{1+k}, v_1^{i_1 p^k}, v_2^{i_2 p^{2k}}, \dots, v_n^{i_n p^{kn}})$  is invariant.  $\square$*

In order to prove this we will need the following.

**4.3.4. LEMMA.** *Let  $B, A_1, A_2, \dots$  be ideals in a commutative ring. Then if*

$$x \equiv y \pmod{pB + \sum_i A_i},$$

then

$$x^{p^n} \equiv y^{p^n} \pmod{p^{n+1}B + \sum_{k=0}^n p^k \sum_i A_i^{p^{n-k}}}.$$

**PROOF.** The case  $n > 1$  follows easily by induction on  $n$  from the case  $n = 1$ . For the latter suppose  $x = y + pb + \sum a_i$ , with  $b \in B$  and  $a_i \in A_i$ . Then

$$x^p = y^p + \sum_{0 < j < p} \binom{p}{j} y^{p-j} (pb + \sum a_i)^j + (pb + \sum a_i)^p$$

and we have

$$\binom{p}{j} y^{p-j} (pb + \sum a_i)^j \in p^2 B + p \sum A_i$$

and

$$(pb + \sum a_i)^p \in p^2 B + p \sum A_i + \sum A_i^p. \quad \square$$

**PROOF OF 4.3.3.** We have  $v_n \equiv \eta_R(v_n) \pmod{I_n}$ , so we apply 4.3.4 to the ring  $BP_*(BP)$  by setting  $B = (1)$ ,  $A_i = (v_i)$ . Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p^m} \pmod{(p^{m+1}) + \sum_{j=0}^m \sum_{i=1}^{n-1} (p^j v_i^{m-j})}.$$

To prove the theorem we must show that the indicated power of  $v_n$  is invariant modulo the ideal generated by the first  $n$  elements. It suffices to replace this ideal by the smaller one obtained by replacing each of  $i_1, \dots, i_{n-1}$  by the smallest power of  $p$  not less than it, i.e., by an ideal of the form

$$(p^{1+k}, v_1^{j_1}, v_2^{j_2}, \dots, v_{n-1}^{j_{n-1}}) = I$$

with  $j_t = p^{t k + k_t}$  where  $0 \leq k_1 \leq k_2 \cdots \leq k_{n-1}$ . Then the hypothesis on  $i_n$  is that it is divisible by  $p^{k_{n-1}}$ , so it suffices to assume that  $i_n = p^{k_{n-1}}$ . Hence we must show

$$v_n^{j_n - 1 p^k} \equiv \eta_R(v_n)^{j_n - 1 p^k} \pmod{I}.$$

We have  $v_n \equiv \eta_R(v_n) \pmod{I_n}$ , so we apply 4.3.4 to the ring  $BP_*(BP)$  by setting  $B = (1)$ ,  $A_i = (v_i)$ . Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p^m} \pmod{(p^{m+1}) + \sum_{j=0}^m \sum_{i=0}^{n-1} (p^j v_i^{p^{m-j}})}.$$

We are interested in the case  $m = kn + k_{n-1}$ . Careful inspection shows that the indicated ideal in this case is contained in  $I$ .  $\square$

Theorem 4.3.3 leads to a list of invariant regular ideals which one might hope is complete. Unfortunately, it is not. For example, it gives  $\{(p^{k+1}, v^{ip^k}) \mid k \geq 0, i \geq 0\}$  as a list of  $I_2$ -primary regular ideals, and this list can be shown to be a complete for  $p > 2$ , but at  $p = 2$  the ideal  $(16, v_1^4 + 8v_1v_2)$  is regular and invariant but not in the list. Similarly, for  $p > 2$  the ideal

$$(p, v_1^{p^2+p-1}, v_2^{2p^2} - 2v_1^{p^2} v_2^{p^2-p} v_3^p - 2v_1^{p^2-1} v_2^{2p^2-p+1})$$

is invariant, regular, and not predicted by 4.3.3. This example and others like it were used by Miller and Wilson [3] to produce unexpected elements in  $\text{Ext}_{BP_*(BP)}^1(BP_*, BP_*/I_n)$  (see Section 5.2).

Now we will make the structure of  $BP_*(BP)$  (4.1.19) more explicit. We start with the formal group law.

Recall the lemma of Witt (see, e.g., Lang [1, pp. 234–235]) which states that there are symmetric integral polynomials  $w_n = w_n(x_1, x_2, \dots)$  of degree  $p^n$  in any number of variables such that

$$(4.3.5) \quad w_0 = \sum x_t \quad \text{and} \quad \sum_t x_t^{p^n} = \sum_j p^j w_j^{p^{n-j}}.$$

For example,

$$(4.3.6) \quad w_1 = \left( \sum (x_t^p) - \left( \sum x_t \right)^p \right) / p$$

and for  $p = 2$  with two variables,

$$w_2 = -x_1^3 x_2 - 2x_1^2 x_2^2 - x_1 x_2^3.$$

Witt's lemma can be restated as follows. Let  $G$  be the formal group law with logarithm  $\sum_{i \geq 0} x^{p^i} / p^i$ . Then

$$(4.3.7) \quad \sum^G x_t = \sum^G w_n.$$

This formula is in some sense more explicit than the usual

$$\log \left( \sum^G x_t \right) = \sum \log x_t.$$

We will derive a similar formula for the universal formal group law.

First we need some notation. Let  $I = (i_1, i_2, \dots, i_m)$  be a finite (possibly empty) sequence of positive integers. Let  $|I| = m$  and  $\|I\| = \sum i_t$ . For positive integers  $n$  let  $\Pi(n) = p - p^{(p^n)}$  and define integers  $\Pi(I)$  recursively by  $\Pi(\emptyset) = 1$  and  $\Pi(I) = \Pi(\|I\|) \Pi(i_1, \dots, i_{m-1})$ . Note that  $\Pi(I) \equiv p^{|I|} \pmod{p^{\|I\|+1}}$ . Given

sequences  $I$  and  $J$  let  $IJ$  denote the sequence  $(i_1, \dots, i_m, j_1, \dots, j_n)$ . Then we have  $|IJ| = |I| + |J|$  and  $\|IJ\| = \|I\| + \|J\|$ . We will need the following analog of Witt's lemma (4.3.5), which we will prove at the end of this section.

4.3.8. LEMMA.

(a) For each sequence  $I$  as above there is a symmetric polynomial of degree  $p^{\|I\|}$  in any number of variables with coefficients in  $\mathbf{Z}_{(p)}$ ,  $w_I = w_I(x_1, x_2, \dots)$  with  $w_\phi = \sum_t x_t$  and

$$\sum_t x_t^{p^{\|K\|}} = \sum_{IJ=K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

(b) Let  $w_i$  be the polynomial defined by 4.3.5. Then

$$w_I \equiv w_{|I|}^{p^{\|I\| - |I|}} \pmod{(p)}. \quad \square$$

Now let  $v_i$  be Araki's generator and define  $v_I$  by  $v_\phi = 1$  and  $v_I = v_{i_1}(v_{I'})^{(p^{i_1})}$  where  $I' = (i_2, i_3, \dots)$ . Hence  $\dim v_I = 2(p^{\|I\|} - 1)$ . Then our analog of 4.3.7 is

4.3.9. THEOREM. With notation as above,

$$\sum_t^F x_t = \sum_I^F v_I w_I(x_1, x_2, \dots).$$

(An analogous formula and proof in terms of Hazewinkel's generators can be obtained, by replacing  $\Pi(I)$  by  $p^{|I|}$  throughout. This requires a different definition of  $w_J$ , which is still congruent to  $w_{|J|}^{p^{\|J\| - |J|}}$  modulo  $p$ .)

PROOF. Araki's formula (A2.2.2) is

$$p\lambda_n = \sum_{0 \leq i \leq n} \lambda_i v_{n-i}^{p^i}$$

which can be written as

$$\Pi(n)\lambda_n = \sum_{0 \leq i < n} \lambda_i v_{n-i}^{p^i}.$$

By a simple exercise this gives

$$\lambda_n = \sum_{\|I\|=n} \frac{v_I}{\Pi(I)},$$

i.e.,

$$(4.3.10) \quad \log(x) = \sum \frac{v_I x^{p^{\|I\|}}}{\Pi(I)}.$$

Therefore we have

$$\begin{aligned}
\log\left(\sum_J^F v_J w_J\right) &= \sum_J \log v_J w_J \\
&= \sum \frac{v_{IJ}}{\Pi(I)} w_J^{p^{\|I\|}} \\
&= \sum_{I,J} \frac{v_K}{\Pi(K)} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \quad (\text{where } K = IJ) \\
&= \sum_{t,K} \frac{v_K}{\Pi(K)} x_t^{p^{\|K\|}} \quad \text{by 4.3.8} \\
&= \sum_t \log x_t \quad \text{by 4.3.10} \\
&= \log \sum_t^F x_t. \quad \square
\end{aligned}$$

In the structure formulas for  $BP_*(BP)$  we encounter expressions of the form  $\sum_{n,i}^F a_{n,i}$ , where  $a_{n,i}$  is in  $BP_*(BP)$  or  $BP_*(BP) \otimes_{BP_*} BP_*(BP)$  (or more generally in some commutative graded  $BP_*$  algebra  $D$ ) and has dimension  $2(p^n - 1)$ . We can use 4.3.9 to simplify such expressions in the following way.

Define subsets  $A_n$  and  $B_n$  of  $D$  as follows.  $A_n = B_n = \phi$  for  $n \leq 0$  and for  $n > 0$ ,  $A_n = \{a_{n,i}\}$  while  $B_n$  is defined recursively by

$$B_n = A_n \cup \bigcup_{\|J\|>0} \{v_J w_J(B_{n-\|J\|})\}.$$

4.3.11. LEMMA. *With notation as above,  $\sum_{n,i}^F a_{n,i} = \sum_{n>0}^F w_\phi(B_n)$ .*

PROOF. We will show by induction on  $m$  that the statement is true in dimensions  $< 2(p^m - 1)$ . Our inductive hypothesis is

$$\sum_{0 < n < m}^F a_{n,i} = \sum_{0 < n < m}^F w_\phi(B_n) +_F \sum_{\substack{n < m \\ \|J\| + n \geq m}}^F v_J w_J(B_n) +_F \sum_{n \geq m}^F a_{n,i},$$

which is trivial for  $m = 1$ . The set of formal summands of dimension  $2(p^m - 1)$  on the right is  $B_m$ . By 4.3.9 the formal sum of these terms is  $\sum^F v_J w_J(B_m)$ , so we get

$$\begin{aligned}
\sum^F a_{n,i} &= \sum_{0 < n < m}^F w_\phi(B_n) +_F \sum_J^F v_J w_J(B_m) +_F \sum_{\substack{n < m \\ \|J\| + n > m}}^F v_J w_J(B_n) +_F \sum_{n > m}^F a_{n,i} \\
&= \sum_{0 < n \leq m}^F w_\phi(B_n) +_F \sum_{\substack{n \leq m \\ \|J\| + n > m}}^F v_J w_J(B_n) +_F \sum_{n > m}^F a_{n,i}.
\end{aligned}$$

which completes the inductive step and the proof.  $\square$

Recall now the coproduct in  $BP_*(BP)$  given by 4.1.18(e), i.e.,

$$\sum_{i \geq 0} \log(\Delta(t_i)) = \sum_{i,j \geq 0} \log(t_i \otimes t_j^i),$$

which can be rewritten as

$$(4.3.12) \quad \sum_{i \geq 0}^F \Delta(t_i) = \sum_{i, j \geq 0}^F t_i \otimes t_j^{p^i}$$

To apply 4.3.11, let  $M_n = \{t_i \otimes t_{n-i}^{p^i} \mid 0 \leq i \leq n\}$  ( $M$  here stands for Milnor since these terms are essentially Milnor's coproduct 3.1.1) and let

$$\Delta_n = M_n \cup \bigcup_{|J| > 0} \{v_J w_J(\Delta_{n-\|J\|})\}.$$

Then we get from 4.3.11 and 4.3.12

4.3.13. THEOREM. *With notation as above,*

$$\Delta(t_n) = w_\phi(\Delta_n) \in BP_*(BP) \otimes_{BP_*} BP_*(BP). \quad \square$$

For future reference we make

4.3.14. DEFINITION. *In  $BP_*(BP) \otimes_{BP_*} BP_*(BP)$  let  $b_{i,j} = w_{(j+1)}(\Delta_i)$ .*  $\square$

For example,

$$b_{1,j} = -\frac{1}{p - p^{(p^{j+1})}} \sum_{0 < i < p^{j+1}} \binom{p^{j+1}}{i} t_1^i \otimes t_1^{p^{j+1}-i}.$$

This  $b_{i,j}$  can be regarded as an element of degree 2 in the cobar complex (A1.2.11)  $C(BP_*)$ . It will figure in subsequent calculations and we will give a formula for its coboundary (4.3.22) below.

If we reduce modulo  $I_n$ , 4.3.13 simplifies as follows.

4.3.15. COROLLARY. *In  $BP_*(BP) \otimes_{BP_*} BP_*(BP)/I_n$  for  $k \leq 2n$*

$$\Delta(t_k) = \sum_{0 \leq i \leq k} t_i \otimes t_{k-i}^{p^i} + \sum_{0 \leq j \leq k-n-1} v_{n+j} b_{k-n-j, n+j-1}. \quad \square$$

Now we will simplify the right unit formula 4.3.1. First we need a lemma. Define  $t_I$  as we defines  $v_I$ , replacing  $v_i$  by  $t_i$ .

4.3.16. LEMMA. *In  $BP_*(BP)$ , for each  $k \geq 0$*

$$\sum_{i, |I| \geq 0}^F [(-1)^{|I|}] \left( t_I t_i^{p^{\|I\|}} \right)^{p^k} = \sum_{i, |I| \geq 0}^F [(-1)^{|I|}] (t_i (t_I)^{p^i})^{p^k} = 1.$$

It is shown in A2.2.5 that for  $p > 2$ ,  $[-1](x) = -x$  for any  $p$ -typical formal group law, and a formulas for it are given for  $p = 2$  in terms of both the Hazewinkel generators of (A2.2.1) and the Araki generators of (A2.2.2).  $[n](x)$  is defined in A2.1.19.

PROOF. In the first expression, for each  $I = (i_1, i_2, \dots, i_n)$  with  $n > 0$ , the expression  $(t_I)^{p^k}$  appears twice: once as  $(t_I t_0)^{p^k}$  and once as  $(t_{I'} (t_{i_n})^{p^{\|I'\|}})^{p^k}$  where  $I' = (i_1, \dots, i_{n-1})$ . These two terms have opposite formal sign and hence cancel, leaving 1 as the value of the first expression. The argument for the second expression is similar.  $\square$

Now we need to use the conjugate formal group law  $c(F)$  over  $BP_*(BP)$ , defined by the homomorphism  $\eta_R: BP_* \rightarrow BP_*(BP)$ . Its logarithm is

$$\log_{c(F)}(x) = \sum_{i \geq 0} \eta_R(\lambda_i) x^{p^i} = \sum_{i, j \geq 0} \lambda_i t_j^{p^i} x^{p^{i+j}}.$$

An analog of 4.3.9 holds for  $c(F)$  with  $v_I$  replaced by  $\eta_R(v_I)$ .

The last equation in the proof of A2.2.6 reads

$$\sum \lambda_i v_j^{p^i} t_k^{p^{i+j}} = \sum \lambda_i t_j^{p^i} \eta_R(v_k)^{p^{i+j}} = \sum \eta_R(\lambda_i) \eta_R(v_j)^{p^i}$$

while 4.3.16 gives

$$\sum \lambda_i = \sum (-1)^{|K|} \lambda_i t_j^{p^i} t_K^{p^{i+j}}.$$

Combining these and reindexing gives

$$\sum (-1)^{|J|} \eta_R(\lambda_i) (t_J(v_k t_l^{p^k})^{p^{\|J\|}})^{p^i} = \sum \eta_R(\lambda_i) \eta_R(v_j)^{p^i},$$

which is equivalent to

$$(4.3.17) \quad \sum_{i \geq 0} {}^{c(F)} \eta_R(v_i) = \sum_{\substack{|I|, j, k \geq 0 \\ |I| \text{ odd}}} {}^{c(F)} [(-1)^{|I|}]_{c(F)} (t_I(v_j t_k^{p^j})^{p^{\|I\|}}).$$

In A2.2.5 we show that for  $p > 2$ ,  $[-1]_{c(F)}(x) = -x$ , and for  $p = 2$ ,

$$[-1]_{c(F)}(x) = x - {}^{c(F)} \sum_{i \geq 0} \eta_R(v_i) x^{2^i}.$$

For  $p > 2$  we define finite subsets of  $BP_*(BP)$  for  $n > 0$

$$N_n = \bigcup_{\|I\|+i+j=n} \left\{ (-1)^{|I|} t^I (v_j t_k^{p^j})^{p^{\|I\|}} \right\}$$

$$R_n = N_n \cup \bigcup_{\substack{\|J\|=i \\ 0 < i < n}} \{ \eta_R(v_J) w_J(R_{n-i}) \}.$$

Then we get

4.3.18. THEOREM. *In  $BP_*(BP)$  for  $p > 2$ , we have  $\eta_R(v_n) = w_\phi(R_n)$ .*  $\square$

For  $p = 2$  we have to rewrite (4.3.17) as

$$\sum_{i \geq 0} {}^{c(F)} \eta_R(v_i) + {}^{c(F)} \sum_{\substack{|I|, j, k \geq 0 \\ |I| \text{ odd}}} {}^{c(F)} t_I (v_j t_k^{p^j})^{p^{\|I\|}} = \sum_{\substack{|I|, j, k \geq 0 \\ |I| \text{ even}}} {}^{c(F)} t_I (v_j t_k^{p^j})^{p^{\|I\|}}.$$

4.3.19. DEFINITION. *In  $BP_*(BP)$ ,  $c_{i,J} = w_J(R_i)$ . For  $J = (j)$  this will be written as  $c_{i,j}$ .*

Again we can simplify further by reducing modulo  $I_n$ .

4.3.20. COROLLARY. *In  $BP_*(BP)/I_n$  for  $0 < k \leq 2n$ ,*

$$\sum_{0 \leq i \leq k} v_{n+i} t_{k-i}^{p^{n+i}} - \eta_R(v_{n+k-i})^{p^i} t_i = \sum_{0 \leq j \leq k-n-1} v_{n+j} c_{k-j, n+j}.$$

(Note that the right-hand side vanishes if  $k \leq n$ .)  $\square$



4.3.21. COROLLARY. In  $BP_*(BP)/I_n$ ,

$$\begin{aligned}
\eta_R(v_{n+1}) &= v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \\
&\quad \text{for } n \geq 1; \\
\eta_R(v_{n+2}) &= v_{n+2} + v_{n+1} t_1^{p^{n+1}} + v_n t_2^{p^n} - v_{n+1}^p t_1 - v_n^p t_2 \\
&\quad + v_n^{p^2} t_1^{1+p} - v_n^p t_1^{1+p^{n+1}} \\
&\quad \text{for } n \geq 2; \\
\eta_R(v_{n+3}) &= v_{n+3} + v_{n+2} t_1^{p^{n+2}} + v_{n+1} t_2^{p^{n+1}} + v_n t_3^{p^n} - v_{n+2}^p t_1 \\
&\quad - v_{n+1}^{p^2} t_2 - v_n^{p^3} t_3 - v_{n+1}^p t_1^{1+p^{n+2}} - v_n^p t_1 t_2^{p^{n+1}} \\
&\quad - v_n^{p^2} t_2 t_1^{p^{n+2}} + v_{n+1}^{p^2} t_1^{1+p} + v_n^{p^3} t_1 t_2^p + v_n^{p^3} t_2 t_1^{p^2} \\
&\quad + v_n^{p^2} t_1^{1+p+p^{n+2}} - v_n^{p^3} t_1^{1+p+p^2} \\
&\quad \text{for } n \geq 3; \\
\eta_R(v_3) &= v_3 + v_2 t_1^{p^2} + v_1 t_2^p - v_2^p t_1 - v_1^{p^2} t_2 - v_1^p t_1^{1+p^2} \\
&\quad + v_1^{p^2} t_1^{1+p} + v_1 w_1(v_2, v_1 t_1^p, -v_1^p t_1) \\
&\quad \text{for } n = 1, p > 2 \text{ (add } v_1^5 t_1^2 \text{ for } p = 2) \\
\text{and } \eta_R(v_5) &= v_5 + v_4 t_1^{p^4} + v_3 t_2^{p^3} + v_2 t_3^{p^2} - v_4^p t_1 - v_3^p t_2 - v_2^p t_3 \\
&\quad - v_3^p t_1^{1+p^4} - v_2^p t_1 t_2^{p^3} - v_2^p t_1^{p^4} t_2 + v_3^p t_1^{1+p} + v_2^p t_1 t_2^p \\
&\quad + v_2^p t_1^{p^2} t_2 + v_2^p t_1^{1+p+p^4} - v_2^p t_1^{1+p+p^2} \\
&\quad + v^2 w_1(v_3, v_2 t_1^{p^2}, -v_2^p t_1^p)^p \\
&\quad \text{for } n = 2, p > 2 \text{ (add } v_2^9 t_1^4 \text{ for } p = 2). \quad \square
\end{aligned}$$

Now we will calculate the coboundaries of  $b_{i,j}$  (4.3.14) and  $c_{i,j}$  (4.3.19) in the cobar complex  $C(BP_*/I_n)$  (A1.2.11).

4.3.22. THEOREM. In  $C(BP_*/I_n)$  for  $0 < i \leq n$  and  $0 \leq j$

$$\begin{aligned}
\text{(a) } d(b_{i,j}) &= \sum_{0 < k < i} b_{k,j} \otimes t_{i-k}^{p^{k+j+1}} - t_k^{p^{1+j}} \otimes b_{i-k,k+j} \quad \text{and} \quad \text{(b) } d(c_{n+i,j+1}) = \\
&\sum_{0 \leq k < i} v_{n+k}^{p^{1+j}} b_{i-k,n+k+j} - v_{n+k}^{p^{i+1+j-k}} b_{i-k,j}.
\end{aligned}$$

PROOF. (a) It suffices to assume  $i = n$ . Recall that in  $C(BP_*/I_n)$ ,  $d(t_i) = t_i \otimes 1 + 1 \otimes t_i - \Delta(t_i)$  and  $d(v_{n+i}) = n_R(v_{n+i}) - v_{n+i}$ .  $\Delta(t_{2n}) - 1 \otimes t_{2n} - t_{2n} \otimes 1$ , given by 4.3.13, is a coboundary and hence a cocycle. Calculating its coboundary term by term using 4.3.13 and 4.3.17 will give the desired formula for  $d(b_{n,n-1})$  and the result will follow. The details are straightforward and left to the reader.

For (b) we assume  $i = n$  if  $i + n$  is even and  $i = n - 1$  if  $i + n$  is odd. Then we use the fact that  $d(v_{2n+i})$  is a cocycle to get the desired formula, as in the proof of (a).  $\square$

Now we will construct an increasing filtration on the Hopf algebroid  $BP_*(BP)/I_n$ . We will use it in Section 6.3.

To do this we first define integers  $d_{n,i}$  by

$$d_{n,i} = \begin{cases} 0 & \text{for } i \leq 0 \\ \max(i, p d_{n,i-n}) & \text{for } i > 0. \end{cases}$$

We then set  $\deg t_i^{p^j} = \deg v_{n+i}^{p^j} = d_{n,i}$  for  $i, j \geq 0$ . The subgroups  $F_r \subset BP_*(BP)/I_n$  are defined to be the smallest possible subgroups satisfying the above conditions.

The associated graded algebra  $E_0BP_*(BP)/I_n$  is defined by  $E_0^iBP_*(BP)/I_n = F_i/F_{i-1}$ . Its structure is given by

4.3.23. PROPOSITION.

$$E_0BP_*(BP)/I_n = T(t_{i,j}, v_{n+i,j} : i > 0, j \geq 0),$$

where  $t_{i,j}$  and  $v_{n+i,j}$  are elements corresponding to  $t_i^{p^j}$  and  $v_{n+1}^{p^j}$ , respectively,  $T(x) = R[x]/(x^p)$  and  $R = \mathbf{Z}(p)[v_n]$ .  $\square$

4.3.24. THEOREM. *With the above filtration,  $BP_*(BP)/I_n$  is a filtered Hopf algebroid, and  $E_0BP_*(BP)/I_n$  is a Hopf algebroid.*

PROOF. For a set of elements  $X$  in  $B_*(BP)/I_n$  or  $BP_*(BP) \otimes_{BP_*} BP_*(BP)/I_n$ , let  $\deg X$  be the smallest integer  $i$  such that  $X \subset F_i$ . It suffices to show then that  $\deg \Delta_i = \deg R_{n+i} = d_{n,i}$ . We do this by induction on  $i$ , the assertion being obvious for  $i = 1$ .

First note that

$$(4.3.25) \quad d_{n,a+b} \geq d_{n,a} + d_{n,b}$$

and

$$(4.3.26) \quad d_{n,a+bn} \geq p^b d_{n,a}.$$

It follows from 4.3.25 that  $\deg M_i = \deg N_{n+i} = d_{n,i}$ . It remains then to show that for  $\|J\| < i$

$$(4.3.27) \quad \deg(v_J w_J(\Delta_{i-\|J\|})) \leq d_{n,i}$$

and

$$(4.3.28) \quad \deg(v_J w_J(R_{n+i-\|J\|})) \leq d_{n,i}.$$

Since

$$(4.3.29) \quad \deg w_J(X) \leq p^{|J|} \deg X,$$

both 4.3.27 and 4.3.28 reduce to showing

$$(4.3.30) \quad d_{n,i} \geq \deg v_J + p^{|J|} d_{n,i-\|J\|}.$$

Now if  $v_J \not\equiv 0 \pmod{I_n}$  we can write

$$J = (n + j'_1, m + j'_2, \dots, n + j'_l)$$

with  $j'_t \geq 0$ , so

$$|J| = l, \quad \|J\| = \ln + \sum_{t=1}^l j'_t, \quad \text{and} \quad \deg v_J = \sum_{t=1}^l d_{n,j'_t}.$$

If we set  $k = \|J\| - n|J|$ , then 4.3.25 implies

$$(4.3.31) \quad d_{n,k} \geq \deg v_J.$$

However, by 4.3.25 and 4.3.26

$$\begin{aligned} d_{n,i} &\geq d_{n,k} + d_{n,i-\|J\|+n|J|} \\ &\geq d_{n,k} + p^{|J|} d_{n,i-\|J\|} \end{aligned}$$

so 4.3.20 follows from 4.3.31.  $\square$

We now turn to the Hopf algebroid structure of  $E_0BP_*(BP)/I_n$ . Let  $\overline{M}_i$ ,  $\overline{\Delta}_i$ ,  $\overline{N}_{n+i}$ , and  $\overline{R}_{n+i}$  denote the associated graded analogs of  $M_i$ ,  $\Delta_i$ ,  $N_{n+i}$ , and  $R_{n+i}$ , respectively, with trivial elements deleted. (An element in one of the latter sets will correspond to a trivial element if its degree is less than  $d_{n,i}$ .) All we have to do is describe these subsets. Let  $\overline{t}_I$ ,  $\overline{v}_I$ , and  $\overline{w}_I(x)$  denote the associated graded elements corresponding to  $t_I$ ,  $v_I$ , and  $w_I(x)$ , respectively.

4.3.32. LEMMA.

$$\overline{M}_i = \begin{cases} \bigcup_{0 \leq j \leq i} \{t_{j,0} \otimes t_{i-j,j}\} & \text{for } i \leq m \\ \{t_{i,0} \otimes 1, 1 \otimes t_{i,0}\} & \text{for } i > m \end{cases}$$

$$\overline{N}_{n+i} = \begin{cases} \bigcup_{\|I\|+j+k=i} \{(-1)^{|I|} \overline{t}_I v_{j,\|I\|} t_{k,j+\|I\|}\} & \text{for } i \leq m \\ \{v_{n+i,0}, v_n t_{i,n}, -v_n^{p^i} t_{i,0}\} & \text{for } i > m \end{cases}$$

where  $m = pn/(p-1)$ .

PROOF. This follows from the fact that equality holds in 4.3.25 if  $a+b \leq m$ .  $\square$

4.3.33. LEMMA.

$$\overline{\Delta}_i = \begin{cases} \overline{M}_i & \text{for } i < m \\ \overline{M}_i \cup \overline{\{v_n w_1^{p^{n-1}}(M_{i-n})\}} & \text{for } i = m \\ \overline{M}_i \cup \bigcup_{\substack{\|J\|=n|J| \\ 0 < \|J\| < i \\ i - \|J\| \geq m - n}} \{v_J w_J(\overline{\Delta_{i-n|J|}})\} & \text{for } i > m \end{cases}$$

$$\overline{R}_{n+i} = \begin{cases} \overline{N}_{n+i} & \text{for } i < m \\ \overline{N}_{n+i} \cup \overline{\{v_n w_1^{p^{n-1}}(R_{i-n})\}} & \text{for } i = m \\ \overline{N}_{n+i} \cup \bigcup_{\substack{\|J\|=n|J| \\ 0 < \|J\| < i \\ i - \|J\| \geq m - n}} \{v_J w_J(\overline{R_{i-n|J|}})\} & \text{for } i > m \end{cases}$$

[Note that the case  $i = m$  occurs only if  $(p-1)|n$ , and that the only  $J$ 's we need to consider for  $i > m$  are those of the form  $(n, n, \dots, n)$ .]

PROOF. We use the observation made in the proof of 4.3.32 along with the fact that equality holds in 4.3.26 if  $a \geq m = n$ .

Now both  $\overline{R}_{n+i}$ , and  $\overline{\Delta}$ , will consist only of the terms associated with those  $J$  for which equality holds in 4.3.30. For  $i > m$  this can occur only if  $\deg v_J = 0$ , i.e., if  $J = (n, n, \dots, n)$ ; the condition  $i - \|J\| \geq m - n$  is necessary to ensure that  $d_{n,i} = p^{|J|} d_{n,i-n|J|}$ . For  $i \leq m$  we still need  $i - \|J\| \geq m - n$ . Since  $\|J\| \geq n$  in all nontrivial terms, the only possibility is  $J = (n)$  when  $i = m$ .  $\square$

Now let  $\Delta_{i,j}$  and  $R_{n+i,j}$  be the subsets obtained from  $\Delta_i$  and  $R_{n+i}$ , respectively, by raising each element to the  $p^j$ th power. The corresponding subsets  $\overline{\Delta}_{i,j}$  and  $\overline{R}_{n+i,j}$  of the appropriate associated graded objects are related to  $\overline{\Delta}_i$  and  $\overline{R}_{n+i}$  in an obvious way. Note that

$$\begin{aligned} \overline{w}_J(\overline{\Delta}_i) &= \overline{w_{|J|}(\overline{\Delta_{i,\|J\|-|J|}})} \\ &= w_{|J|}(\overline{\Delta_{i,\|J\|-|J|}}). \end{aligned}$$

4.3.34. THEOREM. With  $\bar{\Delta}_{i,j}$  and  $\bar{R}_{n+i,j}$  as above, the Hopf algebroid structure of  $E_0BP_*(BP)/I_n$ , is given by

$$\begin{aligned}\Delta(t_{i,j}) &= w_0(\bar{\Delta}_{i,j}) \\ \eta_R(v_{n+i,j}) &= w_0(\bar{R}_{n+i,j}).\end{aligned}\quad \square$$

None of the  $t_{i,j}$  for  $i > 1$  are primitive, so we could not get a Hopf algebroid with  $\deg t_{i,j} < d_{n,i}$  once we have set  $\deg t_{1,j} = 1$ .

Note finally that the structure of  $E_0BP_*(BP)/I_n$  depends in a very essential way on the prime  $p$ .

Theorem 4.3.34 implies that  $E_0BP_*(BP)/I_n$  is cocommutative for  $n = 1$  and  $p > 2$ . For any  $n$  and  $p$  we can use this filtration to construct a spectral sequence as in A1.3.9. The cocommutativity in the case above permits a complete, explicit determination of the  $E_2$ -term, and hence a very promising beginning for a computation of  $\text{Ext}_{BP_*(BP)}(BP_*, BP/I_1)$ . However, after investigating this method thoroughly we found the  $E_2$ -term to be inconveniently large and devised more efficient strategies for computing  $\text{Ext}$ , which will be described in Chapter 7. Conceivably the approach at hand could be more useful if one used a machine to do the bookkeeping. We leave the details to the interested reader.

PROOF OF 4.3.8. We will prove (a) and (b) simultaneously by induction on  $m = |K|$ . If  $K' = (1 + k_1, k_2, \dots, k_m)$  then it follows from (b) that

$$w_{K'} \equiv w_K^p \pmod{(p)}.$$

Let  $K'' = (k_1 + k_2, k_3, \dots, k_m)$  and  $K''' = (k_2, k_3, \dots, k_m)$ . Then by the inductive hypothesis  $w_{K''}$  and  $w_{K'''}$  exist with

$$w_{K''} \equiv w_{K'''}^{a_1} \pmod{(p)},$$

where  $a_1 = p^{k_1}$ . Since  $\|K\| = \|K''\|$  we have

$$\sum_{I \cdot J = K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} = \sum_{I \cdot J = K''} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Expanding both sides partly we get

$$\begin{aligned}\Pi(K)w_K + \frac{\Pi(K)}{\Pi(k_1)} w_{K'''}^{a_1} + \sum_{\substack{|I| \geq 2 \\ I \cdot J = K}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \\ = \Pi(K'')w_{K''} + \sum_{\substack{|I| \geq 1 \\ I \cdot J = K''}} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}.\end{aligned}$$

Note that the same  $w_J^{p^{\|I\|}}$  occur on both sides, and one can use the definition of  $\Pi(k)$  to show that they have the same coefficients so the sums cancel. The remaining terms give

$$\Pi(K) \left( w_K + \frac{w_{K'''}^{a_1}}{\Pi(k_1)} \right) = \Pi(K'')w_{K''}.$$

Since  $\Pi(k_1) \equiv p \pmod{(p^2)}$  and  $w_{K''} = w_{K'''}^{a_1} \pmod{(p)}$ , we get an integral expression when we solve for  $w_K$ .

This completes the proof of (a).

For (b) we have

$$\sum x_t^{p^{\|K\|}} = \Pi(K)w_K + \sum_{\substack{I, J=K \\ |I|>0}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Since  $\Pi(K) \equiv p^{|K|} \pmod{p^{1+|K|}}$  and  $\Pi(I) \equiv p^{|I|} \pmod{p^{1+|I|}}$ , we get  $\Pi(K)/\Pi(I) \equiv p^{|J|} \pmod{p^{1+|J|}}$ . By definition

$$\begin{aligned} \frac{\Pi(K)}{\Pi(I)} &= \Pi(\|K\|)\Pi(\|K\| - j_{|J|}) \cdots \Pi(\|I\| + j_1) \\ &= (p - p^{p^{\|K\|}})(p - p^{p^{\|K\| - j_{|J|}}}) \cdots (p - p^{p^{\|I\| + j_1}}) \\ &\equiv p^{|J|} \pmod{p^{|J| - 1 + p^{\|I\| + j_1}}} \\ &\equiv p^{|J|} \pmod{p^{|K| + 1}} \quad \text{since} \\ &\quad |J| - 1 + p^{\|I\| + j_1} \geq |J| - 1 + \|I\| + 2 \\ &\quad \geq |K| + 1. \end{aligned}$$

By the inductive hypothesis

$$w_J \equiv w_{|J|}^{p^{\|J\| - |J|}} \pmod{p}$$

so  $w_J^{p^{\|I\|}} \equiv w_{|J|}^{p^{\|K\| - |J|}} \pmod{p^{1+\|I\|}}$ . Combining these two statements gives

$$\frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \equiv w_{|J|}^{\|K\| - |J|} \pmod{p^{1+|K|}}.$$

Hence the defining equation for  $w_K$  becomes

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|} w_K + \sum_{\substack{I, J=K \\ |I|>0}} p^{|J|} w_{|J|}^{p^{\|K\| - |J|}} \pmod{p^{1+|K|}}.$$

Let  $n = \|K\| - |K|$ . Substituting  $x_t^{p^n}$  for  $x_t$  in 4.3.5 gives

$$\sum x_t^{p^{\|K\|}} = p^{\|K\|} w_{|K|}(x_t^{p^n}) + \sum_{0 \leq j < |K|} p^j w_j^{p^{|K| - j}}(x_t^{p^n}).$$

Since  $w_j(x_t^{p^n}) \equiv w_j^{p^n} \pmod{p}$ ,

$$w_j^{p^{|K| - j}}(x_t^{p^n}) \equiv w_j^{p^{n+|K| - j}} \pmod{p^{1+|K| - j}},$$

so we get

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|} w_{|K|}(x_t^{p^n}) + \sum_{0 \leq j < |K|} p^j w_j^{p^{\|K\| - j}} \pmod{p^{1+|K|}}.$$

Comparing this with the defining equation above gives

$$w_K \equiv w_{|K|}(x_t^{p^n}) \equiv w_{|K|}^{p^{\|K\| - |K|}} \pmod{p}$$

as claimed.  $\square$

#### 4. Beginning Calculations with the Adams–Novikov Spectral Sequence

In this section we introduce the main object of interest in this book, the Adams–Novikov spectral sequence, i.e., the  $BP_*$ -Adams spectral sequence (2.2.4). There is a different  $BP_*$ -theory and hence a different Adams–Novikov spectral sequence for each prime  $p$ . One could consider the  $MU_*$ -Adams spectral sequence (as Novikov [1] did originally) and capture all primes at once, but there is no apparent advantage in doing so. Stable homotopy theory is a very local (in the arithmetic sense) subject. Even though the structure formulas for  $BP_*(BP)$  are more complicated than those of  $MU_*(MU)$  (both are given in Section 1) the former are easier to work with once one gets used to them. (Admittedly this adjustment has been difficult. We hope this book, in particular the results of Section 3, will make it easier.)

The Adams–Novikov spectral sequence was first constructed by Novikov [1] and the first systematic calculations at the primes 2 and 3 were done by Zahler [1]. In this section we will calculate the  $E_2$ -term for  $t - s \leq 25$  at  $p = 2$  and for  $t - s \leq (p^2 + p)q$  for  $p > 2$ , where  $q = 2p - 2$ . In each case we will compute all the differentials and extensions and thereby find  $\pi(S^0)$  through the indicated range. At  $p = 2$  this will be done by purely algebraic methods based on a comparison of the Adams–Novikov spectral sequence and Adams spectral sequence  $E_2$ -terms. At odd primes we will see that the Adams spectral sequence  $E_2$ -term sheds no light on the Adams–Novikov spectral sequence and one must compute differentials by other means. Fortunately, there is only one differential in this range and it is given by Toda [2, 3]. The more extensive calculations of later chapters will show that in a much larger range all nontrivial differentials follow formally from the first one.

In Section 2.2 we developed the machinery necessary to set up the Adams–Novikov spectral sequence and we have

4.4.1. ADAMS–NOVIKOV SPECTRAL SEQUENCE THEOREM (Novikov [1]). *For any spectrum  $X$  there is a natural spectral sequence  $E_*^{**}(X)$  with  $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$  such that*

- (a)  $E_2 = \text{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$  and
- (b) *if  $X$  is connective and  $p$ -local then  $E_\infty^{**}$  is the bigraded group associated with the following filtration of  $\pi_*(X)$ : a map  $f: S^n \rightarrow X$  has filtration  $\geq s$  if it can be factored with  $s$  maps each of which becomes trivial after smashing the target with  $BP$ .* □

The fact that  $BP_*(BP)$ , unlike the Steenrod algebra, is concentrated in dimensions divisible by  $q = 2p - 2$  has the following consequence.

4.4.2. PROPOSITION: SPARSENESS. Suppose  $BP_*(X)$  is concentrated in dimensions divisible by  $q = 2p - 2$  (e.g.,  $X = S^0$ ). Then in the Adams–Novikov spectral sequence for  $X$ ,  $E_r^{s,t} = 0$  for all  $r$  and  $s$  except when  $t$  is divisible by  $q$ . Consequently  $d_r$  is nontrivial only if  $r \equiv 1 \pmod{q}$  and  $E_{m+2}^{**} = E_{mq+q+1}^{**}$  for all  $m \geq 0$ . □

For  $p = 2$  this leads to the “checkerboard phenomenon”:  $E_r^{s,t} = 0$  if  $t - s$  and  $s$  do not have the same parity.

To compare the Adams spectral sequence and Adams–Novikov spectral sequence we will construct two trigraded spectral sequences converging to the Adams spectral sequence and Adams–Novikov spectral sequence  $E_2$ -terms. The former is a

Cartan–Eilenberg spectral sequence (A1.3.15) for a certain Hopf algebra extension involving the Steenrod algebra, while the latter arises from a filtration of  $BP_*(BP)$  (A1.3.9). The point is that up to reindexing these two spectral sequences have the same  $E_2$ -term. Moreover, at odd primes (but not at  $p = 2$ ) the former spectral sequence collapses, which means that the Adams spectral sequence  $E_2$ -term when suitably reindexed is a trigraded  $E_2$ -term of a spectral sequence converging to the Adams–Novikov spectral sequence  $E_2$ -term. It is reasonable to expect there to be a close relation between differentials in the trigraded filtration spectral sequence, which Miller [2] calls the “algebraic Novikov spectral sequence,” and the differentials in the Adams spectral sequence. Miller [4] has shown that many Adams  $d_2$ ’s can be accounted for in this way. At any rate this indicates that at odd primes the Adams spectral sequence  $E_2$ -term has less information than the Adams–Novikov spectral sequence  $E_2$ -term.

To be more specific, recall (3.1.1) that the dual Steenrod algebra  $A_*$  as an algebra is

$$A_* = \begin{cases} P(\xi_1, \xi_2, \dots) & \text{with } \dim \xi_i = 2^i - 1 \text{ for } p = 2 \\ E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots) & \text{with } \dim \tau_i = 2p^i - 1 \text{ and} \\ & \dim \xi_i = 2p^i - 2 \text{ for } p > 2. \end{cases}$$

Let  $P_* \subset A_*$  be  $P(\xi_1^2, \xi_2^2, \dots)$  for  $p = 2$  and  $P(\xi_1, \xi_2, \dots)$  for  $p > 2$ , and let  $E_* = A_* \otimes_{P_*} \mathbf{Z}/(p)$ , i.e.  $E_* = E(\xi_1, \xi_2, \dots)$  for  $p = 2$  and  $E_* = E(\tau_0, \tau_1, \dots)$  for  $p > 2$ . Then we have

4.4.3. THEOREM. *With notation as above (a)*

$$\text{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1, \dots)$$

with  $a_i \in \text{Ext}^{1, 2p^i - 1}$  represented in the cobar complex (A1.2.11) by  $[\xi_i]$  for  $p = 2$  and  $[\tau_i]$  for  $p > 2$ ,

(b)  $P_* \rightarrow A_* \rightarrow E_*$  is an extension of Hopf algebras (A1.1.15) and there is a Cartan–Eilenberg spectral sequence (A1.3.15) converging to  $\text{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  with

$$E_2^{s_1, s_2, t} = \text{Ext}_{P_*}^{s_1}(\mathbf{Z}/(p), \text{Ext}_{E_*}^{s_2, t}(\mathbf{Z}/(p), \mathbf{Z}(p)))$$

and

$$d_r : E_r^{s_1, s_2, t} \rightarrow E_r^{s_1 + r, s_2 - r + 1, t},$$

(c) the  $P_*$ -coaction on  $\text{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  is given by

$$\psi(a_n) = \begin{cases} \sum_i \xi_{n-i}^{2^{i+1}} \otimes a_i & \text{for } p = 2 \\ \sum_i \xi_{n-i}^{p^i} \otimes a_i & \text{for } p > 2, \text{ and} \end{cases}$$

(d) for  $p > 2$  the Cartan–Eilenberg spectral sequence collapses from  $E_2$  with no nontrivial extensions.

PROOF. Everything is straightforward but (d). We can give  $A_*$  a second grading based on the number of  $\tau_i$ ’s which are preserved by both the product and the coproduct (they do not preserve it at  $p = 2$ ). This translates to a grading of  $\text{Ext}$  by the number of  $a_i$ ’s which must be respected by the differentials, so the spectral sequence collapses.  $\square$

For the algebraic Novikov spectral sequence, let  $I = (p, v_1, v_2, \dots) \subset BP_*$ . We filter  $BP_*(BP)$  by powers of  $I$  and study the resulting spectral sequence (A1.3.9).

4.4.4. ALGEBRAIC NOVIKOV SS THEOREM (Novikov [1], Miller [2]). There is a spectral sequence converging to  $\text{Ext}_{BP_*(BP)}(BP_*, BP_*)$  with

$$E_1^{s,m,t} = \text{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), I^m/I^{m+1})$$

and  $d_r: E_r^{s,m,t} \rightarrow E_r^{s+1,r+m,t}$ . The  $E_1^{***}$  of this spectral sequence coincides with the  $E_2^{***}$  of 4.4.3.

PROOF. A1.3.9 gives a spectral sequence with

$$E_1 = \text{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*).$$

Now we have  $BP_*(BP)/I = E_0BP_*(BP) \otimes_{E_0BP_*} \mathbf{Z}/(p) = P_*$ . We apply the change-of-rings isomorphism A1.3.12 to the Hopf algebroid map  $(E_0BP_*, E_0BP_*(BP)) \rightarrow (\mathbf{Z}/(p), P_*)$  and get

$$\begin{aligned} \text{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*) &= \text{Ext}_{E_0BP_*(BP)}(E_0BP_*, (E_0BP_*(BP) \otimes_{E_0BP_*} \mathbf{Z}/(p)) \square_{P_*} E_0BP_*) \\ &= \text{Ext}_{E_0BP_*(BP)}(E_0BP_*, P_* \square_{P_*} E_0BP_*) \\ &= \text{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*). \end{aligned}$$

The second statement follows from the fact that  $E_0BP_* = \text{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ .  $\square$

In order to use this spectral sequence we need to know its  $E_1$ -term. For  $p > 2$ , 4.4.3(d) implies that it is the cohomology of the Steenrod algebra, i.e., the classical Adams  $E_2$ -term suitably reindexed. This has been calculated in various ranges by May [1], and Liulevicius [2], but we will compute it here from scratch. Theorem 4.4.3(d) fails for  $p = 2$  so we need another method, outlined in Miller [2] and used extensively by Aubry [1].

We start with  $\text{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ . For  $p = 2$  we have  $\text{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \text{Ext}_{P_*}^{s,2t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ , so the latter is known if we know the former through half the range of dimensions being considered. For  $p > 2$  we will make the necessary calculation below.

Then we compute  $\text{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n)$ , by downward induction on  $n$ . To start the induction, observe that through any given finite range of dimensions  $BP_*/I_n \simeq \mathbf{Z}/(p)$  for large enough  $n$ . For the inductive step we use the short exact sequence

$$0 \rightarrow \Sigma^{\dim v_n} E_0BP_*/I_n \rightarrow E_0BP_*/I_n \rightarrow E_0BP_*/I_{n+1} \rightarrow 0,$$

which leads to a Bockstein spectral sequence of the form

$$(4.4.5) \quad P(a_n) \otimes \text{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_{n+1}) \Rightarrow \text{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n).$$

The method we will use in this section differs only slightly from the above. We will compute the groups  $\text{Ext}_{BP_*(BP)}(BP_*, BP_*/I_n)$  by downward induction on  $n$ ; these will be abbreviated by  $\text{Ext}(BP_*/I_n)$ . To start the induction we note that  $\text{Ext}^{s,t}(BP_*/I_n) = \text{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  for  $t < 2(p^n - 1)$ . For the inductive step we analyze the long exact sequence of Ext groups induced by the short exact sequence

$$(4.4.6) \quad 0 \rightarrow \Sigma^{\dim v_n} BP_*/I_n \rightarrow BP_*/I_n \rightarrow BP_*/I_{n+1} \rightarrow 0,$$



either directly or via a Bockstein spectral sequence similar to 4.4.5. The long exact sequence and Bockstein spectral sequence are related as follows. The connecting homomorphism in the former has the form

$$\delta_n: \text{Ext}^s(BP_*/I_{n+1}) \rightarrow \text{Ext}^{s+1}(\Sigma^{2p^n-2}BP_*/I_n).$$

The target is a module over  $\text{Ext}^0(BP_*/I_n)$  which is  $\mathbf{Z}/(p)[v_n]$  for  $n > 0$  and  $\mathbf{Z}_{(p)}$  for  $n = 0$  by 4.3.2. Assume for simplicity that  $n > 0$ . For each  $x \in \text{Ext}(BP_*/I_{n+1})$  there is a maximal  $k$  such that  $\delta_n(x) = v_n^k y$ , i.e., such that  $y \in \text{Ext}(BP_*/I_n)$  is not divisible by  $v_n$ . (This  $y$  is not unique but is only determined modulo elements annihilated by  $v_n^k$ .) Let  $\bar{y} \in \text{Ext}(BP_*/I_{n+1})$  denote the image of  $y$  under the reduction map  $BP_*/I_n \rightarrow BP_*/I_{n+1}$ . Then in the Bockstein spectral sequence there is a differential  $d_{1+k}(x) = a_n^{1+k}\bar{y}$ .

Now we will start the process by computing  $\text{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  for  $p > 2$  and  $t < (p^2 + p + 1)q$ . In this range we have  $P_* = P(\xi_1, \xi_2)$ . We will apply the Cartan–Eilenberg spectral sequence (A1.3.15) to the Hopf algebra extension

$$(4.4.7) \quad P(\xi_1) \rightarrow P(\xi_1, \xi_2) \rightarrow P(\xi_2).$$

The  $E_2$ -term is  $\text{Ext}_{P(\xi_1)}(\mathbf{Z}/(p), \text{Ext}_{P(\xi_2)}(\mathbf{Z}/(p), \mathbf{Z}/(p)))$ . The extension is cocentral (A1.1.15) so we have

$$E_2 = \text{Ext}_{P(\xi_1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \otimes \text{Ext}_{P(\xi_2)}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

By a routine calculation this is in our range of dimensions

$$E(h_{10}, h_{11}, h_{12}, h_{20}, h_{21}) \otimes P(b_{10}, b_{11}, b_{20})$$

with

$$h_{i,j} \in \text{Ext}_{P(\xi_i)}^{1,2p^j(p^i-1)} \quad \text{and} \quad b_{i,j} \in \text{Ext}_{P(\xi_i)}^{2,2p^{j+1}(p^i-1)}.$$

The differentials are (up to sign)  $d_2(h_{2,j}) = h_{1,j}h_{1,j+1}$  and  $d_3(b_{20}) = h_{12}b_{10} - h_{11}b_{11}$  [compare 4.3.22(a)]. The result is

4.4.8. THEOREM. For  $p > 2$  and  $t < (p^2 + p + 1)q$ ,  $\text{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  is a free module over  $P(b_{10})$  on the following 10 generators: 1,  $h_{10}$ ,  $h_{11}$ ,  $g_0 = \langle h_{11}, h_{10}, h_{10} \rangle$ ,  $k_0 = \langle h_{11}, h_{11}, h_{10} \rangle$ ,  $h_{10}k_0 = \pm h_{11}g_0$ ,  $h_{12}$ ,  $h_{10}h_{12}$ ,  $b_{11}$ , and  $h_{10}b_{11}$ . There is a multiplicative relation  $h_{11}b_{11} = h_{12}b_{10}$  and (for  $p = 3$ )  $h_{11}k_0 = \pm h_{10}b_{11}$ .  $\square$

The extra relation for  $p = 3$  follows easily from A1.4.6. For  $p > 3$  there is a corresponding Massey product relation  $\langle k_0, h_{11}, \dots, h_{11} \rangle = h_{10}b_{11}$  up to a nonzero scalar, where there are  $p - 2$  factors  $h_{11}$ .

The alert reader may observe that the restriction  $t < (p^2 + p + 1)q$  is too severe to give us  $\text{Ext}^{s,t}$  for  $t - s < (p^2 + p)q$  because there are elements in this range with  $s > q$ , e.g.,  $b_{10}^p$ . However, one sees easily that in a larger range all elements with  $s > q$  are divisible by  $b_{10}$  and this division gets us back into the range  $t < (p^2 + p + 1)q$ . One could make this more precise, derive some vanishing lines, and prove the following result.

4.4.9. THEOREM. Let  $p > 2$ .

(a)  $\text{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = 0$  for  $t - s < f(s)$  where

$$f(s) = \begin{cases} (p^2 - p - 1)s & \text{for } s \text{ even} \\ 2p - 3 + (p^2 - p - 1)(s - 1) & \text{for } s \text{ odd.} \end{cases}$$

(b) Let  $R_* = P_*/(\xi_1, \xi_2)$ . Then  $\text{Ext}_{R_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}(p)) = 0$  for  $t - s < g(s)$  where

$$g(s) = \begin{cases} (p^4 - p - 1)s & \text{for } s \text{ even} \\ 2p^3 - 3 + (p^4 - p - 1)(s - 1) & \text{for } s \text{ odd.} \end{cases}$$

(c) The map  $P(\xi_1, \xi_2) \rightarrow P_*$  induces an epimorphism in  $\text{Ext}^{s,t}$  for  $(t - s) < h(s)$  and an isomorphism for  $(t - s) < h(s - 1) - 1$ , where

$$\begin{aligned} h(s) &= 2p^3 - 3 + f(s - 1) \\ &= \begin{cases} 2p^3 - 3 + (p^2 - p - 1)(s - 1) & \text{for } s \text{ odd} \\ 2p^3 + 2p - 6 + (p^2 - p - 1)(s - 2) & \text{for } s \text{ even.} \end{cases} \quad \square \end{aligned}$$

This result is far more than we need, and we leave the details to the interested reader.

Now we start feeding in the generators  $v_n$  inductively. In our range 4.4.8 gives us  $\text{Ext}(BP_*/I_3)$ . Each of the specified generators is easily seen to come from a cocycle in the cobar complex  $C(BP_*/I_2)$  so we have

$$\text{Ext}(BP_*/I_2) = \text{Ext}(BP_*/I_3) \otimes P(v_2),$$

i.e., the Bockstein spectral sequence collapses in our range.

The passage to  $\text{Ext}(BP_*/I_1)$  is far more complicated. The following formulas in  $C(BP_*/I_1)$  are relevant.

$$(4.4.10) \quad (\text{a}) \quad d(v_2) = v_1 t_1^p - v_1^p t_1$$

and

$$(\text{b}) \quad d(t_2) = -t_1 |t_1^p - v_1 b_{10}.$$

These follow immediately from 4.3.20 and 4.3.15. From 4.4.10(a) we get

$$(4.4.11) \quad (\text{a}) \quad \delta_1(v_2^i) \equiv i v_2^{i-1} h_{11} \pmod{(v_1)}$$

and

$$(\text{b}) \quad \delta_1(v_2^p) \equiv v_1^{p-1} h_{12} \pmod{(v_1^p)}.$$

Next we look at elements in  $\text{Ext}^1(BP_*/I_2)$ . Clearly,  $h_{10}$ ,  $h_{11}$ , and  $h_{12}$  are in  $\ker \delta_i$  as are  $v_2^i h_{11}$  for  $i < p - 1$  by the above calculation. This leaves  $v_2^i h_{10}$  for  $1 \leq i \leq p - 1$  and  $v_2^{p-1} h_{11}$ . For the former 4.4.10 gives

$$\begin{aligned} d(v_2^i t_1 + i v_1 v_2^{i-1} (t_1^{1+p} - t_2)) &\equiv i v_1^2 v_2^{i-1} b_{10} \\ &+ \binom{i}{2} v_1^2 v_2^{i-1} (t_1^{2p} |t_1 - 2t_1^p |t_2 + 2t_1^p |t_1^{1+p}) \pmod{(v_1^3)}. \end{aligned}$$

The expression in the second term is a multiple of  $k_0$ , so we have

$$(4.4.12) \quad \delta(v_2^i h_{10}) \equiv i v_1 v_2^{i-1} b_{10} \pm \binom{i}{2} v_1 v_2^{i-2} k_0 \pmod{(v_1^2)}.$$

To deal with  $v_2^{p-1} h_{11}$  we use 4.4.10(a) to show

$$d\left(\sum_{0 < i < p} \frac{1}{p} \binom{p}{i} v_2^{p-i} v_1^{i-1} t_1^{pi}\right) \equiv \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} v_1^{p-1} t_1^{pi} |t_1^{p^2-pi} \pmod{(v_1^p)}$$

so

$$(4.4.13) \quad \delta_1(v_2^{p-1}h_{11}) = \pm v_1^{p-2}b_{11}.$$

This is a special case of 4.3.22(b).

Now we move on to the elements in  $\text{Ext}^2(BP_*/I_2)$ . They are  $h_{10}h_{12}$ ,  $b_{11}$ ,  $v_2^i b_{10}$ ,  $v_2^i g_0$ , and  $v_2^i k_0$  for suitable  $i$ . The first two are clearly in  $\ker \delta_1$ . Equation 4.4.12 eliminates the need to consider  $v_2^i b_{10}$  for  $i < p-1$ , so that leaves  $v_2^{p-1} b_{10}$ ,  $v_2^i g_0$ , and  $v_2^i k_0$ . Routine calculation with 4.4.10 gives

$$(a) \quad \delta_1(v_2^i g_0) \equiv \pm(v_2^i h_{10} b_{10} \pm i v_2^{i-1} h_{10} k_0) \pmod{(v_1^2)}$$

and

$$(b) \quad \delta_1(v_2^i k_0) \equiv \pm v_2^i h_{11} b_{10} \pmod{(v_1^2)}.$$

We have to handle  $v_2^{p-1} b_{10}$  more indirectly.

4.4.14. LEMMA.  $\delta_1(v_2^{p-1} b_{10} \pm \frac{1}{2} v_2^{p-2} k_0) = c v_1^{p-3} h_{10} b_{11}$  for some nonzero  $c \in \mathbf{Z}/(p)$ .

PROOF. By 4.4.13,  $v_1^{p-1} b_{11} = 0$  in  $\text{Ext}(BP_*/I_1)$ , so  $v_1^{p-1} h_{10} b_{11} = 0$  and  $v_1^i h_{10} b_{11} = \delta_1(x)$  for some  $i < p-1$  and some  $x \in \text{Ext}^2(BP_*/I_2)$ . The only remaining  $x$  is the indicated one.  $\square$

From 4.4.14 we get  $\delta_1(v_2^{p-1} h_{10} b_{10} \pm \frac{1}{2} v_2^{p-2} h_{10} k_0) \equiv 0 \pmod{(v_1^{p-2})}$ . All other elements in  $\text{Ext}^s(BP_*/I_2)$  for  $s \geq 3$  are divisible by  $h_{10}$  or  $b_{10}$  and they can all be accounted for in such a way that the above element, which we denote by  $\bar{\phi}$ , must be in  $\ker \delta_1$ . Hence  $\delta_1$  is completely determined in our range.

Equivalently, we have computed all of the differentials in the Bockstein spectral sequence. However, there are some multiplicative extensions which still need to be worked out.

4.4.15. THEOREM. For  $p > 2$ ,  $\text{Ext}(BP_*/I_1) = P(v_1) \otimes E(h_{10}) \oplus M$ , where  $M$  is a free module over  $P(b_{10})$  on the following generators:

$$\beta_i = \delta_1(v_2^i), \quad h_{10}\beta_i, \quad \bar{\beta}_i = v_1^{-1}\delta_1(v_2^i h_{10}) \quad (\text{e.g., } \bar{\beta}_1 = \pm b_{10}),$$

$$\text{and } h_{10}\bar{\beta}_i \quad \text{for } 1 \leq i \leq p-1;$$

$$\beta_{p/i} = v_1^{1-i}\delta_1(v_2^p) \quad \text{and } h_{10}\beta_{p/i} \quad \text{for } 1 \leq i \leq p;$$

$$\bar{\beta}_{p/i} = v_1^{2-i}\delta_1(v_2^{p-1} h_{11}) \quad \text{for } 2 \leq i \leq p;$$

$$h_{10}\bar{\beta}_{p/i} \quad \text{for } 3 \leq i \leq p; \quad \bar{\phi} \quad \text{and } \beta_1\bar{\beta}_{p/p}.$$

Here  $\delta_1$  is the connecting homomorphism for the short exact sequence

$$0 \rightarrow \Sigma^q BP_*/I_1 \xrightarrow{v_1} BP_*/I_1 \rightarrow BP_*/I_2 \rightarrow 0.$$

Moreover,

$$h_{10}\beta_i = v_1\bar{\beta}_i, \quad v_1\beta_i = 0, \quad v_1^p\beta_{p/p} = 0,$$

$$v_1^{p-1}\bar{\beta}_{p/p} = 0, \quad \text{and } v_1^{p-2}h_{10}\bar{\beta}_{p/p} = 0.$$

(This description of the multiplicative structure is not complete.)

PROOF. The additive structure of this Ext follows from the above calculations. The relations follow from the way the elements are defined.  $\square$

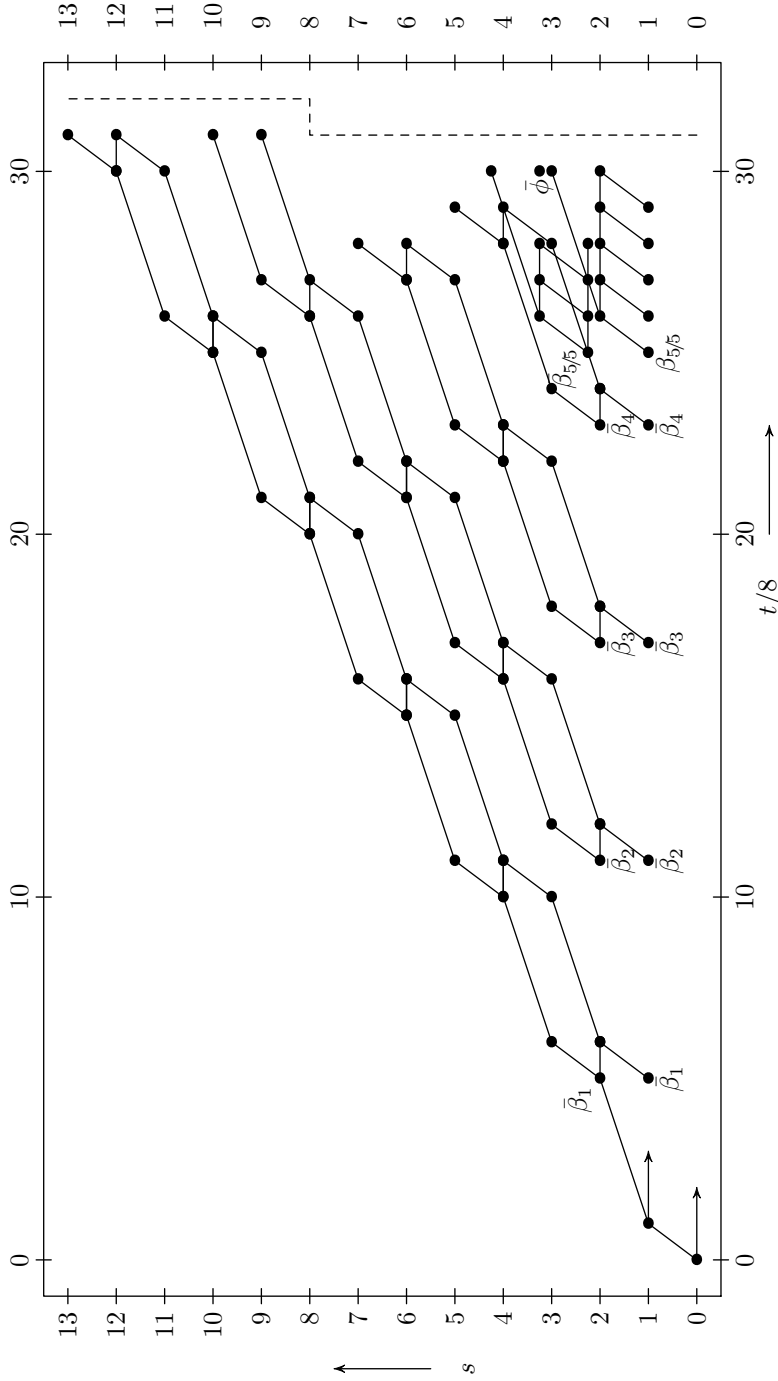


FIGURE 4.4.16.  $\text{Ext}_{BP_*}^{s,t}(BP_*/I_1)$  for  $p = 5$  and  $t - s \leq 240$ .

FIGURE 4.4.16 illustrates this result for  $p = 5$ . Horizontal lines indicate multiplication by  $v_1$ , and an arrow pointing to the right indicates that the element is free over  $P(v_1)$ . A diagonal line which increases  $s$  and  $t/q$  by one indicates multiplication by  $h_{10}$  and one which increases  $t/q$  by 4 indicates the Massey product operation  $\langle -, h_{10}, h_{10}, h_{10}, h_{10} \rangle$ . Thus two successive diagonal lines indicate multiplication by  $b_{10} = \pm \langle h_{10}, h_{10}, h_{10}, h_{10}, h_{10} \rangle$ . The broken line on the right indicates the limit of our calculation.

Now we have to consider the long exact sequence or Bockstein spectral sequence associated with

$$0 \rightarrow BP_* \xrightarrow{p} BP_* \rightarrow BP_*/I_1 \rightarrow 0.$$

First we compute  $\delta_0(v_1^i)$ . Since  $d(v_1) = pt_1$  in  $C(BP_*)$  we have  $d(v_1^i) \equiv ipv_1^{i-1}t_1 \pmod{(ip^2)}$ , so

$$(4.4.17) \quad \delta_0(v_1^i) \equiv iv_1^{i-1}h_{10} \pmod{(ip^2)}.$$

Moving on to  $\text{Ext}^1(BP_*/I_1)$  we need to compute  $\delta_0$  on  $\beta_i$  and  $\beta_{p/i}$ . The former can be handled most easily as follows.  $\delta_0(\bar{\beta}_i) = 0$  because there is no element in the appropriate grading in  $\text{Ext}^3$ .  $\delta_0$  is a derivation mod  $(p)$  so  $\delta_0(v_1\bar{\beta}_i) = h_{10}\bar{\beta}_i$ . Since  $v_1\bar{\beta}_i = h_{10}\beta_i$  we have  $h_{10}\bar{\beta}_i = \delta_0(h_{10}\beta_i) = h_{10}\delta_0(\beta_i)$  so

$$(4.4.18) \quad \delta_0(\beta_i) = \bar{\beta}_i.$$

Now  $\beta_{p/p} = h_{12} - v_1^{p^2-p}h_{11}$  and  $v_1^{p^2-p}h_{11}$  is cohomologous to  $v_1^{p^2-1}h_{10}$ , which by 4.4.16 is in  $\ker \delta_0$ . Hence

$$(4.4.19) \quad \delta_0(\beta_{p/p}) = \delta_0(h_{12}) = b_{11} = \pm \bar{\beta}_{p/p}.$$

It follows that

$$\delta_0(\beta_{p/p-i}) = \delta_0(v_1^i\beta_{p/p}) = iv_1^{i-1}h_{10}\beta_{p/p} \pm v_1^i\bar{\beta}_{p/p}.$$

This accounts for all elements in sight but  $\delta_0(h_{10}\beta_{p/1})$  which vanishes mod  $(p)$ . We will show that it is a unit multiple of  $p\Phi$  below in 5.1.24.

Putting all this together gives

4.4.20. THEOREM. For  $p > 2$  and  $t - s \leq (p^2 + p)q$ ,  $\text{Ext}(BP_*)$  is as follows.  $\text{Ext}^0 = \mathbf{Z}_{(p)}$  concentrated in dimension zero.  $\text{Ext}^{1,qi} = \mathbf{Z}_{(p)}/(pi)$  generated by  $\bar{\alpha}_i = i^{-1}\delta_0(v_1^i)$ , where  $\alpha_1 = h_{10}$ . For  $s \geq 2$   $\text{Ext}^s$  generated by all  $b_{10}^j x$ , where  $x$  is one of the following:  $\beta_i = \delta_0(\beta_i)$  (where  $\beta_1 = \pm b_0$ ) and  $\alpha_1\beta_i$  for  $1 \leq i \leq p - 1$ ;  $\beta_{p/p-i} = \delta_0(\beta_{p/p-i})$  for  $0 \leq i \leq p - 1$ ;  $\alpha_1\beta_{p/p-i}$  for  $0 \leq i \leq p - 3$ ; and  $\phi = p^{-1}\delta_0(h_{10}\beta_{p/1})$  which has order  $p^2$ .  $\phi$  is a unit multiple of  $\langle \beta_{p/2}, \alpha_1, \alpha_1 \rangle$  and  $p\phi$  is a unit multiple  $\alpha_1\beta_{p/1}$ . Here  $\beta_{i/j}$  denotes the image under  $\delta_0$  of the corresponding element in  $\text{Ext}(BP_*/I_1)$ .  $\square$

For  $p = 5$  this is illustrated in FIG. 4.4.21, with notation similar to that of Fig. 4.4.16. It also shows differentials (long arrows originating at  $\beta_{5/5}$  and  $\beta_1\beta_{5/5}$ ), which we discuss now. By sparseness (4.4.2)  $E_2 = E_{2p-1}$  and  $d_{2p-1}: E_{2p-1}^{s,t} \rightarrow E_{2p-1}^{s+2p-1,t-2p+2}$ . It is clear that in our range of dimensions  $E_{2p} = E_\infty$  because any higher (than  $d_{2p-1}$ ) differential would have a target whose filtration (the  $s$ -coordinate) would be too high. Naively, the first possible differential is  $d_{2p-1}(\alpha_{p^2-1}) = c\beta_i^p$ . However,  $d_{2p-1}$  respects multiplication by  $\alpha_1$  and  $\alpha_1\alpha_{p^2-1}$  so  $c\alpha_1\beta_1^p = 0$  and  $c = 0$ . Alternatively one can show (see 5.3.7) that each element in  $\text{Ext}^1$  is a permanent cycle.

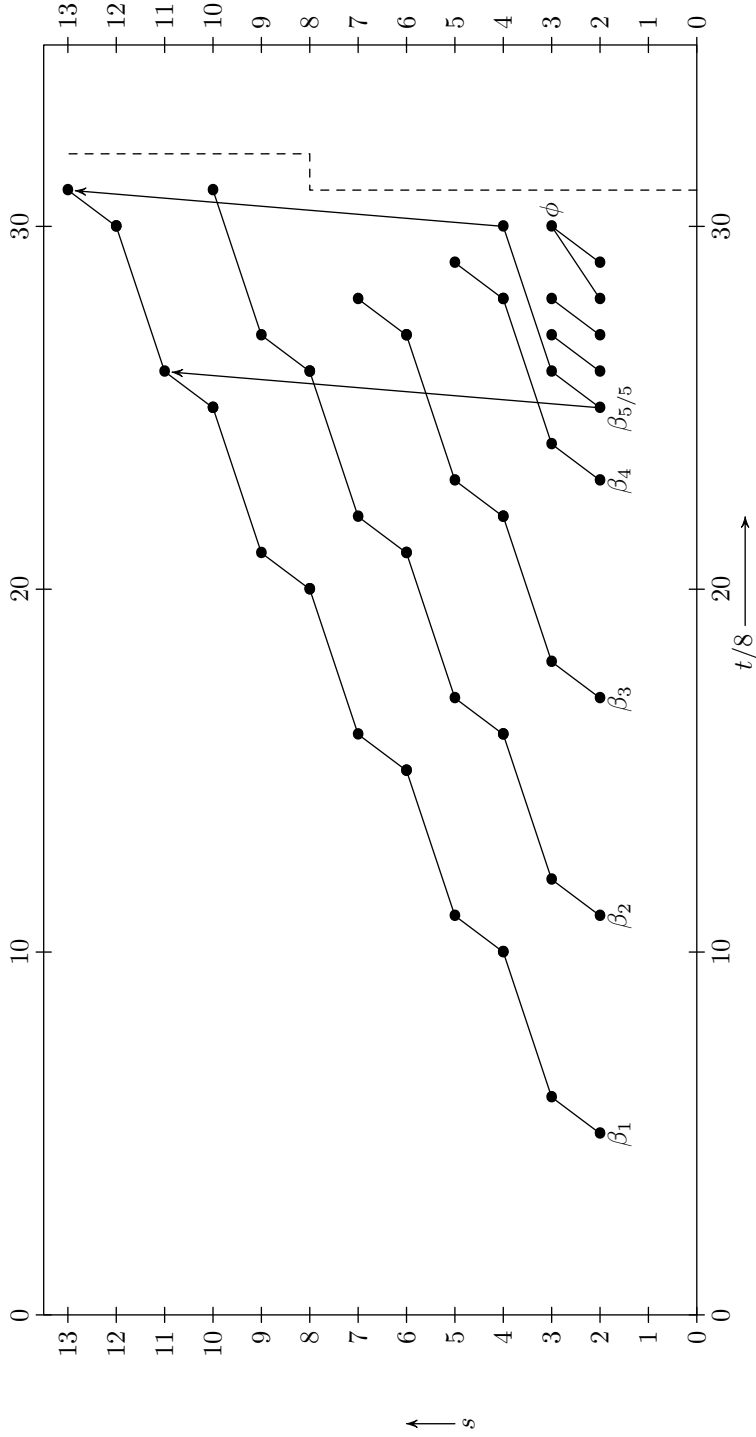


FIGURE 4.4.21. The Adams-Novikov spectral sequence for  $p = 5$ ,  $t - s \leq 240$ , and  $s \geq 2$ .

4.4.22. THEOREM (Toda [2, 3]).  $d_{2p-1}(\beta_{p/p}) = a\alpha_1\beta_1^p$  for some nonzero  $a \in \mathbf{Z}/(p)$ .  $\square$

Toda shows that any  $x \in \pi_*(S)$  of order  $p$  must satisfy  $\alpha_1 x^p = 0$ . For  $x = \beta_1$  this shows  $\alpha_1\beta_1^p = 0$  in homotopy. Since it is nonzero in  $E_2$  it must be killed by a differential and our calculation shows that  $\beta_{p/p}$  is the only possible source for it. We do not know how to compute the coefficient  $a$ , but its value seems to be of little consequence.

Theorem 4.4.22 implies that  $d_{2p-1}(\beta_1\beta_{p/p}) = \alpha_1\beta_1^{p+1}$ . Inspection of 4.4.20 or 4.4.21 shows that there are no other nontrivial differentials.

Notice that the element  $\alpha_1\beta_{p/p}$  survives to  $E_\infty$  even though  $\beta_{p/p}$  does not. Hence the corresponding homotopy element, usually denoted by  $\varepsilon'$ , is indecomposable. It follows easily from the definition of Massey products (A1.4.1) that  $\langle \alpha_1, \alpha_1, \beta_1^p \rangle$  is defined in  $E_{2p}$ , has trivial indeterminacy, and contains a unit multiple of  $\alpha_1\beta_{p/p}$ . It follows from 7.5.4 that  $\varepsilon'$  is the corresponding Toda bracket. Using A1.4.6 we have

$$\langle \alpha_1, \dots, \alpha_1, \varepsilon' \rangle = \langle \alpha_1, \dots, \alpha_1 \rangle \beta_1^p = \beta_1^{p+1}$$

with  $p - 2$   $\alpha_1$ 's on the left and  $p$   $\alpha_1$ 's on the right.

Looking ahead we can see this phenomenon generalize as follows. For  $1 \leq i \leq p - 1$  we have  $d_{2p-1}(\beta_{p/p}^i) = ia\alpha_1\beta_{p/p}^{i-1}$ . For  $i \leq p - 2$  this leads to  $\langle \alpha_1, \dots, \alpha_1\beta_1^{ip} \rangle$  [with  $(i + 1)$   $\alpha_1$ 's] being a unit multiple of  $\varepsilon^{(i)} = \alpha_1\beta_{p/p}^i$ , and  $\langle \alpha_1, \alpha_1, \dots, \alpha_1\varepsilon^{(i)} \rangle$  [with  $(p - i - 1)$   $\alpha_1$ 's] is a unit multiple of  $\beta_1^{1+ip}$ . In particular,  $\alpha_1\varepsilon^{(p-2)}$  is a unit multiple of  $\beta_1^{1+(p-2)p}$ . Since  $\alpha_1\beta_1^p = 0$  (4.4.22),  $\beta_1^{p^2-p+1} = 0$  since it is a unit multiple of  $\alpha_1\beta_1^p\varepsilon^{(p-2)}$ . However, in the  $E_2$ -term all powers of  $\beta_1$  are nonzero (Section 6.4), so  $\beta_1^{p^2-p+1}$  must be killed by a differential, more precisely by  $d_{(p-1)q+1}(\alpha_1\beta_{p/p}^{p-1})$ .

Now we will make an analogous calculation for  $p = 2$ . The first three steps are shown FIG. 4.4.23. In (a) we have  $\text{Ext}_{P_*}(\mathbf{Z}/(2), \mathbf{Z}(2))$ , which is  $\text{Ext}(BP_*/I_4)$  for  $t - s \leq 29$ . Since differentials in the Bockstein spectral sequences and the Adams–Novikov spectral sequence all lower  $t - s$  by 1, we lose a dimension with each spectral sequence. In (a) we give elements the same names they have in  $\text{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}(2))$ . Hence we have  $c_0 = \langle h_{11}, h_{12}^2, h_{11} \rangle$  and  $Px = \langle x, h_{10}^4, h_{12} \rangle$ . Diagonal lines indicate multiplication by  $h_{10}$ ,  $h_{11}$ , and  $h_{12}$ . The arrow pointing up and to the right indicates that all powers of  $h_{10}$  are nontrivial.

The Bockstein spectral sequence for  $\text{Ext}(BP_*/I_3)$  collapses and the result is shown in Fig. 4.4.23(b). The next Bockstein spectral sequence has some differentials. Recall that  $\delta_2$  is the connecting homomorphism for the short exact sequence

$$0 \rightarrow \Sigma^6 BP_*/I_2 \xrightarrow{v_2} BP_*/I_2 \rightarrow BP_*/I_3 \rightarrow 0.$$

Since  $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \pmod{I_2}$  by 4.3.1 we have

$$(4.4.24) \quad \begin{aligned} (a) \quad & \delta_2(v_3h_{10}^i) = (h_{12} + v_2h_{10})h_{10}^i && \text{for } i \leq 2, \\ (b) \quad & \delta_2(v_3h_{10}^i) = v_2h_{10}^{i+1} && \text{for } i \geq 3, \\ (c) \quad & \delta_2(v_3h_{12}^i) = h_{12}^{i+1} && \text{for } i = 1, 2, \\ (d) \quad & \delta_2(v_2^2) = v_2h_{13} + v_2^3h_{11}. \end{aligned}$$

This accounts for all the nontrivial values of  $\delta_2$ . In  $\text{Ext}(BP_*/I_2)$  we denote  $\delta_2(v_3^i)$  by  $\gamma_i$  and  $v_2^{-1}\delta_2(v_3^2)$  by  $\gamma_{2/2}$ . The elements  $v_3h_{1,1}$ ,  $v_3h_{1,0}h_{1,2} \in \text{Ext}(BP_*/I_3)$

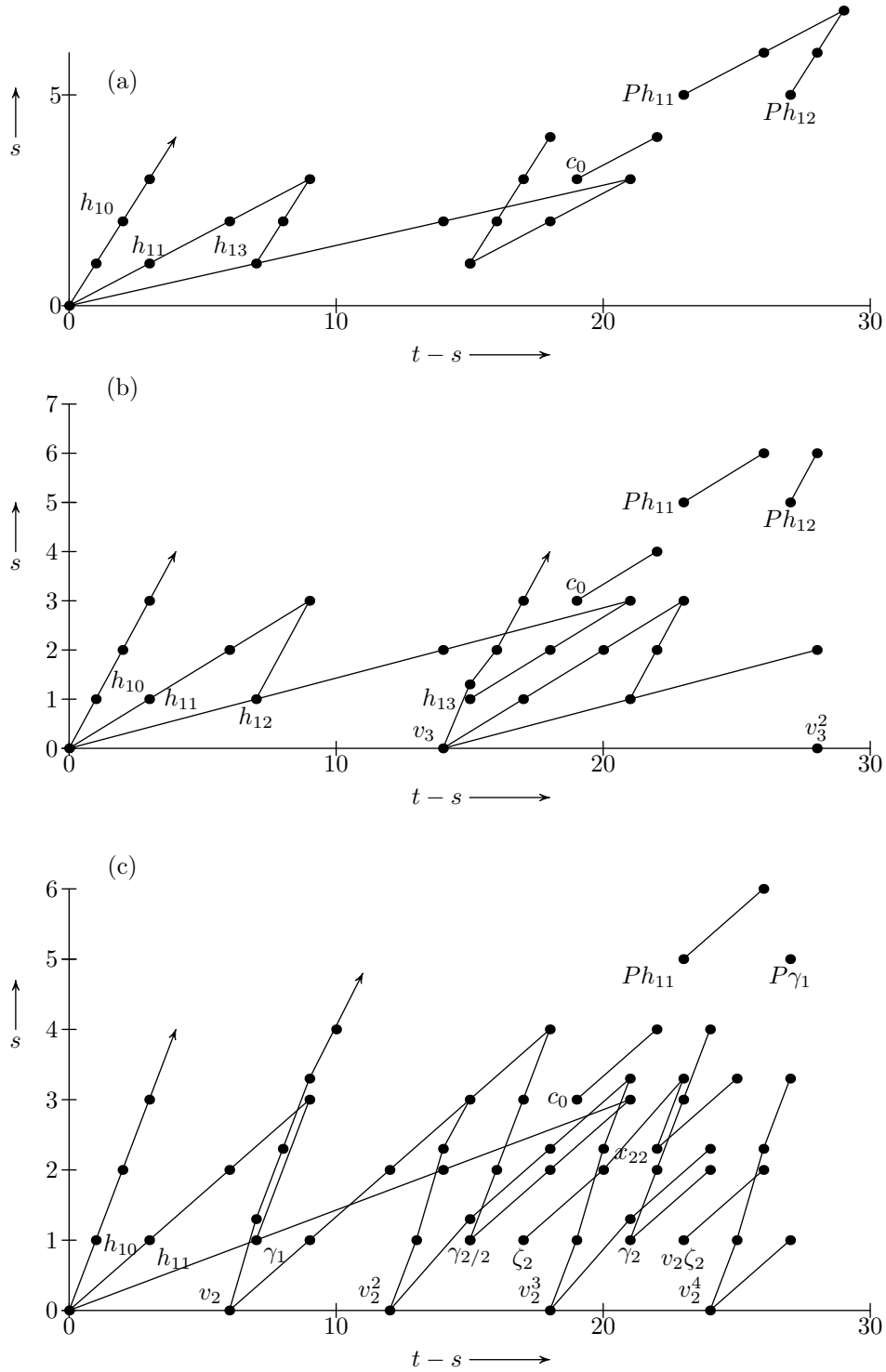


FIGURE 4.4.23. (a)  $\text{Ext}(BP_*/I_4)$  for  $p = 2$  and  $t - s < 29$ .  
 (b)  $\text{Ext}(BP_*/I_3)$  for  $t - s \leq 28$ . (c)  $\text{Ext}(BP_*/I_2)$  for  $t - s \leq 27$ .



are in  $\ker \delta_2$  and hence lift back to  $\text{Ext}(BP_*/I_2)$ , where we denote them by  $\zeta_2$  and  $x_{2,2}$ , respectively. They are represented in  $C(BP_*/I_2)$  by

$$(4.4.25) \quad \begin{aligned} (a) \quad \zeta_2 &= v_3 t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2 t_2 \quad \text{and} \\ (b) \quad x_{22} &= v_3 t_1 |t_1^4 + v_2(t_2 |t_2^2 + t_1 |t_1^2 t_2^2 + t_2 |t_1^6 + t_3 |t_1^2 + t_1^5 |t_1^4) \\ &\quad + v_2^2(t_2 |t_2 + t_1 |t_1^2 t_2 + t_1 t_2 |t_1^2). \end{aligned}$$

Now we pass to  $\text{Ext}(BP_*/I_1)$ . To compute  $\delta_1$  on  $\text{Ext}^0(BP_*/I_2)$  we have  $\eta_R(v_2) \equiv v_2 + v_1 t_1^2 + v_1^2 t_1 \pmod{I_1}$ , so

$$(4.4.26) \quad \begin{aligned} \delta_1(v_2) &\equiv h_{11} && \pmod{(v_1)}, \\ \delta_1(v_2^2) &\equiv v_1 h_{12} \equiv v_1(\gamma_1 + v_2 h_{10}) && \pmod{(v_1^2)}, \\ \delta_1(v_2^3) &\equiv v_2^2 h_{11} && \pmod{(v_1)}, \\ \delta_1(v_2^4) &\equiv v_1^3 h_{13} \equiv v_1^3(\gamma_{2/2} + v_2^2 h_{11}) && \pmod{(v_1^4)}, \\ \delta_1(v_2^5) &\equiv v_2^4 h_{11} && \pmod{(v_1)}. \end{aligned}$$

This means that in  $\text{Ext}^1(BP_*/I_2)$  it suffices to compute  $\delta_1$  on  $v_2 h_{10}$ ,  $\zeta_2$ ,  $v_2^3 h_{10}$ ,  $\gamma_2$ , and  $v_2 \zeta_2$ . We find  $\delta_1(v_2 h_{10})$  and the element pulls back to

$$(4.4.27) \quad x_7 = v_2 t_1 + v_1(t_2 + t_1^3).$$

For  $\zeta_2$  we compute in  $C(BP_*/I_1)$  and get

$$d(\zeta_2 + v_1 t_1^2 t_2^2) \equiv v_1(t_1^4 |t_1^4 + v_2^2 t_1 |t_1) \pmod{(v_1^2)}$$

so

$$(4.4.28) \quad \delta_1(\zeta_2) \equiv \gamma_1^2 \pmod{(v_1)}$$

For  $v_2^3 h_{10}$  we compute

$$d(v_2^3 t_1 + v_1 v_2^2(t_2 + t_1^3) + v_1^2 v_3 t_1) \equiv v_1^2 v_2^2 t_1 |t_1 \pmod{(v_1^3)}$$

so

$$(4.4.29) \quad \delta_1(v_2^3 h_{10}) \equiv v_1 v_2^2 h_{10}^2 \pmod{(v_1^2)}.$$

Similar calculations give

$$(4.4.30) \quad \delta_1(\gamma_2) \equiv h_{11} \gamma_{2/2} \pmod{(v_1^2)}$$

and

$$\delta_1(v_2 \zeta_2) \equiv h_{11} \zeta_2 + v_2^3 h_{10} \pmod{(v_1^2)}$$

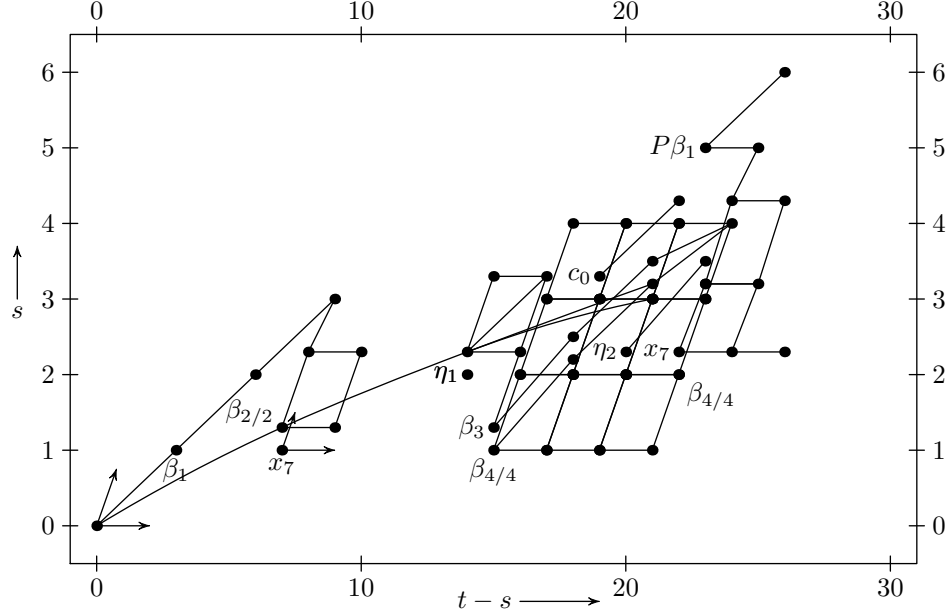
In  $\text{Ext}^2 BP_*/I_2$  it suffices to compute  $\delta_1$  on  $x_{22}$ . We will show

$$(4.4.31) \quad \delta_1(x_{22}) = c_0$$

using Massey products. Since  $x_{22}$  projects to  $v_3 h_{10} h_{12}$  we have  $x_{22} \in \langle v_2, \gamma_1^2, h_{10} \rangle$ , so  $\delta_1(x_{22}) \in \langle \delta_1(v_2), \gamma_1^2, h_{10} \rangle$  by [A1.4.11](#). This is  $\langle h_{11}, \gamma_1^2, h_{10} \rangle$ , which is easily seen to be  $c_0$ .

This completes our calculation of  $\delta_1$ . The resulting value of  $\text{Ext}(BP_*/I_1)$  is shown in [FIG. 4.4.32](#). The elements 1 and  $x_7$  are free over  $P(v_1, h_{10})$ . As usual we denote  $v_1^{1-j} \delta_1(v_2^j)$  by  $\beta_{i/j}$ .  $x_7$  is defined by [4.4.27](#).  $\eta_1$  and  $\eta_2$  (not to be confused with the  $\eta_j$  of Mahowald [[6](#)]) denote  $\delta_1(\zeta_2)$  and  $\delta_1(v_2 \zeta_2)$ .

We must comment on some of the relations indicated in [4.4.32](#).

FIGURE 4.4.32.  $\text{Ext}(BP_*/I_1)$  for  $p = 2$  and  $t - s \leq 26$ 

4.4.33. LEMMA. In  $\text{Ext}(BP_*/I_1)$  for  $p = 2$  the following relations hold.

- (a)  $h_{10}\beta_3 = v_1\beta_{2/2}^2$
- (b)  $\beta_{2/2}^3 = \beta_1^2\beta_{4/4} + h_{10}^2\beta_{4/2}$
- (c)  $h_{10}^3x_7\beta_{4/4} = v_1P\beta_1$ .

PROOF.

(a)  $\beta_{2/2} = h_{12} \pmod{v_1}$  so  $v_1\beta_{2/2}^2 = \delta_1(v_2^2h_{12})$  while  $h_{10}\beta_3 = \delta_1(v_2^3h_{10})$ . Since  $\eta_R(v_2v_3) \equiv v_2v_3 + v_2^2t_1^4 + v_2^3t_1 \pmod{I_2}$ , we have  $v_2^2h_{12} = v_2^3h_{10}$  in  $\text{Ext}(BP_*/I_2)$ .

(b)  $\beta_{2/2} = h_{12} + v_1^2h_{11} = h_{12} + v_1^3h_{10}$  so

$$\begin{aligned} \beta_{2/2}^3 &\equiv h_{12}^3 \equiv h_{11}^2h_{13} \equiv \beta_{4/4}h_{11}^2 \\ &\equiv \beta_{4/4}(\beta_1^2 + v_1^2h_{10}^2) \equiv \beta_{4/4}\beta_1^2 + \beta_{4/2}h_{10}^2 \pmod{v_1^3}. \end{aligned}$$

- (c)  $v_1P\beta_1 = v_1\langle\beta_1, h_{10}^4, \beta_{4/4}\rangle$   
 $= \langle v_1, \beta_1, h_{10}^4 \rangle \beta_{4/4}$  by A1.4.6  
 $= \langle v_1, \beta_1, h_{10} \rangle h_{10}^3 \beta_{4/4}$  by A1.4.6.

The last Massey product is easily seen to contain  $x_7$ .  $\square$

Now we pass to  $\text{Ext}(BP_*)$  by computing  $\delta_0$ , beginning with  $\text{Ext}^0(BP_*/I_1) = P(v_1)$ . By direct calculation we have

$$(4.4.34) \quad \begin{aligned} \delta_0(v_1^{2i+1}) &\equiv v_1^{2i}h_{10} \pmod{2} \\ \delta_0(v_1^2) &= 2\beta_1 \end{aligned}$$

To handle larger even powers of  $v_1$ , consider the formal expression  $u = v_1^2 - 4v_1^{-1}v_2$ . Using the formula (in terms of Hazewinkel's generators A2.2.1)

$$\eta_R(v_2) = v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3,$$

we find that  $d(u) = 8v_1^{-2}x_7$  in  $C(v_1^{-1}BP_*/(2^4))$ . It follows that

$$d(u^2 - 2^4v_1^{-2}v_2^2) \equiv 2^4(x_7 + \beta_{2/2}) \pmod{2^5}$$

and for  $i > 2$

$$d(u^i) \equiv 8iv_1^{2i-4}x_7 \pmod{16i}$$

so

$$(4.4.35) \quad \delta_0(v_1^4) \equiv 2^3(x_7 + \beta_{2/2}) \pmod{2^4}$$

and

$$\delta_0(v_i^{2i}) \equiv 4iv_1^{2i-4}x_7 \pmod{8i} \quad \text{for } i \geq 3.$$

Combining this with

$$(4.4.36) \quad \delta_0(v_1^{2i+1}h_0^j) = v_1^{2i}h_0^{j+1}$$

and

$$\delta_0(v_1^{2i+1}h_0^jx_7) = v_1^{2i}h_0^{j+1}x_7$$

accounts for all elements of the form  $v_1^i h_0^j x_7^\varepsilon$  for  $i, j \geq 0$  and  $\varepsilon = 0, 1$  we have

4.4.37. THEOREM. For  $p = 2$   $\text{Ext}^1(BP_*)$  is generated by  $\bar{\alpha}_i$  for  $i \geq 1$  where

$$\bar{\alpha}_i = \begin{cases} \delta_0(v_1^i) & \text{for } i \text{ odd} \\ \frac{1}{2}\delta_0(v_1^2) & \text{for } i = 2 \\ (1/2i)\delta_0(v_1^i) & \text{for even } i \geq 4. \end{cases}$$

In particular  $\bar{\alpha}_1 = h_{10}$ . Moreover  $\bar{\alpha}_1^j \bar{\alpha}_i \neq 0$  for all  $j > 0$  and  $i \neq 2$ .  $\square$

Moving on to  $\text{Ext}^1(BP_*/I_1)$  we still need to compute  $\delta_0$  on  $h_{12}$ ,  $v_1h_{12}$ ,  $\beta_3$ , and  $v_1^j h_{13}$  for  $0 \leq j \leq 3$ . An easy calculation gives

$$(4.4.38) \quad \begin{aligned} \delta_0(h_{12}) &\equiv h_{11}^2 && \pmod{2}, \\ \delta_0(v_1h_{12}) &\equiv h_{10}h_{12} && \pmod{2}, \\ \delta_0(h_{13}) &\equiv h_{12}^2 && \pmod{2}, \\ \delta_0(v_1h_{13}) &\equiv v_1h_{12}^2 + h_{10}h_{13} && \pmod{2}, \\ \delta_0(v_1^2h_{13}) &\equiv 2(h_{11} + v_1h_{10})h_{13} && \pmod{4}, \end{aligned}$$

and

$$\delta_0(v_1^3h_{13}) \equiv v_1^2h_{10}h_{13} \pmod{2}.$$

For  $\beta_3$  we have

$$(4.4.39) \quad \delta_0(\beta_3) = \beta_{2/2}^2 + \eta_1.$$

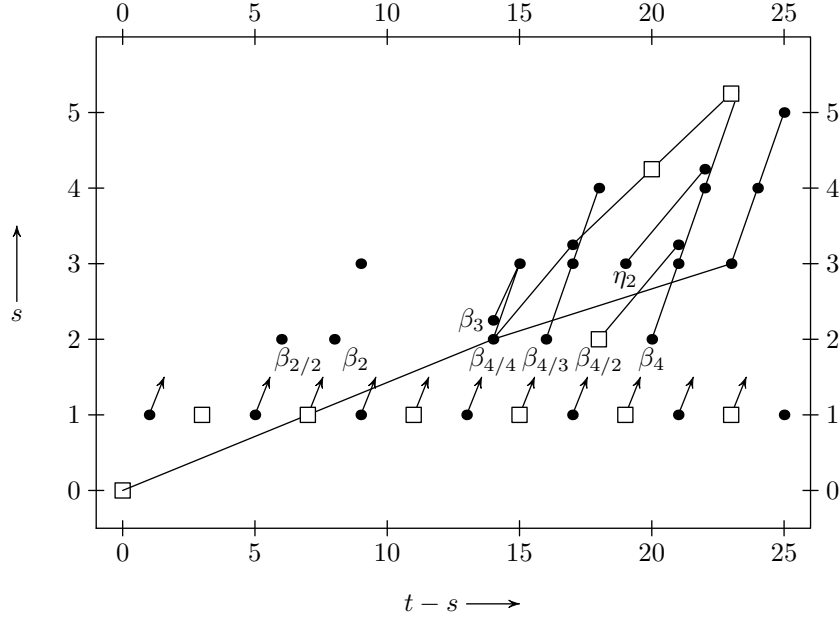
The proof is deferred until the next chapter (5.1.25).

In  $\text{Ext}^2(BP_*/I_1)$  all the calculations are straightforward except  $\eta_2$  and  $x_7\beta_{4/j}$ . The former gives

$$(4.4.40) \quad \delta_0(\eta_2) = c_0,$$

which we defer to 5.1.25. For the latter we have

$$\delta_0(x_7\beta_{4/4}) \equiv x_7h_{12}^2 \pmod{2}.$$

FIGURE 4.4.45.  $\text{Ext}(BP_*)$  for  $p = 2$ ,  $t - s \leq 25$ .

Computing in  $C(BP_*/I_2)$  we get

$$d(t_1^2|t_3 + t_2|t_2^2 + t_1^3|t_2^2 + t_1^2t_2|t_1^4) = t_1|t_1^4|t_1^4 + v_2t_1^2|t_1^2|t_1^2$$

so  $x_7h_{12}^2 \equiv \beta_3\beta_1^2 \pmod{(v_1^2)}$  and

$$(4.4.41) \quad \delta_0(x_7\beta_{4/4}) \equiv \beta_1^2\beta_3 + ch_{10}^2\beta_{4/2} \pmod{(2)}$$

for  $c = 0$  or  $1$ . Note that

$$\delta_0(h_{10}\beta_4) = h_{10}^2\beta_{4/2}.$$

We also get from 4.4.37

$$(4.4.42) \quad \delta_0(x_7\beta_{4/3}) \equiv ch_{10}^2\beta_4 + h_{10}x_7\beta_{4/4} \pmod{(2)}.$$

$\delta_0(x_7\beta_{4/2})$  must be a multiple of  $h_{10}x_7\beta_{4/3}$  but the latter is not in  $\ker \delta_0$  so

$$(4.4.43) \quad \delta_0(x_7\beta_{4/2}) = 0.$$

Of the remaining calculations of  $\delta_0$  all are easy but  $\beta_1^2\beta_{4/4}$  and  $h_{10}^3\beta_4 = \beta_1^3\beta_{4/4}$ . It is clear that  $\delta_0(\beta_1^2\beta_{4/4})$  and  $\delta_0(\beta_1^3\beta_{4/4})$  are multiples of elements which reduce to  $h_{10}^3$  and  $P\beta_1$ , respectively. Since  $\beta_1^2\beta_{2/2} = 0$  and  $\beta_1\beta h_{10}^2\beta_{4/3} = 0$  we have  $\delta_0(\beta_1^2\beta_{4/4}) \equiv 0 \pmod{(2)}$  and  $\delta_0(\beta_1^3\beta_{4/4}) \equiv 0 \pmod{(4)}$ . Thus the simplest possible result is

$$(4.4.44) \quad \begin{aligned} \frac{1}{2}\delta_0(\beta_1^2\beta_{4/4}) &\equiv h_{10}^3\beta_{4/3} \pmod{(2)}, \\ \frac{1}{4}\delta_0(\beta_1^3\beta_{4/4}) &\equiv P\beta_1 \pmod{(2)}. \end{aligned}$$

We will see below that larger values of the corresponding Ext groups would lead to a contradiction.

The resulting value of  $\text{Ext}(BP_*)$  is shown in FIG. 4.4.45. Here squares denote elements of order greater than 2. The order of the elements in  $\text{Ext}^1$  is given in 4.4.37. The generators of  $\text{Ext}^{2,20}$  and  $\text{Ext}^{4,24}$  have order 4 while that of  $\text{Ext}^{5,28}$  has order 8.

We compute differentials and group extensions in the Adams–Novikov spectral sequence for  $p = 2$  by comparing it with the Adams spectral sequence. The  $E_2$ -term of the latter as computed by Tangora [1] is shown in FIG. 4.4.46. This procedure will determine all differentials and extensions in the Adams spectral sequence in this range as well.

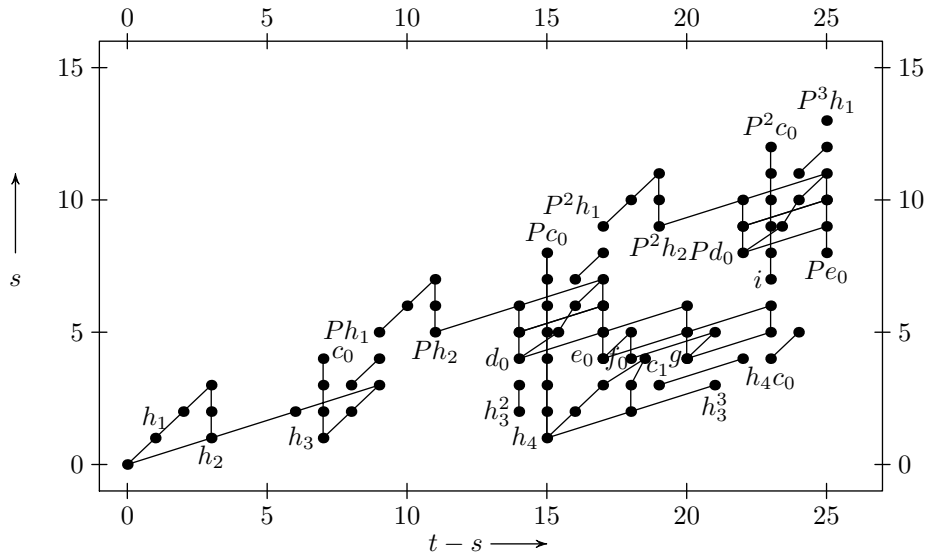


FIGURE 4.4.46.  $\text{Ext}_{A_*}(\mathbf{Z}/2, \mathbf{Z}/2)$  for  $t - s \leq 25$ .

The Adams element  $h_1$  corresponds to the Novikov  $\bar{\alpha}_1$ . Since  $h_1^4 = 0$ ,  $\alpha_1^4$  must be killed by a differential, and it must be  $d_3(\bar{\alpha}_3)$ . It can be shown that the periodicity operator  $P$  in the Adams spectral sequence (see 3.4.6) corresponds to multiplication by  $v_1^4$ , so  $P^i h_1$  corresponds to  $\bar{\alpha}_{4i+1}$ , so  $d_3(\bar{\alpha}_{4i+3}) = \bar{\alpha}_1^3 \bar{\alpha}_{4i+1}$ . The relation  $h_0^2 h_2 = h_1^3$  gives a group extension in the Adams–Novikov spectral sequence,  $2\alpha_{4i+2} = \bar{\alpha}_1^2 \bar{\alpha}_{4i+1}$  in homotopy. The element  $P^i h_2$  for  $i > 0$  corresponds to  $2\bar{\alpha}_{4i+1}$ . This element is not divisible by 2 in the Adams spectral sequence so we deduce  $d_3(\bar{\alpha}_{4i+2}) = \alpha_1^3 \bar{\alpha}_{4i}$  for  $i > 0$ . Summing up we have

4.4.47. THEOREM. *The elements in  $\text{Ext}(BP_*)$  for  $p = 2$  listed in 4.4.37 behave in the Adams–Novikov spectral sequence as follows.  $d_3(\bar{\alpha}_{4i+3}) = \bar{\alpha}_1^3 \bar{\alpha}_{4i+3}$  for  $i \geq 0$  and  $d_3(\bar{\alpha}_{4i+2}) = \bar{\alpha}_1^3 \bar{\alpha}_{4i}$  for  $i \geq 1$ . Moreover the homotopy element corresponding to  $\alpha_{4i+2} = r\bar{\alpha}_{4i+2}$  does not have order 2; twice it is  $\bar{\alpha}_1^2 \bar{\alpha}_{4i}$  for  $i \geq 1$  and  $\bar{\alpha}_1^3$  for  $i = 0$ .  $\square$*

As it happens, there are no other Adams–Novikov spectral sequence differentials in this range, although there are some nontrivial extensions.

These elements in the Adams–Novikov spectral sequence  $E_\infty$ -term correspond to Adams elements near the vanishing line. The towers in dimensions congruent

to 7 mod (8) correspond to the groups generated by  $\bar{\alpha}_{4i}$ . Thus the order of  $\bar{\alpha}_{4i}$  determines how many elements in the tower survive to the Adams  $E_\infty$ -term. For example, the tower in dimension 15 generated by  $h_4$  has 8 elements.  $\bar{\alpha}_8$  has order  $2^5$  so only the top elements can survive. From this we deduce  $d_3(h_0^i h_4) = h_0^i d_0$  for  $i = 1, 2$  and either  $d_3(h_4) = d_0$  or  $d_2(h_4) = h_0 h_3^2$ . To determine which of these two occurs we consult the Adams–Novikov spectral sequence and see that  $\beta_3$  and  $\beta_{4/4}$  must be permanent so  $\pi_{14}^s = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ . If  $d_3(h_4) = d_0$  the Adams spectral sequence would give  $\pi_{14}^s = \mathbf{Z}/(4)$ , so we must have  $d_2(h_4) = h_0 h_3^2$ .

One can also show that  $P^i c_0$  corresponds to  $\alpha_1 \bar{\alpha}_{4i+4}$  for  $i > 1$  and this leads to a nontrivial multiplicative extension in the Adams spectral sequence. For example, the homotopy element corresponding to  $Pc_0$  is  $\alpha_1$  times the one corresponding to  $h_0^3 h_4$ .

The correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles is shown in the following table.

TABLE 4.4.48. Correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles for  $p = 2$ ,  $14 \leq t - s \leq 24$

<i>Adams–Novikov element</i>	<i>Adams element</i>	<i>Adams–Novikov element</i>	<i>Adams element</i>
$\beta_{4/4}$	$h_3^2$	$\beta_4$	$g$
$\beta_3$	$d_0$	$\frac{1}{2} \bar{\alpha}_2^2 \beta_{4/4}$	$h_0 g$
$\beta_{4/3}$	$h_1 h_4$	$\frac{1}{4} \alpha_1^3 \beta_4$	$h_2 g$
$\frac{1}{2} \beta_{4/2}$	$h_2 h_4$	$\beta_{4/4} \bar{\alpha}_4$	$h_4 c_0$
$\eta_2$	$c_1$		

4.4.49. COROLLARY. *The Adams–Novikov spectral sequence has nontrivial group extensions in dimensions 18 and 20 and the homotopy product  $\beta_4 \bar{\alpha}_2$  is detected in filtration 4.* □

4.4.50. COROLLARY. *For  $14 \leq t - s \leq 24$  the following differentials occur in the Adams spectral sequence for  $p = 2$ .*

$$d_2(h_4) = h_0 h_3^2, \quad d_3(h_0 h_4) = h_0 d_0, \quad d_2(e_0) = h_1^2 d_0,$$

$$d_2(f_0) = h_0^2 e_0, \quad d_2(i) = h_0 P d_0, \quad \text{and} \quad d_2(P e_0) = h_1^2 P d_0.$$

*There are nontrivial multiplicative extensions as follows:*

$$h_1 \cdot h_0^3 h_4 = P c_0, \quad h_1 \cdot h_1 g = P d_0, \quad \text{and} \quad h_0 \cdot h_2^2 e_0 = h_1 P d_0 = h_2 \cdot h_2^2 d_0. \quad \square$$

## The Chromatic Spectral Sequence

The spectral sequence of the title is a mechanism for organizing the Adams–Novikov  $E_2$ -term and ultimately  $\pi_*(S^0)$  itself. The basic idea is this. If an element  $x$  in the  $E_2$ -term, which we abbreviate by  $\text{Ext}(BP_*)$  (see 5.1.1), is annihilated by a power of  $p$ , say  $p^i$ , then it is the image of some  $x' \in \text{Ext}(BP_*/p^i)$  under a suitable connecting homomorphism. In this latter group one has multiplication by a suitable power of  $v_1$  (depending on  $i$ ), say  $v_1^m$ .  $x'$  may or may not be annihilated by some power of  $v_1^m$ , say  $v_1^{mj}$ . If not, we say  $x$  is  $v_1$ -periodic; otherwise  $x'$  is the image of some  $x'' \in \text{Ext}(BP_*/(p^i, v_1^{mj}))$  and we say it is  $v_1$ -torsion. In this new Ext group one has multiplication by  $v_2^n$  for some  $n$ . If  $x$  is  $v_1$ -torsion, it is either  $v_2$ -periodic or  $v_2$ -torsion depending on whether  $x''$  is killed by some power of  $v_2^n$ . Iterating this procedure one obtains a complete filtration of the original Ext group in which the  $n$ th subgroup in the  $v_n$ -torsion and the  $n$ th subquotient is  $v_n$ -periodic. This is the *chromatic filtration* and it is associated with the *chromatic spectral sequence* of 5.1.8. The chromatic spectral sequence is like a spectrum in the astronomical sense in that it resolves stable homotopy into periodic components of various types.

Recently we have shown that this algebraic construction has a geometric origin, i.e., that there is a corresponding filtration of  $\pi_*(S^0)$ . The chromatic spectral sequence is based on certain inductively defined short exact sequences of comodules 5.1.5. In Ravenel [9] we show that each of these can be realized by a cofibration

$$N_n \rightarrow M_n \rightarrow N_{n+1}$$

with  $N_0 = S^0$  so we get an inverse system

$$S^0 \leftarrow \Sigma^{-1}N_1 \leftarrow \Sigma^{-2}N_2 \leftarrow \dots$$

The filtration of  $\pi_*(S^0)$  by the images of  $\pi_*(\Sigma^{-n}N^n)$  is the one we want. Applying the Novikov Ext functor to this diagram yields the chromatic spectral sequence, and applying homotopy yields a geometric form of it. For more discussion of this and related problems see Ravenel [8].

The chromatic spectral sequence is useful computationally as well as conceptually. In 5.1.10 we introduce the chromatic cobar complex  $CC(BP_*)$ . Even though it is larger than the already ponderous cobar complex  $C(BP_*)$ , it is easier to work with because many cohomology classes (e.g., the Greek letter elements) have far simpler cocycle representatives in  $CC$  than in  $C$ .

In Section 1 the basic properties of the chromatic spectral sequence are given, most notably the change-of-rings theorem 5.1.14, which equates certain Ext groups with the cohomology of certain Hopf algebras  $\Sigma(n)$ , the  $n$ th Morava stabilizer algebra. This isomorphism enables one to compute these groups and was the original

motivation for the chromatic spectral sequence. These computations will be the subject of the next chapter. Section 1 also contains various computations (5.1.20–5.1.22 and 5.1.24) which illustrate the use of the chromatic cobar complex.

In Section 2 we compute various  $\text{Ext}^1$  groups (5.2.6, 5.2.11, 5.2.14, and 5.2.17) and recover as a corollary the Hopf invariant one theorem (5.2.8), which says almost all elements in the Adams spectral sequence  $E_2^{1,*}$  are not permanent cycles. Our method of proof is to show they are not in the image of the Adams–Novikov  $E_2^{1,*}$  after computing the latter.

In Section 3 we compute the  $v_1$ -periodic part of the Adams–Novikov spectral sequence and its relation to the  $J$ -homomorphism and the  $\mu$ -family of Adams [1]. The main result is 5.3.7, and the resulting pattern in the Adams–Novikov spectral sequence for  $p = 2$  is illustrated in 5.3.8.

In Section 4 we describe  $\text{Ext}^2$  for all primes (5.4.5), referring to the original papers for the proofs, which we cannot improve upon. Corollaries are the nontriviality of  $\gamma_t$ , (5.4.4) and a list of elements in the Adams spectral sequence  $E_2^{2,*}$  which cannot be permanent cycles (5.4.7). This latter result is an analog of the Hopf invariant one theorem. The Adams spectral sequence elements not so excluded include the Arf invariant and  $\eta_j$  families. These are discussed in 5.4.8–5.4.10.

In Section 5 we compile all known results about which elements in  $\text{Ext}^2$  are permanent cycles, i.e., about the  $\beta$ -family and its generalizations. We survey the relevant work of Smith and Oka for  $p \geq 5$ , Oka and Toda for  $p = 3$ , and Davis and Mahowald for  $p = 2$ .

In Section 6 we give some fragmentary results on  $\text{Ext}^s$  for  $s \geq 3$ . We describe some products of  $\alpha$ 's and  $\beta$ 's and their divisibility properties. We close the chapter by describing a possible obstruction to the existence of the  $\delta$ -family.

Since the appearance of the first edition, many computations related to the chromatic spectral sequence have been made by Shimomura. A list of some of them can be found in Shimomura [2]. A description of the first three columns of the chromatic spectral sequence (meaning the rational,  $v_1$ - and  $v_2$ -periodic parts) for the sphere can be found in Shimomura and Wang [3] for  $p = 2$ , in Shimomura and Wang [4] for  $p = 3$ , and in Shimomura and Yabe [5] for  $p \geq 5$ . Analogous computations for the mod  $p$  Moore spectrum can be found in Shimomura [6] for  $p = 2$ , in Shimomura [7] for  $p = 3$  and in Shimomura [8] for  $p \geq 5$ .

### 1. The Algebraic Construction

In this section we set up the chromatic spectral sequence converging to the Adams–Novikov  $E_2$ -term, and use it to make some simple calculations involving Greek letter elements (1.3.17 and 1.3.19). The chromatic spectral sequence was originally formulated by Miller, Ravenel, and Wilson [1]. First we make the following abbreviation in notation, which will be in force throughout this chapter: given a  $BP_*(BP)$  comodule  $M$  (A1.1.2), we define

$$(5.1.1) \quad \text{Ext}(M) = \text{Ext}_{BP_*(BP)}(BP_*, M).$$

To motivate our construction recall the short exact sequence of comodules given by 4.3.2(c)

$$(5.1.2) \quad 0 \rightarrow \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \rightarrow BP_*/I_{n+1} \rightarrow 0$$



and let

$$\delta_n: \text{Ext}^s(BP_*/I_{n+1}) \rightarrow \text{Ext}^{s+1}(BP_*/I_n)$$

denote the corresponding connecting homomorphism.

5.1.3. DEFINITION. For  $t, n > 0$  let

$$\alpha_t^{(n)} = \delta_0 \delta_1 \cdots \delta_{n-1}(v_n^t) \in \text{Ext}^n(BP_*). \quad \square$$

Here  $\alpha^{(n)}$  stands for the  $n$ th letter of the Greek alphabet. The status of these elements in  $\pi_*^S$  is described in 1.3.11, 1.3.15, and 1.3.18. The invariant prime ideals in  $I_n$  in 5.1.2 can be replaced by invariant regular ideals, e.g., those provided by 4.3.3. In particular we have

5.1.4. DEFINITION.  $\alpha_{sp^i/i+1} \in \text{Ext}^{1, qsp^i}(BP_*)$  (where  $q = 2p - 2$ ) is the image of  $v_1^{sp^i}$  under the connecting homomorphism for the short exact sequence

$$0 \rightarrow BP_* \xrightarrow{p^{i+1}} BP_* \rightarrow BP_*/(p^{i+1}) \rightarrow 0. \quad \square$$

We will see below that for  $p > 2$  these elements generate  $\text{Ext}^1(BP_*)$  (5.2.6) and that they are nontrivial permanent cycles in  $\text{im } J$ . We want to capture all of these elements from a single short exact sequence; those of 5.1.4 are related by the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & BP_* & \xrightarrow{p^i} & BP_* & \longrightarrow & BP_*/(p^i) \longrightarrow 0 \\ & & \parallel & & \downarrow p & & \downarrow \\ 0 & \longrightarrow & BP_* & \xrightarrow{p^{i+1}} & BP_* & \longrightarrow & BP_*/(p^{i+1}) \longrightarrow 0 \end{array}$$

Taking the direct limit we get

$$0 \rightarrow BP_* \rightarrow \mathbf{Q} \otimes BP_* \rightarrow \mathbf{Q}/\mathbf{Z}_{(p)} \otimes BP_* \rightarrow 0;$$

we denote these three modules by  $N^0$ ,  $M^0$ , and  $N^1$ , respectively. Similarly, the direct limit of the sequences

$$0 \rightarrow BP_*/(p^{i+1}) \xrightarrow{v_1^{i+j}} \Sigma^{-qp^{i+j}} BP_*/(p^{i+1}) \rightarrow \Sigma^{-1p^{i+j}} BP_*/(p^{i+1}, v_1^{p^{i+j}}) \rightarrow 0$$

gives us

$$0 \rightarrow BP_*/(p^\infty) \rightarrow v_1^{-1}BP_*/(p^\infty) \rightarrow BP_*/(p^\infty, v_1^\infty) \rightarrow 0$$

and we denote these three modules by  $N^1$ ,  $M^1$ , and  $N^2$ , respectively. More generally we construct short exact sequences

$$(5.1.5) \quad 0 \rightarrow N^n \rightarrow M^n \rightarrow N^{n+1} \rightarrow 0$$

inductively by  $M^n = v_n^{-1}BP_* \otimes_{BP_*} N^n$ . Hence  $N^n$  and  $M^n$  are generated as  $\mathbf{Z}_{(p)}$ -modules by fractions  $\frac{x}{y}$  where  $x \in BP_*$  for  $N^n$  and  $v_n^{-1}BP_*$  for  $M^n$  and  $y$  is a monomial in the ideal  $(pv_1 \cdots v_{n-1})$  of the subring  $\mathbf{Z}_{(p)}[v_1, \dots, v_{n-1}]$  of  $BP_*$ . The  $BP_*$ -module structure is such that  $vx/y = 0$  for  $v \in BP_*$  if this fraction when reduced to lowest terms does not have its denominator in the above ideal. For example, the element  $\frac{1}{p^i v_1^j} \in N^2$  is annihilated by the ideal  $(p^i, v_1^j)$ .

5.1.6. LEMMA. 5.1.5 is a short exact sequence of  $BP_*(BP)$ -comodules.

PROOF. Assume inductively that  $N^n$  is a comodule and let  $N' \subset N^n$  be a finitely generated subcomodule. Then  $N'$  is annihilated by some invariant regular ideal with  $n$  generators given by 4.3.3. It follows from 4.3.3 that multiplication by some power of  $v_n$ , say  $v_n^k$ , is a comodule map, so

$$v_n^{-1}N' = \varinjlim_{v_n^k} \Sigma^{-\dim v_n^{ki}} N'$$

is a comodule. Alternatively,  $N'$  is annihilated by some power of  $I_n$ , so multiplication by a suitable power of  $v_n$  is a comodule map by Proposition 3.6 of Landweber [7] and  $v_n^{-1}N'$  is again a comodule. Taking the direct limit over all such  $N'$  gives us a unique comodule structure on  $M^n$  and hence on the quotient  $N^{n+1}$ .  $\square$

5.1.7. DEFINITION. *The chromatic resolution is the long exact sequence of comodules*

$$0 \rightarrow BP_* \rightarrow M^0 \xrightarrow{d_e} M^1 \xrightarrow{d_e} \dots$$

obtained by splicing the short exact sequences of 5.1.5.  $\square$

The associated resolution spectral sequence (A1.3.2) gives us

5.1.8. PROPOSITION. *There is a chromatic spectral sequence converging to  $\text{Ext}(BP_*)$  with  $E_1^{n,s} = \text{Ext}^s(M^n)$  and  $d_r: E_r^{n,s} \rightarrow E_r^{n+r,s+1-r}$  where  $d_1$  is the map induced by  $d_e$  in 5.1.7.*  $\square$

5.1.9. REMARK. There is a chromatic spectral sequence converging to  $\text{Ext}(F)$  where  $F$  is any comodule which is flat as a  $BP_*$ -module, obtained by tensoring the resolution of 5.1.7 with  $F$ .

5.1.10. DEFINITION. *The chromatic cobar complex  $CC(BP_*)$  is given by*

$$CC^u(BP_*) = \bigoplus_{s+n=u} C^s(M^n),$$

where  $C(\ )$  is the cobar complex of A1.2.11, with  $d(x) = d_e^*(x) + (-1)^n d_i(x)$  for  $x \in C^s(M^n)$  where  $d_e^*$  is the map induced by  $d_e$  in 5.1.7 (the external component of  $d$ ) and  $d_i$  (the internal component) is the differential in the cobar complex  $C(M_n)$ .  $\square$

It follows from 5.1.8 and A1.3.4 that  $H(CC(BP_*)) = H(C(BP_*)) = \text{Ext}(BP_*)$ . The embedding  $BP_* \rightarrow M^0$  induces an embedding of the cobar complex  $C(BP_*)$  into the chromatic cobar complex  $CC(BP_*)$ . Although  $CC(BP_*)$  is larger than  $C(BP_*)$ , we will see below that it is more convenient for certain calculations such as identifying the Greek letter elements of 5.1.3.

This entire construction can be generalized to  $BP_*/I_m$  as follows.

5.1.11. DEFINITION. *Let  $N_m^0 = BP_*/I_m$  and define  $BP_*$ -modules  $N_m^n$  and  $M_m^n$  inductively by short exact sequences*

$$0 \rightarrow N_m^n \rightarrow M_m^n \rightarrow N_m^{n+1} \rightarrow 0$$

where  $M_m^n = v_{m+n}^{-1}BP_* \otimes_{BP_*} N_m^n$ .  $\square$

Lemma 5.1.6 can be generalized to show that these are comodules. Splicing them gives a long exact sequence

$$0 \rightarrow BP_*/I_m \rightarrow M_m^0 \xrightarrow{d_e} M_m^1 \xrightarrow{d_e} \dots$$

and a chromatic spectral sequence as in 5.1.8. Moreover  $BP_*/I_m$  can be replaced by any comodule  $L$  having an increasing filtration  $\{F_i L\}$  such that each subquotient  $F_i/F_{i-1}$  is a suspension of  $BP_*/I_m$ , e.g.,  $L = BP_*/I_m^k$ . We leave the details to the interested reader.

Our main motivation here, besides the Greek letter construction, is the computability of  $\text{Ext}(M_n^0)$ ; it is essentially the cohomology of the automorphism group of a formal group law of height  $n$  (1.4.3 and A2.2.18). This theory will be the subject of Chapter 6. We will state the first major result now. We have  $M_n^0 = v_n^{-1}BP_*/I_n$ , which is a comodule algebra (A1.1.2), so  $\text{Ext}(M_n^0)$  is a ring (A1.2.14). In particular it is a module over  $\text{Ext}^0(M_n^0)$ . The following is an easy consequence of the Morava–Landweber theorem, 4.3.2.

5.1.12. PROPOSITION. *For  $n > 0$ ,  $\text{Ext}^0(M_n^0) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ . We denote this ring by  $K(n)_*$ . [The case  $n = 0$  is covered by 5.2.1, so it is consistent to denote  $\mathbf{Q}$  by  $K(0)_*$ .]  $\square$*

5.1.13. DEFINITION. *Make  $K(n)_*$  a  $BP_*$ -module by defining multiplication by  $v_i$  to be trivial for  $i \neq n$ . Then let  $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$ .  $\square$*

$\Sigma(n)$ , the  $n$ th Morava stabilizer algebra, is a Hopf algebroid which will be closely examined in the next chapter. It has previously been called  $K(n)_*K(n)$ , e.g., in Miller, Ravenel, and Wilson [1], Miller and Ravenel [5], and Ravenel [5, 6].  $K(n)_*$  is also the coefficient ring of the  $n$ th Morava  $K$ -theory; see Section 4.2 for references. We have changed our notation to avoid confusion with  $K(n)_*(K(n))$ , which is  $\Sigma(n)$  tensored with a certain exterior algebra.

The starting point of Chapter 6 is

5.1.14. CHANGE-OF-RINGS THEOREM (Miller and Ravenel [5]).

$$\text{Ext}(M_n^0) = \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*). \quad \square$$

We will also show (6.2.10)

5.1.15. MORAVA VANISHING THEOREM. *If  $(p-1) \nmid n$  then  $\text{Ext}^s(M_n^0) = 0$  for  $s > n^2$ .  $\square$*

Moreover this Ext satisfies a kind of Poincaré duality, e.g.,

$$\text{Ext}^s(M_n^0) = \text{Ext}^{n^2-s}(M_n^0),$$

and it is essentially the cohomology of a certain  $n$  stage nilpotent Lie algebra of rank  $n^2$ . If we replace  $\Sigma(n)$  with a quotient by a sufficiently large finitely generated subalgebra, then this Lie algebra becomes abelian and the Ext [even if  $(p-1)$  divides  $n$ ] becomes an exterior algebra over  $K(n)_*$  on  $n^2$  generators of degree one.

To connect these groups with the chromatic spectral sequence we have

5.1.16. LEMMA. *There are short exact sequences of comodules*

$$0 \rightarrow M_{m+1}^{n-1} \xrightarrow{j} \Sigma^{\dim v_m} M_m^n \xrightarrow{v_m} M_m^n \rightarrow 0$$

and Bockstein spectral sequences converging to  $\text{Ext}(M_m^n)$  with

$$E_1^{s,*} = \text{Ext}^s(M_{m+1}^{n-1}) \otimes P(a_m)$$

where multiplication by  $a_m$  in the Bockstein spectral sequence corresponds to division by  $v_m$  in  $\text{Ext}(M_m^n)$ .  $d_r$  is not a derivation but if  $d_r(a_m^r x) = y \neq 0$  then  $d_r(a_m^{r+i} x) = v_m^i y$ .

PROOF. The spectral sequence is that associated with the increasing filtration of  $M_m^n$  defined by  $F_i M_m^n = \ker v_m^i$  (see A1.3.9). Then  $E^0 M_m^n = M_{m+1}^{n-1} \otimes P(a_m)$ .  $\square$

Using 5.1.16  $n$  times we can in principle get from  $\text{Ext}(M_n^0)$  to  $\text{Ext}(M_0^n) = \text{Ext}(M^n)$  and hence compute the chromatic  $E_1$ -term (5.1.8). In practice these computations can be difficult.

5.1.17. REMARK. We will not actually use the Bockstein spectral sequence of 5.1.16 but will work directly with the long exact sequence

$$\rightarrow \text{Ext}^s(M_{m+1}^{n-1}) \xrightarrow{j} \text{Ext}^s(M_m^n) \xrightarrow{v_m} \text{Ext}^s(\Sigma^{-2p^m+2} M_m^n) \xrightarrow{\delta} \text{Ext}^{s+1}(M_{m+1}^{n-1}) \rightarrow \dots$$

by induction on  $s$ . Given an element  $x \in \text{Ext}(M_{m+1}^{n-1})$  which we know not to be in  $\text{im } \delta$ , we try to divide  $j(x)$  by  $v_m$  as many times as possible. When we find an  $x' \in \text{Ext}(M_m^n)$  with  $v_m^r x' = j(x)$  and  $\delta(x') = y \neq 0$  then we will know that  $j(x)$  cannot be divided any further by  $v_m$ . Hence  $\delta$  serves as reduction mod  $I_{m+1}$ . This state of affairs corresponds to  $d_r(a_m^r x) = y$  in the Bockstein spectral sequence of 5.1.16. We will give a sample calculation with  $\delta$  below (5.1.20).

We will now make some simple calculations with the chromatic spectral sequence starting with the Greek letter elements of 5.1.3. The short exact sequence of 5.1.2 maps to that of 5.1.5, i.e., we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & BP_*/I_n & \xrightarrow{v_n} & \Sigma^{-\dim v_n} BP_*/I_n & \longrightarrow & \Sigma^{-\dim v_n} BP_*/I_{n+1} \longrightarrow 0 \\ & & \downarrow i & & \downarrow & & \downarrow i \\ 0 & \longrightarrow & N^n & \longrightarrow & M^n & \longrightarrow & N^{n+1} \longrightarrow 0 \end{array}$$

with

$$i(v_{n+1}^t) = \frac{v_{n+1}^t}{pv_1 \cdots v_n}.$$

Hence  $\alpha_t^{(n)}$  can be defined as the image of  $i(v_n^t)$  under the composite of the connecting homomorphisms of 5.1.5, which we denote by  $\alpha: \text{Ext}^0(N^n) \rightarrow \text{Ext}^n(BP_*)$ . On the other hand, the chromatic spectral sequence has a bottom edge homomorphism

$$\begin{array}{ccccccc} \text{Ext}^0(M^n) & \xlongequal{\quad} & E_1^{n,0} & & & & \\ \uparrow & & \uparrow & & & & \\ \text{Ext}^0(N^n) & \longrightarrow & \ker d_1 & \longrightarrow & E_\infty^{n,0} & \longrightarrow & \text{Ext}^n(BP_*) \end{array}$$

which we denote by

$$\kappa: \text{Ext}^0(N^n) \rightarrow \text{Ext}^n(BP_*).$$

$\kappa$  and  $\alpha$  differ by sign, i.e.,

5.1.18. PROPOSITION.  $\kappa = (-1)^{[(n+1)/2]} \alpha$ , where  $[x]$  is the largest integer not exceeding  $x$ .

PROOF. The image  $y_0$  of  $i(v_n^t)$  in  $M^n$  is an element in the chromatic complex (5.1.10) cohomologous to some class in the cobar complex  $C(BP_*)$ . Inductively we can find  $x_s \in C^s(M^{n-s-1})$ , and  $y_s \in C^s(M^{n-s})$  such that  $d_c(x_s) = y_s$  and  $d_i(x_s) = y_{s+1}$ . Moreover  $y_n \in C^n(M^0)$  is the image of some  $x_n \in C^n(BP_*)$ . It follows from the definition of the connecting homomorphism that  $x_n$  is a co-cycle representing  $\alpha(i(v_n^t)) = \alpha_t^{(n)}$ . On the other hand,  $y_s$  is cohomologous to

$(-1)^{n-s}y_{s+1}$  in  $CC(BP_*)$  by 5.1.10 and  $\prod_{s=0}^{n-1}(-1)^{n-s} = (-1)^{[n+1/2]}$  so  $x_n$  represents  $(-1)^{[n+1/2]}\kappa(i(v_n^t))$ .  $\square$

5.1.19. DEFINITION. *If  $x \in \text{Ext}^0(M^n)$  is in the image of  $\text{Ext}^0(N^n)$  (and hence gives a permanent cycle in the chromatic spectral sequence) and has the form*

$$\frac{v_n^t}{p^{i_0}v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}} \text{ mod } I_n$$

(i.e.,  $x$  is the indicated fraction plus terms with larger annihilator ideals) then we denote  $\alpha(x)$  by  $\alpha_{t/i_{n-1}, \dots, i_0}^{(n)}$ ; if for some  $m < n$ ,  $i_k = 1$  for  $k \leq m$  then we abbreviate  $\alpha(x)$  by  $\alpha_{t/i_{n-1}, \dots, i_{m+1}}^{(n)}$ .  $\square$

5.1.20. EXAMPLES AND REMARKS. We will compute the image of  $\beta_t$  in  $\text{Ext}^2(BP_*/I_2)$  for  $p > 2$  in two ways.

(a) We regard  $\beta_t$  as an element in  $\text{Ext}^0(M^2)$  and compute its image under connecting homomorphisms  $\delta_0$  to  $\text{Ext}^1(M_1^1)$  and then  $\delta_1$  to  $\text{Ext}^2(M_2^0)$ , which is  $E_1^{0,2}$  in the chromatic spectral sequence for  $\text{Ext}(BP_*/I_2)$ . To compute  $\delta_0$ , we pick an element in  $x \in M^2$  such that  $px = \beta_t$ , and compute its coboundary in the cobar complex  $C(M^2)$ . The result is necessarily a cocycle of order  $p$ , so it can be pulled back to  $\text{Ext}^1(M_1^1)$ . To compute  $\delta_1$  on this element we take a representative in  $C^1(M_1^1)$ , divide it by  $v_1$ , and compute its coboundary.

Specifically  $\beta_t$  is  $\frac{v_2^t}{pv_1} \in M^2$ , so we need to compute the coboundary of  $x = \frac{v_2^t}{p^2v_1}$ .

It is convenient to write  $x$  as  $\frac{v_1^{p-1}v_2^t}{p^2v_1^p}$ , then the denominator is the product of elements generating an invariant regular ideal, which means that we need to compute  $\eta_R$  on the numerator only. We have

$$\eta_R(v_1^{p-1}) \equiv v_1^{p-1} - pv_1^{p-2}t_1 \text{ mod } (p^2)$$

and

$$\eta_R(v_2^t) \equiv v_2^t + tv_2^{t-1}(v_1t_1^p + pt_2) \text{ mod } (p^2, pv_1, v_1^2).$$

These give

$$d\left(\frac{v_1^{p-1}v_2^t}{p^2v_1^p}\right) = \frac{-v_2^t t_1}{pv_1^2} + \frac{tv_2^{t-1}}{pv_1}(t_2 - t_1^{1+p}).$$

This is an element of order  $p$  in  $C^1(M^2)$ , so it is in the image of  $C^1(M_1^1)$ . In this group the  $p$  in the denominator is superfluous, since everything has order  $p$ , so we omit it. To compute  $\delta_1$  we divide by  $v_1$  and compute the coboundary; i.e., we need to find

$$d\left(\frac{-v_2^t t_1}{v_1^3} + \frac{tv_2^{t-1}(t_2 - t_1^{1+p})}{v_1^2}\right).$$

Recall (4.3.15)

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 + v_1 b_{10}$$

where

$$b_{10} = - \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i}$$

as in 4.3.14. From this we get

$$\begin{aligned}
d\left(\frac{-v_2^t t_1}{v_1^3} + \frac{tv_2^{t-1}}{v_1^2}(t_2 - t_1^{1+p})\right) &= \frac{-tv_2^{t-1}t_1^p|t_1}{v_1^2} - \binom{t}{2} \frac{v_2^{t-2}}{v_1} t_1^{2p}|t_1 \\
&\quad + t(t-1) \frac{v_2^{t-2}}{t_1^p} |(t_2 - t_1^{1+p}) \\
&\quad + t \frac{v_2^{t-1}}{v_1^2} (-v_1 b_{10} + t_1^p|t_1) \\
&= \binom{t}{2} \frac{v_2^{t-2}}{v_1} (2t_1^p|t_2 - 2t_1^p|t_1^{1+p} - t_1^{2p}|t_1) \\
&\quad - t \frac{v_2^{t-1}}{v_1} b_{10}.
\end{aligned}$$

We will see below that  $\text{Ext}^2(M_2^0)$  has generators  $k_0$  represented by  $2t_1^p|t_2 - 2t_1^p|t_1^{1+p} - t_1^{2p}|t_1$  and  $b_{10}$ . Hence the mod  $I_2$  reduction of  $-\beta_t$  is

$$\binom{t}{2} v_2^{t-2} k_0 + tv_2^{t-1} b_{10}.$$

(b) In the chromatic complex  $CC(BP_*)$  (5.1.10),  $\beta_t \in M^2$  is cohomologous to elements in  $C^1(M^1)$  and  $C^2(M^0)$ . These three elements pull back to  $N^2$ ,  $C^1(N^1)$ , and  $C^2(N^0)$ , respectively. In theory we could compute the element in  $C^2(N^0) = C^2(BP_*)$  and reduce mod  $I_2$ , but this would be very laborious. Most of the terms of the element in  $C^0(BP_*)$  are trivial mod  $I_2$ , so we want to avoid computing them in the first place. The passage from  $C^0(N^2)$  to  $C^2(BP_*)$  is based on the four-term exact sequence

$$0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow N^2 \rightarrow 0.$$

Since  $\frac{v_2^t}{pv_1} \in N^2$  is in the image of  $\Sigma^{-q}BP_*/I_2$ , we can replace this sequence with

$$0 \rightarrow BP_* \xrightarrow{p} BP_* \xrightarrow{v_1} \Sigma^{-q}BP_*/I_1 \rightarrow \Sigma^{-q}BP_*/I_2 \rightarrow 0.$$

We are going to map the first  $BP_*$  to  $BP_*/I_2$ ; we can extend this to a map of sequences to

$$0 \rightarrow BP_*/I_2 \xrightarrow{p} BP_*/(p^2, pv_1, v_1^2) \xrightarrow{v_1} \Sigma^{-q}BP_*/(p, v_1^3) \rightarrow \Sigma^{-q}BP_*/I_2 \rightarrow 0,$$

which is the identity on the last comodule. [The reader may be tempted to replace the middle map by

$$BP_*/(p^2, v_1) \xrightarrow{v_1} \Sigma^{-q}BP_*/(p, v_1^2)$$

but  $BP_*/(p^2, v_1)$  is not a comodule.] This sequence tells us which terms we can ignore when computing in the chromatic complex, as we will see below.

Specifically we find (ignoring signs) that  $\frac{v_2^t}{pv_1} \in M^2$  is cohomologous to

$$\frac{tv_2^{t-1}t_1^p}{p} + \binom{t}{2} \frac{v_1 v_2^{t-2}}{p} t_1^{2p} + \text{higher terms}.$$

Note that the first two terms are divisible by  $v_1$  and  $v_1^2$  respectively in the image of  $C^1(\Sigma^{-q}BP_*/(p))$  in  $C^1(M^1)$ . The higher terms are divisible by  $v_1^3$  and can therefore be ignored.

In the next step we will need to work mod  $I_2^2$  in the image of  $C^2(BP_*)$  in  $C^2(M^0)$  via multiplication by  $p$ . From the first term above we get

$$t(t-1)v_2^{t-2}t_2|t_1^p + tv_2^{t-1}b_{10},$$

while the second term gives

$$\binom{t}{2}v_2^{t-2}t_1|t_1^{2p}$$

and their sum represents the same element obtained in (a).

Our next result is

5.1.21. PROPOSITION. For  $n \geq 3$ ,

$$\alpha_1^{(n)} = (-1)^n \alpha_1 \alpha_{p-1}^{(n-1)}. \quad \square$$

For  $n = 3$  this gives  $\gamma_1 = -\alpha_1 \beta_{p-1}$ . In the controversy over the nontriviality of  $\gamma_1$  (cf. the paragraph following 1.3.18) the relevant stem was known to be generated by  $\alpha_1 \beta_{p-1}$ , so what follows is an easy way (given all of our machinery) to show  $\gamma_1 \neq 0$ .

PROOF OF 5.1.2.  $\alpha_1$  is easily seen to be represented by  $t_1$  in  $C(BP_*)$ , while  $\alpha_1^{(n)}$  and  $\alpha_{p-1}^{(n-1)}$  are represented by

$$(-1)^{[n+1/2]} \frac{v_n}{pv_1 \cdots v_{n-1}} \in M^n \quad \text{and} \quad (-1)^{[n/2]} \frac{v_{n-1}^{p-1}}{pv_1 \cdots v_{n-2}} \in M^{n-1},$$

respectively. Hence  $(-1)^n \alpha_1 \alpha_{p-1}^{(n-1)} = -\alpha_{p-1}^{(n-1)} \alpha_1$  is represented by

$$(-1)^{[n/2]} \frac{v_{n-1}^{p-1} t_1}{pv_1 \cdots v_{n-2}} \in C^1(M^{n-1}) \subset CC^n(BP_*)$$

and it suffices to show that this element is cohomologous to  $\frac{(-1)^{[n+1/2]} v_n}{(pv_1 \cdots v_{n-1})}$  in  $CC(BP_*)$ .

Now consider

$$x = \frac{v_{n-1}^{-1} v_n}{pv_1 \cdots v_{n-2}} - \frac{v_{n-1}^p}{pv_1 \cdots v_{n-3} v_{n-2}^{1+p}} \in M^{n-1}.$$

Clearly

$$d_e(x) = \frac{v_n}{pv_1 \cdots v_{n-1}}.$$

To compute  $d_i(x)$  we need to know  $\eta_R(v_{n-1}^{-1} v_n) \bmod I_{n-1}$  and  $\eta_R(v_{n-1}^p) \bmod (p, v_1, \dots, v_{n-3}, v_{n-2}^{1+p})$  since  $d_i(x) = \eta_R(x) - x$ . We know

$$\eta_R(v_n) \equiv v_n + v_{n-1} t_1^{p^{n-1}} - v_{n-1}^p t_1 \quad \bmod I_{n-1}$$

by 4.3.21, so

$$\eta_R(v_{n-1}^p) \equiv v_{n-1}^p + v_{n-2}^p t_1^{p^{n-1}} - v_{n-2}^{p^2} t_1^p \quad \bmod I_{n-2}.$$

Hence

$$\eta_R(v_{n-1}^{-1} v_n) - v_{n-1}^{-1} v_n \equiv t_1^{p^{n-1}} - v_{n-1}^{p-1} t_1 \quad \bmod I_{n-1}$$

and

$$\eta_R(v_{n-1}^p) - v_{n-1}^p \equiv v_{n-2}^p t_1^{p^{n-1}} \quad \bmod (p, v_1, \dots, v_{n-3}, v_{n-2}^{1+p}).$$

It follows that

$$d_i(x) = \frac{-v_{n-1}^{p-1}t_1}{pv_1 \cdots v_{n-2}}$$

so

$$d(x) = \frac{v_n}{pv_1 \cdots v_{n-1}} + (-1)^n \frac{v_{n-1}^{p-1}t_1}{pv_1 \cdots v_{n-2}}$$

and a simple sign calculation gives the result.  $\square$

For  $p = 2$  5.1.21 says  $\alpha_1^{(n)} = \alpha_1^{n-2}\alpha_1^{(2)}$  for  $n \geq 2$ . We will show that each of these elements vanishes and that they are killed by higher differentials ( $d_{n-1}$ ) in the chromatic spectral sequence. We do not know if there are nontrivial  $d_r$ 's for all  $r \geq 2$  for odd primes.

5.1.22. THEOREM. *In the chromatic spectral sequence for  $p = 2$  there are elements  $x_n \in E_{n-1}^{1, n-2}$  for  $n \geq 2$  such that*

$$d_{n-1}(x_n) = \frac{v_n}{2v_1 \cdots v_{n-1}} \in E_{n-1}^{n, 0}.$$

PROOF. Fortunately we need not worry about signs this time. Equation 4.3.1 gives  $\eta_R(v_1) = v_1 - 2t_1$  and  $\eta_R(v_2) \equiv v_2 + v_1t_1^2 + v_1^2t_1 \pmod{(2)}$ . We find then that

$$x_2 = \frac{v_1^2 + 4v_1^{-1}v_2}{8}$$

has the desired property. For  $n > 2$  we represented  $x_n$  by

$$\frac{[(t_2 - t_1^3 + v_1^{-1}v_2t_1)|t_1| \cdots |t_1|]}{2} \in C^{n-2}(M^1)$$

with  $n - 3$   $t_1$ 's. To compute  $d_{n-1}(x_n)$  let

$$\tilde{x}_n = x_n + \sum_{i=1}^{n-2} \frac{(v_{i+1}^2 - v_i^2v_{i+1}^{-1}v_{i+2})t_1| \cdots |t_1}{2v_1 \cdots v_{i-1}v_i^3} \in CC(BP_*),$$

where the  $i$ th term has  $(n - 2 - i)$   $t_1$ 's. Then one computes

$$d(\tilde{x}_n) = \frac{v_n}{2v_1 \cdots v_{n-1}},$$

so

$$d_{n-1}(x_n) = \frac{v_n}{2v_1 \cdots v_{n-1}}$$

unless this element is killed by an earlier differential, in which case  $x_n$  would represent a nontrivial element in  $\text{Ext}^{n-1, 2n}(BP_*)$ , which is trivial by 5.1.23 below.  $\square$

5.1.23. EDGE THEOREM.

- (a) For all primes  $p$   $\text{Ext}^{s, t}(BP_*) = 0$  for  $t < 2s$ ,
- (b) for  $p = 2$   $\text{Ext}^{s, 2s}(BP_*) = \mathbf{Z}/(2)$  for  $s \geq 1$ , and
- (c) for  $p = 2$   $\text{Ext}^{s, 2s+2}(BP_*) = 0$  for  $s \geq 2$ .

PROOF. We use the cobar complex  $C(BP_*)$  of A1.2.11. Part (a) follows from the fact that  $C^{s, t}$  for  $t < 2s$ .  $C^{s, 2s}$  is spanned by  $t_1| \cdots |t_1$  while  $C^{s, 2s+2}$  is spanned by  $v_1t_1| \cdots |t_1$  and  $e_j = t_1| \cdots |t_1|t_1^2|t_1 \cdots |t_1$  with  $t_1^2$  in the  $j$ th position,  $1 \leq j \leq s$ . Since  $d(t_1^3) = -3t_1|t_1^2 - 3t_1^2|t_1$ , the  $e_j$ 's differ by a coboundary up to sign. Part (b) follows from

$$d(e_1) = 2t_1| \cdots |t_1 = -d(v_1t_1| \cdots |t_1)$$



and (c) follows from

$$d(t_2|t_1|\cdots|t_1) = -v_1 t_1|\cdots|t_1 - e_1.$$

□

We conclude this section by tying up some loose ends in Section 4.4. For  $p > 2$  we need

5.1.24. LEMMA. *For odd primes,  $\alpha_1\beta_p$  is divisible by  $p$  but not by  $p^2$ . (This gives the first element of order  $p^2$  in  $\text{Ext}^s(BP_*)$  for  $s \geq 2$ .)*

PROOF. Up to sign  $\alpha_1\beta_p$  is represented by  $\frac{v_2^p t_1}{pv_1}$ . Now  $\frac{v_2^p t_1}{p^2 v_1}$  is not a cocycle, but if we can get a cocycle by adding a term of order  $p$  then we will have the desired divisibility. It is more convenient to write this element as  $\frac{v_1^{p-1} v_2^p t_1}{p^2 v_1^p}$ ; then the factors of the denominator form an invariant sequence [i.e.,  $\eta_R(v_1^p) \equiv v_1^p \pmod{(p^2)}$ ], so to compute the coboundary it suffices to compute  $\eta_R(v_1^{p-1} v_2)$  mod  $(p^2, v_1^p)$ . We find

$$d\left(\frac{v_1^{p-1} v_2^p}{p^2 v_1^p} t_1\right) = \frac{-v_2^p t_1|t_1}{pv_1^2} = \frac{1}{2} d\left(\frac{v_2^p t_1^2}{pv_1^2}\right)$$

so the desired cocycle is

$$\frac{v_1^{p-1} v_2^p t_1}{p^2 v_1^p} - \frac{1}{2} \frac{v_2^p t_1^2}{pv_1^2}.$$

This divisibility will be generalized in (5.6.2).

To show that  $\alpha_1\beta_p$  is not divisible by  $p^2$  we compute the mod  $(p)$  reduction of our cocycle. More precisely we compute its image under the connecting homomorphism associated with

$$0 \rightarrow M_1^1 \rightarrow M_0^2 \xrightarrow{p} M_0^2 \rightarrow 0$$

(see 5.1.16). To do this we divide by  $p$  and compute the coboundary. Our divided (by  $p$ ) cocycle is

$$\frac{v_1^{p^2-1} v_2^p t_1}{p^3 v_1^{p^2}} - \frac{1}{2} \frac{v_1^{p-2} v_2^p t_1^2}{p^2 v_1^p}$$

and its coboundary is

$$\frac{v_2^p(t_1^2|t_1 + t_1|t_1^2)}{pv_1^3} + \frac{v_2^{p-1} t_2|t_1}{pv_1} - \frac{1}{2} \frac{v_2^{p-1} t_1^p|t_1^2}{pv_1} - \frac{v_2^{p-1} t_1^{1+p}|t_1}{pv_1}$$

We can eliminate the first term by adding  $\frac{1}{3} \frac{v_2^p t_1^3}{pv_1^3}$  (even if  $p = 3$ ). For  $p > 3$  the resulting element in  $C^2(M_1^1)$  is

$$\frac{v_2^{p-1}(t_2|t_1 - t_1^{1+p}|t_1 - t_1^p|t_2)}{v_1}.$$

Reducing this mod  $I_2$  in a similar fashion gives a unit multiple of  $\bar{\phi}$  in 4.1.14. For  $p = 3$  we add  $\frac{-v_2 t_1^6}{3v_1^3}$  to the divided cocycle and get

$$\frac{v_2^{p-1}(t_2|t_1 - \cdots)}{v_1} + \frac{v_2}{v_1} (t_1^3|t_1^6 + t_1^6|t_1^3),$$

which still gives a nonzero element in  $\text{Ext}^2(M_1^1)$ . □

For  $p = 2$  we need to prove 4.4.38 and 4.4.40, i.e.,

5.1.25. LEMMA. *In the notation of 4.4.32 for  $p = 2$  (a)  $\delta_0(\beta_3) \equiv \beta_{2/2}^2 + \eta_1 \pmod{2}$ ,*

(b)  $\delta_0(\eta_2) \equiv c_0 \pmod{2}$ .

PROOF. For (a) we have

$$d\left(\frac{v_1 v_2^3}{4v_1^2} + \frac{v_2 v_3}{2v_1^2}\right) = \frac{v_2^2 t_1^4}{2v_1^2} + \frac{v_3 t_1^2 + v_2 t_2^2 + v_2^2 t_2 + v_2 t_1^6}{2v_1},$$

which gives the result.

For (b) we use Massey products. We have  $\eta_2 \langle \eta_1, v_1, \beta_1 \rangle$  so by A1.4.11 we have  $\delta_0(\eta_2) \equiv \langle \eta_1, h_{10}, \beta_1 \rangle \pmod{2}$ . Hence we have to equate this product with  $c_0$ , which by 4.4.31 is represented by  $\frac{x_{22}}{v_1}$ , where  $x_{22}$  is defined by 4.4.25. To expedite this calculation we will use a generalization of Massey products not given in A1.4 but fully described by May [3]. We regard  $\eta_1$  as an element in  $\text{Ext}^1(M_1^1)$ , and  $h_{10}$ , and  $\beta_1$  as elements in  $\text{Ext}^1(BP_*/I_1)$  and use the pairing  $M_1^1 \otimes BP_*/I_1 \rightarrow M_1^1$  to define the product. Hence the cocycles representing  $\eta_1$ ,  $h_{10}$  and  $\beta_1$  are

$$\frac{v_3 t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2 t_2}{v_1}, \quad t_1, \quad \text{and} \quad t_1^2 + v_1 t_1,$$

respectively. The cochains whose coboundaries are the two successive products are

$$\frac{v_3(t_2 + t_1^3) + v_2(t_3 + t_1 t_2^2 + t_1^4 t_2 + t_1^7) + v_2^2(t_1^4 + t_1 t_2)}{v_1} \quad \text{and} \quad t_2.$$

If we alter the resulting cochain representative of the Massey product by the coboundary of

$$\frac{v_3 t_1^2 t_2 + v_2(t_2^3 + t_2 t_1^6 + t_1^9) + v_2^2(t_1^6 + t_2^2)}{v_1} + \frac{v_2^4(t_2 + t_1^3)}{v_1^4} + \frac{v_2^5 t_1}{v_1^5}$$

we get the desired result.  $\square$

## 2. $\text{Ext}^1(BP_*/I_n)$ and Hopf Invariant One

In this section we compute  $\text{Ext}^1(BP_*/I_n)$  for all  $n$ . For  $n > 0$  our main results are 5.2.14 and 5.2.17. For  $n = 0$  this group is  $E_2^{1,*}$  in the Adams–Novikov spectral sequence and is given in 5.2.6. In 5.2.8 we will compute its image in the classical Adams spectral sequence, thereby obtaining proofs of the essential content of the Hopf invariant one theorems 1.2.12 and 1.2.14. More precisely, we will prove that the specified  $h_i$ 's are not permanent cycles, but we will not compute  $d_2(h_i)$ . The computation of  $\text{Ext}^1(BP_*/I_n)$  is originally due to Novikov [1] for  $n = 0$  and to Miller and Wilson [3] for  $n > 0$  (except for  $n = 1$  and  $p \geq 2$ ).

To compute  $\text{Ext}^1(BP_*)$  with the chromatic spectral sequence we need to know  $\text{Ext}^1(M^0)$  and  $\text{Ext}^0(M^1)$ . For the former we have

$$5.2.1. \text{ THEOREM. (a) } \text{Ext}^{s,t}(M^0) = \begin{cases} \mathbf{Q} & \text{if } s = t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(b) \text{Ext}^{0,t}(BP_*) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}.$$

PROOF. (a) Since  $M^0 = \mathbf{Q} \otimes BP_*$ , we have  $\text{Ext}(M^0) = \text{Ext}_\Gamma(A, A)$  where  $A = M^0$  and  $\Gamma = \mathbf{Q} \otimes BP_*(BP)$ . Since  $t_n$  is a rational multiple of  $\eta_R(v_n) - v_n$  modulo decomposables,  $\Gamma$  is generated by the image of  $\eta_R$  and  $\eta_L$  and is therefore

a unicusorial Hopf algebroid (A1.1.11). Let  $\bar{v}_n = \eta_R(v_n)$ , so  $\Gamma = A[\bar{v}_1, \bar{v}_2, \dots]$ . The coproduct in  $\Gamma$  is given by  $\Delta(v_n) = v_n \otimes 1$  and  $\Delta(\bar{v}_n) = 1 \otimes \bar{v}_n$ . The map  $\eta_R: A \rightarrow \Gamma = A \otimes_A \Gamma$  makes  $A$  a right  $\Gamma$ -comodule. Let  $R$  be the complex  $\Gamma \otimes E(y_1, y_2, \dots)$  where  $E(y_1, y_2, \dots)$  is an exterior algebra on generators  $y_i$  of degree 1 and dimension  $2(p^i - 1)$ . Let the coboundary  $d$  be a derivation with  $d(y_n) = d(\bar{v}_n) = 0$  and  $d(v_n) = y_n$ . Then  $R$  is easily seen to be acyclic with  $H^0(R) = A$ . Hence  $R$  is a suitable resolution for computing  $\text{Ext}_\Gamma(A, A)$  (A1.2.4). We have  $\text{Hom}_\Gamma(A, R) = A \otimes E(y_1, \dots)$  and this complex is easily seen to be acyclic and gives the indicated Ext groups for  $M^0$ .

For (b)  $\text{Ext}^0 BP_* = \ker d_e \subset \text{Ext}^0(M_0^0)$  and  $d_e(x) \neq 0$  if  $x$  is a unit multiple of a negative power of  $p$ .  $\square$

To get at  $\text{Ext}(M^1)$  we start with

5.2.2. THEOREM.

(a) For  $p > 2$ ,  $\text{Ext}(M_1^0) = K(1)_* \otimes E(h_0)$  where  $h_0 \in \text{Ext}^{1,q}$  is represented by  $t_1$  in  $C^1(M_1^0)$  (see 5.1.12) and  $q = 2p - 2$  as usual.

(b) For  $p = 2$ ,  $\text{Ext}(M_1^0) = K(1)_* \otimes P(h_0) \otimes E(\rho_1)$ , where  $h_0$  is as above and  $\rho_1 \in \text{Ext}^{1,0}$  is represented by  $v_1^{-3}(t_2 - t_1^3) + v_1^{-4}v_2t_1$ .  $\square$

This will be proved below as 6.3.21.

Now we use the method of 5.1.17 to find  $\text{Ext}^0(M^1)$ ; in the next section we will compute all of  $\text{Ext}(M^1)$  in this way. From 4.3.3 we have  $\eta_R(v_1^{sp^i}) \equiv v_1^{p^i} \pmod{(p^{i+1})}$ , so  $\frac{v_1^{sp^i}}{p^{i+1}} \in \text{Ext}^0(M^1)$ . For  $p$  odd we have

$$(5.2.3) \quad \eta_R(v_1^{sp^i}) \equiv v_1^{sp^i} + sp^{i+1}v_1^{sp^i-1}t_1 \pmod{(p^{i+2})}$$

so in 5.1.17 we have

$$\delta\left(\frac{v_1^{sp^i}}{p^{i+1}}\right) = sv_1^{sp^i-1}h_0 \in \text{Ext}^1(M_1^0)$$

for  $p \nmid s$ , and we can read off the structure of  $\text{Ext}^0(M_1^0)$  below.

For  $p = 2$ , 5.2.3 fails for  $i > 0$ , e.g.,

$$\eta_R(v_1^2) = v_1^2 + 4v_1t_1 + 4t_1^2 \pmod{(8)}.$$

The element  $t_1^2 + v_1t_1 \in C^1(M_1^0)$  is the coboundary of  $v_1^{-1}v_2$ , so

$$\alpha_{2/3} = \frac{(v_1^2 + 4v_1^{-1}v_2)}{8} \in \text{Ext}^0(M^1);$$

i.e., we can divide by at least one more power of  $p$  than in the odd primary case. In order to show that further division by 2 is not possible we need to show that  $\alpha_{2/3}$  has a nontrivial image under  $\delta$  (5.1.17). This in turn requires a formula for  $\eta_R(v_2) \pmod{(4)}$ . From 4.3.1 we get

$$(5.2.4) \quad \eta_R(v_2) = v_2 + 13v_1t_1^2 - 3v_1^2t_1 - 14t_2 - 4t_1^3.$$

[This formula, as well as  $\eta_R(v_1) = v_1 - 2t_1$ , are in terms of the  $v_i$  defined by Araki's formula A2.2.2. Using Hazewinkel's generators defined by A2.2.1 gives  $\eta_R(v_1) = v_1 + 2t_1$  and  $\eta_R(v_2) = v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3$ .]

Let  $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$ . Then 5.2.4 gives

$$(5.2.5) \quad \eta_R(x_{1,1}) \equiv x_{1,1} + 8(v_1^{-1}t_2 + v_1^{-1}t_1^3 + v_1^{-2}v_2t_1) \pmod{(16)}$$

so  $\delta(\alpha_{2/3}) = v_1^2\rho_1 \neq 0 \in \text{Ext}^1(M_1^0)$ .

5.2.6. THEOREM.

(a) For  $p$  odd

$$\mathrm{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } q \nmid t \text{ where } q = 2p - 2 \\ \mathbf{Q}/\mathbf{Z}_{(p)} & \text{if } t = 0 \\ \mathbf{Z}/(p^{i+1}) & \text{if } t = sp^i q \text{ and } p \nmid s \end{cases}$$

These groups are generated by

$$\frac{v_1^{sp^i}}{p^{i+1}} \in M^1.$$

(b) For  $p$  odd

$$\mathrm{Ext}^{1,t}(BP_*) = \begin{cases} \mathrm{Ext}^{0,t}(M^1) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

(c) For  $p = 2$

$$\mathrm{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } t \text{ is odd} \\ \mathbf{Q}/\mathbf{Z}_{(2)} & \text{if } t = 0 \\ \mathbf{Z}/(2) & \text{if } t \equiv 2 \pmod{4} \\ \mathbf{Z}/(2^{i+3}) & \text{if } t = 2^{i+2}s \text{ for odd } s \end{cases}$$

These groups are generated by  $\frac{v_1^s}{2}$  and  $\frac{x_{1,1}^{2^i s}}{2^{i+3}} \in M^1$  where  $x_{1,1}$  is as in 5.2.5.

(d) For  $p = 2$

$$\mathrm{Ext}^{1,t}(BP_*) = \begin{cases} 0 & \text{if } t \leq 0 \\ \mathrm{Ext}^{0,t}(M^1) & \text{if } t > 0 \text{ and } t \neq 4 \\ \mathbf{Z}/(4) & \text{if } t = 4 \end{cases}$$

and  $\mathrm{Ext}^{1,4}(BP_*)$  is generated by  $\alpha_{2,2} = \pm \frac{v_1^2}{4}$ .

We will see in the next section (5.3.7) that in the Adams–Novikov spectral sequence for  $p > 2$ , each element of  $\mathrm{Ext}^1(BP_*)$  is a permanent cycle detecting an element in the image of the  $J$ -homomorphism (1.1.13). For  $p = 2$  the generators of  $\mathrm{Ext}^{1,2t}$  are permanent cycles for  $t \equiv 0$  and  $1 \pmod{4}$  while for  $t \equiv 2$  and  $3$  the generators support nontrivial  $d_3$ 's (except when  $t = 2$ ) and the elements of order 4 in  $\mathrm{Ext}^{1,8t+4}$  are permanent cycles. The generators of  $E_4^{1,4t} = E_\infty^{1,4t}$  detect elements in  $\mathrm{im} J$  for all  $t > 0$ .

PROOF OF 5.2.6. Part (a) was sketched above. We get  $\mathbf{Q}/\mathbf{Z}_{(p)}$  in dimension zero because  $1/p^i$  is a cocycle for all  $i > 0$ . For (b) the chromatic spectral sequence gives a short exact sequence

$$0 \rightarrow E_\infty^{1,0} \rightarrow \mathrm{Ext}^1(BP_*) \rightarrow E_\infty^{0,1} \rightarrow 0$$

and  $E_\infty^{0,1}$  by 5.2.1.  $E_\infty^{1,0} = E_2^{1,0} = \ker d_e / \mathrm{im} d_e$ . An element in  $E_1^{1,0} = \mathrm{Ext}^0(M^1)$  has a nontrivial image under  $d_e$  iff it has terms involving negative powers of  $v_1$ , so  $\ker d_e \subset E_1^{1,0}$  is the subgroup of elements in nonnegative dimensions. The zero-dimensional summand  $\mathbf{Q}/\mathbf{Z}_{(p)}$  is the image of  $d_e$ , so  $E_2^{1,0} = \mathrm{Ext}^1(BP_*)$  is as stated.

For (c) the computation of  $\mathrm{Ext}^0(M_1^0)$  is more complicated for  $p = 2$  since 5.2.3 no longer holds. From 5.2.5 we get

$$(5.2.7) \quad \eta_R(x_{1,1}^{2^i s}) \equiv x_{1,1}^{2^i s} + 2^{i+3} x_{1,1}^{2^i s-1} (v_1^{-1} t_2 + v_1^{-1} t_1^3 + v_1^{-2} v_2 t_1) \pmod{(2^{i+4})}$$

for odd  $s$ , from which we deduce that  $\frac{x_{1,1}^{2^i s}}{2^{i+3}}$  is a cocycle whose image under  $\delta$  (see 5.1.17) is  $v_1^{2^{i+1}s} \rho_1$ . Equation 5.2.3 does hold for  $p = 2$  when  $i = 0$ , so  $\text{Ext}^{0,2s}(M_1^0)$  is generated by  $\frac{v_1^s}{2}$  for odd  $s$ . This completes the proof of (c).

For (d) we proceed as in (b) and the situation in nonpositive dimensions is the same. We need to compute  $d_e\left(\frac{x_{1,1}^{2^i s}}{2^{i+3}}\right)$ . Since  $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$ , we have

$$\frac{x_{1,1}^{2^i s}}{2^{i+3}} = \frac{v_1^{2^i s} + s - 2^{i+2}v_1^{2^i s-3}v_2}{2^{i+3}}.$$

For  $2^i s = 1$  (but for no  $2^i s > 1$ ) this expression has a negative power of  $v_1$  and we get

$$d_e\left(\frac{x_{1,1}}{8}\right) = \frac{v_2}{2v_1} \in M^2.$$

This gives a chromatic  $d_1$  (compare 5.1.21) and accounts for the discrepancy between  $\text{Ext}^{0,4}(M^1)$  and  $\text{Ext}^{1,4}(BP_*)$ .  $\square$

Now we turn to the Hopf invariant one problem. Theorems 1.2.12 and 1.2.14 say which elements of filtration 1 in the classical Adams spectral sequence are permanent cycles. We can derive these results from our computation of  $\text{Ext}^1(BP_*)$  as follows. The map  $BP \rightarrow H/(p)$  induces a map  $\Phi$  from the Adams–Novikov spectral sequence to the Adams spectral sequence. Since both spectral sequences converge to the same thing there is essentially a one-to-one correspondence between their  $E_\infty$ -terms. A nontrivial permanent cycle in the Adams spectral sequence of filtration  $s$  corresponds to one in the Adams–Novikov spectral sequence of filtration  $\leq s$ .

To see this consider  $BP_*$  and mod  $(p)$  Adams resolutions (2.2.1 and 2.1.3)

$$\begin{array}{ccccccc} S^0 & \longleftarrow & X_0 & \longleftarrow & X_1 & \longleftarrow & \dots \\ & & \downarrow & & \downarrow & & \\ S^0 & \longleftarrow & Y_0 & \longleftarrow & Y_1 & \longleftarrow & \dots \end{array}$$

where the vertical maps are the ones inducing  $\Phi$ . An element  $x \in \pi_*(S^0)$  has Adams filtration  $s$  if it is in  $\text{im } \pi_*(Y_s)$  but not in  $\text{im } \pi_*(Y_{s+1})$ . Hence it is not in  $\text{im } \pi_*(X_{s+1})$  and its Novikov filtration is at most  $s$ .

We are concerned with permanent cycles with Adams filtration 1 and hence of Novikov filtration 0 or 1. Since  $\text{Ext}^0(BP_*)$  is trivial in positive dimensions [5.2.1(b)] it suffices to prove

5.2.8. THEOREM. *The image of*

$$\Phi: \text{Ext}^1(BP_*) \rightarrow \text{Ext}_{A_*}^1(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

*is generated by  $h_1, h_2$ , and  $h_3$ , for  $p = 2$  and by  $h_0 \in \text{Ext}^{1,q}$  for  $p > 2$ . (These elements are permanent cycles; cf. 1.2.11 and 1.2.13.)*

PROOF. Recall that  $A_* = \mathbf{Z}/(p)[t_1, t_2, \dots] \otimes E(e_0, e_1, \dots)$  with

$$\Delta(t_n) = \sum_{0 \leq i \leq n} t_i \otimes t_{n-i}^{p^i} \quad \text{and} \quad \Delta(e_n) = 1 \otimes e_n + \sum_{1 \leq i \leq n} e_i \otimes t_{n-i}^{p^i}$$

where  $t_0 = 1$ . Here  $t_n$  and  $e_n$  are the conjugates of Milnor’s  $\xi_n$  and  $\tau_n$  (3.1.1). The map  $BP_*(BP) \rightarrow A_*$  sends  $t_n \in BP_*(BP)$  to  $t_n \in A_*$ .

Now recall the  $I$ -adic filtration of 4.4.4. We can extend it to the comodules  $M^n$  and  $N^n$  by saying that a monomial fraction  $\frac{v^j}{v^i}$  is in  $F^k$  iff the sum of the exponents in the numerator exceeds that for the denominator by at least  $k$ . (This  $k$  may be negative and there is no  $k$  such that  $F^k M^n = M^n$  or  $F^k N^n = N^n$ . However, there is such a  $k$  for any finitely generated subcomodule of  $M^n$  or  $N^n$ .) For each  $k \in \mathbf{Z}$  the sequence

$$0 \rightarrow F^k N^n \rightarrow F^k M^n \rightarrow F^k N^{n+1} \rightarrow 0$$

is exact. It follows that  $\alpha: \text{Ext}^s(N^n) \rightarrow \text{Ext}^{s+n}(BP_*)$  (5.1.18) preserves the  $I$ -adic filtration and that if  $x \in F^1 \text{Ext}^0(N^1)$  then  $\Phi\alpha(x) = 0$ .

Easy inspection of 5.2.6 shows that the only elements in  $\text{Ext}^0(M^1)$  not in  $F^1$  are  $\alpha_1$  and, for  $p = 2$ ,  $\alpha_{2/2}$ , and  $\alpha_{4/4}$ , and the result follows.  $\square$

Now we turn to the computation of  $\text{Ext}^1(BP_*/I_n)$  for  $n > 0$ ; it is a module over  $\text{Ext}^0(BP_*/I_n)$  which is  $\mathbf{Z}/(p)[v_n]$  by 4.3.2. We denote this ring by  $k(n)_*$ . It is a principal ideal domain and  $\text{Ext}^1(BP_*/I_n)$  has finite type so the latter is a direct sum of cyclic modules, i.e., of free modules and modules of the form  $k(n)_*/(v_n^i)$  for various  $i > 0$ . We call these the  $v_n$ -torsion free and  $v_n$ -torsion summands, respectively. The rank of the former is obtained by inverting  $v_n$ , i.e., by computing  $\text{Ext}^1(M_n^0)$ . The submodule of the  $v_n$ -torsion which is annihilated by  $v_n$  is precisely the image of  $\text{Ext}^0(BP_*/I_{n+1}) = k(n+1)_*$  under the connecting homomorphism for the short exact sequence

$$(5.2.9) \quad 0 \rightarrow \Sigma^{\dim v_n} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \rightarrow BP_*/I_{n+1} \rightarrow 0.$$

We could take these elements in  $\text{Ext}^1(BP_*/I_n)$  and see how far they can be divided by  $v_n$  by analyzing the long exact sequence for 5.2.9, assuming we know enough about  $\text{Ext}^1(BP_*/I_{n+1})$  to recognize nontrivial images of elements of  $\text{Ext}^1(BP_*/I_n)$  when we see them. This approach was taken by Miller and Wilson [3].

The chromatic spectral sequence approach is superficially different but one ends up having to make the same calculation either way. From the chromatic spectral sequence for  $\text{Ext}(BP_*/I_n)$  (5.1.11) we get a short exact sequence

$$(5.2.10) \quad 0 \rightarrow E_\infty^{1,0} \rightarrow \text{Ext}^1(BP_*/I_n) \rightarrow E_\infty^{0,1} \rightarrow 0,$$

where  $E_\infty^{1,0} = E_2^{1,0}$  is a subquotient of  $\text{Ext}^0(M_{n+1}^1)$  and is the  $v_n$ -torsion summand, while  $E_\infty^{0,1} = E_3^{0,1} \text{Ext}^1(M_n^0)$  is the  $v_n$ -torsion free quotient. To get at  $\text{Ext}^0(M_{n+1}^1)$  we study the long exact sequence for the short exact sequence

$$0 \rightarrow M_{n+1}^0 \xrightarrow{j} \Sigma^{\dim v_n} M_n^1 \xrightarrow{v_n} M_n^1 \rightarrow 0$$

as in 5.1.17; this requires knowledge of  $\text{Ext}^0(M_{n+1}^0)$  and  $\text{Ext}^1(M_{n+1}^0)$ . To determine the subgroup  $E_\infty^{0,1}$  of  $\text{Ext}^1(M_n^0)$  we need the explicit representatives of generators of the latter constructed by Moreira [1, 3].

The following result (to be proved later as 6.3.12) then is relevant to both  $E_\infty^{0,1}$  and  $E_\infty^{1,0}$  in 5.2.10.

5.2.11. THEOREM.  $\text{Ext}^1(M_n^0)$  for  $n > 0$  is the  $K(n)_*$ -vector space generated by  $h_i \in \text{Ext}^{1,p^i}$  for  $0 \leq i \leq n-1$  represented by  $t_1^{p^i}$ ,  $\zeta_n \in \text{Ext}^{1,0}$  (for  $n \geq 2$ ) represented for  $n = 2$  by  $v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-1-p}v_3t_1^p$ , and (if  $p = 2$  and  $n \geq 1$ )  $\rho_n \in \text{Ext}^{1,0}$ . ( $\zeta_n$  and  $\rho_n$  will be defined in 6.3.11).  $\square$

5.2.12. REMARK. For  $i \geq n$ ,  $h_i$  does not appear in this list because the equation

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \pmod{I_n}$$

leads to a cohomology between  $h_{n+i}$  and  $v_n^{(p-1)p^i} h_i$ .

Now we will describe  $\text{Ext}^0(M_n^1)$  and  $E_\infty^{1,0}$ . The groups are  $v_n$ -torsion modules. The submodule of the former annihilated by  $v_n$  is generated by  $\left\{ \frac{v_{n+1}^t}{v_n} : t \in \mathbf{Z} \right\}$ . Only those elements with  $t > 0$  will appear in  $E_\infty^{1,0}$ ; if  $t = 0$  the element is in  $\text{im } d_1$ , and  $\ker d_1$  is generated by those elements with  $t \geq 0$ . We need to see how many times we can divide by  $v_n$  and (still have a cocycle). An easy calculation shows that if  $t = sp^i$  with  $p \nmid s$ , then  $\frac{v_{n+1}^t}{v_n^{p^i}}$  is a cocycle whose image in  $\text{Ext}^1(M_{n+1}^0)$  is  $sv_{n+1}^{(s-1)p^i} h_{n+i}$ , but by 5.2.12 these are not linearly independent, so this is not the best possible divisibility result. For example, for  $n = 1$  we find that

$$\frac{v_2^{p^2}}{v_1^{1+p^2}} - \frac{v_2^{p^2-p+1}}{v_1^2} - \frac{v_2^{-p} v_3^p}{v_1}$$

is a cocycle.

The general result is this.

5.2.13. THEOREM. As a  $k(n)_*$ -module,  $\text{Ext}^0(M_n^1)$  is the direct sum of

- (i) the cyclic submodules generated by  $\frac{x_{n+1,i}^s}{v_{n+1,i}}$  for  $i \geq 0$ ,  $p \nmid s$ ; and
- (ii)  $K(n)_*/k(n)_*$ , generated by  $\frac{1}{v_j}$  for  $j \geq 1$ .

The  $x_{n,i}$  are defined as follows.

$$\begin{aligned} x_{1,0} &= v_1, \\ x_{1,1} &= v_1^p \quad \text{if } p > 2 \quad \text{and} \quad v_1^2 + 4v_1^{-1}v_2 \quad \text{if } p = 2, \\ x_{1,i} &= x_{1,i-1}^p \quad \text{for } i \geq 2, \\ x_{2,0} &= v_2, \\ x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3, \\ x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \\ x_{2,i} &= x_{2,i-1}^2 \quad \text{for } i \geq 3 \quad \text{if } p = 2 \end{aligned}$$

and

$$x_{2,i-1}^p - 2v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} \quad \text{for } i \geq 3 \quad \text{if } p > 2,$$

where

$$\begin{aligned} b_{2,i} &= (p+1)(p^{i-1}-1), \\ x_{n,0} &= v_n \quad \text{for } n > 2, \\ x_{n,1} &= v_n^p - v_{n-1}^p v_n^{-1} v_{n+1}, \\ x_{n,i} &= x_{n,i-1}^p \quad \text{for } i > 1 \quad \text{and} \quad i \not\equiv 1 \pmod{n-1}, \\ x_{n,i} &= x_{n,i-1}^p - v_{n-1}^{b_{n,i}} v_n^{p^i - p^{i-1} + 1} \quad \text{for } i > 1, \quad \text{and} \quad i \equiv 1 \pmod{n-1} \end{aligned}$$

where

$$b_{n,i} = \frac{(p^{i-1}-1)(p^n-1)}{p^{n-1}-1} \quad \text{for } i \equiv 1 \pmod{n-1}.$$

The  $a_{n,i}$  are defined by

$$\begin{aligned}
a_{1,0} &= 1 \\
a_{1,i} &= i + 2 && \text{for } p = 2 \quad \text{and } i \geq 1, \\
a_{i,1} &= i + 1 && \text{for } p > 2 \quad \text{and } i \geq 1, \\
a_{2,0} &= 1, \\
a_{2,i} &= p^i + p^{i-1} - 1 && \text{for } p > 2 \quad \text{and } i \geq 1 \quad \text{or } p = 2 \quad \text{and } i = 1, \\
a_{2,i} &= 3 \cdot 2^{i-1} && \text{for } p = 2 \quad \text{and } i > 1, \\
a_{n,0} &= 1 && \text{for } n > 2, \\
a_{n,1} &= p, \\
a_{n,i} &= pa_{n,i-1} && \text{for } i > 1 \quad \text{and } i \not\equiv 1 \pmod{n-1},
\end{aligned}$$

and

$$a_{n,i} = pa_{n,i} + p - 1 \quad \text{for } i > 1 \quad \text{and } i \equiv 1 \pmod{n-1}.$$

□

This is Theorem 5.10 of Miller, Ravenel, and Wilson [1], to which we refer the reader for the proof.

Now we need to compute the subquotient  $E_2^{1,0}$  of  $\text{Ext}^0(M_n^1)$ . It is clear that the summand of (ii) above is in the image of  $d_1$  and that  $\ker d_1$  is generated by elements of the form  $\frac{x_{n+1,i}^s}{v_1^j}$  for  $s \geq 0$ . Certain of these elements for  $s > 0$  are not in  $\ker d_1$ ; e.g., we saw in 5.2.6 that  $d_1\left(\frac{x_{1,1}}{8}\right) \neq 0$ . More generally we find  $d_1\left(\frac{x_{n+1,i,s}}{v_n^j}\right) \neq 0$  iff  $s = 1$  and  $p^i < j \leq a_{n+1,i}$  (see Miller and Wilson [3]), so we have

5.2.14. COROLLARY. *The  $v_n$ -torsion summand of  $\text{Ext}^1(BP_*/I_n)$  is generated by the elements listed in 5.2.13(i) for  $s > 0$  with (when  $s = 1$ )  $\frac{x_{n+1,i}}{v_{n+1,i}}$  replaced by  $\frac{x_{n+1,i}}{v_n^{p^i}}$ .* □

Now we consider the  $k(n)_*$ -free summand  $E_\infty^{0,1} \subset \text{Ext}^1(M_n^0)$ . We assume  $n > 1$  ( $n = 1$  is the subject of 5.2.2); 5.2.11 tells us that  $E_\infty^{0,1}$  has rank  $n + 1$  for  $p > 2$  and  $n + 2$  for  $p = 2$ . We need to determine the image of  $\text{Ext}^1(BP_*/I_n)$  in  $\text{Ext}^1(M_n^0)$ . To show that an element in the former is not divisible by  $v_n$  we must show that it has a nontrivial image in  $\text{Ext}^1(BP_*/I_{n+1})$ . The elements  $h_i \in \text{Ext}^1(M_n^0)$  clearly are in the image of  $\text{Ext}^1(BP_*/I_n)$  and have nontrivial images in  $\text{Ext}^1(BP_*/I_{n+1})$ . The elements  $\zeta_n$  and  $\rho_n$  are more complicated. The formula given in 5.2.11 for  $\zeta_2$  shows that  $v_2^{1+p}\zeta_2$  pulls back to  $\text{Ext}^1(BP_*/I_2)$  and projects to  $v_3h_1 \in \text{Ext}^1(BP_*/I_3)$ . This element figures in the proof of 5.2.13 and in the computation of  $\text{Ext}^2(BP_*)$  to be described in Section 4.

The formula of Moreira [1] for a representative of  $\zeta_n$  is

$$(5.2.15) \quad T_n = \sum_{1 \leq i \leq j \leq k \leq n} u_{2n-k}^p t_j^{p^{n-i}} c(t_{k-j})^{p^{n-i+j}}$$

where the  $u_{n+i} \in M_n^0$  are defined by

$$(5.2.16) \quad u_n = v_n^{-1} \quad \text{and} \quad \sum_{0 \leq i \leq k} u_{n+i} v_{n+k-i}^{p^i} = 0 \quad \text{for } k > 0.$$



One sees from 5.2.16 that  $u_{n+i-1}v_n^{(p^i-1)/(p-1)} \in BP_*/I_n$  so  $\hat{T}_n = v_n^{(p^n-1)/(p-1)}T_n \in BP_*(BP)/I_n$ . In 5.2.15 the largest power of  $v_n^{-1}$  occurs in the term with  $i = j = k = 1$ ; in  $T_n$  this term is  $v_n^{(p^n-1)/(p-1)}u_{2n-1}t_1^{p^{n-1}}$  and its image in  $\text{Ext}^1(BP_*/I_{n+1})$  is  $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$ .

The formula of Moreira [3] for a representative  $U_n$  of  $\rho_n$  is very complicated and we will not reproduce it. From it one sees that  $v_n^{2^{2n-1}+2^{n-1}}U_n \in BP_*(BP)/I_n$  reduces to  $v_{n+1}^{2^{2n-1}-1}t_1^{2^{n-1}} \in BP_*(BP)/I_{n+1}$ .

Combining these results gives

5.3.17. THEOREM. *The  $k(n)_*$ -free quotient  $E_\infty^{0,1}$  of  $\text{Ext}^1(BP_*/I_n)$  for  $n \geq 1$  is generated by  $h_i \in \text{Ext}^{1,p^i q}$  for  $0 \leq i \leq n-1$ ,  $\hat{\zeta} = v_n^{(p^n-1)/(p-1)}\zeta_n$ , and (for  $p = 2$ )  $\hat{\rho}_n = v_n^{2^{2n}+2^{n-1}-1}\rho_n$ . The images of  $\hat{\zeta}_n$  and  $\hat{\rho}_n$  in  $\text{Ext}^1(BP_*/I_{n+1})$  are  $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$  and  $v_{n+1}^{2^{2n-1}-1}h_{n-1}$ , respectively.  $\square$*

### 3. $\text{Ext}(M^1)$ and the $J$ -Homomorphism

In this section we complete the calculation of  $\text{Ext}(M^1)$  begun with 5.2.6 and describe the behavior of the resulting elements in the chromatic spectral sequence and then in the Adams–Novikov spectral sequence. Then we will show that the elements in  $\text{Ext}^1(BP_*)$  (and, for  $p = 2$ ,  $\text{Ext}^2$  and  $\text{Ext}^3$ ) detect the image of the homomorphism  $J: \pi_*(SO) \rightarrow \pi_*^S$  (1.1.12). This proof will include a discussion of Bernoulli numbers. Then we will compare these elements in the Adams–Novikov spectral sequence with corresponding elements in the Adams spectral sequence.

We use the method of 5.1.17 to compute  $\text{Ext}(M^1)$ ; i.e., we study the long exact sequence of Ext groups for

$$(5.3.1) \quad 0 \rightarrow M_1^0 \xrightarrow{j} M^1 \xrightarrow{p} M^1 \rightarrow 0.$$

$\text{Ext}(M_1^0)$  is described in 5.2.6 and the computation of  $\text{Ext}^0(M^1)$  is given in 5.2.6. Let  $\delta$  be the connecting homomorphism for 5.3.1. Then from the proof of 5.2.6 we have

5.3.2. COROLLARY. *The image of  $\delta$  in  $\text{Ext}^1(M_1^0)$  is generated by (a)  $v_1^t h_0$  for all  $t \neq 1$  when  $p$  is odd and*

(b)  $v_1^t h_0$  for all even  $t$  and  $v_1^t \rho_1$  for all  $t \neq 0$  when  $p = 2$ .  $\square$

For odd primes this result alone determines all of  $\text{Ext}(M^1)$ .  $\text{Ext}^s(M_1^0) = 0$  for  $s > 1$  and there is only one basis element of  $\text{Ext}^1(M_1^0)$  not in  $\text{im } \delta$ , namely  $v_1^{-1}h_0$ . Its image under  $j$  is represented by  $\frac{v_1^{-1}t_1}{p}$ . Since  $\text{Ext}^2(M_1^0) = 0$ , there is no obstruction to dividing  $j(v_1^{-1}h_0)$  by any power of  $p$ , so we have

$$(5.3.3) \quad \text{Ext}^{1,t}(M^1) = \begin{cases} \mathbf{Q}/\mathbf{Z}_{(p)} & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

for any odd prime  $p$ . We can construct a representative of an element of order  $p^k$  in  $\text{Ext}^{1,0}(M^1)$  as follows. From 4.3.1 we have  $\eta_R(v_1) = v_1 = put_1$  where  $u = 1 - p^{p-1}$ . Then a simple calculation shows that

$$(5.3.4) \quad y_k = - \sum_{i \geq 1} (-1)^i \frac{v_1^{-i} u^i t_1^i}{i p^{k+1-i}}$$

is the desired cocycle. (This sum is finite although the  $i$ th term for some  $i > k$  could be nonzero if  $p \mid i$ .) The group  $\text{Ext}^{1,0}(M^1) + E_1^{1,1,0}$  cannot survive in the chromatic spectral sequence because it would give a nontrivial  $\text{Ext}^{2,0}(BP_*)$  contradicting the edge theorem, 5.1.23. It can be shown (lemma 8.10 of Miller, Ravenel, and Wilson [1]) that this group in fact supports a  $d_1$  with trivial kernel. Hence we have

5.3.5. THEOREM.

(a) for  $p > 2$  the group  $\text{Ext}^{s,t}(M^1)$  is

$$\begin{aligned} &\mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by } \frac{1}{p^k} \text{ for } (s,t) = (0,0). \\ &\mathbf{Z}/(p^{i+1}) \text{ generated by } \frac{v_1^{rp^i}}{p^{i+1}} \text{ for } p \nmid r \text{ and } (s,t) = (0, rp^i q), \\ &\mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by } y_k \text{ (5.3.4) for } (s,t) = (1,0) \text{ and} \\ &0 \text{ otherwise.} \end{aligned}$$

(b) In the chromatic spectral sequence, where  $\text{Ext}^{s,t}(M^1) = E_1^{1,s,t} E_1^{1,0,0} \subset \text{im } d_1$  and  $\ker d_1 \bigoplus_{t \geq 0} E_1^{1,0,t}$ , so  $E_\infty^{1,*} = \text{Ext}^1(BP_*)$  and  $\ker d_1 = \bigoplus_{t \geq 0} E_1^{1,0,t}$ , so  $E_\infty^{1,*} = \text{Ext}^1(BP_*)$  is generated by the groups  $\text{Ext}^{0,t}(M^1)$  for  $t > 0$ .  $\square$

We will see below that each generator of  $\text{Ext}^1(BP_*)$  for  $p > 2$  is a permanent cycle in the Adams–Novikov spectral sequence detecting an element in the image of  $J$  (1.1.12).

The situation for  $p = 2$  is more complicated because  $\text{Ext}(M_1^0)$  has a polynomial factor not present for odd primes. We use 5.3.2 and 5.2.2 to compute  $\text{Ext}^s(M^1)$  for  $s > 1$ . The elements of order 2 in  $\text{Ext}^{1,0}(M^1)$  are the images under  $j$  (5.3.1) of  $v_1^t h_0$  for  $t$  odd and  $v_1^t \rho_1$  for  $t$  even and  $t = 0$ .

We claim  $j(\rho_1)$  is divisible by any power of 2, so  $\text{Ext}^{1,0}(M^1)$  contains a summand isomorphic to  $\mathbf{Q}/\mathbf{Z}_{(2)}$  as in the odd primary case. To see this use 5.2.4 to compute

$$d\left(\frac{v_1^{-3}v_2 + 2v_1^{-6}v_2^2}{4}\right) = \frac{v_1^{-1}t_1 + v_1^{-2}t_1^2 + 2v_1^{-4}t_1^4}{4} + \frac{v_1^{-4}(v_2t_1 + v_1t_1^3 + v_1t_2)}{2},$$

showing that  $y_2$  (5.3.4) represents  $j(\rho_1)$ ; the same calculation shows that

$$y_1 = \frac{v_1^{-1}t_1 + v_1^{-2}t_1^2}{2}$$

is a coboundary. Hence the  $y_k$  for  $k \geq 2$  give us the cocycles we need.

Next we have to deal with  $j(v_1^t h_0)$  and  $j(v_1^t \rho_1)$  for odd  $t$ . These are not divisible by 2 since an easy calculation gives  $\delta_j(v_1^t x) = v_1^{t-1} h_0 x$  for  $t$  odd and  $x = h_0^{i+1}$  or  $h_0^i \rho_1$  for any  $i \geq 0$ . Indeed this takes care of all the remaining elements in the short exact sequence for 5.3.1 and we get

5.3.6. THEOREM.

(a) For  $p = 2$ ,  $\text{Ext}^{s,t}(M^1)$  is

$$\left\{ \begin{array}{l} \mathbf{Q}/\mathbf{Z}_{(2)} \text{ generated by } \frac{1}{2^j} \\ \quad \text{for } (s, t) = (0, 0), \\ \mathbf{Z}/(2) \text{ generated by } \frac{v_1^r}{2} \\ \quad \text{for } (s, t) = (0, 2r) \text{ and } r \text{ odd}, \\ \mathbf{Z}/(2^{i+3}) \text{ generated by } \frac{x_{1,1}^{r2^i}}{2^{i+3}} \\ \quad \text{for } (s, t) = (0, r2^{i+2}) \text{ and } r \text{ odd}, \\ \mathbf{Q}/\mathbf{Z}_{(2)} \oplus \mathbf{Z}/(2) \text{ generated by } y_k \ (k \geq 2) \text{ and } \frac{v_1^{-1}t_1}{2} \\ \quad \text{for } (s, t) = (1, 0), \\ \mathbf{Z}/(2) \text{ generated by } j(v_1^r h_0^s) \\ \quad \text{for } s > 0, t = 2(r + s), r \text{ odd, and } (s, t) \neq (1, 0) \\ \mathbf{Z}/(2) \text{ generated by } j(v_1^r \rho_1 h_0^{s-1}) \\ \quad \text{for } s > 0, t + 2(r + s - 1), \text{ and } r \text{ odd,} \\ \quad \text{and} \\ 0 \quad \text{otherwise.} \end{array} \right.$$

(b) In the chromatic spectral sequence for  $p = 2$ ,  $E_\infty^{1,2,t}$  is

$$\left\{ \begin{array}{l} \text{Ext}^{s,t}(M^1) \quad \text{for } t = 2s + 2r \text{ and } r \geq 1, r \neq 2, \\ \mathbf{Z}/(4) \text{ generated by } \frac{v_1^2}{4} \quad \text{for } (s, t) = (0, 4), \text{ and } \mu_{2t-1} \in \pi_{8t+1}^S, \\ \quad \text{and} \\ 0 \quad \text{otherwise} \end{array} \right.$$

(See 5.1.22 for a description of differentials originating in  $E_r^{1,s,2s+4}$ .) In other words the subquotient of  $\text{Ext}(BP_*)$  corresponding to  $E_\infty^{1,*}$  is generated by  $\text{Ext}^1(BP_*)$  (5.2.6) and products of its generators (excluding  $\alpha_{2/2} \in \text{Ext}^{1,4}$ ) with all positive powers of  $\alpha_1 \in \text{Ext}^{1,2}$ .

PROOF. Part (a) was proved above. For (b) the elements said to survive, i.e., those in  $E_1^{1,0}$  and  $j(v_1^r \rho_1 h_0^{s-1})$  for  $s > 0$  with odd  $r \geq 5$  and  $j(v_1^r h_0^s)$  for  $s > 0$  with odd  $r \geq 1$ , are readily seen to be permanent cycles. The other elements in  $E_1^{1,s}$  for  $s > 0$  have to support nontrivial differentials by the edge theorem, 5.1.23.  $\square$

Now we describe the behavior of the elements of 5.3.5(b) and 5.3.6(b) in the Adams–Novikov spectral sequence. The result is

### 5.3.7. THEOREM.

(a) For  $p > 2$ , each element in  $\text{Ext}^1(BP_*)$  is a permanent cycle in the Adams–Novikov spectral sequence represented by an element of  $\text{im } J$  (1.1.13) having the same order.

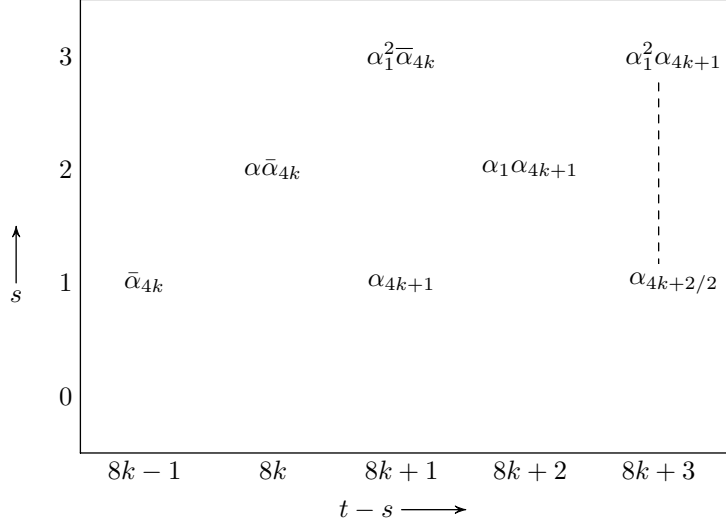
(b) For  $p = 2$  the behavior of  $\text{Ext}^{1,2t}(BP_*)$  in the Adams–Novikov spectral sequence depends on the residue of  $t \bmod (4)$  as follows. If  $t \equiv 1 \pmod{4}$  the generator  $\alpha_t$  is a permanent cycle represented by the element  $\mu_{2t-1} \in \pi_{8t-1}^S$  of order 2 constructed by Adams [1]. In particular  $\alpha_1$  is represented by  $\eta$  (1.1.13).  $\alpha_1 \alpha_t$  is represented by  $\mu_{2t} = \eta \mu_{2t-1}$  and  $\alpha_1^2 \alpha_t$  is represented by an element of order 2 in  $\text{im } J \subset \pi_{2t+1}^S$  (the order of this group is an odd multiple of 8).  $\alpha^{s+3} \alpha_t = d_3(\alpha_1^s \alpha_{t+2})$  for all  $s \geq 0$ .

If  $t \equiv 0 \pmod{4}$  then the generator  $\bar{\alpha}_t$  of  $\text{Ext}^{1,2t}(BP_*)$  is a permanent cycle represented by an element of  $\text{im } J$  having the same order, as are  $\alpha_1 \bar{\alpha}_t$ , and  $\alpha_1^2 \bar{\alpha}_t$ .  $\alpha_1^{s+3} \bar{\alpha}_t = d_3(\alpha_1^s \alpha_{t+2/3})$  for  $s \geq 0$ . In particular  $y_4$  is represented by  $\sigma \in \pi_7^S$  (1.1.13).

If  $t \equiv 2 \pmod{4}$ ,  $\alpha_{t/2}$  (twice the generator except when  $t = 2$ ) is a permanent cycle represented by an element in  $\text{im } J$  of order 8. ( $\alpha_{t/2}$  has order 4 and 4 times the generator of  $\text{im } J$  represents  $\alpha_1^2 \alpha_{t-2}$  as remarked above). In particular  $\alpha_{2/2}$  is represented by  $\nu \in \pi_3^S$  (1.1.13).  $\square$

This result says that the following pattern occurs for  $p = 2$  in the Adams–Novikov spectral sequence  $E_\infty$ -term as a direct summand for all  $k > 0$ :

(5.3.8)



Where all elements have order 2 except  $\alpha_{4k+2/2}$ , which has order 4, and  $\bar{\alpha}_{4k}$ , whose order is the largest power of 2 dividing  $16k$ ; the broken vertical line indicates a nontrivial group extension. The image of  $J$  represents all elements shown except  $\alpha_{4k+1}$  and  $\alpha_1 \alpha_{4k+1}$ .

Our proof of 5.3.7 will be incomplete in that we will not prove that  $\text{im } J$  actually has the indicated order. This is done up to a factor of 2 by [1] Adams [1], where it is shown that the ambiguity can be removed by proving the Adams conjecture, which was settled by Quillen [1] and Sullivan [1].

We will actually use the complex  $J$ -homomorphism  $J: \pi_*(U) \rightarrow \pi_*^S$ , where  $U$  is the unitary group. Its image is known to coincide up to a factor of 2 with that of the real  $J$ -homomorphism. We will comment more precisely on the difference between them in due course.

An element  $x \in \pi_{2t-1}(U)$  corresponds to a stable complex vector bundle  $\xi$  over  $S^{2t}$ . Its Thom spectrum  $T(\xi)$  is a 2-cell CW-spectrum  $S^0 \cup e^{2t}$  with attaching map  $J(x)$  and there is a canonical map  $T(\xi) \rightarrow MU$ . We compose it with the standard map  $MU \rightarrow BP$  and get a commutative diagram

(5.3.9)

$$\begin{array}{ccccc}
 S^0 & \longrightarrow & T(\xi) & \longrightarrow & S^{2t} \\
 \downarrow & & \downarrow & & \downarrow \\
 S^0_{(p)} & \longrightarrow & BP & \longrightarrow & \overline{BP} \\
 & & & & \downarrow \\
 & & & & BP \wedge \overline{BP}
 \end{array}$$

where the two rows are cofibre sequences. The map  $S^{2t} \rightarrow \overline{BP}$  is not unique but we do get a unique element  $e(x) \in \pi_{2t}(BP \wedge \overline{BP})/\text{im } \pi_{2t}(BP)$ . Now  $E_2^{1,2t}$  of the Adams–Novikov spectral sequence is by definition a certain subgroup of this quotient containing  $e(x)$ , so we regard the latter as an element in  $\text{Ext}^{1,2t}(BP_*)$ . Alternatively, the top row in 5.3.9 gives a short exact sequence of comodules which is the extension corresponding to  $e(x)$ . We need to show that if  $x$  generates  $\pi_{2t-1}(U)$  then  $e(x)$  generates  $\text{Ext}^1(BP_*)$  up to a factor of 2.

For a generator  $x_t$  of  $\pi_{2t-1}(U)$  we obtain a lower bound on the order of  $e(x)$  as follows. If  $je(x_t) = 0$  for some integer  $j$  then for the bundle given by  $x = jx_t \in \pi_{2t-1}(U)$  the map  $S^{2t} \rightarrow \overline{BP}$  in 5.3.9 lifts to  $BP$ , so we get an element in  $\pi_{2t}(BP)$ . Now consider the following diagram

$$(5.3.10) \quad \begin{array}{ccc} \pi_*(BU) & & \pi_*(MU) \xrightarrow{\theta} \mathbf{Z} \\ \downarrow & & \downarrow \\ H_*(BU) & \cong & H_*(MU) \longrightarrow \mathbf{Q} \end{array}$$

where the two left-hand vertical maps are the Hurewicz homomorphisms and  $\theta$  is some ring homomorphism; it extends as indicated since  $\pi_*(MU) \otimes \mathbf{Q} \cong H_*(MU) \otimes \mathbf{Q}$  by 3.1.5. Let  $\phi$  be the composite map (not a ring homomorphism) from  $\pi_*(BU)$  to  $\mathbf{Q}$ . If  $\phi(x_t)$  has denominator  $j_t$ , then  $j_t$  divides the order of  $e(x_t)$ .

According to Bott [2] the image of  $x_t$  in  $H_{2t}(BU)$  is  $(t-1)!s_t$  where  $s_t$  is a primitive generator of  $H_{2t}(BU)$ . By Newton’s formula

$$s(z) = \frac{z}{b(z)} \frac{db(z)}{dz},$$

where  $s(z) = \sum_{t \geq 0} s_t z^t$  and  $b(z) = \sum_{t \geq 0} b_t z^t$ , the  $b_t$  being the multiplicative generators of  $H_*(BU) \cong H_*(MU)$  (3.1.4).

Now by Quillen’s theorem, 4.1.6,  $\theta$  defines a formal group law over  $\mathbf{Z}$  (see Appendix 2), and by 4.1.11

$$\theta(b(z)) = \frac{\exp(z)}{z}$$

so

$$\theta(s(z)) = \frac{z}{\exp(z)} \frac{d \exp(z)}{dz} - 1,$$

where  $\exp(z)$  is the exponential series for the formal group law defined by  $\theta$ , i.e., the functional inverse of the logarithm (A2.1.5).

The  $\theta$  we want is the one defining the multiplicative formal group law (A2.1.4)  $x + y + xy$ . An easy calculation shows  $\exp(z) = e^z - 1$  so

$$\theta(s(z)) = \frac{ze^z}{e^z - 1} - 1.$$

This power series is essentially the one used to define Bernoulli numbers (see appendix B of Milnor and Stasheff [5]), i.e., we have

$$\theta(s(z)) = \frac{z}{2} + \sum_{k \geq 1} (-1)^{k+1} \frac{B_k z^{2k}}{(2k)!}$$

where  $B_k$  is the  $k$ th Bernoulli number. Combining this with the above formula of Bott we get

5.3.11. THEOREM. *The image of a generator  $x_t$  of  $\pi_{2t-1}(U) = \pi_{2t}(BU)$  under the map  $\phi: \pi_*BU \rightarrow \mathbf{Q}$  of 5.3.10 is  $\frac{1}{2}$  if  $t = 1$ , 0 for odd  $t > 1$ , and  $\pm B_k/2k$  for  $t = 2k$ . Hence the order of  $x_t$  in  $\text{Ext}^1(BP_*)$  is divisible by 2 for  $t = 1$ , 1 for  $t > 1$ , and the denominator  $j_{2k}$  of  $B_k/2k$  for  $t = 2k$ .  $\square$*

This denominator  $j_{2k}$  is computable by a theorem of von Staudt proved in 1845; references are given in Milnor and Stasheff [5]. The result is that  $p \mid j_{2k}$  iff  $(p-1) \mid 2k$  and that if  $p^i$  is the highest power of such a prime which divides  $2k$  then  $p^{i+1}$  is the highest power of  $p$  dividing  $j_{2k}$ . Comparison with 5.2.6 shows that  $\text{Ext}^{1,4k}(BP_*)$  also has order  $p^{i+1}$  except when  $p = 2$  and  $k > 1$ , in which case it has order  $2^{i+2}$ . This gives

5.3.12. COROLLARY. *The subgroup of  $\text{Ext}^{1,2t}(BP_*)$  generated by  $e(x_t)$  (5.3.9), i.e., by the image of the complex  $J$ -homomorphism, has index 1 for  $t = 1$  and 2, and 1 or 2 for  $t \geq 3$ . Moreover each element in this subgroup is a permanent cycle in the Adams–Novikov spectral sequence.  $\square$*

This completes our discussion of  $\text{im } J$  for odd primes. We will see that the above index is actually 2 for all  $t \geq 3$ , although the method of proof depends on the congruence class of  $t \pmod{4}$ . We use the fact that the complex  $J$ -homomorphism factors through the real one. Hence for  $t \equiv 3 \pmod{4}$ ,  $e(x_t) = 0$  because  $\pi_{2t-1}(SO) = 0$ .

For  $t \equiv 0$  the map  $\pi_{2t-1}(U) \rightarrow \pi_{2t-1}(SO)$  has degree 2 in Bott [1] (and for  $t \equiv 2$  it has degree 1) so  $e(x_t)$  is divisible by 2 and the generator  $y$  of  $\text{Ext}^1(BP_*)$  is as claimed in 5.3.7. This also shows that  $\eta y_t$ , and  $\eta^2 y_t$ , detect elements in  $\text{im } J$ . Furthermore  $\eta^3$  kills the generator of  $\pi_{2t-1}(SO)$  by 3.1.26, so  $\alpha_1^3 y_t$  must die in the Adams–Novikov spectral sequence. It is nonzero at  $E_2$ , so it must be killed by a higher differential and the only possibility is  $d_3(\alpha_{t+2/3}) = \alpha_1^3 y_t$  [here we still have  $t \equiv 0 \pmod{4}$ ].

For  $t \equiv 1$  the generator of  $\pi_{2t-1}(SO) = \mathbf{Z}/(2)$  is detected by  $\eta^2 y_{t-1}$  as observed above, so  $e(x_t) = 0$ . For  $t \equiv 2$  we just saw that the generator  $\alpha_{t/3}$  of  $\text{Ext}^{1,2t}$  supports a nontrivial  $d_3$  for  $t > 2$ , so we must have  $e(x_t) = \alpha_{t/2}$ .

To complete the proof of 5.3.7 we still need to show three things: for  $t \equiv 1 \pmod{4}$ ,  $\alpha_t$  is a permanent cycle, for  $t \equiv 3$ ,  $d_3(\alpha_t) = \alpha_t^3 \alpha_{t-2}$ , and for  $t \equiv 2m \pmod{4}$ ,  $\alpha_t$  is represented by an element of order 4 whose double is detected by  $\alpha_1^2 \alpha_{t-1}$ . To do this we must study the Adams–Novikov spectral sequence for the mod (2) Moore spectrum  $M(2)$ . Since  $BP_*(M(2)) = BP_*/(2)$  is a comodule algebra, the Adams–Novikov  $E_2$ -term for  $M(2)$ ,  $\text{Ext}(BP_*/(2))$ , is a ring (A1.2.14). However, since  $M(2)$  is not a ring spectrum, the Adams–Novikov spectral sequence differentials need not respect this ring structure. The result we need is

5.3.13. THEOREM. (a)  $\text{Ext}(BP_*/(2))$  contains  $\mathbf{Z}/(2)[v_1, h_0] \otimes \{1, u\}$  as a direct summand where  $v_1 \in \text{Ext}^{0,2}$ ,  $h_0 \in \text{Ext}^{1,2}$ , and  $u \in \text{Ext}^{1,8}$  are represented by  $v_1$ ,  $t_1$ , and  $t_1^4 + v_1 t_1^3, v_1^2 t_1^2 + v_1 t_2 + v_2 t_1$  respectively. This summand maps isomorphically to  $E_\infty^{0,*}$  in the chromatic spectral sequence for  $\text{Ext}(BP_*/(2))$  (5.1.11).

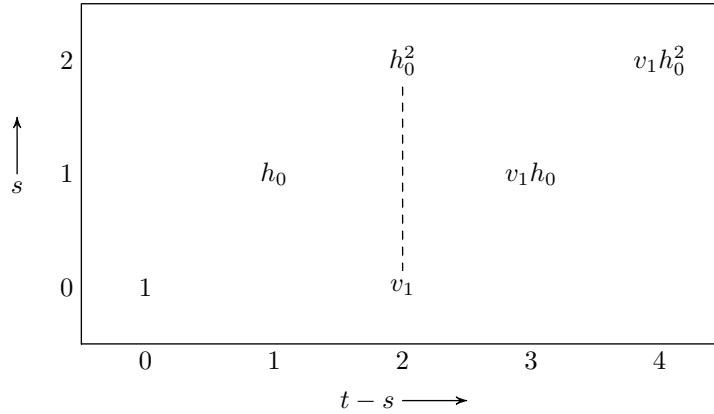
(b) In the Adams–Novikov spectral sequence for  $M(2)$ ,  $v_1^t h_0^s u^e$  is a permanent cycle for  $s \geq 0$ ,  $e = 0, 1$ , and  $t \equiv 0$  or  $1 \pmod{4}$ . If  $t \equiv 2$  or  $3$  then  $d_3(v_1^t h_0^s u^e) = v_1^{t-2} h_0^{s+3} u^e$ . For  $t \equiv 3$ ,  $v_1^t u^e$  is represented by an element of order 4 in  $\pi_{2t+7e}(M(2))$  whose double is detected by  $h_0^2 v_1^{t-1} u^e$ .

(c) Under the reduction map  $BP_* \rightarrow BP_*/(2)$  induced by  $S^0 \rightarrow M(2)$ , if  $t$  is odd then the generator  $\alpha_t$  of  $\text{Ext}^{1,2t}(BP_*)$  maps to  $v_1^{t-1} h_0$ . If  $t$  is even and at least 4 then the generator  $y_t$  of  $\text{Ext}^{1,2t}(BP_*)$  maps to  $v_1^{t-4} u$ .

(d) Under the connecting homomorphism  $\delta: \text{Ext}^s(BP_*/(2)) \rightarrow \text{Ext}^{s+1}(BP_*)$  induced by  $M(2) \rightarrow S^1$  (2.3.4),  $v_1^t$  maps to  $\alpha_t \in \text{Ext}^{1,2t}(BP_*)$  for all  $t > 0$ ;  $uv_1$  maps to  $\alpha_{1y_{t+3}}$  if  $t$  is odd and to 0 if  $t$  is even.  $\square$

In other words, the Adams–Novikov  $E_\infty$ -term for  $M(2)$  has the following pattern as a summand in low dimensions:

(5.3.14)



where the broken vertical line represents a nontrivial group extension. [Compare this with 3.1.28(a) and 5.3.8.] The summand of (a) also contains the products of these elements with  $v_1^{4t}u^e$  for  $t \geq 0$  and  $e = 0, 1$ . The only other generators of  $\text{Ext}^{s,t}(BP_*/(2))$  for  $t - s \leq 13$  are  $\beta_1 \in \text{Ext}^{1,4}$ ,  $\beta_1^2 \in \text{Ext}^{2,8}$ ,  $h_0^s \beta_{2/2} \in \text{Ext}^{1+s,8+2s}$  for  $s = 0, 1, 2$  (where  $h_0^2 \beta_{2/2} = \beta_1^3$ ), and  $h_0^s \beta_2 \in \text{Ext}^{1+s,10+2s}$  for  $s = 0, 1$ .

Before proving this we show how it implies the remaining assertions of 5.3.7 listed above. For  $t \equiv 1 \pmod{4}$ ,  $\alpha_t = \delta(v_1^t)$  by (d) and is therefore a permanent cycle by (b). For  $t \equiv 3$ ,  $\alpha_1 = \delta(v_1^t)$  and  $\delta$  commutes with differentials by 2.3.4, so

$$\begin{aligned} d_3(\alpha_t) &= \delta d_3(v_1^t) = \delta(h_0^3 v_1^{t-2}) \\ &= \alpha_1^3 \alpha_{t-2}. \end{aligned}$$

For the nontrivial group extension note that for  $t \equiv 1$   $\alpha_1^2 \alpha_t$  maps to an element killed by a differential so it is represented in  $\pi_*(S^0)$  by an element divisible by 2. Alternatively,  $\alpha_{t+1}$  is not the image under  $\delta$  of a permanent cycle so it is not represented by an element of order 2.

**PROOF OF 5.3.13.** Recall that in the chromatic spectral sequence converging to  $\text{Ext}(BP_*/(2))$ ,  $\text{Ext}_1^{0,*} = \text{Ext}(M_1^0)$ , which is described in 5.2.2. Once we have determined the subgroup  $E_\infty^{1,*} \subset E_1^{1,*}$  then (c) and (d) are routine calculations, which we will leave to the reader. Our strategy for proving (b) is to make low-dimensional computations by brute force (more precisely by comparison with the Adams spectral sequence) and then transport this information to higher dimensions by means of a map  $\alpha: \Sigma^8 M(2) \rightarrow M(2)$  which induces multiplication by  $v_1^4$  in  $BP$ -homology. [For an odd prime  $p$  there is a map  $\alpha: \Sigma^q M(p) \rightarrow M(p)$  inducing multiplication by  $v_1$ .  $v_1^4$  is the smallest power of  $v_1$  for which such a map exists at  $p = 2$ .]

To prove (a), recall (5.2.2) that  $\text{Ext}(v_1^{-1}BP_*/(2)) = K(1)_*[h_0, \rho_1]/(\rho_1^2)$  with  $h_0 \in \text{Ext}^{1,2}$  and  $\rho_1 \in \text{Ext}^{1,0}$ . We will determine the image of  $\text{Ext}(BP_*/(2))$  in this group. The element  $u$  maps to  $v_1^4 \rho_1$ . [Our representative of  $u$  differs from that

of  $v_1^4 \rho_1$  given in 5.2.2 by an element in the kernel of this map. We choose this  $u$  because it is the mod (2) reduction of  $y_4 \in \text{Ext}^{1,8}(BP_*)$ .] It is clear that the image contains the summand described in (a). If the image contains  $v_1^{-1} h_0^s$  or  $v_1^{4-t} h_0^2 \rho_1$  for any  $t > 0$ , then it also contains that element times any positive power of  $h_0$ . One can show then that such a family of elements in  $\text{Ext}(BP_*/(2))$  would contradict the edge theorem, 5.1.23.

To prove (b) we need some simple facts about  $\pi_*(S^0)$  in dimensions  $\leq 8$  which can be read off the Adams spectral sequence (3.2.11). First we have  $\eta^3 = 4\nu$  in  $\pi_3(S^0)$ . This means  $h_0^3 x$  must be killed by a differential in the Adams–Novikov spectral sequence for  $M(2)$  for any permanent cycle  $x$ . Hence we get  $d_3(v_1^3) = h_0^3$  and  $d_3(v_1^3) = v_1 h_0^3$ . Next, if we did not have  $\pi_2(M(2)) = \mathbf{Z}/(4)$  then  $v_1 \in \pi_1(M(2))$  would extend to a map  $\Sigma^2(M(2)) \rightarrow M(2)$  and by iterating it we could show that all powers of  $v_1$  are permanent cycles, contradicting the above.

Now suppose we can show that  $v_1^4$  and  $u$  are permanent cycles representing elements of order 2 in  $\pi_*(M(2))$ , i.e., maps  $S^n \rightarrow M(2)$  which extend to self-maps  $\Sigma^n M(2) \rightarrow M(2)$ . Then we can iterate the resulting  $\alpha: \Sigma^8 M(2) \rightarrow M(2)$  and compare with the map extending  $u$  to generalize the low-results above to all of (b).

A simple calculation with the Adams spectral sequence shows that  $\pi_7(M(2))$  and  $\pi_8(M(2))$  both have exponent 2 and contain elements representing  $u$  and  $v_1^4$ , respectively, so we have both the desired self-maps.  $\square$

#### 4. $\text{Ext}^2$ and the Thom Reduction

In this section we will describe  $\text{Ext}^2(BP_*)$  and what is known about its behavior in the Adams–Novikov spectral sequence. We will not give all the details of the calculation; they can be found in Miller, Ravenel, and Wilson [1] for odd primes and in Shimomura [1] for  $p = 2$ . The main problem is to compute  $\text{Ext}^0(M^2)$  and the map  $d_e^*$  from it to  $\text{Ext}^0(M^3)$ . From this will follow (5.4.4) that the  $\gamma_t \in \text{Ext}^3(BP_*)$  are nontrivial for all  $t > 0$  if  $p$  is odd. (We are using the notation of 5.1.19.) They are known to be permanent cycles for  $p \geq 7$  (1.3.18).

We will also study the map  $\Phi$  from  $\text{Ext}^2$  to  $E_2^{2,*}$  of the Adams spectral sequence as in 5.2.8 to show that most of the elements in the latter group, since they are not in  $\Phi$ , cannot be permanent cycles (5.4.7). The result is that  $\text{im } \Phi$  is generated by

$$\{\Phi(\beta_{p^n/p^{n-1}}), \Phi(\beta_{p^n/p^n}): n \geq 1\}$$

and a certain finite number of other generators. It is known that for  $p = 2$  the  $\Phi(\beta_{p^n/p^{n-1}})$  are permanent cycles. They are the  $\eta_{n+2} \in \Pi_{2^{n+2}}^S$  constructed by Mahowald [6] using Brown–Gitler spectra. For odd primes it follows that some element closely resembling  $\beta_{p^n/p^{n-i}}$  for  $1 \leq i \leq p^n - 1$  is a nontrivial permanent cycle (5.4.9) and there is a similar more complicated result for  $p = 2$  (5.4.10).

For  $p = 2$ ,  $\Phi(\beta_{2^{2^n}/2^{2^n}}) = b_{n+1}^2$  is known to be a permanent cycle iff there is a framed  $(2^{n+2} - 2)$ -manifold with Kervaire invariant one (Browder [1]), and such are known to exist for  $0 \leq n \leq 4$  (Barratt *et al.* [2]). The resulting element in  $\pi_{2^{j+1}-2}$  is known as  $\theta_j$  and its existence is perhaps the greatest outstanding problem in homotopy theory. It is known to have certain ramifications in the EHP sequence (1.5.29).

For odd primes the situation with  $\Phi(\beta_{p^n/p^n})$  is quite different. We showed in Ravenel [7] that this element is *not* a permanent cycle for  $p \geq 5$  and  $n \geq 1$ , and



that  $\beta_{p^n/p^n}$  itself is *not* a permanent cycle in the Adams–Novikov spectral sequence for  $p \geq 3$  and  $n \geq 1$ ; see 6.4.1.

To compute  $\text{Ext}^2$  with the chromatic spectral sequence we need to know  $E_\infty^{0,2}$ ,  $E_\infty^{1,1}$ , and  $E_\infty^0$ . The first vanishes by 5.2.1; the second is given by 5.3.5 for  $p > 2$  and 5.3.6 for  $p = 2$ . For odd primes  $\text{Ext}^1(M^1) = E_1^{1,1}$  vanishes in positive dimensions; for  $p = 2$  it gives elements in  $\text{Ext}^2(BP_*)$  which are products of  $\alpha_1$  with generators in  $\text{Ext}^2(BP_*)$ . The main problem then is to compute  $E_1^{0,2} = \text{Ext}^0(M^2)$ . We use the short exact sequence

$$0 \rightarrow M_1^1 \rightarrow M^2 \xrightarrow{p} M^2 \rightarrow 0$$

and our knowledge of  $\text{Ext}^0(M_1^1)$  (5.2.13). The method of 5.1.17 requires us to recognize nontrivial elements in  $\text{Ext}^1(M_1^1)$ . This group is not completely known but we have enough information about it to compute  $\text{Ext}^0(M^2)$ . We know  $\text{Ext}^1(M_2^0)$  by 5.2.11, and in proving 5.2.13 one determines the image of  $\text{Ext}^0(M_1^1)$  in it. Hence we know all the elements in  $\text{Ext}^1(M_1^1)$  which are annihilated by  $v_1$ , so any other element whose product with some  $v_1^i$  is one of these must be nontrivial.

To describe  $\text{Ext}^0(M^2)$  we need some notation from 5.2.13. We treat the odd primary case first. There we have

$$\begin{aligned} x_{2,0} &= v_2, \\ x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3, \\ x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \quad \text{and} \\ x_{2,i} &= x_{2,i-1}^p - 2v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} \quad \text{for } i \geq 3, \end{aligned}$$

where  $b_{2,i} = (p+1)(p^{i-1} - 1)$ . Also  $a_{2,0} = 1$  and  $a_{2,i} = p^i + p^{i-1} - 1$  for  $i \geq 1$ . Then

5.4.1. THEOREM (Miller, Ravenel, and Wilson [1]). *For odd primes  $p$ ,  $\text{Ext}^0(M^2)$  is the direct sum of cyclic  $p$ -groups generated by*

- (i)  $\frac{x_{2,i}^s}{p^{k+1}v_1^j}$  with  $p \nmid s$ ,  $j \geq 1$ ,  $k \geq 0$  such that  $p^k \mid j$  and  $j \leq a_{2,i-k}$  and either  $p^{k+1} \nmid j$  or  $a_{i-k-1} < j$ ; and
- (ii)  $\frac{1}{p^{k+1}v_1^j}$  for  $k \geq 0$ ,  $p^k \mid j$ , and  $j \geq 1$ . □

Note that  $s$  may be negative.

For  $p = 2$  we define  $x_{2,i}$  as above for  $0 \leq i \leq 2$ ,  $x_{2,i} = x_{2,i-1}^2$  for  $i \geq 3$ ,  $a_{2,0} = 1$ ,  $a_{2,1} = 2$ , and  $a_{2,i} = 3 \cdot 2^{i-1}$  for  $i \geq 2$ . We also need  $x_{1,0} = v_1$ ,  $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$ , and  $x_{1,i} = x_{1,i-1}^2$  for  $i \geq 2$ . In the following theorem we will describe elements in  $\text{Ext}^0(M^2)$  as fractions with denominators involving  $x_{1,i}$ , i.e., with denominators which are not monomials. These expressions are to be read as shorthand for sums of fractions with monomial denominators. For example, in  $\frac{1}{8x_{1,1}}$  we multiply numerator and denominator by  $x_{1,1}$  to get  $\frac{x_{1,1}}{8x_{1,1}^2}$ . Now  $x_{1,1}^2 \equiv v_1^4 \pmod{(8)}$  so we have

$$\frac{1}{8x_{1,1}} = \frac{v_1^2 + 4v_1^{-1}v_2}{8v_1^4} = \frac{1}{8v_1^2} + \frac{v_2}{2v_1^5}.$$

5.4.2. THEOREM (Shimomura [1]). *For  $p = 2$ ,  $\text{Ext}^0(M^2)$  is the direct sum of cyclic 2-groups generated by*

- (i)  $\frac{v_2^s}{2v_1}, \frac{x_{2,1}^s}{2v_1^j}, \frac{x_{2,2}^s}{2v_1^k},$  and  $\frac{x_{2,2}^s}{8x_{11}}$  for  $s$  odd,  $j = 1$  or  $2$  and  $k = 1, 3, 4, 5,$  or  $6$  ( $k = 2$  is excluded because  $\beta_{4s/2}$  is divisible by  $2$ );
- (ii)  $\frac{x_{2,i}^s}{2v_1^j}$  for  $s$  odd,  $i \geq 3, j \leq a_{2,i},$  and either  $j$  is odd or  $a_{2,i-1} < j$ ;
- (iii)  $\frac{x_{2,j}^s}{2^{k+1}v_1^{j2^k}}$  for  $s$  odd,  $j, k \geq 1, i \geq 3,$  and  $a_{2,i-k-1} < j2^k \leq a_{2,i-k}$ ;
- (iv)  $\frac{x_{2,i}^s}{2^{k+2}x_{1,k}^j}$  for  $s$  odd  $i \geq 3, k \geq 1, j$  odd and  $\geq 1,$  and  $2^k j \leq a_{2,i-k-1}$ ; and
- (v)  $\frac{1}{2v_1^j}, \frac{1}{2^{k+2}x_{1,k}^j}$  for  $j$  odd and  $\geq 1$  and  $k \geq 1.$   $\square$

This result and the subsequent calculation of  $\text{Ext}^2(BP_*)$  for  $p = 2$  were obtained independently by S. A. Mitchell.

These two results give us  $E_1^{2,0}$  in the chromatic spectral sequence. The image of  $d_1$  is the summand of 5.4.1(ii) and 5.4.2(v) and, for  $p = 2,$  the summand generated by  $\beta_1$ ; this is the same  $d_1$  that we needed to find  $\text{Ext}^1(BP_*)$  (5.2.6). We know that  $\text{im } d_2 = 0$  since its source,  $E_2^{0,1},$  is trivial by 5.2.1. The problem then is to compute  $d_1: E_1^{2,0} \rightarrow E_1^{3,0}.$  Clearly it is nontrivial on all the generators with negative exponent  $s.$  The following result is proved for  $p > 2$  as lemma 7.2 in Miller, Ravenel, and Wilson [1] and for  $p = 2$  in section 4 of Shimomura [1].

5.4.3. LEMMA. *In the chromatic spectral sequence,  $d_1: E_1^{2,0} \rightarrow E_1^{3,0}$  is trivial on all of the generators listed in 5.4.1 and 5.4.2 except the following:*

- (i) all generators with  $s < 0$ ;
- (ii)  $\frac{x_{2,i}^s}{pv_1^j}$  with  $p^i < j \leq a_{2,i},$  and  $i \geq 2,$  the image of this generator being  $\frac{-v_3^{p^i-1}}{pv_1^{j-p^i}v_2^{p^i-1}};$  and
- (iii) (for  $p = 2$  only)  $\frac{x_{2,2}}{8x_{1,1}},$  whose image is  $\frac{v_3^2}{2v_1v_2}.$   $\square$

From this one easily read off both the structure of  $\text{Ext}^2(BP_*)$  and the kernel of  $\alpha: \text{Ext}^0(N^3) \rightarrow \text{Ext}^3(BP_*),$  i.e., which Greek letter elements of the  $\gamma$ -family are trivial. We treat the latter case first. The kernel of  $\alpha$  consists of  $\text{im } d_1 \oplus \text{im } d_2 \oplus \text{im } d_3.$  For  $p = 2$  we know that  $\gamma_1 \in \text{im } d_2$  by 5.1.22.  $d_2$  for  $p > 2$  and  $d_3$  for all primes are trivial because  $E_2^{1,1}$  (in positive dimensions) and  $E_3^{0,2}$  are trivial by 5.3.5 and 5.2.1, respectively.

5.4.4. COROLLARY. *The kernel of  $\alpha: \text{Ext}^0(N^3) \rightarrow \text{Ext}^3(BP_*)$  (5.1.18) is generated by  $\gamma_{p^i/p^i,j}$  for  $i \geq 1$  with  $1 \geq j \geq p^i - 1$  for  $p > 2$  and  $1 \leq j \leq p^i$  for  $p = 2$ ; and (for  $p = 2$  only)  $\gamma_1$  and  $\gamma_2.$  In particular  $0 \neq \gamma_t \in \text{Ext}^3(BP_*)$  for all  $t > 0$  if  $p > 2$  and for all  $t > 2$  if  $p = 2.$*

5.4.5. COROLLARY.

(a) *For  $p$  odd,  $\text{Ext}^2(BP_*)$  is the direct sum of cyclic  $p$ -groups generated by  $\beta_{sp^i/j, 1+\phi(i,j)}$  for  $s \geq 1, p \nmid s, j \geq 1, i \geq 0,$  and  $\phi(i,j) \geq 0$  where  $\phi(i,j)$  is the largest integer  $k$  such that  $p^k \mid j$  and*

$$j \leq \begin{cases} a_{2,i-k} & \text{if } s > 1 \text{ or } k > 0 \\ p^i & \text{if } s = 1 \text{ and } k = 0 \end{cases}$$

*This generator has order  $p^{1+\phi(i,j)}$  and internal dimension  $2(p^2 - 1)sp^i - 2(p - 1)j.$  It is the image under  $\alpha$  (5.1.18) of the element  $\frac{x_{2,i}^s}{p^{1+\phi(i,j)}v_1^j}$  of 5.4.1.*

(b) For  $p = 2$ ,  $\text{Ext}^2(BP_*)$  is the direct sum of cyclic 2-groups generated by  $\alpha_1 \bar{\alpha}_t$ , where  $\bar{\alpha}_t$  generates  $\text{Ext}^{1,2t}(BP_*)$  for  $t \geq 1$  and  $t \neq 2$  (see 5.2.6), and by  $\beta_{s2^i/j, 1+\phi(i,j)}$  for  $s \geq 1$ ,  $s$  odd,  $j \geq 1$ ,  $i \geq 0$ , and  $\phi(i, j) \geq 0$  where

$$\phi(i, j) = \begin{cases} 0 & \text{if } 2 \mid j \text{ and } a_{2,i-1} < j \leq a_{2,i}, \\ 0 & \text{if } j \text{ is odd and } j \leq a_{2,i}, \\ 2 & \text{if } j = 2 \text{ and } i = 2, \\ k \geq 2 & \text{if } j \equiv 2^{k-1} \pmod{2^k}, j \leq a_{2,i-k}, \text{ and } i \geq 3, \\ k \geq 1 & \text{if } 2^k \mid j, a_{2,i-k-1} < j \leq a_{2,i-k}, \text{ and } i \geq 3, \\ -1 & \text{otherwise} \end{cases}$$

unless  $s = 1$ , in which case  $a_{2,i}$  is replaced by  $2^i$  in cases above where  $\phi(i, j) = 0$ ,  $\phi(2, 2) = 1$ , and  $\beta_1$  is omitted. The order, internal dimension, and definition of this generator are as in (a).  $\square$

For example when  $p = 2$ ,  $i = 3$  and  $s$  is odd with  $s > 1$ , we have generators

$$\begin{cases} \beta_{8s/j, 2} = \frac{x_{2,2}^{2s}}{4v_1^j} & \text{for } j = 2, 4, 6 \\ \beta_{8s/j} = \frac{x_{2,2}^{2s}}{2v_1^j} & \text{for } 1 \leq j \leq 12 \text{ and } j \neq 2, 4, 6, \end{cases}$$

but  $\beta_{8/j}$  is not defined for  $9 \leq j \leq 12$ . Similarly when  $p > 2$ ,  $i = 4$  and  $s$  is prime to  $p$  with  $s > 1$ , we have generators

$$\begin{cases} \beta_{p^4 s/p^2, 3} = \frac{x_{2,4}^s}{p^3 v_1^{p^2}} \\ \beta_{p^4 s/j, 2} = \frac{x_{2,4}^s}{p^2 v_1^j} & \text{for } p \mid j, j \neq p^2 \text{ and } j \leq p^3 + p^2 - 1 \\ \beta_{p^4 s/j} = \frac{x_{2,4}^s}{p v_1^j} & \text{for other } j \leq p^4 + p^3 - 1, \end{cases}$$

but  $\beta_{p^4/j}$  is not defined for  $p^4 < j \leq p^4 + p^3 - 1$ .

Next we study the Thom reduction map  $\Phi$  from  $\text{Ext}^2(BP_*)$  to  $E_2^{2,*}$  in the classical Adams spectral sequence. This map on  $\text{Ext}^1$  was discussed in 5.2.8. The group  $E_2^{2,*}$  was given in 3.4.1 and 3.4.2. The result is

5.4.6. THEOREM. *The generators of  $\text{Ext}^2(BP_*)$  listed in 5.4.5 map to zero under the Thom reduction map  $\Phi: \text{Ext}(BP_*) \rightarrow \text{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  with the following exceptions.*

(a) (S. A. Mitchell). For  $p = 2$

$$\begin{aligned} \Phi(\alpha_1^2) &= h_1^2, & \Phi(\alpha_1 \alpha_{4/4}) &= h_1 h_3, \\ \Phi(\beta_{2^j/2^j}) &= h_{j+1}^2 & \text{for } j \geq 1, \\ \Phi(\beta_{2^j/2^j-1}) &= h_1 h_{j+2} & \text{for } j \geq 2, \\ \Phi(\beta_{4/2,2}) &= h_2 h_4 & \text{and } \Phi(\beta_{8/6,2}) &= h_2 h_5. \end{aligned}$$

(b) (Miller, Ravenel, and Wilson [1]). For  $p > 2$   $\Phi(\beta_{p^j/p^j}) = -b_j$  for  $j \geq 0$ ;  $\Phi(\beta_{p^j/p^j-1}) = h_0 h_{j+1}$  for  $j \geq 1$ , and  $\Phi(\beta_2) = \pm k_0$ .

PROOF. We use the method of 5.2.8. For (a) we have to consider elements of  $\text{Ext}^1(N^1)$  as well as  $\text{Ext}^0(N^2)$ . Recall (5.3.6) that the former is spanned by  $\frac{v_1^s \rho_1}{2}$  for odd  $s \geq 5$  and  $\frac{v_1^s t_1}{2}$  for odd  $s \geq 1$ . We are looking for elements with  $I$ -adic filtration  $\geq 0$ , and the filtrations of  $t_1$  and  $\rho_1$  are 0 and  $-4$ , respectively. Hence we need to consider only  $\frac{v_1^5 \rho_1}{2}$  and  $\frac{v_1 t_1}{2}$ , which give the first two cases of (a).

The remaining cases come from  $\text{Ext}^0(N^2)$ . The filtration of  $x_{2,i}$  is  $p^i$  so  $\beta_{i/j,k}$  has filtration  $i - j - k$ , and this number is positive in all cases except those indicated above. We will compute  $\Phi(\beta_{2/2})$  and  $\Phi(\beta_{4/2,2})$ , leaving the other cases of (a) and (b) to the reader. [The computation of  $\Phi(\beta_1)$  and  $\Phi(\beta_2)$  for  $p > 2$  were essentially done in 5.1.20.] Using the method of 5.1.20(a), we find that  $\beta_{2/2}$  reduces to  $\frac{v_2 t_1^2}{v_1} \pmod{(2)}$ , which in turn reduces to  $t_1^2 | t_1^2 \pmod{I_2}$ , which maps under  $\Phi$  to  $h_2^2$ . Similarly,  $\beta_{4/2,2}$  reduces to  $\frac{v_2^3 t_1^2}{v_1} + \frac{v_2^4 (t_1^2 + v_1 t_1)}{v_1^4} \pmod{(2)}$  and to  $v_2^2 t_1^2 | t_1^2 + t_1^8 | t_1^2 \pmod{I_2}$ , which maps under  $\Phi$  to  $h_2 h_4$ .  $\square$

This result limits the number of elements in  $\text{Ext}_{A_*}^2(\mathbf{Z}/(p), \mathbf{Z}/(p))$  which can be permanent cycles. As remarked above (5.2.8), any such element must correspond to one having Novikov filtration  $\leq 2$ . Theorem 5.4.6 tells us which elements in  $\text{Ext}(BP_*)^2$  map nontrivially to the Adams spectral sequence. Now we need to see which elements in  $\text{Ext}^1(BP_*)$  correspond to elements of Adams filtration 2. This amounts to looking for elements in  $\text{Ext}^0(N^1)$  with  $I$ -adic filtration 1. From 5.2.8 we see that  $\alpha_{2/2}$  and  $\alpha_{4/4}$  for  $p = 2$  have  $I$ -adic filtration 0, so  $\alpha_2$  and  $\alpha_{4/3}$  have filtration 1 and correspond to  $h_0 h_2$  and  $h_0 h_3$ , respectively. More generally,  $\alpha_t$  for all primes has filtration  $t - 1$  and therefore corresponds to an element with Adams filtration  $\geq t$ . Hence we get

5.4.7. COROLLARY. *Of the generators of  $\text{Ext}_{A_*}^2(\mathbf{Z}/(p), \mathbf{Z}/(p))$  listed in 3.4.1 and 3.4.2, the only ones which can be permanent cycles in the Adams spectral sequence are*

- (a) for  $p = 2$ ,  $h_0^2$ ,  $h_0 h_2$ ,  $h_0 h_3$ ,  $h_j^2$  for  $j \geq 1$ ,  $h_1 h_j$  for  $j \geq 3$ ,  $h_2 h_4$ , and  $h_2 h_5$ ;  
and  
(b) for  $p > 2$ ,  $a_0^2$ ,  $b_j$  for  $j \geq 0$ ,  $a_1$ ,  $a_0 h_1$  for  $p = 3$ ,  $h_0 h_j$  for  $j \geq 2$ , and  $k_0$ .  $\square$

Part (a) was essentially proved by Mahowald and Tangora [8], although their list included  $h_3 h_6$ . In Barratt, Mahowald, and Tangora [1] it was shown that  $h_2 h_5$  is not a permanent cycle. It can be shown that  $d_3(\beta_{8/6,2}) \neq 0$ , while  $\beta_{4/2,2}$  is a permanent cycle. The elements  $h_0^2$ ,  $h_0 h_2$ ,  $h_0 h_3$  for  $p = 2$  and  $a_0^2$ ,  $a_1$ ,  $a_0 h_1$  ( $p = 3$ ) for odd primes are easily seen to be permanent cycles detecting elements in  $\text{im } J$ .

This leaves two infinite families to be considered: the  $b_j$  (or  $h_{j+1}^2$  for  $p = 2$ ) for  $j \geq 0$  and the  $h_0 h_j$  (or  $h_1 h_{j+1}$  for  $p = 2$ ) for  $j \geq 1$ . These are dealt with in 3.4.4 and 4.4.22. In Section 6.4 we will generalize the latter to

5.4.8. THEOREM. (a) *In the Adams–Novikov spectral sequence for  $p \geq 3$ ,*

$$d_{2p-1}(\beta_{p^j/p^j}) \equiv \alpha_1 \beta_{p^{j-1}/p^{j-1}} \neq 0$$

*modulo a certain indeterminacy for  $j \geq 1$ .*

- (b) *In the Adams spectral sequence for  $p \geq 5$ ,  $b_j$  is not a permanent cycle for  $j \geq 1$ .*  $\square$

The restriction on  $p$  in 5.4.8(b) is essential; we will see (6.4.11) that  $b_2$  is a permanent cycle for  $p = 3$ .

The proof of 3.4.4(b) does not reveal which element in  $\text{Ext}^2(BP_*)$  detects the constructed homotopy element. 5.4.5 implies that  $\text{Ext}^{2,(1+p^j)q}$  is a  $\mathbf{Z}/(p)$  vector space of rank  $[j/2]$ ; i.e., it is spanned by elements of the form  $\delta_0(x)$  for  $x \in \text{Ext}^1(BP_*/(p))$ . (This group is described in 5.2.14 and 5.2.17.) The  $x$  that we want must satisfy  $v_1^{p^{j-1}-2}x = \delta_1(v_2^{p^j})$ . ( $\delta_0$  and  $\delta_1$  are defined in 5.1.2.) The fact that the homotopy class has order  $p$ , along with 2.3.4, means that  $x$  itself [as well as  $\delta_0(x)$ ] is a permanent cycle, i.e., that the map  $f: S^m \rightarrow S^0$  for  $m = q(1+p^j) - 3$  given by 3.3.4(d) fits into the diagram

$$\begin{array}{ccc} S^m & \xrightarrow{f} & S^0 \\ \downarrow & & \uparrow \\ \Sigma^m M(p) & \xrightarrow{\tilde{f}} & \Sigma^{-1} M(p) \end{array}$$

where  $M(p)$  denotes the mod  $(p)$  Moore spectrum and the vertical maps are inclusion of the bottom cell and projection onto the top cell. Now  $\tilde{f}$  can be composed with any iterate of the map  $\alpha: \Sigma^q M(p) \rightarrow M(p)$  inducing multiplication by  $v_1$  in  $BP$ -homology, and the result is a map  $S^{m+qt} \rightarrow S^0$  detected by  $\delta_0(v_1^t x)$ . This gives

5.4.9. THEOREM. (R. Cohen [3]) *Let  $\zeta_{j-1} \in \pi_{m-1}^S$  be the element given by 3.4.4(d), where  $m = (1+p^j)q - 2$ . It is detected by an element  $y_{j-1} \in \text{Ext}^{3,2+m}(BP_*)$  congruent to  $\alpha_1 \beta_{p^{j-1}/p^{j-1}}$  modulo elements of higher  $I$ -adic filtration (i.e., modulo  $\ker \Phi$ ). Moreover for  $j \geq 3$  and  $0 < i < p^{j-1} - p^{j-2} - p^{j-3} \zeta_{j-1,i} \in \langle \zeta_{j-1}, p, \alpha_1 \rangle \subset \pi_{m-1+qi}^S$ , obtained as above, is nontrivial and detected by an element in  $\text{Ext}^{3,2+m+qi}(BP_*)$  congruent to  $\alpha_1 \beta_{p^{j-1}/p^{j-1-i}}$ .  $\square$*

The range of  $i$  in 5.4.9(b) is smaller than in (a) because  $\alpha_1 \beta_{p^j/p^{j-1+p^{j-2}}} = 0$  for  $j \geq 2$ . To see this compute the coboundary of  $\frac{v_1 v_2^{p^j}}{p^2 v_1^{(p+1)p^{j-2}}}$ .

The analogous results for  $p = 2$  are more complicated.  $\eta_j \in \pi_{2^j}^S$  is not known to have order 2, so we cannot extend it to a map  $\Sigma^{2^j} M(2) \rightarrow S^0$  and compose with elements in  $\pi_*(M(2))$  as we did in the odd primary case above. In fact, there is reason to believe the order of  $\eta_j$  is 4 rather than 2. To illustrate the results one might expect, suppose  $\beta_{2^j/2^j}$  is a permanent cycle represented by an element of order 2. (This would imply that the Kervaire invariant element  $\theta_{j+1}$  exists; see 1.5.29.) Then we get a map  $f: \Sigma^{2^{j+2}-2} M(2) \rightarrow S^0$  which we can compose with the elements of  $\pi_*(M(2))$  given by 5.3.13. In particular,  $f v_1^{4k}$  would represent  $\beta_{2^j/2^j-4k}$ , which is nontrivial for  $k < 2^{j-2}$ ,  $f v_1$  would represent  $\beta_{2^j/2^j-1}$  (i.e., would be closely related to  $\eta_{j+2}$ ), and  $2f v_1$  would represent  $\alpha_1^2 \beta_{2^j/2^j}$ , leading us to expect  $\eta_{j+2}$  to have order 4. Since the elements of 5.3.13 have filtration  $\leq 3$ , the composites with  $f$  would have filtration  $\leq 5$ . Hence their nontriviality in  $\text{Ext}(BP_*)$  is not obvious.

Now 5.3.13 describes 12 families of elements in  $\text{Ext}(BP_*/(2))$  (each family has the form  $\{v_1^{4k} x: k \geq 0\}$ ) which are nontrivial permanent cycles: the six shown in 5.3.14 and their products with  $u$ . Since we do not know  $\theta_{j+1}$  exists we cannot show that these are permanent cycles directly. However, five of them ( $v_1 \alpha_1$ ,  $v_1 \alpha_1^2$ ,  $u v_1$ ,  $u v_1 \alpha_1$ , and  $u v_1 \alpha_1^2$ ) can be obtained by composing  $v_1$  with mod  $(2)$  reductions of permanent cycles in  $\text{Ext}(BP_*)$ , and hence correspond to compositions of  $\eta_{j+1}$  with elements in  $\pi_*^S$ . Four of these five families have been shown to be nontrivial by Mahowald [10] without use of the Adams–Novikov spectral sequence.

5.4.10. THEOREM (Mahowald [10]). *Let  $\mu_{8k+1} \in \pi_{8k-1}^S$  be the generator constructed by Adams [1] and detected by  $\alpha_{4k+1} \in \text{Ext}^{1,8k+2}(BP_*)$ , and let  $\rho_k \in \pi_{8k-1}^S$  be a generator of  $\text{im } J$  detected by a generator  $y_{4k}$  of  $\text{Ext}^{1,8k}(BP_*)$ . Then for  $0 < k < 2^{j-4}$  the compositions  $\mu_{8k+1}\eta_j$ ,  $\eta\mu_{8k+1}\eta_j$ ,  $\rho_k\eta_j$ , and  $\eta\rho_k\eta_j$  are essential. They are detected in the Adams spectral sequence respectively by  $P^k h_1^2 h_j$ ,  $P^k h_1^3 h_j$ ,  $P^{k-1} c_0 h_j$ , and  $P^{k-1} c_0 h_1 h_j$ .  $\square$*

This result provides a strong counterexample to the “doomsday conjecture”, which says that for each  $s \geq 0$ , only finitely many elements of  $E_2^{s,*}$  are permanent cycles (e.g., 1.5.29 is false). This is true for  $s = 0$  and 1 by the Hopf invariant one theorem, 1.2.12, but 5.4.10 shows it is false for each  $s \geq 2$ .

### 5. Periodic Families in $\text{Ext}^2$

This section is a survey of results of other authors concerning which elements in  $\text{Ext}^2(BP_*)$  are nontrivial permanent cycles. These theorems constitute nearly all of what is known about systematic phenomena in the stable homotopy groups of spheres.

First we will consider elements various types of  $\beta$ 's. The main result is 5.5.5. Proofs in this area tend to break down at the primes 2 and 3. These difficulties can sometimes be sidestepped by replacing the sphere with a suitable torsion-free finite complex. This is the subject of 5.5.6 ( $p = 3$ ) and 5.5.7 ( $p = 2$ ).

In 5.5.8 we will treat decomposable elements in  $\text{Ext}^2$ .

The proof of Smith [1] that  $\beta_t$  is a permanent cycle for  $p \geq 5$  is a model for all results of this type, the idea being to show that the algebraic construction of  $\beta_t$  can be realized geometrically. There are two steps here. First, show that the first two short exact sequences of 5.1.2 can be realized by cofiber sequences, so there is a spectrum  $M(p, v_1)$  with  $BP_*(M(p, v_1)) = BP_*/I_2$ , denoted elsewhere by  $V(1)$ . [Generally if  $I = (q_0, q_1, \dots, q_{n-1}) \in BP_*$  is an invariant regular ideal and there is a finite spectrum  $X$  with  $BP_*(X) = BP_*/I$  then we will denote  $X$  by  $M(q_0, \dots, q_{n-1})$ .] This step is quite easy for any odd prime and we leave the details to the reader. It cannot be done for  $p = 2$ . Easy calculations (e.g., 5.3.13) show that the map  $S^2 \rightarrow M(2)$  realizing  $v_1$  does not have order 2 and hence does not extend to the required map  $\Sigma^2 M(2) \rightarrow M(2)$ . Alternatively, one could show that  $H^*(M(2, v_1); \mathbf{Z}/(2))$ , if it existed, would contradict the Adem relation  $Sq^2 Sq^2 = Sq^3 Sq^1$ .

The second step, which fails for  $p = 3$ , is to show that for all  $t > 0$ ,  $v_1^t \in \text{Ext}^0(BP_*/I_2)$  is a permanent cycle in the Adams–Novikov spectral sequence for  $M(p, v_1)$ . Then 2.3.4 tells us that  $\beta_t = \delta_0 \delta_1(v_2^t)$  detects the composite

$$S^{2t(p^2-1)} \xrightarrow{v_2^t} M(p, v_1) \rightarrow \Sigma^{q+1} M(p) \rightarrow S^{q+2},$$

where  $q = 2p - 2$  as usual. One way to do this is to show that the third short exact sequence of 5.1.2 can be realized, i.e., that there is a map  $\beta: \Sigma^{2(p^2-1)} M(p, v_1) \rightarrow M(p, v_1)$  realizing multiplication by  $v_2$ . This self-map can be iterated  $t$  times and composed with inclusion of the bottom cell to realize  $v_2^t$ . To construct  $\beta$  one must first show that  $v_2$  is a permanent cycle in the Adams–Novikov spectral sequence for  $M(p, v_1)$ . One could then show that the resulting map  $S^{2(p^2-1)} \rightarrow M(p, v_1)$  extends cell by cell to all of  $\Sigma^{2(p^2-1)} M(p, v_1)$  by obstruction theory. However, this would be unnecessary if one knew that  $M(p, v_1)$  were a ring spectrum, which it is

for  $p \geq 5$  but not for  $p = 3$ . Then one could smash  $v_2$  with the identity on  $M(p, v_1)$  and compose with the multiplication, giving

$$\Sigma^{2(p^2-1)}M(p, v_1) \rightarrow M(p, v_1) \wedge M(p, v_1) \rightarrow M(p, v_1),$$

which is the desired map  $\beta$ .

Showing that  $M(p, v_1)$  is a ring spectrum, i.e., constructing the multiplication map, also involves obstruction theory, but in lower dimensions than above.

We will now describe this calculation in detail and say what goes wrong for  $p = 3$ . We need to know  $\text{Ext}^{s,t}(BP_*/I_2)$  for  $t - s \leq 2(p^2 - 1)$ . This deviates from  $\text{Ext}(BP_*/I) = \text{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  only by the class  $v_2 \in \text{Ext}^{0,2(p^2-1)}$ . It follows from 4.4.8 that there are five generators in lower dimensions, namely  $1 \in \text{Ext}^{0,0}$ ,  $h_0 \in \text{Ext}^{1,q}$ ,  $b_0 \in \text{Ext}^{2,pq}$ ,  $h_0b_0 \in \text{Ext}^{3,(p+1)q}$ ,  $h_1 \in \text{Ext}^{1,pq}$ , and  $\text{Ext}^{s,t} = 0$  for  $t - s = 2(p^2 - 1) - 1$  so  $v_2$  is a permanent cycle for any odd prime.

To show  $M(p, v_1)$  is a ring spectrum we need to extend the inclusion  $S^0 \rightarrow M(p, v_1)$  to a suitable map from  $X = M(p, v_1) \wedge M(p, v_1)$ . We now assume  $p = 5$  for simplicity. Then  $X$  has cells in dimensions 0, 1, 2, 9, 10, 11, 18, 19, and 20, so obstructions occur in  $\text{Ext}^{s,t}$  for  $t - s$  one less than any of these numbers. The only one of these groups which is nontrivial is  $\text{Ext}^{0,0} = \mathbf{Z}/(p)$ . In this case the obstruction is  $p$  times the generator (since the 1-cells in  $X$  are attached by maps of degree  $p$ ), which is clearly zero. Hence for  $p \geq 5$   $M(p, v_1)$  is a ring spectrum and we have the desired self-map  $\beta$  needed to construct the  $\beta_t$ 's.

However, for  $p = 3$  obstructions occur in dimensions 10 and 11, where the Ext groups are nonzero. There is no direct method known for calculating an obstruction of this type when it lies in a nontrivial group. In Toda [1] it is shown that the nontriviality of one of these obstructions follows from the nonassociativity of the multiplication of  $M(3)$ .

We will sketch another proof now. If  $M(3, v_1)$  is a ring spectrum then each  $\beta_t$  is a permanent cycle, but we will show that  $\beta_4$  is not. In  $\text{Ext}^{6,84}(BP_*)$  one has  $\beta_1^2\beta_4$  and  $\beta_1\beta_{3/3}^2$ . These elements are actually linearly independent, but we do not need this fact now. It follows from 4.4.22 that  $d_5(\beta_1\beta_{3/3}^2) = \pm\alpha_1\beta_1^4\beta_{3/3} \neq 0$ . The nontriviality of this element can be shown by computing the cohomology of  $P_*$  in this range.

Now  $\beta_2^3 \in \text{Ext}^{6,84}(BP_*)$  is a permanent cycle since  $\beta_2$  is. If we can show

$$(5.5.1) \quad \beta_2^3 = \pm\beta_1\beta_{3/3}^2 \pm \beta_1^2\beta_4$$

then  $\beta_1^2\beta_4$  and hence  $\beta_4$  will have to support a nontrivial  $d_5$ . We can prove 5.5.1 by reducing to  $\text{Ext}(BP_*/I_2)$ . By 5.1.20 the images of  $\beta_1$ ,  $\beta_2$ , and  $\beta_4$  in this group are  $\pm b_{10}$ ,  $\pm v_2 b_{10} \pm k_0$ , and  $\pm v_2^3 b_{10}$ , respectively, and the image of  $\beta_{3/3}$  is easily seen to be  $\pm b_{11}$ . Hence the images of  $\beta_1^2\beta_4$ ,  $\beta_1\beta_{3/3}^2$ , and  $\beta_2^3$  are  $\pm v_2^3 b_{10}^3$ ,  $\pm b_{10} b_{11}^2$  and  $\pm v_2^3 b_{10}^3 \pm k_0^3$  respectively. Thus 5.5.1 will follow if we can show  $k_0^3 = \pm b_{10} b_{11}^2$ . (At any larger prime  $p$  we would have  $k_0^p = 0$ .)  $k_0$  is the Massey product  $\pm\langle h_0, h_1, h_1 \rangle$ .



Using [A1.4.6](#) we have up to sign

$$\begin{aligned}
k_0^2 &= \langle h_0, h_1, h_1 \rangle \langle h_0, h_1, h_1 \rangle \\
&= \langle h_0 \langle h_0, h_1, h_1 \rangle, h_1, h_1 \rangle \\
&= \langle \langle h_0, h_0, h_1 \rangle h_1, h_1, h_1 \rangle \\
&= \langle h_0, h_0, h_1 \rangle \langle h_1, h_1, h_1 \rangle \\
&= g_0 b_{11}
\end{aligned}$$

and

$$\begin{aligned}
k_0^3 &= \langle h_0, h_1, h_1 \rangle \langle h_0, h_0, h_1 \rangle b_{11} \\
&= \langle h_0 \langle h_0, h_0, h_1 \rangle, h_1, h_1 \rangle b_{11} \\
&= \langle \langle h_0, h_0, h_0 \rangle h_1, h_1, h_1 \rangle b_{11} \\
&= \langle h_0, h_0, h_0 \rangle \langle h_1, h_1, h_1 \rangle b_{11} \\
&= b_{10} b_{11}^2 \quad \text{as claimed.}
\end{aligned}$$

5.5.2. THEOREM (Smith [\[1\]](#)). *Let  $p \geq 5$*

(a)  $\beta_t \in \text{Ext}^{2,q((p+1)t-1)}$  *is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for all  $t > 0$ .*

(b) *There is a map  $\beta: \Sigma^{2(p^2-1)}M(p, v_1) \rightarrow M(p, v_1)$  inducing multiplication by  $v_2$  in BP-homology.  $\beta_t$  detects the composite*

$$S^{2t(p^2-1)} \rightarrow \Sigma^{2t(p^2-1)}M(p, v_1) \xrightarrow{\beta^t} M(p, v_1) \rightarrow S^{2p}.$$

(c)  $M(p, v_1)$  *is a ring spectrum.* □

5.5.3. THEOREM (Behrens and Pemmaraju [\[1\]](#)). (a) *For  $p = 3$  the complex  $V(1)$  admits a self-map realizing multiplication by  $v_2^9$  in BP-homology.*

(b) *The element  $\beta_t \in \text{Ext}^{2,q((p+1)t-1)}$  is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for  $t$  congruent to 0, 1, 2, 5, or 6 modulo 9.*

To realize more general elements in  $\text{Ext}^2(BP_*)$  one must replace  $I_2$  in the above construction by an invariant regular ideal. For example a self-map  $\beta$  of  $M(p^2, v_1^p)$  inducing multiplication by  $v_2^{p^2}$  (such a map does not exist) would show that  $\beta_{tp^2/p, 2}$  is a permanent cycle for each  $t > 0$ . Moreover we could compose  $\beta^t$  on the left with maps other than the inclusion of the bottom cell to get more permanent cycles.  $\text{Ext}^0(BP_*/(p^2, v_1^p))$  contains  $pv_1^i$  for  $0 \leq i < p$ , and each of these is a permanent cycle and using it we could show that  $\beta_{tp^2/p-i}$  is a permanent cycle.

It is easy to construct  $M(p^{i+1}, v_1^{sp^i})$  for  $s > 0$  and  $p$  odd. Showing that it is a ring spectrum and constructing the appropriate self-map is much harder. The following result is a useful step.

5.5.4. THEOREM. (a) (Mahowald [\[11\]](#)).  $M(4, v_1^{4t})$  *is a ring spectrum for  $t > 0$ .*

(b) (Oka [\[7\]](#)).  $M(2^{i+2}, v_1^{2^i t} + 2^{i+1}tv_1^{2^i t-3}v_2)$  *is a ring spectrum for  $i \geq 2$  and  $t \geq 2$ .*

(c) (Oka [\[7\]](#)). *For  $p > 2$ ,  $M(p^{i+1}v_1^{p^i t})$  is a ring spectrum for  $i \geq 0$  and  $t \geq 2$  [Recall  $M(p, v_1)$  is a ring spectrum for  $p \geq 5$  by [5.5.2\(c\)](#).] □*

Note that  $M(p^i, v_1^j)$  is not unique; the theorem means that there is a finite ring spectrum with the indicated BP-homology.



Hence we have a large number of four-cell ring spectra available, but it is still hard to show that the relevant power of  $v_2$  is a permanent cycle in  $\text{Ext}^0$ .

### 5.5.5. THEOREM.

(a) (Davis and Mahowald [1], Theorem 1.3). *For  $p = 2$ , there is a map  $\Sigma^{48}M(2, v_1^4) \rightarrow M(2, v_1^4)$  inducing multiplication by  $v_2^8$ , so  $\beta_{8t/4}$  and  $\beta_{8t/3}$  are permanent cycles for all  $t > 0$ .*

(b) *For  $p \geq 5$  the following spectra exist:  $M(p, v_1^{p-1}, v_2^p)$  (Oka [4, 1], Smith [2], Zahler [2]);  $M(p, v_2^p, v_2^{tp})$  for  $t \geq 2$  (Oka [5]);  $M(p, v_1^{2p-2}, v_2^{p^2})$  (Oka [6]);  $M(p, v_2^{2p}, v_2^{tp^2})$  for  $t \geq 2$  (Oka [6]);  $M(p^2, v_1^p, v_2^{tp^2})$  for  $t \geq 2$  (Oka [6]); and consequently the following elements in  $\text{Ext}^2(BP_*)$  are nontrivial permanent cycles:  $\beta_{tp/i}$  for  $t > 0$ ,  $1 \leq i \leq p-1$ ;  $\beta_{tp/p}$  for  $t \geq 2$ ;  $\beta_{tp^2/i}$  for  $t > 0$ ,  $1 \leq i \leq 2p-2$ ;  $\beta_{tp^2/2p}$  and  $\beta_{tp^2/2p-1}$  for  $t \geq 2$ ; and  $\beta_{tp^2/p, 2}$  for  $t \geq 2$ .*

(c) (Oka [10]). *For  $p \geq 5$  the spectra  $M(p, v_1^{2n-2p}, v_2^{p^{nt}})$  for  $t \geq 2$  and  $n \geq 3$ , and  $M(p, v_1^{2^{n-3}p}, v_2^n)$  for  $n \geq 3$  exist. Consequently the following elements are nontrivial permanent cycles:  $\beta_{p^n t/s}$  for  $t \geq 2$ ,  $n \geq 3$ , and  $1 \leq s \leq 2^{n-2}p$ ; and  $\beta_{p^n t/s}$  for  $t \geq 1$ ,  $n \geq 3$ , and  $1 \leq s \leq 2^{n-3}p$ . In particular the  $p$ -rank of  $\pi_k^S$  can be arbitrarily large.  $\square$*

Note that in (a)  $M(2, v_1^4)$  is not a ring spectrum since  $M(2)$  is not, so the proof involves more than just showing that  $v_2^8 \in \text{Ext}^0(BP_*/(2, v_1^4))$  is a permanent cycle.

When a spectrum  $M(p^i, v_1^j, v_2^k)$  for an invariant ideal  $(p^i, v_1^j, v_2^k) \subset BP_*$  does not exist one can look for the following sort of substitute for it. Take a finite spectrum  $X$  with torsion-free homology and look for a finite spectrum  $XM(p^i, v_1^j, v_2^k)$  whose  $BP$  homology is  $BP_*(X) \otimes_{BP_*} BP_*/(p^i, v_1^j, v_2^k)$ . Then the methods above show that the image  $\bar{\beta}_{k/j, i}$  of  $\beta_{k/j, i}$  induced by the inclusion  $S^0 \rightarrow X$  [assuming  $X$  is  $(-1)$ -connected with a single 0-cell] is a permanent cycle. The resulting homotopy class must “appear” on some cell of  $X$ , giving us an element in  $\pi_*^S$  which is related to  $\beta_{k/j, i}$ . The first example of such a result is

5.5.6. THEOREM (Oka and Toda [8]). *Let  $p = 3$  and  $X = S^0 \cup_{\beta_1} e^{11}$ , the mapping cone of  $\beta_1$ .*

(a) *The spectrum  $XM(3, v_1, v_2)$  exists so  $\bar{\beta}_t \in \text{Ext}^2(BP_*(X))$  is a permanent cycle for each  $t > 0$ .*

(b) *The spectrum  $XM(3, v_1^2, v_2^3)$  exists so  $\bar{\beta}_{3t/2} \in \text{Ext}^2(BP_*(X))$  is a permanent cycle for each  $t > 0$ .*

*Let  $p = 5$  and  $X = S^0 \cup_{\beta_1} e^{39}$ .*

(c) *The spectrum  $XM(5, v_1, v_2, v_3)$  exists so  $\bar{\gamma}_t \in \text{Ext}^3(BP_*(X))$  is a permanent cycle for all  $t > 0$ .  $\square$*

Hence  $\bar{\beta}_t$  detects an element in  $\pi_{16t-6}(X)$  which we also denote by  $\bar{\beta}_t$ . The cofibration defining  $X$  gives an long exact sequence

$$\cdots \rightarrow \pi_n(S^0) \xrightarrow{i} \pi_n(X) \xrightarrow{j} \pi_{n-11}(S^0) \xrightarrow{\beta_1} \pi_{n-1}(S^0) \rightarrow \cdots$$

where the last map is multiplication by  $\beta_1 \in \pi_{10}(S^0)$ . If  $\bar{\beta}_t \notin \text{im } i$  then  $j(\beta_t) \neq 0$ , so for each  $t > 0$  we get an element in either  $\pi_{16t-6}^S$  or  $\pi_{16t-17}^S$ . For example, in the Adams–Novikov spectral sequence for the sphere one has  $d_5(\beta_4) = \alpha_1 \beta_1^2 \beta_{3/3}$  so  $\beta_4 \notin \text{im } i$  and  $j(\beta_4) \in \pi_{47}^S$  is detected by  $\alpha_1 \beta_1 \beta_{3/3}$ , i.e.,  $j(\beta_4) = \beta_1 \varepsilon'$  (see 5.1.1). We can regard  $j(\beta_t)$  as a substitute for  $\beta_t$  when the latter is not a permanent cycle.

In the above example we had  $BP_*(X) = BP_* \oplus \Sigma^j BP_*$  as a comodule, so  $\text{Ext}(BP_*)$  is a summand of  $\text{Ext}(BP_*(X))$ . In the examples below this is not the case, so it is not obvious that  $\beta_{k/j,i} \neq 0$ .

5.5.7. THEOREM (Davis and Mahowald [1] and Mahowald [12]). *Let  $p = 2$ ,  $X = S^0 \cup_\eta e^2$ ,  $W = S^0 \cup_\nu e^4$ , and  $Y = X \wedge W$ . Part (a) below is essentially theorem 1.4 of Davis and Mahowald [1], while the numbers in succeeding statements refer to theorems in Mahowald [12]. Their  $Y$  and  $A_1$  are  $XM(2)$  and  $XM(2, v_1)$  in our notation.*

(a)  $XM(2, v_1, v_2^8)$  exists and  $\bar{\beta}_{8t} \in \text{Ext}^2(BP_*(X))$  is a nontrivial permanent cycle.

(b) (1.4) In the Adams–Novikov spectral sequence for  $S^0$ ,  $\beta_{8t}$  is not a permanent cycle and  $\bar{\beta}_{8t} \in \pi_{4qt-4}(X)$  projects under the pinching map  $X \rightarrow S^2$  to an element detected by  $\alpha_1^2 \beta_{8t/3}$  if this element is nontrivial.

(c) (1.5)  $\bar{v}_2^{8t+1} \in \text{Ext}^0(BP_*(X)/I_2)$  and  $\bar{\beta}_{8t+1} \in (BP_*(X))$  are nontrivial permanent cycles.  $\beta_{8t+1} \in \text{Ext}^2(BP_*)$  is not a permanent cycle and  $\beta_{8t+1} \in \pi_{48t+2}(X)$  projects to an element detected by  $\alpha_1 \alpha_{4/4} \beta_{8t/3} \in \text{Ext}^4(BP_*)$  if this element is nontrivial.

PROOF. (a) Davis and Mahowald [1] showed that  $XM(2, v_1)$  admits a self-map realizing  $v_2^8$ . This gives the spectrum and the permanent cycles. To show  $\bar{\beta}_{8t} \neq 0$  it suffices to observe that  $\beta_{8t} \in \text{Ext}^2(BP_*)$  is not divisible by  $\alpha_1$ .

(b) Mahowald [12] shows that  $\bar{\beta}_{8t} \in \pi_{48t-4}(X)$  projects nontrivially to  $\pi_{48t-6}^S$ . By duality there is a map  $f: \Sigma^{48t-4}(X) \rightarrow S^0$  which is nontrivial on the bottom cell. From 5.3.13 one can construct a map  $\Sigma^{48t-4}X \rightarrow \Sigma^{48t-10}M(2)$  which is  $v_1\eta^2$  on the bottom cell and such that the top cell is detected by  $v_1^3 \in \text{Ext}^0(BP_*/(2))$ . Now compose this with the extension of  $\beta_{8t/4} \Sigma^{48t-10}M(2) \rightarrow S^0$  given by 5.5.4(a). The resulting map  $g: \Sigma^{48t-4}X \rightarrow S^0$  is  $\alpha_1^2 \beta_{8t/3}$  on the bottom cell and the top cell is detected by  $\beta_{8t}$ . Hence this map agrees with  $f$  modulo higher Novikov filtration. If  $\alpha_1^2 \beta_{8t/3} \neq 0 \in \text{Ext}^4(BP_*)$  it follows that the bottom cell on  $f$  is detected by that element. [It is likely that  $\alpha_1^3 \beta_{8t/3} = 0$  (this is true for  $t = 1$ ), so the differential on  $\beta_{8t}$  is not a  $d_3$ .]

(c) As in (b) Mahowald [12] shows the projection of  $\beta_{8t+1}$  in  $\pi_{48t}^S$  is nontrivial. To show that  $\alpha_1 \alpha_{4/4} \beta_{8t/3}$  detects our element if it is nontrivial we need to make a low-dimensional computation in the Adams–Novikov spectral sequence for  $M(2, v_1^4)$  where we find that  $v_1^3 v_2 \in \text{Ext}^{0,12}(BP_*/(2, v_1^4))$  supports a differential hitting  $v_1 \alpha_{4/4} \alpha_1^2 \in \text{Ext}^{3,14}$ . It follows that  ${}_1\sigma\eta \in \pi_{11}(M(2, v_1^4))$  extends to a map  $\Sigma^{10}X \rightarrow M(2, v_1^4)$  with the top cell detected by  $v_2 v_1^3$ . Suspending  $48t - 10$  times and composing with the extension of  $\beta_{8t/4}$  to  $\Sigma^{48t-10}M(2, v_1^4)$  gives the result.  $\square$

Now we consider products of elements in  $\text{Ext}^1$ .

5.5.8. THEOREM. *Let  $\bar{\alpha}_t$  be a generator of  $\text{Ext}^{1,qt}(BP_*)$  (see 5.2.6).*

(a) (Miller, Ravenel, and Wilson [1]). *For  $p > 2$ ,  $\bar{\alpha}_s \bar{\alpha}_t = 0$  for all  $s, t > 0$ .*

(b) *For  $p = 2$*

(i) *If  $s$  or  $t$  is odd and neither is 2 then  $\bar{\alpha}_1 \bar{\alpha}_t = \alpha_1 \bar{\alpha}_{s+t-1} \neq 0$ .*

(ii)  $\bar{\alpha}_2^2 = \beta_{2/2}$ .

(iii)  $\bar{\alpha}_4^2 = \beta_{4/4} + \beta_3$ .

(Presumably, all other products of this sort vanish.)

In an earlier version (iii) read  $\bar{\alpha}_4^2 = \beta_{4/4}$  instead of the above. This error was brought to my attention by Hans-Werner Henn in 2016.

PROOF. Part (a) is given in Miller, Ravenel, and Wilson [1] as theorem 8.18. The method used is similar to the proof of (b) below.

For (b)(i) assume first that  $s$  and  $t$  are both odd. Then  $\bar{\alpha}_s = \frac{v_1^s}{2}$  and the mod (2) reduction of  $\bar{\alpha}_t$  is  $v_1^{t-1}t_1$ . Hence  $\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-1}}{2}t_1 = \bar{\alpha}_{s+t-1}\bar{\alpha}_1$ .

For  $s$  odd and  $t = 2$  we have

$$\alpha_s\bar{\alpha}_2 = \frac{v_1^s}{2}(t_1^2 + vt_1) = d\left(\frac{v_1^{s-t}v_2}{2}\right) \quad \text{so} \quad \bar{\alpha}_s\bar{\alpha}_2 = 0.$$

For  $t$  even and  $t > 2$ , recall that

$$\bar{\alpha}_t = \frac{x^{t/2}}{4t} \quad \text{where} \quad x = v_1^2 - 4v_1^{-1}v_2$$

and

$$d(x) = 8\rho$$

where

$$\rho \equiv v_1^{-2}v_2t_1 - v_1^{-1}(t_2 + t_1^3) + 2(v_1t_1 + v_1^{-2}t_1^4 + v_1^{-2}t_1t_2 + v_1^{-3}v_2t_1^3) \quad \text{mod (4)}.$$

Hence for even  $t > 2$  the mod (2) reduction of  $\bar{\alpha}_t$  is  $v_1^{t-2}\rho$  and for odd  $s$

$$\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-2}}{2}\rho = \frac{v_1x^{(s+t-1)/2}}{2}\rho.$$

Since

$$d\left(\frac{v_1x^{(s+t+1)/2}}{2^3(s+t+1)}\right) = \frac{v_1}{2}x^{(s+t-1)/2}\rho + \frac{x^{(s+t+1)/2}t_1}{2^2(s+t+1)}$$

so  $\bar{\alpha}_s\bar{\alpha}_t = \bar{\alpha}_1\bar{\alpha}_{s+t-1}$  as claimed.

For (ii) we have  $\bar{\alpha}_2^2 = \frac{v_1^2(t_1^2 + v_1t_1)}{4}$ . The coboundary of  $\frac{v_1^4}{2^5} + \frac{v_1^{-2}v_2^2}{2}$  shows this is cohomologous to  $\beta_{2/2}$ .

For (iii) we have  $\alpha_{4/4}^2 \in \text{Ext}^{2,16}$  which is  $(\mathbf{Z}/(2))^3$  generated by  $\alpha_1\alpha_7$ ,  $\beta_3$ , and  $\beta_{4/4}$ .  $\alpha_1\alpha_7$  is not a permanent cycle (5.3.7) so  $\alpha_{4/4}^2$  must be a linear combination of  $\beta_{4/4}$  and  $\beta_3$ . Their reductions mod  $I_2$ ,  $t_1^4|t_1^4$  and  $v_2t_1^4|t_1$ , are linearly independent so it suffices to compute  $\alpha_{4/4}^2 \text{ mod } I_2$ . The mod  $I_2$  reduction of  $\alpha_{4/4}$  is  $t_1^4 + v_2t_1$ , so the result follows.  $\square$

## 6. Elements in $\text{Ext}^3$ and Beyond

We begin by considering products of elements in  $\text{Ext}^2$  with those in  $\text{Ext}^1$  and  $\text{Ext}^2$ . If  $x$  and  $y$  are two such elements known to be permanent cycles, then the nontriviality of  $xy$  in  $\text{Ext}$  implies that the corresponding product in homotopy is nontrivial, but if  $xy = 0$  then the homotopy product could still be nontrivial and represent an element in a higher  $\text{Ext}$  group. The same is true of relations among and divisibility of products of permanent cycles; they suggest but do not imply (without further argument) similar results in homotopy.

Ideally one should have a description of the subalgebra of  $\text{Ext}(BP_*)$  generated by  $\text{Ext}^1$  and  $\text{Ext}^2$  for all primes  $p$ . Our results are limited to odd primes and fall into three types (see also 5.5.8). First we describe the subgroup of  $\text{Ext}^3$  generated by products of elements in  $\text{Ext}^1$  with elements of order  $p$  in  $\text{Ext}^2$  (5.6.1). Second we

note that certain of these products are divisible by nontrivial powers of  $p$  (5.6.2). These two results are due to Miller, Ravenel, and Wilson [1], to which we refer for most of the proofs.

Our third result is due to Oka and Shimomura [9] and concerns products of certain elements in  $\text{Ext}^2$  (5.4.4–5.4.7). They show further that in certain cases when a product of permanent cycles is trivial in  $\text{Ext}^4$ , then the corresponding product in homotopy is also trivial.

This brings us to  $\gamma$ 's and  $\delta$ 's. Toda [1] showed that  $\gamma_t$  is a permanent cycle for  $p \geq 7$  (1.3.18), but left open the case  $p = 5$ . In Section 7.5 we will make calculations to show that  $\gamma_3$  does not exist. We sketch the argument here. As remarked in Section 4.4, 4.4.22 implies that  $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$  (up to a nonzero scalar). Calculations show that  $\alpha_1\beta_{5/5}^4$  is a linear combination of  $\beta_1^3\gamma_3$  and  $\beta_1\langle\alpha_1\beta_3, \beta_4, \gamma_2\rangle$ . Hence if the latter can be shown to be a permanent cycle then we must have  $d_{33}(\gamma_3) = \beta_1^{18}$ . Each of the factors in the above Massey product is a permanent cycle, so it suffices to show that the products  $\alpha_1\beta_3\beta_4 \in \pi_{323}(S^0)$  and  $\beta_4\gamma_2 \in \pi_{619}(S^0)$  both vanish. Our calculation shows that both of these stems have trivial 5-torsion.

To construct  $\delta_t$  one could proceed as in the proof of 5.5.2. For  $p \geq 7$  there is a finite complex  $V(3)$  with  $BP_*(V(3)) = BP_*/I_4$ . According to Toda [1] it is a ring spectrum for  $p \geq 11$ . Hence there is a self-map realizing multiplication by  $v_4$  iff there is a corresponding element in  $\pi_*(V(3))$ . We will show (5.6.13) that the group  $\text{Ext}^{2p-1, 2(p^4+p-2)}(BP_*/I_4)$  is nonzero for all  $p \geq 3$ , so it is possible that  $d_{2p-1}(v_4) \neq 0$ .

The following result was proved in Miller, Ravenel, and Wilson [1] as theorem 8.6.

5.6.1. THEOREM. *Let  $m \geq 0$ ,  $p \nmid s$ ,  $s \geq 1$ ,  $1 \leq j \leq a_{2,m}$  (where  $a_{2,m}$  is as in 5.4.1) for  $s > 1$  and  $1 \leq j \leq p^m$  for  $s = 1$ . Then  $\alpha_1\beta_{sp^m/j} \neq 0$  in  $\text{Ext}^3(BP_*)$  iff one of the following conditions holds*

- (i)  $j = 1$  and either  $s \not\equiv -1 \pmod{p}$  or  $s \equiv -1 \pmod{p^{m+2}}$ .
- (ii)  $j = 1$  and  $s = p - 1$ .
- (iii)  $j > 1 + a_{2,m-\nu(j-1)-1}$ .

*In case (ii), we have  $\alpha_1\beta_{p-1} = -\gamma_1$  and for  $m \geq 1$ ,  $2\alpha_1\beta_{(p-1)p^m} = -\gamma_{p^m/p^m, p^m}$ . The only linear relations among these classes are*

$$\alpha_1\beta_{sp^2/p+2} = s\alpha_1\beta_{sp^2-1},$$

and

$$\alpha_1\beta_{sp^{2m+2}/2+a_{2,m+1}} = 2s\alpha_1\beta_{sp^{2m+2}-p^m} \quad \text{for } m \geq 1. \quad \square$$

This result implies that some of these products vanish and therefore certain Massey products (A1.4.1) are defined. For example,  $\alpha_1\beta_{(tp-1)p^m} = 0$  if  $t > 1$  and  $p^{m+2} \nmid t$  so we have Massey products such as  $\langle\beta_{2p-1}, \alpha_1, \alpha_1\rangle$  represented up to nonzero scalar multiplication by

$$\frac{v_2^{2p}t_1}{pv_1^{1+p}} + \frac{v_2^{2p-1}t_1^2 - 2v_2^{p-1}v_3t_1}{pv_1}.$$

This product has order  $p^2$  but many others do not. For example,  $\alpha_1\beta_{p/2} = 0$  and  $\langle\beta_{p/2}, \alpha_1, \alpha_1\rangle$  is represented by

$$\frac{2v_1^{p-1}v_2^p t_1}{p^2 v_1^p} - \frac{v_2^p t_1^2}{pv_1^2}$$

which has order  $p^2$  and  $p\langle\beta_{p/2}, \alpha_1, \alpha_1\rangle = \alpha_1\beta_p$  up to nonzero scalar multiplication. Similarly, one can show

$$\alpha_1\beta_{p^2} = p\langle\beta_{p^2/2}, \alpha_1, \alpha_1\rangle = p^2\langle\beta_{p^2/3}, \alpha_1, \alpha_1, \alpha_1\rangle.$$

The following results were 2.8(c) and 8.17 in Miller, Ravenel, and Wilson [1].

5.6.2. THEOREM. *With notation as in 5.6.1, if  $\alpha_1\beta_{sp^m/j} \neq 0$  in  $\text{Ext}^3(BP_*)$ , then it is divisible by at least  $p^i$  whenever  $0 < i \leq m$  and  $j \geq a_{2,m-i}$ .*  $\square$

5.6.3. THEOREM. *With notation as above and with  $t$  prime to  $p$ ,  $\alpha_{sp^k/k+1}\beta_{tp^m/j} = s\alpha_1\beta_{tp^m/j-sp^k+1}$  in  $\text{Ext}^3(BP_*)$ .*  $\square$

Now we consider products of elements in  $\text{Ext}^2$ , which are studied in Oka and Shimomura [9].

5.6.4. THEOREM. *For  $p \geq 3$  we have  $ij\beta_s\beta_t = st\beta_i\beta_j$  in  $\text{Ext}^4$  for  $i + j = s + t$ .*

PROOF. To compute  $\beta_s\beta_t$  we need the mod  $I_2$  reduction of  $\beta_t$ , which was computed in 5.1.20. Hence we find  $\beta_s\beta_t$  is represented by

$$\frac{-tv_2^{s+t-1}b_{10} + \binom{t}{2}v_2^{s+t-2}k_0}{pv_1}.$$

Now let

$$u_m = \frac{v_2^m v_1^{p-1} t_1^p}{p^2 v_1^p} - \frac{v_2^m t_2}{pv_1^2} + \frac{kv_2^{k-1}}{pv_1} (t_1^p t_2 - t_1^{2p+1}).$$

A routine computation gives

$$d\left(\frac{t}{2} u_{s+t-1}\right) = \frac{sv_2^{s+t-1}b_{10}}{pv_1} - \frac{s}{2}(s+t-1)v_2^{s+t-2}k_0$$

and hence  $\beta_s\beta_t$  is represented by  $-\frac{st}{2} \frac{v_2^{s+t-2}k_0}{pv_1}$  and the result follows.  $\square$

The analogous result in homotopy for  $p \geq 5$  was first proved by Toda [7]. The next three results are 6.1, A, and B of Oka and Shimomura [9].

5.6.5. THEOREM. *For  $p \geq 3$  the following relations hold in  $\text{Ext}^4$  for  $s, t > 0$ .*

- (i)  $\beta_s\beta_{tp^k/r} = 0$  for  $k \geq 1$ ,  $t \geq 2$  and  $r < a_{2,k}$ .
- (ii)  $\beta_s\beta_{tp^2/p,2} = \beta_{s+t(p^2-p)}\beta_{tp/p}$ .
- (iii) For  $t, k \geq 2$ ,

$$\begin{aligned} \beta_2\beta_{tp^k/a_{2,k}} &= \beta_{s+(tp-1)(p^k-1-p)}\beta_{tp^2/a_{2,2}} \\ &= (t/2)\beta_{s+(tp-1)p^{k-1}-(2p-1)p}\beta_{2p^2/a_{2,2}}. \quad \square \end{aligned}$$

5.6.6. THEOREM. *For  $p \geq 5$ ,  $0 < r \leq p$ , with  $r \leq p-1$  if  $t = 1$ , the element  $\beta_s\beta_{tp/r}$  is trivial in  $\pi_*(S^0)$  if one of the following conditions holds.*

- (i)  $r \leq p-2$ .
- (ii)  $r = p-1$  and  $s \not\equiv -1 \pmod{p}$ .
- (iii)  $r = p-1$  or  $p$  and  $t \equiv 0 \pmod{p}$ .  $\square$

5.6.7. THEOREM. *For  $p \geq 5$ ,  $s \not\equiv 0$  or  $1$ ,  $t \not\equiv 0 \pmod{p}$ , and  $t \geq 2$ , the elements  $\beta_s\beta_{tp/p}$  and  $\beta_s\beta_{tp^2/p,2}$  are nontrivial.*  $\square$

Now we will display the obstruction to the existence of  $V(4)$ , i.e., a nontrivial element in  $\text{Ext}^{2p-1, 2(p^4+p-2)}(BP_*/I_4)$ . This group is isomorphic to the corresponding Ext group for  $P_* = P[t_1, t_2, \dots]$ , the dual to the algebra of Steenrod reduced powers. To compute this Ext we use a method described in Section 3.5. Let  $P(1)_* = P/(t_1^{p^2}, t_2^p, t_3, \dots)$ , the dual to the algebra generated by  $P^1$  and  $P^p$ . We will give  $P_*$  a decreasing filtration so that  $P(1)_*$  is a subalgebra of  $E_0P_*$ . We let  $t_1, t_2 \in F_0$ , and  $t_i^{p^2}, t_{i+1}^p, t_{i+2} \in F^{(p^i-1)/(p-1)}$  for  $i \geq 1$ . Then as an algebra we have

$$(5.6.8) \quad E_0P_* = P(1)_* \otimes T(t_{i+2,0}, t_{i+1,1}) \otimes P(t_{i,2}),$$

where  $i \geq 1$ ,  $t_{i,j}$  corresponds to  $t_i^{p^j}$ , and  $T$  denotes the truncated polynomial algebra of height  $p$ . Let  $R$  denote the tensor product of the second two factors in 5.6.8. Then

$$(5.6.9) \quad P(1)_* \rightarrow E_0P_* \rightarrow R$$

is an extension of Hopf algebras (A1.1.5) for which there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to

$$\text{Ext}_{E_0P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

with

$$(5.6.10) \quad E_2 = \text{Ext}_{P(1)_*}(\mathbf{Z}/(p), \text{Ext}_R(\mathbf{Z}/(p), \mathbf{Z}/(p))).$$

The filtration of  $P_*$  gives a spectral sequence (A1.3.9) converging to

$$\text{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

with

$$(5.6.11) \quad E_2 = \text{Ext}_{E_0P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

In the range of dimensions we need to consider, i.e., for  $t-s \leq 2(p^4-1)$   $\text{Ext}_R$  is easy to compute. We leave it to the reader to show that it is the cohomology of the differential  $P(1)_*$ -comodule algebra

$$E(h_{12}, h_{21}, h_{30}, h_{13}, h_{22}, h_{31}, h_{40}) \otimes P(b_{12}, b_{21}, b_{30})$$

with  $d(h_{22}) = h_{12}h_{13}$ ,  $d(h_{31}) = h_{21}h_{13}$ , and  $d(h_{40}) = h_{30}h_{13}$ . In our range this cohomology is

$$(5.6.12) \quad E(h_{12}, h_{21}, h_{30}, h_{13})/h_{13}(h_{12}, h_{21}, h_{30}) \otimes P(b_{12}, b_{21}, b_{30}),$$

where the nontrivial action of  $P(1)$  is given by

$$P^1h_{30} = h_{21}, \quad P^ph_{21} = h_{12}, \quad \text{and} \quad P^pb_{30} = b_{21}.$$

We will not give all of the details of the calculations since our aim is merely to find a generator of  $\text{Ext}_*^{2p-1, 2(p^4+p-2)}$ . The element in question is

$$(5.6.13) \quad b_{20}^{p-3}h_{11}h_{20}h_{12}h_{21}h_{30}.$$

We leave it to the interested reader to decipher this notation and verify that it is a nontrivial cocycle.

## Morava Stabilizer Algebras

In this chapter we develop the theory which is the mainspring of the chromatic spectral sequence. Let  $K(n)_* = \mathbf{Z}/(p)[v_n, v_n^{-1}]$  have the  $BP_*$ -module structure obtained by sending all  $v_i$ , with  $i \neq n$  to 0. Then define  $\Sigma(n)$  to be the Hopf algebra  $K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$ . We will describe this explicitly as a  $K(n)_*$ -algebra below. Its relevance to the Adams–Novikov spectral sequence is the isomorphism (6.1.1)

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*),$$

which is input needed for the chromatic spectral sequence machinery described in Section 5.1. In combination with 6.2.4, this is the result promised in 1.4.9. Since  $\Sigma(n)$  is much smaller than  $BP_*(BP)$ , this result is a great computational aid. We will prove it along with some generalizations in Section 1, following Miller and Ravenel [5] and Morava [2].

In Section 2 we study  $\Sigma(n)$ , the  $n$ th Morava stabilizer algebra. We will show (6.2.5) that it is closely related to the  $\mathbf{Z}/(p)$ -group algebra of a pro- $p$ -group  $S_n$  (6.2.3 and 6.2.4).  $S_n$  is the strict automorphism group [i.e., the group of automorphisms  $f(x)$  having leading term  $x$ ] of the height  $n$  formal group law  $F_n$  (see A2.2.18 for a description of the corresponding endomorphism ring). We use general theorems from the cohomology of profinite groups to show  $S_n$  is either  $p$ -periodic (if  $(p-1) \mid n$ ) or has cohomological dimension  $n^2$  (6.2.10).

In Section 3 we study this cohomology in more detail. The filtration of 4.3.24 leads to a May spectral sequence studied in 6.3.3 and 6.3.4. Then we compute  $H^1$  (6.3.12) and  $H^2$  (6.3.14) for all  $n$  and  $p$ . The section concludes with computations of the full cohomology for  $n = 1$  (6.3.21),  $n = 2$  and  $p > 3$  (6.3.22),  $n = 2$  and  $p = 3$  (6.3.24),  $n = 2$  and  $p = 2$  (6.3.27), and  $n = 3$ ,  $p > 3$  (6.3.32).

The last two sections concern applications of this theory. In Section 4 we consider certain elements  $\beta_{p^i/p^i}$  in  $\mathrm{Ext}^2(BP_*)$  for  $p > 2$  analogous to the Kervaire invariant elements  $\beta_{2^i/2^i}$  for  $p = 2$ . We show (6.4.1) that these elements are *not* permanent cycles in the Adams–Novikov spectral sequence. A crucial step in the proof uses the fact that  $S_{p-1}$  has a subgroup of order  $p$  to detect a lot of elements in  $\mathrm{Ext}$ . Theorem 6.4.1 provides a test that must be passed by any program to prove the Kervaire invariant conjecture: it must not generalize to odd primes!

In Section 5 we construct ring spectra  $T(m)$  satisfying  $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$  as comodules. The algebraic properties of these spectra will be exploited in the next chapter. We will show (6.5.5, 6.5.6, 6.5.11, and 6.5.12) that its Adams–Novikov  $E_2$ -term has nice properties.

### 1. The Change-of-Rings Isomorphism

Our first objective is to prove

6.1.1. THEOREM (Miller and Ravenel [5]). *Let  $M$  be a  $BP_*(BP)$ -comodule annihilated by  $I_n = (p, v_1, \dots, v_{n-1})$ , and let  $\overline{M} = M \otimes_{BP_*} K(n)_*$ . Then there is a natural isomorphism*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \mathrm{Ext}_{\Sigma(n)}(K(n)_*, \overline{M}).$$

Here  $v_n^{-1}M$  denotes  $v_n^{-1}BP_* \otimes_{BP_*} M$ , which is a comodule (even though  $v_n^{-1}BP_*$  is not) by 5.1.6.  $\square$

This result can be generalized in two ways. Let

$$E(n)_* = v_n^{-1}BP_*/(v_{n+i}: i > 0)$$

and

$$E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*.$$

It can be shown, using the exact functor theorem of Landweber [3], that  $E(n)_* \otimes_{BP_*} BP_*(X)$  is a homology theory on  $X$  represented by a spectrum  $E(n)$  with  $\pi_*(E(n)) = E(n)_*$ , and with  $E(n)_*(E(n))$  being the object defined above. We can generalize 6.1.1 by replacing  $\Sigma(n)$  with  $E(n)_*(E(n))$  and relaxing the hypothesis on  $M$  to the condition that it be  $I_n$ -nil, i.e., that each element (but not necessarily the entire comodule) be annihilated by some power of  $I_n$ . For example,  $N^n$  of Section 5.1 is  $I_n$ -nil. Then we have

6.1.2. THEOREM (Miller and Ravenel [5]). *Let  $M$  be  $I_n$ -nil and let  $\overline{M} = M \otimes_{BP_*} E(n)_*$ . Then there is a natural isomorphism*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \mathrm{Ext}_{E(n)_*(E(n))}(E(n)_*, \overline{M}).$$

There is another variation due to Morava [2]. Regard  $BP_*$  as a  $\mathbf{Z}/2(p^n - 1)$ -graded object and consider the homomorphism  $\theta: BP_* \rightarrow \mathbf{Z}/(p)$  given by  $\theta(v_n) = 1$  and  $\theta(v_i) = 0$  for  $i \neq n$ . Let  $I \subset BP_*$  be  $\ker \theta$  and let  $V_\theta$  and  $VT_\theta$  denote the  $I$ -adic completions of  $BP_*$  and  $BP_*(BP)$ . Let  $E_\theta = V_\theta(v_{n+i}: i > 0)$  and  $EH_\theta = E_\theta \otimes_{V_\theta} VT_\theta \otimes_{V_\theta} E_\theta$ .

6.1.3. THEOREM (Morava [2]). *With notation as above there is a natural isomorphism*

$$\mathrm{Ext}_{VT_\theta}(V_\theta, M) \cong \mathrm{Ext}_{EH_\theta}(E_\theta, \overline{M})$$

where  $M$  is a  $VT_\theta$ -comodule and  $\overline{M} = M \otimes_{V_\theta} E_\theta$ .  $\square$

Of these three results only 6.1.1 is relevant to our purposes so we will not prove the others in detail. However, Morava's proof is more illuminating than that of Miller and Ravenel [5] so we will sketch it first.

Morava's argument rests on careful analysis of the functors represented by the Hopf algebroids  $VT_\theta$  and  $EH_\theta$ . First we need some general nonsense.

Recall that a groupoid is a small category in which every morphism is invertible. Recall that a Hopf algebroid  $(A, \Gamma)$  over  $K$  is a cogroupoid object in the category of commutative  $K$ -algebras; i.e., it represents a covariant groupoid-valued functor. Let  $\alpha, \beta: \mathbf{G} \rightarrow \mathbf{H}$  be maps (functors) from the groupoid  $\mathbf{G}$  to the groupoid  $\mathbf{H}$ . Since  $\mathbf{G}$  is a category it has a set of objects,  $Ob(\mathbf{G})$ , and a set of morphisms,  $Mor(\mathbf{G})$ , and similarly for  $\mathbf{H}$ .



6.1.4. DEFINITION. *The functors  $\alpha, \beta: G \rightarrow H$  are equivalent if there is a map  $\theta: \text{Ob}(\mathbf{G}) \rightarrow \text{Mor}(\mathbf{H})$  such that for any morphism  $g: g_1 \rightarrow g_2$  in  $\mathbf{G}$  the diagram*

$$\begin{array}{ccc} \alpha(g_1) & \xrightarrow{\alpha(g)} & \alpha(g_2) \\ \theta(g_1) \downarrow & & \theta(g_2) \downarrow \\ \beta(g_1) & \xrightarrow{\beta(g)} & \beta(g_2) \end{array}$$

*commutes. Two maps of Hopf algebroids  $a, b: (A, \Gamma) \rightarrow (B, \Sigma)$  are naturally equivalent if the corresponding natural transformations of groupoid-valued functors are naturally equivalent in the above sense. Two Hopf algebroids  $(A, \Gamma)$  and  $(B, \Sigma)$  are equivalent if there are maps  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  and  $h: (B, \Sigma) \rightarrow (A, \Gamma)$  such that  $hf$  and  $fh$  are naturally equivalent to the appropriate identity maps.  $\square$*

Now we will show that a Hopf algebroid equivalence induces an isomorphism of certain Ext groups. Given a map  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  and a left  $\Gamma$ -comodule  $M$ , define a  $\Sigma$ -comodule  $f^*(M)$  to be  $B \otimes_A M$  with coactions

$$B \otimes_A M \rightarrow B \otimes_A \Gamma \otimes_A M \rightarrow B \otimes_B \Sigma \otimes_A M = \Sigma \otimes_B B \otimes_A M.$$

6.1.5. LEMMA. *Let  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  a Hopf algebroid equivalence. Then there is a natural isomorphism  $\text{Ext}_\Gamma(A, M) \cong \text{Ext}_\Sigma(B, f^*(M))$  for any  $\Gamma$ -comodule  $M$ .*

PROOF. It suffices to show that equivalent maps induce the same homomorphisms of Ext groups. An equivalence between the maps  $a, b: (A, \Gamma) \rightarrow (B, \Sigma)$  is a homomorphism  $\phi: \Gamma \rightarrow B$  with suitable properties, including  $\phi\eta_R = a$  and  $\phi\eta_L = b$ . Since  $\eta_R$  and  $\eta_L$  are related by the conjugation in  $\Gamma$ , it follows that the two  $A$ -module structures on  $B$  are isomorphic and that  $a^*(M)$  is naturally isomorphic to  $b^*(M)$ . We denote them interchangeably by  $M'$ . The maps  $a$  and  $b$  induce maps of cobar complexes (A1.2.11)  $C_\Gamma(M) \rightarrow C_\Sigma(M')$ . A tedious routine verification shows that  $\phi$  induces the required chain homotopy.  $\square$

Now we consider the functors represented by  $VT_\theta$  and  $EH_\theta$ . Recall that an *Artin local ring* is a commutative ring with a single maximal ideal satisfying the descending chain condition, i.e., the maximal ideal is nilpotent. If  $A$  is such a ring with finite residue field  $k$  then it is  $W(k)$ -module, where  $W(k)$  is the Witt ring of A2.2.16. Let  $\mathbf{Art}_\theta$  denote the category of  $\mathbf{Z}/(2(p^n - 1))$ -graded Artin local rings whose residue field is an  $\mathbf{F}_p$ -algebra. Now let  $m_\theta = \ker \theta \subset BP_*$ . Then  $BP_*/m_\theta^n$  with the cyclic grading is an object in  $\mathbf{Art}_\theta$ , so  $V_\theta = \varprojlim BP_*/m_\theta^n$  is an inverse limit of such objects as is  $VT_\theta$ . For any  $A \in \mathbf{Art}_\theta$ , we can consider  $\text{Hom}^c(VT_\theta, A)$ , the set of continuous ring homomorphisms from  $VT_\theta$  to  $A$ . It is a groupoid-valued functor on  $\mathbf{Art}_\theta$  pro-represented by  $VT_\theta$ . (We have to say “pro-represented” rather than “represented” because  $VT_\theta$  is not in  $\mathbf{Art}_\theta$ .)

6.1.6. PROPOSITION.  *$VT_\theta$  pro-represents the functor  $\mathbf{lifts}_\theta$  from  $\mathbf{Art}_\theta$  to the category of groupoids, defined as follows. Let  $A \in \mathbf{Art}_\theta$  have residue field  $k$ . The objects in  $\mathbf{lifts}_\theta(A)$  are  $p$ -typical liftings to  $A$  of the formal group law over  $k$  induced by the composite  $BP_* \xrightarrow{\theta} \mathbf{F}_p \rightarrow k$ , and morphisms in  $\mathbf{lifts}_\theta(A)$  are strict isomorphisms between such liftings.  $\square$*

6.1.7. DEFINITION. *Let  $m_A \subset A$  be the maximal ideal for  $A \in \mathbf{Art}_\theta$ . Given a homomorphism  $f: F \rightarrow G$  of formal group laws over  $A$ , let  $\bar{f}: \bar{F} \rightarrow \bar{G}$  denote their reductions mod  $m_A$ .  $f$  is a  $*$ -isomorphism if  $\bar{f}(x) = x$ .*

6.1.8. LEMMA. *Let  $F$  and  $G$  be objects in  $\mathbf{lifts}_\theta(A)$ . Then the map  $\mathrm{Hom}(F, G) \rightarrow \mathrm{Hom}(\overline{F}, \overline{G})$  is injective.*

PROOF. Suppose  $\bar{f} = 0$ , i.e.,  $f(x) = 0 \pmod{m_A}$ . We will show that  $f(x) \equiv 0 \pmod{m_A^r}$  implies  $f(x) \equiv 0 \pmod{m_A^{r+1}}$  for any  $r > 0$ , so  $f(x) = 0$  since  $m_A$  is nilpotent. We have

$$G(f(x), f(y)) \equiv f(x) + f(y) \pmod{m_A^{2r}}$$

since

$$G(x, y) \equiv x + y \pmod{(x, y)^2}.$$

Consequently,

$$[p]_G(f(x)) \equiv pf(x) \pmod{m_A^{2r}} \equiv 0 \pmod{m_A^{r+1}}$$

since  $p \in m_A$ . On the other hand

$$[p]_G(f(x)) = f([p]_F(x))$$

and we know  $[p]_F(x) \equiv x^{p^n} \pmod{m_A}$  by A2.2.4. Hence  $f([p]_F(x)) \equiv 0 \pmod{m_A^{r+1}}$  gives the desired congruence  $f(x) \equiv 0 \pmod{m_A^{r+1}}$ .  $\square$

Now suppose  $f_1, f_2: F \rightarrow G$  are  $*$ -isomorphisms (6.1.7) as is  $g: G \rightarrow F$ . Then  $gf_1 = gf_2$  by 6.1.8 so  $f_1 = f_2$ ; i.e.  $*$ -isomorphisms are unique. Hence we can make

6.1.9. DEFINITION.  $\mathbf{lifts}_\theta^*(A)$  is the groupoid of  $*$ -isomorphism classes of objects in  $\mathbf{lifts}_\theta A$ .

6.1.10. LEMMA. *The functors  $\mathbf{lifts}_\theta$  and  $\mathbf{lifts}_\theta^*$  are naturally equivalent.*

PROOF. There is an obvious natural transformation  $\alpha: \mathbf{lifts}_\theta \rightarrow \mathbf{lifts}_\theta^*$ , and we need to define  $\beta: \mathbf{lifts}_\theta^* \rightarrow \mathbf{lifts}_\theta$ , of each  $*$ -isomorphism class. Having done this,  $\alpha\beta$  will be the identity on  $\mathbf{lifts}_\theta^*$  and we will have to prove  $\beta\alpha$  is equivalent (6.1.4) to the identity on  $\mathbf{lifts}_\theta$ .

The construction of  $\beta$  is essentially due to Lubin and Tate [3]. Suppose  $G_1 \in \mathbf{lifts}_\theta(A)$  is induced by  $\theta_1: BP_* \rightarrow A$ . Using A2.1.26 and A2.2.6 one can show that there is a unique  $G_2 \in \mathbf{lifts}_\theta(A)$   $*$ -isomorphic to  $G_1$  and induced by  $\theta_2$  satisfying  $\theta(v_{n+i}) = 0$  for all  $i > 0$ . We leave the details to the interested reader. As remarked above, the  $*$ -isomorphism from  $G_1$  to  $G_2$  is unique. The verification that  $\beta\alpha$  is equivalent to the identity is straightforward.  $\square$

To prove 6.1.3, it follows from 6.1.5 and 6.1.10 that it suffices to show  $EH_\theta$  pro-represents  $\mathbf{lifts}_\theta^*$ . In the proof of 6.1.10 it was claimed that any suitable formal group law over  $A$  is canonically  $*$ -isomorphic to one induced by  $\theta: BP_* \rightarrow A$  which is such that it factors through  $E_\theta$ . In the same way it is clear that the morphism set of  $\mathbf{lifts}_\theta^*(A)$  is represented by  $EH_\theta$ , so 6.1.3 follows.

Now we turn to the proof of 6.1.1. We have a map  $BP_*(BP) \rightarrow \Sigma(n)$  and we need to show that it satisfies the hypotheses of the general change-of-rings isomorphism theorem A1.3.12, i.e., of A1.1.19. These conditions are

- (6.1.11) (i) the map  $\Gamma' = BP_*(BP) \otimes_{BP_*} K(n)_* \rightarrow \Sigma(n)$  is onto and  
(ii)  $\Gamma' \square_{\Sigma(n)} K(n)_*$  is a  $K(n)_*$ -summand of  $\Gamma'$ .

Part (i) follows immediately from the definition  $\Sigma(n) = K(n)_* \otimes_{BP_*} \Gamma'$ . Part (ii) is more difficult. We prefer to replace it with its conjugate,

(ii)  $K(n)_* \square_{\Sigma(n)} K(n)_*(BP)$  is a  $K(n)_*$  summand of  $K(n)_*BP$  which is defined to be  $K(n)_* \otimes_{BP_*} BP_*(BP)$ . Let  $B(n)_*$  denote  $v_n^{-1}BP_*/I_n$ . Then the right  $BP_*$ -module structure on  $K(n)_*(BP)$  induces a right  $B(n)_*$ -module structure.

6.1.12. LEMMA. *There is a map*

$$K(n)_*BP \rightarrow \Sigma(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of  $\Sigma(n)$ -comodules and of  $B(n)_*$ -modules, and which carries 1 to 1.

PROOF. Our proof is a counting argument, and in order to meet requirements of connectivity and finiteness, we pass to suitable “valuation rings”. Thus let

$$\begin{aligned} k(0)_* &= \mathbf{Z}_{(p)} \subset K(0)_*, \\ k(n)_* &= \mathbf{F}_p[v_n] \subset K(n)_*, \quad n > 0, \\ k(n)_*BP &= k(n)_* \otimes_{BP_*} BP_*(BP) \subset K(n)_*BP, \\ b(n)_* &= k(n)_*[u_1, u_2, \dots] \subset B(n)_*, \end{aligned}$$

where  $u_k = v_n^{-1}v_{n+k}$ .

It follows from A2.2.6 that in  $k(n)_*BP$ ,

$$(6.1.13) \quad \eta_R(v_{n+k}) \equiv v_n t_k^{p^n} - v_n^{p^k} t_k \pmod{(\eta_R(v_{n+1}), \dots, \eta_R(v_{n+k-1}))}.$$

Hence  $\eta_R: BP_* \rightarrow k(n)_*BP$  factors through an algebra map  $b(n)_* \rightarrow k(n)_*BP$ . It is clear from 6.1.13 that as a right  $b(n)_*$ -module,  $k(n)_*BP$  is free on generators  $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \dots$  where  $0 \leq \alpha_i < p^n$  and all but finitely many  $\alpha_i$  are 0; in particular, it is of finite type over  $b(n)_*$ .

Now define

$$\sigma(n) = k(n)_*BP \otimes_{b(n)_*} k(n)_* \subset \Sigma(n);$$

by the above remarks  $\sigma(n) = k(n)_*[t_1, t_2, \dots]/(t_k^{p^n} - v_n^{p^k-1}t_k: k \geq 1)$  as an algebra.  $(k(n)_*, \sigma(n))$  is clearly a sub-Hopf algebroid of  $(K(n)_*, \Sigma(n))$ , so  $\sigma(n)$  is a Hopf algebra over the principal ideal domain  $k(n)_*$ .

The natural map  $BP_*(BP) \rightarrow \sigma(n)$  makes  $BP_*(BP)$  a left  $\sigma(n)$ -comodule, and this induces a left  $\sigma(n)$ -comodule structure on  $k(n)_*BP$ . We will show that the latter is an *extended* left  $\sigma(n)$ -comodule.

Define a  $b(n)_*$ -linear map  $f: k(n)_*BP \rightarrow b(n)_*$  by

$$f(t^\alpha) = \begin{cases} 1 & \text{if } \alpha = (0, 0, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

Then  $f$  satisfies the equations

$$\begin{aligned} f\bar{\eta}_R &= id: b(n)_* \rightarrow b(n)_*, \\ f \otimes_{b(n)_*} k(n)_* &= \varepsilon: \sigma(n) \rightarrow k(n)_*. \end{aligned}$$

Now let  $\tilde{f}$  be the  $\sigma(n)$ -comodule map lifting  $f$ :

$$(6.1.14) \quad \begin{array}{ccc} k(n)_*BP & \xrightarrow{\psi} & \sigma(n) \otimes_{k(n)_*} k(n)_*BP \\ & \searrow \tilde{f} & \downarrow \sigma(n) \otimes f \\ & & \sigma(n) \otimes_{k(n)_*} b(n)_* \end{array}$$

Since  $\psi\bar{\eta}_R(x) = 1 \otimes \bar{\eta}_R(x)$ ,  $\psi$  is  $b(n)_*$ -linear, so  $\tilde{f}$  is too. We claim  $\tilde{f}$  is an isomorphism. Since both sides are free of finite type over  $b(n)_*$  it suffices to prove that  $\tilde{f} \otimes_{b(n)_*} k(n)$  is an isomorphism. But 6.1.14 is then reduced to

$$\begin{array}{ccc} \sigma(n) & \xrightarrow{\Delta} & \sigma(n) \otimes_{k(n)_*} \sigma(n) \\ & \searrow \tilde{f} \otimes_{b(n)_*} k(n)_* & \downarrow 1 \otimes \varepsilon \\ & & \sigma(n) \otimes_{k(n)_*} k(n)_* \end{array}$$

so the claim follows from unitarity of  $\Delta$ .

Now the map  $K(n)_* \otimes_{k(n)_*} \tilde{f}$  satisfies the requirements of the lemma.  $\square$

6.1.15. COROLLARY.  $\bar{\eta}_R: B(n)_* \rightarrow K(n)_* \square_{\Sigma(n)} K(n)_* BP$  is an isomorphism of  $B(n)_*$ -modules.

PROOF. The natural isomorphism

$$B(n)_* \rightarrow K(n)_* \square_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*)$$

is  $B(n)_*$ -linear and carries 1 to 1. Hence

$$\begin{array}{ccc} & & K(n)_* \square_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*) \\ & \nearrow \cong & \uparrow \cong \\ B(n)_* & & K(n)_* \square_{\Sigma(n)} K(n)_* BP \\ & \searrow \bar{\eta}_R & \end{array}$$

commutes, and  $\bar{\eta}_R$  is an isomorphism.  $\square$

Hence 6.1.11(ii) follows from the fact that  $K(n)_*$  is a summand of  $\Sigma(n)$ , and 6.1.1 is proved. From the proof of 6.1.12 we get an explicit description of  $\Sigma(n)$ , namely

6.1.16. COROLLARY. As an algebra

$$\Sigma(n) = K(n)_*[t_1, t_2, \dots] / (v_n t_i^{p^n} - v_n^{p^i} t_i : i > 0).$$

Its coproduct is inherited from  $BP_*(BP)$ , i.e., a suitable reduction of 4.3.13 holds.

## 2. The Structure of $\Sigma(n)$

To study  $\Sigma(n)$  it is convenient to pass to the corresponding object graded over  $\mathbf{Z}/2(p^n - 1)$ . Make  $\mathbf{F}_p$  a  $K(n)_*$ -module by sending  $v_n$  to 1, and let  $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_p$ . For a  $\Sigma(n)$ -comodule  $M$  let  $\bar{M} = M \otimes_{K(n)_*} \mathbf{F}_p$ , which is easily seen to be an  $S(n)$ -comodule. The categories of  $\Sigma(n)$ - and  $S(n)$ -comodules are equivalent and we have

6.2.1. PROPOSITION. For a  $\Sigma(n)$ -comodule  $M$ ,

$$\text{Ext}_{\Sigma(n)}(K(n)_*, M) \otimes_{K(n)_*} \mathbf{F}_p \cong \text{Ext}_{S(n)}(\mathbf{F}_p, \bar{M}). \quad \square$$

We will see below (6.2.5) that if we regard  $S(n)$  and  $\overline{M}$  as graded merely over  $\mathbf{Z}/(2)$ , there is a way to recover the grading over  $\mathbf{Z}/2(p^n - 1)$ . If  $M$  is concentrated in even dimensions (which it is in most applications) then we can regard  $\overline{M}$  and  $S(n)$  as ungraded objects. Our first major result is that  $S(n) \otimes \mathbf{F}_{p^n}$  (ungraded) is the continuous linear dual of the  $\mathbf{F}_{p^n}$ -group algebra of a certain profinite group  $S_n$  to be defined presently.

6.2.2. DEFINITION. *The topological linear dual  $S(n)^*$  of  $S(n)$  is as follows. [In Ravenel [5]  $S(n)^*$  and  $S(n)$  are denoted by  $S(n)$  and  $S(n)_*$ , respectively.] Let  $S(n)_{(i)}$  be the sub-Hopf algebra of  $S(n)$  generated by  $\{t, \dots, t_i\}$ . It is a vector space of rank  $p^{ni}$  and  $S(n) = \varinjlim S(n)_{(i)}$ . Then  $S(n)^* = \varprojlim \text{Hom}(S(n)_{(i)}, \mathbf{F}_p)$ , equipped with the inverse limit topology. The product and coproduct in  $S(n)$  give maps of  $S(n)^*$  to and from the completed tensor product*

$$S(n)^* \widehat{\otimes} S(n)^* = \varprojlim \text{Hom}(S(n)_{(i)} \otimes S(n)_{(j)}, \mathbf{F}_p).$$

To define the group  $S_n$  recall the  $\mathbf{Z}_p$ -algebra  $E_n$  of A2.2.17, the endomorphism ring of a height  $n$  formal group law. It is a free  $\mathbf{Z}_p$ -algebra of rank  $n^2$  generated by  $\omega$  and  $S$ , where  $\omega$  is a primitive  $(p^n - 1)$ th root of units,  $S\omega = \omega^p S$ , and  $S^n = p$ .  $S_n \subset E_n^\times$ , is the group of units congruent to 1 mod  $(S)$ , the maximal ideal in  $E_n$ .  $S_n$  is a profinite group, so its group algebra  $\mathbf{F}_{p^n}[S_n]$  has a topology and is a profinite Hopf algebra.  $S_n$  is also a  $p$ -adic Lie group; such groups are studied by Lazard [4].

6.2.3. THEOREM.  *$S(n)^* \otimes \mathbf{F}_q \cong \mathbf{F}_q[S_n]$  as profinite Hopf algebras, where  $q = p^n$ ,  $S_n$  is as above, and we disregard the grading on  $S(n)^*$ .*

PROOF. First we will show  $S(n)^* \otimes \mathbf{F}_q$  is a group algebra. According to Sweedler [1], Proposition 3.2.1, a cocommutative Hopf algebra is a group algebra iff it has a basis of group-like elements, i.e., of elements  $x$  satisfying  $\Delta x = x \otimes x$ . This is equivalent to the existence of a dual basis of idempotent elements  $\{y\}$  satisfying  $y_i^2 = y_i$ , and  $y_i y_j = 0$  for  $i \neq j$ . Since  $S(n) \otimes \mathbf{F}_q$  is a tensor product of algebras of the form  $R = \mathbf{F}_q[t]/(t^q - t)$ , it suffices to find such a basis for  $R$ . Let  $a \in \mathbf{F}_q^\times$  be a generator and let

$$r_i = \begin{cases} - \sum_{0 < j < q} (a^i t)^j & \text{for } 0 < i < q, \\ 1 - t^{q-1} & \text{for } i = 0. \end{cases}$$

Then  $\{r_i\}$  is such a basis, so  $S(n)^* \otimes \mathbf{F}_q$  is a group algebra.

Note that tensoring with  $\mathbf{F}_q$  cannot be avoided, as the basis of  $R$  is not defined over  $\mathbf{F}_p$ .

For the moment let  $G_n$  denote the group satisfying  $\mathbf{F}_p[G_n] \cong S(n)^* \otimes \mathbf{F}_q$ . To get at it we define a completed left  $S(n)$ -comodule structure on  $\mathbf{F}_q[[x]]$ , thereby defining a left  $G_n$ -action. Then we will show that it coincides with the action of  $S_n$  as formal group law automorphisms given by A2.2.18.

We now define the comodule structure map

$$\psi: \mathbf{F}_q[[x]] \rightarrow S(n) \widehat{\otimes} \mathbf{F}_q[[x]]$$

to be an algebra homomorphism given by

$$\psi(x) = \sum_{i \geq 0}^F t_i \otimes x^{p^i},$$

where  $t_0 = 1$  as usual. To verify that this makes sense we must show that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{F}_q[[x]] & \xrightarrow{\psi} & S(n) \widehat{\otimes} \mathbf{F}_q[[x]] \\ \psi \downarrow & & \downarrow \Delta \otimes 1 \\ S(n) \widehat{\otimes} \mathbf{F}_q[[x]] & \xrightarrow{1 \otimes \psi} & S(n) \widehat{\otimes} S(n) \widehat{\otimes} \mathbf{F}_q[[x]] \end{array}$$

for which we have

$$\begin{aligned} (\Delta \otimes 1)\psi(x) &= (\Delta \otimes 1) \sum_{i \geq 0}^F t_i \otimes x^{p^i} \\ &= \sum_{i \geq 0}^F \left( \sum_{j-k=i}^F t_j \otimes t_k^{p^j} \right) \otimes x^{p^i} \\ &= \sum_{j,k \geq 0}^F t_j \otimes t_k^{p^j} \otimes x^{p^{j+k}} \end{aligned}$$

This can be seen by inserting  $x$  as a dummy variable in 4.3.12. We also have

$$\begin{aligned} (1 \otimes \psi)\psi(x) &= (1 \otimes \psi) \left( \sum_{j \geq 0}^F t_j \otimes x^{p^j} \right) \\ &= \sum_{j \geq 0}^F t_j \otimes \left( \sum_{j \geq 0}^F t_j \otimes x^{p^j} \right)^{p^i} \\ &= \sum_{i,j \geq 0}^F t_i \otimes t_j^{p^i} \otimes x^{p^{i+j}}. \end{aligned}$$

The last equality follows from the fact that  $F(x^p, y^p) = F(x, y)^p$ . The linearity of  $\psi$  follows from A2.2.21(b), so  $\psi$  defines an  $S(n) \otimes \mathbf{F}_q$ -comodule structure on  $\mathbf{F}_q[[x]]$ .

We can regard the  $t_i$ , as continuous  $\mathbf{F}_q$ -valued functions on  $G_n$  and define an action of  $G_n$  on the algebra  $\mathbf{F}_q[[x]]$  by

$$g(x) = \sum_{i \geq 0}^F t_i(g) x^{p^i}$$

for  $g \in G_n$ . Hence  $G(x) = x$  iff  $g = 1$ , so our representation is faithful.

We can embed  $G_n$  in the set of all power series of the form  $\sum_{i \geq 0}^F a_i x^{p^i}$  which is  $E_n$  by A2.2.21 so the result follows.  $\square$

6.2.4. COROLLARY. *If  $M$  is an ungraded  $S(n)$ -comodule, then 6.2.3 gives a continuous  $S_n$ -action on  $M \otimes \mathbf{F}_q$ , and*

$$\mathrm{Ext}_{S(n)}^*(\mathbf{F}_p, M) \otimes \mathbf{F}_q = H_c^*(G_n, M \otimes \mathbf{F}_q)$$

where  $H_c^*$  denotes continuous group cohomology.  $\square$

To recover the grading on  $S(n) \otimes M$ , we have an action of the cyclic group of order  $q - 1$  generated by  $\bar{\omega}^i \omega^i$  via conjugation in  $E_n$ .

6.2.5. PROPOSITION. *The eigenspace of  $S(n) \otimes \mathbf{F}_q$  with eigenvalue  $\bar{\omega}^i$  is the component  $S(n)_{2i} \otimes \mathbf{F}_q$  of degree  $2i$ .*

PROOF. The eigenspace decomposition is multiplicative in the sense that if  $x$  and  $y$  are in the eigenspaces with eigenvalues  $\bar{\omega}^i$  and  $\bar{\omega}^j$ , respectively, the  $xy$  is in the eigenspace with eigenvalue  $\bar{\omega}^{i+j}$ . Hence it suffices to show that  $t_k$  is in the eigenspace with eigenvalue  $\bar{\omega}^{p^k-1}$ .

To see this we compute the conjugation of  $t_k S^k \in E_n$  by  $\omega$  and we have  $\omega^{-1}(t_k S^k)\omega = \omega^{-1}t_k \omega^{p^k} S^k = \omega^{p^k-1}t_k S^k$ .  $\square$

Corollary 6.2.4 enables us to apply certain results from group cohomology theory to our situation. First we give a matrix representation of  $E_n$  over  $W(\mathbf{F}_q)$ .

6.2.6. PROPOSITION. *Let  $e = \sum_{0 \leq i < n} e_i S^i$  with  $e_i \in W(\mathbf{F}_q)$  be an element of  $E_n$ . Define an  $n \times n$  matrix  $(e_{i,j})$  over  $W(\mathbf{F}_q)$  by*

$$e_{i+1,j+1} = \begin{cases} e_{j-i}^{\sigma^i} & \text{for } i \leq j \\ p e_{j+n-i}^{\sigma^i} & \text{for } i > j. \end{cases}$$

Then (a) this defines a faithful representation of  $E_n$ ; (b) the determinant  $|e_{i,j}|$  lies in  $\mathbf{Z}_p$ .

PROOF. Part (a) is straightforward. For (b) it suffices to check that  $\omega$  and  $S$  give determinants in  $\mathbf{Z}_p$ .  $\square$

We can now define homomorphisms  $c: \mathbf{Z}_p \rightarrow S_n$  and  $d: S_n \rightarrow \mathbf{Z}_p$  for  $p > 2$ , and  $c: \mathbf{Z}_2^\times \rightarrow S_n$  and  $d: \mathbf{Z}_2^\times \rightarrow \mathbf{Z}_2^\times$  for  $p = 2$  by identifying  $S_n$  with the appropriate matrix group. ( $\mathbf{Z}_p$  is to be regarded here as a subgroup of  $\mathbf{Z}_p^\times$ .) Let  $d$  be the determinant for all primes. For  $p > 2$  let  $c(x) = \exp(px)I$ , where  $I$  is the  $n \times n$  identity matrix and  $x \in \mathbf{Z}_p$ ; for  $p = 2$  let  $c(x) = xI$  for  $x \in \mathbf{Z}_2^\times$ .

6.2.7. THEOREM. *Let  $S_n^1 = \ker d$ .*

(a) *If  $p > 2$  and  $p \nmid n$  then  $S_n \cong \mathbf{Z}_p \oplus S_n^1$ .*

(b) *If  $p = 2$  and  $n$  is odd then  $S_n \cong S_n^1 \oplus \mathbf{Z}_2^\times$ .*

PROOF. In both cases one sees that  $\text{im } c$  lies in the center of  $S_n$  (in fact  $\text{im } c$  is the center of  $S_n$ ) and is therefore a normal subgroup. The composition  $dc$  is multiplication by  $n$  which is an isomorphism for  $p \nmid n$ , so we have the desired splitting.  $\square$

We now describe an analogous splitting for  $S(n)$ . Let  $A^* = \mathbf{F}_p[\mathbf{Z}_p]$  for  $p > 2$  and  $A^* = \mathbf{F}_2[\mathbf{Z}_2^\times]$  for  $p = 2$ . Let  $A_*$  be the continuous linear dual of  $A$ .

6.2.8. PROPOSITION. *As an algebra  $A = \mathbf{F}_p[u_1, u_2, \dots]/(u_i - u_i^p)$ . The coproduct  $\Delta$  is given by*

$$\sum_{i \geq 0}^G \Delta(u_i) = \sum_{i,j \geq 0}^G u_i \otimes u_j$$

where  $u_0 = 1$  and  $G$  is the formal group law with

$$\log_G(X) = \sum \frac{x^{p^i}}{p^i}.$$

PROOF. Since  $A \cong \mathbf{F}_p[S_1]$ , this follows immediately from 6.2.3.  $\square$

We can define Hopf algebra homomorphisms  $c_*: S(n) \otimes \mathbf{F}_q \rightarrow A \otimes \mathbf{F}_q$  and  $d_*: A \otimes \mathbf{F}_q \rightarrow S(n) \otimes \mathbf{F}_q$  dual to the group homomorphisms  $c$  and  $d$  defined above.

6.2.9. THEOREM. *There exist maps  $c_*: S(n) \rightarrow A$  and  $d_*: A \rightarrow S(n)$  corresponding to those defined above, and for  $p \nmid n$ ,  $S(n) \cong A \otimes B$ , where  $B \otimes \mathbf{F}_q$  is the continuous linear dual of  $\mathbf{F}_q[S_n^1]$ , where  $S_n^1$  is defined in 6.2.7.*

PROOF. We can define  $c_*$  explicitly by

$$c_* t_i = \begin{cases} u_{i/n} & \text{if } n \mid i \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is a homomorphism corresponding to the  $c_*$  defined above. In lieu of defining  $d_*$  explicitly we observe that the determinant of  $\sum_{i \geq 0} t_i S^i$ , where  $t_i \in W(\mathbf{F}_q)$  and  $t_i = t_i^q$ , is a power series in  $p$  whose coefficients are polynomials in the  $t_i$  over  $\mathbf{Z}_p$ . It follows that  $d_*$  can be defined over  $\mathbf{F}_p$ . The splitting then follows as in 6.2.7.  $\square$

Our next result concerns the size of  $\text{Ext}_{S(n)}(\mathbf{F}_p, \mathbf{F}_p)$ , which we abbreviate by  $H^*(S(n))$ .

6.2.10. THEOREM.

- (a)  $H^*(S(n))$  is finitely generated as an algebra.
- (b) If  $(p-1) \nmid n$ , then  $H^i(S(n)) = 0$  for  $i > n^2$  and  $H^i(S(n)) = H^{n^2-i}(S(n))$  for  $0 \leq i \leq n^2$ , i.e.,  $H^*(S(n))$  satisfies Poincaré duality.
- (c) If  $(p-1) \mid n$ , then  $H^*(S(n))$  is  $p$ -periodic, i.e., there is some  $x \in H^i(S(n))$  such that  $H^*(S(n))$  above some finite dimension is a finitely generated free module over  $\mathbf{F}_p[x]$ .  $\square$

We will prove 6.2.10(a) below as a consequence of the open subgroup theorem (6.3.6), which states that every sufficiently small open subgroup of  $S_n$  has the same cohomology as  $\mathbf{Z}_p^{n^2}$ . Then (c) and the statement in (b) of finite cohomological dimension are equivalent to saying that the Krull dimension of  $H^*(S(n))$  is 1 or 0, respectively. Recall that the Krull dimension of a Noetherian ring  $R$  is the largest  $d$  such that there is an ascending chain  $p_0 \subset p_1 \subset \cdots \subset p_d$  of nonunit prime ideals in  $R$ . Roughly speaking,  $d$  is the number of generators of the largest polynomial algebra contained in  $R$ . Thus  $d = 0$  iff every element in  $R$  is nilpotent, which in view of (a) implies (b). If  $d = 1$  and  $R$  is a graded  $\mathbf{F}_p$ -algebra, then every element in  $R$  has a power in  $\mathbf{F}_p[x]$  for a fixed  $x \in R$ .  $R$  is a module over  $\mathbf{F}_p[x]$ , which is a principal ideal domain. Since  $H^*(S(n))$  is graded and finitely generated, it is a direct sum of cyclic modules over  $\mathbf{F}_p[x]$ . More specifically it is a direct sum of a torsion module (where each element is annihilated by some power of  $x$ ) and a free module. Since it is finitely generated, the torsion must be confined to low dimensions, and  $H^*(S(n))$  is therefore a free  $\mathbf{F}_p[x]$ -module in high dimensions, so (a) implies (c).

The following result helps determine the Krull dimension.

6.2.11. THEOREM (Quillen [3]). *For a profinite group  $G$  the Krull dimension of  $H^*(G; \mathbf{F}_p)$  is the maximal rank of an elementary abelian  $p$ -subgroup of  $G$ , i.e., subgroup isomorphic to  $(\mathbf{Z}/(p))^d$ .  $\square$*

To determine the maximal elementary abelian subgroup of  $S_n$ , we use the fact that  $D_n = E_n \otimes \mathbf{Q}$  is a division algebra over  $\mathbf{Q}_p$  (A2.2.17), so if  $G \subset S_n$  is abelian, then the  $\mathbf{Q}_p$ -vector space in  $D_n$  spanned by the elements of  $G$  is a subfield  $K \subset D_n$ . Hence the elements of  $G$  are all roots of unity,  $G$  is cyclic, and the Krull dimension is 0 or 1.



6.2.12. THEOREM. *A degree  $m$  extension  $K$  of  $\mathbf{Q}_p$  embeds in  $D_n$  iff  $m \mid n$ .*

PROOF. See Serre [1, p. 138] or Cassels and Fröhlich [1, p. 202],  $\square$

By 6.2.11  $H^*(S(n))$  has Krull dimension 1 iff  $S_n$  contains  $p$ th roots of unity. Since the field  $K$  obtained by adjoining such roots to  $\mathbf{Q}_p$  has degree  $p - 1$ , 6.2.12 gives 6.2.10(c) and the finite cohomological dimension statement in (b). For the rest of (b) we rely on theorem V.2.5.8 of Lazard [4], which says that if  $S_n$  (being an analytic pro- $p$ -group of dimension  $n^2$ ) has finite cohomological dimension, then that dimension is  $n^2$  and Poincaré duality is satisfied.

The following result identifies some Hopf algebra quotients of  $S(n) \otimes \mathbf{F}_{p^n}$ . These are related to the graded Hopf algebras  $\Sigma_A(n)$  discussed in Ravenel [10]. More precisely,  $S(d, f)_a$  is a nongraded form of  $\Sigma_A(d/f)$ , where  $A$  is the ring of integers in an extension  $K$  (depending on  $a$ ) of  $\mathbf{Q}_p$  of degree  $fn/d$  and residue degree  $f$ .

6.2.13. THEOREM. *Let  $a \in \overline{\mathbf{F}_p}$  be a  $(p^n - 1)$ th root of unity, let  $d$  divide  $n$ , and let  $f$  divide  $d$ . Then there is a Hopf algebra*

$$S(d, f)_a = \mathbf{F}_{p^n}[t_f, t_{2f}, \dots] / (t_{if}^{p^d} - a_i t_{if} : i > 0)$$

where  $a_i = a^{p^{id} - 1}$ , and a surjective homomorphism

$$\theta : S(n) \otimes \mathbf{F}_{p^n} \rightarrow S(d, f)_a$$

given by

$$t_i \mapsto \begin{cases} t_i & \text{if } f \mid i \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct on  $S(d, f)_a$  is determined by the one on  $S(n)$ . This Hopf algebra is cocommutative when  $f = d$ .

PROOF. We first show that the algebra structure on  $S(d, f)_a$  is compatible with that on  $S(n)$ . The relation  $t_{if}^{p^d} = a_i t_{if}$  implies

$$\begin{aligned} t_{if}^{p^{2d}} &= (a_i t_{if})^{p^d} = a_i^{(p^{2d} - 1)/(p^d - 1)} t_{if} = a^{(p^{id} - 1)(p^{2d} - 1)/(p^d - 1)} t_{if} \\ t_{if}^{p^{3d}} &= a^{(p^{id} - 1)(p^{3d} - 1)/(p^d - 1)} t_{if} \\ &\vdots \\ t_{if}^{p^n} &= a^{(p^{id} - 1)(p^n - 1)/(p^d - 1)} t_{if} = t_{if}, \end{aligned}$$

so  $\theta$  exists as an algebra map.

For the coproduct in  $S(n)$  we have

$$\sum_{i \geq 0}^F \Delta(t_i) x^{p^i} = \sum_{i, j \geq 0}^F t_i \otimes t_j^{p^i} x^{p^{i+j}}$$

(where  $x$  is a dummy variable) which induces

$$\sum_{i \geq 0}^F \Delta(t_{if}) x^{p^{if}} = \sum_{i, j \geq 0}^F t_{if} \otimes t_{jf}^{p^{if}} x^{p^{(i+j)f}}$$

in  $S(d, f)_a$ . We need to show that this is compatible with the multiplicative relations. We can write  $if = kd + \ell f$  with  $0 \leq \ell f < d$ , so we can rewrite the above

as

$$\begin{aligned}
\sum_{i \geq 0}^F \Delta(t_{if})x^{p^{if}} &= \sum_{i,j \geq 0}^F t_{if} \otimes t_{jf}^{p^{kd+\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j \geq 0}^F a_j^{p^{\ell f}(p^{kd-1})/(p^d-1)} t_{if} \otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j \geq 0}^F a^{p^{\ell f}(p^{jd-1})(p^{kd-1})/(p^d-1)} t_{if} \otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}},
\end{aligned}$$

which gives a well defined coproduct in  $S(d, f)_a$ .

If  $f = d$  then the right hand side simplifies to

$$\sum_{i,j \geq 0}^F a^{(p^{jd-1})(p^{if-1})/(p^d-1)} t_{if} \otimes t_{jf} x^{p^{(i+j)f}},$$

which is cocommutative as claimed.  $\square$

### 3. The Cohomology of $\Sigma(n)$

In this section we will use a spectral sequence (A1.3.9) based on the filtration of  $\Sigma(n)$  induced by the one on  $BP_*(BP)/I_n$  given in 4.3.24. We have

6.3.1. THEOREM. Define integers  $d_{n,i}$  by

$$d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i > 0. \end{cases}$$

Then there is a unique increasing filtration of the Hopf algebra  $S(n)$  with  $\deg t_i^{p^j} = d_{n,i}$  for  $0 \leq j < n$ .  $\square$

The following is a partial description of the coproduct in the associated graded object  $E^0 S(n)$ . For large  $i$  we need only partial information about the coproduct on  $t_{i,j}$  in order to prove Theorem 6.3.3. I am grateful to Agnes Beaudry for finding an error in an earlier version of the following.

6.3.2. THEOREM. Let  $E^0 S(n)$  denote the associated bigraded Hopf algebra. Its algebra structure is

$$E^0 S(n) = T(t_{i,j} : i > 0, \quad j \in \mathbf{Z}/(n)),$$

where  $T(\cdot)$  denotes the truncated polynomial algebra of height  $p$  on the indicated elements and  $t_{i,j}$  corresponds to  $t_i^{p^j}$ . The coproduct is induced by the one given in 4.3.34. Explicitly, let  $m = pn/(p-1)$ . Then

$$\Delta(t_{i,j}) = \begin{cases} \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} & \text{if } i < m, \\ \sum_{0 \leq k \leq i} t_{k,j} \otimes t_{i-k,k+j} + \bar{b}_{i-n,j+n-1} & \text{if } i = m, \\ t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \bar{b}_{i-n,j+n-1} \\ \quad \text{mod}(t_{k,\ell} : k \leq i - n - 1) & \text{if } i > m, \end{cases}$$

where  $t_{0,j} = 1$  and  $\bar{b}_{i,j}$  corresponds to the  $b_{i,j}$  of 4.3.14.  $\square$

As in the case of the Steenrod algebra, the dual object  $E_0S(n)^*$  is primitively generated and is the universal enveloping algebra of a restricted Lie algebra  $L(n)$ .  $L(n)$  has basis  $\{x_{i,j} : i > 0, j \in \mathbf{Z}/(n)\}$ , where  $x_{i,j}$  is dual to  $t_{i,j}$ .

6.3.3. THEOREM.  $E_0S(n)^*$  is the restricted enveloping algebra on primitives  $x_{i,j}$  with bracket

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta_{i+j}^l x_{i+k,j} - \delta_{k+1}^j x_{i+k,l} & \text{for } i+k \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

where  $m$  is the largest integer not exceeding  $pn/(p-1)$ , and  $\delta_t^s = 1$  iff  $s \equiv t \pmod{n}$  and  $\delta_t^s = 0$  otherwise. The restriction  $\xi$  is given by

$$\xi(x_{i,j}) = \begin{cases} x_{i+n,j+1} & \text{if } i > n/(p-1) \\ & \text{or } i = n/(p-1) \text{ and } p > 2 \\ x_{2n,j} + x_{2n,j+1} & \text{if } i = n \text{ and } p = 2 \\ 0 & \text{if } i < n/p - 1. \quad \square \end{cases}$$

The formula for the restriction was given incorrectly in the first edition, and this error led to an incorrect description in 6.3.24 of the multiplicative structure of  $H^*(S(2))$  for  $p = 3$ . The correct description is due to Henn [1] and will be given below. The corrected restriction formula was given to me privately by Ethan Devinatz.

PROOF OF 6.3.3. The formula for the bracket can be derived from 6.3.2 as follows. The primitive  $x_{i,j}$  is dual to  $t_{i,j}$ . The bracket has the form

$$[x_{i,j}, x_{k,l}] = \sum_{m,n} c_{i,j,k,\ell}^{a,b} x_{a,b},$$

where the coefficient  $c_{i,j,k,\ell}^{a,b}$  is nonzero only if the coproduct expansion on  $t_{a,b}$  contains a term of the form  $t_{i,j} \otimes t_{k,\ell}$  or  $t_{k,\ell} \otimes t_{i,j}$ . This can happen only when the two expressions have the same bidegree. This means that

$$d_{n,a} = d_{n,i} + d_{n,k} \\ \text{and} \quad 2p^b(p^a - 1) \equiv 2p^j(p^i - 1) + 2p^\ell(p^k - 1) \pmod{2(p^n - 1)}$$

This happens only when  $a = i+k \leq m$  and  $b = j$  or  $\ell$ . Inspection of the coproduct formula leads to indicated Lie bracket.

The restriction requires more care. For finding the restriction on  $x_{i,j}$  it suffices to work in the subalgebra of  $E_0S(n)^*$  generated by  $x_{k,\ell}$  for  $k \geq i$ .

It is also dual to passing to the quotient of  $E^0S(n)$  obtained by killing  $t_{k,\ell}$  for  $k < i$ . Hence description of  $\Delta(t_{i,j})$  for  $i > m$  given in 6.3.2 is sufficient for our purposes.

When  $i > m$  we have

$$\begin{aligned} \Delta(t_{i,j}) &= t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \bar{b}_{i-n,j-1} \\ &= t_{i,j} \otimes 1 + 1 \otimes t_{i,j} - \sum_{0 < \ell < p} p^{-1} \binom{p}{\ell} t_{i-n,j-1}^\ell \otimes t_{i-n,j-1}^{p-\ell}, \end{aligned}$$

so for  $i > n/(p-1)$ ,

$$\begin{aligned} \Delta(t_{i+n,j+1}) &\equiv t_{i+n,j+1} \otimes 1 + 1 \otimes t_{i+n,j+1} - \sum_{0 < \ell < p} p^{-1} \binom{p}{\ell} t_{i,j}^\ell \otimes t_{i,j}^{p-\ell} \\ &\quad \text{mod } (t_{k,\ell} : k \leq i-1). \end{aligned}$$

For brevity let  $B = E^0 S(n)/(t_{k,\ell} : k \leq i-1)$  and let  $\bar{B} = B/\mathbf{F}_p$  denote the unit coideal, the dual of the augmentation ideal in  $B^*$ .

It follows that under the reduced iterated coproduct

$$B \xrightarrow{\Delta^{p-1}} B^{\otimes p} \longrightarrow \bar{B}^{\otimes p}$$

we have

$$t_{i+n,j+1} \mapsto t_{i,j} \otimes t_{i,j} \otimes \cdots \otimes t_{i,j},$$

which leads to the desired value of  $\xi(x_{i,j})$  for  $i > n/(p-1)$ . The argument for  $i = n/p - 1$  and  $p$  odd is similar.

For the case  $p = 2$  and  $i = n$ , 6.3.2 gives

$$\begin{aligned} \Delta(t_{2n,j}) &= \sum_{0 \leq k \leq 2n} t_{k,j} \otimes t_{2n-k,k+j} + \bar{b}_{n,j-1} \\ &= t_{n,j-1} \otimes t_{n,j-1} + \sum_{0 \leq k \leq 2n} t_{k,j} \otimes t_{2n-k,k+j} \\ &= t_{n,j-1} \otimes t_{n,j-1} + t_{n,j} \otimes t_{n,j} \\ &\quad + \sum_{0 \leq k < n} (t_{k,j} \otimes t_{2n-k,j+k} + t_{2n-k,j} \otimes t_{k,j-k}), \end{aligned}$$

and the formula for  $\xi(x_{n,j})$  follows.

For  $i < n/(p-1)$  there are no terms in  $\Delta(t_{i+n,k})$  for any  $k$  that would lead to a nontrivial restriction on  $x_{i,j}$ .  $\square$

Recall that Theorem 6.2.3 identifies  $S(n)^* \otimes \mathbf{F}_q$  with the group ring  $\mathbf{F}_q[S_n]$  and that  $S_n$  is the group of units in the  $\mathbf{Z}_p$ -algebra  $E_n$  congruent to 1 modulo the maximal ideal  $(S)$ . Killing the first few  $t_i$ s in  $S(n)$  as we did in the proof above corresponds to replacing the group  $S_n$  by the subgroup of units congruent to 1 modulo a power of  $(S)$ .

Let  $L(n)$  be the Lie algebra without restriction with basis  $x_{i,j}$  and bracket as above. We now recall the main results of May [2].

6.3.4. THEOREM. *There are spectral sequences*

$$(a) E_2 = H^*(L(n)) \otimes P(b_{i,j}) \Rightarrow H^*(E_0 S(n)^*),$$

$$(b) E_2 = H^*(E_0 S(n)^*) \Rightarrow H^*(S(n)),$$

where  $b_{i,j} \in H^{2pd_i}(E_0 S(n)^*)$  with internal degree  $2p^{j+1}(p^i - 1)$  and  $P(\cdot)$  is the polynomial algebra on the indicated generators.  $\square$

Now let  $L(n, k)$  be the quotient of  $L(n)$  obtained by setting  $x_{i,j} = 0$  for  $i > k$ . Then our first result is

6.3.5. THEOREM. *The  $E_2$ -term of the first May spectral sequence [6.3.4(a)] may be replaced by  $H^*(L(n, m)) \otimes P(b_{i,j} : i \leq m - n)$ , where  $m = [pn/(p-1)]$  as before.*

PROOF. By 6.3.3  $L(n)$  is the product of  $L(n, m)$  and an abelian Lie algebra, so

$$H^*(L(n)) \cong H^*(L(n, m)) \otimes E(h_{i,j} : i > m),$$

where  $E(\cdot)$  denotes the exterior algebra on the indicated generators and  $h_{i,j} \in H^1 L(n)$  is the element corresponding to  $x_{i,j}$ . It also follows from 6.3.4 that the appropriate differential will send  $h_{i,j}$  to  $-b_{i-n,j-1}$  for  $i > m$ . It follows that the entire spectral sequence decomposes as a tensor product of two spectral sequences, one with the  $E_2$ -term indicated in the statement of the theorem, and the other having  $E_2 = E(h_{i,j}) \otimes P(b_{i-n,j})$  with  $i > m$  and  $E_\infty = \mathbf{F}_p$ .  $\square$

If  $n < p - 1$  then 6.3.5 gives a spectral sequence whose  $E_2$ -term is  $H^*(L(n, n))$ , showing that  $H^*(S(n))$  has cohomological dimension  $n^2$  as claimed in 6.2.10(b).

In Ravenel [6] we claimed erroneously that the spectral sequence of 6.3.4(b) collapses for  $n < p - 1$ . The argument given there is incorrect. For example, we have reason to believe that for  $p = 11$ ,  $n = 9$  the element

$$(h_{1,0}h_{2,0} \cdots h_{7,0})(h_{2,8}h_{3,7} \cdots h_{7,3})$$

supports a differential that hits a nonzero multiple of

$$h_{1,0}h_{2,0}(h_{1,8}h_{2,7} \cdots h_{6,3})(h_{2,1}h_{3,1} \cdots h_{6,1}).$$

We know of no counterexample for smaller  $n$  or  $p$ .

Now we will prove 6.2.10(a), i.e., that  $H^*(S(n))$  is finitely generated as an algebra. For motivation, the following is a special case of a result in Lazard [4].

**6.3.6. OPEN SUBGROUP THEOREM.** *Every sufficiently small open subgroup of  $S_n$  is cohomologically abelian in the sense that it has the same cohomology as  $\mathbf{Z}_p^{n^2}$ , i.e., an exterior algebra on  $n^2$  generators.*  $\square$

We will give a Hopf algebra theoretic proof of this for a cofinal set of open subgroups, namely the subgroups of elements in  $E_n$  congruent to 1 modulo  $(S^i)$  for various  $i > 0$ . The corresponding quotient group (which is finite) is dual the subalgebra of  $S(n)$  generated by  $\{t_k : k < i\}$ . Hence the  $i$ th subgroup is dual to  $S(n)/(t_k : k < i)$ , which we denote by  $S(n, i)$ .

The filtration of 6.3.1 induces one on  $S(n, i)$  and analogs of the succeeding four theorems hold for it.

**6.3.7. THEOREM.** *If  $i \geq n$  and  $p > 2$ , or  $i > n$  and  $p = 2$ , then*

$$H^*(S(n, i)) = E(h_{k,j} : i \leq k < i + n, j \in \mathbf{Z}/(n)).$$

**PROOF.** The condition on  $i$  is equivalent to  $i > n - 1$  and  $i > m/2$ , where as before  $m = pn/(p - 1)$ . In the analog of 6.3.3 we have  $i, k > m/2$  so  $i + k > m$  so the Lie algebra is abelian. We also see that the restriction  $\xi$  is injective, so the spectral sequence of 6.3.5 has the  $E_2$ -term claimed to be  $H^*(S(n, i))$ . This spectral sequence collapses because  $h_{k,j}$  corresponds to  $t_k^{p^j} \in S(n, i)$ , which is primitive for each  $k$  and  $j$ .  $\square$

**PROOF OF 6.2.10(a).** Let  $A(i)$  be the Hopf algebra corresponding to the quotient of  $S_n$  by the  $i$ th congruence subgroup, so we have a Hopf algebra extension (A1.1.15)

$$A(i) \rightarrow S(n) \rightarrow S(n, i).$$

The corresponding Cartan–Eilenberg spectral sequence (A1.3.14) has

$$E_2 = \text{Ext}_{A(i)}(\mathbf{F}_p, H^*(S(n, i)))$$

and converges to  $H^*(S(n))$  with  $d_r: E_r \rightarrow E_r^{s+r, t-r+1}$ . Each  $E_r$ -term is finitely generated since  $A(i)$  and  $H^*(S(n, i))$  are finite-dimensional for  $i > m/2$ . Moreover,  $E_{n^2} = E_\infty$ , so  $E_\infty$  and  $H^*(S(n))$  are finitely generated.  $\square$

Now we continue with the computation of  $H^*(S(n))$ . Theorem 6.3.5 indicates the necessity of computing  $H^*(L(n, k))$  for  $k \leq m$ , and this may be done with the Koszul complex, i.e.,

6.3.8. THEOREM.  $H^*(L(n, k))$  for  $k \leq m$  is the cohomology of the exterior complex  $E(h_{i,j})$  on one-dimensional generators  $h_{i,j}$  with  $i \leq k$  and  $j \in \mathbf{Z}/(n)$ , with coboundary

$$d(h_{i,j}) = \sum_{0 < s < i} h_{s,j} h_{i-s, s+j}.$$

The element  $h_{i,j}$  corresponds to the element  $x_{i,j}$  and therefore has filtration degree  $i$  and internal degree  $2p^j(p^i - 1)$ .

PROOF. This follows from standard facts about the cohomology of Lie algebras (Cartan and Eilenberg [1, XII, Section 7]).  $\square$

Since  $L(n, k)$  is nilpotent its cohomology can be computed with a sequence of change-of-rings spectral sequences analogous to A1.3.14.

6.3.9. THEOREM. There are spectral sequences with

$$E_2 = E(h_{k,j}) \otimes H^*(L(n, k-1)) \Rightarrow H^*(L(n, k))$$

and  $E_3 = E_\infty$ .

PROOF. The spectral sequence is that of Hochschild–Serre (see Cartan and Eilenberg [1, pp. 349–351] for the extension of Lie algebras

$$A(n, k) \rightarrow L(n, k) \rightarrow L(n, k-1)$$

where  $A(n, k)$  is the abelian Lie algebra on  $x_{k,j}$ . Hence  $H^*(A(n, k)) = E(h_{k,j})$ . The  $E_2$ -term,  $H^*(L(n, k-1))$ ,  $H^*(A(n, k))$  is isomorphic to the indicated tensor product since the extension is central.

For the second statement, recall that the spectral sequence can be constructed by filtering the complex of 6.3.8 in the obvious way. Inspection of this filtered complex shows that  $E_3 = E_\infty$ .  $\square$

In addition to the spectral sequence of 6.3.4(a), there is an alternative method of computing  $H^*(E_0 S(n)^*)$ . Define  $\tilde{L}(n, k)$  for  $k \leq m$  to be the quotient of  $PE_0 S(n)^*$  by the restricted sub-Lie algebra generated by the elements  $x_{i,j}$  for  $k < i \leq m$ , and define  $F(n, k)$  to be the kernel of the extension

$$0 \rightarrow F(n, k) \rightarrow \tilde{L}(n, k) \rightarrow \tilde{L}(n, k-1) \rightarrow 0.$$

Let  $H^*(\tilde{L}(n, k))$  denote the cohomology of the restricted enveloping algebra of  $\tilde{L}(n, k)$ . Then we have

6.3.10. THEOREM. There are change-of-rings spectral sequences converging to  $H^*(\tilde{L}(n, k))$  with

$$E_2 = H^*(F(n, k)) \otimes H^*(\tilde{L}(n, k-1))$$

where

$$H^*(F(n, k)) = \begin{cases} E(h_{k,j}) & \text{for } k > m-n \\ E(h_{k,j}) \otimes P(b_{k,j}) & \text{for } k \leq m-n \end{cases}$$

and  $H^*(\tilde{L}(n, m)) = H^*(E_0S(n)^*)$ .

PROOF. Again the spectral sequence is that given in Theorem XVI.6.1 of Cartan and Eilenberg [1]. As before, the extension is cocentral, so the  $E_2$ -term is the indicated tensor product. The structure of  $H^*(F(n, k))$  follows from 6.3.3 and the last statement is a consequence of 6.3.5.  $\square$

We begin the computation of  $H^1(S(n))$  with:

6.3.11. LEMMA.  $H^1(E_0S(n)^*) = H^1(E^0S(n))$  is generated by

$$\zeta_n = \sum_j h_{n,j} \quad \text{and} \quad \rho_n = \sum_j h_{2n,j} \quad \text{for } p = 2;$$

and for  $n > 1$ ,  $h_{1,j}$  for each  $j \in \mathbf{Z}/(n)$ .

PROOF. By 6.3.4(a) and 6.3.5  $H^1(E_0S(n)) = H^1L(n, m)$ . The indicated elements are nontrivial cycles by 6.3.8. It follows from 6.3.3 that  $L(n, m)$  can have no other generators since  $[x_{1,j}, x_{i-1,j+1}] = x_{i,j} - \delta_{i+j}^j x_{i,j+1}$ .  $\square$

In order to pass to  $H^1(S(n))$  we need to produce primitive elements in  $S(n)_*$  corresponding to  $\zeta_n$  and  $\rho_n$  (the primitive  $t_1^{p^j}$  corresponds to  $h_{1,j}$ ). We will do this with the help of the determinant of a certain matrix. Recall from (6.2.3) that  $S(n) \otimes \mathbf{F}_{p^n}$  was isomorphic to the dual group ring of  $S_m$  which has a certain faithful representation over  $W(\mathbf{F}_{p^n})$  (6.2.6). The determinant of this representation gave a homomorphism of  $S(n)$  into  $\mathbf{Z}_p^\times$ , the multiplicative group of units in the  $p$ -adic integers. We will see that in  $H^1$  this map gives us  $\zeta_n$  and  $\rho_n$ .

More precisely, let  $M = (m_{i,j})$  be the  $n$  by  $n$  matrix over  $\mathbf{Z}_p[t_1, t_2, \dots]/(t_i - t_i^{p^n})$  given by

$$m_{i,j} = \begin{cases} \sum_{k \geq 0} p^k t_{kn+j-i}^{p^i} & \text{for } i \leq j \\ \sum_{k \geq 0} p^{k+1} t_{kn+j-i}^{p^i} & \text{for } i > j \end{cases}$$

where  $t_0 = 1$ .

Now define  $T_n \in S(n)_*$  to be the mod  $(p)$  reduction  $p^{-1}(\det M - 1)$  and for  $p = 2$  define  $U_n \in S(n)_*$  to be the mod  $(2)$  reduction of  $\frac{1}{8}(\det M^2 - 1)$ . Then we have

6.3.12. THEOREM. *The elements  $T_n \in S(n)$  and, for  $p = 2$ ,  $U_n \in S(n)$  are primitive and represent the elements  $\zeta_n$  and  $\rho_n + \zeta_n \in H^1(S(n))$ , respectively. Hence  $H^1(S(n))$  is generated by these elements and for  $n > 1$  by the  $h_{1,j}$  for  $j \in \mathbf{Z}/(n)$ .*

PROOF. The statement that  $T_n$  and  $U_n$  are primitive follows from 6.2.6. That they represent  $\zeta_n$  and  $\rho_n + \zeta_n$  follows from the fact that

$$T_n \equiv \sum_j t_n^{p^j} \pmod{(t_1, t_2, \dots, t_{n-1})}$$

and

$$U_n \equiv \sum_j t_{2n}^{2^j} + t_n^{2^j} \pmod{(t_1, t_2, \dots, t_{n-1})}. \quad \square$$

EXAMPLES.

$$\begin{aligned} T_1 &= t_1, & U_1 &= t_1 + t_2, & T_2 &= t_2 + t_2^p - t_1^{1+p}, \\ U_2 &= t_4 + t_4^2 + t_1 t_3^2 + t_1^2 t_3 + t_1^3 t_2 + t_1^3 t_2^2, \end{aligned}$$

and

$$T_3 = t_3 + t_3^p + t_3^{p^2} + t_1^{1+p+p^2} - t_1 t_2^p - t_1^p t_2^p - t_1^{p^2} t_2.$$

Moreira [1, 3] has found primitive elements in  $BP_*(BP)/I_n$  which reduce to our  $T_n$ . The following result is a corollary of 6.2.7.

6.3.13. PROPOSITION. *If  $p \nmid n$ , then  $H^*(S(n))$  decomposes as a tensor product of an appropriate subalgebra with  $E(\zeta_n)$  for  $p > 2$  and  $P(\zeta_n) \otimes E(\rho_n)$  for  $p = 2$ .  $\square$*

We now turn to the computation of  $H^2(S(n))$  for  $n > 2$ . We will compute all of  $H^*S(n)$  for  $n = 2$  below.

6.3.14. THEOREM. *Let  $n > 2$*

(a) *For  $p = 2$ ,  $H^2(S(n))$  is generated as a vector space by the elements  $\zeta_n^2$ ,  $\rho_n \zeta_n$ ,  $\zeta_n$ ,  $\zeta_n h_{1,j}$ ,  $\rho_n h_{1,j}$ , and  $h_{1,i} h_{1,j}$  for  $i \neq j \pm 1$ , where  $h_{1,i} h_{1,j} = h_{1,j} h_{1,i}$  and  $h_{1,i}^2 \neq 0$ .*

(b) *For  $p > 2$ ,  $H^2(S(n))$  is generated by the elements*

$$\zeta_n h_{1,i}, b_{1,i}, g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i} \rangle, \quad k_i = \langle h_{1,i+1}, h_{1,i+1}, h_{1,i} \rangle$$

and  $h_{1,i} h_{1,j}$  for  $i \neq j \pm 1$ , where  $h_{1,i} h_{1,j} + h_{1,j} h_{1,i} = 0$ .  $\square$

Both statements require a sequence of lemmas. We treat the case  $p = 2$  first.

6.3.15. LEMMA. *Let  $p = 2$  and  $n > 2$ .*

(a)  *$H^1(L(n, 2))$  is generated by  $h_{1,i}$  for  $i \in \mathbf{Z}/(n)$ .*

(b)  *$H^2(L(n, 2))$  is generated by the elements  $h_{1,i} h_{1,j}$  for  $i \neq j \pm 1$ ,  $g_i$ ,  $k_i$ , and  $e_{3,i} = \langle h_{1,i}, h_{1,i+1}, h_{1,i+2} \rangle$ . The latter elements are represented by  $h_{1,i} h_{2,i}$ ,  $h_{1,i+1} h_{2,i}$ , and  $h_{1,i} h_{2,i+1} + h_{2,i} h_{1,i+2}$ , respectively.*

(c)  *$e_{3,i} h_{1,i+1} = h_{1,i} e_{3,i+1} + e_{3,i} h_{1,i} h_{1,i+3} = 0$ , and these are the only relations among the elements  $h_{1,i} e_{3,j}$ .*

PROOF. We use the spectral sequence of 6.3.9 with  $E_2 = E(h_{1,i}, h_{2,i})$  and  $d_2(h_{2,i}) = h_{1,i} h_{1,i+1}$ . All three statements can be verified by inspection.  $\square$

6.3.16. LEMMA. *Let  $p = 2$ ,  $n > 2$ , and  $2 < k \leq 2n$ .*

(a)  *$H^1(L(n, k))$  is generated by the elements  $h_{1,i}$  along with  $\zeta_n$  for  $k \geq n$  and  $\rho_n$  for  $k = 2n$ .*

(b)  *$H^2(L(n, k))$  is generated by products of elements in  $H^1(L(n, k))$  subject to  $h_{1,i} h_{1,i+1} = 0$ , along with*

$$\begin{aligned} g_i &= \langle h_{1,i}, h_{1,i}, h_{1,i+1} \rangle, & k_i &= \langle h_{1,i}, h_{1,i+1}, h_{1,i+1} \rangle, \\ \alpha_i &= \langle h_{1,i}, h_{1,i+1}, h_{1,i+2}, h_{1,i+1} \rangle, & \text{and} \\ e_{k+1,i} &= \langle h_{1,i}, h_{1,i+1}, \dots, h_{1,i+k} \rangle. \end{aligned}$$

The last two families of elements can be represented by  $h_{3,i} h_{1,i+1} + h_{2,i} h_{2,i+1}$  and  $\Sigma_s h_{s,i} h_{k+1-s,i+1}$  respectively.

(c)  *$h_{1,i} e_{k+1,i+1} + e_{k+1,i} h_{1,i+1+k} = 0$  and no other relations hold among products of the  $e_{k+1,i}$  with elements of  $H^1$ .*



PROOF. Again we use 6.3.9 and argue by induction on  $k$ , using 6.3.15 to start the induction. We have  $E_2 = E(h_{k,i}) \otimes H^*(L(n, k-1))$  with  $d_2(h_{k,i}) = e_{k,i}$ . The existence of the  $\alpha_i$  follows from the relation  $e_{3,i}h_{1,i+1} = 0$  in  $H^3(L(n, 2))$  and that of  $e_{k+1,i}$  from  $h_{1,i}e_{k,i+1}h_{1,i+k} = 0$  in  $H^3(L(n, k-1))$ . The relation (c) for  $k < 2$  is formal; it follows from a Massey product identity A1.4.6 or can be verified by direct calculation in the complex of 6.3.8. No combination of these products can be in the image of  $d_2$  for degree reasons.  $\square$

6.3.17. *Let  $p = 2$  and  $n > 2$ . Then  $H^2(E_0S(n)^*)$  is generated by the elements  $\rho_n\zeta_n$ ,  $\rho_n h_{1,i}$ ,  $\zeta_n h_{1,i}$ ,  $h_{1,i}h_{1,j}$  for  $i \neq j \pm 1$ ,  $\alpha_i$ , and  $h_{i,j}^2 = b_{i,j}$  for  $1 \leq i \leq n$ ,  $j \in \mathbf{Z}/(n)$ .*

PROOF. We use the modified first May spectral sequence of 6.3.5. We have  $m = 2n$  and  $H^2(L(n, m))$  is given by 6.3.16. By easy direct computation one sees that  $d_2(g_i) = b_{1,i}h_{1,i+1}$  and  $d_2(k_i) = h_{1,i}b_{1,i+1}$ . We will show that  $d_2(e_{2n+1,i}) = h_{1,i}b_{n,i} + h_{1,i+n}b_{n,i-1}$ .

$$\Delta(t_{2n+1}) = \sum t_j \otimes t_{2n+1-j}^{p^j} + b_{n+1,n-1}$$

modulo terms of lower filtration by 4.3.15. Then by 4.3.22

$$d(b_{n+1,n-1}) = t_1 \otimes b_{n,n} + b_{n,n-1} \otimes t_1$$

modulo terms of lower filtration and the nontriviality of  $d_2(e_{2n+1,i})$  follows.  $\square$

PROOF OF 6.3.14(a). We now consider the second May spectral sequence (6.3.4(b)). By 4.3.22 we have  $d_2(b_{i,j}) = h_{1,j+1}b_{i-1,j+1} + h_{1,i+j}b_{i-1,j} \neq 0$  for  $i > 1$ . The remaining elements of  $H^2E_0S(n)$  survive either for degree reasons or by 6.3.12.  $\square$

For  $p > 2$  we need an analogous sequence of lemmas. We leave the proofs to the reader.

6.3.18. LEMMA. *Let  $n > 2$  and  $p > 2$ .*

- (a)  $H^1(L(n, 2))$  is generated by  $h_{1,i}$ .
- (b)  $H^2(L(n, 2))$  is generated by the elements  $h_{1,i}h_{1,j}$  (with  $h_{1,i}h_{1,i+1} = 0$ ).
- $g_i = h_{1,i}h_{2,i}$ ,  $k_i = h_{1,i+1}h_{2,i}$  and  $e_{3,i} = h_{1,i}h_{2,i+1}h_{2,i}h_{1,i+2}$ .
- (c) *The only relations among the elements  $h_{1,i}e_{3,j}$  are  $h_{1,i}e_{3,i+1} - e_{3,i}h_{1,i+3} = 0$ .*  $\square$

6.3.19. LEMMA. *Let  $n > 2$ ,  $p > 2$ , and  $2 < k \leq m$ . Then*

- (a)  $H^1(L(n, k))$  is generated by  $h_{1,i}$  and, for  $k \geq n$ ,  $\zeta_n$ .
- (b)  $H^2(L(n, k))$  is generated by  $h_{1,i}h_{1,j}$  (with  $h_{1,i}h_{1,i+1} = 0$ ),  $g_i$ ,  $h_i$ ,

$$e_{k+1,i} = \sum_{0 < j < k+1} h_{j,i}h_{k+1-j,i+j},$$

and, for  $k \geq n$ ,  $\zeta_n h_{1,i}$ .

- (c) *The only relations among products of elements in  $H^1$  with the  $e_{k+1,i}$  are  $h_{1,i}e_{k+1,i+1} - e_{k+1,i}h_{1,k+1} = 0$ .*  $\square$

6.3.20. LEMMA. *Let  $n > 2$  and  $p > 2$ . Then  $H^2(E_0S(n)^*)$  is generated by the elements  $b_{i,j}$  for  $i \leq m - n$  and by the elements of  $H^2(L(n, m))$ .*

PROOF OF 6.3.14(b). Again we look at the spectral sequence of 6.3.4(b). By arguments similar to those for  $p = 2$  one can show that

$$d_p(b_{i,j}) = h_{1,i+j}b_{i-1,j} - h_{1,j+1}b_{i-1,j+1} \quad \text{for } i > 1$$

and

$$d_s(e_{m+1,i}) = h_{1,m+1+i-n}b_{m-n,i-1} - h_{1,i}b_{m-n,j} \quad \text{where } s = 1 + pn - (p-1)m,$$

and the remaining elements of  $H^2(E_0S(n)^*)$  survive as before.  $\square$

Now we will compute  $H^*(S(n))$  at all primes for  $n \leq 2$  and at  $p > 3$  for  $n = 3$ .

6.3.21. THEOREM.

(a)  $H^*(S(1)) = P(h_{1,0}) \otimes E(\rho_1)$  for  $p = 2$ ;

(b)  $H^*(S(1)) = E(h_{1,0})$  for  $p > 2$

[note that  $S(1)$  is commutative and that  $\zeta_1 = h_{1,0}$ ].

PROOF. This follows immediately from 6.3.3, 6.3.5, and routine calculation.  $\square$

6.3.22. THEOREM. For  $p > 3$ ,  $H^*(S(2))$  is the tensor product of  $E(\zeta_2)$  with the subalgebra with basis  $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0h_{1,1}\}$  where

$$\begin{aligned} g_i &= \langle h_{1,i}, h_{1,i+1}, h_{1,i} \rangle, \\ h_{1,0}g_1 &= g_0h_{1,1}, \quad h_{1,0}g_0 = h_{1,1}g_1 = 0, \end{aligned}$$

and

$$h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = 0.$$

In particular, the Poincaré series is  $(1+t)^2(1+t+t^2)$ .

PROOF. The computation of  $H^*(L(2,2))$  by 6.3.8 or 6.3.9 is elementary, and there are no algebra extension problems for the spectral sequences of 6.3.9 or 6.3.4(b).  $\square$

We will now compute  $H^*(S(2))$  for  $p = 3$ . Our description of it in the first edition was incorrect, as was pointed out by Henn [1]. The computation given here is influenced by Henn but self-contained. Henn showed that there are two conjugacy classes of subgroups of order 3 in the group  $S_2$ . In each case the centralizer is the group of units congruent to one modulo the maximal ideal in the ring of integers of an embedded copy of the field  $K = \mathbf{Q}_3[\zeta]$ , where  $\zeta$  is a primitive cube root of unity. Let  $C_1$  and  $C_2$  denote these two centralizers. Henn showed that the resulting map

$$H^*(S_2) \rightarrow H^*(C_1) \oplus H^*(C_2)$$

is a monomorphism.

We will describe this map in Hopf algebraic terms. Choose a fourth root of unity  $i \in \mathbf{F}_9$ , let  $a = \pm i$ , and consider the two quotients

$$\overline{S(2)}_+ = S(1,1)_i \quad \text{and} \quad \overline{S(2)}_- = S(1,1)_{-i},$$

where  $S(1,1)_a$  is the quotient of  $S(2) \otimes \mathbf{F}_9$  described in 6.2.13. Henn's map is presumably equivalent to

$$(6.3.23) \quad H^*(S(2)) \otimes \mathbf{F}_9 \rightarrow H^*(\overline{S(2)}_+) \oplus H^*(\overline{S(2)}_-).$$

In any case we will show that this map is a monomorphism.

We have the following reduced coproducts in  $\overline{S(2)}_{\pm}$ .

$$\begin{aligned}\bar{t}_1 &\mapsto 0 \\ \bar{t}_2 &\mapsto a\bar{t}_1 \otimes \bar{t}_1 \\ \bar{t}_3 &\mapsto \bar{t}_1 \otimes \bar{t}_2 + \bar{t}_2 \otimes \bar{t}_1 - a^3(\bar{t}_1^2 \otimes \bar{t}_1 + \bar{t}_1 \otimes \bar{t}_1^2)\end{aligned}$$

It follows that  $\bar{t}_2 + a\bar{t}_1^2$  and  $\bar{t}_3 - \bar{t}_1\bar{t}_2$  are primitive. The filtration of 6.3.1 induces one on  $\overline{S(2)}_{\pm}$ , and the methods of this section lead to

$$H^*(\overline{S(2)}_{\pm}) = E(\bar{h}_{1,0}, \bar{h}_{2,0}, \bar{h}_{3,0}) \otimes P(\bar{b}_{1,0})$$

with the evident notation.

6.3.24. THEOREM. For  $p = 3$ ,  $H^*(S(2))$  is a free module over

$$E(\zeta_2) \otimes P(b_{1,0})$$

on the generators

$$\{1, h_{1,0}, h_{1,1}, b_{1,1}, \xi, a_0, a_1, b_{1,1}\xi\},$$

where the elements  $\xi \in H^2$  and  $a_0, a_1 \in H^3$  will be defined below. The algebra structure is indicated in the following multiplication table.

1	$h_{1,0}$	$h_{1,1}$	$b_{1,1}$	$\xi$	$a_0$	$a_1$
$h_{1,0}$	0	0	$-b_{1,0}h_{1,1}$	0	$-b_{1,1}\xi$	$-b_{1,0}\xi$
$h_{1,1}$		0	$b_{1,0}h_{1,0}$	0	$-b_{1,0}\xi$	$b_{1,1}\xi$
$b_{1,1}$			$-b_{1,0}^2$	$b_{1,1}\xi$	$-b_{1,0}a_1$	$b_{1,0}a_0$
$\xi$				0	0	0
$a_0$					0	0
$a_1$						0

In particular, the Poincaré series is

$$(1+t)^2(1+t^2)/(1-t).$$

Moreover the map of (6.3.23) is a monomorphism.

PROOF. Our basic tools are the spectral sequences of 6.3.10 and some Massey product identities from A1.4. We have  $H^*(\tilde{L}(2,1)) \cong E(h_{1,0}, h_{1,1}) \otimes P(b_{1,0}, b_{1,1})$ , and a spectral sequence converging to  $H^*(\tilde{L}(2,2))$  with  $E_2 = E(\zeta_2, \eta) \otimes H^*(\tilde{L}(2,1))$ , where

$$\begin{aligned}\zeta_2 &= h_{2,0} + h_{2,1}, & \eta &= h_{2,1} - h_{2,0}, \\ d_2(\zeta_2) &= 0, & d_2(\eta) &= h_{1,0}h_{1,1},\end{aligned}$$

and  $E_3 = E_{\infty}$ . Hence  $E_{\infty}$  is a free module over  $E(\zeta_2) \otimes P(b_{1,0}, b_{1,1})$  on generators

$$\{1, h_{1,0}, h_{1,1}, g_0, g_1, h_{1,0}g_1 = h_{1,1}g_0, \},$$

where  $g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i} \rangle$ . This determines the additive structure of  $H^*(\tilde{L}(2,2))$ , but there are some nontrivial extensions in the multiplicative structure. We know by 6.3.13 that we can factor out  $E(\zeta_2)$ , and we can write  $b_{1,i}$  as the Massey product  $-\langle h_{1,i}, h_{1,i}, h_{1,i} \rangle$ . Then by A1.4.6 we have  $h_{1,i}g_i = -b_{1,i}h_{1,i+1}$ ,  $g_i^2 = -b_{1,i}g_{i+1}$ ,  $g_i g_{i+1} = b_{1,i}b_{1,i+1}$ . These facts along with the usual  $h_{1,i}^2 = h_{1,0}h_{1,1} = 0$  determine  $H^*(\tilde{L}(2,2))$  as an algebra.

This algebra structure allows us to embed  $H^*(\tilde{L}(2,2))$  in the ring

$$R = E(\zeta_2, h_{1,0}, h_{1,1}) \otimes P(s_0, s_1) / (h_{1,0}h_{1,1}, h_{1,0}s_1 - h_{1,1}s_0)$$

by sending  $\zeta_2$  and  $h_{1,i}$  to themselves and

$$\begin{aligned} b_{1,i} &\mapsto -s_i^3 \\ g_0 &\mapsto s_0^2 s_1 \\ g_1 &\mapsto s_0 s_1^2. \end{aligned}$$

Here the cohomological degree of  $s_i$  is  $2/3$ , and  $H^*(\tilde{L}(2,2))$  maps isomorphically to the subring of  $R$  consisting of elements of integral cohomological degree.

Next we have the spectral sequence of 6.3.10 converging to

$$H^*(\tilde{L}(2,3)) \cong H^*(E^0 S(2))$$

with  $E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2,2))$ , and  $d_2(h_{3,i}) = g_i - b_{1,i+1}$ . We will see shortly that  $E_3 = E_\infty$  for formal reasons. Tensoring this over  $H^*(\tilde{L}(2,2))$  with  $R$  gives a spectral sequence with

$$\begin{aligned} E_2 &= E(h_{3,0}, h_{3,1}) \otimes R \\ \text{and } d_2(h_{3,0}) &= s_1(s_0^2 + s_1^2) \\ d_2(h_{3,1}) &= s_0(s_0^2 + s_1^2). \end{aligned}$$

This can be simplified by tensoring with  $\mathbf{F}_9$  (which contains  $i = \sqrt{-1}$ ) and defining

$$\begin{aligned} x_0 &= h_{1,0} + ih_{1,1} & x_1 &= h_{1,0} - ih_{1,1} \\ y_0 &= s_0 + is_1 & y_1 &= s_0 - is_1 \\ z_0 &= ih_{3,0} + h_{3,1} & z_1 &= -ih_{3,0} + h_{3,1} \end{aligned}$$

The Galois group of  $\mathbf{F}_9$  over  $\mathbf{F}_3$  acts here by conjugating scalars and permuting the two subscripts. Then we have

$$R \otimes \mathbf{F}_9 = E(\zeta_2, x_0, x_1) \otimes P(y_0, y_1)/(x_0 x_1, x_0 y_1 - x_1 y_0),$$

where the cohomological degrees of  $x_i$  and  $y_i$  are 1 and  $2/3$  respectively. In the spectral sequence we have

$$(6.3.25) \quad d_2(z_0) = y_0^2 y_1 \quad \text{and} \quad d_2(z_1) = y_0 y_1^2.$$

The image of  $H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$  in  $R \otimes \mathbf{F}_9$  is a free module over the ring

$$B = E(\zeta_2) \otimes P(y_0^3, y_1^3)$$

on the following set of six generators.

$$C = \{1, x_0, x_1, y_0^2 y_1, y_0 y_1^2, x_0 y_0 y_1^2 = x_1 y_0^2 y_1\}$$

Hence the image of  $E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$  is a free  $B$ -module on the set

$$\{1, z_0, z_1, z_0 z_1\} \otimes C,$$

but it is convenient to replace this basis by the set of elements listed in the following table.

1	$z_0$	$z_1$	$z_0 z_1$
$x_0$	$x_0 z_0$	$\beta = x_0 z_1 - x_1 z_0$	$-x_0 z_0 z_1$
$x_1$	$\delta = -x_1 z_0 - x_0 z_1$	$x_1 z_1$	$x_1 z_0 z_1$
$y_0^2 y_1$	$\alpha_1 = y_0^2 y_1 z_0 - y_0^3 z_1$	$\varepsilon = y_0^2 y_1 z_1 - y_0 y_1^2 z_0$	$y_0^2 y_1 z_0 z_1$
$y_0 y_1^2$	$\gamma = -y_0 y_1^2 z_0 - y_0^2 y_1 z_1$	$\alpha_0 = y_0 y_1^2 z_1 - y_1^3 z_0$	$-y_0 y_1^2 z_0 z_1$
$x_0 y_0 y_1^2$	$-x_0 \varepsilon$	$x_1 \varepsilon$	$x_0 y_0 y_1^2 z_0 z_1$

This basis is Galois invariant up to sign, i.e., the Galois image of each basis element is another basis element. The elements  $1$ ,  $x_0y_0y_1^2$ ,  $\delta$ , and  $\gamma$  are self-conjugate, while  $\beta$ ,  $\varepsilon$ ,  $z_0z_1$  and  $x_0y_0y_1^2z_0z_1$  are antiself-conjugate. The remaining elements form eight conjugate pairs.

In the spectral sequence the following twelve differentials (listed as six Poincaré dual pairs) are easily derived from (6.3.25) and account for each of these 24 basis elements.

$$\begin{array}{ll} d_2(z_0) &= y_0^2y_1 & d_2(x_1z_0z_1) &= x_1\varepsilon \\ d_2(z_1) &= y_0y_1^2 & d_2(-x_0z_0z_1) &= -x_0\varepsilon \\ d_2(z_0z_1) &= \varepsilon & d_2(\delta) &= x_0y_0y_1^2 \\ d_2(x_0z_0) &= y_0^3(x_1) & d_2(y_0^2y_1z_0z_1) &= y_0^3(\alpha_0) \\ d_2(x_1z_1) &= y_1^3(x_0) & d_2(-y_0y_1^2z_0z_1) &= y_1^3(\alpha_1) \\ d_2(\gamma) &= y_0^3y_1^3(1) & d_2(x_0y_0y_1^2z_0z_1) &= y_0^3y_1^3(\beta) \end{array}$$

The spectral sequence collapses from  $E_3$  since there are no elements in  $E_3^{*,t}$  for  $t > 1$ . The image of  $H^*(\tilde{L}(2,3)) \otimes \mathbf{F}_9$  in the  $E_\infty$ -term is the  $B$ -module generated by

$$\{1, x_0, x_1, \alpha_0, \alpha_1, \beta\}$$

subject to the module relations

$$\begin{array}{ll} y_0^3y_1^3(1) &= 0, & y_0^3y_1^3(\beta) &= 0, \\ y_0^3(x_1) &= 0, & y_0^3(\alpha_0) &= 0, \\ y_1^3(x_0) &= 0, & \text{and } y_1^3(\alpha_1) &= 0. \end{array}$$

The only nontrivial products among these six elements are

$$x_0\alpha_1 = -y_0^3\beta \quad \text{and} \quad x_1\alpha_0 = y_1^3\beta.$$

Equivalently the image is the free module over  $E(\zeta_2) \otimes P(y_0^3 + y_1^3)$  on the eight generators

$$(6.3.26) \quad \{1, x_0, x_1, y_1^3, \beta, \alpha_0, \alpha_1, y_1^3\beta\}$$

with suitable algebra relations.

It follows that  $H^*(E^0S(2))$  itself is a free module over  $E(\zeta_2) \otimes P(b_{1,0})$  on the eight generators

$$\{1, h_{1,0}, h_{1,1}, b_{1,1}, \xi, a_0, a_1, b_{1,1}\xi\}.$$

where

$$\xi = i\beta, \quad a_0 = \alpha_0 + \alpha_1, \quad \text{and} \quad a_1 = i(\alpha_0 - \alpha_1).$$

It also follows that  $E^0H^*(S(2))$  has the relations stated in the theorem. The absence of nontrivial multiplicative extensions in  $H^*(S(2))$  will follow from the fact that the map of (6.3.23) is monomorphic and there are no extensions in its target.

Now we will determine the images of the elements of (6.3.26) under the map of (6.3.23). Recall that

$$H^*(\overline{S(2)}_\pm) = E(\overline{h}_{1,0}, \overline{h}_{2,0}, \overline{h}_{3,0}) \otimes P(\overline{b}_{1,0})$$

As before it is convenient to adjoin a cube root  $\overline{\varepsilon}_0$  of  $-\overline{b}_{1,0}$  and let

$$\overline{R}_\pm = E(\overline{h}_{1,0}, \overline{h}_{2,0}) \otimes P(\overline{\varepsilon}_0).$$

The map

$$H^*(S(2)) \otimes \mathbf{F}_9 \rightarrow E(\overline{h}_{3,0}) \otimes \overline{R}_+ \oplus E(\overline{h}_{3,0}) \otimes \overline{R}_-$$

behaves as follows.

$$\begin{array}{ll}
x_0 \mapsto (0, -\bar{h}_{1,0}) & x_1 \mapsto (-\bar{h}_{1,0}, 0) \\
y_0 \mapsto (0, -\bar{s}_0) & y_1 \mapsto (-\bar{s}_0, 0) \\
z_0 \mapsto (-i\bar{h}_{3,0}, 0) & z_1 \mapsto (0, i\bar{h}_{3,0}) \\
\beta \mapsto (-i\bar{h}_{1,0}\bar{h}_{3,0}, -i\bar{h}_{1,0}\bar{h}_{3,0}) & \\
\alpha_0 \mapsto (i\bar{s}_0^3\bar{h}_{3,0}, 0) & \alpha_1 \mapsto (0, -i\bar{s}_0^3\bar{h}_{3,0})
\end{array}$$

It follows that Henn's map is a monomorphism.  $\square$

We now turn to the case  $n = p = 2$ . We will only compute  $E^0H^*(S(2))$ , so there will be some ambiguity in the multiplicative structure of  $H^*(S(2))$ . In order to state our result we need to define some classes. Recall (6.3.12) that  $H^1(S(2))$  is the  $\mathbf{F}_2$ -vector space generated by  $h_{1,0}$ ,  $h_{1,1}$ ,  $\zeta_2$  and  $\rho_2$ . Let

$$\alpha_0 \in \langle \zeta_2, h_{1,0}, h_{1,1} \rangle, \quad \beta \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle, \quad g = \langle h, h^2, h, h^2 \rangle,$$

where  $h = h_{1,0} + h_{1,1}$ ,  $\tilde{x} = \langle x, h, h^2 \rangle$  for  $x = \zeta_2, \alpha_0, \zeta_2^2$ , and  $\alpha_0\zeta_2$  (more precise definitions of  $\alpha_0$  and  $\beta$  will be given in the proof).

**6.3.27. THEOREM.**  *$E^0H^*(S(2))$  for  $p = 2$  is a free module over  $P(g) \otimes E(\rho_2)$  on 20 generators:  $1, h_{1,0}, h_{1,1}, h_{1,0}^2, h_{1,1}^2, h_{1,0}^3, \beta, \beta h_{1,0}, \beta h_{1,1}, \beta h_{1,0}^2, \beta h_{1,1}^2, \beta h_{1,0}^3$ ,  $\zeta_2, \alpha_0, \zeta_2^2, \alpha_0\zeta_2, \tilde{\zeta}_2, \tilde{\alpha}_0, \tilde{\zeta}_2^2, \alpha_0\tilde{\zeta}_2$ , where  $\alpha_0 \in H^2(S(2))$  and has filtration degree 4,  $\beta \in H^3(S(2))$  and has filtration degree 8,  $g \in H^4(S(2))$  and has filtration degree 8, and the cohomological and filtration degrees of  $\tilde{x}$  exceed those of  $x$  by 2 and 4, respectively. Moreover  $h_{1,0}^3 = h_{1,1}^3$ ,  $\alpha_0^2 = \tilde{\zeta}_2^2$ , and all other products are zero. The Poincaré series is  $(1+t)^2(1-t^5)/(1-t)^2(1+t^2)$ .*

**PROOF.** We will use the same notation for corresponding classes in the various cohomology groups we will be considering along the way.

Again our basic tool is 6.3.10. It follows from 6.3.5 that  $H^*(E_0S(2)^*)$  is the cohomology of the complex

$$P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0}) \otimes E(h_{3,0}, h_{3,1}, \rho_2, h_{4,0})$$

with

$$\begin{aligned}
d(h_{1,i}) &= d(\zeta_2) = d(\rho_2) = 0, \\
d(h_{3,i}) &= h_{1,i}\zeta_2, \quad d(h_{2,0}) = h_{1,0}h_{1,1},
\end{aligned}$$

and

$$d(h_{4,0}) = h_{1,0}h_{3,1} + h_{1,1}h_{3,0} + \zeta_2^2.$$

This fact will enable us to solve the algebra extension problems in the spectral sequences of 6.3.10.

For  $H^*(\tilde{L}(2, 2))$  we have a spectral sequence with  $E_2 = P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0})$  with  $d_2(\zeta_2) = 0$  and  $d_2(h_{2,0}) = h_{1,0}h_{1,1}$ . It follows easily that

$$H^*(\tilde{L}(2, 2)) = P(h_{1,0}, h_{1,1}, \zeta_2, b_{2,0}) / (h_{1,0}h_{1,1})$$

where  $b_{2,0} = h_{2,0}^2 = \langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle$ .

For  $H^*(\tilde{L}(2, 3))$  we have a spectral sequence with

$$E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2, 2))$$

and  $d_2(h_{3,i}) = h_{1,i}\zeta_2$ . Let

$$\alpha_i = h_{1,i+1}h_{3,i} + \zeta_2h_{2,i} \in \langle \zeta_2, h_{1,i}, h_{1,i+1} \rangle.$$

Then  $H^*(\tilde{L}(2,3))$  as a module over  $H^*(\tilde{L}(2,2))$  is generated by 1,  $\alpha_0$ , and  $\alpha_1$  with

$$\zeta_2h_{1,i} = \zeta_2(\alpha_0 + \alpha_1 + \zeta_2^2) = h_{1,i}\alpha_i = \zeta_2h_{1,i+1}\alpha_i = 0$$

and

$$\alpha_0^2 = \zeta_2^2b_{2,0}, \quad \alpha_1^2 = \zeta_2^2(\zeta_2^2 + b_{2,0}), \quad \alpha_0\alpha_1 = \zeta_2^2(\alpha_0 + b_{2,0}).$$

The Poincaré series for  $H^*(\tilde{L}(2,3))$  is  $(1+t+t^2)/(1-t^2)$ .

For  $H^*(\tilde{L}(2,4))$  we have a spectral sequence with

$$E_2 = E(h_{4,0}, \rho_2) \otimes H^*(\tilde{L}(2,3)),$$

$d_2(\rho_2) = 0$ , and  $d_2(h_{4,0}) = \alpha_0 + \alpha_1$ . Define  $\beta \in H^3(\tilde{L}(2,4))$  by

$$\beta = h_{4,0}(\alpha_0 + \alpha_1 + \zeta_2^2) + \zeta_2h_{3,0}h_{3,1} \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle.$$

Then  $H^*(\tilde{L}(2,4))$  is a free module over  $E(\rho_2) \otimes P(b_{2,0})$  on generators 1,  $h_{1,i}^t$ ,  $\zeta_2$ ,  $\zeta_2^2$ ,  $\alpha_0$ ,  $\alpha_0\zeta_2$ ,  $\beta$ , and  $\beta h_{1,i}^t$ , where  $t > 0$ . As a module over  $H^*(\tilde{L}(2,3)) \otimes E(p_2)$  it is generated by 1 and  $\beta$ , with  $(\alpha_0 + \alpha_1)1 = \zeta_2^3(1) = \alpha_0\zeta_2^2(1) = 0$ . To solve the algebra extension problem we observe that  $\beta\zeta_2 = 0$  for degree reasons;  $\beta\alpha_i = \beta\langle \zeta_2, h_{1,i}, h_{1,i+1} \rangle = \langle \beta, \zeta_2, h_{1,i} \rangle h_{1,i+1} = 0$  since  $\langle \beta, \zeta_2, h_{1,i} \rangle = 0$  for degree reasons; and  $E(\rho_2)$  splits off multiplicatively by the remarks at the beginning of the proof.

This completes the computation of  $H^*(E_0S(2)^*)$ . Its Poincaré series is  $(1+t)^2/(1-t)^2$ . We now use the second May spectral sequence [6.3.4(b)] to pass to  $E^0H^*(S(2))$ .  $H^*(E_0S(2)^*)$  is generated as an algebra by the elements  $h_{1,0}$ ,  $h_{1,1}$ ,  $\zeta_2$ ,  $\rho_2$ ,  $\alpha_0$ ,  $b_{2,0}$ , and  $\beta$ . The first four of these are permanent cycles by 6.3.12.

By direct computation in the cobar resolution we have

$$(6.3.28) \quad d(t_3 + t_1t_2^2) = \zeta_2 \otimes t_1,$$

so the Massey product for  $\alpha_0$  is defined in  $H^*(S(2))$  and the  $\alpha_0$  is a permanent cycle. We also have

$$d(t_2 \otimes t_2 + t_1 \otimes t_1^2t_2 + t_1t_2 \otimes t_1^2) = t_1 \otimes t_1 \otimes t_1 + t_1^2 \otimes t_1^2 \otimes t_1^2,$$

so  $d_2(b_{2,0}) = h_{1,0}^2 + h_{1,1}^3$ . Inspection of the  $E_3$  term shows that  $b_{2,0}^2 = \langle h, h^2, h, h^2 \rangle$ , (where  $h = h_{1,0} + h_{1,1}$ ) is a permanent cycle for degree reasons.

We now show that  $\beta = \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle$  is a permanent cycle by showing that its Massey product expression is defined in  $E^0H^*(S(2))$ . The products  $h_{1,0}\zeta_2$  and  $\zeta_2^2h_{1,1}$  are zero by 6.3.28 and we have

$$(6.3.29) \quad \begin{aligned} d(\tilde{t}_3 \otimes \tilde{t}_3^2 + T_2\tilde{t}_3 \otimes t_1^2 + T_2 \otimes t_4 + T_2 \otimes t_2^3 + T_2 \otimes t_1^3(1 + t_2 + t_2^2)) \\ = T_2 \otimes T_2 \otimes T_2, \end{aligned}$$

where  $\tilde{t}_3 = t_3 + t_1t_2^2$  and  $T_2 = t_2 + t_2^2 + t_1^3$ , so  $\zeta_2^3 = 0$  in  $H^*(S(2))$ . Inspection of  $H^3(E_0S(2)^*)$  shows there are no elements of internal degree 2 or 4 and filtration degree  $> 7$ , so the triple products  $\langle h_{1,0}, \zeta_2, \zeta_2^2 \rangle$  and  $\langle \zeta_2, \zeta_2^2, h_{1,1} \rangle$  must vanish and  $\beta$  is a permanent cycle.

Now the  $E_3$  term is a free module over  $E(\rho_2) \otimes P(b_{2,0}^2)$  on 20 generators: 1,  $h_{1,0}$ ,  $h_{1,1}$ ,  $h_{1,0}^2$ ,  $h_{1,1}^2$ ,  $h_{1,0}^3 = h_{1,1}^3$ ,  $\beta$ ,  $\beta h_{1,0}$ ,  $\beta h_{1,1}$ ,  $\beta h_{1,1}^2$ ,  $\beta j_{10}^2$ ,  $\beta h_{1,0}^3$ ,  $\zeta_2$ ,  $\alpha_0$ ,  $\zeta_2^2$ ,  $\alpha_0\zeta_2$ ,  $\zeta_2b_{1,0}$ ,  $\alpha_0b_{2,0}$ ,  $\zeta_2^2b_{2,0}$ ,  $\zeta_2\alpha_0b_{2,0}$ . The last four in the list now have Massey product expressions  $\langle \zeta_2, h, h^2 \rangle$ ,  $\langle \alpha_0, h, h^2 \rangle$ ,  $\langle \zeta_2^2, h, h^2 \rangle$ , and  $\langle \alpha_0, \zeta_2, h, h^2 \rangle$ , respectively. These

elements have to be permanent cycles for degree reasons, so  $E_3 = E_\infty$ , and we have determined  $E^0H^*(S(2))$ .  $\square$

We now describe an alternative method of computing  $H^*(S(2) \otimes \mathbf{F}_4)$ , which is quicker than the previous one, but yields less information about the multiplicative structure. By 6.3.4, this group is isomorphic to  $H^*(S_2; \mathbf{F}_4)$ , the continuous cohomology of certain 2-adic Lie group with trivial coefficients in  $\mathbf{F}_4$ ,  $S_2$  is the group of units in the degree 4 extension  $E_2$  of  $\mathbf{Z}_2$  obtained by adjoining  $\omega$  and  $S$  with  $\omega^2 + \omega + 1 = 0$ ,  $S^2 = 2$  and  $S\omega = \omega^2S$ .

Let  $Q$  denote the quaternion group, i.e., the multiplicative group (with 8 elements) of quaternionic integers of modulus 1.

6.3.30. PROPOSITION. *There is a split short exact sequence of groups*

$$(6.3.31) \quad 1 \rightarrow G \xrightarrow{i} S_2 \xrightarrow{j} Q \rightarrow 1.$$

*The corresponding extension of dual group algebras over is*

$$Q_* \xrightarrow{j_*} S(2) \xrightarrow{i_*} G_*$$

where  $Q_* \cong \mathbf{F}_4[x, y]/(x^4 - x, y^2 - y)$  and  $G_* \cong S(2)/(t_1, t_2 + \omega t_2^2)$  as algebras where  $j_*(x) = t_1$ ,  $j_*(y) = \bar{\omega}t_2 + \bar{\omega}x^2t_2^2$ , and  $\bar{\omega}$  is the residue class of  $\omega$ .

PROOF. The splitting follows the theory of division algebras over local fields (Cassels and Fröhlich [1, pp. 137–138]) which implies that  $E_2 \otimes \mathbf{Q}_2$  is isomorphic to the 2-adic quaternions. We leave the remaining details to the reader.  $\square$

6.3.32.

- (a)  $H^*(Q; \mathbf{F}_2) = P(h_{1,0}, h_{1,1}, g)/(h_{1,0}h_{1,1}, h_{1,0}^3 + h_{1,1}^3)$ .
- (b)  $H^*(G; \mathbf{F}_2) = E(\zeta_2, \rho_2, h_{3,0}, h_{3,1})$ .

PROOF. Part (a) is an easy calculation with the change-to-rings spectral sequence (A1.3.14) for  $\mathbf{F}_2[x]/(x^4 + x) \rightarrow Q_* \rightarrow \mathbf{F}_2[y]/(y^2 + y)$ . For (b) the filtration of  $S(2)$  induces one on  $G_*$ . It is easy to see that  $E^0G_*$  is cocommutative and the result follows with no difficulty.  $\square$

6.3.33. PROPOSITION. *In the Cartan–Eilenberg spectral sequence for 6.3.31,  $E_3 = E_\infty$  and we get the same additive structure for  $H^*(S(2))$  as in 6.3.27.*

PROOF. We can take  $H^*(G) \otimes H^*(Q)$  as our  $E_1$ -term. Each term is a free module over  $E(\rho_2) \otimes P(g)$ . We leave the evaluation of the differentials to the reader.  $\square$

Finally, we consider the case  $n = 3$  and  $p \geq 5$ . We will not make any attempt to describe the multiplicative structure. An explicit basis of  $E^0H^*(S(3))$  will be given in the proof, from which the multiplication can be read off by the interested reader. It seems unlikely that there are any nontrivial multiplicative extensions.

6.3.34. THEOREM. *For  $p \geq 5$ ,  $H^*(S(3))$  has the following Poincaré series:  $(1+t)^3(1+t+6t^2+3t^3+6t^4+t^5+t^6)$ .*

PROOF. We use the spectral sequences of 6.3.9 to compute  $H^*(L(3, 2))$  and  $H^*(L(3, 3))$ . For the former the  $E_2$ -term is  $H(h_{1,i}) \otimes E(h_{2,i})$  with  $i \in \mathbf{Z}/(3)$ ,  $d_2(h_{1,i}) = 0$  and  $d_2(h_{2,i}) = h_{1,i}h_{1,i+1}$ . The Poincaré series for  $H^*(L(3, 2))$  is  $(1+t)^2(1+t+5t^2+t^3+t^4)$  and it is generated as a vector space by the following elements and their Poincaré duals:  $1, h_{1,i}, g_i = h_{1,i}h_{2,i}, k_i = h_{2,i}h_{1,i+1}, e_{3,i} =$



$h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$  (where  $\sum_i e_{3,i} = 0$ ),  $g_i h_{1,i+1} = h_{1,i}k_i = h_{1,i}h_{2,i}h_{1,i+1}$ , and  $h_{1,i}e_{3,i} = g_i h_{1,i+2} = h_{1,i}h_{2,i}h_{1,i+2}$ .

For  $H^*(L(3, 3))$  we have  $E_2 = E(h_{3,i} \otimes H^*(L(3, 2)))$  with  $d_2(h_{3,i}) = e_{3,i}$ , so  $d_2(\sum h_{3,i}) = 0$ .  $H^*(L(3, 3))$  has the indicated Poincaré series and is a free module over  $E(\zeta_3)$ , where  $\zeta_3 = \sum h_{3,i}$ , on the following 38 elements and the duals of their products with  $\zeta_3$ :

$$\begin{aligned} &1, h_{1,i}, g_i, k_i, b_{1,i+2} = h_{1,i}h_{3,i} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i}, \\ &g_i h_{1,i+1} = h_{1,i}, k_i, h_{1,i}h_{2,i}h_{2,i+2}, h_{1,i}h_{2,i}h_{2,i+1} + h_{1,i}h_{1,i+1}h_{3,i}, \\ &h_{1,i}h_{2,i}h_{3,i}, h_{1,i}h_{2,i+2}h_{3,i+1}, \sum_i (h_{1,i}h_{2,i+1} - h_{1,i+1}h_{2,i+2})h_{3,i}, h_{1,i}k_i h_{3,j} \end{aligned}$$

(where  $h_{1,i}k_i \sum_j h_{3,j}$  is divisible by  $\zeta_3$ ), and  $h_{1,i+2}h_{1,i}h_{2,i}(h_{3,i} + h_{3,i+1}) \pm h_{1,i}h_{2,0}h_{2,1}h_{2,2}$ . □

### 4. The Odd Primary Kervaire Invariant Elements

The object of this section is to apply the machinery above to show that the Adams–Novikov element  $\beta_{p^i/p^i} \in \text{Ext}^2$  (see 5.1.19) is not a permanent cycle for  $p > 2$  and  $i > 0$ . This holds for the corresponding Adams element  $b_i$  (4.3.2) for  $p > 3$  and  $i > 0$ ; by 5.4.6 we know  $\beta_{p^i/p^i}$  maps to  $b_i$ . The latter corresponds to the secondary cohomology operation associated with the Adem relation  $P^{(p-1)p^i} P^{p^i} = \dots$ . The analogous relation for  $p = 2$  is  $Sq^{2^i} Sq^{2^i} = \dots$ , which leads to the element  $h_i^2$ , which is related to the Kervaire invariant by Browder’s theorem, hence the title of the section. To stress this analogy we will denote  $\beta_{p^i/p^i}$  by  $\theta_i$ .

We know by direct calculation (e.g., 4.4.20) that  $\theta_0$  is a permanent cycle corresponding to the first element in coker  $J$ . By Toda’s theorem (4.4.22) we know  $\theta_1$  is not a permanent cycle; instead we have  $d_{2p-1}(\theta_1) = \alpha_1 \theta_0^p$  (up to nonzero scalar multiplication) and this is the first nontrivial differential in the Adams–Novikov spectral sequence. Our main result is

**6.4.1. ODD PRIMARY KERVAIRE INVARIANT THEOREM.** *In the Adams–Novikov spectral sequence for  $p > 2$   $d_{2p-1}(\theta_{i+1}) \equiv \alpha_1 \theta_i^p \pmod{\ker \theta_0^{\alpha_i}}$  (up to nonzero scalar multiplication) where  $\alpha_i = p(p^i - 1)/(p - 1)$  and  $\alpha_1 \theta_i^p$  is nonzero modulo this indeterminacy.* □

Our corresponding result about the Adams spectral sequence fails for  $p = 3$ , where  $b_2$  is a permanent cycle even though  $b_1$  is not.

**6.4.2. THEOREM.** *In the Adams spectral sequence for  $p \geq 5$   $b_i$  is not a permanent cycle for  $i \geq 1$ .* □

From 6.4.1 we can derive the nonexistence of certain finite complexes which would be useful for constructing homotopy elements with Novikov filtration 2.

**6.4.3. THEOREM.** *There is no connective spectrum  $X$  such that*

$$BP_*(X) = BP_*/(p, v_1^{p^i}, v_2^{p^i})$$

for  $i > 0$  and  $p > 2$ .

**PROOF.** Using methods developed by Smith [1], one can show that such an  $X$  must be an 8-cell complex and that there must be cofibrations

- (i)  $\Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \rightarrow X$ ,
- (ii)  $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g} V(0) \rightarrow Y$ ,
- (iii)  $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g'} V(0) \rightarrow Y'$ ,

where  $V(0)$  is the mod  $(p)$  Moore spectrum,  $g$  and  $g'$  induce multiplication by  $v_1^{p^i}$  in  $BP_*(V(0)) = BP_*/(p)$ , and  $f$  induces multiplication by  $v_2^{p^i}$  in

$$BP_*(Y) = BP_*(Y') = BP_*/(p, v_1^{p^i}).$$

$V(0)$  and the maps  $g, g'$  certainly exist; e.g., Smith showed that there is a map

$$\alpha: \Sigma^{2(p-1)}V(0) \rightarrow V(0)$$

which includes multiplication by  $v_1$ , hence  $\alpha^{p^i}$  induces multiplication by  $v_1^{p^i}$ , but it may not be the only map that does so.

Hence we have to show that the existence of  $f$  leads to a contradiction. Consider the composite

$$S^{2p^i(p^2-1)} \xrightarrow{j} \Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \xrightarrow{k} S^{2+2p^i(p-1)},$$

where  $j$  is the inclusion of the bottom cell and  $k$  is the collapse onto the top cell. We will show that the resulting element in  $\pi_{2p^i+1(p-1)-2}^S$  would be detected in the Novikov spectral sequence by  $\theta_i$ , thus contradicting 6.4.1. The cofibrations (ii) and (iii) induce the following short exact sequence of  $BP^*$  modules

$$0 \rightarrow \Sigma^{2p^i(p-1)}BP_*/(p) \xrightarrow{v_1^{p^i}} BP_*/(p) \rightarrow BP_*/(p, v_1^{p^i}) \rightarrow 0,$$

and the cofibration

$$S^0 \xrightarrow{p} S^0 \rightarrow V(0)$$

induces

$$0 \rightarrow BP_* \xrightarrow{p} BP_* \rightarrow BP_*/(p) \rightarrow 0.$$

Hence we get connecting homomorphisms

$$\delta_1: \text{Ext}^0(BP_*/(p, v_1^{p^i})) \rightarrow \text{Ext}^1(BP_*/(p))$$

and

$$\delta_0: \text{Ext}^1(BP_*/(p)) \rightarrow \text{Ext}^2(BP_*).$$

The element  $fj \in \pi_{2p^i(p^2-1)}(Y')$  is detected by  $v_2^{p^i} \in \text{Ext}^0(BP_*/(p, v_1^{p^i}))$ . We know (5.1.19) that

$$\theta_i = \delta_0 \delta_1(v_2^{p^i}) \in \text{Ext}^2(BP_*)$$

detects the element  $kfj \in \pi_{2p^i+1(p-1)-2}^S$ .  $\square$

The statement in 6.4.1 that  $\alpha_1 \theta_i^p$  is nonzero modulo the indeterminacy is a corollary of the following result, which relies heavily on the results of the previous three sections.

**6.4.4. DETECTION THEOREM.** *In the Adams–Novikov  $E_2$ -term for  $p > 2$  let  $\theta^I$  be a monomial in the  $\theta_i$ . Then each  $\theta^I$  and  $\alpha_1 \theta^I$  is nontrivial.  $\square$*

We are not asserting that these monomials are linearly independent, which indeed they are not. Certain relations among them will be used below to prove 6.4.1. Assuming 6.4.4, we have

PROOF OF 6.4.1. We begin with a computation in  $\text{Ext}(BP_*/(p))$ . We use the symbol  $\theta_i$  to denote the mod  $p$  reduction of the  $\theta_i$  defined above in  $\text{Ext}(BP_*)$ . We also let  $h_i$  denote the element  $-[t_1^{p^i}]$ . In the cobar construction we have

$$d[t_2] = -[t_1|t_1^p] + v_1 \sum_{0 < j < p} \frac{1}{p} \binom{p}{j} [t_1^j|t_1^{p-j}]$$

so

$$(6.4.5) \quad v_1\theta_0 = -h_0h_1.$$

May [5] developed a general theory of Steenrod operations which is applicable to this Ext group (see A1.5). His operations are similar to the classical ones in ordinary cohomology, except for the fact that  $P^0 \neq 1$ . Rather we have  $P^0(h_i) = h_{i+1}$  and  $P^0(\theta_i) = \theta_{i+1}$ . We also have  $\beta P^0(h_i) = \theta_i$ ,  $\beta P^0(\theta_i) = 0$ ,  $\beta P^0(v_1) = 0$ ,  $P^1(\theta_i) = \theta_i^p$  and the Cartan formula implies that  $P^{p^j}(\theta_i^{p^j}) = \theta_i^{p^{j+1}}$ . Applying  $\beta P^0$  to (6.4.6) gives

$$(6.4.6) \quad 0 = \theta_0h_2 - h_1\theta_1.$$

If we apply the operation  $P^{p^{i-1}}P^{p^{i-2}} \cdots P^1$  to (6.4.5) we get

$$(6.4.7) \quad h_{1+i}\theta_1^{p^i} = h_{2+i}\theta_0^{p^i}.$$

Now associated with the short exact sequence

$$0 \rightarrow BP_* \xrightarrow{p} BP_* \rightarrow BP_*/(p) \rightarrow 0$$

there is a connecting homomorphism

$$\delta: \text{Ext}^{s,*}(BP_*/(p)) \rightarrow \text{Ext}^{s+1,*}(BP_*)$$

with  $\delta(h_{i+1}) = \theta_i$ . Applying  $\delta$  to 6.4.7 gives

$$(6.4.8) \quad \theta_i\theta_1^{p^i} = \theta_{i+1}\theta_0^{p^i} \in \text{Ext}_{(BP_*BP_*)}(BP_*, BP_*).$$

We can now prove the theorem by induction on  $i$ , using 4.4.22 to start the induction. We have for  $i > 0$

$$\begin{aligned} d_{2p-1}(\theta_{i+1})\theta_0^{p^i} &= d_{2p-1}(\theta_{i+1}\theta_0^{p^i}) \\ &= d_{2p-1}(\theta_i\theta_1^{p^i}) \\ &= d_{2p-1}(\theta_i)\theta_1^{p^i} \\ &\equiv h_0\theta_{i-1}^p\theta_1^{p^i} \pmod{\ker \theta_0^{a_{i-1}}} \\ &\equiv h_0(\theta_{i-1}\theta_1^{p^{i-1}})^p \\ &\equiv h_0(\theta_i\theta_0^{p^{i-1}})^p \\ &\equiv h_0\theta_i^p\theta_0^{p^i} \end{aligned}$$

so

$$d_{2p-1}(\theta_{i+1}) \equiv h_0\theta_i^p \pmod{\ker \theta_0^{a_i}}. \quad \square$$

We now turn to the proof of 6.4.4. We map  $\text{Ext}(BP_*)$  to  $\text{Ext}(v_n^{-1}BP_*/I_n)$  with  $n = p - 1$ . By 6.1.1 this group is isomorphic to  $\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$ , which is essentially the cohomology of the profinite group  $S_n$  by 6.2.4. By 6.2.12  $S_n$  has a subgroup of order  $p$  since the field  $K$  obtained by adjoining  $p$ th roots of unity to  $\mathbf{Q}_p$  has degree  $p - 1$ . We will show that the elements of 6.4.4 have nontrivial images under the resulting map to the cohomology of  $\mathbf{Z}/(p)$ . In other words, we will consider the composite

$$BP_*(BP) \rightarrow \Sigma(n) \rightarrow S(n) \otimes \mathbf{F}_{p^n} \rightarrow C,$$

where  $C$  is the linear dual of the group ring  $\mathbf{F}_{p^n}[\mathbf{Z}/(p)]$ .

6.4.9. LEMMA. *Let  $C$  be as above. As a Hopf algebra*

$$C = \mathbf{F}_{p^n}[t]/(t^p - t) \quad \text{with} \quad \Delta t = t \otimes 1 + 1 \otimes t.$$

PROOF. As a Hopf algebra we have  $\mathbf{F}_{p^n}[\mathbf{Z}/(p)] = \mathbf{F}_{p^n}[u]/(u^p - 1)$  with  $\Delta u = u \otimes u$ , where  $u$  corresponds to a generator of the group  $\mathbf{Z}/(p)$ . We define an element  $t \in C$  by its Kronecker pairing  $\langle u^i, t \rangle = i$ . Since the product in  $C$  is dual to the coproduct in the group algebra, we have

$$\langle u^i, t^k \rangle = \langle \Delta(u^i), t \otimes t^{k-1} \rangle = \langle u^i, t \rangle \langle u^i, t^{k-1} \rangle$$

so by induction on  $k$

$$(6.4.10) \quad \langle u^i, t^k \rangle = i^k.$$

We also have  $\langle u^i, 1 \rangle = 1$ .

We show that  $\{1, t, t^2, \dots, t^{p-1}\}$  is a basis for  $C$  by relating it to the dual basis of the group algebra. Define  $x_j \in C$  by

$$x_j = \sum_{0 < k < p} (jt)^k$$

for  $0 < j < p$  and  $x_0 = 1 + \sum_{0 < j < p} x_j$ . Then

$$\begin{aligned} \langle u^i, x_j \rangle &= \left\langle u^i, \sum_{0 < k < p} (jt)^k \right\rangle = \sum_{0 < k < p} j^k t^{ik} \\ &= \sum_{0 < k < p} (ij)^k = \begin{cases} -1 & \text{if } ij = 1 \pmod{p} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\langle u^i, x_0 \rangle = \left\langle u^i, 1 + \sum_{0 < j < p} x_j \right\rangle = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

so  $\{x_0, -x_1, -x_2, \dots, -x_{p-1}\}$  is the dual basis up to permutation.

Moreover, 6.4.10 implies that  $t^p = t$  so  $C$  has the desired algebra structure.

For the coalgebra structure we use the fact that the coproduct in  $C$  is dual the product in the group algebra. We have

$$\langle u^i \otimes u^j, t \otimes 1 + 1 \otimes t \rangle = i + j$$

and

$$\langle u^i \otimes u^j, \Delta(t) \rangle = \langle u^{i+j}, t \rangle = i + j$$

so  $\Delta t = t \otimes 1 + 1 \otimes t$ . □

To proceed with the proof of 6.4.4; we now show that under the epimorphism

$$f: \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_{p^n} \rightarrow C \quad (\text{where } n = p - 1), \quad f(t_1) \neq 0.$$

From the proof of 6.2.3,  $t_1$  can be regarded as a continuous function from  $S_n$  to  $\mathbf{F}_{p^n}$ . It follows then that the nontriviality of  $f(t_1)$  is equivalent to the nonvanishing of the function  $t_1$  on the nontrivial element of order  $p$  in  $S_n$ . Suppose  $x \in S_{p-1}$  is such an element. We can write

$$x = 1 + \sum_{i>0} e_i S^i$$

with  $e_i \in W(\mathbf{F}_{p^n})$  and  $e_i^{p^n} = e_i$ . Recalling that  $S^{p-1} = p$ , we compute

$$1 = x^p \equiv 1 + pe_1 S + (e_1 S)^p \pmod{(S)^{1+p}}$$

and

$$(e_1 S)^p \equiv e_1^{(p^p-1)/(p-1)} S^p \pmod{(S)^{1+p}}$$

so it follows that

$$e_1 + e_1^{(p^p-1)/(p-1)} \equiv 0 \pmod{(p)}.$$

[Remember that  $t_1(x)$  is the mod  $(p)$  reduction of  $e_1$ .] Clearly, one solution to this equation is  $e_1 \equiv 0 \pmod{(p)}$  and hence  $e_1 = 0$ . We exclude this possibility by showing that it implies that  $x = 1$ . Suppose inductively that  $e_i = 0$  for  $i < k$ . Then  $x \equiv 1 + e_k S^k \pmod{(S^{k+1})}$  and  $x^p \equiv 1 + pe_k S^k \pmod{(S^{k+p})}$  so  $e_k \equiv 0 \pmod{(p)}$ . Since  $e_k^{p^n} - e_k = 0$ , this implies  $e_k = 0$ .

Hence,  $f$  is a map of Hopf algebras,  $f(t_1)$  primitive, so  $f(t_1) = ct$  where  $c \in \mathbf{F}_{p^n}$  is nonzero. Now recall that

$$\text{Ext}_C(\mathbf{F}_{p^n}, \mathbf{F}_{p^n}) = H^*(\mathbf{Z}/(p); \mathbf{F}_{p^n}) = E(h) \otimes P(b),$$

where  $E()$  and  $P()$  denote exterior and polynomial algebras over  $\mathbf{F}_{p^n}$ , respectively,  $h = [t] \in H^1$ , and

$$b = \sum_{0 < j < p} \frac{1}{p} \binom{p}{j} [t^j | t^{p-j}] \in H^2.$$

Let  $f^*$  denote the composition

$$\begin{aligned} \text{Ext}(BP_*) &\rightarrow \text{Ext}(v_n^{-1}BP_*/I_n) \\ &\xrightarrow{\cong} \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)) \rightarrow \text{Ext}_C(\mathbf{F}_{p^n}, \mathbf{F}_{p^n}) \xrightarrow{\cong} H^*(\mathbf{Z}/(p); \mathbf{F}_{p^n}). \end{aligned}$$

Then it follows that  $f^*(h_0) = -ch$  and  $f^*(b_i) = -c^{p^{i+1}}b$  and 6.4.4 is proved.

Note that the scalar  $c$  must satisfy  $1 + c^{(p^p-p)/(p-1)} = 0$ . Since  $c^{p^{p-1}-1} = 1$ , the equation is equivalent to  $1 + c^{(p^{p-1}-1)/(p-1)} = 0$ . It follows that  $c = w^{(p-1)/2}$  for some generator  $w$  of  $\mathbf{F}_{p^{p-1}}^\times$ , so  $c$  is not contained in any proper subfield of  $\mathbf{F}_{p^{p-1}}$ . Hence tensoring with this field is essential to the construction of the detecting map  $f$ .

Now we examine the corresponding situation in the Adams spectral sequence. The relations used to prove 6.4.1 (apart from the assertion of nontriviality) are also valid here, but the machinery used to prove 6.4.4 is, of course, not available. Indeed the monomials vanish in some cases. The following result was first proved by May [1].

6.4.11. PROPOSITION. For  $p = 3$ ,  $h_0 b_1^3 = 0$  in  $\text{Ext}_{A_3}(\mathbf{Z}/(3), \mathbf{Z}/(3))$ ; i.e.,  $b_2$  cannot support the expected nontrivial differential.

PROOF. We use a certain Massey product identity (A1.4.6) and very simple facts about  $\text{Ext}_{A_3}(\mathbf{Z}/(3), \mathbf{Z}/(3))$  to show  $h_0 b_1^2 = 0$ . We have

$$h_0 b_1^2 = -h_0 \langle h_1, h_1, h_1 \rangle b_1 = -\langle h_0, h_1, h_1 \rangle h_1 b_1.$$

By (6.4.7)  $h_1 b_1 = h_2 b_0$ , so

$$h_0 b_1^2 = -\langle h_0, h_1, h_1 \rangle h_2 b_0 = -\langle h_1, h_0, h_1 \rangle h_2 b_0 = -h_1 \langle h_0, h_1, h_2 \rangle b_0.$$

The element  $\langle h_0, h_1, h_2 \rangle$  is represented in the cobar construction by  $\xi_1^9 | \xi_2 + \xi_2^3 | \xi_1$ , which is the coboundary of  $\xi_3$ , so  $h_0 b_1^2 = 0$ .  $\square$

The case of  $b_2$  at  $p = 3$  is rather peculiar. One can show in the Adams–Novikov spectral sequence that  $d_5(\beta_7) = \pm \alpha_1 \beta_{3/3}^3$ . (This follows from the facts that  $d_5(\beta_4) = \pm \alpha_1 \beta_1^2 \beta_{3/3}$ ,  $\beta_4^2 = \pm \beta_1 \beta_7$ ,  $\beta_4 \beta_{3/3} = \pm \beta_1 \beta_{6/3}$ , and  $\beta_{3/3}^3 = \pm \beta_1^2 \beta_{6/3}$ . We leave the details to the reader.) Hence  $\beta_{9/9} \pm \beta_7$  is a permanent cycle mapping to  $b_2$ . The elements  $\beta_7$  and  $\alpha_1 \beta_{3/3}^3 = \pm \alpha_1 \beta_1^2 \beta_{6/3}$  correspond to Adams elements in filtrations 8 and 10 which are linked by a differential. We do not know the fate of the  $b_i$  at  $p = 3$  for  $i > 2$ .

To prove 6.4.2 we will need two lemmas.

6.4.12. LEMMA. For  $p \geq 3$

(i)  $\text{Ext}^{2, qp^{i+2}}(BP_*)$  is generated by the  $[(i+3)/2]$  elements  $\beta_{\alpha_{i,j}/p^{i+3-2j}}$ , where  $j = 1, 2, \dots, [(i+3)/2]$ ,  $\alpha_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1)$ , and  $[(i+3)/2]$  is the largest integer  $\leq (i+3)/2$ . Each of these elements has order  $p$ .

(i) Each of these elements except  $\beta_{p^{i+1}/p^{i+1}}$  reduces to zero in

$$\text{Ext}^{2, qp^{i+2}}(BP^*/I_3). \quad \square$$

6.4.13. LEMMA. For  $p \geq 5$ , any element of  $\text{Ext}^{2, qp^{i+2}}(BP_*)$  (for  $i \geq 0$ ) which maps to  $b_{i+1}$  in the Adams  $E_2$ -term supports a nontrivial differential  $d_{2p-1}$ .  $\square$

We have seen above that 6.4.13 is false for  $p = 3$ .

Theorem 6.4.2 follows immediately from 6.4.13 because a permanent cycle in the Adams spectral sequence of filtration 2 must correspond to one in the Adams–Novikov spectral sequence of filtration  $\leq 2$ . By sparseness (4.4.2) the Novikov filtration must also be 2, but 6.4.13 says that no element in  $\text{Ext}^2(BP_*)$  mapping to  $b_i$  for  $i \geq 1$  can be a permanent cycle.

PROOF OF 6.4.12. Part (i) can be read off from the description of  $\text{Ext}^{2,*}(BP_*)$  given in 5.4.5.

To prove (ii) we recall the definition of the elements in question. We have short exact sequences of  $BP_*(BP)$ -comodules

$$(6.4.14) \quad 0 \rightarrow BP_* \rightarrow BP_* \xrightarrow{p} BP_*/(p) \rightarrow 0.$$

$$(6.4.15) \quad 0 \rightarrow BP_*/(p) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p) \rightarrow BP_*/(p, v_1^{p^{i+3-2j}}) \rightarrow 0.$$

Let  $\delta_0$  and  $\delta_1$ , denote the respective connecting homomorphisms. Then we have  $v_2^{\alpha_{i,j}} \in \text{Ext}_{BP_*BP}^0(BP_*, BP_*/(p, v_1^{i+3-2j}))$  and  $\beta_{\alpha_{i,j}/p^{i+3-2j}} = \delta_0 \delta_1(v_2^{\alpha_{i,j}})$ . The element  $\beta_{p^{i+1}/p^{i+1}}$  the above element for  $j = 1$ ) can be shown to be  $b_{i+1}$  as follows. The right unit formula 4.3.21 gives

$$(6.4.16) \quad \begin{aligned} \eta_R(v_2) &\equiv v_2 + v_1 t_1^p - v_1^p t_1 \pmod{(p)}, \\ \delta_1(v_2^{p^{i+1}}) &= t_1^{p^{i+2}} - v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \end{aligned}$$

and  $\delta_0(t_1^{p^{i+2}}) = b_{i+1}$ . Moreover 6.4.16 implies that in  $\text{Ext}(BP_*/(p))$ ,

$$v_1^{p^j} t_1^{p^{j+1}} \cong v_1^{p^{j+1}} t_1^{p^j}, \quad \text{so} \quad v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \cong v_1^{p^{i+2}-1} t_1.$$

This element is the mod  $(p)$  reduction of  $p^{-i-2} \delta_0(v_1^{p^{i+2}})$  and is therefore in  $\ker \delta_0$ . Hence  $\delta_0 \delta_1(v_2^{p^{i+1}}) = \delta_0(t_1^{p^{i+2}}) = b_{i+1}$ .

This definition of  $\beta_{p^{i+1}/p^{i+1}}$  differs from that of 5.4.5, where for  $i > 0$  it is defined to be  $\delta_0 \delta_1(v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1})^{p^{i-1}}$ .

In principle one can compute this element explicitly in the cobar complex (A1.2.11) and reduce mod  $I_3$ , but that would be very messy. A much easier method can be devised using Yoneda's interpretation of elements in Ext groups as equivalence classes of exact sequences (see, for example, Chapter IV of Hilton and Stambach [1]) as in 5.1.20(b). Consider the following diagram.

(6.4.17)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BP_* & \xrightarrow{p} & BP_* & \xrightarrow{v_1^{i+3-3j}} & BP_*/(p) & \xrightarrow{\rho_2} & BP_*/(p, v_1^{p^{i+3-2j}}) & \longrightarrow & 0 \\ & & \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \parallel & & \\ 0 & \longrightarrow & BP_*/(p, v_1, v_2) & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & BP_*/(p, v_1^{p^{i+3-2j}}) & \longrightarrow & 0. \end{array}$$

The top row is obtained by splicing 6.4.14 and 6.4.15 and it corresponds to an element in  $\text{Ext}^2(BP_*/(p, v_1^{p^{i+3-2j}}), BP_*)$ . Composing this element with

$$v_2^{a_{i,j}} \in \text{Ext}^0(BP_*/(p, v_1^{i+3-2j}))$$

gives  $\beta_{a_{i,j}/p^{i+3-2j}}$ .

We let  $p_1$  be the standard surjection. It follows from Yoneda's result that if we choose  $BP_*BP$ -comodules  $M_1$  and  $M_2$ , and comodule maps  $p_2$  and  $p_3$  such that the diagram commutes and the bottom row is exact, then the latter will determine the element of

$$\text{Ext}_{BP_*BP}^2(BP_*/(p, v_1^{i+3-2j}), BP_*/(p, v_1, v_2))$$

which, when composed with  $v_2^{a_{i,j}}$ , will give the mod  $I_3$  reduction of  $\beta_{a_{i,j}/p^{i+3-2j}}$ . We choose  $M_1 = BP_*/(p^2, pv_1, v_1^2, pv_2)$  and  $M_2 = BP_*/(p, v_1^{2+p^{i+3-2j}})$  and let  $p_2$  and  $p_3$  be the standard surjections. It is easy to check that  $M_1$  and  $M_2$  are comodules over  $BP_*(BP)$ , i.e., that the corresponding ideals in  $BP_*$  are invariant. (The ideal used to define  $M_1$  is simply  $I_2^2 + I_1 I_3$ .) Moreover, the resulting diagram has the desired properties.

The resulting bottom row of 6.4.17 is the splice of the two following short exact sequences.

$$(6.4.18) \quad 0 \rightarrow BP_*/(p, v_1, v_2) \xrightarrow{p} BP_*/(p^2, pv_1, pv_2, v_1^2) \rightarrow BP_*/(p, v_1^2) \rightarrow 0,$$

$$(6.4.19) \quad 0 \rightarrow BP_*/(p, v_1^2) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p, v_1^{2+p^{i+3-2j}}) \rightarrow BP_*/(p, v_1^{i+3-2j}) \rightarrow 0.$$

Let  $\delta'_0, \delta'_1$  denote the corresponding connecting homomorphisms. The elements we are interested in then are  $\delta'_0 \delta'_1(v_2^{a_{i,j}})$ .

To compute  $\delta'_1(v_2^{a_{i,j}})$  we use the formula  $d(v_2^n) = (v_2 + v_1 v_1 t_1^p - v_1^p t_1)^n - v_2^n$ , implied by 6.4.16, in the cobar construction for  $BP_*/(p, v_1^{2+p^{i+3-2j}})$ . Recall that

$$a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1) \quad 1 \leq j \leq [(i+3)/2].$$

Hence  $a_{i,j} = p^{i+3-2j} \pmod{(p^{i+4-2j})}$  and  $d(v_2^{a_{i,j}}) = v_2^{b_{i,j}} v_1^{p^{i+3-2j}} [t_1^{p^{i+4-2j}}]$ , so

$$\delta'_1(v_2^{a_{i,j}}) = v_2^{b_{i,j}} [t_1^{p^{i+3-2j}}],$$

where  $b_{i,j} = a_{i,j} - p^{i+3-2j} = (p^{i+2} - p^{i+4-2j})/(p+1)$ .

For  $j = 1$ ,  $b_{i,1} = 0$  and

$$\delta'_0 \delta'_1(v_2^{a_{i,1}}) = - \sum_{0 < k < p} \frac{1}{p} \binom{p}{k} [t_1^{kp^i} | t_1^{(p-k)p^i}] = -b_{i+1}.$$

For  $j > 1$ ,  $b_{i,j}$  is divisible by  $p$  and  $d(v_2^{b_{i,j}}) \equiv 0 \pmod{(p^2, pv_1, v_1^2)}$  and

$$v_2^{b_{i,j}} d(t_1^{p^{i+4-2j}}) \equiv 0 \pmod{(pv_2)},$$

so  $\delta'_1 v_2^{a_{i,j}} \in \text{Ext}^1(BP_*/(p, v_1^2))$  pulls back in 6.4.17 to an element of

$$\text{Ext}^1(BP_*/(p^2, pv_1, pv_2, v_2^2)) \quad \text{and} \quad \delta'_0 \delta'_1(v_2^{a_{i,j}}) = 0,$$

completing the proof.  $\square$

**PROOF OF 6.4.13.** Any element of  $\text{Ext}^{2,qp^{i+2}}(BP_*)$  can be written uniquely as  $cb_{i+1} + x$  where  $x$  is in the subgroup generated by the elements  $\beta_{a_{i,j}/p^{i+3-2j}}$  for  $j > 1$ . In 5.4.6, we showed that  $x$  maps to zero in the classical Adams  $E_2$ -term. Hence it suffices to show that no such  $x$  can have the property

$$d_{2p-1}(x) = d_{2p-1}(b_{i+1})$$

By 5.5.2 for  $p \geq 5$  there is an 8-cell spectrum  $V(2) = M(p, v_1, v_2)$  with  $BP_*(V(2)) = BP_*/(p, v_1, v_2)$ , and a map  $f: S^0 \rightarrow V(2)$  inducing a surjection in  $BP$  homology.  $f$  also induces the standard map

$$f_*: \text{Ext}(BP_*) \rightarrow \text{Ext}(BP_*/I_3).$$

Lemma 6.4.12 asserts that  $f_*(\beta_{a_{i,j}/p^{i+3-2j}}) = 0$  for  $j > 1$ , so  $f_*(d_{2p-1}(x)) = 0$  where  $x$  is as above. However, 6.4.1 and the proof of 6.4.4 show that

$$g_*(d_{2p-1}(b_{i+1})) \neq 0,$$

where  $g_*$  is induced by the obvious map

$$g: BP_* \rightarrow v_{p-1}^{-1} BP_*/I_{p-1}.$$

Since  $g$  factors through  $BP_*/I_3$ , this shows that  $f_*(d_{2p-1}(b_{i+1})) \neq 0$ , completing the proof.  $\square$

## 5. The Spectra $T(m)$

In this section will we construct certain spectra  $T(m)$  and study the corresponding chromatic spectral sequence.  $T(m)$  satisfies

$$BP_*(T(m)) = BP_*[t_1, t_2, \dots, t_m] \subset BP_*(BP_*)$$

as a comodule algebra. These are used in Chapter 7 in a computation of the Adams–Novikov  $E_2$ -term. We will see there that the Adams–Novikov spectral sequence for  $T(m)$  is easy to compute through a range of dimensions that grows rapidly with  $m$ , and here we will show that its chromatic spectral sequence is very regular.

To construct the  $T(m)$  recall that  $BU = \Omega SU$  by Bott periodicity, so we have maps  $\Omega SU(k) \rightarrow BU$  for each  $k$ . Let  $X(k)$  be the Thom spectrum of



the corresponding vector bundle over  $\Omega SU(k)$ . An easy calculation shows that  $H_*(X(k)) = \mathbf{Z}[b_1, b_2, \dots, b_{k-1}] \subset H_*(MU)$ . Our first result is

**6.5.1. SPLITTING THEOREM.** *For any prime  $p$ ,  $X(k)_{(p)}$  is equivalent to a wedge of suspensions of  $T(m)$  with  $m$  chosen so that  $p^m \leq k < p^{m+1}$ , and  $BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP)$ . Moreover  $T(m)$  is a homotopy associative commutative ring spectrum.*  $\square$

From this we get a diagram

$$S_{(p)}^0 = T(0) \rightarrow T(1) \rightarrow T(2) \rightarrow \cdots \rightarrow BP.$$

In Ravenel [8, §3] we show that after  $p$ -adic completion there are no essential maps from  $T(i)$  to  $T(j)$  if  $i > j$  or from  $BP$  to  $T(i)$ .

This theorem is an analog of 4.1.12, which says that  $MU_{(p)}$  splits into a wedge of suspensions of  $BP$ , as is its proof. We start with the following generalization of 4.1.1.

**6.5.2. DEFINITION.** *Let  $E$  be an associative commutative ring spectrum. A complex orientation of degree  $k$  for  $E$  is a class  $x_E \in \tilde{E}^2(\mathbf{C}P^k)$  whose restriction to  $\tilde{E}^2(\mathbf{C}P^1) \cong \pi_0(E)$  is 1.*  $\square$

A complex orientation as in 4.1.1 is of degree  $k$  for all  $k > 0$ . This notion is relevant in view of

**6.5.3. LEMMA.**  *$X(k)$  admits a complex orientation of degree  $k$ .*

**PROOF.**  $X(k)$  is a commutative associative ring spectrum (up to homotopy) because  $\Omega SU(k)$  is a double loop space. The standard map  $\mathbf{C}P^{k-1} \rightarrow BU$  lifts to  $\Omega SU(k)$ . Thomifying gives a stable map  $\mathbf{C}P^k \rightarrow X(k)$  with the desired properties.  $\square$

$X(k)$  plays the role of  $MU$  in the theory of spectra with orientation of degree  $k$ . The generalizations of lemmas 4.1.4, 4.1.7, 4.1.8, and 4.1.13 are straightforward. We have

**6.5.4. PROPOSITION.** *Let  $E$  be an associative commutative ring spectrum with a complex orientation  $x_E \in \tilde{E}_2(\mathbf{C}P^k)$  of degree  $k$ .*

(a)  $E^*(\mathbf{C}P^k) = \pi_*(E)[x_E]/(x_E^{k+1})$ .

(b)  $E^*(\mathbf{C}P^k \times \mathbf{C}P^k) = \pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E^{k+1} \otimes 1, 1 \otimes x_E^{k+1})$ .

(c) *For  $0 < i < k$  the map  $t: \mathbf{C}P^i \times \mathbf{C}P^{k-i} \rightarrow \mathbf{C}P^k$  induces a formal group law  $k$ -chunk; i.e., the element  $t^*(x_E)$  in the truncated power series ring*

$$\pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E \otimes 1, 1 \otimes x_E)^{k+1}$$

*has properties analogous to an formal group law (A2.1.1).*

(d)  $E_*(X(k)) = \pi_*(E)[b_1^E, \dots, b_{k-1}^E]$  where  $b_i^E \in E_{2i}(X(k))$  is defined as in 4.1.7.

(e) *With notation as in 4.1.8, in  $(E \wedge X(k))^2(\mathbf{C}P^k)$  we have*

$$\hat{x}_{X(k)} = \sum_{0 \leq i \leq k-1} b_i^E \hat{x}_E^{i+1} \quad \text{where } b_0 = 1.$$

*This power series will be denoted by  $g_E(\hat{x}_E)$ .*

(f) *There is a one-to-one correspondence between degree  $k$  orientations of  $E$  and multiplicative maps  $X(k) \rightarrow E$  as in 4.1.13.*  $\square$

We do not have a generalization of 4.1.15, i.e., a convenient way of detecting maps  $X(k) \rightarrow X(k)$ , but we can get by without it. By 6.5.4(f) a multiplicative map  $g: X(k)_{(p)} \rightarrow X(k)_{(p)}$  is determined by a polynomial  $f(x) = \sum_{0 \leq i \leq k-1} f_i x^{i+1}$  with  $f_0 = 1$  and  $f_i \in \pi_{2i}(X(k)_{(p)})$ . In this range of dimensions  $\pi_*(\overline{X(k)})$  is isomorphic to  $\pi_*(MU)$ , so we can take  $f(x)$  to be the truncated form of the power series of A2.1.23. Then the calculations of 4.1.12 show that  $g$  induces an idempotent in ordinary or  $BP_*$ -homology. In the absence of 4.1.15 it does not follow that  $g$  itself is idempotent. Nevertheless we can define

$$T(m) = \lim_g X(k)_{(p)},$$

i.e.,  $T(m)$  is the mapping telescope of  $g$ . Then we can compose the map  $X(k)_{(p)} \rightarrow T(m)$  with various self-maps of  $X(k)_{(p)}$  to construct the desired splitting, thereby proving 6.5.1.

Now we consider the chromatic spectral sequence for  $T(m)$ . Using the change-of-rings isomorphism 6.1.1, the input needed for the machinery of Section 5.1 is  $\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m)))$  where  $K(n)_*(T(m)) = K(n)_*[t_1, \dots, t_m]$ . Using notation as in 6.3.7, let  $\Sigma(n, m+1) = \Sigma(n)/(t_1, \dots, t_m)$ . Then we have

6.5.5. THEOREM. *With notation as above we have*

$$\begin{aligned} \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) \\ = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes_{K(n)_*} \text{Ext}_{\Sigma(n, m+1)}(K(n)_*, K(n)), \end{aligned}$$

where  $\dim u_j = \dim v_j$ . Moreover  $u_j$  maps to  $v_j$  under the map to  $\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(BP)) = B(n)_*$  (6.1.11) induced by  $T(m) \rightarrow BP$ . In other words its image in  $K(n)_*(BP)$  coincides with that of  $\eta_R(v_j) \in BP_*(BP)$  under the map  $BP_*(BP) \rightarrow K(n)_*(BP)$ .  $\square$

Applying 6.3.7 gives

6.5.6. COROLLARY. *If  $n < m + 2$  and  $n < 2(p - 1)(m + 1)/p$  then*

$$\begin{aligned} \text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) \\ = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes E(h_{k,j}: m+1 \leq k \leq m+n, \quad j \in \mathbf{Z}/(n)). \end{aligned}$$

$\square$

PROOF OF 6.5.5. The images of  $\eta_R(v_{n+j})$  (for  $1 \leq j \leq m$ ) in  $K(n)_*(T(m))$  are primitive and give the  $u_{n+j}$ . The image of  $BP_*(T(m)) \rightarrow BP_*(BP) \rightarrow \Sigma(n)$  is the subalgebra generated by  $\{t_n: n \leq m\}$ . The result follows by a routine argument.  $\square$

Now we will use the chromatic spectral sequence to compute  $\text{Ext}^s(BP_*(T(m)))$  for  $s = 0$  and 1. We assume  $m > 0$  since  $T(0) = S^0$ , which was considered in 5.2.1 and 5.2.6. By 6.5.5 and 6.5.6 we have

$$\begin{aligned} (6.5.7) \quad \text{Ext}_{\Sigma(0)}(K(0)_*, K(0)_*(T(m))) &= \mathbf{Q}[u_1, \dots, u_m] \quad \text{and} \\ \text{Ext}_{\Sigma(1)}(K(1)_*, K(1)_*(T(m))) &= K(1)_*[u_2, \dots, u_{m+1}] \otimes E(h_{m+1,0}). \end{aligned}$$

The short exact sequence

$$(6.5.8) \quad 0 \rightarrow M_1^0 \otimes BP_*(T(m)) \xrightarrow{i} M^1 \otimes BP_*(T(m)) \xrightarrow{p} M^1 \otimes BP_*(T(m)) \rightarrow 0$$

induces a six-term exact sequence of Ext groups with connecting homomorphism  $\delta$ . For  $j \leq m$ ,  $\eta_R(v_j) \in BP_*(T(m)) \subset BP_*(BP)$ , so if  $u$  is any monomial in these elements then  $\delta(u/p^i) = 0$  for all  $i > 0$  and  $\text{Ext}^0(M^1 \otimes BP_*(T(m)))$  has a corresponding summand isomorphic to  $\mathbf{Q}/\mathbf{Z}(p)$ . Hence in the chromatic spectral sequence,  $E_1^{1,0}$  has a summand isomorphic to  $\mathbf{Z}_{(p)}[u_1, \dots, u_m] \otimes \mathbf{Q}/\mathbf{Z}(p)$ , which is precisely the image of  $d_1: E_1^{0,0} \rightarrow E_1^{1,0}$ , giving

6.5.9. PROPOSITION.

$$\text{Ext}^0(BP_*(T(m))) = \mathbf{Z}_{(p)}[u_1, \dots, u_m]. \quad \square$$

Next we need to consider the divisibility of  $u_{m+1}^t/p \in \text{Ext}^0(M^1 \otimes BP_*(T(m)))$ . Note that  $\eta_R(v_{m+1})$  is not in  $BP_*(T(m))$  but  $\eta_R(v_{m+1}) - pt_{m+1}$  (where  $v_{m+1}$  is Hazewinkel's generator given by A2.2.1) is, so we call this element  $u_{m+1}$ . It follows that in the cobar complex  $C(BP_*(T(m)))$  (A1.2.11)  $d(u_{m+1}) = pt_{m+1}$  and

$$(6.5.10) \quad d(u_{m+1}^t) \equiv ptu_{m+1}^{t-1}t_{m+1} + p^2 \binom{t}{2} u_{m+1}^{t-2}t_{m+1}^2 \pmod{(p^2t)},$$

where the second term is nonzero only when  $p = 2$  and  $t$  is even. Thus the situation is similar to that for  $m = 0$  where we have  $v_1 = u_1$ . Recall that in that case the presence of the second term caused  $\text{Ext}^1$  to behave differently at  $p = 2$ . We will show that this does not happen for  $m \geq 1$  and we have

6.5.11. THEOREM. For  $m \geq 1$  and all primes  $p$

$$\text{Ext}^1(BP_*(T(m))) = \text{Ext}^0(BP_*(T(m))) \otimes \{u_{m+1}^t/pt : t > 0\}.$$

PROOF. For  $p > 2$  the result follows from 6.5.10 as in 5.2.6. Now recall the situation for  $m = 0$ ,  $p = 2$ . For  $t = 2$ , 6.5.10 gives  $d(v_1^2) = 4(v_1t_1 + t_1^2)$  and we have  $d(4v_1^{-1}v_2) \equiv 4(v_2t_1 + t_1^2) \pmod{(8)}$ , so we get a cocycle  $(v_1^2 + 4v_1^{-1}v_2)/8$ . The analogous cocycle for  $m \geq 1$  would be something like

$$(u_{m+1}^2 + 4v_1^{-1}u_{m+2})/q$$

where  $u_{m+2}$  is related somehow to  $v_{m+2}$ . However, the relevant terms in  $\eta_R(v_{m+2}) \pmod{(2)}$  are  $v_1t_{m+1}^2 + v_2^{2^{m+1}}t_{m+1}$ , which does not bear the resemblance to 6.5.10 for  $m \geq 1$  that it does for  $m = 0$ . In other words  $u_{m+1}^{t-2}t_{m+1}^2$  is not cohomologous  $\pmod{(2)}$  to  $u_{m+1}^{t-1}t_{m+1}$ , so the calculation for  $p = 2$  can proceed as it does for  $p > 2$ .  $\square$

Our last result is useful for computing the Adams–Novikov  $E_2$ -term for  $T(m)$  by the method used in Section 4.4.

6.5.12. THEOREM. For  $t < 2(p^{2m+2} - 1)$

$$\text{Ext}(BP_*(T(m))/I_{m+1}) = \mathbf{Z}/(p)[u_{m+1}, u_{m+2}, \dots, u_{2m+1}] \otimes E(h_{i,j}) \otimes P(b_{i,j})$$

with  $i \geq m + 1$ ,  $i + j \leq 2m + 2$ ,  $h_{i,j} \in \text{Ext}^{1,2p^j(p^i-1)}$  and  $b_{i,j} \in \text{Ext}^{2,2p^{j+1}(p^i-1)}$ .

6.5.13. EXAMPLE. For  $m = 1$  we have

$$\begin{aligned} \text{Ext}(BP_*(T(1))/I_2) &= \mathbf{Z}/(p)[u_2, u_3] \otimes E(h_{2,0}, h_{2,1}, h_{2,2}, h_{3,0}, h_{3,1}) \\ &\quad \otimes P(b_{2,0}, b_{2,1}, b_{3,0}) \end{aligned}$$

in 6.5.1 for  $t \leq 2(p^4 - 1)$

PROOF OF 6.5.12. By a routine change-of-rings argument (explained in Section 7.1) the Ext in question is the cohomology of  $C_\Gamma(BP_*/I_{m+1})$  (A1.2.11) where  $\Gamma = BP_*(BP)/(t_1, \dots, t_m)$ . Then from 4.3.15 and 4.3.20 we can deduce that  $v_i$  and  $t_i$  are primitive for  $m+1 \leq i \leq 2m+1$ .  $h_{i,j}$  corresponds to  $t_i^{p^j}$  and  $b_{i,j}$  to  $-\sum_{0 < k < p} p^{-1} \binom{p}{k} t_i^{kp^j} | t_i^{(p-k)p^j}$ . The result follows by routine calculation.  $\square$

## Computing Stable Homotopy Groups with the Adams–Novikov Spectral Sequence

In this chapter we apply the Adams–Novikov spectral sequence to the motivating problem of this book, the stable homotopy groups of spheres. Our main accomplishment is to find the first thousand stems for  $p = 5$ , the previous record being 760 by Aubry [1]. In Section 1 we describe the method of infinite descent for computing the Adams–Novikov spectral sequence  $E_2$ -term in a range of dimensions, namely to find it for the spectra  $T(m)$  of Section 6.5 by downward induction on  $m$ . Recall  $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$  as a comodule, so  $T(m)$  is equivalent to  $BP$  in dimensions less than  $|v_{m+1}|$ . This starts our downward induction since we always restrict our attention to a finite range of dimensions.

In Section 2 we construct a resolution enabling us in theory to extract the Adams–Novikov  $E_2$ -term for  $S^0$  from that for  $T(1)$ . In practice we must proceed more slowly, computing for skeleta  $T(1)^{(p^i-1)q}$  by downward induction on  $i$ . In Section 3 we do this down to  $i = 1$ ; see 7.5.1.  $T(1)^{(p-1)q}$  is a complex with  $p$  cells, its Adams–Novikov spectral sequence collapses in our range, and its homotopy is surprisingly regular.

In Section 4 we take the final step from  $T(1)^{(p-1)q}$  to  $S^0$ . We have a spectral sequence (7.1.16) for this calculation and a practical procedure (7.1.18) for the required bookkeeping. We illustrate this method for  $p = 3$ , but here our range of dimensions is not new; see Tangora [6] and Nakamura [3].

In Section 5 we describe the calculations for  $p = 5$ , giving a running account of the more difficult differentials in the spectral sequence of 7.1.16 for that case. The results are tabulated in Appendix 3 and range up to the 1000-stem.

In more detail, the method in question involves the connective  $p$ -local ring spectra  $T(m)$  of 6.5, which satisfy

$$BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP).$$

$T(0)$  is the  $p$ -local sphere spectrum, and there are maps

$$S^0 = T(0) \rightarrow T(1) \rightarrow T(2) \rightarrow \dots \rightarrow BP.$$

The map  $T(m) \rightarrow BP$  is an equivalence below dimension  $|v_{m+1}| - 1 = 2p^{m+1} - 3$ .

To descend from  $\pi_*(T(m))$  to  $\pi_*(T(m-1))$  we need some spectra interpolating between  $T(m-1)$  and  $T(m)$ . Note that  $BP_*(T(m))$  is a free module over  $BP_*(T(m-1))$  on the generators  $\{t_m^j : j \geq 0\}$ . In Lemma 7.1.11 we show that for each  $h$  there is a  $T(m-1)$ -module spectrum  $T(m-1)_h$  with

$$BP_*(T(m-1)_h) = BP_*(T(m-1))\{t_m^j : 0 \leq j \leq h\}.$$

We will be most interested in the case where  $h$  is one less than a power of  $p$ , and we will denote  $T(m)_{p^i-1}$  by  $T(m)_{(i)}$ .

We have inclusions

$$T(m-1) = T(m-1)_{(0)} \rightarrow T(m-1)_{(1)} \rightarrow T(m-1)_{(2)} \rightarrow \cdots \rightarrow T(m)$$

and the map  $T(m-1)_{(i)} \rightarrow T(m)$  is an equivalence below dimension  $p^i |t_m| - 1 = 2(h+1)(p^m - 1) - 1$ .

For example when  $m = i = 0$ , the spectrum  $T(m)_{(i)}$  is  $S^0$  while  $T(m)_{p^{i+1}-1}$  is the  $p$ -cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where  $q = 2p - 2$ .

In Theorem 7.1.16 we give a spectral sequence for computing  $\pi_*(T(m-1)_{(i)})$  in terms of  $\pi_*(T(m-1)_{(i+1)})$ . Its  $E_1$ -term is

$$E(h_{m,i}) \otimes P(b_{m,i}) \otimes \pi_*(T(m-1)_{(i+1)})$$

where the elements

$$\begin{aligned} h_{m,i} &\in E_1^{1, 2p^i(p^m-1)} \\ \text{and } b_{m,i} &\in E_1^{2, 2p^{i+1}(p^m-1)} \end{aligned}$$

are permanent cycles.

In the case  $m = i = 0$  cited above, the  $E_1$ -term of this spectral sequence is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where  $h_{1,0}$  and  $b_{1,0}$  represent the homotopy elements  $\alpha_1$  and  $\beta_1$  ( $\alpha_1^2$  for  $p = 2$ ) respectively.

Thus to compute  $\pi_*(S^0)$  below dimension  $p^3(2p-2)$  we could proceed as follows. In this range we have

$$BP \cong T(3) \cong T(2)_{(1)}.$$

We then use the spectral sequence of 7.1.16 to get down to  $T(2)$ , which is equivalent in this range to  $T(1)_{(2)}$ , then use it twice to get down to  $T(1) \cong T(0)_{(3)}$ , and so on. This would make for a total of six applications of 7.1.16. Fortunately we have some shortcuts that make this process easier.

The Adams–Novikov  $E_2$ -term for  $T(m)$  is

$$\text{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))).$$

From now on we will drop the first variable when writing such Ext groups, since we will never consider any value for it other than  $BP_*$ . There is a change-of-rings isomorphism that equates this group with

$$\text{Ext}_{\Gamma(m+1)}(BP_*)$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots].$$

Using our knowledge of  $\text{Ext}_{\Gamma(m+1)}^0(BP_*)$  (Proposition 7.1.24) and  $\text{Ext}_{\Gamma(m+1)}^1(BP_*)$  (Theorem 7.1.31) in all dimensions, we will construct a 4-term exact sequence

$$0 \rightarrow BP_* \rightarrow D_{m+1}^0 \rightarrow D_{m+1}^1 \rightarrow E_{m+1}^2 \rightarrow 0$$

of  $\Gamma(m+1)$ -comodules. The two  $D_{m+1}^i$  are weak injective, meaning that all of their higher Ext groups (above  $\text{Ext}^0$ ) vanish (we study such comodules systematically at the end of Section 1), and below dimension  $p^2|v_{m+1}|$

$$\text{Ext}_{\Gamma(m+1)}^0(D_{m+1}^i) \cong \text{Ext}_{\Gamma(m+1)}^i(BP_*).$$

It follows that in that range

$$\text{Ext}_{\Gamma(m+1)}^s(E_{m+1}^2) \cong \text{Ext}_{\Gamma(m+1)}^{s+2}(BP_*) \quad \text{for all } s \geq 0.$$

The comodule  $E_{m+1}^2$  is  $(2p^{m+2} - 2p - 1)$ -connected. In Theorem 7.2.6 we determine its Ext groups (and hence those of  $BP_*$ ) up to dimension  $p^2|v_{m+1}|$ . There are no Adams–Novikov differentials or nontrivial group extensions in this range (except in the case  $m = 0$  and  $p = 2$ ), so this also determines  $\pi_*(T(m))$  in the same range.

Thus Theorem 7.2.6 gives us the homotopy of  $T(0)_{(3)}$  in our range directly without any use of 7.1.16. In a future paper with Hirofume Nakai we will study the homotopy of  $T(m)_{(2)}$  and the spectral sequence of 7.1.16 for the homotopy of  $T(m)_{(1)}$  below dimension  $p^3|v_{m+1}|$ . There are still no room for Adams–Novikov differentials, so the homotopy and Ext calculations coincide. For  $m = 0$  this computation was the subject of Ravenel [11].

It is only when we pass from  $T(m)_{(1)}$  to  $T(m)_{(0)} = T(m)$  that we encounter Adams–Novikov differentials below dimension  $p^3|v_{m+1}|$ . For  $m = 0$  the first of these is the Toda differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1 \beta_1^p$$

of Toda [3] and Toda [2].

## 1. The method of infinite descent

First we define some Hopf algebroids that we will need.

7.1.1. DEFINITION.  $\Gamma(m+1)$  is the quotient  $BP_*(BP)/(t_1, t_2, \dots, t_m)$ ,

$$A(m) = BP_* \square_{\Gamma(m+1)} BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_m]$$

and

$$G(m+1, k-1) = \Gamma(m+1) \square_{\Gamma(m+k+1)} BP_* = A(m+k)[t_{m+1}, t_{m+1} \dots, t_{m+k}]$$

We abbreviate  $G(m+1, 0)$  by  $G(m+1)$ , and is understood that  $G(m+1, \infty) = \Gamma(m+1)$ .

In particular,  $\Gamma(1) = BP_*(BP)$ .

7.1.2. PROPOSITION.  $G(m+1, k-1) \rightarrow \Gamma(m+1) \rightarrow \Gamma(m+k+1)$  is a Hopf algebroid extension (A1.1.15). Given a left  $\Gamma(m+1)$ -comodule  $M$  there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to  $\text{Ext}_{\Gamma(m+1)}(BP_*, M)$  with

$$\tilde{E}_2^{s,t} = \text{Ext}_{G(m+1, k-1)}^s(A(m+k)), \text{Ext}_{\Gamma(m+k+1)}^t(BP_*, M)$$

and  $d_r : E_r^{s,t} \rightarrow \tilde{E}_r^{s+r, t-r+1}$ . (We use the notation  $\tilde{E}_r^{s,t}$  to distinguish the Cartan–Eilenberg spectral sequence from the resolution spectral sequence.)

7.1.3. COROLLARY. *Let  $M$  be a  $\Gamma(m+1)$ -comodule concentrated in nonnegative dimensions. Then*

$$\mathrm{Ext}_{\Gamma(m+k+1)}(BP_*, M) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, G(m+1, k-1) \otimes_{A(m+k)} M).$$

*In particular,  $\mathrm{Ext}_{\Gamma(m+1)}^{s,t}(BP_*, M)$  for  $t < 2(p^{m+1} - 1)$  is isomorphic to  $M$  for  $s = 0$  and vanishes for  $s > 0$ . Moreover for the spectrum  $T(m)$  constructed in 6.5 and having  $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$ ,*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*).$$

The following characterization of the Cartan–Eilenberg spectral sequence is a special case of (A1.3.16).

7.1.4. LEMMA. *The Cartan–Eilenberg spectral sequence of 7.1.2 is the one associated with the decreasing filtration of the cobar complex  $C_{\Gamma(m+1)}(BP_*, M)$  (see below) defined by saying that*

$$\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \in C_{\Gamma(m+1)}^s(BP_*, M)$$

*is in  $F^i$  if  $i$  of the  $\gamma$ 's project trivially to  $\Gamma(m+k+1)$ .*

The *method of infinite descent* for computing  $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$  for a connective comodule  $M$  (e.g. the BP-homology of a connective spectrum) is to compute over  $\mathrm{Ext}$  over  $\Gamma(m+1)$  by downward induction on  $m$ . To calculate through a fixed range of dimensions  $k$ , we choose  $m$  so that  $k \leq 2(p^{m+1} - 1)$  and use 7.1.3 to start the induction. For the inductive step we could use the Cartan–Eilenberg spectral sequence of 7.1.2, but it is more efficient to use a different spectral sequence, which we now describe.

7.1.5. DEFINITION. *A comodule  $M$  over a Hopf algebroid  $(A, \Gamma)$  is **weak injective** (through a range of dimensions) if  $\mathrm{Ext}^s(M) = 0$  for  $s > 0$  (through the same range).*

We will study such comodules in the at the end of this section.

7.1.6. DEFINITION. *For a left  $G(m+1, k-1)$ -comodule  $M$  let*

$$\widehat{r}_j : M \rightarrow \Sigma^{j|t_{m+1}|} M$$

*be the group homomorphism defined by*

$$M \xrightarrow{\psi_M} G(m+1, k-1) \otimes M \xrightarrow{\rho_j \otimes M} \Sigma^{j|t_{m+1}|} M$$

*where  $\rho_j : G(m+1, k-1) \rightarrow A(m+k)$  is the  $A(m+k)$ -linear map sending  $t_{m+1}^j$  to 1 and all other monomials in the  $t_{m+i}$  to 0.*

*We will refer to this map as a **Quillen operation**. When  $m = 0$  we denote it simply by  $r_j$ .*

It follows that

$$\psi(x) = \sum_j t_{m+1}^j \otimes \widehat{r}_j(x) + \dots,$$

where the missing terms involve  $t_\ell$  for  $\ell > m+1$ .

The following is proved in Ravenel [12] as Lemma 1.10.

7.1.7. LEMMA. *The Quillen operation  $\widehat{r}_j$  of 7.1.6 is a comodule map and for  $j > 0$  it induces the trivial endomorphism in  $\mathrm{Ext}$ .*



7.1.8. DEFINITION. Let  $T_m^h \subset G(m+1, k-1)$  denote the sub- $A(m+k)$ -module generated by  $\{t_{m+1}^j : 0 \leq j \leq h\}$ . We will denote  $T_m^{p^i-1}$  by  $T_m^{(i)}$ . A  $G(m+1, k-1)$ -comodule  $M$  is  $i$ -**free** if the comodule tensor product  $T_m^{(i)} \otimes_{A(m+k)} M$  is weak injective.

We have suppressed the index  $k$  from the notation  $T_m^h$  because it will usually be clear from the context. In the case  $k = \infty$  the Ext group has the topological interpretation given in Lemma 7.1.11 below. The following lemma is useful in dealing with such comodules. It is proved in Ravenel [12] as Lemma 1.12.

7.1.9. LEMMA. For a left  $G(m+1)$ -comodule  $M$ , the group

$$\text{Ext}_{G(m+1)}^0(A(m+1), T_m^{(i)} \otimes_{A(m+k)} M)$$

is isomorphic as an  $A(m)$ -module to

$$L = \bigcap_{j \geq p^i} \ker \hat{r}_j \subset M.$$

The following is proved in Ravenel [12] as Lemma 1.14.

7.1.10. LEMMA. Let  $D$  be a weak injective comodule over  $\Gamma(m+1)$ . Then  $T_m^{(i)} \otimes D$  is also weak injective with

$$\text{Ext}_{\Gamma(m+1)}^0(T_m^{(i)} \otimes D) \cong A(m) \left\{ t_{m+1}^j : 0 \leq j \leq p^i - 1 \right\} \otimes \text{Ext}_{\Gamma(m+1)}^0(D).$$

Given  $x_0 \in \text{Ext}_{\Gamma(m+1)}^0(D)$ , the element isomorphic to  $t_{m+1}^j \otimes x_0$  is

$$\sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_{m+1}^k \otimes x_{j-k} \in T_m^{(i)} \otimes D$$

where  $x_j \in D$  satisfies

$$\psi(x_j) = \sum_{0 \leq k \leq j} \binom{j}{k} t_{m+1}^{j-k} \otimes x_k.$$

The following is proved in Ravenel [12] as Lemma 1.15. The only case of it that we will need here is for  $m = 0$ , where  $T(0)_h$  is the  $2(p-1)h$ -skeleton of  $T(1)$ .

7.1.11. LEMMA. For each nonnegative  $m$  and  $h$  there is a spectrum  $T(m)_h$  where  $BP_*(T(m)_h) \subset BP_*(BP)$  is a free module over

$$BP_*(T(m)) = BP_*[t_1, \dots, t_m]$$

on generators  $\{t_{m+1}^j : 0 \leq j \leq h\}$ . Its Adams–Novikov  $E_2$ -term is

$$\text{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)_h)) \cong \text{Ext}_{\Gamma(m+1)}(BP_*, T_m^h).$$

We will denote  $T(m)_{p^i-1}$  by  $T(m)_{(i)}$ .

To pass from  $\text{Ext}_{G(m+1, k-1)}(T_m^{(i+1)} \otimes M)$  to  $\text{Ext}_{G(m+1, k-1)}(T_m^{(i)} \otimes M)$  we can make use of the tensor product (over  $A(m+k)$ ) of  $M$  with the long exact sequence

$$(7.1.12) \quad 0 \longrightarrow T_m^{(i)} \xrightarrow{i} R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} R^2 \xrightarrow{d^2} \dots,$$

where

$$R^{2s+e} = \sum^{(ps+e)2p^i(p^{m+1}-1)} T_m^{(i)} \quad \text{for } e = 0, 1$$

$$\text{and } d^s = \begin{cases} \widehat{r}_p^i & \text{for } s \text{ even} \\ \widehat{r}_{(p-1)p^i} & \text{for } s \text{ odd,} \end{cases}$$

which leads to a spectral sequence as in (A1.3.2).

7.1.13. THEOREM. *For a  $G(m+1, k-1)$ -comodule  $M$  there is a spectral sequence converging to  $\text{Ext}_{G(m+1, k-1)}(M \otimes T_m^{(i)})$  with*

$$E_1^{s,t} = E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \text{Ext}_{G(m+1, k-1)}^t(T_m^{(i)} \otimes M)$$

with  $h_{m+1,i} \in E_1^{1,0}$ ,  $b_{m+1,i} \in E_1^{2,0}$ , and  $d_r : E_r^{s,t} \rightarrow E_r^{s+r, t-r+1}$ . If  $M$  is  $(i+1)$ -free in a range of dimensions, then the spectral sequence collapses from  $E_2$  in the same range.

Moreover  $d_1$  is induced by the action on  $M$  of  $\widehat{r}_{p^i \Delta_{m+1}}$  for  $s$  even and  $\widehat{r}_{(p-1)p^i}$  for  $s$  odd.

The action of  $d_1$  is as follows. Let

$$x = \sum_{0 \leq j < p^{i+1}} t_{m+1}^j \otimes m_j \in T_m^{(i)} \otimes M$$

Then  $d_1$  is induced by the endomorphism

$$x \mapsto \begin{cases} - \sum_{0 \leq k < p^i} \sum_{k \leq j < p^{i+1}} \binom{j}{k} t_{m+1}^{j-k} \otimes \widehat{r}_{(p^i-k)}(m_j) & \text{for } s \text{ even} \\ - \sum_{0 \leq k < (p-1)p^i} \sum_{k \leq j < p^{i+1}} \binom{j}{k} t_{m+1}^{j-k} \otimes \widehat{r}_{((p-1)p^i-k)}(m_j) & \text{for } s \text{ odd.} \end{cases}$$

We will refer to this as the **small descent spectral sequence**.

PROOF. Additively this spectral sequence is a special case of the one in (A1.3.2) associated with  $M$  tensored with the long exact sequence (7.1.12), and the collapsing for  $(i+1)$ -free  $M$  follows from the fact that the spectral sequence is in that case concentrated on the horizontal axis.

For the identification of  $d_1$ , note that by (7.1.12) it is induced by the endomorphism

$$x \mapsto \begin{cases} \sum_{0 \leq j < p^{i+1}} \widehat{r}_{p^i}(t_{m+1}^j) \otimes m_j & \text{for } s \text{ even} \\ \sum_{0 \leq j < p^{i+1}} \widehat{r}_{(p-1)p^i}(t_{m+1}^j) \otimes m_j & \text{for } s \text{ odd} \end{cases}$$

$$= \begin{cases} \sum_{p^i \leq j < p^{i+1}} \binom{j}{p^i} t_{m+1}^{j-p^i} \otimes m_j & \text{for } s \text{ even} \\ \sum_{(p-1)p^i \leq j < p^{i+1}} \binom{j}{(p-1)p^i} t_{m+1}^{j-(p-1)p^i} \otimes m_j & \text{for } s \text{ odd.} \end{cases}$$

It follows from Lemma 7.1.7 that  $\widehat{r}_{p^i \Delta_{m+1}}$  and  $\widehat{r}_{(p-1)p^i \Delta_{m+1}}$  each induce trivial endomorphisms in Ext, so  $d_1$  is also induced by

$$x \mapsto \begin{cases} -\widehat{r}_{p^i}(x) + \sum_{0 \leq j < p^{i+1}} \widehat{r}_{p^i}(t_{m+1}^j) \otimes m_j & \text{for } s \text{ even} \\ -\widehat{r}_{(p-1)p^i}(x) + \sum_{0 \leq j < p^{i+1}} \widehat{r}_{(p-1)p^i}(t_{m+1}^j) \otimes m_j & \text{for } s \text{ odd,} \end{cases}$$

which leads to the stated formula.

The multiplicative structure requires some explanation. The elements  $h_{m+1,i}$  and  $b_{m+1,i}$  correspond under Yoneda's isomorphism Hilton and Stambach [1, page 155] to the tensor product of  $M$  with the exact sequences

$$\begin{aligned} 0 \rightarrow T_m^{(i)} \rightarrow T_m^{2p^i-1} \rightarrow \Sigma^{p^i|t_{m+1}|} T_m^{(i)} \rightarrow 0 \\ \text{and} \\ 0 \rightarrow T_m^{(i)} \rightarrow T_m^{(i+1)} \rightarrow \Sigma^{p^i|t_{m+1}|} T_m^{(i+1)} \rightarrow \Sigma^{p^{i+1}|t_{m+1}|} T_m^{(i)} \rightarrow 0 \end{aligned}$$

respectively. Products of these elements correspond to the splices of these. It follows that these two elements are permanent cycles and that the spectral sequence is one of modules over the algebra  $E(h_{m+1,i}) \otimes P(b_{m+1,i})$ .  $\square$

In practice we will find higher differentials in this spectral sequence by computing in the cobar complex  $C_{G(m+1,k-1)}(M \otimes T_m^{(i)})$  or its subcomplex  $C_{G(m+1,k-1)}(M)$ . As explained in the proof of (A1.3.2), it can be embedded by a quasi-isomorphism (i.e., a map inducing an isomorphism in cohomology) into the double complex  $B = \bigoplus_{s,t \geq 0} B^{s,t}$  defined by

$$B^{s,t} = C_{G(m+1,k-1)}^t(M \otimes R^s)$$

with coboundary

$$\partial = d + (-1)^s d^s,$$

where  $d$  is the coboundary operator in the cobar complex. Our spectral sequence is obtained from the filtration of  $B$  by horizontal degree, i.e., the one defined by

$$F^r B = \bigoplus_{s \geq r, t \geq 0} B^{s,t}.$$

Theorem 7.1.13 also has a topological counterpart in the case  $M = BP_*$ . Before stating it we need to define topological analogs of the operations  $\widehat{r}_{p^i}$  and  $\widehat{r}_{(p-1)p^i}$ . One can show that there are cofiber sequences

$$(7.1.14) \quad T(m)_{(i)} \rightarrow T(m)_{(i+1)} \rightarrow \Sigma^{p^i|t_{m+1}|} T(m)_{p^i(p-1)-1}$$

and

$$(7.1.15) \quad T(m)_{p^i(p-1)-1} \rightarrow T(m)_{(i+1)} \rightarrow \Sigma^{(p-1)p^i|t_{m+1}|} T(m)_{(i)}.$$

We define

$$T(m)_{(i+1)} \xrightarrow{\rho_{p^i}} \Sigma^{p^i|t_{m+1}|} T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)} \xrightarrow{\rho_{p^i(p-1)}} \Sigma^{(p-1)p^i|t_{m+1}|} T(m)_{(i+1)}$$

to be the composites

$$T(m)_{(i+1)} \rightarrow \Sigma^{p^i |t_{m+1}|} T(m)_{p^i(p-1)-1} \rightarrow \Sigma^{p^i |t_{m+1}|} T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)} \rightarrow \Sigma^{(p-1)p^i |t_{m+1}|} T(m)_{(i)} \rightarrow \Sigma^{(p-1)p^i |t_{m+1}|} T(m)_{(i+1)}.$$

7.1.16. THEOREM. Let  $T(m)_{(i)}$  be the spectrum of Lemma 7.1.11. There is a spectral sequence converging to  $\pi_*(T(m)_{(i)})$  with

$$E_1^{s,t} = E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{(i+1)}) \quad \text{and} \quad d_r : E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$$

with  $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$  and  $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$ . Moreover  $d_1$  is  $\rho_p^i$  for  $s$  even and  $\rho_{(p-1)p^i}$  for  $s$  odd. The elements  $h_{m+1,i}$  and  $b_{m+1,i}$  are permanent cycles, and the spectral sequence is one of modules over the ring

$$R = E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

We will refer to this as the **topological small descent spectral sequence**.

PROOF. This is the spectral sequence based on the Adams diagram

$$\begin{array}{ccccccc} X & \longleftarrow & \Sigma^a X' & \longleftarrow & \Sigma^b X & \longleftarrow & \Sigma^{a+b} X' \longleftarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & & \Sigma^a Y & & \Sigma^b Y & & \Sigma^{a+b} Y \end{array}$$

where

$$\begin{aligned} a &= 2p^i(p^{m+1} - 1) - 1, \\ b &= 2p^{i+1}(p^{m+1} - 1) - 2, \\ X &= T(m)_{(i)}, \\ X' &= T(m)_{p^i(p-1)-1}, \\ \text{and } Y &= T(m)_{(i+1)}. \end{aligned}$$

We will show that the elements  $h_{m+1,i}$  and  $b_{m+1,i}$  can each be realized by maps of the form

$$S^0 \longrightarrow X \xrightarrow{f} \Sigma^{-?} X$$

For  $h_{m+1,i}$ ,  $f$  is the boundary map for the cofiber sequence

$$T(m)_{(i)} \rightarrow T(m)_{2p^i-1} \rightarrow \Sigma^{h+1} T(m)_{(i)},$$

and for  $b_{m+1,i}$  it is the composite (in either order) of the ones for (7.1.14) and (7.1.15).  $\square$

7.1.17. EXAMPLE. When  $m = i = 0$ , the spectrum  $T(0)_{(0)}$  is  $S^0$  while  $T(0)_{(1)}$  is the  $p$ -cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \dots \cup_{\alpha_1} e^{(p-1)q},$$

where  $q = 2p - 2$ . The  $E_1$ -term of the spectral sequence of Theorem 7.1.16 is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where  $h_{1,0}$  and  $b_{1,0}$  represent the homotopy elements  $\alpha_1$  and  $\beta_1$  ( $\alpha_1^2$  for  $p = 2$ ) respectively.

We will use this spectral sequence through a range of dimensions in the following way. For each spectrum  $T(m)_{(i+1)}$  the elements of Adams–Novikov filtration 0 and 1 are all permanent cycles, so we ignore them, replacing  $\pi_*(T(m)_{(i+1)})$  by an appropriate subquotient of  $\text{Ext}_{\Gamma(m+1)}(T_m^{(i)} \otimes E_{m+1}^2)$ . Let  $\mathbf{N}$  be a list of generators of this group arranged by dimension. When an element  $x$  has order greater than  $p$ , we also list its nontrivial multiples by powers of  $p$ . Thus

$$\mathbf{N} \otimes E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

contains a list of generators of the  $E_1$ -term in our range. Rather than construct similar lists for each  $E_r$  term we use the following method.

7.1.18. INPUT/OUTPUT PROCEDURE. *We make two lists  $\mathbf{I}$  (input) and  $\mathbf{O}$  (output).  $\mathbf{I}$  is the subset of  $\mathbf{N} \otimes E(h_{m+1,i})$  that includes all elements in our range. Then  $\mathbf{O}$  is constructed by dimensional induction starting with the empty list as follows. Assuming  $\mathbf{O}$  has been constructed through dimensions  $k - 1$ , add to it the  $k$ -dimensional elements of  $\mathbf{I}$ . If any of them supports a nontrivial differential in the spectral sequence, remove both the source and target from  $\mathbf{O}$ . (It may be necessary to alter the list of  $(k - 1)$ -dimensional elements by a linear transformation so that each nontrivial target is a “basis” element.) Then if  $k > |b_{m+1,i}|$ , we append the product of  $b_{m+1,i}$  with each element of  $\mathbf{O}$  in dimension  $k - |b_{m+1,i}|$ . This completes the inductive step.*

Note that each element in  $E_1$  of filtration greater than 1 is divisible by  $b_{m+1,i}$ . Since the spectral sequence is one of  $R$ -modules, that same is true of each  $E_r$ . In 7.1.18 we compute the differentials originating in filtrations 0 and 1. If  $d_r(x) = y$  is one of them, there is no chance that for some minimal  $t > 0$

$$d_{r'}(x') = b_{m+1,i}^t y \quad \text{with } r' < r$$

because such an  $x'$  would have to be divisible by  $b_{m+1,i}$ . This justifies the removal of  $b_{m+1,i}^t x$  and  $b_{m+1,i}^t y$  for all  $t \geq 0$  from consideration.

We will consider various  $\Gamma(m + 1)$ -comodules  $M$  and will abbreviate  $\text{Ext}_{\Gamma(m+1)}(BP_*, M)$  by  $\text{Ext}_{\Gamma(m+1)}(M)$  or simply  $\text{Ext}(M)$ .

Excluding the case  $m = 0$  and  $p = 2$ , we will construct a diagram of 4-term exact sequences of  $\Gamma(m + 1)$ -comodules

$$(7.1.19) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & BP_* & \longrightarrow & D_{m+1}^0 & \longrightarrow & D_{m+1}^1 & \longrightarrow & E_{m+1}^2 & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & BP_* & \longrightarrow & D_{m+1}^0 & \longrightarrow & v_1^{-1}E_{m+1}^1 & \longrightarrow & E_{m+1}^1/(v_1^\infty) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & BP_* & \longrightarrow & M^0 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow & 0 \end{array}$$

where each vertical map is a monomorphism,  $M^i$  and  $N^2$  are as in 5.1.5, the  $D_{m+1}^i$  are weak injectives with  $\text{Ext}^0(D_{m+1}^0) = \text{Ext}^0(BP_*)$ ,  $\text{Ext}^0(D_{m+1}^1)$  contains  $\text{Ext}^1(BP_*)$  (with equality holding for  $m = 0$  and  $p$  odd), and  $E_{m+1}^1 = D_{m+1}^0/BP_*$ .  $\text{Ext}^0(BP_*)$  and  $\text{Ext}^1(BP_*)$  are given in 7.1.24 and 7.1.31 respectively.

It follows that for  $m = 0$  and  $p$  odd, there is an isomorphism

$$\text{Ext}_{\Gamma(m+1)}^s(E_{m+1}^2) \cong \text{Ext}_{\Gamma(m+1)}^{s+2}(BP_*),$$

and for  $m > 0$  there is a similar isomorphism below dimension  $p^2|v_{m+1}|$  for all primes.  $E_{m+1}^2$  is locally finite and  $(p|v_{m+1}| - 1)$ -connected, which means that  $\text{Ext}_{\Gamma(m+1)}^s$  for  $s > 1$  vanishes below dimension  $p|v_{m+1}|$ .

We will construct the map from  $BP_*$  to the weak injective  $D_{m+1}^0$ , inducing an isomorphism in  $\text{Ext}^0$ , explicitly in Theorem 7.1.28. For  $m > 0$  we cannot construct a similar map out of  $E_{m+1}^1 = D_{m+1}^0/BP_*$ . Instead we will construct a map to a weak injective  $D_{m+1}^1$  which enlarges  $\text{Ext}^0$  by as little as possible. We will do this by producing a comodule  $E_{m+1}^2 \subset E_{m+1}^1/v_1^\infty$  and using the induced extension

$$(7.1.20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_{m+1}^1 & \longrightarrow & v_1^{-1}E_{m+1}^1 & \longrightarrow & E_{m+1}^1/(v_1^\infty) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & E_{m+1}^1 & \longrightarrow & D_{m+1}^1 & \longrightarrow & E_{m+1}^2 \longrightarrow 0 \end{array}$$

The comodule  $E_{m+1}^2$  for  $m > 0$  will be described in the next section. For  $m = 0$  and  $p$  odd, a map from  $E_1^1$  to a weak injective  $D_1^1$  inducing an isomorphism in  $\text{Ext}^0$  will be constructed in below in Lemma 7.2.1.

We will use the following notations for  $m > 0$ . We put hats over the symbols in order to distinguish this notation from the usual one for elements in  $\text{Ext}_{BP_*(BP)}$ . For  $m = 0$  we will use similar notation without the hats.

$$(7.1.21) \quad \left\{ \begin{array}{l} \hat{v}_i = v_{m+i}, \quad \hat{t}_i = t_{m+i}, \quad \omega = p^m, \\ \hat{h}_{i,j} = h_{m+i,j}, \quad \text{and} \quad \hat{b}_{i,j} = b_{m+i,j}. \end{array} \right.$$

We will show that in dimensions below  $p^2|\hat{v}_1|$ ,  $E_{m+1}^2$  is the  $A(m+1)$ -module generated by the set of chromatic fractions

$$(7.1.22) \quad \left\{ \frac{\hat{v}_2^{e_2}}{p^{e_0}v_1^{e_1}} : e_0, e_1 > 0, e_2 \geq e_0 + e_1 - 1 \right\},$$

and its Ext group in this range is

$$(7.1.23) \quad A(m+1)/I_2 \otimes E(\hat{h}_{1,0}) \otimes P(\hat{b}_{1,0}) \otimes \left\{ \frac{\hat{v}_2^{e_2}}{pv_1} : e_2 \geq 1 \right\},$$

where  $\hat{h}_{1,0} \in \text{Ext}^{1,2(p\omega-1)}$  corresponds to the primitive  $\hat{t}_1 \in \Gamma(m+1)$ , and  $\hat{b}_{1,0} \in \text{Ext}^{1,2p(p\omega-1)}$  is its transpotent. In both cases there is no power of  $v_1$  in the numerator when  $m = 0$ . These statements will be proved below as Theorem 7.2.6.

An Adams–Novikov differential for  $T(m)$  originating in the 2-line would have to land in filtration  $2p+1$ , which is trivial in the is range of dimensions, so by proving 7.2.6 we have determined  $\pi_*(T(m))$  in this range.

Our first goal here is to compute  $\text{Ext}^0$  and  $\text{Ext}^1$ . The following generalization of the Morava–Landweber theorem (4.3.2) is straightforward.

7.1.24. PROPOSITION.

$$\text{Ext}_{\Gamma(m+1)}^0(BP_*/I_n) = A(n+m)/I_n.$$

For  $n = 0$  each of the generators is a permanent cycle.

PROOF. The indicated elements are easily seen to be invariant in  $\Gamma(m+1)$ . In dimensions less than  $|\widehat{v}_1|-1$ ,  $T(m)$  is homotopy equivalent to  $BP$ , so the generators  $v_i$  for  $i \leq m$  are permanent cycles as claimed.  $\square$

Now we will describe a map from  $BP_*$  to a weak injective  $D_{m+1}^0$  inducing an isomorphism in  $\text{Ext}^0$ .  $D_{m+1}^0$  is the sub- $A(m)$ -algebra of  $p^{-1}BP_*$  generated by certain elements  $\widehat{\lambda}_i$  for  $i > 0$  congruent to  $\widehat{v}_i/p$  modulo decomposables.

To describe them we need to recall the formula of Hazewinkel [4] (see A2.2.1) relating polynomial generators  $v_i \in BP_*$  to the coefficients  $\ell_i$  of the formal group law, namely

$$(7.1.25) \quad p\ell_i = \sum_{0 \leq j < i} \ell_j v_{i-j}^{p^j} \quad \text{for } i > 0.$$

This recursive formula expands to

$$\begin{aligned} \ell_1 &= \frac{v_1}{p} \\ \ell_2 &= \frac{v_2}{p} + \frac{v_1^{p+1}}{p^2} \\ \ell_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3} \\ &\vdots \end{aligned}$$

We need to define reduced log coefficients  $\widehat{\ell}_i$  for  $i > 0$  obtained from the  $\ell_{m+i}$  by subtracting the terms which are monomials in the  $v_j$  for  $j \leq m$ . Thus for  $m > 0$  we have

$$\begin{aligned} \widehat{\ell}_1 &= \frac{\widehat{v}_1}{p} \\ \widehat{\ell}_2 &= \frac{\widehat{v}_2}{p} + \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{\widehat{v}_1 v_1^{p\omega}}{p^2} \\ &\vdots \end{aligned}$$

The analog of Hazewinkel's formula for these elements is

$$(7.1.26) \quad p\widehat{\ell}_i = \sum_{0 \leq j < i} \ell_j \widehat{v}_{i-j}^{p^j} + \sum_{0 < j < \min(i, m+1)} \widehat{\ell}_{i-j} v_j^{p^{i-j}\omega}.$$

We use these to define our generators  $\widehat{\lambda}_i$  recursively for  $i > 0$  by

$$(7.1.27) \quad \widehat{\lambda}_i = \widehat{\ell}_i - \sum_{0 < j < i} \ell_j \widehat{\lambda}_{i-j}^{p^j}.$$

For  $m = 0$  we will denote these by  $\lambda_i$ .

The following is proved as Theorem 3.12 and equation (3.15) in Ravenel [12].

7.1.28. THEOREM. *The  $BP_*$ -module  $D_{m+1}^0 \subset p^{-1}BP_*$  described above is a subcomodule over  $\Gamma(m+1)$  that is weak injective (7.1.5) with  $\text{Ext}^0 = A(m)$ . In it we have*

$$\eta_R(\widehat{\lambda}_i) \equiv \widehat{\lambda}_i + \widehat{t}_i \text{ mod decomposables.}$$

Before proceeding further we need the following technical tool.

7.1.29. DEFINITION. Let  $H$  be a graded connected torsion abelian  $p$ -group of finite type, and let  $H_i$  have order  $p^{h_i}$ . Then the **Poincaré series** for  $H$  is

$$g(H) = \sum h_i t^i.$$

7.1.30. EXAMPLE. Let  $I \subset BP_*$  be the maximal ideal so that  $BP_*/I = \mathbf{Z}/(p)$ . Then the Poincaré series for  $\Gamma(m+1)/I$  is

$$G_m(t) = \prod_{i>0} (1 - t^{|v_{m+i}|})^{-1}.$$

We will abbreviate  $t^{|v_{m+i}|}$  by  $x_i$  and denote  $x_1$  simply by  $x$ . When  $m > 0$  we will denote  $t^{|v_i|}$  for  $i \leq m$  by  $y_i$  and  $t^{|v_1|}$  simply by  $y$ .

For  $\text{Ext}^1$  we have

7.1.31. THEOREM. Unless  $m = 0$  and  $p = 2$  (which is handled in (5.2.6)),  $\text{Ext}_{\Gamma(m+1)}^1(BP_*, BP_*)$  is the  $A(m)$ -module generated by the set

$$\left\{ \delta_0 \left( \frac{\widehat{v}_1^j}{jp} \right) : j > 0 \right\},$$

where  $\delta_0$  is the connecting homomorphism for the short exact sequence

$$0 \rightarrow BP_* \rightarrow M^0 \rightarrow N^1 \rightarrow 0$$

as in (5.1.5). Its Poincaré series is

$$g_m(t) = \sum_{i>0} \frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}},$$

where  $x = t^{|v_{m+1}|}$ . Moreover each of these elements is a permanent cycle.

PROOF. The  $\text{Ext}$  calculation follows from (6.5.11) and (7.1.3). For the Poincaré series, note that the set of  $A(m)$ -module generators of order  $p^i$  is

$$\left\{ \delta_0 \left( \frac{\widehat{v}_1^{jp^{i-1}}}{p^i} \right) : j > 0 \right\},$$

and its Poincaré series is

$$\frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}}.$$

To show that each of these elements is a permanent cycle, we will study the Bockstein spectral sequence converging to  $\pi_*(T(m))$  with

$$E_1 = \mathbf{Z}/(p)[v_0] \otimes \pi_*(V(0) \wedge T(m)).$$

$V(0) \wedge T(m)$  is a ring spectrum in all cases except  $m = 0$  and  $p = 2$ . We know that  $T(m)$  is a ring spectrum for all  $m$  and  $p$  and that  $V(0)$  is one for  $p$  odd. The case  $p = 2$  and  $m > 0$  is dealt with in Lemma 3.18 of Ravenel [12].

Low dimensional calculations reveal that  $\widehat{v}_1 \in \text{Ext}^0(BP_*/p)$  is a homotopy element. The elements  $\widehat{\alpha}_j = \frac{\widehat{v}_1^j}{p}$  can then be constructed in the usual way using the self-map of  $V(0) \wedge T(m)$  inducing multiplication by  $\widehat{v}_1^j$  followed by the pinch map

$$V(0) \wedge T(m) \rightarrow \Sigma T(m).$$

In the Bockstein spectral sequence it follows that  $\widehat{v}_1^{sp^i}$  survives to  $E_{i+1}$ , so  $\widehat{\alpha}_{sp^i}$  is divisible (as a homotopy element) by  $p^i$ .  $\square$



Now we will recall some results about weak injective comodules  $M$  over a general Hopf algebroid  $(A, \Gamma)$  over  $\mathbf{Z}_{(p)}$ . In most cases we will refer to Ravenel [12] for the proofs. We will abbreviate  $\text{Ext}_\Gamma(A, M)$  by  $\text{Ext}(M)$ .

The definition 7.1.5 of a weak injective should be compared with other notions of injectivity. A comodule  $I$  (or more generally an object in an abelian category) is *injective* if any homomorphism to it extends over monomorphisms, i.e., if one can always fill in the following diagram.

$$\begin{array}{ccc} & I & \\ & \uparrow & \swarrow \\ 0 & \longrightarrow M & \xrightarrow{i} N \end{array}$$

This definition is rather limiting. For example if  $A$  is a free  $\mathbf{Z}_{(p)}$ -module, then an injective  $I$  must be  $p$ -divisible since a homomorphism  $A \rightarrow I$  must extend over  $A \otimes \mathbf{Q}$ .

There is also a notion of relative injectivity (A1.2.7) requiring  $I$  to be a summand of  $\Gamma \otimes_A I$ , which implies that the diagram above can always be completed when  $i$  is split over  $A$ . This implies weak injectivity as we have defined it here (see (A1.2.8)(b)), but we do not know if the converse is true. In any case the requirements of our definition can be said to hold only through a range of dimensions. The following is proved in Ravenel [12] as Lemma 2.1.

7.1.32. LEMMA. *A connective comodule  $M$  over  $(A, \Gamma)$  is weak injective in a range of dimensions iff  $\text{Ext}^1(M) = 0$  in the same range.*

The following is proved in Ravenel [12] as Lemma 2.2.

7.1.33. LEMMA. *Let*

$$(D, \Phi) \rightarrow (A, \Gamma) \rightarrow (A, \Sigma)$$

*be an extension (A1.1.15) of graded connected Hopf algebroids of finite type, and suppose that  $M$  is a weak injective comodule over  $\Gamma$ . Then  $M$  is also weak injective over  $\Sigma$ , and  $\text{Ext}_\Sigma^0(A, M)$  is weak injective over  $\Phi$  with*

$$\text{Ext}_\Phi^0(D, \text{Ext}_\Sigma^0(A, M)) \cong \text{Ext}_\Gamma^0(A, M).$$

Here is a method of recognizing weak injectives without computing any higher Ext groups. The following is proved in Ravenel [12] as Theorem 2.6.

7.1.34. THEOREM. *Let  $(A, \Gamma)$  be a graded connected Hopf algebroid over  $\mathbf{Z}_{(p)}$ , and let  $M$  be a connected torsion  $\Gamma$ -comodule of finite type. Let  $I \subset A$  be the maximal ideal (so that  $A/I = \mathbf{Z}/(p)$ ). Then*

$$g(M) \leq g(\text{Ext}^0(M))g(\Gamma/I),$$

*meaning that each coefficient of the power series on the left is dominated by the corresponding one on the right, with equality holding if and only if  $M$  is a weak injective (7.1.5).*

It would be nice if for any comodule  $M$  one could find a map  $M \rightarrow W$  to a weak injective inducing an isomorphism in  $\text{Ext}^0$ , but this is not always possible. In Ravenel [12, Example 2.8] we showed that it fails when  $(A, \Gamma) = (A(1), G(1))$  and  $M = A/(p^2)$ .

For future reference will need the Poincaré series of  $E_{m+1}^1 = D_{m+1}^0/BP_*$ . The following is proved as Lemma 3.16 in Ravenel [12].

7.1.35. LEMMA. *Let*

$$g_m(t) = \prod_{1 \leq i \leq m} \frac{1}{1 - y_i}$$

and

$$G_m(t) = \prod_{i > 0} \frac{1}{1 - x_i},$$

(with  $x_i$  and  $y_i$  as in 7.1.30) the series for  $A(m)/(p)$  and  $\Gamma(m+1)/I$  respectively. Then the Poincaré series for  $E_{m+1}^1 = D_{m+1}^0/BP_*$  is

$$g_m(t)G_m(t) \sum_{i > 0} \frac{x_i}{1 - x_i}.$$

## 2. The comodule $E_{m+1}^2$

In this section we will describe the comodule  $E_{m+1}^2$  needed above in (7.1.20) below dimension  $p^2|\widehat{v}_1|$ . This will determine  $\pi_*(T(m))$  below dimension  $p^2|\widehat{v}_1| - 3$ . For  $m = 0$  and  $p$  odd we can construct  $D_1^1$  in all dimensions directly as follows.

7.2.1. LEMMA. *For  $p$  odd there is a map  $E_1^1 \rightarrow D_1^1$  to a weak injective inducing an isomorphism in  $\text{Ext}^0$ .*

PROOF.  $M^1 = v_1^{-1}E_1^1$  is not a weak injective for  $m = 0$  since  $\text{Ext}_{\Gamma(1)}^1(M^1) = \mathbf{Q}/\mathbf{Z}$  concentrated in degree 0.

We will construct  $D_1^1$  as a union of submodules of  $M^1$  as follows. Let  $K_0 = E_1^1 \subset M^1$ . For each  $i \geq 0$  we will construct a diagram

$$\begin{array}{ccccc}
 & & L_{i+1} & \xlongequal{\quad} & L_{i+1} \\
 & & \uparrow & & \uparrow \\
 K_i & \longrightarrow & M^1 & \longrightarrow & L_i \\
 \parallel & & \uparrow & & \uparrow \\
 K_i & \longrightarrow & K_{i+1} & \longrightarrow & L'_i
 \end{array}$$

in which each row and column is exact.  $L'_i$  will be the sub- $BP_*$ -module of  $L_i = M^1/K_i$  generated by the positive dimensional part of  $\text{Ext}^0(L_i)$ . It is a subcomodule of  $L_i$ ,  $K_{i+1}$  is defined to be the induced extension by  $K_i$ , and  $L_{i+1} = M^1/K_{i+1}$ . Hence  $K_i$ ,  $K_{i+1}$ , and  $L'_i$  are connective while  $L_i$  and  $L_{i+1}$  are not.

We know that in positive dimensions  $K_0 = E_1^1$  has the same  $\text{Ext}^0$  as  $M^1$ . We will show by induction that the same is true for each  $K_i$ . In the long exact sequence of Ext groups associated with the right column, the map  $\text{Ext}^0(L'_i) \rightarrow \text{Ext}^0(L_i)$  is an isomorphism in positive dimensions, so the positive dimensional part of  $\text{Ext}^0(L_{i+1})$  is contained in  $\text{Ext}^1(L'_i)$ , which has higher connectivity than  $\text{Ext}^0(L_i)$ .

It follows that the connectivity of  $L'_i$  increases with  $i$ , and therefore the limit  $K_\infty$  has finite type. The connectivity of the positive dimensional part of  $\text{Ext}^0(L_i)$

also increases with  $i$ , so  $\text{Ext}^0(L_\infty)$  is trivial in positive dimensions. From the long exact sequence of Ext groups for the short exact sequence

$$0 \rightarrow K_\infty \rightarrow M^1 \rightarrow L_\infty \rightarrow 0$$

we deduce that  $\text{Ext}^1(K_\infty) = 0$ , so  $K_\infty$  is a weak injective by Lemma 7.1.32. It has the same  $\text{Ext}^0$  as  $E_1^1$ , so it is our  $D_1^1$ .  $\square$

Now we are ready to study the hypothetical comodule  $E_{m+1}^2$  of (7.1.19) for  $m > 0$ .

7.2.2. LEMMA. *The Poincaré series for  $E_{m+1}^2$  is at least*

$$g_m(t)G_m(t) \sum_{i>0} \frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}$$

(where  $g_m(t)$  and  $G_m(t)$  are as in Lemma 7.1.35), with equality holding for  $m = 0$  and  $p > 2$ . In dimensions less than  $p^2|\widehat{v}_1|$  it simplifies to

$$g_{m+2}(t) \left( \frac{x^p(1-y)}{(1-x_2)(1-x^p)} \right),$$

where  $x$ ,  $y$ ,  $x_i$  and  $y_i$  are as in 7.1.30.

We will see in Theorem 7.2.6 below that equality also holds in dimensions less than  $p^2|\widehat{v}_1|$ .

PROOF OF 7.2.2. The relevant Poincaré series (excluding the case  $m = 0$  and  $p = 2$ ) are

$$\begin{aligned} g(E_{m+1}^1) &= g_m(t)G_m(t) \sum_{i>0} \frac{x_i}{(1-x_i)} && \text{by 7.1.35} \\ &= g_m(t)G_m(t) \left( \frac{x}{1-x} + \sum_{i>0} \frac{x_{i+1}}{1-x_{i+1}} \right), \\ \text{and } g(\text{Ext}^0(E_{m+1}^1)) &= g(\text{Ext}^1(BP_*)) \\ &= g_m(t) \sum_{i>0} \frac{x^{p^{i-1}}}{1-x^{p^{i-1}}} && \text{by 7.1.31} \\ &= g_m(t) \left( \frac{x}{1-x} + \sum_{i>0} \frac{x^{p^i}}{1-x^{p^i}} \right). \end{aligned}$$

If there were a map  $E_{m+1}^1 \rightarrow D_{m+1}^1$  to a weak injective inducing an isomorphism in  $\text{Ext}^0$  (which there is for  $m = 0$  and  $p$  odd by 7.2.1), we would have

$$\begin{aligned} g(D_{m+1}^1) &= G_m(t)g(\text{Ext}^0(E_{m+1}^1)) && \text{by 7.1.34} \\ &= G_m(t)g(\text{Ext}^1(BP_*)) \\ &= g_m(t)G_m(t) \left( \frac{x}{1-x} + \sum_{i>0} \frac{x^{p^i}}{1-x^{p^i}} \right). \end{aligned}$$

It follows that

$$\begin{aligned}
g(E_{m+1}^2) &\geq g_m(t)G_m(t) \left( \frac{x}{1-x} + \sum_{i>0} \frac{x^{p^i}}{1-x^{p^i}} \right) - g(E_{m+1}^1) \\
&= g_m(t)G_m(t) \sum_{i>0} \left( \frac{x^{p^i}}{1-x^{p^i}} - \frac{x_{i+1}}{1-x_{i+1}} \right) \\
&= g_m(t)G_m(t) \sum_{i>0} \frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}.
\end{aligned}$$

In our range of dimensions we can replace  $g_m(t)G_m(t)$  by  $g_{m+2}(t)$ , and only the first term of the last sum is relevant. Hence we have

$$g(E_{m+1}^2) \equiv g_{m+2}(t) \left( \frac{x^p(1-y)}{(1-x_2)(1-x^p)} \right) \pmod{(t^{p^2|\widehat{v}_1|})}. \quad \square$$

7.2.3. COROLLARY. *For a locally finite bounded below subcomodule*

$$E \subset E_{m+1}^1/(v_1^\infty),$$

let  $D$  denote the induced (as in (7.1.20)) extension by  $E_{m+1}^1$  shown in the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & E_{m+1}^1 & \longrightarrow & v_1^{-1}E_{m+1}^1 & \longrightarrow & E_{m+1}^1/(v_1^\infty) \longrightarrow 0 \\
& & \parallel & & \uparrow & & \uparrow \\
0 & \longrightarrow & E_{m+1}^1 & \longrightarrow & D & \longrightarrow & E \longrightarrow 0
\end{array}$$

Let  $K$  denote the kernel of the connecting homomorphism

$$\delta : \text{Ext}^0(E) \rightarrow \text{Ext}^1(E_{m+1}^1) = \text{Ext}^2(BP_*).$$

Then  $D$  is weak injective if and only if the Poincaré series  $g(E)$  is  $g(K)G_m(t)$  plus the series specified in Lemma 7.2.2. In particular it is weak injective if  $\delta$  is a monomorphism and  $g(E)$  is the specified series.

PROOF. The specified series is  $G_m(t)g(\text{Ext}^0(E_{m+1}^1)) - g(E_{m+1}^1)$ , and

$$g(\text{Ext}^0(D)) = g(\text{Ext}^0(E_{m+1}^1)) + g(K).$$

Hence our hypothesis implies

$$\begin{aligned}
g(D) &= g(E_{m+1}^1) + g(E) \\
&= g(E_{m+1}^1) + G_m(t)g(\text{Ext}^0(E_{m+1}^1)) - g(E_{m+1}^1) + g(K)G_m(t) \\
&= G_m(t)(g(\text{Ext}^0(E_{m+1}^1)) + g(K)) \\
&= G_m(t)g(\text{Ext}^0(D)),
\end{aligned}$$

which is equivalent to the weak injectivity of  $D$  by Theorem 7.1.34.  $\square$

Now we need to identify some elements in  $E_{m+1}^1/(v_1^\infty)$ .

7.2.4. LEMMA. *The comodule  $E_{m+1}^1/(v_1^\infty)$  contains the sets*

$$(a) \quad \left\{ \begin{array}{l} \left\{ \frac{1}{p^{1+e_0}v_1^{e_1-e_0}} : e_1 > e_0 \geq 0 \right\} \quad \text{for } m = 0 \\ \left\{ \frac{\widehat{v}_1^{1+e_0}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0 \right\} \quad \text{for } m > 0 \end{array} \right.$$

$$(b) \quad \left\{ \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0 \right\}.$$

These generators will be discussed further in Theorem 7.2.6 below.

PROOF. (i) The element in question is the image of  $v_1^{-1-e_1}\widehat{\lambda}_1^{1+e_0}$ .

(ii) In  $D_{m+1}^0$  we have

$$\begin{aligned} \widehat{\lambda}_2 &= \widehat{\ell}_2 - \ell_1 \widehat{\lambda}_1^p \\ &= \frac{\widehat{v}_2}{p} - \frac{v_1 \widehat{\lambda}_1^p}{p} + \begin{cases} \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{v_1^{p\omega} \widehat{v}_1}{p^2} & \text{for } m = 0 \\ \frac{v_1^{p+1}}{p^2} & \text{for } m > 0 \end{cases} \\ &= \frac{\widehat{v}_2}{p} + \frac{v_1}{p} (1 - p^{p-1}) \widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m = 0 \\ v_1^{p\omega} \widehat{\lambda}_1 & \text{for } m > 0 \end{cases} \end{aligned}$$

so

$$(7.2.5) \quad \frac{\widehat{v}_2}{p} = \widehat{\lambda}_2 + \frac{v_1}{p} \mu$$

where

$$\mu = (1 - p^{p-1}) \widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m = 0 \\ v_1^{\omega-1} \widehat{\lambda}_1 & \text{for } m > 0. \end{cases}$$

Hence in  $p^{-1}v_1^{-1}BP_*$  we have

$$\begin{aligned} \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} &= \frac{p^{e_1}}{v_1^{1+e_1}} \left( \frac{\widehat{v}_2}{p} \right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}} \left( \widehat{\lambda}_2 + \frac{v_1}{p} \mu \right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}} \sum_{k \geq 0} \binom{1+e_0+e_1}{k} \widehat{\lambda}_2^{1+e_0+e_1-k} \frac{v_1^k}{p^k} \mu^k \\ &= \sum_{k \geq 0} \binom{1+e_0+e_1}{k} \frac{p^{e_1-k}}{v_1^{1+e_1-k}} \widehat{\lambda}_2^{1+e_0+e_1-k} \mu^k. \end{aligned}$$

The image of this element in  $p^{-1}BP_*/(v_1^\infty)$  is

$$\sum_{0 \leq k \leq e_1} \binom{1+e_0+e_1}{k} \frac{p^{e_1-k}}{v_1^{1+e_1-k}} \widehat{\lambda}_2^{1+e_0+e_1-k} \mu^k.$$

The coefficient of each term is an integer, so the expression lies in  $D_{m+1}^0/(v_1^\infty)$ , and its image in  $E_{m+1}^1/(v_1^\infty)$  is the desired element.  $\square$

We will now construct a comodule  $E_{m+1}^2 \subset E_{m+1}^1/(v_1^\infty)$  satisfying the conditions of Corollary 7.2.3 with  $\delta$  monomorphic below dimension  $p^2|\widehat{v}_1|$ .

7.2.6. THEOREM. *Let  $E_{m+1}^2 \subset E_{m+1}^1/(v_1^\infty)$  be the  $A(m+2)$ -module generated by the set*

$$\left\{ \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0 \right\}.$$

*Below dimension  $p^2|\widehat{v}_1|$  it has the Poincaré series specified in Lemma 7.2.2, it is a comodule, it is 1-free, and its Ext group is*

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{ \frac{\widehat{v}_2^{e_2}}{pv_1} : e_2 \geq 1 \right\}.$$

*In particular  $\text{Ext}^0$  maps monomorphically to  $\text{Ext}^2(BP_*)$  in that range.*

PROOF. Recall that the Poincaré series specified in Lemma 7.2.2 in this range is

$$g_{m+2}(t) \left( \frac{x^p(1-y)}{(1-x_2)(1-x^p)} \right) = g(BP_*/I_2) \frac{x^p}{(1-x_2)(1-x^p)}.$$

Each generator of  $E_{m+1}^2$  can be written as

$$x_{e_0, e_1} = \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} = \frac{\widehat{v}_2}{pv_1} \left( \frac{\widehat{v}_2}{p} \right)^{e_0} \left( \frac{\widehat{v}_2}{v_1} \right)^{e_1}$$

with  $e_0, e_1 \geq 0$ . Since  $|\frac{\widehat{v}_2}{pv_1}| = p|\widehat{v}_1|$ , the Poincaré series for this set of generators is

$$\frac{x^p}{(1-x_2)(1-x^p)}.$$

We can filter  $E_{m+1}^2$  by defining  $F_i$  to be the submodule generated by the  $x_{e_0, e_1}$  with  $e_0 + e_1 \leq i$ . Then each subquotient is a direct sum of suspensions of  $BP_*/I_2$ , so the Poincaré series is as claimed.

To see that  $E_{m+1}^2$  is a comodule, we will use the  $I$ -adic valuation as defined in the proof of Lemma 7.1.35. In our our range the set of elements with valuation at least  $-1$  is the  $A(m)$ -submodule  $M$  generated by

$$\left\{ \frac{\widehat{v}_1^i \widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0, i+j \geq 1+e_0+e_1 \right\},$$

while  $E_{m+1}^2$  is generated by a similar set with  $j \geq 1+e_0+e_1$ . Thus it suffices to show that the decreasing filtration on  $M$  defined by letting  $F^k M$  be the submodule generated by all such generators with  $j - e_0 - e_1 \geq k$  is a comodule filtration. For this observe that modulo  $\Gamma(m+1) \otimes F^{1+j-e_0-e_1} M$  we have

$$\frac{\eta_R(\widehat{v}_1^i \widehat{v}_2^j)}{p^{1+e_0}v_1^{1+e_1}} \equiv \frac{\widehat{v}_1^i (\widehat{v}_2 + v_1 \widehat{t}_1^p + p \widehat{t}_2)^j}{p^{1+e_0}v_1^{1+e_1}} \in \Gamma(m+1) \otimes F^{j-e_0-e_1} M,$$

so  $E_{m+1}^2 = F^1 M$  is a subcomodule.

We use the same filtration for the Ext computation. Assuming that  $j \geq 1 + e_0 + e_1 > 1$  we have

$$\begin{aligned} \frac{\eta_R(\widehat{v}_1^i \widehat{v}_2^j) - \widehat{v}_1^i \widehat{v}_2^j}{p^{1+e_0} v_1^{1+e_1}} &\equiv \frac{\widehat{v}_1^i (\widehat{v}_2 + v_1 \widehat{t}_1^p + p \widehat{t}_2)^j - \widehat{v}_1^i \widehat{v}_2^j}{p^{1+e_0} v_1^{1+e_1}} \\ &\equiv \binom{j}{e_0 + e_1} \frac{\widehat{v}_1^i \widehat{v}_2^{j-e_0-e_1} (v_1 \widehat{t}_1^p + p \widehat{t}_2)^{e_0+e_1}}{p^{1+e_0} v_1^{1+e_1}} + \dots \\ &\equiv (e_0, e_1, j - e_0 - e_1) \frac{\widehat{v}_1^i \widehat{v}_2^{j-e_0-e_1} \widehat{t}_1^{pe_1} \widehat{t}_2^{e_0}}{p v_1} + \dots \end{aligned}$$

where the missing terms involve higher powers of  $\widehat{v}_2$ . The multinomial coefficient  $(e_0, e_1, j - e_0 - e_1)$  is always nonzero since  $j < p$ . This means no linear combination of such elements is invariant, and the only invariant generators are the ones with  $e_0 = e_1 = 0$ , so  $\text{Ext}^0$  is as claimed.

We will use this to show that  $E_{m+1}^2$  is 1-free (as defined in 7.1.8), i.e., that  $T_m^{p-1} \otimes_{BP_*} E_{m+1}^2$  is weak injective in this range. For  $0 \leq k \leq p-1$  we have

$$\frac{\psi(\widehat{v}_1^i \widehat{v}_2^j \widehat{t}_1^k) - \widehat{v}_1^i \widehat{v}_2^j \widehat{t}_1^k}{p^{1+e_0} v_1^{1+e_1}} = (e_0, e_1, j - e_0 - e_1) \widehat{t}_1^{pe_1+k} \widehat{t}_2^{e_0} \otimes \frac{\widehat{v}_1^i \widehat{v}_2^{j-e_0-e_1}}{p v_1} + \dots$$

This means that

$$\text{Ext}^0(T_m^{p-1} \otimes_{BP_*} E_{m+1}^2) = \text{Ext}^0(E_{m+1}^2).$$

It follows that

$$\begin{aligned} g(\text{Ext}^0) &= g_{m+1}(t)(1-y) \frac{x^p}{1-x_2} \\ \text{so } g(E_{m+1}^2) &= g(\text{Ext}^0) \frac{1}{(1-x^p)(1-x_2)}, \\ \text{and } g(T_m^{p-1} \otimes_{BP_*} E_{m+1}^2) &= g(\text{Ext}^0) \frac{1}{(1-x)(1-x_2)} \\ &= g(\text{Ext}^0) G_m(t) \end{aligned}$$

This makes  $T_m^{p-1} \otimes_{BP_*} E_{m+1}^2$  weak injective in this range by Theorem 7.1.34.

We can use the small descent spectral sequence of Theorem 7.1.13 to pass from  $\text{Ext}(T_m^{p-1} \otimes_{BP_*} E_{m+1}^2)$  to  $\text{Ext}(E_{m+1}^2)$ . It collapses from  $E_1$  since the two comodules have the same  $\text{Ext}^0$ . This means that  $\text{Ext}(E_{m+1}^2)$  is as claimed.

To show that  $\text{Ext}^0(E_{m+1}^2)$  maps monomorphically to  $\text{Ext}^2(BP_*)$ , the chromatic method tells us that  $\text{Ext}^2(BP_*)$  is a certain subquotient of  $\text{Ext}^0(M^2)$ , namely the kernel of the map to  $\text{Ext}^0(M^3)$  modulo the image of the map from  $\text{Ext}^0(M^1)$ . We know that the latter is the  $A(m)$ -module generated by the elements  $\frac{\widehat{v}_1^i}{p^i}$ , and the elements in question, the  $A(m+1)$  multiples of  $\frac{\widehat{v}_2^i}{p v_1}$  are not in the image. The latter map trivially to  $\text{Ext}^0(M^3)$  because they involve no negative powers of  $v_2$ .  $\square$

7.2.7. COROLLARY. *Excluding the case  $(p, m) = (2, 0)$ , below dimension  $p^2|\widehat{v}_1|$ ,*

$$\text{Ext}_{\Gamma(m+1)}^s = \begin{cases} A(m) & \text{for } s = 0 \\ A(m) \left\{ \frac{\widehat{v}_1^j}{pj} : j > 0 \right\} & \text{for } s = 1 \\ \text{Ext}_{\Gamma(m+1)}^{s-2}(E_{m+1}^2) & \text{for } s \geq 2. \end{cases}$$

*The Adams–Novikov spectral sequence collapses with no nontrivial extensions in this range, so  $\pi_*(T(m))$  has a similar description below dimension  $p^2|\widehat{v}_1| - 3$ .*

The group  $\text{Ext}_{\Gamma(m+1)}(E_{m+1}^2)$  was described in Theorem 7.2.6.

We now specialize to the case  $m = 0$  and  $p$  odd. Using Lemma 7.2.1 we get the 4-term exact sequence

$$(7.2.8) \quad 0 \rightarrow BP_* \rightarrow D_1^0 \rightarrow D_1^1 \rightarrow E_1^2 \rightarrow 0,$$

for which the resolution spectral sequence (A1.3.2) collapses from  $E_1$ .

We could get at  $\text{Ext}_{\Gamma(1)}(E_1^2)$  via the Cartan–Eilenberg spectral sequence for the extension

$$(A(1), G(1)) \rightarrow (BP_*, \Gamma(1)) \rightarrow (BP_*, \Gamma(2))$$

if we knew the value of  $\text{Ext}_{\Gamma(2)}(E_1^2)$  as a  $G(1)$ -comodule. For this we need to consider (7.2.8) as an exact sequence of  $\Gamma(2)$ -comodules and study the resulting resolution spectral sequence. By Lemma 7.1.33 we know that  $D_1^0$  and  $D_1^1$  are weak injectives over  $\Gamma(2)$ . It follows that the resolution spectral sequence collapses from  $E_2$  and that the connecting homomorphism

$$\delta : \text{Ext}_{\Gamma(2)}^s(E_1^2) \rightarrow \text{Ext}_{\Gamma(2)}^{s+1}(E_1^1) = \text{Ext}_{\Gamma(2)}^{s+2}$$

is an isomorphism for  $s > 0$ . This implies that

$$\text{Ext}_{\Gamma(2)}^s(E_1^2) \cong \text{Ext}_{\Gamma(2)}^{s+2},$$

which is described in our range by Theorem 7.2.6.

For  $s = 0$ , the situation is only slightly more complicated. Recall that the 4-term exact sequence (7.2.8) is the splice of two short exact sequences,

$$0 \rightarrow BP_* \rightarrow D_1^0 \rightarrow E_1^1 \rightarrow 0$$

(where  $E_1^1 = D_1^0/BP_*$ ) and

$$0 \rightarrow E_1^1 \rightarrow D_1^1 \rightarrow E_1^2 \rightarrow 0,$$

Thus we have a short exact sequence

$$(7.2.9) \quad 0 \rightarrow \text{Ext}_{\Gamma(2)}^0(E_1^1) \rightarrow \text{Ext}_{\Gamma(2)}^0(D_1^1) \rightarrow L \rightarrow 0$$

and this  $L$  is the kernel of  $\delta$  for  $s = 0$ . Thus there is a short exact sequence

$$(7.2.10) \quad 0 \rightarrow L \rightarrow \text{Ext}_{\Gamma(2)}^0(E_1^2) \rightarrow U \rightarrow 0,$$

where  $U = \text{Ext}_{\Gamma(2)}^2$ , which is described in our range by Theorem 7.2.6.

7.2.11. THEOREM. *The comodule  $L$  of (7.2.9) is the  $A(1)$ -submodule  $B \subset N^2$  generated by the set*

$$\left\{ \frac{v_2^i}{ipv_1^i} : i > 0 \right\}.$$

We will denote the element  $\frac{v_2^i}{ipv_1^i}$  by  $\beta'_{i/i}$ . Theorem 7.2.11 implies



7.2.12. THEOREM. *In the resolution spectral sequence for (7.2.8) we have*

$$\begin{aligned} E_1^{0,s} = E_\infty^{0,s} &= \begin{cases} \mathbf{Z}_{(p)} & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases} \\ E_2^{1,s} = E_\infty^{1,s} &= \begin{cases} \text{Ext}_{\Gamma(1)}^1 & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases} \end{aligned}$$

and for

$$E_1^{2,*} = \text{Ext}_{\Gamma(1)}(E_1^2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14) for this group we have

$$\tilde{E}_2^{s,t} = \text{Ext}_{G(1)}^s(\text{Ext}_{\Gamma(2)}^t(E_2^1)).$$

For  $t > 0$ ,

$$\text{Ext}_{G(1)}^s(\text{Ext}_{\Gamma(2)}^t(E_2^1)) = \text{Ext}_{G(1)}^s(\text{Ext}_{\Gamma(2)}^{t+2})$$

and for  $t = 0$  there is long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}_{G(1)}^0(B) & \longrightarrow & \tilde{E}_2^{0,0} & \longrightarrow & \text{Ext}_{G(1)}^0(U) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{G(1)}^1(U) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{G(1)}^2(U) \\ & & & & & & \downarrow \\ & & & & & & \dots \end{array}$$

associated with the short exact sequence (7.2.10).

We will also need to consider the tensor product of (7.2.8) with  $T_0^{(j)}$ , and we will denote the resulting resolution spectral sequence by  $\{E_r^{s,t}(T_0^{(j)})\}$ . Let  $\{\tilde{E}_r^{s,t}(T_0^{(j)})\}$  denote the Cartan–Eilenberg spectral sequence for  $\text{Ext}_{\Gamma(1)}(T_0^{(j)} \otimes E_1^2)$ . For a  $\Gamma(1)$ -comodule  $M$ , we have

$$\text{Ext}_{\Gamma(2)}(T_0^{(j)} \otimes_{BP_*} M) \cong \bar{T}_0^{(j)} \otimes_{A(1)} \text{Ext}_{\Gamma(2)}(M),$$

where  $T_0^{(j)} \subset \Gamma(1)$  and  $\bar{T}_0^{(j)} \subset G(1)$ , since  $T_0^{(j)}$  is isomorphic over  $\Gamma(2)$  to a direct sum of  $p^j$  suspensions of  $BP_*$ . It follows that we have a short exact sequence

$$0 \rightarrow \text{Ext}_{\Gamma(2)}^0(T_0^{(j)} \otimes E_1^1) \rightarrow \text{Ext}_{\Gamma(2)}^0(T_0^{(j)} \otimes D_1^1) \rightarrow \bar{T}_0^{(j)} \otimes B \rightarrow 0$$

and the long exact sequence of Theorem 7.2.12 generalizes to

$$(7.2.13) \quad \begin{array}{c} 0 \longrightarrow \text{Ext}_{G(1)}^0(\overline{T}_0^{(j)} \otimes B) \longrightarrow \tilde{E}_2^{0,0}(T_0^{(j)}) \longrightarrow \text{Ext}_{G(1)}^0(\overline{T}_0^{(j)} \otimes U) \\ \longrightarrow \text{Ext}_{G(1)}^1(\overline{T}_0^{(j)} \otimes B) \longrightarrow \tilde{E}_2^{1,0}(T_0^{(j)}) \longrightarrow \text{Ext}_{G(1)}^1(\overline{T}_0^{(j)} \otimes U) \\ \longrightarrow \text{Ext}_{G(1)}^2(\overline{T}_0^{(j)} \otimes B) \longrightarrow \dots \end{array}$$

The following is helpful in proving Theorem 7.2.11.

7.2.14. LEMMA. *Let  $M \subset \text{Ext}_{\Gamma(2)}^0(E_1^1/(v_1^\infty))$  be a  $G(1)$ -subcomodule with trivial image (under the connecting homomorphism) in*

$$\text{Ext}_{\Gamma(2)}^1(E_1^1) = \text{Ext}_{\Gamma(2)}^2;$$

*equivalently let*

$$M \subset E/(v_1^\infty).$$

*where  $E = \text{Ext}_{\Gamma(2)}^0(E_1^1)$ . Then it is a subcomodule of  $\text{Ext}_{\Gamma(2)}^0(E_1^2)$  if it has a preimage*

$$\tilde{M} \subset \text{Ext}_{\Gamma(2)}^0(v_1^{-1}E_1^1) \subset v_1^{-1}E_1^1$$

*that is obtained from  $E$  by adjoining elements divisible by the ideal  $J = (\hat{\lambda}_2, \hat{\lambda}_3, \dots)$ .*

PROOF. We have a diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & \tilde{M} & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & v_1^{-1}E & \longrightarrow & E/v_1^\infty \longrightarrow 0 \end{array}$$

We need to verify that the monomorphism

$$\text{Ext}_{\Gamma(1)}^0(E_1^1) = \text{Ext}_{G(1)}^0(E) \rightarrow \text{Ext}_{G(1)}^0(\tilde{M})$$

is an isomorphism. If an element  $x \in \tilde{M}$  is invariant, then some  $v_1$ -multiple of it must lie in  $\text{Ext}_{\Gamma(1)}^0(E_1^1)$ , which has no elements divisible by  $J$ . Hence  $x$  has trivial image in  $M$  and therefore lies in  $E$ , and we have our isomorphism.

Now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & \tilde{M} & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_1^1 & \longrightarrow & D_1^1 & \longrightarrow & E_1^2 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_1^1 & \longrightarrow & v_1^{-1}E_1^1 & \longrightarrow & E_1^1/(v_1^\infty) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & v_1^{-1}E_1^1/D_1^1 & = & v_1^{-1}E_1^1/D_1^1 \end{array}$$

We have shown that the map  $\tilde{M} \rightarrow v_1^{-1}E_1^1/D_1^1$  is trivial in  $\text{Ext}^0$ , so it is trivial. It follows that  $\tilde{M}$  maps to  $D_1^1$ , so  $M$  maps to  $E_1^2$ .  $\square$

7.2.15. LEMMA. *Let  $L$  be as in (7.2.9). Then*

$$g(L) = \frac{1}{1-x} \sum_{i \geq 0} \frac{x^{p^{i+1}}(1-x^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})},$$

where  $x = t^{|v_1|}$  and  $x_2 = t^{|v_2|}$ .

PROOF. Since  $D_1^0$  is weak injective, applying the functor  $\text{Ext}_{\Gamma(2)}$  to the short exact sequence

$$0 \rightarrow BP_* \rightarrow D_1^0 \rightarrow E_1^1 \rightarrow 0$$

yields a 4-term exact sequence

$$0 \rightarrow A(1) \rightarrow \text{Ext}_{\Gamma(2)}^0(D_1^0) \rightarrow \text{Ext}_{\Gamma(2)}^0(E_1^1) \rightarrow \text{Ext}_{\Gamma(2)}^1 \rightarrow 0$$

and hence a short exact sequence

$$0 \rightarrow \text{Ext}_{\Gamma(2)}^0(D_1^0)/A(1) \rightarrow \text{Ext}_{\Gamma(2)}^0(E_1^1) \rightarrow \text{Ext}_{\Gamma(2)}^1 \rightarrow 0,$$

where

$$\text{Ext}_{\Gamma(2)}^0(D_1^0) = A(1)[p^{-1}v_1].$$

A calculation similar to that of Lemma 7.1.35 shows that

$$g(\text{Ext}_{\Gamma(2)}^0(D_1^0)/A(1)) = \frac{x}{(1-x)^2}$$

so

$$(7.2.16) \quad \text{Ext}_{\Gamma(2)}^0(E_1^1) = \frac{x}{1-x} \left( \frac{x}{1-x} + \sum_{i \geq 0} \frac{x_2^{p^i}}{1-x_2^{p^i}} \right).$$

Now consider the short exact sequence

$$(7.2.17) \quad 0 \rightarrow \text{Ext}_{\Gamma(2)}^0(E_1^1) \rightarrow \text{Ext}_{\Gamma(2)}^0(D_1^1) \rightarrow L \rightarrow 0.$$

Since  $D_1^1$  is weak injective over  $\Gamma(1)$ , Lemma 7.1.33 tells us that  $\text{Ext}_{\Gamma(2)}^0(D_1^1)$  is weak injective over  $G(1,0)$  with

$$\text{Ext}_{G(1,0)}^0(\text{Ext}_{\Gamma(2)}^0(D_1^1)) = \text{Ext}_{\Gamma(1)}^0(D_1^1) = \text{Ext}_{\Gamma(1)}^1$$

so

$$(7.2.18) \quad g(\text{Ext}_{\Gamma(2)}^0(D_1^1)) = \frac{x}{1-x} \sum_{i \geq 0} \frac{x^{p^i}}{1-x^{p^i}}.$$

Combining (7.2.16), (7.2.17), and (7.2.18) gives

$$\begin{aligned}
g(L) &= g(\mathrm{Ext}_{\Gamma(2)}^0(D_1^1)) - g(\mathrm{Ext}_{\Gamma(2)}^0(E_1^1)) \\
&= \frac{x}{1-x} \left( \sum_{i \geq 0} \frac{x^{p^i}}{1-x^{p^i}} \frac{x}{1-x} - \sum_{i \geq 0} \frac{x_2^{p^i}}{1-x_2^{p^i}} \right) \\
&= \frac{x}{1-x} \sum_{i \geq 0} \left( \frac{x^{p^{i+1}}}{1-x^{p^{i+1}}} - \frac{x_2^{p^i}}{1-x_2^{p^i}} \right) \\
&= \frac{x}{1-x} \sum_{i \geq 0} \frac{x^{p^{i+1}}(1-x_1^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})}.
\end{aligned}$$

□

7.2.19. LEMMA. *Let  $B$  be as in Theorem 7.2.11. Its Poincaré series is the same as the one for  $L$ , as given in Lemma 7.2.15.*

PROOF. Let  $F_k B \subset B$  denote the submodule of exponent  $p^k$ , with  $B_0 = \phi$ . Then we find that

$$F_k B = F_{k-1} B + A(1) \left\{ \beta_{ip^{k-1}/ip^{k-1}, k} : i > 0 \right\}$$

so

$$F_k B / F_{k-1} B = A(1) / I_1 \left\{ \beta_{ip^{k-1}/ip^{k-1}, k} : i > 0 \right\},$$

and

$$\begin{aligned}
F_k B &= F_{k-1} B + g(F_k B / F_{k-1} B) \\
&= g(A(1) / I_2) \sum_{i > 0} \frac{x^{ip^k} - x_2^{ip^{k-1}}}{1-x} \\
&= \frac{x}{1-x} \sum_{i > 0} (x^{ip^k} - x_2^{ip^{k-1}}) \\
&= \frac{x}{1-x} \left( \frac{x^{p^k}}{1-x^{p^k}} - \frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}} \right) \\
&= \frac{x}{1-x} \frac{x^{p^k}(1-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}.
\end{aligned}$$

Summing this for  $k \geq 1$  gives the desired Poincaré series of  $B$ . □

PROOF OF THEOREM 7.2.11. We first show that  $B$  is a  $G(1)$ -comodule by showing that it is invariant over  $\Gamma(2)$ . In  $\Gamma(2)$  we have

$$\eta_R(v_2) = v_2 + pt_2,$$

so for each  $i > 0$ , the elements

$$\frac{v_2^i}{ip} \in N^1 \quad \text{and hence} \quad \frac{v_2^i}{ipv_1^i} \in N^2$$

are invariant.

Next we show that  $B \subset E_1^1/(v_1^\infty)$ . Note that

$$\begin{aligned}
 v_1^{-1}v_2 &= pv_1^{-1}\widehat{\lambda}_2 + (1 - p^{p-1})\widehat{\lambda}_1^p \\
 &= pv_1^{-1}(\widehat{\lambda}_2 + \widehat{\lambda}_1 w) \\
 \text{where } w &= (1 - p^{p-1})\widehat{\lambda}_1^p \\
 \text{so } \beta'_{i/i} &= \frac{p^i(\widehat{\lambda}_2 + \widehat{\lambda}_1 w)^i}{ipv_1^i} = \frac{p^{i-1}(\widehat{\lambda}_2 + \widehat{\lambda}_1 w)^i}{iv_1^i}.
 \end{aligned}
 \tag{7.2.20}$$

The coefficient  $p^{i-1}/i$  in this expression is always a  $p$ -local integer, so

$$\beta'_{i/i} \in E_1^1/(v_1^\infty).$$

Let

$$\tilde{\beta}'_{i/i} = \frac{v_1^{-i}v_2^i - w^i}{pi}.$$

Then we have

$$\begin{aligned}
 \tilde{\beta}'_{i/i} &= \frac{v_1^{-i}(p\widehat{\lambda}_2 + v_1 w)^i - w^i}{ip} \\
 &= \sum_{j>0} \binom{i}{j} \frac{(pv_1^{-1}\widehat{\lambda}_2)^j w^{i-j}}{ip} \\
 &\in v_1^{-1}E_1^1,
 \end{aligned}$$

so  $\tilde{\beta}'_{i/i}$  is a lifting of  $\beta'_{i/i}$  to  $v_1^{-1}E_1^1$ . Let  $\tilde{B} \subset \text{Ext}_{\Gamma(2)}^0(v_1^{-1}E_1^1)$  be the  $A(1)$ -submodule obtained by adjoining the elements  $\tilde{\beta}'_{i/i}$  to  $\text{Ext}_{\Gamma(2)}^0(E_1^1)$ ; it projects to  $B \subset E_1^1/(v_1^\infty)$ . Since each  $\tilde{\beta}'_{i/i}$  is divisible by  $\widehat{\lambda}_2$ , it follows from Lemma 7.2.14 that  $B \subset E_1^2$ .

$B$  and  $L$  have the same Poincaré series by 7.2.15 and 7.2.19, so they are equal. □

### 3. The homotopy of $T(0)_{(2)}$ and $T(0)_{(1)}$

In this section we will determine the Adams–Novikov  $E_2$ -term

$$\text{Ext}_{\Gamma(1)}(BP_*(T(0)_{(2)}))$$

and  $\pi_*(T(0)_{(2)})$  in dimensions less than  $(p^3 + p)|v_1| - 3$ . This is lower than the range of the previous section for reasons that will be explained below in Lemma 7.3.5. All assertions about Ext groups and related objects will apply only in that range.

We will then state a theorem (7.3.15) about differentials in the spectral sequence of (7.2.13) for  $j = 1$ , which we will prove in the next section.

Our starting point is the determination in Corollary 7.2.7 of  $\pi_*(T(1))$  and its Adams–Novikov  $E_2$ -term through a larger range, roughly  $p^2|v_2|$ . There is an equivalence

$$T(1) \cong T(0)_{p^3+p^2-1},$$

so we could use the small descent spectral sequence of Theorem 7.1.13 and the topological small descent spectral sequence 7.1.16 (which turn out to be the same up to regrading) to get what we want. It turns out that we can finesse this by standard algebra.

Theorem 7.2.12 gives a Cartan–Eilenberg spectral sequence converging to  $\text{Ext}_{\Gamma(1)}$  whose  $E_2$ -term is expressed in terms of  $\text{Ext}_{G(1)}(B)$  and  $\text{Ext}_{G(1)}(\text{Ext}_{\Gamma(2)}^s)$  for  $s \geq 2$ .

First we have the following partial result about  $\text{Ext}_{G(1)}(B)$ .

7.3.1. LEMMA. *For each  $j > 0$ , the  $G(1)$ -comodule  $B$  of Theorem 7.2.11 is  $j$ -free below dimension  $p^j|v_2|$ , and  $\text{Ext}_{G(1)}^0(T_0^{(j)} \otimes B)$  is additively isomorphic in this range to the  $A(1)$ -submodule of  $E_1^1/(v_1^\infty)$  generated by the set*

$$\left\{ \beta'_{i/\min(i,p^{j-1})} : i > 0 \right\} \cup \left\{ \beta_{i/p^j} : p^j \leq i < p^j + p^{j-1} \right\}.$$

*In particular it is 2-free in our range of dimensions.*

PROOF. We will denote the indicated group by  $H^0(B)$ . Given a  $G(1)$ -comodule  $M$ , let  $M' = T_0^{(j)} \otimes_{A(1)} M$ . According to Theorem 7.1.34,  $M$  is  $j$ -free (i.e.  $M'$  is weak injective) if

$$g(M') = \frac{g(\text{Ext}^0(M'))}{1-x},$$

where as before  $x = t^{|v_1|}$ . We also know that

$$g(M') = g(M) \frac{1-x^{p^j}}{1-x},$$

so the condition for weak injectivity can be rewritten as

$$g(M) = \frac{g(\text{Ext}^0(M'))}{1-x^{p^j}}.$$

Now in  $B$  we have

$$(7.3.2) \quad \begin{cases} r_{kp^j}(\beta'_{i/i}) &= \binom{i}{kp^{j-1}} \frac{v_2^{i-kp^{j-1}}}{ipv_1^{i-kp^{j-1}}} = \binom{i-1}{kp^{j-1}} \beta'_{i-kp^{j-1}/i-kp^{j-1}} \\ r_{kp^j}(\beta_{i/i}) &= \binom{i}{kp^{j-1}} \beta_{i-kp^{j-1}/i-kp^{j-1}}. \end{cases}$$

For  $p^{j-1} < i < p^j + p^{j-1}$ , choose  $k$  so that  $0 < i - kp^{j-1} \leq p^{j-1}$ . Then the coefficients of  $\beta$  and  $\beta'$  above are units in every case except for  $r_{kp^j}(\beta_{p^j/p^j})$ . It follows that for each element in  $B$ , applying  $r_{kp^j}$  for some  $k$  will yield an element in  $H^0(B)$ . This means that in our range we have

$$g(B) = \frac{g(H^0(B))}{1-x^{p^j}},$$

so  $B$  is  $j$ -free if  $H^0(B)$  is additively isomorphic to  $\text{Ext}^0(B')$ .

Each element in  $H^0(B)$  is killed by  $r_i$  for  $i \geq p^j$ , so there is a corresponding invariant element in  $T_0^{(j)} \otimes B'$  by Lemma 7.1.9. On the other hand, (7.3.2) implies that no element in  $B'$  outside of  $T_0^{(j)} \otimes H^0(B)$  is invariant, so  $\text{Ext}^0(B')$  is as desired.  $\square$

The groups  $\text{Ext}_{\Gamma(2)}^s$  for  $s \geq 2$  in our range were determined in Theorem 7.2.6. Translated to the present context, it reads as follows.

7.3.3. THEOREM. Below dimension  $p^2|v_2|$ , the group  $\text{Ext}_{\Gamma(2)}^{2+*}$  is

$$E(h_{2,0}) \otimes P(b_{2,0}) \otimes U$$

(where  $U = \text{Ext}_{\Gamma(2)}^2$ ), or more explicitly

$$A(1)/I_2 \otimes E(h_{2,0}) \otimes P(b_{2,0}) \otimes \left\{ \delta_0 \delta_1 \left( \frac{v_3^i v_2^j}{pv_1} \right) : i > 0, j \geq 0 \right\},$$

where  $\delta_0$  and  $\delta_1$  are the connecting homomorphisms for the short exact sequences

$$0 \rightarrow BP_* \rightarrow M^0 \rightarrow N^1 \rightarrow 0$$

and

$$0 \rightarrow N^1 \rightarrow M^1 \rightarrow N^2 \rightarrow 0$$

respectively.

7.3.4. THEOREM. For  $i, j \geq 0$ , let

$$u_{i,j} = v_2^j \left( \frac{v_3^i}{i!pv_1} - \frac{v_2^{i+ip}}{c_{i,j}pv_1^{1+ip}} \right) \in N^2$$

where

$$c_{i,j} = \prod_{1 \leq k \leq i} \binom{i+j+kp}{p}.$$

Then  $u_{i,j}$  has the following properties.

(i)  $u_{i,j}$  lies in  $E_1^1/(v_1^\infty)$  and is invariant over  $\Gamma(2)$ , i.e., it lies in

$$\text{Ext}_{\Gamma(2)}^0(E_1^1/(v_1^\infty)).$$

(ii) Its image in  $U$  is that of

$$\frac{v_3^i v_2^j}{i!pv_1}.$$

(iii) For  $i > 0$

$$r_{p^2}(u_{i,j}) = u_{i-1,j+1},$$

where  $u_{0,j} = 0$ .

(iv) For  $j \geq 0$ ,

$$r_p(u_{1,j}) = -\frac{j+1}{(p,j)} \beta_{j+p/p}.$$

We will denote  $u_{1,j}$  by  $u_j$ . The coefficients  $i!$ ,  $c_{i,j}$  and  $(p,j)$  are always nonzero modulo  $p$  in our range except in the case

$$u_{p^2-p-1} = \frac{v_2^{p^2-p-1} v_3}{pv_1} - \frac{v_2^{p^2}}{p^2 v_1^{p+1}}.$$

*Proof of Theorem 7.3.4.* (i) Recall (7.2.5) that

$$\frac{v_2}{p} = \lambda_2 + (1 - p^{p-1}) \lambda_1^{p+1},$$

while the definition of  $\lambda_3$  implies that

$$\frac{v_3}{p} \equiv \lambda_3 \pmod{(v_1)}.$$

Hence

$$\begin{aligned} \frac{v_2^j v_3^i}{pv_1} &= \frac{p^{i+j-1} \lambda_2^j \lambda_3^i}{v_1} \\ &\in E_1^1 / (v_1^\infty), \\ \text{and } \frac{v_2^{j+i+ip}}{pv_1^{1+ip}} &= p^{j+i+ip-1} \frac{(\lambda_2 + (1-p^{p-1})\lambda_1^{p+1})^{j+i+ip}}{v_1^{1+ip}} \\ &\in E_1^1 / (v_1^\infty), \end{aligned}$$

so  $u_{i,j} \in E_1^1 / (v_1^\infty)$ .

The invariance of  $u_{i,j}$  over  $\Gamma(2)$  follows from the fact (Proposition 7.1.24) that  $v_2$  is invariant modulo  $(p)$  and  $v_3$  is invariant modulo  $I_2$ .

(ii) We will show that the difference between the two elements maps trivially to  $U$ . It is a scalar multiple of

$$e = \frac{v_2^{i+ip}}{pv_1^{1+ip}},$$

which is the image of

$$\frac{v_1^{-1-ip} v_2^{i+ip}}{p} \in M^1.$$

This is invariant over  $\Gamma(2)$ , so our element  $e \in \text{Ext}_{\Gamma(2)}^0(N^2)$  is in the image of  $\text{Ext}_{\Gamma(2)}^0(M^1)$ , so it maps trivially to  $\text{Ext}_{\Gamma(2)}^2 = U$ .

(iii) Since

$$\begin{aligned} \eta_R(v_3) &\equiv v_3 + v_2 t_1^{p^2} - v_2^p t_1 \pmod{I_2} \\ \text{and } \eta_R(v_2) &\equiv v_2 + v_1 t_1^p - v_1^p t_1 \pmod{I_1}, \end{aligned}$$

we have

$$\begin{aligned} \eta_R \left( \frac{v_2^j v_3^i}{i! p v_1} \right) &= \frac{v_2^j (v_3^i + i v_2 v_3^{i-1} t_1^{p^2} + \dots)}{i! p v_1}, \\ \text{so } r_{p^2} \left( \frac{v_2^j v_3^i}{i! p v_1} \right) &= \frac{v_2^{j+1} v_3^{i-1}}{(i-1)! p v_1}. \end{aligned}$$

For the second term we have

$$\begin{aligned} \eta_R \left( \frac{v_2^{j+i+ip}}{c_{i,j} p v_1^{1+ip}} \right) &= \frac{(v_2 + v_1 t_1^p - v_1^p t_1)^{j+i+ip}}{c_{i,j} p v_1^{1+ip}} \\ &= \sum_{0 \leq k \leq ip} \binom{j+i+ip}{k} \frac{v_2^{j+i+ip-k} (t_1^p - v_1^{p-1} t_1)^k}{c_{i,j} p v_1^{1+ip-k}} \\ &= \sum_{0 \leq k \leq ip} \binom{j+i+ip}{k} \sum_{0 \leq \ell \leq k} (-1)^\ell \binom{k}{\ell} \frac{v_2^{j+i+ip-k} t_1^{p(k-\ell)+\ell}}{c_{i,j} p v_1^{1+ip-k-(p-1)\ell}}. \end{aligned}$$

We need to collect the terms in which the exponent of  $t_1$  is  $p^2$ , i.e. for which  $(p-1)\ell = p(k-p)$ . Hence  $k-p$  must be divisible by  $p-1$ , so we can write



$k = p + (p - 1)k'$  and  $\ell = pk'$ . This gives

$$\begin{aligned} r_{p^2} \left( \frac{v_2^{j+i+ip}}{c_{i,j}pv_1^{1+ip}} \right) &= \sum_{0 \leq k' \leq p} (-1)^{pk'} \binom{j+i+ip}{p+(p-1)k'} \binom{p+(p-1)k'}{pk'} \frac{v_2^{j+i+ip-p-(p-1)k'}}{c_{i,j}pv_1^{1+ip-p-(p^2-1)k'}} \\ &= \binom{j+i+ip}{p} \frac{v_2^{j+i+ip-p}}{c_{i,j}pv_1^{1+ip-p}} \\ &= \frac{v_2^{j+i+(i-1)p}}{c_{i-1,j+1}pv_1^{1+(i-1)p}} \end{aligned}$$

and the result follows.

(iv) Using the methods of (iii), we find that

$$\begin{aligned} r_p(u_{1,j}) &= r_p \left( \frac{v_2^j v_3}{pv_1} \right) - r_p \left( \frac{v_2^{j+p+1}}{c_{1,j}pv_1^{p+1}} \right) \\ &= - \binom{j+p+1}{1} \frac{v_2^{j+p}}{c_{1,j}pv_1^p} \\ &= - \frac{j+1}{(p,j)} \frac{v_2^{j+p}}{c_{1,j}pv_1^p}. \end{aligned}$$

□

In order to use the Cartan–Eilenberg spectral sequence of (7.2.13) we need to know  $\text{Ext}_{G(1)}(T_0^{(j)} \otimes U)$ . We will compute it by downward induction on  $j$  using the small descent spectral sequence of Theorem 7.1.13. Recall (Theorem 7.3.3) that  $U$  in our range is generated as an  $A(1)$ -module by the elements

$$\delta_0 \delta_1(u_{i,j}) = \delta_0 \delta_1 \left( \frac{v_3^i v_2^j}{pv_1} \right).$$

We start with the following.

7.3.5. LEMMA. *Let  $U = \text{Ext}_{\Gamma(2)}^2$  as before. In dimensions less than  $(p^3 + p)|v_1|$ , there is a short exact sequence of  $G(1)$ -comodules*

$$(7.3.6) \quad 0 \rightarrow U \rightarrow U_0 \rightarrow U_1 \rightarrow 0$$

where  $U_0 \subset v_2^{-1}U$  is the  $A(2)$ -submodule generated by

$$\left\{ \delta_0 \delta_1 \left( \frac{v_2^{1-i} v_3^i}{pv_1} \right) : i > 0 \right\}.$$

$U_0$  and  $U_1$  are each 2-free (7.1.8) as  $G(1)$ -comodules, and we have

$$\begin{aligned} \text{Ext}_{G(1)}^0(T_0^{(2)} \otimes U_0) &= A(1) \left\{ \delta_0 \delta_1(u_{1,j}) : j \geq 0 \right\} \\ \text{and } \text{Ext}_{G(1)}^0(T_0^{(2)} \otimes U_1) &= A(1) \left\{ \delta_0 \delta_1 \delta_2 \left( \frac{u_{i,j}}{v_2} \right) : i \geq 2, j \geq 0 \right\} \end{aligned}$$

(where  $\delta_2$  is the connecting homomorphism for 7.3.6) so

$$\mathrm{Ext}_{G(1)}^s(T_0^{(2)} \otimes U) = \begin{cases} A(1) \left\{ \delta_0 \delta_1(u_{1,j}) : j \geq 0 \right\} & \text{for } s = 0 \\ A(1) \left\{ \gamma_i : i \geq 2 \right\} & \text{for } s = 1 \\ 0 & \text{for } s > 1, \end{cases}$$

where

$$\gamma_i = \delta_0 \delta_1 \delta_2 \left( \frac{u_{i,0}}{v_2} \right).$$

Note that we have reduced our range of dimensions from  $(p^3 + p^2)|v_1|$  to  $(p^3 + p)|v_1|$ . A 2-free submodule of  $M^2$  containing  $U$  must contain the element

$$x = \frac{v_2^{1-p} v_3^p}{p^2 v_1},$$

and  $|x| = (p^3 + p)|v_1|$ .  $v_2^{p-1}x$  is in  $\mathrm{Ext}_{\Gamma(2)}^2$ , but is out of the range of Theorems 7.2.6 and 7.3.3.

PROOF. We will construct the desired extension of  $\mathrm{Ext}_{\Gamma(2)}^2$  by inducing from one of  $\mathrm{Ext}_{\Gamma(2)}^0(E_1^2)$  as in the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_{\Gamma(2)}^0(E_1^2) & \longrightarrow & U'_0 & \longrightarrow & U_1 \longrightarrow 0 \\ & & \downarrow \delta_0 \delta_1 & & \downarrow & & \parallel \\ 0 & \longrightarrow & U & \longrightarrow & U_0 & \longrightarrow & U_1 \longrightarrow 0 \end{array}$$

We can extend the definition of  $u_{i,j}$  to negative  $j$  and we have  $u_{i,j-i} = v_2^{j-1}u_{i,0}$  for  $1 \leq j \leq i$ .  $U'_0$  is the  $A(1)$ -submodule of  $v_2^{-1}\mathrm{Ext}_{\Gamma(2)}^0(E_1^2)$  obtained by adjoining the elements

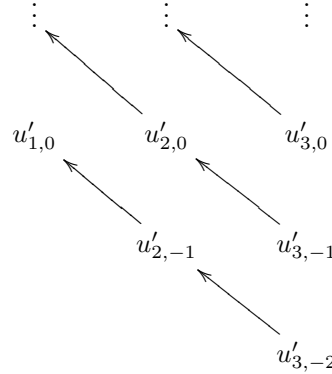
$$\left\{ u_{i,j-i} : i > 0, 1 \leq j \leq i \right\}$$

Theorem 7.3.4(v) implies that  $U'_0$  and hence  $U_1$  and  $U_0$  are comodules.

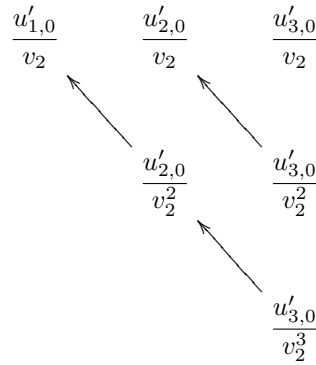
It follows that  $U_0 \subset v_2^{-1}\mathrm{Ext}_{\Gamma(2)}^2$  is as claimed. The 2-freeness of  $U_0$  and  $U_1$  follows from Theorem 7.3.4(iii).

For the computation of  $\mathrm{Ext}_{G(1)}^0(T_0^{(2)} \otimes U_k)$  for  $k = 0$  and 1, the following pictures for  $p = 3$  may be helpful. We denote  $\delta_0 \delta_1(u_{i,j})$  by  $u'_{i,j}$  and each diagonal

arrow represents the action of  $r_{p^2}$ . For  $U_0$  (which is  $v_2$ -torsion free) we have



where the missing elements have higher second subscripts. For  $U_1$  (which is all  $v_2$ -torsion) we denote  $\delta_0\delta_1\delta_2(v_2^{-k}u_{i,j})$  by  $\frac{u'_{i,j}}{v_2^k}$ , and the picture is



In each case  $\text{Ext}^0$  is generated by the elements not supporting an arrow, i.e., the ones in the left column of the first picture and the top row of the second.  $\square$

Now consider the Cartan–Eilenberg spectral sequence of (7.2.13) for  $j = 2$ . For  $t > 2$ ,  $\text{Ext}^t_{\Gamma(2)}$  is a suspension of  $U = \text{Ext}^2_{\Gamma(2)}$ , so  $\tilde{E}_r^{s,t}(T_0^{p^{2-1}}) = 0$  for  $s > 1$ . More precisely for  $t = \varepsilon + 2t'$  with  $\varepsilon = 0$  or  $1$ ,

$$\text{Ext}^{t+2}_{\Gamma(2)} = h_{2,0}^\varepsilon b_{2,0}^{t'} U,$$

which we abbreviate by  $\Sigma^{(t)}U$ . Then we have

7.3.7. COROLLARY. *In the resolution spectral sequence we have the following short exact sequences for the groups  $E_2^{2,t}(T_0^{(2)})$ : for  $t = 0$*

$$0 \longrightarrow \text{Ext}^0(T_0^{(2)} \otimes B) \longrightarrow E_2^{2,0}(T_0^{(2)}) \longrightarrow \text{Ext}^0(T_0^{(2)} \otimes U) \longrightarrow 0,$$

and for  $t > 0$

$$0 \longrightarrow \text{Ext}^1(T_0^{(2)} \otimes \Sigma^{(t-1)}U) \longrightarrow E_2^{2,t}(T_0^{(2)}) \longrightarrow \text{Ext}^0(T_0^{(2)} \otimes \Sigma^{(t)}U) \longrightarrow 0,$$

where  $\Sigma^{(t)}U$  is as above and the *Ext* groups are over  $G(1)$ .

The groups  $\text{Ext}_{G(1)}^0(T_0^{(2)} \otimes B)$  and  $\text{Ext}_{G(1)}(T_0^{(2)} \otimes U)$  are described in Lemmas 7.3.1 and 7.3.5 respectively.

PROOF. The long exact sequence of (7.2.13) and Lemma 7.3.1 imply that

$$\tilde{E}_2^{s,t}(T_0^{(2)}) = \text{Ext}_{G(1)}^s(T_0^{(2)} \otimes \Sigma^{(t)}U) \quad \text{for } t > 0.$$

For  $t = 0$  there is a short exact sequence

$$0 \longrightarrow \text{Ext}_{G(1)}^0(T_0^{(2)} \otimes B) \longrightarrow \tilde{E}_2^{0,0}(T_0^{p^{2-1}}) \longrightarrow \text{Ext}_{G(1)}^0(T_0^{(2)} \otimes U) \longrightarrow 0$$

and

$$\tilde{E}_2^{s,0}(T_0^{p^{2-1}}) = \begin{cases} \text{Ext}_{G(1)}^1(T_0^{(2)} \otimes \Sigma^{(t)}U) & \text{for } s = 1 \\ 0 & \text{for } s > 1. \end{cases}$$

Since  $\tilde{E}_2^{s,t}$  vanishes for  $s > 1$ , this spectral sequence collapses from  $E_2$  and reduces to the indicated collection of short exact sequences for the groups  $E_2^{2,*}(T_0^{p^{2-1}})$  in the resolution spectral sequence.  $\square$

7.3.8. COROLLARY. *The Adams–Novikov spectral sequence for  $\pi_*(T(0)_{(2)})$  collapses in our range of dimensions, i.e., below dimension  $(p^2)|v_2| - 3$ .*

PROOF. This will follow by a sparseness argument if we can show that in this range  $E_2^{s,*}$  (for the Adams–Novikov spectral sequence) vanishes for  $s < 2p + 1$ . We can rull out differentials originating in filtrations 0 or 1 by the usual arguments, and by sparseness the each nontrivial differential  $d_r$  has  $r \equiv 1 \pmod{2p - 2}$ . Thus the shortest possible one is  $d_{2p-1}$ , for which the filtration of the target would be too high.

For the vanishing statement the first element in filtration  $2p + 1$  is  $u_1 b_{2,0}^{p-1} h_{2,0}$ , and we have

$$\begin{aligned} |u_1| = |b_{2,0}| &= p|v_2| - 2 \\ &= p(2p^2 - 2) - 2 = 2p^3 - 2p - 2 \\ \text{and } |h_{2,0}| &= |v_2| - 1 = 2p^2 - 3 \\ \text{so } |u_1 b_{2,0}^{p-1} h_{2,0}| &= p(2p^3 - 2p - 2) + 2p^2 - 3 \\ &= 2p^4 - 2p - 3 \\ &> p^2|v_2| - 3. \end{aligned}$$

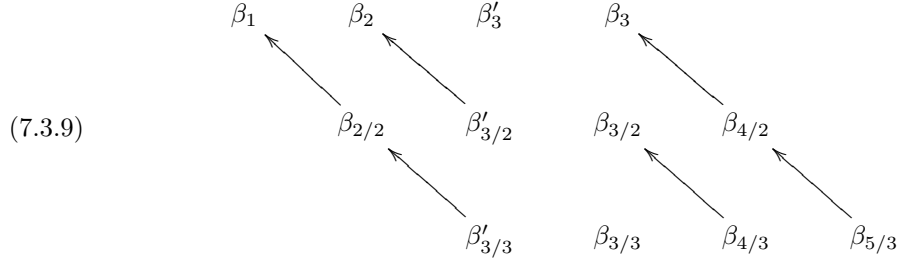
$\square$

Now we will analyze the Cartan–Eilenberg spectral sequence of (7.2.13) for  $j = 1$ . It has a rich pattern of differentials. This (in slightly different language) was the subject of Ravenel [11]. In order to use this spectral sequence we need to know  $\text{Ext}_{G(1)}(T_0^{(1)} \otimes B)$  and  $\text{Ext}_{G(1)}(T_0^{(1)} \otimes U)$ . We will derive these from the corresponding Ext groups for  $T_0^{(2)}$  given in Lemmas 7.3.1 and 7.3.5 using the the small descent spectral sequence of Theorem 7.1.13.

The former collapses from  $E_2$  since  $\text{Ext}_{G(1)}(T_0^{(2)} \otimes B)$  is concentrated in degree 0. The action of  $r_p$  on  $\text{Ext}_{G(1)}^0(T_0^{(2)} \otimes B)$  is given by

$$\begin{aligned} r_p(\beta'_{i/e_1}) &= \beta_{i-1/e_1-1} \\ \text{and } r_p(\beta_{pi/e_1}) &= 0. \end{aligned}$$

In order to understand this, the following picture for  $p = 3$  may be helpful.



Each arrow represents the action of  $r_p$  up to unit scalar. Thought of as a graph, this picture has  $2p$  components, two of which have maximal size. Each component corresponds to an  $A(m)$ -summand of our  $E_2$ -term, with the caveat that  $p\beta'_{p/e_1} = \beta_{p/e_1}$  and  $v_1\beta'_{i/e} = \beta'_{i/e-1}$ .

In the summand containing  $\beta_1$ , the subset of  $E_1$

$$\left\{ \beta_1, \beta_{2/2}, \beta'_{3/3} \right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces on passage to  $E_2$  to simply  $\{\beta_1\}$ . Similarly

$$\left\{ \beta_2, \beta'_{3/2} \right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces to

$$\left\{ \beta_2, \beta'_{3/2}h_{1,1} \right\} \otimes P(b_{1,1}),$$

where

$$\begin{aligned} \beta'_{3/2}h_{1,1} &= \langle h_{1,1}, h_{1,1}, \beta_2 \rangle \\ \text{and } h_{1,1}(\beta'_{3/2}h_{1,1}) &= h_{1,1}\langle h_{1,1}, h_{1,1}, \beta_2 \rangle \\ &= \langle h_{1,1}, h_{1,1}, h_{1,1} \rangle \beta_2 \\ &= b_{1,1}\beta_2. \end{aligned}$$

The entire configuration is  $v_2^p$ -periodic. This leads to the following.

7.3.10. PROPOSITION. *In dimensions less than  $p^2|v_2|$ ,  $\text{Ext}_{G(1)}(T_0^{(1)} \otimes B)$  has  $\mathbf{Z}/(p)$  basis*

$$P(b_{1,1}) \otimes \left\{ \begin{aligned} &\left\{ \beta_{1+pi}, \beta_{p+pi}; \beta_{p^2/p^2-p+1}, \dots, \beta_{p^2/p+1} \right\} \\ &\oplus \\ &\left\{ \beta'_{2+pi}, \dots, \beta'_{p+pi}; \beta_{p+pi/p}, \dots, \beta_{p+pi/2}; \beta_{p^2/p^2}, \dots, \beta_{p^2/p^2-p+2} \right\} \\ &\oplus \\ &h_{1,1} \left\{ \beta'_{p+pi/p-1}, \dots, \beta'_{p+pi}; \beta_{p+pi/p}, \dots, \beta_{2p-1+pi/p}; \right. \\ &\quad \left. \beta_{p^2/p^2}, \dots, \beta_{p^2+p-2/p^2} \right\}, \end{aligned} \right.$$

where  $0 \leq i < p$ , subject to the caveat that  $v_1\beta_{p/e} = \beta_{p/e-1}$  and  $p\beta'_{p/e} = \beta_{p/e}$ . In particular  $\text{Ext}_{G(1)}^0(T_0^{(1)} \otimes B)$  has basis

$$\left\{ \beta'_{1+pi}, \dots, \beta'_{p+pi}; \beta_{p+pi/p}, \dots, \beta_{p+pi/1}; \beta_{p^2/p^2}, \dots, \beta_{p^2/p+1} \right\}.$$

The action of  $r_p$  on  $U$  is trivial, so  $E_1 = E_2$  in the small descent spectral sequence for  $\text{Ext}_{G(1)}(T_0^{(1)} \otimes U)$ . In theory there could be a nontrivial differential

$$d_2 : E_2^{s,1} \rightarrow E_2^{s+2,0},$$

but this cannot happen since  $E_2^{s,1}$  is  $v_2$ -torsion while  $E_2^{s,0}$  is  $v_2$ -torsion free. Hence the spectral sequence collapses and we have

$$(7.3.11) \quad \text{Ext}_{G(1)}(T_0^{(1)} \otimes U) = E(h_{1,1}) \otimes P(b_{1,1}) \otimes \text{Ext}_{G(1)}(T_0^{(2)} \otimes U),$$

where  $\text{Ext}_{G(1)}(T_0^{(2)} \otimes U)$  is as in Lemma 7.3.5.

We now have the ingredients needed to study the Cartan–Eilenberg spectral sequence  $\left\{ \tilde{E}_r^{s,t}(T_0^{(1)}) \right\}$  of (7.2.13). We first need to analyze the connecting homomorphism  $\delta$  in the long exact sequence for  $t = 0$ . Since the target groups of it are  $v_2$ -torsion free,  $\delta$  is trivial on the  $v_2$ -torsion module

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \text{Ext}_{G(1)}^1(T_0^{(2)} \otimes U).$$

For its behavior on

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \text{Ext}_{G(1)}^0(T_0^{(2)} \otimes U)$$

we have

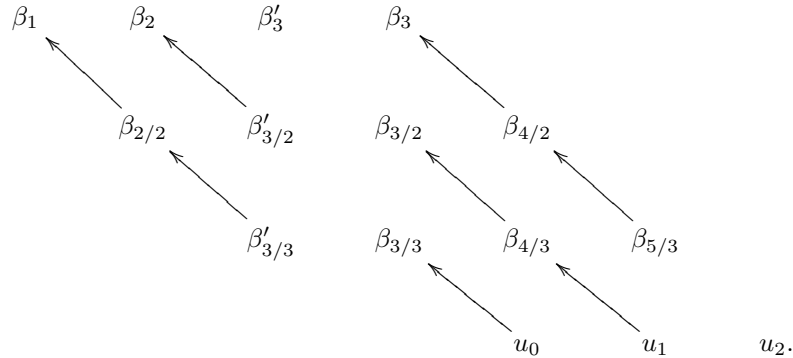
7.3.12. LEMMA. *In the long exact sequence of (7.2.13) for  $j = 1$  we have (up to unit scalar)*

$$\begin{aligned} \delta^{2k}(b_{1,1}^k u_i) &= (i+1)h_{1,1}b_{1,1}^k \beta_{i+p/p}, \\ \text{and } \delta^{2k+1}(h_{1,1}b_{1,1}^k u_i) &= \binom{i+1}{p-1} b_{1,1}^{k+1} \beta_{i+2/2} \end{aligned}$$

for all  $i, k \geq 0$ .

This means that  $\tilde{E}_2^{*,0}(T_0^{(1)})$  looks like the Ext group one would have if the picture of (7.3.9) were replaced by

(7.3.13)



The graph now has  $2p + 1$  instead of  $2p$  components, three of which are maximal.

*Proof of Lemma 7.3.12.* It suffices to show that  $r_p(u_i)$  is as indicated in the picture above. We have (using Theorem 7.3.4)

$$u_i = u_{1,i} = v_2^i \left( \frac{v_3}{pv_1} - \frac{v_2^{p+1}}{pc_{1,i}v_1^{p+1}} \right)$$

so

$$r_p(u_i) = -(i+p+1) \frac{v_2^{i+p}}{pc_{1,i}v_1^p} = -\frac{i+1}{(p,i)} \beta_{i+p/p}.$$

□

7.3.14. COROLLARY. *In the Cartan–Eilenberg spectral sequence of (7.2.13),  $\tilde{E}_2(T_0^{(1)})$  has  $\mathbf{Z}/(p)$ -basis*

$$\begin{aligned} & \left\{ \beta_{1+pi}, \beta_{p+pi}, \beta_{p+pi/2}; \beta_{p^2/p^2-p+1}, \dots, \beta_{p^2/p+1} \right\} \\ & \oplus \\ & P(b_{1,1}) \otimes \left\{ \begin{array}{l} \{ \beta'_{2+pi}, \dots, \beta'_{p+pi}; \beta_{p+pi/p}, \dots, \beta_{p+pi/2}; \\ u_{pi+p-1}; \beta_{p^2/p^2}, \dots, \beta_{p^2/p^2-p+2} \} \\ \oplus \\ h_{1,1} \{ \beta'_{p+pi/p-1}, \dots, \beta'_{p+pi}; u_{pi}, \dots, u_{pi+p-2}; \\ \beta_{p^2/p^2}, \dots, \beta_{p^2+p-2/p^2} \} \end{array} \right\} \\ & \oplus \\ & E(h_{1,1}) \otimes P(b_{1,1}, b_{2,0}) \otimes \{ h_{2,0}u_j, b_{2,0}u_j : j \geq 0 \} \\ & \oplus \\ & E(h_{1,1}, h_{2,0}) \otimes P(b_{1,1}, b_{2,0}) \otimes \{ \gamma_2, \gamma_3, \dots \}, \end{aligned}$$

where  $0 \leq i < p$ , (omitting unnecessary subscripts)

$$u, v, \beta, \in \tilde{E}_2^{0,0} \quad \text{and} \quad \gamma \in \tilde{E}_2^{0,1},$$

and the operators  $h_{i,j}$ ,  $b_{i,j}$ , etc. behave as if they had the following bidegrees.

$$\begin{aligned} h_{2,0} & \in \tilde{E}_2^{0,1}, & h_{1,1} & \in \tilde{E}_2^{1,0}, \\ b_{2,0} & \in \tilde{E}_2^{0,2}, & \text{and } b_{1,1} & \in \tilde{E}_2^{2,0}. \end{aligned}$$

Now we need to study higher differentials.

7.3.15. THEOREM. *The Cartan–Eilenberg spectral sequence of (7.2.13) for  $j = 1$  has the following differentials and no others in dimensions less than  $(p^3 + p)|v_1|$ .*

(i)

$$d_2(h_{2,0}u_i) = b_{1,1}\beta'_{i+2}.$$

(ii)

$$d_3(h_{2,0}^\varepsilon b_{2,0}^k u_i) = (k+i+1)h_{1,1}h_{2,0}^\varepsilon b_{1,1}b_{2,0}^{k-1}u_i \quad \text{for } k > 0 \text{ and } \varepsilon = 0 \text{ or } 1.$$

(iii)

$$d_{2k+2}(h_{1,1}h_{2,0}b_{2,0}^k u_{pi'-2-k}) = h_{1,1}b_{1,1}^{k+1}\beta'_{pi'/k+1} \quad \text{for } k < p-1.$$

(iv)

$$d_{2k+1}(h_{1,1}b_{2,0}^k u_{pi'-2-k}) = b_{1,1}^{k+1}\beta_{pi'/k+2} \quad \text{for } k > 0.$$

(v)

$$d_3(h_{2,0}^\varepsilon b_{2,0}^k \gamma_j) = kh_{1,1} h_{2,0}^\varepsilon b_{1,1} b_{2,0}^{k-1} \gamma_j.$$

We will prove Theorem 7.3.15 in the next section. For a more explicit description of the resulting Ext group, see Theorem 7.5.1. An illustration of it for  $p = 5$  can be found in Figure 7.3.17. There are no Adams–Novikov differentials in this range. In the figure

- $\text{Ext}^0$  and  $\text{Ext}^1$  are not shown.
- Short vertical and horizontal lines indicate multiplication by  $p$  and  $v_1$ .
- Diagonal lines indicate multiplication by  $h_{1,1}$ ,  $h_{2,0}$  and the Massey product operations  $\bar{5}i$  of 7.4.12.

Now that we have computed  $\text{Ext}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2)$ , it is a simple matter to get to  $\text{Ext}_{\Gamma(1)}(T_0^{(1)})$  itself. We have the 4-term exact sequence

$$(7.3.16) \quad 0 \rightarrow T_0^{(1)} \rightarrow T_0^{(1)} \otimes D_1^0 \rightarrow T_0^{(1)} \otimes D_1^1 \rightarrow T_0^{(1)} \otimes E_1^2 \rightarrow 0$$

in which the two middle terms are weak injectives by Lemma 7.1.10 with

$$\begin{aligned} \text{Ext}_{\Gamma(1)}^0(T_0^{(1)} \otimes D_1^i) &\cong \mathbf{Z}_{(p)} \left\{ t_1^j : 0 \leq j < p \right\} \otimes \text{Ext}_{\Gamma(1)}^0(D_1^i) \\ &\cong \mathbf{Z}_{(p)} \left\{ t_1^j : 0 \leq j < p \right\} \otimes \text{Ext}_{\Gamma(1)}^i(BP_*). \end{aligned}$$

We will compute  $\text{Ext}^0$  of the middle map of (7.3.16) using the description of the groups given in 7.1.10. Recall that  $D_1^0$  contains all powers of  $\lambda_1 = p^{-1}v_1$ . Then  $\text{Ext}_{\Gamma(1)}^0(T_0^{(1)} \otimes D_1^0)$  is the free  $\mathbf{Z}_{(p)}$ -module on the set  $\{z_j : 0 \leq j < p\}$  where

$$z_j = \sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_1^k \otimes \lambda_1^{j-k} = t_1^j \otimes 1 + \dots$$

The image of

$$\begin{aligned} p^t z_j &= \sum_{0 \leq k \leq j} (-1)^k p^t \binom{j}{k} t_1^k \otimes \lambda_1^{j-k} \\ &= \sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_1^k \otimes p^{t+k-j} v_1^{j-k} \end{aligned}$$

in  $\text{Ext}_{\Gamma(1)}^0(T_0^{(1)} \otimes D_1^1)$  is

$$\begin{aligned} \sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_1^k \otimes \frac{v_1^{j-k}}{p^{j-k-t}} &= (-1)^{j-1-t} \binom{j}{t+1} t_1^{j-1-t} \otimes \alpha_{t+1} + \dots \\ &= 0 \quad \text{if } t \geq j. \end{aligned}$$

From this we deduce that

$$\text{Ext}^0(T_0^{(1)}) = \mathbf{Z}_{(p)} \left\{ p^j z_j : 0 \leq j < p \right\},$$

and  $\text{Ext}^0$  of the third map of (7.3.16) sends

$$t_1^{(1)} \otimes \alpha_1 + \dots \mapsto 1 \otimes \beta_1.$$



Thus the map

$$\mathrm{Ext}^0(T_0^{(1)} \otimes E_1^2) \rightarrow \mathrm{Ext}^2(T_0^{(1)})$$

has a kernel, namely the  $\mathbf{Z}/(p)$ -summand generated by  $\beta_1$ , and for  $s > 2$ ,

$$\mathrm{Ext}_{\Gamma(1)}^s(T_0^{(1)}) \cong \mathrm{Ext}_{\Gamma(1)}^{s-2}(T_0^{(1)} \otimes E_1^2),$$

which can be read off from Theorem 7.3.15.

#### 4. The proof of Theorem 7.3.15

Recall that our range of dimensions is now  $(p^3 + p)|v_1|$ .

It is easy to see that all of the elements in Corollary 7.3.14 save those involving  $u_j$  or  $b_{2,0}$  are permanent cycles. Establishing the indicated differentials will ultimately be reduced to computing Ext groups for certain comodules over the Hopf algebra

$$P(1)_* = \mathbf{Z}/(p)[c(t_1), c(t_2)]/(c(t_1^2), c(t_2)^p)$$

with coproduct inherited from that of  $BP_*(BP)$ , i.e., with

$$\begin{aligned} \Delta(c(t_1)) &= c(t_1) \otimes 1 + 1 \otimes c(t_1) \\ \text{and } \Delta(c(t_2)) &= c(t_2) \otimes 1 + c(t_1)^p \otimes c(t_1) + 1 \otimes c(t_2). \end{aligned}$$

It is dual to the subalgebra  $P(1)$  of the Steenrod algebra generated by the reduced power operations  $P^1$  and  $P^p$ . For a  $P(1)_*$ -comodule  $M$ , we will abbreviate  $\mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p), M)$  by  $\mathrm{Ext}_{P(1)_*}(M)$ , or, when  $M = \mathbf{Z}/(p)$ , by simply  $\mathrm{Ext}_{P(1)_*}$ .

In principle one could get at  $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2)$  in our range of dimensions (i.e., below dimension  $p^3|v_1|$ ) by finding  $\mathrm{Ext}_{\Gamma(3)}(T_0^{(1)} \otimes E_1^2)$  and using the Cartan–Eilenberg spectral sequence for the extension

$$G(1, 1) \rightarrow \Gamma(1) \rightarrow \Gamma(3).$$

(Recall that  $G(1, 1) = A(2)[t_1, t_2]$ .)

Consider our 4-term exact sequence

$$0 \rightarrow BP_* \rightarrow D_1^0 \rightarrow D_1^1 \rightarrow E_1^2 \rightarrow 0$$

The two middle terms are weak injective over  $\Gamma(1)$  and hence over  $\Gamma(3)$ . For the last term we have,

$$\mathrm{Ext}_{\Gamma(3)}^s(E_1^2) = \mathrm{Ext}_{\Gamma(3)}^{s+2} \quad \text{for } s > 0.$$

The first generator for  $s = 1$  is  $\frac{v_4 t_3}{p v_1}$ , which is out of our range. This means that the fourth term is also weak injective over  $\Gamma(3)$  in our range.

For a  $\Gamma(1)$ -comodule  $M$ , we will denote the  $G(1, 1)$ -comodule  $\mathrm{Ext}_{\Gamma(3)}^0(M)$  by  $\tilde{M}$ . Applying  $\mathrm{Ext}_{\Gamma(3)}^0(\cdot)$  to our 4-term exact sequence yields a 4-term exact (in our range) sequence of  $G(1, 1)$ -comodules

$$0 \rightarrow A(2) \rightarrow \tilde{D}_1^0 \rightarrow \tilde{D}_1^1 \rightarrow \tilde{E}_1^2 \rightarrow 0.$$

Let  $\tilde{D}_1^2$  be the  $A(2)$ -submodule of  $\tilde{M}^2$  (where  $M^2$  is the chromatic comodule) obtained by adjoining the elements

$$\left\{ \frac{v_2^{-j} v_3^k}{p v_1^i} : i, j > 0, k \geq i + j \right\}$$

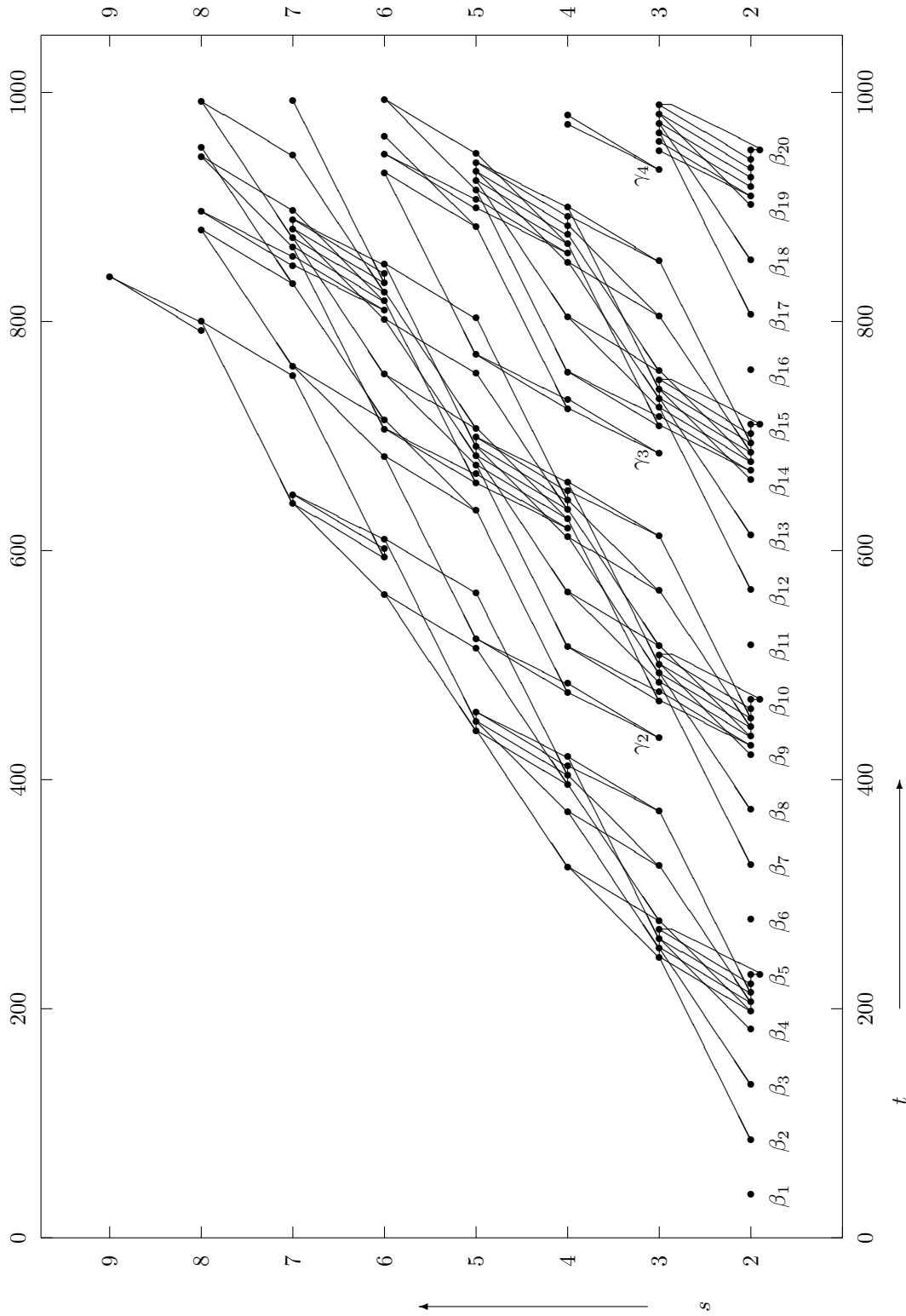


FIGURE 7.3.17.  $\text{Ext}(T_0^{(1)})$  for  $p = 5$  in dimensions below 998.

to  $\tilde{E}_1^2$ , so we have a short exact sequence of  $G(1, 1)$ -comodules

$$(7.4.1) \quad 0 \rightarrow \tilde{E}_1^2 \rightarrow \tilde{D}_1^2 \rightarrow \tilde{E}_1^3 \rightarrow 0,$$

where  $\tilde{E}_1^3$  is the  $A(2)$ -submodule of  $\tilde{N}^3$  generated by

$$\left\{ \frac{v_3^{2+e_1+e_2+e_3}}{pv_1^{1+e_1}v_2^{1+e_2}} : e_1, e_2, e_3 \geq 0 \right\}.$$

Its Poincaré series is

$$(7.4.2) \quad g(\tilde{E}_1^3) = \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)}.$$

7.4.3. DEFINITION. *Let  $P$  be the left  $G(1, 1)$ -comodule*

$$\begin{aligned} P &= A(2) \left\{ c(t_1^i t_2^j) : 0 \leq i, pj < p^2 \right\} \\ &= A(2) \left\{ t_1^i (t_2 - t_1^{p+1})^j : 0 \leq i, pj < p^2 \right\} \subset G(1, 1). \end{aligned}$$

A  $G(1, 1)$ -comodule  $M$  is  **$P$ -free** (in a range of dimensions) if  $P \otimes_{A(2)} M$  is weak injective (in the same range).

7.4.4. LEMMA.  $\tilde{D}_1^2$  and  $\tilde{E}_1^3$  are  $P$ -free in our range, i.e. below dimension  $p^2|v_2|$ .

PROOF. For  $\tilde{E}_1^3$  we can show this by direct calculation. Up to unit scalar we have

$$r_{(j-1)p^2\Delta_1+(i-1)p\Delta_2} \left( \frac{v_3^k}{pv_1^i v_2^j} \right) = \frac{v_3^{k+2-i-j}}{pv_1 v_2} = \gamma_{k+2-i-j},$$

so these elements form a basis for  $\text{Ext}_{G(1,1)}^0(\tilde{E}_1^3)$  and for  $\text{Ext}_{G(1,1)}^0(P \otimes \tilde{E}_1^3)$ . (Here  $r_{a,b}$  denotes the Quillen operation dual to  $t_1^a t_2^b$ .) The Poincaré series for this  $\text{Ext}^0$  is

$$\frac{x^{2p^2+p}}{1-x_3}.$$

Meanwhile we have

$$\begin{aligned} g(P \otimes_{A(2)} \tilde{E}_1^3) &= g(P(1))g(\tilde{E}_1^3) \\ &= \frac{(1-x^{p^2})(1-x_2^p)}{(1-x)(1-x_2)} \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)} \\ &= \frac{x^{2p^2+p}}{(1-x)(1-x_2)(1-x_3)} \\ &= g_2(t)g(\text{Ext}_{G(1,1)}^0(\tilde{E}_1^3)) \end{aligned}$$

so  $\tilde{E}_1^3$  is  $P$ -free as claimed.

For  $\tilde{D}_1^2$  we will first show that  $P \otimes \tilde{D}_1^2$  is weak injective over  $G(2)$ . Then it will suffice to show that

$$\text{Ext}_{G(2)}^0(P \otimes \tilde{D}_1^2)$$

is weak injective over  $G(1)$ , i.e. that  $\text{Ext}_{G(2)}^0(\tilde{D}_1^2)$  is 2-free.

As a  $G(2)$ -comodule,  $P$  is isomorphic to a direct sum of certain suspensions of  $T_1^{(1)}$ . We know by Theorem 7.2.6 that  $T_1^{(1)} \otimes E_2^2$  is weak injective over  $\Gamma(2)$  in our

range. The same is true of  $T_1^{(1)} \otimes E_1^2$  since it has the same positively graded Ext groups over  $\Gamma(2)$ . Thus the same goes for  $T_1^{(1)} \otimes \tilde{E}_1^2$  and  $P \otimes \tilde{E}_1^2$  over  $G(2)$ . Since we already know that  $P \otimes \tilde{E}_1^3$  is weak injective over  $G(1, 1)$  and hence over  $G(2)$ , this implies that  $P \otimes \tilde{D}_1^2$  is weak injective over  $G(2)$ .

This means that it suffices to show that  $\text{Ext}_{G(2)}^0(\tilde{D}_1^2)$  is 2-free. For this we have the following diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & B & \longrightarrow & \text{Ext}_{G(2)}^0(\tilde{E}_1^2) & \longrightarrow & U \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B & \longrightarrow & \text{Ext}_{G(2)}^0(\tilde{D}_1^2) & \longrightarrow & U_0 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \text{Ext}_{G(2)}^0(\tilde{E}_1^3) & \equiv & U_1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0,
 \end{array}$$

where  $B$  is as in Theorem 7.2.11 and the column on the right is as in Lemma 7.3.5. Since  $B$  and  $U_0$  are both 2-free in our range, so is  $\text{Ext}_{G(2)}^0(\tilde{E}_1^2)$ .  $\square$

We will show that  $\text{Ext}_{G(1,1)}^0(P \otimes \tilde{D}_1^2)$  and  $\text{Ext}_{G(1,1)}^0(P \otimes \tilde{E}_1^3)$  each admit filtrations whose associated bigraded objects are comodules over  $P(1)_*$ , and analyzing them will lead to a proof of Theorem 7.3.15.

As in the above lemma,  $\tilde{E}_1^3$  is easier to handle. We have

$$(7.4.5) \quad \text{Ext}_{G(1,1)}^0(P \otimes \tilde{E}_1^3) = \text{Ext}_{G(1,1)}^0(\tilde{E}_1^3) = \mathbf{Z}/(p) \left\{ \gamma_k : k \geq 2 \right\}.$$

No filtration is necessary here since it is annihilated by  $I_2$ , and we have

$$\text{Ext}_{G(1,1)}(\tilde{E}_1^3) = \mathbf{Z}/(p) \left\{ \gamma_k : k \geq 2 \right\} \otimes \text{Ext}_{P(1)_*}(\mathbf{Z}/(p)).$$

The case of  $\tilde{D}_1^2$  is more complicated.

7.4.6. LEMMA. *Let*

$$M = \text{Ext}_{G(1,1)}^0(P \otimes \tilde{D}_1^2).$$

*In our range it is generated by the following set.*

$$\begin{aligned}
 & \left\{ \beta_{i/j,k} : 1 \leq j, k \leq p, i \geq j + k - 1 \right\} \\
 & \cup \left\{ \tilde{\beta}_{i/j, \min(p+1, i+2-j)} : 1 \leq j \leq p, i \geq p \right\} \\
 & \cup \left\{ \tilde{\beta}_{i/p+1} : i \geq p + 1 \right\} \cup \left\{ \beta_{p^2/p^2-j} : 0 \leq j < p \right\}
 \end{aligned}$$

Here

$$\begin{aligned}\tilde{\beta}_{i/j,k} &= \frac{v_2^i}{p^k v_1^j} (1+x) \left( 1 - \binom{i}{p} y + \binom{i}{2p} y^2 \right) \\ &\quad \text{where } x = p^p v_1^{-1-p} v_2 \text{ and } y = v_1^p v_2^{-1-p} v_3 \\ &= \frac{v_2^i}{p^k v_1^j} + \begin{cases} 0 & \text{for } j, k < p+1 \\ -\binom{i}{p} \frac{v_2^{i-p-1} v_3}{p v_1} & \text{for } (j, k) = (p+1, 1) \\ \frac{v_2^{i+1}}{p v_1^{j+p+1}} - \binom{i}{p} \frac{v_2^{i-p} v_3}{p v_1^{j+1}} + \binom{i}{2p} \frac{v_2^{i-2p-1} v_3^2}{p v_1^{j+1-p}} & \text{for } k = p+1. \end{cases}\end{aligned}$$

It has a decreasing filtration defined by

$$\|\tilde{\beta}_{i/j,k}\| = i + [i/p] - j - k.$$

The above set is a  $\mathbf{Z}/(p)$ -basis for the associated bigraded object, which is a  $P(1)_*$ -comodule. Its structure as a  $P(1)$ -module is given by

$$\begin{aligned}r_1(\tilde{\beta}_{i/j,k}) &= j \tilde{\beta}_{i/j+1,k-1} \\ r_p(\tilde{\beta}_{i/j,k}) &= \begin{cases} \binom{i}{p} \tilde{\beta}_{i-1/j-1,k-1} & \text{for } p|i \\ \begin{cases} i \tilde{\beta}_{i-1/j-1,k} & \text{for } j > 1 \\ i \tilde{\beta}_{i/j+p,1} & \text{for } (j, k) = (1, p+1) \\ 0 & \text{for } j = 1 \text{ and } k < p+1. \end{cases} & \text{for } p \nmid i. \end{cases}\end{aligned}$$

Note that  $\tilde{\beta}_{i/p+1}$  is a unit multiple of the element  $u_{i-p-1}$  of Theorem 7.3.4.

PROOF. Recall that  $g(E_1^2)$  was determined in Lemma 7.2.2, which implies that in our range,

$$\begin{aligned}g(\tilde{E}_1^2) &= g_2(t) \sum_{i \geq 1} \frac{x^{p^i} (1-x_i)}{(1-x^{p^i})(1-x_{i+1})} \\ &= \frac{1}{(1-x)(1-x_2)} \left( \frac{x^p(1-x)}{(1-x^p)(1-x_2)} + \frac{x^{p^2}(1-x_2)}{(1-x^{p^2})(1-x_3)} \right. \\ &\quad \left. + \frac{x^{p^3}(1-x_3)}{(1-x^{p^3})(1-x_4)} \right) \\ &= \frac{x^p}{(1-x^p)(1-x_2)^2} + \frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)} + \frac{x^{p^3}}{1-x},\end{aligned}$$

so we have

$$\begin{aligned}
g(M) &= \frac{g(P(1))}{g_2(t)} \left( g(\tilde{E}_1^2) + g(\tilde{E}_1^3) \right) \\
&= (1-x^{p^2})(1-x_2^p) \left( \frac{x^p}{(1-x^p)(1-x_2)^2} + \frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)} \right. \\
&\quad \left. + \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)} + \frac{x^{p^3}}{1-x} \right) \\
&= \frac{x^p(1-x^{p^2})(1-x_2^p)}{(1-x^p)(1-x_2)^2} + \frac{x^{p^2}}{1-x} + \frac{x^{p^3}}{1-x} \\
&= \sum_{1 \leq j \leq p} x^{jp} \frac{1-x_2^p}{(1-x_2)^2} + \sum_{1 \leq j \leq p} \frac{x^{p^2+p-j}}{1-x_2} + \frac{x_2^p}{1-x_2} + \frac{x^{p^3}}{1-x}.
\end{aligned}$$

The four indicated subsets correspond to these four terms.

In order to show that we have the right elements, we need to show that for each indicated generator  $z$ , the invariant element

$$\tilde{z} = \sum_{a,b \geq 0} \tilde{t}_1^a \tilde{t}_2^b \otimes r_{a,b}(z) \in G(1,1) \otimes M$$

actually lies in  $P \otimes M$ . For dimensional reasons we need only consider the cases where  $a < p^3$  and  $b < p^2$ . Then if  $a \geq p^2$  or  $b \geq p$ ,  $r_{a,b}(z)$  vanishes if both  $r_{p^2}(z)$  and  $r_{0,p}(z)$  do. But for each of our generators, the correcting terms (i.e.  $\tilde{\beta}_{i/j,k} - \beta_{i/j,k}$ ) are chosen to insure that  $r_{p^2}$  and  $r_{0,p}$  act trivially.

Our putative filtration is similar to the  $I$ -adic one, which is given by

$$||\tilde{\beta}_{i/j,k}|| = i - j - k.$$

Note that we are not assigning a filtration to each chromatic monomial, but to each of the generators listed in Lemma 7.4.6.

Roughly speaking, it suffices to show that an operation  $r_{a,b}$  raises this filtration by the amount by which it lowers the value of  $[i/p]$ . Since  $r_{p^2}$  and  $r_{0,p}$  act trivially, it suffices to consider the action of  $r_1$  and  $r_p$ . The actions of  $r_1$  on  $v_2$  and  $v_3$ , and the action of  $r_p$  on  $v_3$  raise  $I$ -adic filtration by at least  $p-1$  and can therefore be

ignored. It follows that modulo such terms, we have

$$\begin{aligned}
 r_1 \left( \tilde{\beta}_{i/j,k} \right) &= -j \frac{v_2^i}{p^{k-1} v_1^{j+1}} = j \tilde{\beta}_{i/j+1,k-1} \\
 &= \begin{cases} 0 & \text{for } p|j \\ -j \tilde{\beta}_{i/j+1,k-1} & \text{for } p \nmid j \end{cases} \\
 r_p \left( \tilde{\beta}_{i/j,k} \right) &= i \frac{v_2^{i-1}}{p^k v_1^{j-1}} + \binom{-j}{p} \frac{v_2^i}{p^{k-p} v_1^{j+p}} \\
 &\quad + \begin{cases} 0 & \text{for } k < p+1 \\ (i+1) \frac{v_2^i}{p v_1^{j+p}} - i \binom{i}{p} \frac{v_2^{i-p-1} v_3}{p v_1^j} & \text{for } k = p+1 \end{cases} \\
 &= i \left( \frac{v_2^{i-1}}{p^k v_1^{j-1}} + \begin{cases} 0 & \text{for } k < p+1 \\ \frac{v_2^i}{p v_1^{j+p}} - \binom{i}{p} \frac{v_2^{i-p-1} v_3}{p v_1^j} & \text{for } k = p+1 \end{cases} \right) \\
 &= \begin{cases} \binom{i}{p} \tilde{\beta}_{i-1/j-1,k-1} & \text{for } p|i \\ \begin{cases} i \tilde{\beta}_{i-1/j-1,k} & \text{for } j > 1 \\ i \tilde{\beta}_{i/j+p,1} & \text{for } (j,k) = (1,p+1) \\ 0 & \text{for } j = 1 \text{ and } k < p+1. \end{cases} & \text{for } p \nmid i. \end{cases}
 \end{aligned}$$

Note that  $r_1$  never changes the value of  $i$  or the  $I$ -adic filtration, while  $r_p$  raises the latter by 1 precisely when lowers the value of  $[i/p]$  by 1. It follows that the indicated filtration is preserved by  $r_1$  and  $r_p$ .

The associated bigraded is killed by  $I_2$  because multiplication by it always raises filtration.  $\square$

In what follows we will ignore the elements

$$\left\{ \beta_{p^2/p^2-j} : 0 \leq j < p \right\}.$$

They are clearly permanent cycles and will thus have no bearing on the proof of Theorem 7.3.15. From now on,  $M$  will denote the quotient of  $M$  (as defined previously) by the subspace spanned by these elements.

To explore the structure of  $E_0 M$  further, we need to introduce some auxiliary  $P(1)_*$ -comodules. For  $0 \leq i < p$  let

$$C_i = \mathbf{Z}/(p) \left\{ t_1^{pj} : 0 \leq j \leq i \right\},$$

and let  $C_{-1} = 0$ . Let

$$H = P(1)_* \square_{P(0)_*} \mathbf{Z}/(p).$$

7.4.7. LEMMA. (i) For  $i \geq 0$ , let  $c(i) = p \left[ \frac{i+p}{p} \right] - i - 1$ . There is a 4-term exact sequence

$$0 \longrightarrow \Sigma^e C_{c(i)-1} \longrightarrow \Sigma^e H \longrightarrow E_0^{\|u_i\|} M \longrightarrow \Sigma^{|u_i|} C_{c(i)} \longrightarrow 0,$$

where

$$e = \begin{cases} |\beta_{i+2}| & \text{for } i \equiv -1 \pmod{p} \\ |\beta_{i+1}| = |u_i| - |b_{1,1}| & \text{otherwise.} \end{cases}$$

When  $i$  is congruent to  $-1$  modulo  $p$ , then  $c(i) = 0$  so the first term is trivial. The sequence splits in that case, i.e. for  $j > 0$

$$E_0^{(p+1)j-1}M = \Sigma^{|\beta_{pj+1}|}H \oplus \Sigma^{|u_{pj-1}|}\mathbf{Z}/(p).$$

The value of  $\|u_i\|$  is never congruent to  $-2$  modulo  $p+1$ , and for  $j > 0$

$$E_0^{(p+1)j-2}M = \Sigma^{|\beta_{pj}|}H.$$

(ii) For  $i$  not congruent to  $-1$  modulo  $p$ , there is are maps of 4-term sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^e C_{c(i)-1} & \longrightarrow & \Sigma^e H & \longrightarrow & E_0^{\|u_i\|} M & \longrightarrow & \Sigma^{|u_i|} C_{c(i)} & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^e C_{c(i)-1} & \longrightarrow & \Sigma^e C_{p-1} & \xrightarrow{r_{pc(i)}} & \Sigma^{e+|t_1^{pc(i)}} C_{p-1} & \longrightarrow & \Sigma^{|u_i|} C_{c(i)-1} & \longrightarrow & 0 \\ & & \uparrow & & \parallel & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^e \mathbf{Z}/(p) & \longrightarrow & \Sigma^e C_{p-1} & \xrightarrow{r_p} & \Sigma^{e+|t_1^p} C_{p-1} & \longrightarrow & \Sigma^{|u_i|} \mathbf{Z}/(p) & \longrightarrow & 0 \end{array}$$

in which each vertical map is a monomorphism. The bottom sequence is a Yoneda representative for the class  $b_{1,1} \in \text{Ext}_{P(1)_*}^2$ .

PROOF. (i) Let  $r_{0,1} = r_p r_1 - r_1 r_p \in P(1)$ . It generates a truncated polynomial algebra of height  $p$  which we denote by  $T(r_{0,1})$ . It follows from 7.4.6 that

$$r_{0,1}(\tilde{\beta}_{i/j,k}) = i\tilde{\beta}_{i-1/j,k-1}.$$

The element on the right is nonzero modulo higher filtration when  $i/p^{k-1}$  is not a  $p$ -local integer, i.e. when  $k$  is not too small. Thus up to unit scalar we get

$$\begin{aligned} r_{0,1}^{p-1}(\tilde{\beta}_{i+p-1/j,p+1}) &= \binom{i+p-1}{p-1} \tilde{\beta}_{i/j,2} \\ &= \begin{cases} \tilde{\beta}_{i/j} & \text{for } p \nmid i \\ \tilde{\beta}_{i/j,2} & \text{for } p \mid i \end{cases} \\ r_{0,1}^{p-1}(\tilde{\beta}_{pi+p-1/j,p}) &= \tilde{\beta}_{pi/j} \end{aligned}$$

This means that each element in the first two subsets in Lemma 7.4.6 is part of free module over  $T(r_{0,1})$ .

In  $P(1)$ ,  $r_p$  commutes with  $r_{0,1}$ , and  $H$  is free as a module over  $T(r_p, r_{0,1})$  on its top element  $x$ . It is characterized as a cyclic  $P(1)$ -module by  $r_1(x) = 0$  and  $r_{p(p-1),p}(x) \neq 0$ .

In  $E_0^{(p+1)j-2}M$ , the top element is  $\tilde{\beta}_{pj+2p-2/p,p+1}$ . It is killed by  $r_1$ , and up to unit scalar,

$$r_{p(p-1),p}(\tilde{\beta}_{pj+2p-2/p,p+1}) = \beta_{pj},$$

so  $E_0^{(p+1)j-2}M$  has the indicated structure.

In  $E_0^{(p+1)j-2}M$  for  $j > 0$ ,  $u_{pj-1}$  is killed by both  $r_1$  and  $r_p$  and generates a  $P(1)$ -summand. It is not present for  $j = 0$ . For  $j \geq 0$ , the class  $\tilde{\beta}_{pj+2p-1/p,p+1}$  generates a summand isomorphic to a suspension of  $H$  as claimed.



In  $E_0^{(p+1)j}M$  consider the sub- $P(1)$ -module generated by the element  $x = \beta_{pj+2p-1/p,p}$ . Up to unit scalar we have

$$\begin{aligned} r_1(x) &= 0 \\ r_{0,p-1}(x) &= \beta_{pj+p/p} \\ r_{(p-1)p,p-2}(x) &= \beta_{pj+2} \\ r_{p,p-1}(x) &= 0. \end{aligned}$$

Thus there is a homomorphism from the indicated suspension of  $H$  to  $E_0^{(p+1)j}M$  sending the top element to  $x$  with kernel isomorphic to  $C_{p-2}$ . Its cokernel is a copy of  $C_{p-1}$  in which top element is the image of  $\tilde{\beta}_{pj+2p-1/p-1,p+1}$  and the bottom element is the image of  $\tilde{\beta}_{pj+p+1/p+1}$ .

The remaining cases,  $E_0^{(p+1)j+k}M$  for  $1 \leq k \leq p-2$ , are similar. The top element in the image of  $H$  is  $\tilde{\beta}_{pj+k+2p-1/p,p+1}$ , and the top and bottom elements in the cokernel are the images of  $\tilde{\beta}_{pj+k+2p-1/p-1,p+1}$  and  $\tilde{\beta}_{pj+p+1+k/p+1}$  respectively.

(ii) The existence of the map of follows by inspection. Consider the case  $p = 3$  and  $i = 0$ . Then the diagram is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{12}C_1 & \longrightarrow & \Sigma^{12}H & \longrightarrow & E_0^0M & \longrightarrow & \Sigma^{48}C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^{12}C_0 & \longrightarrow & \Sigma^{12}C_2 & \longrightarrow & \Sigma^{24}C_2 & \longrightarrow & \Sigma^{48}C_0 & \longrightarrow & 0 \end{array}$$

The following diagram may be helpful in understanding the vertical maps.

$$\begin{array}{ccccccc} * & \longleftarrow & * & \longleftarrow & \beta_{3/3} & \longleftarrow & \tilde{\beta}_{4/4} \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \beta_2 & \longleftarrow & \beta_{3/2,2} & \longleftarrow & \beta_{4/3,2} \\ & & & & \uparrow & & \uparrow \\ & & & & \beta_{3/1,3} & \longleftarrow & \beta_{4/2,3} & \longleftarrow & \beta_{5/3,3} \\ & & & & & & \uparrow & & \uparrow \\ & & & & & & \tilde{\beta}_{4/1,4} & \longleftarrow & \tilde{\beta}_{5/2,4} \end{array}$$

Here the short vertical arrows represent the action of  $r_1$ , and the longer arrows represent  $r_3$ . The named elements form a basis of  $E_0^0M$  and the asterisks are elements in  $\Sigma^{12}H$  which map trivially to  $E_0^0M$ .  $H$  consists of all elements in the first three rows except  $\tilde{\beta}_{4/4}$ . □

We will use Lemma 7.4.7 to determine  $\text{Ext}_{P(1)_*}(E_0M)$  in the following way. We regard the 4-term sequence of 7.4.7(i) as a resolution of 0, apply the functor  $\text{Ext}_{P(1)_*}(\bar{T}_0^{(1)} \otimes \cdot)$ , and get a 4-column spectral sequence converging to 0. It turns out to have a  $d_3$  that is determined by 7.4.7(ii), and this information will determine our Ext group.

In order to proceed further we need to know

$$\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes H) \quad \text{and} \quad \mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes C_i),$$

where

$$\overline{T}_0^h = T_0^h \otimes_{BP_*} \mathbf{Z}/(p) \quad \text{with} \quad \overline{T}_0^{(i)} = \overline{T}_0^{p^i-1}.$$

This is a comodule over  $P(1)_*$ .

We will abbreviate  $\mathrm{Ext}_{P(1)_*}^s(\overline{T}_0^{(1)} \otimes N)$  by  $F^s(N)$ .

Since  $\overline{T}_0^{(1)} \otimes H = P(1)_*$ , we have

$$(7.4.8) \quad F^{s,t}(H) = \begin{cases} \mathbf{Z}/(p) & \text{for } (s,t) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Next we compute  $F^*(\mathbf{Z}/(p))$ . There is a Hopf algebra extension

$$(7.4.9) \quad \mathbf{Z}/(p)[t_1]/(t_1^{p^2}) \rightarrow P(1)_* \rightarrow \mathbf{Z}/(p)[t_2]/(t_2^p)$$

and we have

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0})$$

where

$$h_{2,0} \in \mathrm{Ext}^{1,2(p^2-1)} \quad \text{and} \quad b_{2,0} \in \mathrm{Ext}^{2,2p(p^2-1)}.$$

In particular  $\overline{T}_0^{(2)} = \mathbf{Z}/(p)[t_1]/(t_1^{p^2})$ , so

$$(7.4.10) \quad \mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(2)}) = \mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0})$$

where  $h_{2,0} \in \mathrm{Ext}^{1,2p^2-2}$  and  $b_{2,0} \in \mathrm{Ext}^{2,2p^3-2p}$ .

To compute  $F^*(\mathbf{Z}/(p))$ , we will use the long exact sequence

$$(7.4.11) \quad 0 \longrightarrow \overline{T}_0^{(1)} \longrightarrow \overline{T}_0^{(2)} \xrightarrow{r_p} \Sigma^{p|v_1|} \overline{T}_0^{(2)} \xrightarrow{r_{p^2-p}} \Sigma^{p^2|v_1|} \overline{T}_0^{(2)} \xrightarrow{r_p} \dots$$

This leads to a resolution spectral sequence converging to  $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$  with

$$E_1^{s,t} = E(h_{1,1}, h_{2,0}) \otimes P(b_{1,1}, b_{2,0}),$$

where

$$\begin{aligned} h_{1,1} &\in E_1^{1,0}, & h_{2,0} &\in E_1^{0,1}, \\ b_{1,1} &\in E_1^{2,0}, & \text{and } b_{2,0} &\in E_1^{0,2}. \end{aligned}$$

Alternatively, one could use the same resolution to show that

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_1]/(t_1^{p^2})}(\overline{T}_0^{(1)}) = E(h_{1,1}) \otimes P(b_{1,1})$$

and then use the Cartan–Eilenberg spectral sequence for (7.4.9). It is isomorphic to the resolution spectral sequence above.

Before describing this spectral sequence we need some notation for certain Massey products.

7.4.12. DEFINITION. *Let  $i$  be an integer with  $0 < i < p$ . Then  $\underline{i}x$  denotes the Massey product (when it is defined)*

$$\langle h_{1,0}, \dots, h_{1,0}, x \rangle$$

with  $i$  factors  $h_{1,0}$ , and  $\underline{pix}$  denotes the Massey product (when it is defined)

$$\langle h_{1,1}, \dots, h_{1,1}, x \rangle$$

with  $i$  factors  $h_{1,1}$ .

Under suitable hypotheses we have  $b_{1,0}x \in \underline{p-i} \cdot \underline{ix}$  and  $b_{1,1}x \in \underline{p(p-i)} \cdot \underline{pix}$ .

7.4.13. THEOREM. *The differentials in the above spectral sequence are as follows:*

$$(a) \quad d_3(h_{2,0}^\varepsilon b_{2,0}^i) = ih_{1,1}h_{2,0}^\varepsilon b_{11}b_{2,0}^{i-1};$$

$$(b) \quad d_{2p-1}(h_{2,0}^\varepsilon h_{1,1}b_{2,0}^{p+i-1}) = h_{2,0}^\varepsilon b_{1,1}^p b_{2,0}^{pi},$$

where  $\varepsilon = 0$  or  $1$ . These differentials commute with multiplication by  $h_{2,0}$ ,  $h_{1,1}$ , and  $b_{1,1}$ , and all other differentials are trivial. Consequently  $\text{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$  is a free module over

$$P(b_{2,0}^p) \otimes E(h_{2,0})$$

on the set

$$\{b_{1,1}^i : 0 \leq i \leq p-1\} \cup \{h_{1,1}b_{2,0}^i : 0 \leq i \leq p-2\}.$$

There are Massey product relations

$$h_{1,1}b_{2,0}^i \in \underline{p(i+1)}b_{1,1}^i \quad \text{and} \quad b_{1,1}^{i+1} \in \underline{p(p-i-1)}h_{1,1}b_{2,0}^i$$

for  $0 \leq i \leq p-2$ . We will denote this object by  $R$ .

PROOF. In the Cartan–Eilenberg spectral sequence for (7.4.9) one has

$$d_2(h_{2,0}) = \pm h_{1,0}h_{1,1}$$

since the reduced diagonal on  $t_2$  is  $t_1 \otimes t_1^p$ . Now we use the theory of algebraic Steenrod operations of A1.5 and the Kudo transgression theorem A1.5.7. Up to sign we have  $\beta P^0(h_{2,0}) = b_{2,0}$ , so

$$d_3(b_{2,0}) = \beta P^0(h_{1,0}h_{1,1}) = \beta(h_{1,1}h_{1,2}) = h_{1,1}b_{1,1}$$

as claimed in (a). Then A1.5.7 implies that

$$d_{2p-1}(h_{1,1}b_{1,1}b_{2,0}^{p-1}) = \beta(h_{1,2}b_{1,1}^p) = b_{1,1}^{p+1},$$

so  $d_{2p-1}(h_{1,1}b_{2,0}^{p-1}) = b_{1,1}^p$  as claimed in (b). The stated Massey product relations follow easily from (a) and (b).  $\square$

To compute  $F^*(C_i)$  for  $0 < i < p$ , we use the spectral sequence associated with the skeletal filtration of  $C_i$ . In it we have

$$E_1^{j,k} = F^k(\Sigma^{pj|t_1}\mathbf{Z}/(p)) \quad \text{for } 0 \leq j \leq i \quad \text{and} \quad d_r : E_r^{j,k} \rightarrow E_r^{j-r,k+1}.$$

We will denote the generator of  $E_1^{j,0}$  by  $x_{pj}$  and write  $x_0$  as 1. Since

$$\overline{T}_0^{(1)} \otimes C_{p-1} = \overline{T}_0^{(2)},$$

its Ext group is given by (7.4.10). There is a pattern of differentials implied by the Massey product relations of Theorem 7.4.13.

7.4.14. PROPOSITION. *In the skeletal filtration spectral sequence for*

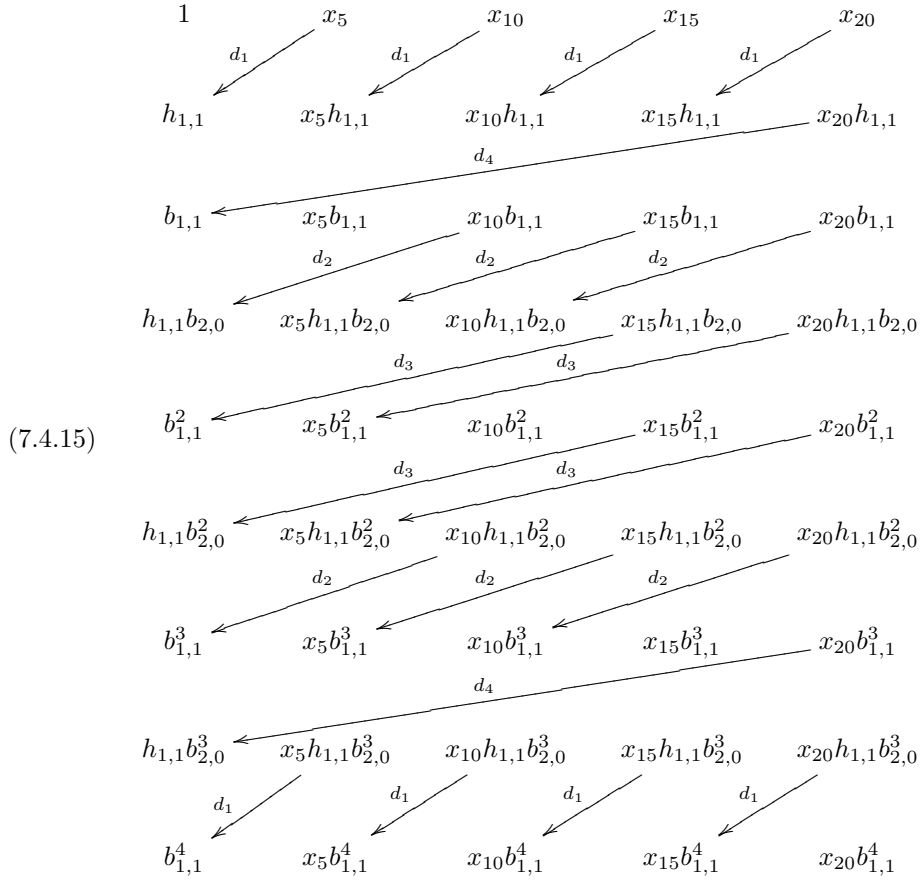
$$F^*(C_i) = \text{Ext}_{P(1)_*}^*(\overline{T}_0^{(1)} \otimes C_i)$$

*we have the following differentials and no others.*

$$\begin{aligned} d_{k+1}(x_{pj}h_{2,0}^\varepsilon b_{1,1}^k) &= x_{p(j-k-1)}h_{2,0}^\varepsilon h_{1,1}b_{2,0}^k \quad \text{for } 0 \leq k < j \leq i \\ d_{p-1-k}(x_{pj}h_{2,0}^\varepsilon h_{1,1}b_{2,0}^k) &= x_{p(j+k+1-p)}h_{2,0}^\varepsilon b_{1,1}^{k+1} \quad \text{for } p-1-j \leq k \leq p-1-j+i \\ &\quad \text{and } 0 \leq j \leq i, \end{aligned}$$

where  $\varepsilon = 0$  or  $1$ .

The following diagram illustrates this for  $p = 5$ .



Each row and column corresponds to a different value of  $k$  and  $j$  respectively. The skeletal filtration spectral sequence for  $C_{p-1}$  is obtained by tensoring the pattern indicated above with  $E(h_{2,0}) \otimes P(b_{2,0}^p)$ . Note that the only element in the  $j$ th column not on either end of a differential is  $x_{pj}b_{1,1}^j$ , which represents  $b_{2,0}^j$ .

The skeletal filtration spectral sequence for  $C_i$  is obtained from that for  $C_{p-1}$  by looking only at the first  $i + 1$  columns.

Now we consider the resolution spectral sequence converging to 0 associated the 4-term exact sequence of Lemma 7.4.7(i). In it we have

$$\begin{aligned} E_1^{0,s} &= F^s(\Sigma^e C_{c(i)-1}) \\ E_1^{1,s} &= F^s(\Sigma^e H) \\ E_1^{2,s} &= F^s(\Sigma^e E_0^{b(i)} M) \\ E_1^{3,s} &= F^s(\Sigma^{|u_i|} C_{c(i)}); \end{aligned}$$

each of these groups is graded by dimension. The last differential here is

$$d_3 : E_3^{0,s} \rightarrow E_3^{3,s-2}.$$

It is an isomorphism and hence has an inverse since the spectral sequence converges to 0. The bottom dimension is  $e = |\beta_{b(i)+1}|$ . By (7.4.8) we have

$$E_1^{1,s} = \begin{cases} \mathbf{Z}/(p) & \text{concentrated in dimension } e \text{ for } s = 0 \\ 0 & \text{for } s > 0. \end{cases}$$

The bottom class here is killed by a  $d_1$  coming from the one in  $E_1^{1,s}$ . Above the bottom dimension, the only differentials in addition to the  $d_3$  above are

$$d_2 : E_2^{0,s} \rightarrow E_2^{2,s-1} \quad \text{and} \quad d_1 : E_1^{2,s} \rightarrow E_1^{3,s}.$$

It follows that above dimension  $e$  there is a short exact sequence

$$(7.4.16) \quad 0 \longrightarrow \text{coker } \tilde{d}_3^{-1} \xrightarrow{d_2} F^s(\Sigma^e E_0^{b(i)} M) \xrightarrow{d_1} \ker \tilde{d}_3^{-1} \longrightarrow 0,$$

where  $\tilde{d}_3^{-1}$  denotes the composite

$$F^s(\Sigma^{b_{1,1}} C_{c(i)}) \longrightarrow E_3^{3,s} \xrightarrow[\cong]{\tilde{d}_3^{-1}} E_3^{0,s+2} \longrightarrow F^{s+2}(C_{c(i)-1})$$

Here  $\text{coker } \tilde{d}_3^{-1}$  is a quotient of  $F^{s+1}(\Sigma^e C_{c(i)-1})$  and  $\ker \tilde{d}_3^{-1}$  is a subgroup of  $F^s(\Sigma^{|u_i|} C_{c(i)})$ . Note that  $|u_i| - e = |b_{1,1}|$  in all cases. Lemma 7.4.7(ii) implies that  $\tilde{d}_3^{-1}$ , roughly speaking, multiplication by  $b_{1,1}$ .

We illustrate this for the case  $p = 5$  and  $i = 0$ . The 4-term sequence is

$$0 \longrightarrow \Sigma^{40} C_3 \longrightarrow \Sigma^{40} H \longrightarrow E_0^0 M \longrightarrow \Sigma^{240} C_4 \longrightarrow 0.$$

Referring to (7.4.15) we see that the product of  $b_{1,1}$  with any element in  $F^*(C_4)$  (except  $x_{20} b_{1,1}^4$ , which is out of our range) is killed by a differential originating in the last column, which means that it is alive in  $F^*(C_3)$ . Thus  $\tilde{d}_3^{-1}$  is a monomorphism in our range, so its kernel is trivial and the  $d_2$  in (7.4.16) is an isomorphism. The cokernel of  $\tilde{d}_3^{-1}$  is the quotient of

$$\Sigma^{40} E(h_{2,0}) \otimes \left\{ 1, x_5 h_{1,1}, x_5 b_{1,1}, x_{10} h_{1,1} b_{2,0}, x_{10} b_{1,1}^2, x_5 h_{1,1} b_{2,0}^2, x_{15} b_{1,1}^3, h_{1,1} b_{2,0}^3 \right\}$$

obtained by killing the bottom class. The classes  $h_{2,0}$  and  $x_{15} h_{1,1}$  map to  $\beta_2$  and  $\beta_{5/5}$ . By inspection this leads to the desired value of  $F^*(E_0^0 M)$ .

For  $i = 1$ , the 4-term sequence is

$$0 \longrightarrow \Sigma^{88} C_2 \longrightarrow \Sigma^{88} H \longrightarrow E_0^1 M \longrightarrow \Sigma^{288} C_3 \longrightarrow 0.$$

Again  $\tilde{d}_3^{-1}$  is a monomorphism in our range. The cokernel of  $\tilde{d}_3^{-1}$  is the quotient of

$$\Sigma^{88} E(h_{2,0}) \otimes \left\{ 1, x_{10}h_{1,1}, x_5b_{1,1}, x_5h_{1,1}b_{2,0}, x_{10}b_{1,1}^2, h_{1,1}b_{2,0}^2 \right\}$$

obtained by killing the bottom class. The classes  $h_{2,0}$  and  $x_{10}h_{1,1}$  map to  $\beta_3$  and  $\beta_{5/4}$ . By inspection this leads to the desired value of  $F^*(E_0^1 M)$ .

In order to see that this works in general it is useful to compare the comodules  $\overline{T}_0^{(1)} \otimes E_0^i M$  with certain others with known Ext groups. Let

$$(7.4.17) \quad 0 \longrightarrow \overline{T}_0^{(1)} \longrightarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \dots$$

be a minimal free resolution of  $\overline{T}_0^{(1)}$ . Its structure is as follows.

7.4.18. PROPOSITION. *The free  $P(1)_*$ -comodule  $F_i$  above is*

$$F_i = \begin{cases} P(1)_* & \text{for } i = 0 \\ \Sigma^{i'|b_{1,1}|} P(1)_* \oplus \Sigma^{(i'-1)|b_{2,0}|+|t_1^p t_2|} P(1)_* & \text{for } i = 2i' \text{ and } 0 < i' < p \\ \Sigma^{i'|b_{2,0}|+|t_1^p|} P(1)_* \oplus \Sigma^{i'|b_{1,1}|+|t_2|} P(1)_* & \text{for } i = 2i' + 1 \text{ and} \\ & 0 < i' < p - 1 \\ \Sigma^{|t_2|+(p-1)|b_{1,1}|} P(1)_* & \text{for } i = 2p - 1 \\ \Sigma^{p|b_{2,0}|} F_{i-2p} & \text{for } i \geq 2p. \end{cases}$$

In  $P(1)$  let  $x = P^1$ ,  $y = P^p$ ,  $z = yx - xy$ . Then there are relations

$$x^p = 0, [x, z] = 0, [y, z] = 0, \text{ and } y^p = xz^{p-1},$$

(corresponding to the four generators of  $\text{Ext}_{P(1)_*}^2$ ) which imply that  $z^p = 0$ . Then  $d_i$  is represented (via left multiplication) by a matrix  $M_i$  over  $P(1)$  as follows.

$$M_i = \begin{cases} \begin{bmatrix} y \\ z \end{bmatrix} & \text{for } i = 0 \\ \begin{bmatrix} y^{p-i'} & -xz^{p-2} \\ z & -y^{i'} \end{bmatrix} & \text{for } i = 2i' - 1 \text{ with } 0 < i' < p \\ \begin{bmatrix} y^{i'+1} & -(xy + (i'+1)z)z^{p-2} \\ z & -y^{p-i'} \end{bmatrix} & \text{for } i = 2i' \text{ with } 0 < i' < p - 1 \\ \begin{bmatrix} z & -y \end{bmatrix} & \text{for } i = 2p - 2 \\ \begin{bmatrix} y^{p-1} z^{p-1} \end{bmatrix} & \text{for } i = 2p - 1 \\ M_{i-2p} & \text{for } i \geq 2p. \end{cases}$$

Let  $K_i$  denote the kernel of  $d_i$ , and consider the following diagram with exact rows and columns for  $0 < i < p$ .

$$(7.4.19) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_i & \longrightarrow & \Sigma^{a_i} P(1)_* & \longrightarrow & K_{2i} \longrightarrow Y_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & K_{2i-1} & \xrightarrow{d_{2i-2}} & F_{2i-1} & \xrightarrow{d_{2i-1}} & K_{2i} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \Sigma^{b_i} P(1)_* & \equiv & \Sigma^{b_i} P(1)_* & & \\ & & \downarrow & & \downarrow & & \\ & & Y_i & & 0 & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where  $a_i = (i-1)|b_{2,0}| + |t_1^p|$ ,  $b_i = (i-1)|b_{1,1}| + |t_2|$ , and the middle column is split. We will see that the top row (up to reindexing and suspension) is the 4-term sequence of Lemma 7.4.7(i) tensored with  $\overline{T}_0^{(1)}$ . For this we need to identify  $X_i$  and  $Y_i$ .

$X_i$  is the kernel of the map represented by the first column of  $M_{2i-1}$ , namely

$$\begin{bmatrix} y^{p-i} \\ z \end{bmatrix}.$$

This kernel is the ideal generated by  $y^i z^{p-1}$ , which is

$$\Sigma^{a_i} \overline{T}_0^{p(p-i)-1} = \Sigma^{i|b_{2,0}| - |b_{1,1}|} \overline{T}_0^{(1)} \otimes C_{p-1-i}.$$

$Y_i$  is the cokernel of the map to  $\Sigma^{b_i} P(1)_*$  represented by the bottom row of  $M_{2i-2}$ , namely

$$\begin{cases} \begin{bmatrix} z \end{bmatrix} & \text{for } i = 1 \\ \begin{bmatrix} z & -y^{p+1-i} \end{bmatrix} & \text{for } 1 < i < p. \end{cases}$$

This cokernel is

$$\Sigma^{b_i + |y^{i-1} z^{p-1}|} \overline{T}_0^{(p+1-i)p-1} = \Sigma^{i|b_{2,0}|} \overline{T}_0^{(1)} \otimes C_{p-i}.$$

This enables us to prove the following analog of Lemma 7.4.7.

7.4.20. LEMMA. *For  $0 < i < p$  there are maps of 4-term exact sequences*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{a_i} \overline{T}_0^k & \longrightarrow & \Sigma^{a_i} P(1)_* & \longrightarrow & K_{2i} \longrightarrow \Sigma^{a_i + |b_{1,1}|} \overline{T}_0^{k+p} \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Sigma^{a_i} \overline{T}_0^k & \longrightarrow & \Sigma^{a_i} \overline{T}_0^{(2)} & \xrightarrow{r_{k+1}} & \Sigma^{i|b_{1,1}|} \overline{T}_0^{(2)} \longrightarrow \Sigma^{a_i + |b_{1,1}|} \overline{T}_0^k \longrightarrow 0 \\ & & \uparrow & & \parallel & & \uparrow \\ 0 & \longrightarrow & \Sigma^{a_i} \overline{T}_0^{(1)} & \longrightarrow & \Sigma^{a_i} \overline{T}_0^{(2)} & \xrightarrow{r_p} & \Sigma^{a_i + |t_1^p|} \overline{T}_0^{(2)} \longrightarrow \Sigma^{a_i + |b_{1,1}|} \overline{T}_0^{(1)} \longrightarrow 0 \\ & & & & & & \uparrow \\ & & & & & & \Sigma^{i|b_{1,1}|} \overline{T}_0^{(2)} \end{array}$$

where  $k = p(p-i) - 1$ , the top row is the same as that in (7.4.19), and each vertical map is a monomorphism.

PROOF. The statement about the top row is a reformulation of our determination of  $X_i$  and  $Y_i$  above. Each vertical map is obvious except the one to  $K_{2i}$ .  $K_{2i}$  is the kernel of the map  $d_{2i}$  from

$$F_{2i} = \Sigma^{i|b_{1,1}|} P(1)_* \oplus \Sigma^{(i-1)|b_{2,0}|+|t_1^p t_2|} P(1)_*$$

(note that  $i|b_{1,1}| = a_i + |t_1^{p(p-i)}|$ ) to

$$F_{2i+1} = \Sigma^{i|b_{2,0}|+|t_1^p|} P(1)_* \oplus \Sigma^{i|b_{1,1}|+|t_2|} P(1)_*$$

represented by the matrix

$$M_{2i} = \begin{bmatrix} y^{i+1} & -(xy + (i+1)z)z^{p-2} \\ z & -y^{p-i} \end{bmatrix}.$$

The map  $\Sigma^{a_i} P(1)_*$  is the restriction of  $d_{2i-1}$ , under which we have

$$t_1^{p(p-i)} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so this is the image of the bottom element in  $\Sigma^{i|b_{1,1}|} \overline{T}_0^{(2)}$  in  $K_{2i}$ . This means that the top element in  $\Sigma^{i|b_{1,1}|} \overline{T}_0^{(2)}$  must map to an element of the form

$$\begin{bmatrix} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are each killed by  $y^{p-1}$ . We also need this element to be in  $K_{2i}$ , so it must satisfy

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= M_{2i} \begin{bmatrix} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \\ &= \begin{bmatrix} ct_1^{p(p-2-i)} + y^{i+1}(\varepsilon_1) - (xy + (i+1)z)z^{p-2}(\varepsilon_2) \\ z(\varepsilon_1) - y^{p-i}(\varepsilon_2) \end{bmatrix} \end{aligned}$$

for a certain unit scalar  $c$ . We can get this by setting  $\varepsilon_1 = 0$  and making  $\varepsilon_2$  a linear combination of  $t_1^{p(p-2-i)}t_2^{p-1}$  and  $t_1^{1+p(p-1-i)}t_2^{p-2}$  chosen to make the element in the top row vanish. Such an  $\varepsilon_2$  will be killed by  $y^{p-i}$ , so the element in the bottom row will vanish as well.  $\square$

This means that the Ext computation for the  $K_{2i}$  is essentially identical to that  $\overline{T}_0^{(1)} \otimes E_0 M$  described above. It follows from the way the  $K_i$  were constructed that for all  $i$  and  $s$ ,

$$(7.4.21) \quad \text{Ext}_{P(1)_*}^s(K_i) = \text{Ext}_{P(1)_*}^{s+i}(K_0) = \text{Ext}_{P(1)_*}^{s+i}(\overline{T}_0^{(1)}).$$

These groups are known by Theorem 7.4.13.

We need the following analog of Theorem 7.4.13 for these comodules, whose proof we leave as an exercise for the reader.



7.4.22. THEOREM. *In the Cartan–Eilenberg spectral sequence converging to  $\text{Ext}_{P(1)_*}(K_{2i})$  based on the extension (7.4.9) for  $0 < i < p$ ,  $\tilde{E}_2$  is a subquotient (determined by the  $d_1$  indicated below) of*

$$P(b_{1,1}) \otimes \left\{ \begin{array}{c} \{b_{2,0}^i\} \otimes E(h_{2,0}, h_{1,1}) \otimes P(b_{2,0}) \\ \oplus \\ \{h_{1,1}h_{2,0}b_{2,0}^{i-1}, \underline{p(p-i)}(h_{1,1}h_{2,0}b_{2,0}^{i-1}), b_{1,1}, \underline{pib_{1,1}^i}\} \end{array} \right\}.$$

Here we are using the isomorphism of (7.4.21) to name the generators in the two indicated sets. Thus we have

$$\begin{array}{ll} b_{2,0}^i, h_{1,1}h_{2,0}b_{2,0}^{i-1}, b_{1,1} & \in \tilde{E}_2^{0,0} \\ \underline{p(p-i)}h_{1,1}h_{2,0}b_{2,0}^{i-1}, \underline{pib_{1,1}^i} & \in \tilde{E}_2^{1,0} \\ h_{1,1} \in E_2^{1,0} & b_{1,1} \in E_2^{2,0} \\ h_{2,0} \in E_2^{0,1} & b_{2,0} \in E_2^{0,2}, \end{array}$$

and the differentials are (up to unit scalar)

$$\begin{aligned} d_1(b_{2,0}^i) &= \underline{pib_{1,1}^i} \\ d_2(h_{2,0}b_{2,0}^i) &= b_{1,1} \cdot h_{1,1}h_{2,0}b_{2,0}^{i-1} \\ d_3(h_{2,0}^\varepsilon b_{2,0}^k \cdot b_{2,0}^i) &= (i+k)h_{1,1}h_{2,0}^\varepsilon b_{1,1}b_{2,0}^{k-1} \cdot b_{2,0}^i \\ &\quad \text{for } k > 0 \text{ and } \varepsilon = 0 \text{ or } 1 \\ d_{2p-2i-1}(h_{1,1}b_{2,0}^{p-1-i} \cdot b_{2,0}^i) &= b_{1,1}^{p-i} \cdot b_{1,1}^i \\ d_{2p-2i-2}(h_{1,1}h_{2,0}b_{2,0}^{p-1-i} \cdot b_{2,0}^i) &= b_{1,1}^{p-i} \cdot \underline{p(p-i)}h_{1,1}h_{2,0}b_{2,0}^{i-1}. \end{aligned}$$

The last four differentials listed above should be compared with the first four listed in Theorem 7.3.15. The first differential of Theorem 7.4.13 corresponds to the last one of 7.3.15, while the second differential of 7.4.13 would correspond to one in 7.3.15 that is out of our range.

Thus Theorem 7.3.15 is a consequence of the relation between the  $K_{2i}$  and  $E_0M$ .

### 5. Computing $\pi_*(S^0)$ for $p = 3$

We begin by recalling the results of the previous sections. We are considering groups  $\text{Ext}^{s,t} = 0$  for  $t < p^3|v_1|$  (where  $|v_1| = 2p - 2$ ) with  $p > 2$ . For each odd prime  $p$ , we have the 4-term exact sequence (7.1.19) of comodules over  $BP_*(BP)$

$$0 \longrightarrow BP_* \longrightarrow D_1^0 \longrightarrow D_1^1 \longrightarrow E_1^2 \longrightarrow 0$$

in which  $D_1^0$  and  $D_1^1$  are weak injective (meaning that their higher Ext groups vanish, see 7.1.5) and the maps

$$\text{Ext}^0(D_1^0) \longrightarrow \text{Ext}^0(D_1^1) \longrightarrow \text{Ext}^0(E_1^2)$$

are trivial. This means that the resolution spectral sequence collapses from  $E_1$  and we have isomorphisms

$$\text{Ext}^s = \begin{cases} \text{Ext}^0(D_1^0) & \text{for } s = 0 \\ \text{Ext}^0(D_1^1) & \text{for } s = 1 \\ \text{Ext}^{s-2}(E_1^2) & \text{for } s \geq 2 \end{cases}$$

We have determined  $\text{Ext}(T_0^{(1)} \otimes E_1^2)$  in Theorem 7.3.15, which can be reformulated as follows.

7.5.1. ABC THEOREM. For  $p > 2$  and  $t < (p^3 + p)|v_1|$

$$\text{Ext}(T_0^{(1)} \otimes E_1^2) = A \oplus B \oplus C$$

where  $A$  is the  $\mathbf{Z}/(p)$ -vector space spanned by

$$\left\{ \beta_i = \frac{v_2^i}{pv_1} : i > 0 \text{ and } i \equiv 0, 1 \pmod{p} \right\} \cup \left\{ \beta_{p^2/p^2-j} : 0 \leq j < p \right\},$$

$$B = R \otimes \left\{ \gamma_k \in \text{Ext}^{1, 2k(p^3-1)-2(p^2+p-2)} : k \geq 2 \right\}$$

and

$$C^{s,t} = \bigoplus_{i \geq 0} R^{2+s+2i, t+i(p^2-1)q}.$$

Here  $R = \text{Ext}_{P(1)_*}(\mathbf{Z}/(p), \overline{T}_0^{(1)})$  as described in Theorem 7.4.13. □

This result is illustrated for  $p = 5$  in Figure 7.3.17. Each dot represents a basis element. Vertical lines represent multiplication by 5 and horizontal lines represent the Massey product operation  $\langle -, 5, \alpha_1 \rangle$ , corresponding to multiplication by  $v_1$ . The diagonal lines correspond either to multiplication by  $h_{2,0}$  or to Massey product operations  $\langle -, h_{11}, h_{11}, \dots, h_{11} \rangle$ .

The next step is to pass from this group to  $\text{Ext}(E_1^2)$  using the small descent spectral sequence of Theorem 7.1.13. Alternatively one could observe that  $E_1^2 = BP_*(\text{coker } J)$  and that the Adams–Novikov spectral sequence for  $\pi_*(T(0)_{(1)} \wedge \text{coker } J)$  collapses for dimensional reasons. We can then use the topological small descent spectral sequence of Theorem 7.1.16 to pass from this group to  $\pi_*(\text{coker } J)$ . We will do this using the input/output procedure of 7.1.18.

We give a basis for  $\mathbf{N}$ . Recall the the input  $\mathbf{I}$  in this case is  $\mathbf{N} \otimes E(h_{1,0})$ .

7.5.2. PROPOSITION. For  $p = 3$ ,  $\mathbf{N}$  as in 7.1.18 has basis elements in dimensions indicated below.

10 $\beta_1$	72 $b_{2,0}\beta_2$	93 $h_{2,0}u_2$
26 $\beta_2$	74 $\beta_5$	96 $h_{2,0}\gamma_2$
34 $\beta_{3/3} = b_{1,1}$	78 $u_2$	97 $h_{2,0}\beta_{6/3}$
38 $\beta_{3/2}$	81 $\gamma_2$	97 $h_{2,0}\beta_{6/3}$
42 $\beta_3, \beta_{3/1,2}$	82 $\beta_{6/3}$	101 $h_{11}\beta_{6/1,2}$
49 $h_{2,0}b_{1,1}$	83 $\beta_2\eta_1 = h_{2,0}b_{1,1}^2$	104 $\beta_2u_2$
53 $h_{11}\beta_{3/1,2}$	86 $\beta_{6/2}$	105 $\eta_4 = h_{11}u_3 = \underline{6}\beta_{6/3}$
57 $\eta_1 = h_{11}u_0 = \underline{6}\beta_{3/3}$	89 $\eta_3 = h_{11}u_2$	106 $\beta_7, \beta_{9/9}$
58 $\beta_4$	90 $\beta_6, \beta_{6/1,2}$	107 $\gamma_2\beta_2$
68 $b_{1,1}^2$	92 $h_{11}\gamma_2$	□

The notation  $\underline{nx}$  for an integer  $n$  denotes a certain Massey product involving  $x$  as in 7.4.12.  $\underline{3x}$  and  $\underline{6x}$  denote  $h_{11}x$  and  $\langle h_{11}, h_{11}, x \rangle$ , respectively.

Now we turn to the list **O** of 7.1.18, shown in 7.5.3. Elements from **N** are underlined. A differential is indicated by enclosing the target in square brackets and indicating the source on the right. Hence such pairs are to be omitted from the final output. The computation of the differentials will be described below.

7.5.3. THEOREM. *With notation as above the list **O** of 7.1.18 for  $p = 3$  is as follows.*

10 $\underline{\beta_1}$	57 $[\underline{2\beta_1^5}]\underline{\beta_4}$	91 $\beta_1\gamma_2$
13 $\alpha_1\beta_1$	59 $[\alpha_1\beta_1^3\beta_2]\alpha_1\eta_1$	$\underline{2\beta_1}\beta_5$
20 $\beta_1^2$	60 $[\beta_1^6]\alpha_1\beta_4$	92 $\beta_1\beta_{6/3}$
23 $\alpha_1\beta_1^2$	62 $\beta_1\beta_2^2$	$\underline{h_{11}\gamma_2}$
26 $\underline{\beta_2}$	65 $\alpha_1\beta_1\beta_2^2$	$[\alpha_1u_3 + \beta_1\beta_{6/3}]\underline{h_{2,0}u_2}$
29 $\alpha_1\beta_2$	68 $\underline{\beta_{3/3}^2} \pm \beta_4\beta_1 = x_{68}$	93 $\underline{2\beta_{6/2}}$
30 $\beta_1^3$	71 $[\alpha_1x_{68}]\underline{b_{2,0}\beta_2}$	$\alpha_1\beta_6$
33 $[\alpha_1\beta_1^3]\underline{\beta_{3/3}}$	72 $\beta_1^2\beta_2^2$	94 $\alpha_1\beta_1\gamma_2$
36 $\beta_1\beta_2$	74 $\underline{\beta_5}$	$\beta_1^2\beta_5$
37 $\underline{2\beta_1^3}$	75 $\underline{2x_{68}}$	95 $[\alpha_1\underline{3\gamma_2}]\underline{h_{2,0}\gamma_2}$
38 $\underline{\beta_{3/2}}$	$\alpha_1\beta_1^2\beta_2^2$	$\alpha_1\beta_1\beta_{6/3}$
39 $\alpha_1\beta_1\beta_2$	77 $[\alpha_1\beta_5]\underline{u_2}$	$[\underline{2\beta_1}\beta_2^3]\alpha_1\underline{h_{2,0}u_2}$
40 $\beta_1^4$	78 $\beta_2^3 = \beta_1x_{68}$	96 $[\beta_1\beta_{6/2}]\underline{h_{2,0}\beta_{6/3}}$
41 $[\alpha_1\beta_{3/2}]\underline{\beta_{3/1,2}}$	81 $\underline{\gamma_2}$	99 $\underline{2} \cdot \underline{3}\gamma_2$
42 $\underline{\beta_3}$	$\underline{2\beta_5}$	100 $\beta_2\beta_5 = \underline{2} \cdot \underline{2}\beta_{6/2}$
45 $\underline{2\beta_{3/2}}$	82 $\underline{\beta_{6/3}}$	$[\beta_1\beta_6]\underline{h_{11}\beta_{6/1,2}}$
$\alpha_1\beta_3$	$[\beta_1^3\beta_2^2]\underline{\beta_2\eta_1}$	$\underline{2\beta_1^2}\beta_5$
46 $\beta_1^2\beta_2$	84 $\alpha_1\gamma_2$	101 $\underline{2\beta_1^2}\beta_5$
47 $\underline{2\beta_1^4}$	$\beta_1\beta_5$	102 $\beta_1^2\beta_{6/3} = \pm\beta_{3/3}^3$
48 $[\beta_1\beta_{3/2}]\underline{h_{2,0}b_{1,1}}$	85 $\underline{2\beta_2^3}$	$\underline{3}\beta_1\gamma_2$
49 $\alpha_1\beta_1^2\beta_2$	$\alpha_1\beta_{6/3}$	103 $[\alpha_1\beta_2\beta_5]\underline{\beta_2u_2}$
50 $\beta_1^5$	$[\alpha_1\beta_1^3\beta_2^2]\alpha_1\beta_2\eta_1$	$[\alpha_1\beta_1\beta_6]\alpha_1\underline{h_{11}\beta_{6/1,2}}$
52 $\beta_2^2$	86 $\underline{\beta_{6/2}}$	104 $\alpha_1\beta_1^2\gamma_2$
$[\beta_1\beta_3]\underline{h_{11}\beta_{3/1,2}}$	88 $[\beta_1\beta_2^3]\underline{\eta_3}$	$[\beta_1^3\beta_5]\underline{\eta_4}$
55 $\alpha_1\beta_2^2$	89 $[\alpha_1\beta_{6/2}]\underline{\beta_{6/1,2}}$	105 $[\alpha_1\beta_{3/3}^3]\underline{\beta_{9/9}}$
$[\alpha_1\beta_1\beta_3]\alpha_1\underline{h_{11}\beta_{3/1,2}}$	90 $\underline{\beta_6}$	106 $\underline{\beta_7} \pm \beta_{9/9}$
56 $[\beta_1^3\beta_2]\eta_1$		

7.5.4. REMARK. In the calculations below we shall make use of *Toda brackets* (first defined by Toda [6]) and their relation to Massey products. Suppose we have spaces (or spectra) and maps  $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$  with  $gf$  and  $hg$  null-homotopic. Let  $\bar{f}: CW \rightarrow Y$  and  $\bar{g}: CX \rightarrow Z$  be null homotopies. Define a map  $k: \Sigma W \rightarrow Z$  by regarding  $\Sigma W$  as the union of two copies of  $CW$ , and letting the restrictions of  $k$  be  $h\bar{f}$  and  $\bar{g}(Cf)$ .  $k$  is not unique up to homotopy as it depends on the choice of the null homotopies  $\bar{f}$  and  $\bar{g}$ . Two choices of  $\bar{f}$  differ by a map  $\Sigma W \rightarrow Y$  and similarly for  $\bar{g}$ . Hence we get a certain coset of  $[\Sigma W, Z]$  denoted in Toda [6] by  $\{f, g, h\}$ , but here by  $\langle f, g, h \rangle$ . Alternatively, let  $C_g$  be the cofiber of  $g$ ,  $\bar{h}: C_g \rightarrow Z$  an extension of  $h$  and  $\bar{f}: \Sigma W \rightarrow C_g$  a lifting of  $\Sigma f$ . Then  $k$  is the composite  $\bar{h}\bar{f}$ .

Recall (A1.4.1) that for a differential algebra  $C$  with  $a, b, c \in H^*C$  satisfying  $ab = bc = 0$  the Massey product  $\langle a, b, c \rangle$  is defined in a similar way. The interested reader can formulate the definition of higher matrix Toda brackets, but any such map can be given as the composite of two maps to and from a suitable auxiliary spectrum (such as  $C_g$ ). For example, given

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$$

satisfying suitable conditions with each  $X_i$  a sphere, the resulting  $n$ -fold Toda bracket is a composite  $\Sigma^{n-2}X_0 \rightarrow Y \rightarrow X_n$ , where  $Y$  is a complex with  $(n - 1)$  cells.

The relation between Toda brackets and Massey products and their behavior in the Adams spectral sequence is studied by Kochman [2, 4, 5]. The basic idea of Kochman [4] is to show that the Adams spectral sequence arises from a filtered complex, so the spectral sequence results of A1.4 apply. Given Kochman’s work we will use Toda brackets and Massey products interchangeably.

7.5.5. REMARK. In the following discussions we will not attempt to keep track of nonzero scalars mod  $(p)$ . For  $p = 3$  this means that a  $\pm$  should appear in front of every symbol in an equation. The reader does not have the right to sue for improper coefficients.

Now we provide a running commentary on this list. The notation  $\underline{2}x$  denotes the Massey product  $\langle \alpha_1, \alpha_1, x \rangle$ . If  $d_r(y) = \alpha_1 x$  then  $\alpha_1 y$  represents  $\underline{2}x$ . Also note that  $\alpha_1 \underline{2}x = \pm \beta_1 x$ .

In the 33-stem we have the Toda differential of 4.4.22. The element  $\alpha_1 \beta_{3/2}$  is a permanent cycle giving  $\underline{2}\beta_1^3$ . The coboundary of  $\frac{v_1 v_2^3}{9v_1^3}$  gives

$$(7.5.6) \quad \langle \alpha_1, \alpha_1, \beta_1^3 \rangle \langle \beta_2, 3, \beta_1 \rangle.$$

The differentials shown in the 41-, 48-, 52-, and 55-stems can be computed algebraically; i.e., they correspond to relations in Ext. The elements  $\alpha_1 \beta_{3/2}$ ,  $\beta_1 \beta_{3/2}$ , and  $\beta_1 \beta_3$  are the coboundaries of

$$\frac{v_2^3}{9v_1}, \quad \frac{v_2^3 t_1^2}{9v_1} + \frac{v_2^3 t_2}{3v_1^3}, \quad \text{and} \quad \frac{v_1^2 v_2^3 t_1^3}{9v_1^3} - \frac{v_2^3 t_2}{3v_1^2},$$

respectively. We also have  $3(\underline{2}\beta_{3/2}) = \alpha_1 \beta_3$ , i.e.,  $\pi_{45}(S^0) = \mathbf{Z}/(9)$ .

For the differential in the 56-stem we claim  $\alpha_1 \eta_1 = \pm \beta_2 \beta_{3/3}$ , forcing  $d_5(\eta_1) = \pm \beta_1^3 \beta_2$  in the Adams–Novikov spectral sequence. The claim could be verified by direct calculation, but the following indirect argument is easier.  $\beta_2 \beta_{3/3}$  must be

nonzero and hence a multiple of  $\alpha_1\eta_1$  because  $\alpha_1\beta_1^3\beta_2 \neq 0$  in Ext and must be killed by a differential.

However, we will need the direct calculation in the future, so we record it now for general  $p$ . Consider the element

$$\frac{v_2^{i-1}v_3(t_2 - t_1^{1+p}) - v_2^i(t_3 - t_1t_2^p - t_2t_1^{p^2} + t_1^{1+p+p^2}) + v_2^{i+p-1}(t_1^{p+2} - t_1t_2)}{pv_1} + \frac{2v_2^{i+p}}{p^2v_1^p(i+p)} \sum_{0 < j < p} \frac{(-1)^j}{j} p^{j-1} v_1^{p-j} t_1^j \quad \text{with } i > 0.$$

Straightforward calculation shows the coboundary is

$$(7.5.7) \quad \frac{v_2^{i+1}b_{1,1}}{pv_1} + \frac{2p^{p-3}v_2^{i+p}b_{1,0}}{(i+p)v_1^p} - \frac{v_2^{i-1}(v_2t_2^p + v_2^p t_2 - v_2t_1^{p+p^2} - v_3t_1^p)|t_1}{pv_1},$$

which gives the desired result since the third term represents  $\eta_i$ . The second term is nonzero in our range only in the case  $i = p = 3$ , where we have  $\pm\alpha_1\eta_3 = \beta_4b_{1,1} + \beta_{6/3}b_{1,0}$ . This element is also the coboundary of  $\frac{v_2^6t_1^3}{9v_1^3} + \frac{v_2^5t_1^6}{3v_1^2}$ , so  $\alpha_3\eta_3 = 0$ .

For the differentials in the 57- and 60-stems we claim  $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_{3/3}\beta_1$  in Ext. This must be a permanent cycle since  $\beta_2$  is. It is straightforward that  $d_5(\beta_{3/3}^2) = \pm 2\beta_1^6$  in the Adams–Novikov spectral sequence, so we get  $d_5(\beta_4) = \pm 2\beta_1^5$ . Then  $\beta_1^6 = \alpha_1 2\beta_1^5 = 0$  in  $\pi_*(S^0)$ , so  $d_9(\alpha_1\beta_4) = \beta_1^6$ .

To verify our claim that  $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_1\beta_{3/3}^2$ , it suffices to compute in  $\text{Ext}(BP_*/I_2)$ . The mod  $I_2$  reductions of  $\beta_2$ ,  $\beta_4$ , and  $\beta_{3/3}$  are  $v_2b_{1,0} \pm k_0$ ,  $v_2^3b_{1,0}$ , and  $b_{1,1}$ , respectively, where  $k_0 = \langle h_{10}, h_{11}, h_{11} \rangle$ . A Massey product manipulation shows  $k_0^3 = b_{1,0}b_{1,1}^2$  and the result follows.

Now we will show

$$(7.5.8) \quad x_{68} = \langle \alpha_1, \beta_{3/2}, \beta_2 \rangle.$$

We can do this calculation in Ext and work mod  $I$ , i.e., in  $\text{Ext}_P$ , and it suffices to show that the indicate product is nonzero. We have

$$\begin{aligned} \langle h_{10}, h_{10}h_{12}, \langle h_{11}, h_{11}, h_{10} \rangle \rangle &= \langle h_{10}, h_{10}, h_{12} \langle h_{11}, h_{11}, h_{10} \rangle \rangle \\ &= \langle h_{10}, h_{10}, \langle h_{12}, h_{11}, h_{11} \rangle h_{10} \rangle \\ &= b_{1,0} \langle h_{12}, h_{11}, h_{11} \rangle \\ &= \langle b_{1,0}h_{12}, h_{11}, h_{11} \rangle = \langle b_{1,1}h_{11}, h_{11}, h_{11} \rangle \\ &= b_{1,1}^2 \neq 0. \end{aligned}$$

This element satisfies  $\beta_1x_{68} = \beta_2^3$ . To show  $\alpha_1x_{68} = 0$ , consider the coboundary of

$$\frac{v_2^2b_{2,0} \pm v_2v_3b_{1,0}}{3v_1} \pm \frac{v_1v_2^3b_{1,0}}{9v_1^3}.$$

Next we show that there is a nontrivial extension in the 75-stem. We have

$$\begin{aligned} \beta_1^3 \langle \alpha_1, \alpha_1, \beta_{3/2} \rangle &= \langle \beta_1^3, \alpha_1, \alpha_1 \rangle \beta_{3/2} \\ &= \langle \beta_2, 3, \beta_1 \rangle \beta_{3/2} \quad \text{by 7.5.6} \\ &= 3 \langle \beta_1, \beta_{3/2}, \beta_2 \rangle = 3 \langle \langle \alpha_1, \alpha_1, \alpha_1 \rangle, \beta_{3/2}, \beta_2 \rangle \\ &= 3 \langle \alpha_1, \beta_{3/2}, \beta_2 \rangle = 3 \langle x_{68} \rangle. \end{aligned}$$

For the differential in the 77-stem note that  $\alpha_1\beta_5$  is the coboundary of  $u_2 = \frac{v_2^6}{3v_1^4} + \frac{v_2^2v_3}{3v_1}$ .

This brings us to the 88-stem, where we need to show  $\beta_1^2x_{68} = 0$ . Since  $x_{68} = \langle \alpha_1, \alpha_1, \beta_1^5, \beta_1 \rangle$  we can show  $\beta_1^3x_{68} = 0$ . There is no element in the 99-stem other than  $\beta_1\eta_3$  to kill it, so the differential follows.

The differential in the 89-stem is similar to that in the 41-stem. The one in the 92-stem follows from 7.5.7.

In the 95-stem  $\alpha_1h_{11}\gamma_2$  the coboundary of  $\frac{v_3^2t_2}{3v_1v_2}$ . The differential in the 105-stem is a special case of 6.4.1. The others are straightforward. The resulting homotopy groups are shown in Table A3.4.

### 6. Computations for $p = 5$

We will apply the results and techniques of Section 9 to compute up to the 1000-stem for  $p = 5$ . Naturally the lists **I** and **O** are quite long. The length of **O**, i.e., the number of additive generators in coker  $J$  through dimension  $k$ , appears to be roughly a quadratic function of  $k$  in our range. The conventions of 7.5.4 and 7.5.5 are still in effect.

The highlight of the 5-primary calculation is the following result

7.6.1. THEOREM. *For  $p=5$ ,  $\beta_1^{17} \neq 0$  and there are Adams–Novikov differentials  $d_{33}(\gamma_3) = \beta_1^{18}$ . Consequently the Smith–Toda complex  $V(3)$  does not exist, and  $V(2)$  is not a ring spectrum.*  $\square$

7.6.2. CONJECTURE. *For  $p \geq 7$ ,  $\beta_1^{p^2-p} \neq 0$  and  $\beta_1^{p^2-p+1} = 0$ . Moreover  $\langle \gamma_3, \gamma_2, \dots, \gamma_2 \rangle = \beta_1^{(2p-1)(p-1)/2}$  where  $\gamma_2$  appears in the bracket  $(p-5)/2$  times.*  $\square$

We will prove 7.6.1 modulo certain calculations to be carried out below. First we give a classical argument due to Toda for  $\beta_1^{p^2-p+1} = 0$ . We know  $\alpha_1\beta_1^p = 0$  from Toda [2, 3]. It follows by bracket manipulations that  $w_i = \langle \alpha_1, \alpha_1, \dots, \alpha_1, \beta_1^{ip} \rangle$  is defined with  $(i+1)$  factors  $\alpha_1$  and  $1 \leq i \leq p-2$ . The corresponding ANSS element is  $\alpha_1\beta_{p/p}^i$ . Now since  $\beta_1 = \langle \alpha_1, \dots, \alpha_1 \rangle$  with  $p$  factors we have [using A1.4.6(c)]

$$\alpha_1w_{p-2} = \langle \alpha_1, \dots, \alpha_1 \rangle \beta_1^{p^2-2p} = \beta_1^{p^2-2p+1}.$$

Hence  $\beta_1^{p^2-p+1}$  is divisible by  $\alpha_1\beta_1^p$  and is therefore zero. The corresponding Adams–Novikov differential is  $d_r(\alpha_1\beta_{p/p}^{p-1}) = \beta_1^{p^2-p+1}$  with  $r = 2p^2 - 4p + 3$ .

We will give a more geometric translation of this argument for  $p = 5$ . Let  $X_i = T(1)^{iq} = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} \dots \cup_{\alpha_1} e^{iq}$ . The Toda bracket definition of  $\beta_1$  means there is a diagram

$$(7.6.3) \quad \begin{array}{ccc} S^{31} & \longrightarrow & X_3 \\ \alpha_1 \uparrow & & \uparrow \\ S^{38} & \xrightarrow{\beta_1} & S^0 \end{array}$$

where the cofiber of the top map is  $X_4$ . From  $\alpha_1\beta_1^5 = 0$  we get a diagram

$$\begin{array}{ccc} & \Sigma^{190} X_1 & \\ \uparrow & \searrow f & \\ S^{190} & \xrightarrow{\beta_1^5} & S^0. \end{array}$$

We smash this with itself three times and use the fact that  $X_3$  is a retract of  $X_1^3$  to get

$$\begin{array}{ccc} & \Sigma^{570} X_3 & \\ \uparrow & \searrow g & \\ S^{570} & \xrightarrow{\beta_1^{15}} & S^0. \end{array}$$

Combining this with 7.6.3 we get

$$\begin{array}{ccccc} & & S^{601} & \longrightarrow & \Sigma^{570} X_3 \\ & \nearrow 0 & \uparrow \alpha_1 & & \uparrow \\ S^{79q} & \xrightarrow{\beta_1^5} & S^{608} & \xrightarrow{\beta_1} & S^{570} \xrightarrow{\beta_1^{15}} S^0 \\ & & & & \uparrow \\ & & & & \Sigma^{570} X_3 \end{array}$$

so  $\beta_1^{21} = 0$ .

The calculation below shows that  $\alpha_1\beta_{5/5}^4$  is a linear combination of  $\beta_1^3\gamma_3$ ,  $\beta_1^3\beta_{14}$ , and  $\beta_1x_{761}$ , where

$$x_{761} = \langle \alpha_1\beta_3, \beta_4, \gamma_2 \rangle \in \text{Ext}^{7,768}.$$

Each factor of  $x_{761}$  is a permanent cycle, so  $x_{761}$  can fail to be one only if one of the products  $\alpha_1\beta_3\beta_4$  and  $\beta_4\gamma_2$  is nonzero in homotopy. But these products lie in stems 323 and 619 which are trivial, so  $x_{761}$  is a permanent cycle, as is  $\beta_1^3\beta_{14}$ . Since  $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$ , we must have  $d_{33}(\gamma_3) = \beta_1^{18}$  as claimed.

The nonexistence of  $\gamma_3$  as a homotopy element shows the Smith–Toda complex  $V(3)$  [satisfying  $BP_*(V(3)) = BP_*/I_4$ ] cannot exist for  $p = 5$ . If one computes the Adams–Novikov spectral sequence for  $V(2)$  through dimension 248, one finds that  $v_3 \in \text{Ext}^0$  is a permanent cycle; i.e.,  $v_3$  is realized by a map  $S^{248} \xrightarrow{f} V(2)$ . If  $V(2)$  were a ring spectrum we could use the multiplication to extend  $f$  to a self-map with cofiber  $V(3)$ , giving a contradiction.

Now we proceed with the calculation for  $p = 5$ .

**7.6.4. THEOREM.** *For  $p = 5$   $\mathbf{N}$  as in 7.1.18 has basis elements in dimensions indicated below, with notation as in 7.5.2.  $\eta_i$  denotes  $h_{1,1}u_{i-1}$ .*

38 $\beta_1$	222 $\beta_{5/2}$	277 $\eta_1$
86 $\beta_2$	230 $\beta_5$	278 $\beta_6$
134 $\beta_3$	$\beta_{5/1,2}$	324 $b_{2,0}\beta_2$
182 $\beta_4$	245 $h_{2,0}b_{1,1}$	325 $\eta_2$
198 $\beta_{5/5} = b_{1,1}$	253 $h_{2,0}\beta_{5/4}$	326 $\beta_7$
206 $\beta_{5/4}$	261 $h_{2,0}\beta_{5/3}$	372 $b_{2,0}\beta_3$
214 $\beta_{5/3}$	269 $h_{11}\beta_{5/1,2}$	373 $\eta_3$

374 $\beta_8$	594 $\beta_{5/5}^3$	717 $h_{2,0}u_9$
396 $\beta_{5/5}^2$	602 $\beta_{5/5}^2\beta_{5/4}^2$	724 $h_{11}\gamma_3$
404 $\beta_{5/5}\beta_{5/4}$	610 $b_{2,0}^2\beta_3$	725 $h_{2,0}\beta_{15/5}$
412 $\beta_{5/4}^2$	612 $b_{2,0}\beta_8$	732 $h_{2,0}\gamma_3$
420 $b_{2,0}\beta_4$	613 $\eta_8$	733 $h_{2,0}\beta_{15/4}$
422 $\beta_9$	614 $\beta_{13}$	741 $h_{2,0}\beta_{15/3}$
430 $u_4$	620 $b_{2,0}u_3$	749 $h_{11}\beta_{15/1,2}$
437 $\gamma_2$	628 $b_{1,1}u_4$	753 $b_{2,0}^2\eta_1$
438 $\beta_{10/5}$	635 $b_{1,1}\gamma_2$	754 $h_{2,0}b_{2,0}\eta_5$
443 $h_{2,0}b_{1,1}^2$	636 $b_{1,1}\beta_{10/5}$	755 $b_{2,0}\eta_6$
446 $\beta_{10/4}$	641 $h_{2,0}b_{1,1}^3$	756 $h_{2,0}\eta_{10}$
451 $h_{2,0}b_{1,1}\beta_{5/4}$	644 $\beta_{5/4}\beta_{10/5}$	757 $\eta_{11}$
454 $\beta_{10/3}$	649 $h_{2,0}b_{1,1}^2\beta_{5/4}$	758 $\beta_{16}$
459 $h_{2,0}b_{1,1}\beta_{5/3}$	652 $\beta_{10/5}\beta_{5/3}$	761 $b_{2,0}\beta_2\gamma_2$
462 $\beta_{10/2}$	659 $h_{2,0}b_{2,0}\beta_8$	771 $\beta_2\gamma_3$
469 $\eta_5$	660 $b_{2,0}\beta_9$	792 $\beta_{5/5}^4$
470 $\beta_{10}$	662 $\beta_{14}$	800 $b_{2,0}^3\beta_2$
$\beta_{10/1,2}$	667 $h_{2,0}b_{2,0}u_3$	802 $b_{2,0}^2\beta_7$
477 $h_{2,0}u_4$	670 $u_9$	803 $b_{2,0}\eta_7$
476 $h_{11}\gamma_2$	675 $h_{2,0}b_{1,1}u_4$	804 $b_{2,0}\beta_{12}$
484 $h_{2,0}\gamma_2$	678 $\beta_{15/5}$	805 $\eta_{12}$
485 $h_{2,0}\beta_{10/5}$	682 $h_{2,0}b_{1,1}\gamma_2$	806 $\beta_{17}$
493 $h_{2,0}\beta_{10/4}$	683 $h_{2,0}b_{1,1}\beta_{10/5}$	810 $b_{2,0}^2u_2$
501 $h_{2,0}\beta_{10/3}$	685 $\gamma_3$	818 $b_{1,1}b_{2,0}u_3$
509 $h_{11}\beta_{10/1,2}$	686 $\beta_{15/4}$	826 $b_{1,1}^2u_4$
515 $b_{2,0}\eta_1$	691 $h_{2,0}b_{1,1}\beta_{10/4}$	833 $b_{1,1}^2\gamma_2$
516 $h_{2,0}\eta_5$	694 $\beta_{15/3}$	834 $b_{1,1}^2\beta_{10/5}$
517 $\eta_6$	699 $\eta_1\beta_9$	839 $h_{2,0}b_{1,1}^4$
518 $\beta_{11}$	702 $\beta_{15/2}$	842 $b_{1,1}^2\beta_{10/4}$
523 $\beta_2\gamma_2$	706 $h_{2,0}b_{1,1}b_{2,0}u_3$	849 $h_{2,0}b_{2,0}^2\beta_7$
562 $b_{2,0}^2\beta_2$	707 $b_{2,0}\eta_5$	850 $b_{2,0}^2\beta_8$
563 $b_{2,0}\eta_2$	709 $\eta_{10}$	852 $b_{2,0}\beta_{13}$
564 $b_{2,0}\beta_7$	710 $\beta_{15}$	873 $\eta_{13}$
565 $\eta_7$	$\beta_{15/1,2}$	854 $\beta_{18}$
566 $\beta_{12}$	714 $h_{11}b_{2,0}\gamma_2$	857 $h_{2,0}b_{2,0}^2u_2$



860 $b_{2,0}u_8$	918 $\beta_{20/5}$	962 $h_{11}b_{2,0}\gamma_3$
865 $h_{2,0}b_{1,1}b_{2,0}u_3$	923 $h_{2,0}b_{1,1}\beta_{15/5}$	965 $h_{2,0}\beta_{20/5}$
868 $b_{1,1}u_9$	926 $\beta_{20/4}$	972 $h_{11}\gamma_4$
873 $h_{2,0}b_{1,1}^2u_4$	930 $h_{2,0}b_{1,1}\gamma_3$	980 $h_{2,0}\gamma_4$
876 $\beta_{10/5}^2$	931 $h_{2,0}b_{1,1}\beta_{15/4}$	981 $h_{2,0}\beta_{20/3}$
880 $h_{2,0}b_{1,1}^2\gamma_2$	933 $\gamma_4$	989 $h_{11}\beta_{10/1,2}$
881 $h_{2,0}b_{1,1}^2\beta_{10/5}$	934 $\beta_{20/3}$	992 $h_{2,0}b_{2,0}^2\eta_5$
884 $\beta_{5/4}\beta_{15/5}$	939 $\eta_1\beta_{14}$	993 $b_{2,0}^2\eta_6$
889 $h_{2,0}b_{1,1}^2\beta_{10/4}$	942 $\beta_{20/2}$	994 $h_{2,0}b_{2,0}\eta_{10}$
892 $\beta_{10/4}^2$	944 $h_{2,0}h_{11}b_{2,0}^2u_3$	995 $b_{2,0}\eta_{11}$
896 $h_{2,0}h_{11}b_{2,0}^2u_2$	945 $b_{2,0}^2\eta_5$	996 $h_{2,0}\eta_{15}$
897 $h_{11}b_{2,0}^2u_3$	946 $h_{2,0}h_{11}b_{2,0}u_8$	997 $\eta_{16}$
899 $h_{2,0}b_{2,0}\beta_{13}$	947 $b_{2,0}\eta_{10}$	998 $\beta_{21}$
900 $b_{2,0}\beta_{14}$	949 $\eta_{15}$	$\beta_{25/25}$
902 $\beta_{19}$	950 $\beta_{20}$	999 $b_{2,0}^2\beta_2\gamma_2$
907 $h_{2,0}b_{2,0}u_8$	$\beta_{20/1,2}$	1000 $b_{2,0}^2u_1$ <span style="float: right;">□</span>
910 $u_{14}$	952 $h_{11}b_{2,0}^2\gamma_2$	
915 $h_{2,0}b_{1,1}u_9$	957 $h_{2,0}u_{14}$	

Now we will describe the list  $\mathbf{O}$ , i.e., the analog of 7.5.3. The notation of that result is still in force, and we assume the reader is familiar with techniques used there. We will not comment on differentials with an obvious 3-primary analog, in particular on those following from 7.5.7. Many differentials we encounter are periodic under  $v_2$  or  $v_2^p$ .

Since the list  $\mathbf{O}$  is quite long, we will give it in six installments, pausing for comments and proofs when appropriate.

7.6.5. THEOREM. *For  $p = 5$  the list  $\mathbf{O}$  (7.1.18) is as follows. (First installment)*

38 $\underline{\beta}_1$	131 $\alpha_1\beta_1\beta_2$	182 $\underline{\beta}_4$
45 $\alpha_1\beta_1$	134 $\underline{\beta}_3$	189 $\alpha_1\beta_4$
76 $\beta_1^2$	141 $\alpha_1\beta_3$	190 $\beta_1^5$
83 $\alpha_1\beta_1^2$	152 $\beta_1^4$	197 $[\alpha_1\beta_1^5]\beta_{5/5}$
86 $\underline{\beta}_2$	159 $\alpha_1\beta_1^4$	200 $\beta_1^3\beta_2$
93 $\alpha_1\beta_2$	162 $\beta_1^2\beta_2$	205 $\underline{2}\beta_1^5$
114 $\beta_1^3$	169 $\alpha_1\beta_1^2\beta_2$	206 $\underline{\beta}_{5/4}$
121 $\alpha_1\beta_1^3$	172 $\beta_1\beta_3$	207 $\alpha_1\beta_1^3\beta_2$
124 $\beta_1\beta_2$	179 $\alpha_1\beta_1\beta_3$	210 $\beta_1^2\beta_3$

213 $\alpha_1\beta_{5/4}$	276 $[\beta_1^5\beta_2]\eta_1$	371 $[\alpha_1\beta_1\beta_7]\underline{b_{2,0}\beta_3}$
214 $\underline{\beta_{5/3}}$	278 $\underline{\beta_6}$	372 $[\beta_1^5\beta_4]\eta_3$
217 $\alpha_1\beta_1^2\beta_3$	281 $\underline{2}\beta_1^7$	374 $\underline{\beta_8}$
220 $\beta_1\beta_4$	283 $[\alpha_1\beta_1^4\beta_2]\alpha_1\eta_1$	379 $[\alpha_1\beta_1^5\beta_4]\alpha_1\eta_3$
221 $\alpha_1\beta_{5/3}$	285 $\alpha_1\beta_6$	$\underline{2}\beta_1\beta_7$
222 $\underline{\beta_{5/2}}$	286 $\beta_1^4\beta_3$	380 $\beta_1^{10}$
227 $\alpha_1\beta_1\beta_4$	293 $\alpha_1\beta_1^4\beta_3$	381 $\alpha_1\beta_8$
228 $\underline{\beta_1^6}$	296 $\beta_1^3\beta_4$	382 $\beta_1^3\beta_2\beta_4$
229 $[\alpha_1\beta_{5/2}]\underline{\beta_{5/1,2}}$	303 $\alpha_1\beta_1^3\beta_4$	389 $\alpha_1\beta_1^3\beta_2\beta_4$
230 $\underline{\beta_5}$	304 $\beta_1^8$	392 $\beta_1^3\beta_6$
237 $\underline{2}\beta_{5/2}$	306 $\beta_1\beta_2\beta_4$	395 $[\underline{2}\beta_1^{10}]\underline{\beta_{5/5}^2}$
$\alpha_1\beta_5$	313 $\alpha_1\beta_1\beta_2\beta_4$	402 $\beta_1^2\beta_7$
238 $\beta_1^4\beta_2$	316 $\beta_1\beta_6$	403 $\underline{3}\beta_1^{10}$
243 $\underline{2}\beta_1^6$	319 $\underline{2}\beta_1^8$	404 $\underline{\beta_{5/5}\beta_{5/4}}$
244 $[\beta_1\beta_{5/4}]\underline{h_{2,0}b_{1,1}}$	323 $[\alpha_1\beta_1\beta_6]\underline{b_{2,0}\beta_2}$	$= \langle \alpha_1\beta_1^4, \beta_1, \beta_{5/4} \rangle$
245 $\alpha_1\beta_1^4\beta_2$	324 $[\beta_1^5\beta_3]\underline{\eta_2}$	$= x_{404}$
248 $\beta_1^3\beta_3$	326 $\beta_7$	407 $\underline{2}\beta_1^3\beta_6$
251 $[\alpha_1\beta_1\beta_{5/4}]\alpha_1h_{2,0}b_{1,1}$	331 $[\alpha_1\beta_1^5\beta_3]\alpha_1\eta_2$	411 $\alpha_1x_{404} = \beta_{5/4}\underline{2}\beta_1^5$
252 $[\beta_1\beta_{5/3}]\underline{h_{2,0}\beta_{5/4}}$	$\underline{2}\beta_1\beta_6$	412 $\beta_1\beta_8$
255 $\alpha_1\beta_1^3\beta_3$	333 $\alpha_1\beta_7$	$\beta_1\beta_8 + \underline{\beta_{5/4}^2} = x_{412}$
258 $\beta_1^2\beta_4$	334 $\beta_1^4\beta_4$	417 $\underline{2}\beta_1^2\beta_7$
259 $[\alpha_1\beta_1\beta_{5/3}]\alpha_1h_{2,0}\beta_{5/4}$	341 $\alpha_1\beta_1^4\beta_4$	418 $\beta_1^{11}$
260 $[\beta_1\beta_{5/2}]\underline{h_{2,0}\beta_{5/3}}$	342 $\beta_1^9$	419 $[\alpha_1x_{412}]\underline{b_{2,0}\beta_4}$
265 $\alpha_1\beta_1^2\beta_4$	344 $\beta_1^2\beta_2\beta_4$	$\alpha_1\beta_1\beta_8$
266 $\beta_1^7$	351 $\alpha_1\beta_1^2\beta_2\beta_4$	420 $\beta_1^4\beta_2\beta_4$
268 $[\beta_1\beta_5]\underline{h_{11}\beta_{5/1,2}}$	354 $\beta_1^2\beta_6$	422 $\beta_9$
$\beta_2\beta_4$	357 $\underline{2}\beta_1^9$	427 $\underline{2}x_{412}$
275 $[\alpha_1\beta_1\beta_5]\alpha_1h_{11}\beta_{5/1,2}$	364 $\beta_1\beta_7$	$\alpha_1\beta_1^4\beta_2\beta_4$
	369 $\underline{2}\beta_1^2\beta_6$	429 $[\alpha_1\beta_9]\underline{u_4}$

□

7.6.6. REMARK. The small descent spectral sequences of 7.1.13 and 7.1.16 have some useful multiplicative structure even though  $T(0)_{(1)}$  (the complex with  $p$  cells) is not a ring spectrum and its  $BP$ -homology is not a comodule algebra. Recall that  $T(0)_i$  is the  $iq$ -skeleton of  $T(1)$ . Then  $\pi_*(T(0)_{(1)})$  is filtered by the images of  $\pi_*(T(0)_i)$  for  $i \leq p-1$ . One has maps  $T(0)_i \wedge T(0)_j \rightarrow T(0)_{i+j}$  inducing pairings  $F_i \otimes F_j \rightarrow F_{i+j}$  for  $i+j \leq p-1$ . spectral sequence differentials always lower this filtration degree and respect this pairing. The filtration can be dualized as follows.

A map  $S^m \rightarrow T(0)_i$  is dual to a map  $\Sigma^{m-iq}T(0)_i \rightarrow S^0$  since  $DT(0)_i = \Sigma^{-iq}T(0)_i$  for  $i \leq p-1$ . An element in  $\pi_m(T(0)_{p-1})$  is in  $F_i$  iff the diagram

$$\begin{array}{ccc} S^m & \longrightarrow & T(0)_{p-1} \\ \uparrow & & \uparrow \\ \Sigma^{m-qi}T(0)_i & \dashrightarrow & S^0 \end{array}$$

can be completed. The pairing  $T(0)_i \wedge T(0)_j \rightarrow T(0)_{i+j}$  dualizes to  $DT(0)_{i+j} \rightarrow DT(0)_i \wedge DT(0)_j$ . If  $\alpha \in \pi_m(T(0)_{p-1})$  is in  $F_i$  and  $\beta \in \pi_n(T(0)_{p-1})$  is in  $F_j$  with  $i+j = p$ , then we get a map  $\Sigma^{m+n}DT(0)_p \rightarrow S^0$ . If this map is trivial on the bottom cell then it factors through  $\Sigma^{m+n}DT(0)_{p-1} = \Sigma^{m+n-q(p-1)}T(0)_{p-1}$ . This factorization will often lead to a differential in our spectral sequence.

For the differentials in dimensions 323, 371, and 419 recall (4.3.22) that there is an element  $b_{2,0} \in C(BP_*/I_2)$  with  $d(b_{2,0}) = (b_{1,0}|t_1^{p^2}) - (t_1^9|b_{1,1})$ . Since  $b_{1,0}$  and  $t_1^{p^2}$  are both cycles there is a  $y \in C(BP_*/I_2)$  such that  $d(y) = (b_{1,0}|t_1^{p^2}) - (t_1^{p^2}|b_{1,0})$ . Hence the coboundary of

$$\frac{v_2^{i-p}v_1b_{1,0} + v_2^{i+p-1}(y - b_{2,0})}{pv_1} - \frac{v_2^{i+2-p}b_{1,1}}{(1+2-p)pv_1^2} \quad \text{for } i \geq p$$

is

$$\frac{v_2^i t_1 | b_{1,0}}{pv_1} + \frac{2v_2^{i+2-p} | b_{1,1}}{(i+2-p)v_1^3},$$

where the second term is nonzero only if  $i \equiv -2 \pmod{p}$ . This gives

$$(7.6.7) \quad \alpha_1 \beta_1 \beta_i = \begin{cases} 0 & \text{for } i \geq p, i \not\equiv -2 \pmod{p} \\ \alpha_1 \beta_{(i+2-p)/4} \beta_{p/p-1} & \text{for } i \equiv -2. \end{cases}$$

Remember (7.5.5) we are not keeping track of nonzero scalar coefficients. The differentials in question follow.

Next we show that there is a nontrivial group extension in the 427-stem, similar to that for  $p = 3$  in the 75-stem. We want to prove  $\alpha_1 \beta_1^4 \beta_2 \beta_4 = 5\underline{2}x_{412}$ . Since  $\alpha_1 \beta_2 \beta_4 = \beta_1 \underline{2} \beta_{5/2}$  we need to look at  $\beta_1^5 \underline{2} \beta_{5/2}$ . We have

$$\begin{aligned} \beta_1^5 \underline{2} \beta_{5/2} &= \beta_1^5 \langle \alpha_1, \alpha_1, \beta_{5/2} \rangle = \beta_1^5 \langle \alpha_1, \alpha_3, \beta_{5/4} \rangle = \langle \beta_1^5, \alpha_1, \alpha_3 \rangle \beta_{5/4} \\ &= \alpha_1 \beta_{5/3} \beta_{5/4} = \alpha_1 \langle \alpha_1, 5, \beta_{5/4} \rangle \beta_{5/4} = \alpha_1 \langle \alpha_1, 5, \beta_{5/4}^2 \rangle \\ &= \alpha_1 \langle \alpha_1, 5, x_{412} \rangle = x_{412} \langle \alpha_1, \alpha_1, 5 \rangle = 5\underline{2}x_{412}. \end{aligned}$$

More generally one has

$$(7.6.8) \quad \beta_1^p \underline{2} \beta_{p/2} = p\underline{2}(\beta_{p/4} \beta_{p/p-1} + \beta_1 \beta_{2p-2}).$$

Since

$$\alpha_1 \langle \alpha_1, 5, x_{412} \rangle = \beta_1^5 \underline{2} \beta_{5/2} = \alpha_1 \beta_1^4 \beta_2 \beta_4$$

we have

$$\langle \alpha_1, 5, x_{412} \rangle = \beta_1^4 \beta_2 \beta_4$$

7.6.5 (Second installment)

430 $\beta_1^4 \beta_6$	479 $\underline{3} \beta_1^{12}$
437 $\underline{2} \beta_9$	482 $\alpha_1 \beta_1 \gamma_2$
$\gamma_2$	483 $\alpha_1 \beta_1 \beta_{10/5}$
438 $\underline{\beta_{10/5}}$	$[2\beta_1^5 \beta_6] \alpha_1 h_{2,0} u_4$
440 $\underline{3} \beta_1^4$	483 $[\alpha_1 \underline{5} \gamma_2] h_{2,0} \gamma_2$
442 $[\beta_1 x_{404}] h_{2,0} b_{1,1}^2$	484 $[\beta_1 \beta_{10/4}] \underline{h_{2,0} \beta_{10/5}}$
444 $\alpha_1 \gamma_2$	488 $\beta_1^3 \beta_8$
445 $\alpha_1 \beta_{10/5}$	491 $\underline{2} \underline{5} \gamma_2$
$\underline{2} \beta_1^4 \beta_6$	$[\alpha_1 \beta_1 \beta_{10/4}] \alpha_1 h_{2,0} \beta_{10/5}$
446 $\underline{\beta_{10/4}}$	492 $[\beta_1 \beta_{10/3}] \underline{h_{2,0} \beta_{5/4}}$
449 $[\alpha_1 \beta_1 x_{404}] \alpha_1 h_{2,0} b_{1,1}^2$	493 $\underline{2} \beta_1^4 \beta_7$
450 $\beta_1^2 \beta_8$	494 $\beta_1^{13}$
$[\beta_1 \beta_{5/4}^2] \underline{h_{2,0} b_{1,1} \beta_{5/4}}$	498 $\beta_1^2 \beta_9$
453 $\alpha_1 \beta_{10/4}$	499 $[\alpha_1 \beta_1 \beta_{10/3}] \alpha_1 h_{2,0} \beta_{5/4}$
454 $\underline{\beta_{10/3}}$	500 $[\beta_1 \beta_{10/2}] \underline{h_{2,0} \beta_{10/3}}$
455 $\underline{2} \beta_1^3 \beta_7$	503 $\underline{2} \beta_1^3 \beta_8$
456 $\beta_1^{12}$	508 $\alpha_1 h_{2,0} \beta_{10/3} = \beta_2 \beta_9$
457 $[\alpha_1 \beta_1^2 \beta_8] \alpha_1 h_{2,0} b_{1,1} \beta_{5/4}$	$[\beta_1 \beta_{10}] \underline{h_{11} \beta_{10/1,2}}$
458 $[\beta_1^5 \beta_2 \beta_4] \underline{h_{2,0} b_{1,1} \beta_{5/3}}$	513 $\beta_1^2 \gamma_2$
460 $\beta_1 \beta_9$	$\underline{2} \beta_1^2 \beta_9$
461 $\alpha_1 \beta_{10/3}$	514 $\beta_1^2 \beta_{10/5}$
462 $\underline{\beta_{10/2}}$	515 $[\alpha_1 \beta_2 \beta_9] \underline{h_{2,0} \eta_5}$
465 $\underline{2} \beta_1^2 \beta_8$	$[\beta_1^2 \beta_{10/5} + \beta_1 \underline{5} \gamma_2] \underline{b_{2,0} \eta_1}$
$[\alpha_1 \beta_1^5 \beta_2 \beta_4] \alpha_1 h_{2,0} b_{1,1} \beta_{5/3}$	$[\alpha_1 \beta_1 \beta_{10}] \alpha_1 h_{11} \beta_{10/1,2}$
468 $[\beta_1^5 \beta_6] \eta_5$	516 $[\beta_1^5 \beta_7] \eta_6$
469 $[\alpha_1 \beta_{10/2}] \underline{\beta_{10/1,2}}$	517 $\underline{3} \beta_1^{13}$
470 $\underline{\beta_{10}}$	518 $\underline{\beta_{11}}$
475 $\beta_1 \gamma_2$	520 $\alpha_1 \beta_1^2 \gamma_2$
$\underline{2} \beta_1 \beta_9$	521 $[\alpha_1 \beta_1^2 \beta_{10/5}] \alpha_1 b_{2,0} \eta_1$
476 $[\alpha_1 \eta_5 + \beta_1 \beta_{10/5}] h_{2,0} u_4$	523 $\underline{2} \beta_2 \beta_9$
$\beta_1 \beta_{10/5}$	$\underline{\beta_2 \gamma_2}$
$\underline{5} \gamma_2$	524 $\alpha_1 \eta_6 = \beta_2 \beta_{10/5}$
477 $\underline{2} \beta_{10/2}$	525 $\alpha_1 \beta_{11}$
$\alpha_1 \beta_{10}$	526 $\beta_1^4 \beta_8$
478 $\beta_1^4 \beta_7$	529 $\underline{2} \underline{5} \beta_1 \gamma_2$

530 $\alpha_1\beta_2\gamma_2$	556 $\beta_1\beta_{11}$	
531 $\underline{2}\beta_1^5\beta_7 = \alpha_1\beta_2\beta_{10/5}$	561 $\underline{2}\beta_1\beta_2\beta_9$	
532 $\beta_1^{14}$	$[\beta_1\beta_2\gamma_2 + \underline{2}\beta_1\beta_2\beta_9]\underline{b_{2,0}^2\beta_2}$	
536 $\beta_1^3\beta_9$	562 $[\beta_1\beta_2\beta_{10/5}]\underline{b_{2,0}\eta_2}$	
541 $\underline{2}\beta_1^4\beta_8$	563 $[\alpha_1\beta_1\beta_{11}]\underline{b_{2,0}\beta_7}$	
546 $\beta_1\beta_2\beta_9$	564 $[\beta_1^5\beta_8]\underline{\eta_7}$	
551 $\beta_1^3\gamma_2$	566 $\underline{\beta_{12}}$	
$\underline{2}\beta_1^3\beta_9$	567 $\underline{2}\underline{5}\beta_1^2\gamma_2$	
552 $\underline{5}\beta_1^2\gamma_2$	568 $[\alpha_1\beta_1\beta_2\gamma_2]\alpha_1\underline{b_{2,0}^2\beta_2}$	
555 $\underline{3}\beta_1^{14}$	569 $[\alpha_1\beta_1\beta_2\beta_{10/5}]\alpha_1\underline{b_{2,0}\eta_2}$	□
558 $\alpha_1\beta_1^3\gamma_2$		

For the differential in the 514-stem, note that in the corresponding spectral sequence for  $\text{Ext}_P(\mathbf{Z}/(p), \mathbf{Z}/(p))$  the image of  $b_{2,0}\eta_1$  kills that of  $\beta_1\underline{5}\gamma_2$ , so the target in our spectral sequence of  $b_{2,0}\eta_1$  is  $\beta_1\underline{5}\gamma_2$  plus some multiple of  $\beta_1^2\beta_{10/5}$ . On the other hand, we have

$$\begin{aligned}
 \alpha_1\beta_1^2\beta_{10/5} &= \alpha_1\beta_1\beta_6\beta_{5/5} \\
 &= \alpha_1\beta_1\langle\beta_6, \alpha_1\beta_1, \beta_1^4\rangle \\
 &= \langle\alpha_1\beta_1\beta_6, \alpha_1\beta_1, \beta_1^4\rangle \\
 &= 0
 \end{aligned}$$

and the result follows.

The relation in the 524-stem follows from 7.5.7. The differential in dimension 561 is  $h_{2,0}$  times that in the 514-stem. The one in dimension 562 comes from a relation in Ext, i.e.,  $\beta_1^2\beta_2\beta_{10/5} = \beta_1^2\beta_7\beta_{5/5} = \beta_1\beta_6\alpha_1\eta_1 = 0$  since  $\alpha_1\beta_1\beta_6 = 0$ . Theorem 7.6.4 shows that there is no element in dimension 601 to give this relation, so we must have  $\beta_1\beta_2\beta_{10/5}$  as indicated.

More generally, we have in Ext for  $1 < i < p$  and  $j > 1$

$$\begin{aligned}
 (7.6.9) \quad \beta_1^2\beta_i\beta_{pi/p} &= \beta_1\beta_{i+p}\beta_{i+pj-2p}\beta_{p/p} \\
 &= \beta_1\beta_{1+p}\alpha_1\eta_{i+pj-1-2p} && \text{by 7.5.7} \\
 &= 0 && \text{by 7.6.7.}
 \end{aligned}$$

In some cases this result along with inspection of  $\underline{I}$  implies  $\beta_1\beta_i\beta_{pj/p} = 0$ .

7.6.5 (Third installment)

570 $\beta_1^{15}$	584 $\beta_1^2\beta_2\beta_9$
571 $\underline{2}\beta_1\beta_{11}$	589 $\beta_1^4\gamma_2$
572 $\alpha_1\eta_7 = \beta_3\beta_{10/5}$	$\underline{2}\beta_1^4\beta_9$
573 $\alpha_1\beta_{12}$	590 $\underline{5}\beta_1^3\gamma_2$
574 $\beta_1^4\beta_9$	593 $[\underline{3}\beta_1^{15}]\underline{\beta_{5/5}^3}$
579 $\underline{2}\beta_1^5\beta_8 = \alpha_1\beta_3\beta_{10/5}$	594 $\beta_1^2\beta_{11}$

596 $\alpha_1\beta_1^4\gamma_2$	617 $x_{617} = \alpha_1\beta_3b_{2,0}^2$
599 $\underline{2}\beta_1^2\beta_2\beta_9$	$\alpha_1\beta_1\beta_3\beta_{10/5}$
601 $\underline{4}\beta_1^{15}$	619 $[2\beta_1\beta_{12}]\underline{b_{2,0}u_3}$
602 $\underline{\beta_{5/5}^2\beta_{5/4}} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4} \rangle = x_{602}$	620 $\alpha_1\eta_8 = \beta_4\beta_{10/5}$
604 $\beta_1\beta_{12}$	621 $\alpha_1\beta_{13}$
605 $\underline{2}\underline{5}\beta_1^3\gamma_2$	622 $\beta_1^3\beta_2\beta_9$
608 $\beta_1^{16}$	627 $\underline{3}\beta_1\beta_{12}$
609 $\alpha_1x_{602}$	$\beta_1^5\gamma_2$
$[\underline{2}\beta_1^2\beta_{11} + \alpha_1x_{602}]b_{2,0}^2\beta_3$	$[\alpha_1\beta_4\beta_{10/5}]\underline{b_{1,1}u_4}$
610 $\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, 5, x_{602} \rangle$	628 $\underline{5}\beta_1^4\gamma_2$
611 $[\alpha_1\beta_1\beta_{12}]\underline{b_{2,0}\beta_8}$	632 $\beta_1^3\beta_{11}$
612 $[\beta_1^5\beta_9]\underline{\eta_8}$	634 $[\alpha_1\beta_1^5\gamma_2]\underline{b_{1,1}\gamma_2}$
614 $\underline{\beta_{13}}$	

□

The differential in the 609-stem is an Ext relation derived as follows. Since  $x_{602}$  is divisible in Ext by  $\beta_{5/4}$  we have  $d(h_{2,0}b_{1,1}^3) = \beta_1x_{602}$ , so  $d(\alpha_1h_{2,0}b_{1,1}^3) = \alpha_1\beta_1x_{602}$ . On the other hand, whenever  $\alpha_1x = \alpha_1y = 0$ ,  $\underline{2}xy = 0$ , e.g.,  $\underline{2}(\beta_1\beta_6)^2 = \underline{2}\beta_1^3\beta_{11} = 0$ , forcing the image of  $b_{2,0}^2\beta_3$  to contain a nonzero multiple of  $\underline{2}\beta_1^2\beta_{11}$ . Similary

$$(7.6.10) \quad \underline{2}\beta_1^3\beta_k = 0 \quad \text{for all } k \geq 2p+1.$$

In many cases (such as  $k = 12$ ) inspection of  $\underline{N}$  (7.6.4) shows  $\underline{2}\beta_1^2\beta_k = 0$ . To get the other term we compute modulo filtration 2 in our spectral sequence (7.1.16), i.e., mod  $\beta_{10}$ . Then we get  $\langle h_{11}, h_{11}, b_{1,1}^2 \rangle$  is killed by  $b_{2,0}^2$  in  $\text{Ext}(BP_*/I_2)$ , so  $\beta_3b_{2,0}^2$  kills  $\langle \beta_3, h_{11}, h_{11} \rangle b_{1,1}^2$  and the coboundary of  $\frac{v_2^5v_1^2}{25v_1^5}$  shows  $\langle \beta_3, h_{11}, h_{11} \rangle = \beta_{5/4}\alpha_1$ .

There is a nontrivial group extension in dimension 617 similar to the one in the 427-stem. We have

$$x_{617} = \left\langle \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6), \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \right\rangle$$

so

$$\begin{aligned} 5x_{617} &= \langle 5, \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6) \rangle \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \\ &= \langle 5, \alpha_1, \alpha_1 \rangle x_{602} = \alpha_1x_{602} \\ &= \alpha_2 \langle \alpha_1, \beta_1^5, x_{404} \rangle \\ &= \langle \alpha_2, \alpha_1, \beta_1^5 \rangle x_{404}. \end{aligned}$$

On the other hand

$$\begin{aligned} \beta_1^5 \langle \alpha_1, \alpha_1, x_{412} \rangle &= \beta_1^5 \left\langle \alpha_1, (\alpha_2\alpha_1\beta_2), \begin{pmatrix} x_{404} \\ \beta_7 \end{pmatrix} \right\rangle \\ &= \langle \beta_1^5, \alpha_1, (\alpha_2\alpha_1\beta_2) \rangle \begin{pmatrix} x_{404} \\ \beta_7 \end{pmatrix} \\ &= \langle \beta_1^5, \alpha_1, \alpha_2 \rangle x_{404} \end{aligned}$$

so the result follows. We also have  $\alpha_2 x_{602} = \alpha_1 \langle \alpha_1, 5, x_{602} \rangle$  so  $\langle \alpha_1, 5, x_{602} \rangle = \beta_1 \beta_3 \beta_{10/5}$ .

In the 627-stem we have an Ext relation

$$\alpha_1 \beta_4 \beta_{10/5} = \alpha_1 \beta_9 \beta_{5/5} = 0.$$

### 7.6.5 (Fourth installment)

635 $\underline{2}\beta_4 \beta_{10/5} = \langle \beta_9, \alpha_1, \underline{2}\beta_1^5 \rangle$	674 $[\beta_1 x_{636}] \underline{h_{2,0} b_{1,1} u_4}$
636 $\underline{\beta_{5/5} \beta_{10/5}} = \langle \beta_1^3, \alpha_1 \beta_1^2, \beta_{10/5} \rangle = x_{636}$	675 $\underline{2}\beta_1^4 \beta_2 \beta_9$
637 $\underline{2}\beta_1^3 \beta_2 \beta_9$	677 $\underline{2}\beta_{14}$
639 $\underline{4}\beta_1^{16}$	$\underline{4}\beta_1^{17}$
640 $[\beta_1 x_{602}] \underline{h_{2,0} b_{1,1}^3}$	678 $\underline{\beta_{15/5}}$
642 $\beta_1^2 \beta_{12}$	680 $\beta_1^3 \beta_{12}$
643 $\underline{2}\underline{5}\beta_1^4 \gamma_2 = \beta_{5/4} \gamma_2$	$\underline{2}\beta_1^6 \gamma_2$
$\alpha_1 x_{636}$	681 $[\alpha_1 \beta_1 x_{636}] \alpha_1 h_{2,0} b_{1,1} u_4$
644 $\underline{\beta_{5/4} \beta_{10/5}}$	$[\underline{2}\underline{5}\beta_1^5 \gamma_2] \underline{h_{2,0} b_{1,1} \gamma_2}$
646 $\beta_1^{17}$	682 $[\beta_1 \beta_{5/4} \beta_{10/5}] \underline{h_{2,0} b_{1,1} \beta_{10/5}}$
647 $[\underline{2}\beta_1^3 \beta_{11}] \alpha_1 h_{2,0} b_{1,1}^3$	684 $[\beta_1^{18}] \gamma_3$
648 $[\beta_1^2 \beta_3 \beta_{10/5}] \underline{h_{2,0} b_{1,1}^2 \beta_{5/4}}$	$\alpha_1 \beta_1 \beta_{13}$
651 $\alpha_1 \beta_{5/4} \beta_{10/5}$	685 $\alpha_1 \beta_{15/5}$
652 $\beta_1 \beta_{13}$	686 $\underline{\beta_{15/4}}$
$\underline{\beta_{5/3} \beta_{10/5}} + \beta_1 \beta_{13} = x_{652}$	689 $\underline{3}\underline{5}\beta_1^6 \gamma_2$
655 $[\alpha_1 \beta_1^2 \beta_3 \beta_{10/5}] \alpha_1 h_{2,0} b_{1,1}^2 \beta_{5/4}$	$[\alpha_1 \beta_1 \beta_{5/4} \beta_{10/5}] \alpha_1 h_{2,0} b_{1,1} \beta_{10/5}$
$\beta_1 x_{617}$	690 $\beta_1^2 \beta_{13}$
658 $[\beta_1 \beta_4 \beta_{10/5}] \underline{h_{2,0} b_{2,0} \beta_8}$	$[\beta_1 \beta_{5/3} \beta_{10/5}] \underline{h_{2,0} b_{1,1} \beta_{10/4}}$
659 $[\alpha_1 x_{652}] b_{2,0} \beta_9$	692 $\alpha_1 \gamma_3 = \langle \alpha_1, \beta_1^5, \beta_1^{13} \rangle = x_{692}$
660 $\beta_1^4 \beta_2 \beta_9$	693 $\alpha_1 \beta_{15/4}$
662 $\underline{\beta_{14}}$	$\underline{3}\beta_1^4 \beta_{11}$
665 $\underline{3}\beta_1^2 \beta_{12}$	694 $\underline{\beta_{15/3}}$
$\beta_1^6 \gamma_2$	697 $[\alpha_1 \beta_1 \beta_{5/3} \beta_{10/5}] \alpha_1 h_{2,0} b_{1,1} \beta_{10/4}$
666 $\underline{5}\beta_1^5 \gamma_2$	698 $[\beta_1^5 \beta_2 \beta_9] \eta_1 \beta_9$
$[\alpha_1 h_{2,0} b_{2,0} \beta_8] \underline{h_{2,0} b_{2,0} u_3}$	700 $\beta_1 \beta_{14}$
$\underline{2}\beta_1^5 \gamma_2$	701 $\alpha_1 \beta_{15/3}$
667 $\underline{2}x_{652}$	702 $\underline{\beta_{15/2}}$
669 $[\alpha_1 \beta_{14}] \underline{u_9}$	703 $\underline{3}\beta_1^3 \beta_{12}$
670 $\beta_1^4 \beta_{11}$	$\beta_1^7 \gamma_2$
673 $[\underline{2}\beta_1 \beta_4 \beta_{10/5}] \alpha_1 h_{2,0} b_{2,0} u_3$	704 $\underline{5}\beta_1^6 \gamma_2$

705 $\underline{2}\beta_1^2\beta_{13}]h_{2,0}h_{11}b_{2,0}u_3$	742 $\underline{5}\beta_1^7\gamma_2$
706 $[\alpha_1\eta_1\beta_9]b_{2,0}\eta_5$	748 $\beta_2\beta_{14}$
708 $[\beta_1^5\beta_{11}]\eta_{10}$	$[\beta_1\beta_{15}]h_{11}\beta_{15/1,2}$
709 $[\alpha_1\beta_{15/2}]\beta_{15/1,2}$	751 $\underline{3}\beta_1^3\beta_{13}$
710 $\beta_{15}$	752 $[\beta_1x_{714}]b_{2,0}\eta_1$
713 $\underline{3}\beta_1^2\beta_{13}$	753 $[\underline{2}\beta_1^2\beta_{14}]h_{2,0}b_{2,0}\eta_5$
$[\underline{2}\beta_1^5\beta_2\beta_9]\alpha_1b_{2,0}\beta_5$	$\underline{3}\beta_1x_{692}$
714 $\underline{\eta_1\gamma_2} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9 \rangle = x_{714}$	754 $[\beta_1^2\beta_{15/5}]b_{2,0}\eta_6$
715 $\underline{3}x_{692}$	755 $[\alpha_1\beta_2\beta_{14}]h_{2,0}\eta_{10}$
$\underline{2}\beta_1\beta_{14}$	$[\alpha_1\beta_1\beta_{15}]\alpha_1h_{11}\beta_{15/1,2}$
716 $\beta_1\beta_{15/5}$	756 $\underline{2}\beta_1^8\gamma_2$
$[\alpha_1\eta_{10} + \beta_1\beta_{15/5}]h_{2,0}u_9$	$[\beta_1^5\beta_{12}]\eta_{11}$
717 $\underline{2}\beta_{15/2}$	758 $\beta_{16}$
$\alpha_1\beta_{15}$	759 $[\alpha_1\beta_1x_{714}]\alpha_1b_{2,0}\eta_1$
718 $\beta_1^4\beta_{12}$	761 $[\alpha_1\beta_1^2\beta_{15/5}]\alpha_1b_{2,0}\eta_6$
$\underline{2}\beta_1^7\gamma_2$	$b_{2,0}\beta_2\gamma_2 = \langle \alpha_1\beta_1, \beta_6, \gamma_2 \rangle = x_{761}$
721 $\alpha_1x_{714}$	$\underline{3}\beta_1^2\beta_{14}$
723 $\alpha_1\beta_1\beta_{15/5}$	762 $\beta_1x_{724}$
724 $\underline{5}\gamma_3 = \langle \beta_1, 5, \beta_1, \beta_1^{17} \rangle = x_{724}$	763 $\underline{2}\beta_2\beta_{14}$
$\alpha_1h_{2,0}u_9 = \langle \beta_1^2, \beta_1^3\beta_{11}, \alpha_1, \alpha_1 \rangle$	$\beta_1x'_{724}$
$= x'_{724}$	764 $\alpha_1\eta_{11} = \beta_2\beta_{15/5}$
$[\beta_1\beta_{15/4}]h_{2,0}\beta_{15/5}$	765 $\alpha_1\beta_{16}$
727 $\underline{3}\underline{5}\beta_1^6\gamma_2$	$\underline{3}\underline{5}\beta_1^7\gamma_2$
728 $\beta_1^3\beta_{13}$	766 $\beta_1^4\beta_{13}$
730 $\beta_1x_{692}$	768 $\alpha_1x_{761} = \gamma_2\underline{2}\beta_1\beta_6$
731 $\underline{3}\beta_1^5\beta_{11}$	$\beta_1^2x_{692}$
$[\alpha_1\beta_1\beta_{15/4}]\alpha_1h_{2,0}\beta_{15/5}$	769 $\alpha_1\beta_1x'_{724}$
$[\alpha_1x_{724}]h_{2,0}\gamma_3$	771 $\underline{\beta_2\gamma_3} = \langle \beta_2, \beta_1^5, \beta_1^{13} \rangle = x_{771}$
732 $[\beta_1\beta_{15/3}]h_{2,0}\beta_{15/4}$	776 $\beta_1^3\beta_{14}$
738 $\beta_1^2\beta_{14}$	777 $\underline{2}\beta_1x_{724}$
739 $\underline{2}x_{724}$	778 $\alpha_1x_{771}$
$[\alpha_1\beta_1\beta_{15/3}]\alpha_1h_{2,0}\beta_{15/4}$	779 $\beta_1^9\gamma_2$
740 $[\beta_1\beta_{15/2}]h_{2,0}\beta_{15/3}$	$\underline{2}\beta_2\beta_{15/5}$
741 $\underline{3}\beta_1^4\beta_{12}$	780 $\underline{5}\beta_1^8\gamma_2$
$\beta_1^8\gamma_2$	786 $\beta_1\beta_2\beta_{14}$



789 $\underline{3}\beta_1^4\beta_{13}$	807 $\alpha_1\beta_1^2x'_{724}$
791 $[\underline{3}\beta_1^3x_{692}]\beta_{5/5}^4$	809 $[\underline{3}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2}u_2$
796 $\beta_1\beta_{16}$	794 $\underline{2}\beta_1^9\gamma_2 \beta_1x_{771}$
799 $\underline{3}\beta_1^3\beta_{14}$	810 $\alpha_1b_{2,0}\eta_7 =$
$\beta_1x_{761}$	$\langle \alpha_1, \alpha_1, \beta_1\beta_{12}, \beta_1^5 \rangle = \beta_2x'_{724}$
$[\underline{4}\beta_1^2x_{692}+?]\underline{b_{2,0}^3}\beta_2$	811 $\underline{2}\beta_1\beta_{16}$
800 $\beta_1^2x_{724}$	812 $\alpha_1\eta_{12} = \beta_3\beta_{15/5}$
$\beta_1^2x'_{724}$	813 $\alpha_1\beta_{17}$
801 $[\underline{2}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2}\beta_7$	814 $\beta_1^4\beta_{14}$
802 $[\beta_1\beta_2\beta_{15/5}]\underline{b_{2,0}}\eta_7$	815 $\underline{2}\beta_1^2x_{724}$
803 $\underline{3}\underline{5}\beta_1^8\gamma_2$	816 $\alpha_1\beta_1x_{771}$
$[\alpha_1\beta_1\beta_{16}]\underline{b_{2,0}}\beta_{12}$	817 $\beta_1^{10}\gamma_2$
804 $[\beta_1^5\beta_{13}]\underline{\eta}_{12}$	$\underline{4}\beta_1\beta_2\beta_{14}$
806 $\underline{\beta}_{17}$	$[\underline{2}\beta_1\beta_2\beta_{15/5}]\underline{b_{1,1}b_{2,0}}u_3$ <span style="float: right;">□</span>
$[\beta_1^3x_{692} + \alpha_1\beta_1x_{761}]\alpha_1b_{2,0}^3\beta_2$	
$\alpha_1\beta_1x_{761}$	

For the relation in the 643-stem we have

$$\begin{aligned}
 \beta_{p/p-1} &= \langle \alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1 \rangle \quad \text{and} \\
 \underline{2}p\gamma_2 &= \langle \alpha_1, \alpha_1\beta_1, p, \gamma_2 \rangle \quad \text{so} \\
 \beta_1^{p-1}\underline{2}p\gamma_2 &= \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2 \rangle \\
 &= \alpha_1 \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2 \rangle \\
 &= \alpha_1 \langle \alpha_1\beta_1^{p-1}, \beta_1, p, \gamma_2 \rangle \\
 &= \alpha_1 \langle \alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1 \rangle \gamma_2 \\
 &= \beta_{p/p-1}\gamma_2.
 \end{aligned}$$

This generalizes immediately to

7.6.11. PROPOSITION. *Let  $x$  be an element satisfying  $px = 0$ ,  $\langle \alpha_1\beta_1, p, x \rangle = 0$ , and  $\alpha_1x \neq 0$ . Then  $\beta_{p/p-1}x = \beta_1^{p-1}\underline{2}px$ . □*

For the differentials in dimensions 666 and 673 it suffices to show  $\beta_1^2\underline{2}\beta_4\beta_{10/5} = 0$ . We have  $\beta_4\beta_{10/5} = \beta_9\beta_{5/5} = \langle \beta_9, \alpha_1, \beta_1^5 \rangle$  so  $\underline{2}\beta_4\beta_{10/5} = \langle \beta_9, \alpha_1, \underline{2}\beta_1^5 \rangle$ . Then

$$\begin{aligned}
 \beta_1^2\underline{2}\beta_4\beta_{10/5} &= \langle \beta_1^2\beta_9, \alpha_1, \underline{2}\beta_1^5 \rangle \\
 &= \langle \alpha_1\beta_1\beta_6, \beta_4, \underline{2}\beta_1^5 \rangle = 0.
 \end{aligned}$$

The differential on  $\gamma_3$  is explained in 7.6.1. Recall that the key point was that  $\alpha_1\beta_{5/5}^4$  in Ext is a linear combination of the three elements  $\underline{3}\beta_1^3k_{14}$ ,  $\beta_1x_{761}$  and  $\beta_1^3\gamma_3$ . In our setting this relation is given by the differential on  $b_{2,0}^3\beta_2$ , whose target is some linear combination of the four elements (including  $\alpha_1\beta_{5/5}^4$ ) in question. This target is difficult to compute precisely, but it suffices to show that it includes a nontrivial

multiple of  $\alpha_1\beta_{5/5}^4$ . Knowing then that  $\underline{3}\beta_1^3\beta_{14}$  and  $\beta_1x_{761}$  are permanent cycles and  $\alpha_1\beta_{5/5}^4$  is not, we can conclude that the linear combination also includes  $\beta_1^3\gamma_3$  and that the latter is not a permanent cycle in the Adams–Novikov spectral sequence.

To make this calculation we map to the spectral sequence going from

$$\text{Ext}_{P(1)}(\mathbf{Z}/(p), P(0))$$

(this is the  $R$  of 7.5.1 and 7.4.13) to  $\text{Ext}_{P(1)}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ . The elements  $\underline{3}\beta_1^3\beta_{14}$ ,  $\gamma_3$  and  $x_{761}$  all have trivial images, while  $b_{2,0}^3\beta_2$  and  $\alpha_1\beta_{5/5}^4$  do not, and it suffices to show that  $\alpha_1\beta_{5/5}^4 = h_{10}b_{1,1}^4$  vanishes in  $\text{Ext}_{P(1)}$ .  $h_{11}b_{1,1}$  is killed by  $b_{2,0}$ , so  $\langle b_{1,1}^3, h_{11}, h_{11}, h_{11} \rangle$  is killed by  $b_{2,0}^3$  so we have

$$\begin{aligned} 0 &= \langle b_{1,1}^3, h_{11}, h_{11}, h_{11} \rangle \langle h_{11}, h_{11}, h_{10} \rangle \\ &= b_{1,1}^3 \langle h_{11}, h_{11}, h_{11}, \langle h_{11}, h_{11}, h_{10} \rangle \rangle \\ &= b_{1,1}^3 \langle h_{11}, h_{11}, h_{11}, h_{11}, h_{11} \rangle h_{10} \\ &= b_{1,1}^4 h_{10}. \end{aligned}$$

Given this situation the target of the differential from  $\beta_{5/5}$ ,  $\underline{4}\beta_1^{20}$ , is the same as  $\underline{3}\beta_1^2x_{692}$ , and  $\alpha_1\beta_{5/5}^4$  is  $\underline{4}\beta_1^2x_{692}$  which accounts for the indicated differentials in dimensions 791 and 799.

The differential in the 752-stem can be recovered from the corresponding spectral sequence for  $\text{Ext}_P$ . The images of  $\eta_1$  and  $\gamma_2$  are the Massey products  $\langle h_{11}, \underline{h}, \underline{b} \rangle$  and  $\langle h_{12}, \underline{h}, \underline{b} \rangle$  where  $\underline{h}$  and  $\underline{b}$  denote the matrices

$$\begin{pmatrix} h_{11} & h_{12} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{1,1} \\ b_{1,0} \end{pmatrix},$$

respectively. Then we have  $\beta_1\eta_1\gamma_2 = \langle h_{12}b_{1,0}\eta_1, \underline{h}, \underline{b} \rangle = \langle h_{11}b_{1,1}\eta_1, \underline{h}, \underline{b} \rangle = 0$  since  $h_{11}\eta_1 = 0$ .

### 7.6.5 (Fifth installment)

818 $\beta_1^{10}\beta_{10/5}$	841 $\alpha_1x_{834}$
824 $\beta_1^2\beta_2\beta_{14}$	$\underline{3}\beta_1^{10}\beta_{10/5}$
825 $\alpha_1b_{1,1}b_{2,0}u_3 = \underline{2}\beta_2x'_{724}$	842 $\underline{b}_{1,1}^2\beta_{10/4} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{10/4} \rangle = x_{842}$
826 $\underline{b}_{1,1}^4u_4 = \langle \alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5} \rangle = x_{826}$	844 $\alpha_1\beta_1^2x_{761}$
827 $\underline{2}\beta_3\beta_{15/5}$	$\beta_1\beta_{17}$
832 $[\underline{2}\beta_1^{10}\gamma_2]\underline{b}_{1,1}^2\gamma_2$	845 $[\alpha_1\beta_1^3x'_{724}]\alpha_1h_{2,0}b_{1,1}^4$
833 $\alpha_1x_{826}$	847 $\beta_1^2x_{771}$
834 $\beta_1^2\beta_{16}$	848 $[\beta_1x_{810}]\underline{h}_{2,0}b_{2,0}^2\beta_7$
$\underline{b}_{1,1}^2\beta_{10/5} = \langle \beta_1^4, \underline{2}\beta_1^6, \beta_{10/5} \rangle = x_{834}$	849 $[\underline{2}\beta_1^2\beta_{16} + \alpha_1x_{842}]\underline{b}_{2,0}^2\beta_8$
837 $\underline{3}\beta_1^4\beta_{14}$	$2\beta_1^2\beta_{16}$
$\beta_1^2x_{761}$	850 $\beta_1\beta_3\beta_{15/3}$
838 $\beta_1^3x_{724}$	851 $[\alpha_1\beta_1\beta_{17}]\underline{b}_{2,0}\beta_{13}$
$[\beta_1^3x'_{724}]\underline{h}_{2,0}b_{1,1}^4$	852 $[\beta_1^5\beta_{14}]\underline{\eta}_{13}$
840 $\underline{3}\beta_1^{10}\gamma_2$	853 $\underline{2}\beta_1^3x_{724}$

854 $\underline{\beta}_{18}$	859 $[\underline{2}\beta_1\beta_{17}]\underline{b_{2,0}u_8}$
$\alpha_1\beta_1^2x_{771}$	860 $\alpha_1\eta_{13} = \beta_4\beta_{15/5}$
855 $\beta_1^{11}\gamma_2$	861 $\alpha_1\beta_{18}$
$\underline{4}\beta_1^2\beta_2\beta_{14}$	862 $\beta_1^3\beta_2\beta_{14}$
856 $[\underline{4}\underline{2}x_{810}]\underline{h_{2,0}b_{2,0}^2u_2}$	853 $[\underline{2}\beta_1x_{810}]\alpha_1h_{2,0}b_{2,0}^2u_2$
856 $\beta_1^{11}\beta_{10/5}$	864 $[\beta_1x_{826}]\underline{h_{2,0}b_{1,1}b_{2,0}u_3}$
857 $\alpha_1b_{2,0}^2\beta_8 = x_{857}$	865 $\underline{2}\beta_1\beta_2\beta_{15/5}$
	867 $\underline{3}\beta_1\beta_{17}$ <span style="float: right;">□</span>

For the differential in the 838-stem we use the method of 7.6.6. We have maps  $f: \Sigma^{190}T(0)_1 \rightarrow S^0$  and  $g: \Sigma^{609}T(0)_4 \rightarrow S^0$  where  $f$  is  $\beta_1^5$  on the bottom cell, and  $g$  is  $\alpha_1x_{602}$  on the bottom cell and  $x_{617}$  on the second cell. The smash product vanishes on the bottom cell so we have a map  $\Sigma^{807}T(0)_4 \rightarrow S^0$  which is  $\beta_1^5x_{617} + 2\beta_1^5x_{602}$  on the bottom cell. The second term vanishes because  $\beta_1^5x_{602} \in \pi_{792} = 0$ . We have

$$x_{617} = \left\langle \alpha_1, (\underline{2}\beta_1\beta_{11}\alpha_1), \left( \begin{array}{c} \beta_1 \\ x_{602} \end{array} \right) \right\rangle.$$

A routine calculation gives  $\beta_1x_{617} = \underline{3}\beta_1^3\beta_{11}$  and  $\beta_1^3x_{617} = \alpha_1x'_{724}$ . Our map gives  $0 = \underline{4}\beta_1^5x_{617} = \beta_1^3x'_{724}$ , hence the desired differential.

We use a similar argument in the 848-stem. We start with the maps

$$\Sigma^{316}T(0)_1 \rightarrow S^0 \quad \text{and} \quad \Sigma^{531}T(0)_4 \rightarrow S^0$$

carrying  $\beta_1\beta_6$ , and  $\alpha_1\beta_2\beta_{10/5}$  on the bottom cells. The resulting relation is  $\beta_1^2x_{810} = 0$ . From 7.6.4 we see that  $\mathbf{N}$  is vacuous in dimensions 887 and 856, so the indicated differential is the only one which can give this relation.

The argument in dimension 849 is similar to that in dimension 609.

In dimension 864 we use 7.6.6 again starting with the extensions of  $\beta_1^5$  and  $\underline{2}\beta_4\beta_{10/5}$  to  $T(0)_1$  and  $T(0)_4$ .

7.6.5 (*Sixth installment*)

868 $\underline{b_{1,1}u_9} = x_{868}$	879 $[\alpha_1\beta_1x_{834}]\alpha_1h_{2,0}b_{1,1}^2u_4$
871 $[\alpha_1\beta_1x_{826}]\alpha_1h_{2,0}b_{1,1}b_{2,0}u_3$	$[\underline{3}\beta_1^{11}\beta_{10/5}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$
872 $\beta_1^3\beta_{16}$	880 $[\beta_1x_{842}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$
$[\beta_1x_{834}]\underline{h_{2,0}b_{1,1}^2u_4}$	882 $[\alpha_1\beta_1^3x_{761}]\underline{b_{1,1}\gamma_3}$
875 $\alpha_1x_{868}$	$\beta_1^2\beta_{17}$
$\underline{2}\beta_4\beta_{15/5}$	883 $\alpha_1\beta_{10/5}^2$
$\beta_1^3x_{761} = \beta_{10/5}\gamma_2$	884 $\beta_{5/4}\beta_{15/5} = \beta_{10/4}\beta_{10/5}$
876 $\beta_{10/5}^2$	885 $\beta_1^3x_{771}$
$\beta_1^4x_{724}$	887 $\underline{4}\beta_1^{11}\beta_{10/5}$
878 $\underline{3}\beta_1^{11}\gamma_2$	$[\underline{2}\beta_1^3\beta_{16}]\alpha_1h_{2,0}b_{1,1}^2\gamma_2$
	888 $[\beta_1^2\beta_3\beta_{15/5}]\underline{h_{2,0}h_{11}^2\beta_{10/4}}$

890 $\alpha_1 b_{1,1} \gamma_3 = \langle \alpha_1, \alpha_1, \beta_1^{10}, \beta_1^{13} \rangle \underline{2\beta_1^3 x_{761}}$	923 $\beta_1^4 x_{771}$
891 $\underline{2\beta_1^4 x_{724}}$ $\alpha_1 \beta_{5/4} \beta_{15/5}$	925 $\alpha_1 \beta_{20/5}$ $\underline{4\beta_1^{12} \beta_{10/5}}$
892 $\beta_1 \beta_{18}$ $\alpha_1 \beta_1^3 x_{771}$ $\underline{\beta_{10/4}^2 + \beta_1 \beta_{18} = x_{892}}$	926 $\underline{\beta_{20/4}}$
893 $\beta_1^{12} \gamma_2$ $\underline{4\beta_1^3 \beta_2 \beta_{14}}$	928 $\underline{2\beta_1^4 x_{761}}$
894 $\beta_1^{12} \beta_{10/5}$	929 $\underline{[2\beta_1^5 x_{724}] h_{2,0} b_{1,1} \gamma_3}$ $[\alpha_1 \beta_1 \beta_{5/4} \beta_{15/5}] \alpha_1 h_{2,0} b_{1,1} \beta_{15/5}$
895 $[\beta_1 x_{857}] h_{2,0} h_{11} b_{2,0}^2 u_2$	930 $\alpha_1 \beta_1^4 x_{771}$ $\beta_1^2 \beta_{18}$ $[\beta_1 x_{892}] \underline{h_{2,0} b_{1,1} \beta_{15/4}}$
896 $\underline{[4 \underline{2\beta_1 \beta_3 \beta_{15/5}}] h_{11} b_{2,0}^2 u_3}$	931 $\beta_1^{13} \gamma_2$ $\underline{4\beta_1^4 \beta_2 \beta_{14}}$
898 $[\beta_1 \beta_4 \beta_{15/5}] \underline{h_{2,0} b_{1,0} \beta_{13}}$	932 $\beta_1^{13} \beta_{10/5}?$
899 $[\alpha_1 x_{892}] \underline{h_{2,0} \beta_{14}}$ $\alpha_1 \beta_1 \beta_{18}$	933 $\gamma_4?$ $\alpha_1 \beta_{20/4}$
900 $\beta_1^4 \beta_2 \beta_{14}$	934 $\beta_{20/3}$
902 $\beta_{19}$	937 $\underline{3\beta_1^5 x_{724}}$ $[\alpha_1 \beta_1 \beta_{10/4}^2] \alpha_1 h_{2,0} b_{1,1} \beta_{15/4}$
903 $\underline{4\beta_1^3 \beta_{16}}$ $[\underline{2\beta_1^2 \beta_3 \beta_{15/5}}] \alpha_1 h_{11} b_{2,0}^2 u_3$	938 $[\beta_1^5 \beta_2 \beta_{14}] \underline{\eta_1 \beta_{14}}$
905 $\underline{3\beta_1^2 \beta_{17}}$	940 $\alpha_1 \gamma_4$ $\beta_1 \beta_{19}$
906 $[\alpha_1 h_{2,0} b_{2,0} \beta_{13}] \underline{h_{2,0} b_{2,0} u_8}$ $\beta_1 x_{868}$	941 $\alpha_1 \beta_{20/3}$ $\underline{4\beta_1^4 \beta_{16}}$
907 $\underline{2x_{892}}$	942 $\beta_{20/2}$
909 $[\alpha_1 \beta_{19}] u_{14}$	943 $[\underline{3\beta_1^3 \beta_{17}}] h_{2,0} h_{11} b_{2,0}^2 u_3$
910 $\beta_1^4 \beta_{16}$	944 $[\beta_1^2 x_{868}] \underline{b_{2,0}^2 \eta_5}$
913 $\alpha_1 \beta_1 x_{868}$ $\beta_1^4 x_{761}$ $[\underline{2\beta_1 \beta_4 \beta_{15/5}}] \alpha_1 h_{2,0} b_{2,0} u_8$	945 $[\underline{2\beta_1^2 \beta_{18}}] h_{2,0} h_{11} b_{2,0} u_8$
914 $\beta_1^5 x_{724}$ $[\beta_1 \beta_{10/5}^2] \underline{h_{2,0} b_{1,1} u_9}$	946 $[\beta_2 \beta_4 \beta_{15/5}] \underline{b_{2,0} \eta_{10}}$
916 $\underline{3\beta_1^{12} \gamma_2}$	948 $[\beta_1^5 \beta_{16}] \underline{\eta_{15}}$
917 $\underline{2\beta_{19}}$	949 $[\alpha_1 \beta_{20/2}] \underline{\beta_{20/1,2}}$
918 $\underline{\beta_{20/5}}$	950 $\beta_{20}$
920 $\beta_1^3 \beta_{17}$	951 $\beta_1^5 x_{761}$ $\underline{4\beta_1^3 \beta_{17}}$ $[\alpha_1 \beta_1^2 x_{868}] \alpha_1 b_{2,0}^2 \eta_5$
921 $[\alpha_1 \beta_1 \beta_{10/5}^2] \alpha_1 \underline{h_{2,0} b_{1,1} u_9}$	
922 $[\beta_1 \beta_{5/4} \beta_{15/5}] \underline{h_{2,0} b_{1,1} \beta_{15/5}}$	

952 $\beta_1^6 x_{724}$	972 $\underline{5}\gamma_4$
$\underline{b_{2,0}\eta_1\gamma_2} = \langle \beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2 \rangle$	$[\beta_1\beta_{20/3}]\underline{h_{2,0}\beta_{20/4}}$
$= x_{952}$	975 $\underline{3}\beta_1^5 x_{724}$
953 $\underline{3}\beta_1^2 \beta_{18}$	978 $\alpha_1 \beta_1 \gamma_4$
954 $\underline{3}\beta_1^{13} \gamma_2$	$\beta_1^2 \beta_{19}$
$\underline{34}\beta_1^4 \beta_2 \beta_{14} +$	979 $\underline{2}x_{964}$
$\langle \alpha_1, \alpha_1, \beta_1 \beta_2 \beta_{14}, \beta_1^4 \rangle = x_{954}$	$[\alpha_1 \underline{5}\gamma_4]\underline{h_{2,0}\gamma_4}$
955 $\underline{2}\beta_1 \beta_{19}$	$[\alpha_1 \beta_1 \beta_{20/3}]\alpha_1 h_{2,0} \beta_{20/4}$
956 $\beta_1 \beta_{20/5}$	980 $[\beta_1 \beta_{20/2}]\underline{h_{2,0}\beta_{20/3}}$
$[\alpha_1 \eta_{15} + \beta_1 \beta_{20/5}]\underline{h_{2,0}u_{14}}$	987 $\underline{2} \cdot \underline{5}\gamma_4$
957 $\underline{2}\beta_{20/2}$	988 $\beta_2 \beta_{19}$
$\alpha_1 \beta_{20}$	$[\beta_1 \beta_{20}]\underline{h_{11}\beta_{20/1,2}}$
958 $\beta_1^4 \beta_{17}$	989 $\beta_1^6 x_{761}$
959 $\alpha_1 x_{952}$	$\underline{4}\beta_1^4 \beta_{17}$
961 $[\beta_1^5 x_{771}]\underline{h_{11}b_{2,0}\gamma_3}$	990 $\beta_1^7 x_{724}$
963 $\alpha_1 \beta_1 \beta_{20/5}$	$\beta_1 x_{952} = \beta_5^5$
$\underline{4}\beta_1^{13} \beta_{10/5}$	991 $[\underline{3}\beta_1^3 \beta_{18}]\underline{h_{2,0}b_{2,0}^2 \eta_5}$
964 $\alpha_1 h_{2,0} u_{14} = \underline{34}\beta_1^4 \beta_{16} = x_{964}$	992 $\underline{3}\beta_1^{14} \gamma_2$
$[\beta_1 \beta_{20/4}]\underline{h_{2,0}\beta_{20/5}}$	$[\beta_1 x_{954}]\underline{b_{2,0}^2 \eta_6}$
966 $\underline{2}\beta_1^5 x_{761}$	993 $[\underline{2}\beta_1^2 \beta_9]\underline{h_{2,0}b_{2,0}\eta_{10}}$
968 $\beta_1^3 \beta_{18}$	994 $[\beta_1^2 \beta_{20/5}]\underline{b_{2,0}\eta_{11}}$
$[\alpha_1 \beta_1^5 x_{771}]\alpha_1 h_{11} b_{2,0} \gamma_3$	995 $[\alpha_1 \beta_2 \beta_{19}]\underline{h_{2,0}\eta_{15}}$
969 $\beta_1^{14} \gamma_2$	$[\alpha_1 \beta_1 \beta_{20}]\alpha_1 h_{11} \beta_{20/1,2}$
$\underline{2}x_{954}$	996 $[\beta_1^5 \beta_{17}]\underline{\eta_{16}}$
970 $\beta_1^{14} \beta_{10/5}?$	997 $[\alpha_1 \beta_1 x_{952}]\underline{\beta_{25/25}}$
971 $\beta_1 \gamma_4?$	998 $\underline{\beta_{21}}$
$[\alpha_1 \beta_1 \beta_{20/4}]\alpha_1 h_{2,0} \beta_{20/5}$	999 $\underline{b_{2,0}^2 \beta_2 \gamma_2} = \langle \beta_1 \beta_2, \gamma_1, \gamma_2 \rangle = x_{999}$
	$[\underline{4}\beta_1^3 \beta_{18}]\underline{b_{2,0}^3 u_1}$
	1000 $[\alpha_1 \beta_1^2 \beta_{20/5}]\alpha_1 b_{2,0} \eta_{11}$ <span style="float: right;">□</span>

The element  $x_{868}$  is constructed as follows. There is a commutative diagram

$$\begin{array}{ccc}
 S^{860} & \xrightarrow{f} & \Sigma^{837} T(0)_2 \\
 \downarrow \beta_q & & \downarrow g \\
 S^{438} & \xrightarrow{\beta_{10/5}} & S^0
 \end{array}$$

where the cofiber of  $f$  is  $\Sigma^{837}T(0)_3$  and  $g$  is an extension of  $\underline{3}\beta_1^4\beta_{14}$ . Both  $f$  and  $\beta_9$  extend to  $\Sigma^{860}T(0)_1$ . The difference of the composite extensions of  $\beta_{10/5}\beta_9$  and  $gf$  gives  $x_{868}$  on the top cell. In other words  $x_{868}$  is the Toda bracket for

$$S^{867} \xrightarrow{\alpha_1} S^{860} \rightarrow S^{438} \vee \Sigma^{837}T(0)_2 \rightarrow S^0.$$

We will see below that  $\beta_1^2g = 0$  and  $\beta_1\beta_{10/5}\beta_9 = 0$  so it follows that  $\beta_1^2x_{868}$  is divisible by  $\alpha_1$  and hence trivial.

For the relation in dimension 875 we have  $x_{761} = \langle \alpha_1\beta_1, \beta_6, \gamma_2 \rangle$  and  $\beta_1\beta_{10/5} = \langle \beta_1^4, \alpha_1\beta_1, \beta_6 \rangle$  so  $\beta_1^4x_{761} = \gamma_2\beta_1\beta_{10/5}$ .

For dimension 879 we have, using 7.6.11,

$$\begin{aligned} \gamma_2x_{404} &= \gamma_2\langle \beta_{5/4}, \beta_1, \alpha_1\beta_1^4 \rangle = \langle \gamma_2\beta_{5/4}, \beta_1, \alpha_1\beta_1^4 \rangle \\ &= \langle \underline{2} \cdot \underline{5}\gamma_2\beta_1^4, \beta_1, \alpha_1\beta_1^4 \rangle = \underline{3} \cdot \underline{5}\beta_1\gamma_2 = \underline{3}\beta_1^{10}\beta_{10/5} \end{aligned}$$

so

$$\underline{3}\beta_1^{11}\beta_{10/5} = \gamma_2\beta_1x_{404} = 0.$$

In dimension 888 we have

$$\beta_1^2\beta_3\beta_{15/5} = \beta_1^2\beta_8\beta_{10/5} = \beta_8\beta_1\underline{5}\gamma_2 = \beta_2\beta_7\underline{5}\gamma_2 = 0.$$

For the 896 stem we have

$$\beta_1\underline{4}\underline{2}\beta_1\beta_3\beta_{15/5} = \beta_1\beta_2\underline{4}\underline{2}\beta_2\beta_{15/5} = 0,$$

which (by inspection 7.6.4) implies  $\underline{4}\underline{2}\beta_1\beta_3\beta_{15/5} = 0$ .

We are not sure about  $\gamma_4$ . A possible approach to it is this. Extrapolating 7.6.4 slightly we see that  $\text{Ext}^{7,1016}$  has two generators,  $\beta_1^2\gamma_4$  and  $\langle \gamma_3, \gamma_1, \beta_3 \rangle$ . The latter supports a differential hitting  $\beta_1^{15}\beta_{10/5} = \langle \beta_1^{18}, \gamma_1, \beta_2 \rangle$ . The same Ext group contains  $\langle \gamma_2, \gamma_2, \beta_3 \rangle$ , which is a permanent cycle. Hence if it is nonzero it is neither  $\beta_1^2\gamma_4$ , in which case  $\gamma_4$  is a permanent cycle, or  $\beta_1^2\gamma_4 + \langle \gamma_3, \gamma_1, \beta_3 \rangle$ , in which case  $d_{25}(\gamma_4) = \beta_1^{13}\beta_{10/5}$ .

In the 992-stem we have  $\beta_1x_{954} = \beta_4x_{810}$  so  $\beta_1^2x_{954} = \beta_1\beta_4x_{810} = 0$ . Extrapolating the pattern in 7.6.4 we find that the only element in the appropriate dimension is  $b_{1,1}^3\gamma_2$ , which kills  $\underline{3}\beta_1^{15}\gamma_2$ .

## Hopf Algebras and Hopf Algebroids

Commutative, noncocommutative Hopf algebras, such as the dual of the Steenrod algebra  $A$  (3.1.1), are familiar objects in algebraic topology and the importance of studying them is obvious. Computations with the Adams spectral sequence require the extensive use of homological algebra in the category of  $A$ -modules or, equivalently, in the category of  $A_*$ -comodules. In particular there are several change-of-rings theorems (A1.1.18, A1.1.20, and A1.3.13) which are major labor-saving devices. These results are well known, but detailed proofs (which are provided here) are hard to find.

The use of generalized homology theories such as  $MU$ - and  $BP$ -theory requires a generalization of the definition of a Hopf algebra to that of a *Hopf algebroid*. This term is due to Haynes Miller and its rationale will be explained below. The dual Steenrod algebra  $A_*$  is defined over  $\mathbf{Z}/(p)$  and has a coproduct  $\Delta: A_* \rightarrow A_* \otimes_{\mathbf{Z}/(p)} A_*$  dual to the product on  $A$ . The  $BP$ -theoretic analog  $BP_*(BP)$  has a coproduct  $\Delta: BP_*(BP) \rightarrow BP_*(BP) \otimes_{\pi_*(BP)} BP_*(BP)$ , but the tensor product is defined with respect to a  $\pi_*(BP)$ -bimodule structure on  $BP_*(BP)$ ; i.e.,  $\pi_*(BP)$  acts differently on the two factors. These actions are defined by two different  $\mathbf{Z}/(p)$ -algebra maps  $\eta_L, \eta_R: \pi_*(BP) \rightarrow BP_*(BP)$ , known as the left and right units. In the case of the Steenrod algebra one just has a single unit  $\eta: \mathbf{Z}/(p) \rightarrow A_*$ . Hence  $BP_*(BP)$  is not a Hopf algebra, but a more general sort of object of which a Hopf algebra is a special case.

The definition of a Hopf algebroid A1.1.1 would seem rather awkward and unnatural were it not for the following category theoretic observation, due to Miller. A Hopf algebra such as  $A_*$  is a *cogroup object* in the category of graded  $\mathbf{Z}/(p)$ -algebras. In other words, given any such algebra  $R$ , the coproduct  $\Delta: A_* \rightarrow A_* \otimes A_*$  induces a set map  $\text{Hom}(A_*, R) \times \text{Hom}(A_*, R) \rightarrow \text{Hom}(A_*, R)$  which makes  $\text{Hom}(A_*, R)$  into a group. Now the generalization of Hopf algebras to Hopf algebroids corresponds precisely to that from groups to *groupoids*. Recall that a group can be thought of as a category with a single object in which every morphism is invertible; the elements in the group are identified with the morphisms in the category. A groupoid is a small category in which every morphism is invertible and a Hopf algebroid is a cogroupoid object in the category of commutative algebras over a commutative ground ring  $K$  [ $\mathbf{Z}/(p)$  in the case of  $BP_*(BP)$ ]. The relation between the axioms of a groupoid and the structure of a Hopf algebroid is explained in A1.1.1.

The purpose of this appendix is to generalize the standard tools used in homological computations over a Hopf algebra to the category of comodules over a Hopf algebroid. It also serves as a self-contained (except for Sections 4 and 5) account of the Hopf algebra theory itself. These standard tools include basic definitions (Section 1), some of which are far from obvious; resolutions and homological functors such as  $\text{Ext}$  and  $\text{Cotor}$  (Section 2); spectral sequences of various sorts (Section 3),

including that of Cartan and Eilenberg [1, p. 349]; Massey products (Section 4); and algebraic Steenrod operations (Section 5). We will now describe these five sections in more detail.

In Section 1 we start by defining Hopf algebroids (A1.1.1), comodules and primitives (A1.1.2), cotensor products (A1.1.4), and maps of Hopf algebroids (A1.1.7). The category of comodules is shown to be abelian (A1.1.3), so we can do homological algebra over it in Section 2. Three special types of groupoid give three corresponding types of Hopf algebroid. If the groupoid has a single object (or if all morphisms have the same source and target) we get an ordinary Hopf algebra, as remarked above. The opposite extreme is a groupoid with many objects but at most a single morphism between any pair of them. From such groupoids we get unicursal Hopf algebroids (A1.1.11). A third type of groupoid can be constructed from a group action on a set, and a corresponding Hopf algebroid is said to be split (A1.1.22).

The most difficult definition of Section 1 (which took us quite a while to formulate) is that of an extension of Hopf algebroids (A1.1.15). An extension of Hopf algebras corresponds to an extension of groups, for which one needs to know what a normal subgroup is. We are indebted to Higgins [1] for the definition of a normal subgroupoid. A groupoid  $C_0$  is normal in  $C_1$  if

- (i) the objects of  $C_0$  are the same as those of  $C_1$ ,
- (ii) the morphisms in  $C_0$  form a subset of those in  $C_1$ , and
- (iii) if  $g: X \rightarrow Y$  and  $h: Y \rightarrow Y$  are morphisms in  $C_1$  and  $C_0$ , respectively, then  $g^{-1}hg: X \rightarrow X$  is a morphism in  $C_0$ .

This translates to the definition of a normal map of Hopf algebroids (A1.1.10). The quotient groupoid  $C = C_1/C_0$  is the one

- (i) whose objects are equivalence classes of objects in  $C_1$ , where two objects are equivalent if there is a morphism between them in  $C_0$ , and
- (ii) whose morphisms are equivalence classes of morphisms in  $C_1$ , where two morphisms  $g$  and  $g'$  are equivalent if  $g' = h_1gh_2$  where  $h_1$  and  $h_2$  are morphisms in  $C_0$ .

The other major result of Section 1 is the comodule algebra structure theorem (A1.1.17) and its corollaries, which says that a comodule algebra (i.e., a comodule with a multiplication) which maps surjectively to the Hopf algebroid  $\Sigma$  over which it is defined is isomorphic to the tensor product of its primitives with  $\Sigma$ . This applies in particular to a Hopf algebroid  $\Gamma$  mapping onto  $\Sigma$  (A1.1.19). The special case when  $\Sigma$  is a Hopf algebra over a field was first proved by Milnor and Moore [3].

In Section 2 we begin our study of homological algebra in the category of comodules over a Hopf algebroid. We show (A1.2.2) that there are enough injectives and define Ext and Cotor (A1.2.3). For our purposes Ext can be regarded as a special case of Cotor (A1.1.6). We find it more convenient here to state and prove our results in terms of Cotor, although no use of it is made in the text. In most cases the translation from Cotor to Ext is obvious and is omitted. After defining these functors we discuss resolutions (A1.2.4, A1.2.10) that can be used to compute them, especially the cobar resolution (A1.2.11). We also define the cup product in Cotor (A1.2.14).

In Section 3 we construct some spectral sequences for computing the Cotor and Ext groups we are interested in. First we have the spectral sequence associated with an LES of comodules (A1.3.2); the example we have in mind is the chromatic



spectral sequence of Chapter 5. Next we have the spectral sequence associated with a (decreasing or increasing) filtration of a Hopf algebroid (A1.3.9); examples include the classical May spectral sequence (3.2.9), the spectral sequence of 3.5.2, and the so-called algebraic Novikov spectral sequence (4.4.4).

In A1.3.11 we have a spectral sequence associated with a map of Hopf algebroids which computes Cotor over the target in terms of Cotor over the source. When the map is surjective the spectral sequence collapses and we get a change-of-rings isomorphism (A1.3.12). We also use this spectral sequence to construct a Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.15) for an extension of Hopf algebroids.

In Section 4 we discuss Massey products, an essential tool in some of the more intricate calculations in the text. The definitive reference is May [3] and this section is little more than an introduction to that paper. We refer to it for all the proofs and we describe several examples designed to motivate the more complicated statements therein. The basic definitions of Massey products are given as A1.4.1, A1.4.2, and A1.4.3. The rules for manipulating them are the juggling theorems A1.4.6, A1.4.8, and A1.4.9. Then we discuss the behavior of Massey products in spectral sequences. Theorem A1.4.10 addresses the problem of convergence; A1.4.11 is a Leibnitz formula for differentials on Massey products; and A1.4.12 describes the relation between differentials and extensions.

Section 5 treats algebraic Steenrod operations in suitable Cotor groups. These are defined in the cohomology of any cochain complex having certain additional structure and a general account of them is given by May [5]. Our main result (A1.5.1) here (which is also obtained by Bruner *et al.* [1]) is that the cobar complex (A1.2.11) has the required structure. Then the theory of May [5] gives the operations described in A1.5.2. Our grading of these operations differs from that of other authors including May [5] and Bruner *et al.* [1]; our  $\mathcal{P}^i$  raises cohomological (as opposed to topological) degree by  $2i(p-1)$ .

## 1. Basic Definitions

A1.1.1. DEFINITION. A Hopf algebroid over a commutative ring  $K$  is a co-groupoid object in the category of (graded or bigraded) commutative  $K$ -algebras, i.e., a pair  $(A, \Gamma)$  of commutative  $K$ -algebras with structure maps such that for any other commutative  $K$ -algebra  $B$ , the sets  $\text{Hom}(A, B)$  and  $\text{Hom}(\Gamma, B)$  are the objects and morphisms of a groupoid (a small category in which every morphism is an equivalence). The structure maps are

$$\begin{array}{ll} \eta_L: A \rightarrow \Gamma & \text{left unit or source,} \\ \eta_R: A \rightarrow \Gamma & \text{right unit or target,} \\ \Delta: \Gamma \rightarrow \Gamma \otimes_A \Gamma & \text{coproduct or composition,} \\ \varepsilon: \Gamma \rightarrow A, & \text{counit or identity,} \\ c: \Gamma \rightarrow \Gamma & \text{conjugation or inverse.} \end{array}$$

Here  $\Gamma$  is a left  $A$ -module via  $\eta_L$  and a right  $A$ -module via  $\eta_R$ ,  $\Gamma \otimes_A \Gamma$  is the usual tensor product of bimodules, and  $\Delta$  and  $\varepsilon$  are  $A$ -bimodule maps. The defining properties of a groupoid correspond to the following relations among the structure maps:

- (a)  $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$ , the identity map on  $A$ . (The source and target of an identity morphism are the object on which it is defined.)
- (b)  $(\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes \Gamma)\Delta = 1_\Gamma$ . (Composition with the identity leaves a morphism unchanged.)
- (c)  $(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta$ . (Composition of morphisms is associative.)
- (d)  $c\eta_R = \eta_L$  and  $c\eta_L = \eta_R$ . (Inverting a morphism interchanges source and target.)
- (e)  $cc = 1_\Gamma$ . (The inverse of the inverse is the original morphism.)
- (f) Maps exist which make the following commute

$$\begin{array}{ccccc}
 & & \Gamma & \xleftarrow{c \cdot \Gamma} & \Gamma \otimes_K \Gamma & \xrightarrow{\Gamma \cdot c} & \Gamma & & \\
 & & \uparrow & \swarrow & \downarrow & \searrow & \uparrow & & \\
 & & \Gamma & & \Gamma \otimes_A \Gamma & & \Gamma & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 \eta_R & & \Gamma & & \Gamma & & \Gamma & & \eta_L \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A & \xleftarrow{\varepsilon} & \Gamma & \xrightarrow{\varepsilon} & A & & 
 \end{array}$$

where  $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$  and  $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1c(\gamma_2)$ . (Composition of a morphism with its inverse on either side gives an identity morphism.)

If our algebras are graded the usual sign conventions are assumed; i.e., commutativity means  $xy = (-1)^{|x||y|}yx$ , where  $|x|$  and  $|y|$  are the degrees or dimensions of  $x$  and  $y$ , respectively.

A graded Hopf algebra is *connected* if the right and left sub- $A$ -modules generated by  $\Gamma_0$  are both isomorphic to  $A$ .

In most cases the algebra  $A$  will be understood and the Hopf algebra will be denoted simply by  $\Gamma$ .

Note that if  $\eta_R = \eta_L$ , then  $\Gamma$  is a commutative Hopf algebra over  $A$ , which is to say a cogroup object in the category of commutative  $A$ -algebras. This is the origin of the term Hopf algebra. More generally if  $D \subset A$  is the subalgebra on which  $\eta_R = \eta_L$ , then  $\Gamma$  is also a Hopf algebra over  $D$ .

The motivating example of a Hopf algebra is  $(\pi_*(E), E_*(E))$  for a suitable spectrum  $E$  (see Section 2.2).

**A1.1.2. DEFINITION.** A left  $\Gamma$ -comodule  $M$  is a left  $A$ -module  $M$  together with a left  $A$ -linear map  $\psi: M \rightarrow \Gamma \otimes_A M$  which is counitary and coassociative, i.e., such that  $(\varepsilon \otimes M)\psi = M$  (i.e., the identity on  $M$ ) and  $(\Delta \otimes M)\psi = (\Gamma \otimes \psi)\psi$ . A right  $\Gamma$ -comodule is similarly defined. An element  $m \in M$  is primitive if  $\psi(m) = 1 \otimes m$ .

A comodule algebra  $M$  is a comodule which is also a commutative associative  $A$ -algebra such that the structure map  $\psi$  is an algebra map. If  $M$  and  $N$  are left  $\Gamma$ -comodules, their comodule tensor product is  $M \otimes_A N$  with structure map being the composite

$$M \otimes N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes M \otimes \Gamma \otimes N \rightarrow \Gamma \otimes \Gamma \otimes M \otimes N \rightarrow \Gamma \otimes M \otimes N,$$

where the second map interchanges the second and third factors and the third map is the multiplication on  $\Gamma$ . All tensor products are over  $A$  using only the left  $A$ -module structure on  $A$ . A differential comodule  $C^*$  is a cochain complex in which each  $C^s$  is a comodule and the coboundary operator is a comodule map.

A1.1.3. THEOREM. *If  $\Gamma$  is flat as an  $A$ -module then the category of left  $\Gamma$ -comodules is abelian (see Hilton and Stammbach [1]).*

PROOF. If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $A$ -modules, then since  $\Gamma$  is flat over  $A$ ,

$$0 \rightarrow \Gamma \otimes_A M' \rightarrow \Gamma \otimes_A M \rightarrow \Gamma \otimes_A M'' \rightarrow 0$$

is also exact. If  $M$  is a left  $\Gamma$ -comodule then a comodule structure on either  $M'$  or  $M''$  will determine such a structure on the other one. From this fact it follows easily that the kernel or cokernel (as an  $A$ -module) of a map of comodules has a unique comodule structure, i.e., that the category has kernels and cokernels. The other defining properties of an abelian category are easily verified.  $\square$

In view of the above, we assume from now on that  $\Gamma$  is flat over  $A$ .

A1.1.4. DEFINITION. *Let  $M$  and  $N$  be right and left  $\Gamma$ -comodules, respectively. Their cotensor product over  $\Gamma$  is the  $K$ -module defined by the exact sequence*

$$0 \rightarrow M \square_\Gamma N \rightarrow M \otimes_A N \xrightarrow{\psi \otimes N - M \otimes \psi} M \otimes_A \Gamma \otimes_A N,$$

where  $\psi$  denotes the comodule structure maps for both  $M$  and  $N$ .

Note that  $M \square_\Gamma N$  is not a comodule or even an  $A$ -module but merely a  $K$ -module.

A left comodule  $M$  can be given the structure of a right comodule by the composition

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M \otimes c} M \otimes \Gamma,$$

where  $T$  interchanges the two factors and  $c$  is the conjugation map (see A1.1.1). A right comodule can be converted to a left comodule by a similar device. With this in mind we have

A1.1.5. PROPOSITION.  $M \square_\Gamma N = N \square_\Gamma M$ .

The following relates the cotensor product to  $\text{Hom}$ .

A1.1.6. LEMMA. *Let  $M$  and  $N$  be left  $\Gamma$ -comodules with  $M$  projective over  $A$ . Then*

- (a)  $\text{Hom}_A(M, A)$  is a right  $\Gamma$ -comodule and
- (b)  $\text{Hom}_\Gamma(M, N) = \text{Hom}_A(M, A) \square_\Gamma N$ , e.g.,  $\text{Hom}_\Gamma(A, N) = A \square_\Gamma N$ .

PROOF. Let  $\psi_M: M \rightarrow \Gamma \otimes_A M$  and  $\psi_N: N \rightarrow \Gamma \otimes_A N$  be the comodule structure maps. Define

$$\psi_M^*, \psi_N^*: \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, \Gamma \otimes_A N)$$

by

$$\psi_M^*(f) = (\Gamma \otimes f)\psi_M \quad \text{and} \quad \psi_N^*(f) = \psi_N f$$

for  $f \in \text{Hom}_A(M, N)$ . Since  $M$  is projective we have a canonical isomorphism,

$$\text{Hom}_A(M, A) \otimes_A N \approx \text{Hom}_A(M, N).$$

Hence for  $N = A$  we have

$$\psi_M^*: \text{Hom}_A(M, A) \rightarrow \text{Hom}_A(M, A) \otimes_A \Gamma.$$

To show that this is a right  $\Gamma$ -comodule structure we need to show that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_A(M, A) & \xrightarrow{\psi_M^*} & \mathrm{Hom}_A(M, \Gamma) \\ \psi_M^* \downarrow & & \downarrow \mathrm{Hom}(M, \Delta) \\ \mathrm{Hom}_A(M, \Gamma) & \xrightarrow{\psi_M^*} & \mathrm{Hom}_A(M, \Gamma \otimes \Gamma), \end{array}$$

i.e., that  $\psi_M^*$  is coassociative.

We have a straightforward calculation

$$\begin{aligned} \psi_M^* \psi_M^*(f) &= (\Gamma \otimes \psi_M^*(f)) \psi_M \\ &= (\Gamma \otimes ((\Gamma \otimes f) \psi_M)) \psi_M \\ &= (\Gamma \otimes \Gamma \otimes f) (\Gamma \otimes \psi_M) \psi_M \\ &= (\Gamma \otimes \Gamma \otimes f) (\Delta \otimes M) \psi_M \\ &= (\Delta \otimes A) (\Gamma \otimes f) \psi_M \\ &= (\Delta \otimes A) \psi_M^* f \end{aligned}$$

so the diagram commutes and (a) follows.

For (b) note that by definition

$$\mathrm{Hom}(M, N) = \ker(\psi_M^* - \psi_M^*) \subset \mathrm{Hom}_A(M, N)$$

while

$$\begin{aligned} \mathrm{Hom}_A(M, A) \square_\Gamma N &= \ker(\psi_M^* \otimes N - \mathrm{Hom}_A(M, A) \otimes \psi_N) \\ &\subset \mathrm{Hom}_A(M, A) \otimes_A N \end{aligned}$$

and the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}(M, A) \otimes N & \xrightarrow{\simeq} & \mathrm{Hom}_A(M, N) & \square \\ \psi_M^* \otimes N \downarrow \Downarrow \mathrm{Hom}(M, A) \otimes \psi_N & & \psi_M^* \downarrow \Downarrow \psi_N^* & \\ \mathrm{Hom}(M, A) \otimes \Gamma \otimes N & \xrightarrow{\simeq} & \mathrm{Hom}_A(M, \Gamma \otimes_A N) & \end{array}$$

The next few definitions and lemmas lead up to that of an extension of Hopf algebroids given in [A1.1.15](#). In [A1.3.14](#) we will derive a corresponding Cartan–Eilenberg spectral sequence.

**A1.1.7. DEFINITION.** *A map of Hopf algebroids  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  is a pair of  $K$ -algebra maps  $f_1: A \rightarrow B$ ,  $f_2: \Gamma \rightarrow \Sigma$  such that*

$$\begin{aligned} f_1 \varepsilon &= \varepsilon f_2, & f_2 \eta_R &= \eta_R f_1, & f_2 \eta_L &= \eta_L f_1, \\ f_2 c &= c f_2, & \text{and } \Delta f_2 &= (f_2 \otimes f_2) \Delta. \end{aligned}$$

**A1.1.8. LEMMA.** *Let  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  be a map of Hopf algebroids. Then  $\Gamma \otimes_A B$  is a right  $\Sigma$ -comodule and for any left  $\Sigma$ -comodule  $N$ ,  $(\Gamma \otimes_A B) \square_\Sigma N$  is a sub-left  $\Gamma$ -comodule of  $\Gamma \otimes_A N$ , where the structure map for the latter is  $\Delta \otimes N$ .*

**PROOF.** The map  $(\Gamma \otimes f_2) \Delta: \Gamma \rightarrow \Gamma \otimes_A \Sigma = (\Gamma \otimes_A B) \otimes_B \Sigma$  extends uniquely to  $\Gamma \otimes_A B$ , making it a right  $\Sigma$ -comodule. By definition  $(\Gamma \otimes_A B) \square_\Sigma N$  is the kernel in the exact sequence

$$0 \rightarrow (\Gamma \otimes_A B) \square_\Sigma N \rightarrow \Gamma \otimes_A N \rightarrow \Gamma \otimes_A \Sigma \otimes_B N$$

where the right-hand arrow is the difference between  $(\Gamma \otimes f_2)\Delta \otimes N$  and  $\Gamma \otimes \psi$ . Since  $\Gamma \otimes_A N$  and  $\Gamma \otimes_A \Sigma \otimes_B N$  are left  $\Gamma$ -comodules it suffices to show that the two maps respect the comodule structure. This is clear for  $\Gamma \otimes \psi$ , and for  $(\Gamma \otimes f)\Delta \otimes N$  we need the commutativity of the following diagram, tensored over  $B$  with  $N$ .

$$\begin{array}{ccc} \Gamma \otimes_A B & \xrightarrow{(\Gamma \otimes f_2)\Delta \otimes B} & \Gamma \otimes_A \Sigma \\ \Delta \otimes B \downarrow & & \downarrow \Delta \otimes \Sigma \\ \Gamma \otimes_A \Gamma \otimes_A B & \xrightarrow{\Gamma \otimes (\Gamma \otimes f_2)\Delta \otimes B} & \Gamma \otimes_A \Gamma \otimes_A \Sigma \end{array}$$

It follows from the fact that  $f$  is a Hopf algebroid map.  $\square$

**A1.1.9. DEFINITION.** *If  $(A, \Gamma)$  is a Hopf algebroid the associated Hopf algebra  $(A, \Gamma')$  is defined by  $\Gamma' = \Gamma / (\eta_L(a) - \eta_R(a) \mid a \in A)$ . (The easy verification that a Hopf algebra structure is induced on  $\Gamma'$  is left to the reader.)*

*Note that  $\Gamma'$  may not be flat over  $A$  even though  $\Gamma$  is.*

**A1.1.10. DEFINITION.** *A map of Hopf algebroids  $f: (A, \Gamma) \rightarrow (A, \Sigma)$  is normal if  $f_2: \Gamma \rightarrow \Sigma$  is surjective,  $f_1: A \rightarrow A$  is the identity, and  $\Gamma \square_{\Sigma'} A = A \square_{\Sigma'} \Gamma$  in  $\Gamma$ .*

**A1.1.11. DEFINITION.** *A Hopf algebroid  $(A, U)$  is unicursal if it is generated as an algebra by the images of  $\eta_L$  and  $\eta_R$ , i.e., if  $U = A \otimes_D A$  where  $D = A \square_U A$  is a subalgebra of  $A$ . (The reader can verify that the Hopf algebroid structure of  $U$  is unique.)  $\square$*

This term was taken from page 9 of Higgins [1].

**A1.1.12. LEMMA.** *Let  $M$  be a right comodule over a unicursal Hopf algebroid  $(A, U)$ . Then*

- (a)  $M$  is isomorphic as a comodule to  $M \otimes_A A$  with structure map  $M \otimes \eta_R$  and
- (b)  $M = (M \square_U A) \otimes_D A$  as  $A$ -modules.

**PROOF.** For  $m \in M$  let  $\psi(m) = m' \otimes m''$ . Since  $U$  is unicursal we can assume that each  $m''$  is in the image of  $\eta_R$ . It follows that

$$(\psi \otimes U)\psi(m) = (M \otimes \Delta)\psi(m) = m' \otimes 1 \otimes m''$$

so each  $m'$  is primitive. Let  $\tilde{m} = m' \varepsilon(m'')$ . Then  $\psi(\tilde{m}) = m' \otimes m'' = \psi(m)$ , so  $m = \tilde{m}$  since  $\psi$  is a monomorphism; Hence  $M$  is generated as an  $A$ -module by primitive elements and (a) follows. For (b) we have, using (a),

$$(M \square_U A) \otimes_D A = M \otimes_A (A \square_U A) \otimes_D A = M \otimes_A D \otimes_D A = M. \quad \square$$

**A1.1.13. LEMMA.** *Let  $(A, \Sigma)$  be a Hopf algebroid,  $(A, \Sigma')$  the associated Hopf algebra (A1.1.7)  $D = A \square_{\Sigma} A$ , and  $(A, U)$  the unicursal Hopf algebroid (A1.1.9) with  $U = A \otimes_D A$ . Then*

- (a)  $U = \Sigma \square_{\Sigma'} A$  and
- (b) for a left  $\Sigma$ -comodule  $M$ ,  $A \square_{\Sigma'} M$  is a left  $U$ -comodule and  $A \square_{\Sigma} M = A \square_U (A \square_{\Sigma'} M)$ .

**PROOF.** By definition,  $\Sigma' = A \otimes_U \Sigma$ , where the  $U$ -module structure on  $A$  is given by  $\varepsilon: U \rightarrow A$ , so we have

$$\Sigma \otimes_A \Sigma' = \Sigma \otimes_A A \otimes_U \Sigma = \Sigma \otimes_U \Sigma.$$

By [A1.1.3](#), there is a short exact sequence

$$0 \rightarrow \Sigma \square_{\Sigma'} A \rightarrow \Sigma \otimes_U \Sigma$$

where the last map is induced by  $\Delta - \Sigma \otimes \eta_L$ . An element  $\sigma \in \Sigma$  has  $\Delta(\sigma) = \sigma \otimes 1$  in  $\Sigma \otimes_U \Sigma$  iff  $\sigma \in U$ , so (a) follows.

For (b) we have

$$A \square_{\Sigma} M = A \square_U (U \square_{\Sigma} M)$$

and

$$U \square_{\Sigma} M = (A \square_{\Sigma'} \Sigma) \square_{\Sigma} M = A \square_{\Sigma'} M. \quad \square$$

The following example may be helpful. Let  $(A, \Gamma) = (\pi_*(BP), BP_*(BP))$  ([4.1.19](#)), i.e.,  $A = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  and  $\Gamma = A[t_1, t_2, \dots]$  where  $\dim v_i = \dim t_i = 2(p^i - 1)$ . Let  $\Sigma = A[t_{n+1}, t_{n+2}, \dots]$  for some  $n \geq 0$ . The Hopf algebroid structure on  $\Sigma$  is that of the quotient  $\Gamma/(t_1, \dots, t_n)$ . The evident map  $(A, \Gamma) \rightarrow (A, \Sigma)$  is normal ([A1.1.10](#)).  $D = A \square_{\Sigma} A$  is  $\mathbf{Z}_{(p)}[v_1, \dots, v_n]$  and  $\Phi = A \square_{\Sigma} \Gamma \square_{\Sigma} A$  is  $D[t_1, \dots, t_n]$ .  $(D, \Phi)$  is a sub-Hopf algebroid of  $(A, \Gamma)$  and  $(D, \Phi) \rightarrow (A, \Gamma) \rightarrow (A, \Sigma)$  is an extension ([A1.1.15](#) below).

**A1.1.14. THEOREM.** *Let  $f: (A, \Gamma) \rightarrow (A, \Sigma)$  be a normal map of Hopf algebroids and let  $D = A \square_{\Sigma} A$  and  $\Phi = A \square_{\Sigma} \Gamma \square_{\Sigma} A$ . Then  $(D, \Phi)$  is a sub-Hopf algebroid of  $(A, \Gamma)$ .*

(Note that by [A1.1.8](#),  $A \square_{\Sigma} \Gamma$  and  $\Gamma \square_{\Sigma} A$  are right and left  $\Gamma$ -comodules, respectively, so the expressions  $(A \square_{\Sigma} \Gamma) \square_{\Sigma} A$  and  $A \square_{\Sigma} (\Gamma \square_{\Sigma} A)$  make sense. It is easy to check, without using the normality of  $f$ , that they are equal, so  $\Phi$  is well defined.)

**PROOF.** By definition an element  $a \in A$  is in  $D$  iff  $f_2 \eta_L(a) = f_2 \eta_R(a)$  and is in  $\Phi$  iff  $(f_2 \otimes \Gamma \otimes f_2) \Delta^2(\gamma) = 1 \otimes \gamma \otimes 1$ . To see that  $\eta_R$  sends  $D$  to  $\Phi$ , we have for  $d \in D$

$$\begin{aligned} (f_2 \otimes \Gamma \otimes f_2) \Delta^2 \eta_R(d) &= 1 \otimes 1 \otimes f_2 \eta_R(d) \\ &= 1 \otimes 1 \otimes f_2 \eta_L(d) = 1 \otimes \eta_R(d) \otimes 1. \end{aligned}$$

The argument for  $\eta_L$  is similar. It is clear that  $\Phi$  is invariant under the conjugation  $c$ . To show that  $\varepsilon$  sends  $\Phi$  to  $D$  we need to show  $f_2 \eta_R \varepsilon(\phi) = f_2 \eta_L \varepsilon(\phi)$  for  $\phi \in \Phi$ . But  $f_2 \eta_R \varepsilon(\phi) = \eta_R \varepsilon f_2(\phi)$  and since  $\Delta^2 f_2(\phi) = 1 \otimes f_2(\phi) \otimes 1$  we have  $\Delta f_2(\phi) = 1 \otimes f_2(\phi) = f_2(\phi) \otimes 1$  so  $f_2(\phi) \in D$ , and  $(\eta_R - \eta_L) \varepsilon f_2(\phi) = 0$ .

To define a coproduct on  $\Phi$  we first show that the natural map from  $\Phi \otimes_D \Phi$  to  $\Gamma \otimes_A \Gamma$  is monomorphic. This amounts to showing that  $a\phi \in \Phi$  iff  $a \in D$ . Now by definition  $a\phi \in \Phi$  iff

$$f_2(a\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes a\phi \otimes 1 = f_2 \eta_R(a) \otimes \phi \otimes 1.$$

Since  $\phi \in \Phi$  we have

$$f_2(\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes \phi \otimes 1,$$

so the criterion is

$$f_2(a) \otimes 1 \otimes 1 = f_2 \eta_R(a) \otimes 1 \otimes 1,$$

i.e.,  $a \in D$ .

Now consider the commutative diagram

$$\begin{array}{ccccc}
 (D, \Phi) & \longrightarrow & (A, \tilde{\Phi}) & \xrightarrow{g} & (A, U) \\
 \parallel & & \downarrow & & \downarrow \\
 (D, \Phi) & \longrightarrow & (A, \Gamma) & \xrightarrow{f} & (A, \Sigma) \\
 & & \downarrow f' & & \downarrow \\
 & & (A, \Sigma') & \equiv & (A, \Sigma')
 \end{array}$$

where  $\Sigma'$  is the Hopf algebra associated to  $\Sigma$  (A1.1.9),  $f'$  is the induced map,  $U$  is the unicursal Hopf algebroid (A1.1.11)  $A \otimes_D A$ ,  $\tilde{\Phi} = A \square_{\Sigma'} \Gamma \square_{\Sigma'} A$ , and  $g$  will be constructed below. We will see that  $\Phi$  and  $\tilde{\Phi}$  are both Hopf algebroids.

Now the map  $f'$  is normal since  $f$  is and  $A \square_{\Sigma'} A = A$ , so the statement that  $\tilde{\Phi}$  is a Hopf algebroid is a special case of the theorem. Hence we have already shown that it has all of the required structure but the coproduct. Since  $\Gamma \square_{\Sigma'} A = A \square_{\Sigma'} \Gamma$ , we have  $\tilde{\Phi} = A \square_{\Sigma'} \Gamma \square_{\Sigma'} A = A \square_{\Sigma'} A \square_{\Sigma'} \Gamma = A \square_{\Sigma'} \Gamma$ . One easily verifies that the image of  $\Delta: \Gamma \rightarrow \Gamma \otimes_A \Gamma$  is contained in  $\Gamma \square_{\Gamma} \Gamma$  and hence in  $\Gamma \square_{\Sigma'} \Gamma$ . There  $\Delta$  sends  $\tilde{\Phi} = A \square_{\Sigma'} \Gamma \square_{\Sigma'} A$  to  $A \square_{\Sigma'} \Gamma \square_{\Sigma'} \Gamma \square_{\Sigma'} A = \tilde{\Phi} \square_{\Sigma'} \tilde{\Phi} \subset \tilde{\Phi} \otimes_A \tilde{\Phi}$ , so  $\tilde{\Phi}$  is a Hopf algebroid.

Since  $\tilde{\Phi} = \Gamma \square_{\Sigma'} A$  and  $U = \Sigma \square_{\Sigma'} A$  [A1.1.13(a)] we can define  $g$  to be  $f_2 \square A$ . It follows from A1.1.13(b) that

$$\begin{aligned}
 \Phi &= A \square_{\Sigma} \Gamma \square_{\Sigma} A = A \square_U (A \square_{\Sigma'} \Gamma \square_{\Sigma'} A) \square_U A \\
 &= A \square_U \tilde{\Phi} \square_U A.
 \end{aligned}$$

By A1.1.12(b) we have  $\tilde{\Phi} = A \otimes_D \Phi \otimes_D A$ , so  $\tilde{\Phi} \otimes_A \tilde{\Phi} = A \otimes_D \tilde{\Phi} \otimes_D A \otimes_D \Phi \otimes_D A$ . The coproduct  $\Delta$  sends  $\tilde{\Phi}$  to  $\tilde{\Phi} \square_U \tilde{\Phi} \subset \tilde{\Phi} \otimes_A \tilde{\Phi}$  and we have

$$\begin{aligned}
 \tilde{\Phi} \square_U \tilde{\Phi} &= \tilde{\Phi} \otimes_A (A \square_U A) \otimes_A \tilde{\Phi} \quad \text{by A1.1.12(a)} \\
 &= A \otimes_D \tilde{\Phi} \otimes_D (A \square_U A) \otimes_D \tilde{\Phi} \otimes_D A \\
 &= A \otimes_D \tilde{\Phi} \otimes_D D \otimes_D \tilde{\Phi} \otimes_D A \\
 &= A \otimes_D \tilde{\Phi} D \otimes \tilde{\Phi} \otimes_D A.
 \end{aligned}$$

Since  $\Delta$  is  $A$ -bilinear it sends  $\Phi$  to  $\Phi \otimes_D \Phi$  and  $\Phi$  is a Hopf algebroid.  $\square$

A1.1.15. DEFINITION. *An extension of Hopf algebroids is a diagram*

$$(D, \Phi) \xrightarrow{i} (A, \Gamma) \xrightarrow{f} (A, \Sigma)$$

where  $f$  is normal (A1.1.10) and  $(D, \Phi)$  is as in A1.1.14.

The extension is cocentral if the diagram

$$\begin{array}{ccc}
 & \Gamma \otimes \Sigma & \\
 (\Gamma \otimes f_2) \Delta \nearrow & \downarrow t & \\
 \Gamma & & \Sigma \otimes \Gamma \\
 (f_2 \otimes \Gamma) \Delta \searrow & & 
 \end{array}$$

(where  $t$  interchanges factors) commutes up to the usual sign. In particular  $\Sigma$  must be cocommutative.

A nice theory of Hopf algebra extensions is developed by Singer [5] and in Section II 3 of Singer [6].

Note that (as shown in the proof of A1.1.14) if  $\Sigma$  is a Hopf algebra then  $\Phi = A \square_{\Sigma} \Gamma = \Gamma \square_{\Sigma} A$ . More generally we have

A1.1.16. LEMMA. *With notation as above,  $A \square_{\Sigma} \Gamma = \Phi \otimes_D A$  as right  $\Gamma$ -comodules.*

PROOF. Using A1.1.12 and A1.1.13 we have

$$\begin{aligned}
 \Phi \otimes_D A &= A \square_{\Sigma} \Gamma \square_{\Sigma} A \otimes_D A \\
 &= A \square_{\Sigma} \Gamma \square_{\Sigma'} A \square_I A \otimes_D A \\
 &= A \square_{\Sigma} \Gamma \square_{\Sigma'} A \\
 &= A \square_{\Sigma} A \square_{\Sigma'} \Gamma \\
 &= A \square_U A \square_{\Sigma'} A \square_{\Sigma'} \Gamma \\
 &= A \square_U A \square_{\Sigma'} \Gamma \\
 &= A \square_{\Sigma} \Gamma. \quad \square
 \end{aligned}$$

A1.1.17. COMODULE ALGEBRA STRUCTURE THEOREM. *Let  $(B, \Sigma)$  be a graded connected Hopf algebroid,  $M$  a graded connected right  $\Sigma$ -comodule algebra, and  $C = M \square_{\Sigma} B$ . Suppose*

- (i) *there is a surjective comodule algebra map  $f: M \rightarrow \Sigma$  and*
- (ii)  *$C$  is a  $B$ -module and as such it is a direct summand of  $M$ .*

*Then  $M$  is isomorphic to  $C \otimes_B \Sigma$  simultaneously as a left  $C$ -module and a right  $\Sigma$ -comodule.*  $\square$

We will prove this after listing some corollaries. If  $\Sigma$  is a Hopf algebra over a field  $K$  then the second hypothesis is trivial so we have the following result, first proved as Theorem 4.7 of Milnor and Moore [3].

A1.1.18. COROLLARY. *Let  $(K, \Sigma)$  be a commutative graded connected Hopf algebra over a field  $K$ . Let  $M$  be a  $K$ -algebra and a right  $\Sigma$ -comodule and let  $C = M \square_{\Sigma} K$ . If there is a surjection  $f: M \rightarrow \Sigma$  which is a homomorphism of algebras and  $\Sigma$ -comodules, then  $M$  is isomorphic to  $C \otimes \Sigma$  simultaneously as a left  $C$ -module and as a right  $\Sigma$ -comodule.*  $\square$

A1.1.19. COROLLARY. *Let  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  be a map of graded connected Hopf algebroids (A1.1.7) and let  $\Gamma' = \Gamma \otimes_A B$  and  $C = \Gamma' \square_{\Sigma} B$ . Suppose*

- (i)  *$f'_2: \Gamma' \rightarrow \Sigma$  is onto and*
- (ii)  *$C$  is a  $B$ -module and there is a  $B$ -linear map  $g: \Gamma' \rightarrow C$  split by the inclusion of  $C$  in  $\Gamma'$ .*

*Then there is a map  $\tilde{g}: \Gamma' \rightarrow C \otimes_B \Sigma$  defined by  $\tilde{g}(\gamma) = g(\gamma') \otimes f'_2(\gamma'')$  which is an isomorphism of  $C$ -modules and  $\Sigma$ -comodules.*  $\square$

A1.1.20. COROLLARY. *Let  $K$  be a field and  $f: (K, \Gamma) \rightarrow (K, \Sigma)$  a map of graded connected commutative Hopf algebras and let  $C = \Gamma \square_{\Sigma} K$ . If  $f$  is surjective then  $\Gamma$  is isomorphic to  $C \otimes \Sigma$  simultaneously as a left  $C$ -module and as a right  $\Sigma$ -comodule.*  $\square$



In [A1.3.12](#) and [A1.3.13](#) we will give some change-of-rings isomorphisms of Ext groups relevant to the maps in the previous two corollaries.

PROOF OF [A1.1.17](#). Let  $i: C \rightarrow M$  be the natural inclusion and let  $g: M \rightarrow C$  be a  $B$ -linear map such that  $gi$  is the identity. Define  $\tilde{g}: M \rightarrow C \otimes_B \Sigma$  to be  $(g \otimes \Sigma)\psi$ ; it is a map of  $\Sigma$ -comodules but not necessarily of  $C$ -modules and we will show below that it is an isomorphism.

Next observe that  $f \square B: C \rightarrow B$  is onto. In dimension zero it is simply  $f$ , which is onto by assumption, and it is  $B$ -linear and therefore surjective. Let  $j: B \rightarrow C$  be a  $B$ -linear splitting of  $f \square B$ . Then  $h = \tilde{g}^{-1}(j \otimes \Sigma): \Sigma \rightarrow M$  is a comodule splitting of  $f$ .

Define  $\tilde{h}: C \otimes_B \Sigma \rightarrow M$  by  $\tilde{h}(c \otimes \sigma) = i(c)h(\sigma)$  for  $c \in C$  and  $\sigma \in \Sigma$ . It is clearly a  $C$ -linear comodule map and we will show that it is the desired isomorphism. We have

$$\tilde{g}\tilde{h}(c \otimes \sigma) = \tilde{g}(i(c)h(\sigma)) = g(i(c)h(\sigma')) \otimes \sigma'' = c \otimes \sigma$$

where the second equality holds because  $i(c)$  is primitive in  $M$  and the congruence is modulo elements of lower degree with respect to the following increasing filtration (A1.2.7) on  $C \otimes_B \Sigma$ . Define  $F_n(C \otimes_B \Sigma) \subset C \otimes_B \Sigma$  to be the sub- $K$ -module generated by elements of the form  $c \otimes \sigma$  with  $\dim \sigma \leq n$ . It follows that  $\tilde{g}\tilde{h}$  and hence  $h$  are isomorphisms.

We still need to show that  $\tilde{g}$  is an isomorphism. To show that it is 1-1, let  $\tilde{m} \otimes \sigma$  be the leading term (with respect to the above filtration of  $M \otimes \Sigma$ ) of  $\psi(m)$ . It follows from coassociativity that  $\tilde{m}$  is primitive, so  $g(\tilde{m}) \neq 0$  if  $m \neq 0$  and  $\ker \tilde{g} = 0$ . To show that  $\tilde{g}$  is onto, note that for any  $c \otimes \sigma \in C \otimes_B \Sigma$  we can choose  $m \in f^{-1}(\sigma)$  and we have

$$\tilde{g}(i(c)m) = g(i(c)m') \otimes m'' = gi(c) \otimes \sigma = c \otimes \sigma$$

so  $\text{coker } \tilde{g} = 0$  by standard arguments. □

A1.1.21. DEFINITION. *An ideal  $I \subset A$  is invariant if it is a sub- $\Gamma$ -comodule, or equivalently if  $\eta_R(I) \subset I\Gamma$ .*

The following definition is intended to mimic that of a split groupoid, which is derived from the action of a group  $G$  acting on a set  $X$ . Here the set of objects is  $X$  and the set of morphism is  $G \times X$ , where  $(g, x)$  is a morphism from the object  $x \in X$  to the object  $g(x)$ .

A1.1.22. DEFINITION. *A Hopf algebroid  $(A, \Gamma)$  is split if there is a Hopf algebroid map  $i: (K, \Sigma) \rightarrow (A, \Gamma)$  (A1.1.19) such that  $i'_2: \Sigma \otimes A \rightarrow \Gamma$  is an isomorphism of  $K$ -algebras.*

Note that composing  $\eta_R: A \rightarrow \Gamma$  with the inverse of  $i'_2$  defines a left  $\Sigma$ -comodule structure on  $A$ .

## 2. Homological Algebra

Recall ([A1.1.3](#)) that the category of comodules over a Hopf algebroid  $(A, \Gamma)$  is abelian provided  $\Gamma$  is flat over  $A$ , which means that we can do homological algebra in it. We want to study the derived functors of Hom and cotensor product ([A1.1.4](#)). Derived functors are discussed in most books on homological algebra, e.g., Cartan and Eilenberg [[1](#)], Hilton and Stammbach [[1](#)], and Mac Lane [[1](#)]. In order to define them we must be sure that our category has enough injectives, i.e., that each  $\Gamma$ -comodule can be embedded in an injective one. This can be seen as follows.

A1.2.1. DEFINITION. Given an  $A$ -module  $N$ , define a comodule structure on  $\Gamma \otimes_A N$  by  $\psi = \Delta \otimes N$ . Then for any comodule  $M$ ,

$$\theta: \text{Hom}_A(M, N) \rightarrow \text{Hom}_\Gamma(M, \Gamma \otimes_A N)$$

is the isomorphism given by  $\theta(f) = (\Gamma \otimes f)\psi_M$  for  $f \in \text{Hom}_A(M, N)$ . For  $g \in \text{Hom}_\Gamma(M, \Gamma \otimes_A N)$ ,  $\theta^{-1}(g)$  is given by  $\theta^{-1}(g) = (\varepsilon \otimes N)g$ .

A1.2.2. LEMMA. If  $I$  is an injective  $A$ -module then  $\Gamma \otimes_A I$  is an injective  $\Gamma$ -comodule. Hence the category of  $\Gamma$ -comodules has enough injectives.

PROOF. To show that  $\Gamma \otimes_A I$  is injective we must show that if  $M$  is a subcomodule of  $N$ , then a comodule map from  $M$  to  $\Gamma \otimes_A I$  extends to  $N$ . But  $\text{Hom}_\Gamma(M, \Gamma \otimes_A I) = \text{Hom}_A(M, I)$  which is a subgroup of  $\text{Hom}_A(N, I) = \text{Hom}_\Gamma(N, \Gamma \otimes_A I)$  since  $I$  is injective as an  $A$ -module. Hence the existence of enough injectives in the category of  $A$ -modules implies the same in the category of  $\Gamma$ -comodules.  $\square$

This result allows us to make

A1.2.3. DEFINITION. For left  $\Gamma$ -comodules  $M$  and  $N$ ,  $\text{Ext}_\Gamma^i(M, N)$  is the  $i$ th right derived functor of  $\text{Hom}_\Gamma(M, N)$ , regarded as a functor of  $N$ . For  $M$  a right  $\Gamma$ -comodule,  $\text{Cotor}_\Gamma^i(M, N)$ , is the  $i$ th right derived functor of  $M \square_\Gamma N$  (A1.1.4), also regarded as a functor of  $N$ . The corresponding graded groups will be denoted simply by  $\text{Ext}_\Gamma(M, N)$  and  $\text{Cotor}_\Gamma(M, N)$ , respectively.

In practice we shall only be concerned with computing these functors when the first variable is projective over  $A$ . In that case the two functors are essentially the same by A1.1.6. We shall therefore make most of our arguments in terms of  $\text{Cotor}$  and list the corresponding statements about  $\text{Ext}$  as corollaries without proof.

Recall that the zeroth right derived functor is naturally equivalent to the functor itself if the latter is left exact. The cotensor product is left exact in the second variable if the first variable is flat as an  $A$ -comodule.

One knows that right derived functors can be computed using an injective resolution of the second variable. In fact the resolution need only satisfy a weaker condition.

A1.2.4. LEMMA. Let

$$0 \rightarrow N \rightarrow R^0 \rightarrow R^1 \rightarrow \dots$$

be a long exact sequence of left  $\Gamma$ -comodules such that  $\text{Cotor}_\Gamma^n(M, R^i) = 0$  for  $n > 0$ . Then  $\text{Cotor}_\Gamma(M, N)$  is the cohomology of the complex

$$(A1.2.5) \quad \text{Cotor}_\Gamma^0(M, R^0) \xrightarrow{\delta_0} \text{Cotor}_\Gamma^0(M, R^1) \xrightarrow{\delta_1} \dots$$

PROOF. Define comodules  $N^i$  inductively by  $N^0 = N$  and  $N^{i+1}$  is the quotient in the short exact sequence

$$0 \rightarrow N^i \rightarrow R^i \rightarrow N^{i+1} \rightarrow 0.$$

These give long exact sequences of  $\text{Cotor}$  groups which, because of the behavior of  $\text{Cotor}_\Gamma(M, R^i)$ , reduce to four-term sequences

$$\begin{aligned} 0 \rightarrow \text{Cotor}_\Gamma^0(M, N^i) &\rightarrow \text{Cotor}_\Gamma^0(M, R^i) \\ &\rightarrow \text{Cotor}_\Gamma^0(M, N^{i+1}) \rightarrow \text{Cotor}_\Gamma^1(M, N^i) \rightarrow 0 \end{aligned}$$

and isomorphisms

$$(A1.2.6) \quad \text{Cotor}_{\Gamma}^n(M, N^{i+1}) \approx \text{Cotor}_{\Gamma}^{n+1}(M, N^i) \quad \text{for } n > 0.$$

Hence in A1.2.5,  $\ker \delta_i = \text{Cotor}_{\Gamma}^0(M, N^i)$  while  $\text{im } \delta_i$  is the image of  $\text{Cotor}_{\Gamma}^0(M, R^i)$  in  $\text{Cotor}_{\Gamma}^0(M, N^{i+1})$  so

$$\ker \delta_i / \text{im } \delta_{i-1} = \text{Cotor}_{\Gamma}^1(M, N^{i-1}) = \text{Cotor}_{\Gamma}^i(M, N)$$

by repeated use of A1.2.6. This quotient by definition is  $H^i$  of A1.2.5.  $\square$

For another proof see A1.3.2.

We now introduce a class of comodules which satisfy the Ext condition of A1.2.4 when  $M$  is projective over  $A$ .

A1.2.7. DEFINITION. *An extended  $\Gamma$ -comodule is one of the form  $\Gamma \otimes_A N$  where  $N$  is an  $A$ -module. A relatively injective  $\Gamma$ -comodule is a direct summand of an extended one.*

This terminology comes from relative homological algebra, for which the standard references are Eilenberg and Moore [1] and Chapter IX of Mac Lane [1]. Our situation is dual to theirs in the following sense. We have the category  $\mathbf{\Gamma}$  of left (or right)  $\Gamma$ -comodules, the category  $\mathbf{A}$  of  $A$ -modules, the forgetful functor  $G$  from  $\mathbf{\Gamma}$  to  $\mathbf{A}$ , and a functor  $F: \mathbf{A} \rightarrow \mathbf{\Gamma}$  given by  $F(M) = \Gamma \otimes_A M$  (A1.2.1). Mac Lane [1] then defines a resolvent pair to be the above data along with a natural transformation from  $GF$  to the identity on  $\mathbf{A}$ , i.e., natural maps  $M \rightarrow \Gamma \otimes_A M$  with a certain universal property. We have instead maps  $\varepsilon \otimes M: \Gamma \otimes_A M \rightarrow M$  such that for any  $A$ -homomorphism  $\mu: C \rightarrow M$  where  $C$  is a  $\Gamma$ -comodule there is a unique comodule map  $\alpha: C \rightarrow \Gamma \otimes_A M$  such that  $\mu = (\varepsilon \otimes M)\alpha$ . Thus we have what Mac Lane might call a coresolvent pair. Our  $F$  produces relative injectives while his produces relative projectives. This duality is to be expected because the example he had in mind was the category of modules over an algebra, while our category  $\mathbf{\Gamma}$  is more like that of comodules over a coalgebra. The following lemma is comparable to Theorem IX.6.1 of Mac Lane [1].

A1.2.8. LEMMA.

(a) *If  $i: M \rightarrow N$  is a monomorphism of comodules which is split over  $A$ , then any map  $f$  from  $M$  to a relatively injective comodule  $S$  extends to  $N$ . (If  $i$  is not assumed to be split, then this property would make  $S$  injective.)*

(b) *If  $M$  is projective as an  $A$ -module and  $S$  is a relatively injective comodule, then  $\text{Cotor}_{\Gamma}^i(M, S) = 0$  for  $i > 0$  and if  $S = \Gamma \otimes_A N$  then  $\text{Cotor}_{\Gamma}^0(M, S) = M \otimes_A N$ .*

PROOF. (a) Let  $j: N \rightarrow M$  be a splitting of  $i$ . Then  $(\Gamma \otimes f)(\Gamma \otimes j)\psi = g$  is a comodule map from  $N$  to  $\Gamma \otimes_A S$  such that  $gi = \psi f: M \rightarrow \Gamma \otimes_A S$ . It suffices then to show that  $S$  is a direct summand of  $\Gamma \otimes_A S$ , for then  $g$  followed by the projection of  $\Gamma \otimes_A S$  onto  $S$  will be the desired extension of  $f$ . By definition  $S$  is a direct summand of  $\Gamma \otimes_A T$  for some  $A$ -module  $T$ . Let  $k: S \rightarrow \Gamma \otimes_A T$  and  $k^{-1}: \Gamma \otimes_A T \rightarrow S$  be the splitting maps. Then  $k^{-1}(\Gamma \otimes \varepsilon \otimes T)(\Gamma \otimes k)$  is the projection of  $\Gamma \otimes_A S$  onto  $S$ .

(b) One has an isomorphism  $\phi: M \otimes_A N \rightarrow M \square_{\Gamma}(\Gamma \otimes_A N)$  given by  $\phi(m \otimes n) = \psi(m) \otimes n$ . Since  $S$  is a direct summand of  $\Gamma \otimes_A N$ , it suffices to replace the former by the latter. Let

$$0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

be a resolution of  $N$  by injective  $A$ -modules. Tensoring over  $A$  with  $\Gamma$  gives a resolution of  $\Gamma \otimes_A N$  by injective  $\Gamma$ -comodules.  $\text{Cotor}_\Gamma(M, \Gamma \otimes_A N)$  is the cohomology of the resolution cotensored with  $M$ , which is isomorphic to

$$M \otimes_A I^0 \rightarrow M \otimes_A I^1 \rightarrow \cdots$$

This complex is acyclic since  $M$  is projective over  $A$ .  $\square$

Compare the following with Theorem IX.4.3 of Mac Lane [1].

A1.2.9. LEMMA. (a) *Let*

$$0 \rightarrow M \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \xrightarrow{d_1} \cdots$$

and

$$0 \rightarrow N \xrightarrow{d_{-1}} R^0 \xrightarrow{d_0} R^1 \xrightarrow{d_1} \cdots$$

be long exact sequences of  $\Gamma$ -comodules in which each  $P^i$  and  $R^i$  is relatively injective and the image of each map is a direct summand over  $A$ . Then a comodule map  $f: M \rightarrow N$  extends to a map of long exact sequences.

(b) *Applying  $L \square_\Gamma (\cdot)$  (where  $L$  is a right  $\Gamma$ -comodule projective over  $A$ ) to the two sequences and taking cohomology gives  $\text{Cotor}_\Gamma(L, M)$  and  $\text{Cotor}_\Gamma(L, N)$ , respectively. The induced map from the former to the latter depends only on  $f$ .*

PROOF. That the cohomology indicated in (b) is  $\text{Cotor}$  follows from A1.2.4 and A1.2.8(b). The proof of the other assertions is similar to that of the analogous statements about injective resolutions. Define comodules  $M^i$  and  $N^i$  inductively by  $M^0 = M$ ,  $N^0 = N$ , and  $M^{i+1}$  and  $N^{i+1}$  are the quotients in the short exact sequences

$$0 \rightarrow M^i \rightarrow P^i \rightarrow M^{i+1} \rightarrow 0$$

and

$$0 \rightarrow N^i \rightarrow R^i \rightarrow N^{i+1} \rightarrow 0.$$

These sequences are split over  $A$ . Assume inductively that we have a suitable map from  $M^i$  to  $N^i$ . Then A1.2.8(a) gives us  $f_i: P^i \rightarrow R^i$ , and this induces a map from  $M^{i+1}$  to  $N^{i+1}$ , thereby proving (a).

For (b) it suffices to show that the map of long exact sequences is unique up to chain homotopy, i.e., given two sets of maps  $f_i, f'_i: P^i \rightarrow R^i$  we need to construct  $h_i: P^i \rightarrow R^{i-1}$  (with  $h_0 = 0$ ) such that  $h_{i+1}d_i + d_{i-1}h_i = f_i - f'_i$ . Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^i & \xrightarrow{d_{i-1}} & P^i & \xrightarrow{d_i} & M^{i+1} & \longrightarrow & 0 \\ & & \downarrow g_{i-1} & & \downarrow g_i & & \downarrow g_i & & \\ 0 & \longrightarrow & N^i d_{i-1} & \longrightarrow & R^i d_i & \longrightarrow & N^{i+1} & \longrightarrow & 0 \end{array}$$

where  $g_i = f_i - f'_i: P^i \rightarrow R^i$  and we use the same notation for the map induced from the quotient  $M^{i+1}$ . Assume inductively that  $h_i: P^i \rightarrow R^{i-1}$  has been constructed. Projecting it to  $N^i$  we get  $h_i: P^i \rightarrow N^i$  with  $h_i d_{i-1} = g_{i-1}$ . Now we want a map  $h_{i+1}: M^{i+1} \rightarrow R^i$  such that  $h_{i+1}d_i = g_i - d_{i-1}h_i$ . By the exactness of the top row,  $h_{i+1}$  exists iff  $(g_i - d_{i-1}h_i)d_{i-1} = 0$ . But we have  $g_i d_{i-1} - d_{i-1}(h_i d_{i-1}) = g_i d_{i-1} - d_i g_{i-1} = 0$ , so  $h_{i+1}$  exists. By A1.2.8(a) it extends from  $M^{i+1}$  to  $P^{i+1}$  giving the desired  $h_{i+1}$ .  $\square$

Resolution of the above type serve as a substitute for injective resolutions. Hence we have

A1.2.10. DEFINITION. A resolution by relative injectives of a comodule  $M$  is a long exact sequence

$$0 \rightarrow M \rightarrow R^0 \rightarrow R^1 \rightarrow \dots$$

in which each  $R^i$  is a relatively injective and the image of each map is a direct summand over  $A$ . We now give an important example of such a resolution.

A1.2.11. DEFINITION. Let  $M$  be a left  $\Gamma$ -comodule. The cobar resolution  $D_\Gamma^*(M)$  is defined by  $D_\Gamma^s(M) = \Gamma \otimes_A \bar{\Gamma}^{\otimes s} \otimes_A M$ , where  $\bar{\Gamma}$  is the unit coideal (the cokernel of  $\eta_L$ ), with coboundary  $d_s: D_\Gamma^s(M) \rightarrow D_\Gamma^{s+1}(M)$  given by

$$\begin{aligned} d_s(\gamma_0 \otimes \gamma_1 \otimes \dots \otimes \gamma_s \otimes m) &= \sum_{i=0}^s (-1)^i \gamma_0 \otimes \dots \otimes \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \dots \otimes m \\ &\quad + (-1)^{s+1} \gamma_0 \otimes \dots \otimes \gamma_s \otimes \psi(m) \end{aligned}$$

for  $\gamma_0 \in \Gamma$ ,  $\gamma_1, \dots, \gamma_s \in \bar{\Gamma}$ , and  $m \in M$ . For a right  $\Gamma$ -comodule  $L$  which is projective over  $A$ , the cobar complex  $C_\Gamma^*(L, M)$  is  $L \square_\Gamma D_\Gamma^*(M)$ , so  $C_\Gamma^s(L, M) = L \otimes_A \bar{\Gamma}^{\otimes s} \otimes_A M$ , where  $\Gamma^{\otimes s}$  denotes the  $s$ -fold tensor product of  $\Gamma$  over  $A$ . Whenever possible the subscript  $\Gamma$  will be omitted, and  $C_\Gamma^*(A, M)$  will be abbreviated to  $C_\Gamma^*(M)$ . The element  $a \otimes \gamma_1 \otimes \dots \otimes \gamma_n \otimes m \in C_\Gamma(L, M)$ , where  $a \in L$ , will be denoted by  $a\gamma_1|\gamma_2|\dots|\gamma_n m$ . If  $a = 1$  or  $m = 1$ , they will be omitted from this notation.

A1.2.12. COROLLARY.  $H(C_\Gamma^*(L, M)) = \text{Cotor}_\Gamma(L, M)$  if  $L$  is projective over  $A$ , and  $H(C_\Gamma^*(M)) = \text{Ext}_\Gamma(A, M)$ .

PROOF. It suffices by A1.2.9 to show that  $D_\Gamma(M) = C_\Gamma(\Gamma, M)$  is a resolution of  $M$  by relative injectives. It is clear that  $D_\Gamma^s(M)$  is a relative injective and that  $d^s$  is a comodule map. To show that  $D_\Gamma(M)$  is acyclic we use a contracting homotopy  $S: D_\Gamma^s(M) \rightarrow D_\Gamma^{s-1}(M)$  defined by  $S(\gamma\gamma_1|\dots|\gamma_s m) = \varepsilon(\gamma)\gamma_1\gamma_2|\dots|\gamma_s m$  for  $s > 0$  and  $S(\gamma m) = 0$ . Then  $Sd + dS$  is the identity on  $D_\Gamma^s(M)$  for  $s > 0$ , and  $1 - \phi$  on  $D_\Gamma^0(M)$ , where  $\phi(\gamma m) = \varepsilon(\gamma)m'm''$ . Hence

$$H^s(D_\Gamma(M)) = \begin{cases} 0 & \text{for } s > 0, \\ \text{im } \phi = M & \text{for } s = 0. \end{cases} \quad \square$$

Our next job is to define the external cup product in  $\text{Cotor}$ , which is a map  $\text{Cotor}_\Gamma(M_1, N_1) \otimes \text{Cotor}_\Gamma(M_2, N_2) \rightarrow \text{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$  (see A1.1.2 for the definition of the comodule tensor product). If  $M_1 = M_2 = M$  and  $N_1 = N_2 = N$  are comodule algebras (A1.1.2) then composing the above with the map in  $\text{Cotor}$  induced by  $M \otimes_A M \rightarrow M$  and  $N \otimes_A N \rightarrow N$  gives a product on  $\text{Cotor}_\Gamma(M, N)$ . Let  $P_1^*$  and  $P_2^*$  denote relative injective resolutions of  $N_1$  and  $N_2$ , respectively. Then  $P_1^* \otimes_A P_2^*$  is a resolution of  $N_1 \otimes_A N_2$ . We have canonical maps

$$\text{Cotor}_\Gamma(M_1, N_1) \otimes \text{Cotor}_\Gamma(M_2, N_2) \rightarrow H(M_1 \square_\Gamma P_1^* \otimes M_2 \square_\Gamma P_2^*)$$

(with tensor products over  $K$ ) and

$$M_1 \square_\Gamma P_1^* \otimes M_2 \square_\Gamma P_2^* \rightarrow (M_1 \otimes_A M_2) \square_\Gamma (P_1^* \otimes_A P_2^*).$$

A1.2.13. DEFINITION. *The external cup product*

$$\text{Cotor}_\Gamma(M_1, N_1) \otimes \text{Cotor}_\Gamma(M_2, N_2) \rightarrow \text{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$$

and the internal cup product on  $\text{Cotor}_\Gamma(M, N)$  for comodule algebras  $M$  and  $N$  are induced by the maps described above.

Note that A1.2.9(b) implies that these products are independent of the choices made. Since the internal product is the composition of the external product with the products on  $M$  and  $N$  and since the latter are commutative and associative we have

A1.2.14. COROLLARY. *If  $M$  and  $N$  are comodule algebras then  $\text{Cotor}_\Gamma(M, N)$  is a commutative (in the graded sense) associative algebra.*  $\square$

It is useful to have an explicit pairing on cobar complexes

$$C_\Gamma(M_1, N_1) \otimes C_\Gamma(M_2, N_2) \rightarrow C_\Gamma(M_1 \otimes M_2, N_1 \otimes N_2).$$

This can be derived from the definitions by tedious straightforward calculation. To express the result we need some notation. For  $m_2 \in M_2$  (a right  $\Gamma$ -comodule) and  $n_1 \in N_1$  (a left  $\Gamma$ -comodule) let

$$m_2^{(0)} \otimes \cdots \otimes m_2^{(s)} \in M_2 \otimes_A \Gamma^{\otimes s},$$

with  $m_2^{(0)} \in M_2$  and  $m_2^{(i)} \in \Gamma$  for  $1 \leq i \leq s$ , and

$$n_1^{(1)} \otimes \cdots \otimes n_1^{(t+1)} \in \Gamma^{\otimes t} \otimes_A N_1,$$

for  $n_1^{(j)} \in \Gamma$  for  $1 \leq j \leq t$  and  $n_1^{(1+t)} \in N_1$ , denote the iterated coproducts. Each of these expressions is shorthand for a certain sum of monomials in the indicated tensor product. Then the pairing is given by

$$(A1.2.15) \quad (m_1 \gamma_1 | \cdots | \gamma_s n_1) \otimes (m_2 \gamma_{s+1} | \cdots | \gamma_{s+t} n_2) \\ \mapsto (-1)^\tau \left( m_1 \otimes m_2^{(0)} \right) \left( \gamma_1 m_2^{(1)} \right) | \cdots | \gamma_s m_2^{(s)} | n_1^{(1)} \gamma_{s+1} | \cdots | \left( n_1^{(t)} \gamma_{s+t} \right) \left( n_1^{(1+t)} \otimes n_2 \right)$$

where

$$\tau = \deg m_2 \deg n_1 + \sum_{i=0}^s \deg m_2^{(i)} \left( s - i + \sum_{j=i+1}^s \deg \gamma_j \right) \\ + \sum_{i=1}^{t+1} \deg n_1^{(i)} \left( i - 1 + \sum_{j=1}^{i-1} \deg \gamma_{j+s} \right).$$

Note that this is natural in all variables in sight.

Finally, we have two easy miscellaneous results.

A1.2.16. PROPOSITION. (a) *If  $I \subset A$  is invariant (A1.2.12) then  $(A/I, \Gamma/I\Gamma)$  is a Hopf algebroid.*

(b) *If  $M$  is a left  $\Gamma$ -comodule annihilated by  $I$  as above, then*

$$\text{Ext}_\Gamma(A, M) = \text{Ext}_{\Gamma/I\Gamma}(A/I, M).$$

PROOF. Part (a) is straightforward. For (b) observe that the complexes  $C_\Gamma(M)$  and  $C_{\Gamma/I\Gamma}(M)$  are identical.  $\square$

A1.2.17. PROPOSITION. *If  $(A, \Gamma)$  is split (A1.1.22) then  $\text{Ext}_\Gamma(A, M) = \text{Ext}_\Sigma(K, M)$  where the left  $\Sigma$ -comodule structure on the left  $\Gamma$ -comodule  $M$  comes from the isomorphism  $\Gamma \otimes_A M = \Sigma \otimes M$ .*

PROOF.  $C_\Gamma(M) = C_\Sigma(M)$ .  $\square$

### 3. Some Spectral Sequences

In this section we describe several spectral sequences useful for computing  $\text{Ext}$  over a Hopf algebroid. The reader is assumed to be familiar with the notion of a spectral sequence; the subject is treated in each of the standard references for homological algebra (Cartan and Eilenberg [1], Mac Lane [1] and Hilton and Stammback [1]) and in Spanier [1]. The reader is warned that most spectral sequences can be indexed in more than one way. With luck the indexing used in this section will be consistent with that used in the text, but it may differ from that appearing elsewhere in the literature and from that used in the next two sections.

Suppose we have a long exact sequence of  $\Gamma$ -comodules

$$(A1.3.1) \quad 0 \rightarrow M \rightarrow R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} R^2 \rightarrow \dots$$

Let  $S^{i+1} = \text{im } d^i$  and  $S^0 = M$  so we have short exact sequences

$$0 \rightarrow S^i \xrightarrow{a^i} R^i \xrightarrow{b^i} S^{i+1} \rightarrow 0$$

for all  $i \geq 0$ . Each of these gives us a connecting homomorphism

$$\delta^i: \text{Cotor}_\Gamma^{s,t}(L, S^i) \rightarrow \text{Cotor}_\Gamma^{s+1,t}(L, S^{i-1}).$$

Let  $\delta_{(i)}: \text{Cotor}_\Gamma^{s,t}(L, S^i) \rightarrow \text{Cotor}_\Gamma^{s+i,t}(L, S^0)$  be the composition  $\delta^1 \delta^2 \cdots \delta^i$ . Define a decreasing filtration on  $\text{Cotor}_\Gamma^{s,*}(L, M)$  by  $F^i = \text{im } \delta_{(i)}$  for  $i \leq s$ , where  $\delta_{(0)}$  is the identity and  $F^i = 0$  for  $i \leq 0$ .

A1.3.2. THEOREM. *Given a long exact sequence of  $\Gamma$ -comodules A1.3.1 there is a natural trigraded spectral sequence  $(E_*^{***})$  (the resolution spectral sequence) such that*

- (a)  $E_1^{n,s,t} = \text{Cotor}_\Gamma^{s,t}(L, R^n)$ ;
- (b)  $d_r: E_r^{n,s,t} \rightarrow E_r^{n+r,s-r+1,t}$  and  $d_1$  is the map induced by  $d^*$  in A1.3.1 and
- (c)  $E_\infty^{n,s,t}$  is the subquotient  $F^n/F^{n+1}$  of  $\text{Cotor}_\Gamma^{n+s,t}(L, M)$  defined above.

PROOF. We will give two constructions of this spectral sequence. For the first define an exact couple (2.1.6) by

$$\begin{aligned} E_1^{s,t} &= \text{Cotor}_\Gamma^{-t,*}(L, R^s), \\ D_1^{s,t} &= \text{Cotor}_\Gamma^{-t,*}(L, S^s), \end{aligned}$$

$i_1 = \delta^*$ ,  $j_1 = a^*$ , and  $k_1 = b^*$ . Then the associated spectral sequence is the one we want.

The second construction applies when  $L$  is projective over  $A$  and is more explicit and helpful in practice; we get the spectral sequence from a double complex as described in Cartan and Eilenberg [1], Section XV.6 or Mac Lane [1], Section XI.6. We will use the terminology of the former. Let

$$\begin{aligned} B^{n,s,*} &= C_\Gamma^s(L, R^n) \quad (\text{A1.2.11}), \\ \partial_1^{n,s,*} &= (-1)^n C_\Gamma^s(d^n): B^{n,s,*} \rightarrow B^{n+1,s,*}, \end{aligned}$$

and

$$\partial_2^{n,s,*} = d^s : B^{n,s,*} \rightarrow B^{n,s+1,*}.$$

(Our  $\partial_1, \partial_2$  correspond to the  $d_1, d_2$  in Cartan and Eilenberg [1], IV.4.) Then  $\partial_2^{n+1,s,*} \partial_1^{n,s,*} + \partial_1^{n,s+1,*} \partial_2^{n,s,*} = 0$  since  $d^s$  commutes with  $C_\Gamma^s(d^n)$ . The associated complex  $(B^{p,*}, \partial)$  is defined by

$$(A1.3.3) \quad B^{p,*} = \bigoplus_{n+s=p} B^{n,s,*} = \bigoplus_{n+s=p} C_\Gamma^s(L, R^n)$$

with  $\partial = \partial_1 + \partial_2 : B^{p,*} \rightarrow B^{p+1,*}$ .

This complex can be filtered in two ways, i.e.,

$$\begin{aligned} F_I^p B &= \bigoplus_{r \geq p} \bigoplus_q B^{r,q,*}, \\ F_{II}^q B &= \bigoplus_{s \geq q} \bigoplus_p B^{p,s,*} \end{aligned}$$

and each of these filtrations leads to a spectral sequence. In our case the functor  $C_\Gamma^s(L, \cdot)$  is exact since  $\Gamma$  is flat over  $A$ , so  $H^{s,*}(F_{II} B) = C_\Gamma^s(L, M)$ . Hence in the second spectral sequence

$$E_1^{n,s,*} = \begin{cases} C_\Gamma^s(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_2^{n,s,*} = E_\infty^{n,s,*} = \begin{cases} \text{Cotor}_\Gamma^{s,*}(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The two spectral sequences converge to the same thing, so the first one, which is the one we want, has the desired properties.  $\square$

A1.3.4. COROLLARY. *The cohomology of the complex  $B^{**}$  of A1.3.3 is  $\text{Cotor}_\Gamma^{**}(L, M)$ .*  $\square$

Note that A1.2.4 is a special case of A1.3.3 in which the spectral sequence collapses.

Next we discuss spectral sequences arising from increasing and decreasing filtration of  $\Gamma$ .

A1.3.5. DEFINITION. *An increasing filtration on a Hopf algebroid  $(A, \Gamma)$  is an increasing sequence of sub- $K$ -modules*

$$K = F_0\Gamma \subset F_1\Gamma \subset F_2\Gamma \subset \dots$$

with  $\Gamma = \bigcup F_s\Gamma$  such that

- (a)  $F_s\Gamma \cdot F_t\Gamma \subset F_{s+t}\Gamma$ ,
- (b)  $c(F_s\Gamma) \subset F_s\Gamma$ , and
- (c)  $\Delta F_s\Gamma \subset \bigoplus_{p+q=s} F_p\Gamma \otimes_A F_q\Gamma$ .

A decreasing filtration on  $(A, \Gamma)$  is a decreasing sequence of sub- $K$ -modules

$$\Gamma = F^0\Gamma \supset F^1\Gamma \supset F^2\Gamma \supset \dots$$



with  $0 = \bigcap F^s \Gamma$  such that conditions similar to (a), (b), and (c) above (with the inclusion signs reversed) are satisfied. A filtered Hopf algebroid  $(A, \Gamma)$  is one equipped with a filtration. Note that a filtration on  $\Gamma$  induces one on  $A$ , e.g.,

$$F_s A = \eta_L(A) \cap F_s \Gamma = \eta_R(A) \cap F_s \Gamma = \varepsilon(F_s \Gamma).$$

A1.3.6. DEFINITION. Let  $(A, \Gamma)$  be filtered as above. The associated graded object  $E^0 \Gamma$  (or  $E_0 \Gamma$ ) is defined by

$$E_s^0 \Gamma = F_s \Gamma / F_{s-1} \Gamma$$

or

$$E_0^s \Gamma = F^s \Gamma / F^{s-1} \Gamma.$$

The graded object  $E_*^0 A$  (or  $E_0^* A$ ) is defined similarly.

A1.3.7. DEFINITION. Let  $M$  be a  $\Gamma$ -comodule. An increasing filtration on  $M$  is an increasing sequence of sub- $K$ -modules

$$0 = F_1 M \subset F_2 M \subset \dots$$

such that  $M = \bigcup F_s M$ ,  $F_s A \cdot F_t M \subset F_{s+t} M$ , and

$$\psi(F_s M) \subset \bigoplus_{p+q=s} F_p \Gamma \otimes F_q M.$$

A decreasing filtration on  $M$  is similarly defined, as is the associated graded object  $E_*^0 M$  or  $E_0^* M$ . A filtered comodule  $M$  is a comodule equipped with a filtration.

A1.3.8. PROPOSITION.  $(E^0 A, E^0 \Gamma)$  or  $(E_0 A, E_0 \Gamma)$  is a graded Hopf algebroid and  $E^0 M$  or  $E_0 M$  is a comodule over it.  $\square$

Note that if  $(A, \Gamma)$  and  $M$  are themselves graded than  $(E^0 A, E^0 \Gamma)$  and  $E^0 M$  are bigraded.

We assume from now on that  $E^0 \Gamma$  or  $E_0 \Gamma$  is flat over  $E^0 A$  or  $E_0 A$ .

A1.3.9. THEOREM. Let  $L$  and  $M$  be right and left filtered comodules, respectively, over a filtered Hopf algebroid  $(A, \Gamma)$ . Then there is a natural spectral sequence converging to  $\text{Cotor}_\Gamma(L, M)$  such that

(a) in the increasing case

$$E_1^{s,*} = \text{Cotor}_{E_0 \Gamma}^s(E^0 L, E^0 M)$$

where the second grading comes from the filtration and

$$d_r: E_r^{s,t} \rightarrow E_r^{s+1,t-r};$$

(b) in the decreasing case

$$E_1^{s,*} = \text{Cotor}_{E_0 \Gamma}^s(E_0 L, E_0 M)$$

and

$$d_r: E_r^{s,t} \rightarrow E_r^{s+1,t+r}.$$

Note that our indexing differs from that of Cartan and Eilenberg [1] and Mac Lane [1].

PROOF. The filtrations on  $\Gamma$  and  $M$  induce one on the cobar complex (A1.2.11)  $C_\Gamma M$  and we have  $E_0 C_\Gamma(L, M) = C_{E_0 \Gamma}(E_0 L, E_0 M)$  or  $E^0 C_\Gamma(L, M) = C_{E^0 \Gamma}(E^0 L, E^0 M)$ . The associated spectral sequence is the one we want.  $\square$

The following is an important example of an increasing filtration.

A1.3.10. EXAMPLE. Let  $(K, \Gamma)$  be a Hopf algebra. Let  $\bar{\Gamma}$  be the unit coideal, i.e., the quotient in the short exact sequence

$$0 \rightarrow K \xrightarrow{\eta} \Gamma \rightarrow \bar{\Gamma} \rightarrow 0,$$

The coproduct map  $\Delta$  can be iterated by coassociativity to a map  $\Delta^s: \Gamma \rightarrow \Gamma^{\otimes s+1}$ . Let  $F_s \Gamma$  be the kernel of the composition

$$\Gamma \xrightarrow{\Delta^s} \Gamma^{\otimes s+1} \rightarrow \bar{\Gamma}^{\otimes s+1}.$$

This is the *filtration of  $\Gamma$  by powers of the unit coideal*.

Next we treat the spectral sequence associated with a map of Hopf algebroids.

A1.3.11. THEOREM. *Let  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  be a map of Hopf algebroids (A1.1.18),  $M$  a right  $\Gamma$ -comodule and  $N$  a left  $\Sigma$ -comodule.*

(a)  *$C_\Sigma(\Gamma \otimes_A B, N)$  is a complex of left  $\Gamma$ -comodules, so  $\text{Cotor}_\Sigma(\Gamma \otimes_A B, N)$  is a left  $\Gamma$ -comodule.*

(b) *If  $M$  is flat over  $A$ , there is a natural spectral sequence converging to  $\text{Cotor}_\Sigma(M \otimes_A B, N)$  with*

$$E_2^{s,t} = \text{Cotor}_\Gamma^s(M, \text{Cotor}_\Sigma^t(\Gamma \otimes_A B, N))$$

and  $d_r: E_r^{s,t} \rightarrow E_r^{s+r, t-r+1}$ .

(c) *If  $N$  is a comodule algebra then so is  $\text{Cotor}_\Sigma(\Gamma \otimes_A B, N)$ . If  $M$  is also a comodule algebra, then the spectral sequence is one of algebras.*

PROOF. For (a) we have  $C_\Sigma^s(\Gamma \otimes_A B, N) = \Gamma \otimes_A \bar{\Sigma}^{\otimes s} \otimes_B N$  with the coboundary  $d_s$  as given in A1.2.11. We must show that  $d_s$  commutes with the coproduct on  $\Gamma$ . For all terms other than the first in the formula for  $d_s$  this commutativity is clear. For the first term consider the diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\Delta} & \Gamma \otimes_A \Gamma & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_A \Sigma \\ \Delta \downarrow & & \Delta \otimes \Gamma \downarrow & & \Delta \otimes \Sigma \downarrow \\ \Gamma \otimes_A \Gamma & \xrightarrow{\Gamma \otimes \Delta} & \Gamma \otimes_A \Gamma \otimes_A \Gamma & \xrightarrow{\Gamma \otimes \Gamma \otimes f} & \Gamma \otimes_A \Gamma \otimes_A \Sigma \end{array}$$

The left-hand square commutes by coassociativity and other square commutes trivially. The top composition when tensored over  $B$  with  $\Sigma^{\otimes s} \otimes_B N$  is the first term in  $d_s$ . Hence the commutativity of the diagram shows that  $d_s$  is a map of left  $\Gamma$ -comodules.

For (b) consider the double complex

$$C_\Gamma^*(M, C_\Sigma^*(\Gamma \otimes_A B, N)),$$

which is well defined because of (a). We compare the spectral sequences obtained by filtering by the two degrees. Filtering by the first gives

$$E_1 = C_\Gamma^*(M, \text{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

so

$$E_2 = \text{Cotor}_\Gamma(M, \text{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

which is the desired spectral sequence. Filtering by the second degree gives a spectral sequence with

$$\begin{aligned} E_1^{s,t} &= \text{Cotor}_\Gamma^s(M, C_\Sigma^t(\Gamma \otimes_A B, N)) \\ &= \text{Cotor}_\Gamma(M, \Gamma \otimes_A \bar{\Sigma}^{\otimes t} \otimes_B N) \\ &= M \otimes_A \bar{\Sigma}^{\otimes t} \otimes_B N \quad \text{by A1.2.8(b)} \\ &= C_\Sigma^t(M \otimes_A B, N) \end{aligned}$$

so  $E_2 = E_\infty = \text{Cotor}_\Sigma(M \otimes_A B, N)$ .

For (c) note that  $\Gamma \otimes_A B$  as well as  $N$  is a  $\Sigma$ -comodule algebra. The  $\Gamma$ -coaction on  $C_\Sigma(\Gamma \otimes_A B, N)$  is induced by the map

$$\begin{aligned} C(\Delta \otimes B, N): C_\Sigma(\Gamma \otimes_A B, N) &\rightarrow C_\Sigma(\Gamma \otimes_A \Gamma \otimes_A B, N) \\ &= \Gamma \otimes_A C_\Sigma(\Gamma \otimes_A B, N). \end{aligned}$$

Since the algebra structure on  $C_\Sigma(\cdot, \cdot)$  is functorial,  $C(\Delta \otimes B, N)$  induces an algebra map in cohomology and  $\text{Cotor}_\Sigma(\Gamma \otimes_A B, N)$  is a  $\Gamma$ -comodule algebra.

To show that we have a spectral sequence of algebras we must define an algebra structure on the double complex used in the proof of (b), which is  $M \square_\Gamma D_\Gamma(\Gamma \otimes_A B \square_\Sigma D_\Sigma(N))$ . Let  $\tilde{N} = \Gamma \otimes_A B \square_\Sigma D_\Sigma(N)$ . We have just seen that it is a  $\Gamma$ -comodule algebra. Then this algebra structure extends to one on  $D_\Gamma(\tilde{N})$  by A1.2.9 since  $D_\Gamma(\tilde{N}) \otimes_A D_\Gamma(\tilde{N})$  is a relatively injective resolution of  $\tilde{N} \otimes_A \tilde{N}$ . Hence we have maps

$$\begin{aligned} M \square_\Gamma D_\Gamma(\tilde{N}) \otimes M \square_\Gamma D_\Gamma(\tilde{N}) &\rightarrow M \otimes_A M \square_\Gamma D_\Gamma(\tilde{N}) \otimes_A D_\Gamma(\tilde{N}) \\ &\rightarrow M \square_\Gamma D_\Gamma(\tilde{N}) \otimes_A D_\Gamma(\tilde{N}) \rightarrow M \square_\Gamma D_\Gamma(\tilde{N}), \end{aligned}$$

which is the desired algebra structure.  $\square$

Our first application of this spectral sequence is a change-of-rings isomorphism that occurs when it collapses.

**A1.3.12. CHANGE-OF-RINGS ISOMORPHISM THEOREM.** *Let  $f: (A, \Gamma) \rightarrow (B, \Sigma)$  be a map of graded connected Hopf algebroids (A1.1.7) satisfying the hypotheses of A1.1.19; let  $M$  be a right  $\Gamma$ -comodule and let  $N$  be a left  $\Sigma$ -comodule which is flat over  $B$ . Then*

$$\text{Cotor}_\Gamma(M, (\Gamma \otimes_A B) \square_\Sigma N) = \text{Cotor}_\Sigma(M \otimes_A B, N).$$

*In particular*

$$\text{Ext}_\Gamma(A, (\Gamma \otimes_A B) \square_\Sigma N) = \text{Ext}_\Sigma(B, N),$$

**PROOF.** By A1.1.19 and A1.2.8(b) we have

$$\text{Cotor}_\Sigma^s(\Gamma \otimes_A B, N) = 0 \quad \text{for } s > 0.$$

A1.3.11(b) gives

$$\text{Cotor}_\Gamma(M, \text{Cotor}_\Sigma^0(\Gamma \otimes_A B, N)) = \text{Cotor}_\Sigma(M \otimes_A B, N).$$

Since  $N$  is flat over  $B$ ,

$$\text{Cotor}_\Sigma^0(\Gamma \otimes_A B, N) = (\Gamma \otimes_A B) \square_\Sigma N$$

and the result follows.  $\square$

A1.3.13. COROLLARY. *Let  $K$  be a field and  $f: (K, \Gamma) \rightarrow (K, \Sigma)$  be a surjective map of Hopf algebras. If  $N$  is a left  $\Sigma$ -comodule then*

$$\mathrm{Ext}_\Gamma(K, \Gamma \square_\Sigma N) = \mathrm{Ext}_\Sigma(K, N). \quad \square$$

Next we will construct a change-of-rings spectral sequence for an extension of Hopf algebroids (A1.1.15) similar to that of Cartan and Eilenberg [1, XVI 6.1], which we will refer to as the Cartan–Eilenberg spectral sequence.

A1.3.14. CARTAN–EILENBERG SPECTRAL SEQUENCE THEOREM. *Let*

$$(D, \Phi) \xrightarrow{i} (A, \Gamma) \xrightarrow{f} (A, \Sigma)$$

*be an extension of graded connected Hopf algebroids (A1.1.15). Let  $M$  be a right  $\Phi$ -comodule and  $N$  a left  $\Gamma$ -comodule.*

(a)  *$\mathrm{Cotor}_\Sigma(A, N)$  is a left  $\Phi$ -comodule. If  $N$  is a comodule algebra, then so is this  $\mathrm{Cotor}$ .*

(b) *There is a natural spectral sequence converging to  $\mathrm{Cotor}_\Gamma(M \otimes_D A, N)$  with*

$$E_2^{s,t} = \mathrm{Cotor}_\Phi^s(M, \mathrm{Cotor}_\Sigma^t(A, N))$$

*and*

$$d_r: E_r^{s,t} \rightarrow E_r^{s+r, t-r+1}.$$

(c) *If  $M$  and  $N$  are comodule algebras, then the spectral sequence is one of algebras.*

PROOF. Applying A1.3.11 to the map  $i$  shows that  $\mathrm{Cotor}_\Gamma(\Phi \otimes_D A, N)$  is a left  $\Phi$ -comodule algebra and there is a spectral sequence converging to  $\mathrm{Cotor}_\Gamma(M \otimes_D A, N)$  with

$$E_2 = \mathrm{Cotor}_\Phi(M, \mathrm{Cotor}_\Gamma(\Phi \otimes_A D, N)).$$

Hence the theorem will follow if we can show that  $\mathrm{Cotor}_\Gamma(\Phi \otimes_D A, N) = \mathrm{Cotor}_\Sigma(A, N)$ . Now  $\Phi \otimes_D A = A \square_\Sigma \Gamma$  by A1.1.16. We can apply A1.3.12 to  $f$  and get  $\mathrm{Cotor}_\Gamma(P \square_\Sigma \Gamma, R) = \mathrm{Cotor}_\Sigma(P, R)$  for a right  $\Sigma$ -comodule  $P$  and left  $\Gamma$ -comodule  $R$ . Setting  $P = A$  and  $R = N$  gives the desired isomorphism

$$\mathrm{Cotor}_\Gamma(\Phi \otimes_D A, N) = \mathrm{Cotor}_\Gamma(A \square_\Sigma \Gamma, N) = \mathrm{Cotor}_\Sigma(A, N). \quad \square$$

The case  $M = D$  gives

A1.3.15. COROLLARY. *With notation as above, there is a spectral sequence of algebras converging to  $\mathrm{Ext}_\Gamma(A, N)$  with  $E_2 = \mathrm{Ext}_\Phi(D, \mathrm{Ext}_\Sigma(A, N))$ .*  $\square$

Now we will give an alternative formulation of the Cartan–Eilenberg spectral sequence (A1.3.14) suggested by Adams [12], 2.3.1 which will be needed to apply the results of the next sections on Massey products and Steenrod operations. A similar formulation was given by Hochschild–Serre [1] in the context of group cohomology. Using the notation of A1.2.14, we define a decreasing filtration on  $C_\Gamma(M \otimes_D A, N)$  by saying that  $m\gamma_1 | \dots | \gamma_s n \in F^i$  if  $i$  of the  $\gamma$ 's are in  $\ker f_2$ .

A1.3.16. THEOREM. *The spectral sequence associated with the above filtration of  $C_\Gamma(M \otimes_D A, N)$  coincides with the Cartan–Eilenberg spectral sequence of A1.3.14.*

PROOF. The Cartan–Eilenberg spectral sequence is obtained by filtering the double complex  $C_{\Phi}^*(M, C_{\Gamma}^*(\Phi \otimes_D A, N))$  by the first degree. We define a filtration-preserving map  $\theta$  from this complex to  $C_{\Gamma}(M \otimes_D A, N)$  by

$$\begin{aligned} \theta(m \otimes \phi_1 \otimes \cdots \otimes \phi_s \otimes \phi \otimes \gamma_{s+1} \otimes \cdots \otimes \gamma_{s+t} \otimes n) \\ = m \otimes i_2(\phi_1) \otimes \cdots \otimes i_2(\phi_s) i_1 \varepsilon(\phi) \otimes \gamma_{s+1} \otimes \cdots \otimes \gamma_{s+t} \otimes n. \end{aligned}$$

Let  $E_1^{s,t}(M, N) = C_{\Phi}^s(M, \text{Cotor}_{\Gamma}^t(\Phi \otimes_D A, N)) = C_{\Phi}^s(M, \text{Cotor}_{\Sigma}^t(A, N))$  be the  $E_1$ -term of the Cartan–Eilenberg spectral sequence and  $\tilde{E}_1(M, N)$  the  $E_1$ -term of the spectral sequence in question. It suffices to show that

$$\theta_* : E_1(M, N) \rightarrow \tilde{E}_1(M, N)$$

is an isomorphism.

First consider the case  $s = 0$ . We have

$$F_0/F^1 = C_{\Sigma}(M \otimes_D A, N) = M \otimes_D C_{\Sigma}(A, N)$$

so this is the target of  $\theta$  for  $s = 0$ . The source is  $M \otimes_D C_{\Gamma}(\phi \otimes_D A, N)$ . The argument in the proof of Theorem A1.3.14 showing that

$$\text{Cotor}_{\Gamma}(\Phi \otimes_D A, N) = \text{Cotor}_{\Sigma}(A, N)$$

shows that our two complexes are equivalent so we have the desired isomorphism for  $s = 0$ .

For  $s > 0$  we use the following argument due to E. Ossa.

The differential

$$d_0 : E_0^{s,t}(M, N) \rightarrow E_0^{s,t+1}(M, N)$$

depends only on the  $\Sigma$ -comodule structures of  $M$  and  $N$ . In fact we may define a complex  $\tilde{D}_{\Sigma}(N)$  formally by

$$\tilde{D}_{\Sigma}^{s,t}(N) = \tilde{E}_0^{s,t}(\Sigma, N).$$

Then we have

$$\tilde{E}_0^{s,t}(M, n) = M \square_{\Sigma} \tilde{D}_{\Sigma}^{s,t}(N).$$

Observe that

$$\tilde{D}_{\Sigma}^{0,t}(N) = C_{\Sigma}^t(\Sigma, N).$$

Now let  $G = \ker f$  and

$$C^{s+1} = G^s \square_{\Sigma} G = G \square_{\Sigma} G \square_{\Sigma} \cdots \square_{\Sigma} G$$

with  $s + 1$  factors.

Note that

$$G = \Sigma \otimes \bar{\Phi} \quad \text{and hence}$$

$$G^s = \Sigma \otimes \bar{\Phi}^{\otimes s}$$

as left  $\Sigma$ -comodules, where the tensor products are over  $D$ .

Define

$$\beta_s : G^s \square_{\Sigma} \tilde{D}_{\Sigma}^{0,t}(N) \rightarrow \tilde{D}_{\Sigma}^{s,t}(N)$$

by

$$\begin{aligned} \beta_s((g_1 \otimes \cdots \otimes g_s) \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n) \\ = \Sigma f(g'_1) g''_1 \otimes g_2 \cdots \otimes g_s \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n. \end{aligned}$$

Then  $\beta_s$  is a map of differential  $\Sigma$ -comodules and the diagram

$$\begin{array}{ccc}
\tilde{E}_0^{s,t}(M, N) & \xlongequal{\quad} & \tilde{E}_0^{0,t}(M \otimes \Phi^{\otimes s}, N) \\
\downarrow \theta^{s,t} & & \downarrow \theta^{0,t} \\
\tilde{E}_0^{s,t}(M, N) & & \tilde{E}_0^{0,t}(M \otimes \Phi^{\otimes s}, N) \\
\parallel & & \parallel \\
M \square_{\Sigma} \tilde{D}_{\Sigma}^{s,t}(N) & \xleftarrow{\beta_s} & M \square_{\Sigma} G^s \square_{\Sigma} \tilde{D}_{\Sigma}^{s,t}(N)
\end{array}$$

commutes.

We know that  $\theta^{0,t}$  is a chain equivalence so it suffices to show that  $\beta_s$  is one by induction on  $s$ . To start this induction note that  $\beta_0$  is the identity map by definition.

Let

$$F^{s,t}(\Gamma, N) = F^s C_{s+t}(\Gamma, N)$$

and

$$\begin{aligned}
F^{s,t}(\Gamma, N) &= F^{s,t}(G, N) + F^{s+1,t-1}(\Gamma, N) \\
&= F^{s,t}(\Gamma, N).
\end{aligned}$$

Then  $F^{s,*}(\Gamma, N)$  is a  $\Sigma$ -comodule subcomplex of  $C_{\Gamma}(\Gamma, N)$  which is invariant under the contraction

$$S(\gamma \otimes \gamma_1 \dots \gamma_s \otimes n) = \varepsilon(\gamma) \otimes \gamma_1 \dots \gamma_s \otimes n.$$

Since  $H_0(F^{s,*}(\Gamma, N)) = 0$ , the complex  $F^{s,*}(\Gamma, N)$  is acyclic.

Now look at the short exact sequence of complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{F^{s+1}(\Gamma, N)}{\tilde{F}^{s+1}(\Gamma, N)} & \longrightarrow & \frac{\tilde{F}^s(\Gamma, N)}{\tilde{F}^{s+1}(\Gamma, N)} & \longrightarrow & \frac{\tilde{F}^s(\Gamma, N)}{F^{s+1}(\Gamma, N)} \longrightarrow 0 \\
& & \phi \downarrow \cong & & & & \psi \downarrow \cong \\
& & \tilde{D}_{\Sigma}^{s+1}(N) & & & & G \square_{\Sigma} \tilde{D}_{\Sigma}^s(N)
\end{array}$$

The connecting homomorphism in cohomology is an isomorphism.

We use this for the inductive step. By the inductive hypothesis, the composite

$$G \square_{\Sigma} (G^s \square_{\Sigma} N) \rightarrow G \square_{\Sigma} (G^s \square_{\Sigma} D_{\Sigma}^0(N)) \rightarrow G \square_{\Sigma} D_{\Sigma}^s(N)$$

is an equivalence. If we follow it by  $\phi \partial \psi$  we get  $\beta_{s+1}$ . This completes the inductive step and the proof.  $\square$

**A1.3.17. THEOREM.** *Let  $\Phi \rightarrow \Gamma \rightarrow \Sigma$  be a cocentral extension (A1.1.15) of Hopf algebras over a field  $K$ ;  $M$  a left  $\Phi$ -comodule and  $N$  a trivial left  $\Gamma$ -comodule. Then  $\text{Ext}_{\Sigma}(K, N)$  is trivial as a left  $\Phi$ -comodule, so the Cartan–Eilenberg spectral sequence (A1.3.14)  $E_2$ -term is  $\text{Ext}_{\Phi}(M, K) \otimes \text{Ext}_{\Sigma}(K, K) \otimes N$ .*

**PROOF.** We show first that the coaction of  $\Phi$  on  $\text{Ext}_{\Sigma}(K, N)$  is essentially unique and then give an alternative description of it which is clearly trivial when the extension is cocentral. The coaction is defined for any (not necessarily trivial) left  $\Gamma$ -comodule  $N$ . It is natural and determined by its effect when  $N = \Gamma$  since

we can use an injective resolution of  $N$  to reduce to this case. Hence any natural  $\Phi$ -coaction on  $\text{Ext}_\Sigma(K, N)$  giving the standard coaction on  $\text{Ext}_\Sigma(K, \Gamma) = \Phi$  must be identical to the one defined above.

Now we need some results of Singer [5]. Our Hopf algebra extension is a special case of the type he studies. In Proposition 2.3 he defines a  $\Phi$ -coaction on  $\Sigma$ ,  $\rho_\Sigma: \Sigma \rightarrow \Phi \otimes \Sigma$  via a sort of coconjugation. Its analog for a group extension  $N \rightarrow G \rightarrow H$  is the action of  $H$  on  $N$  by conjugation. This action is trivial when the extension is central, as is Singer's coaction in the cocentral case.

The following argument is due to Singer.

Since  $\Sigma$  is a  $\Phi$ -comodule it is a  $\Gamma$ -comodule so for any  $N$  as above  $\Sigma \otimes_K N$  is a  $\Gamma$ -comodule. It follows that the cobar resolution  $D_\Sigma N$  is a differential  $\Gamma$ -comodule and that  $\text{Hom}_\Sigma(K, D_\Sigma N)$  is a differential comodule over  $\text{Hom}_\Sigma(K, \Gamma) = \Phi$ . Hence we have a natural  $\Phi$ -coaction on  $\text{Ext}_\Sigma(K, N)$  which is clearly trivial when  $N$  has the trivial  $\Gamma$ -comodule structure and the extension is cocentral.

It remains only to show that this  $\Phi$ -coaction is identical to the standard one by evaluating it when  $N = \Gamma$ . In that case we can replace  $D_\Sigma N$  by  $N$ , since  $N$  is an extended  $\Sigma$ -comodule. Hence we have the standard  $\Gamma$ -coaction on  $\Gamma$  inducing the standard  $\Phi$ -coaction on  $\text{Hom}_\Sigma(K, \Gamma) = \Phi$ .  $\square$

### 4. Massey Products

In this section we give an informal account of Massey products, a useful structure in the Ext over a Hopf algebroid which will figure in various computations in the text. A parallel structure in the ASS is discussed in Kochman [4] and Kochman [2, Section 12]. These products were first introduced by Massey [3], but the best account of them is May [3]. We will give little more than an introduction to May's paper, referring to it for all the proofs and illustrating the more complicated statements with simple examples.

The setting for defining Massey products is a differential graded algebra (DGA)  $C$  over a commutative ring  $K$ . The relevant example is the cobar complex  $C_\Gamma(L, M)$  of A1.2.11, where  $L$  and  $M$  are  $\Gamma$ -comodule algebras and  $\Gamma$  is a Hopf algebroid (A1.1.1) over  $K$ . The product in this complex is given by A1.2.15.

We use the following notation to keep track of signs. For  $x \in C$ , let  $\bar{x}$  denote  $(-1)^{1+\text{deg } x}x$ , where  $\text{deg } x$  is the total degree of  $x$ ; i.e., if  $C$  is a complex of graded objects,  $\text{deg } x$  is the sum of the internal and cohomological degrees of  $x$ . Hence we have  $d(\bar{x}) = -\bar{d}(x)$ ,  $(\bar{x}\bar{y}) = -\bar{x}\bar{y}$ , and  $d(xy) = d(x)y - \bar{x}d(y)$ .

Now let  $\alpha_i \in H^*(C)$  be represented by cocycles  $a_i \in C$  for  $i = 1, 2, 3$ . If  $\alpha_i \alpha_{i+1} = 0$  then there are cochains  $u_i$  such that  $d(u_i) = \bar{a}_i a_{i+1}$ , and  $\bar{u}_1 a_3 + \bar{a}_1 u_2$  is a cocycle. The corresponding class in  $H^*(C)$  is the *Massey product*  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ . If  $\alpha_i \in H^{s_i}$  then this  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{s-1}$ , where  $s = \sum s_i$ . Unfortunately, this triple product is not well defined because the choices made in its construction are not unique. The choices of  $a_i$  do not matter but the  $u_i$  could each be altered by adding a cocycle, which means  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  could be altered by any element of the form  $x\alpha_3 + \alpha_1 y$  with  $x \in H^{s_1+s_2-1}$  and  $y \in H^{s_2+s_3-1}$ . The group  $\alpha_1 H^{s_3+s_2-1} \oplus \alpha_3 H^{s_1+s_2-1}$  is called the *indeterminacy*, denoted by  $\text{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ . It may be trivial, in which case  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is well defined.

A1.4.1. DEFINITION. *With notation as above,  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^s(C)$  is the coset of  $\text{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  represented by  $\bar{a}_1 u_2 + \bar{u}_1 a_3$ . Note that  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is only defined when  $\bar{\alpha}_1 \alpha_2 = \bar{\alpha}_2 \alpha_3 = 0$ .*

This construction can be generalized in two ways. First the relations  $\alpha_i \bar{\alpha}_{i+1} = 0$  can be replaced by

$$\sum_{j=1}^m (\bar{\alpha}_1)_j (\alpha_2)_{j,k} = 0 \quad \text{for } 1 \leq k \leq n$$

and

$$\sum_{k=1}^n (\bar{\alpha}_2)_{j,k} (\alpha_3)_k = 0 \quad \text{for } 1 \leq j \leq m.$$

Hence the  $\alpha_i$  become matrices with entries in  $H^*(C)$ . We will denote the set of matrices with entries in a ring  $R$  by  $MR$ . For  $x \in MC$  or  $MH^*(C)$ , define  $\bar{x}$  by  $(\bar{x})_{j,k} = \bar{x}_{j,k}$ .

As before, let  $a_i \in MC$  represent  $\alpha_i \in MH^*(C)$  and let  $u_1 \in MC$  be such that  $d(u_1) = \bar{a}_1 a_{i+1}$ . Then  $u_1$  and  $u_2$  are  $(1 \times n)$ - and  $(m \times 1)$ -matrices, respectively, and  $\bar{a}_1 u_2 + \bar{u}_1 a_3$  is a cocycle (not a matrix thereof) that represents the coset  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ .

Note that the matrices  $\alpha_i$  need not be homogeneous (i.e., their entries need not all have the same degree) in order to yield a homogeneous triple product. In order to multiply two such matrices we require that, in addition to having compatible sizes, the degrees of their entries be such that the entries of the product are all homogeneous. These conditions are easy to work out and are given in 1.1 of May [3]. They hold in all of the applications we will consider and *will be tacitly assumed in subsequent definitions*.

A1.4.2. DEFINITION. *With notation as above, the matrix Massey product  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is the coset of  $\text{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  represented the cocycle  $\bar{a}_1 u_2 + \bar{u}_1 a_3$ , where  $\text{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is the group generated by elements of the form  $x\alpha_3 + \alpha_1 y$  where  $x, y \in MH^*(C)$  have the appropriate form.*

The second generalization is to higher (than triple) order products. The Massey product  $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$  for  $\alpha_i \in MH^*(C)$  is defined when all of the lower products  $\langle \alpha_i, \alpha_{i+1}, \dots, \alpha_j \rangle$  for  $1 \leq i < j \leq n$  and  $j - i < n - 1$  are defined and contain zero. Here the double product  $\langle \alpha_i, \alpha_{i+1} \rangle$  is understood to be the ordinary product  $\alpha_i \alpha_{i+1}$ . Let  $a_{i-1,i}$  be a matrix of cocycles representing  $\alpha_i$ . Since  $\alpha_i \alpha_{i+1} = 0$  there are cochains  $a_{i-1,i+1}$  with  $d(a_{i-1,i+1}) = \bar{a}_{i-1,i} a_{i,i+1}$ . Then the triple product  $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2} \rangle$  is represented by  $b_{i-1,i+2} = \bar{a}_{i-1,i+1} a_{i+1,i+2} + \bar{a}_{i-1,i} a_{i,i+2}$ . Since this triple product is assumed to contain zero, the above choices can be made so that there is a matrix of cochains  $a_{i-1,i+2}$  whose coboundary is  $b_{i-1,i+2}$ .

Then the fourfold product  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$  is represented by the cocycle  $\bar{a}_{0,3} a_{3,4} + \bar{a}_{0,2} a_{2,4} + \bar{a}_{0,1} a_{1,4}$ . More generally, we can choose elements  $a_{i,j}$  and  $b_{i,j}$  by induction on  $j - i$  satisfying  $b_{i,j} = \sum_{i < k < j} \bar{a}_{i,k} a_{k,j}$  and  $d(a_{i,j}) = b_{i,j}$  for  $i - j < n - 1$ .

A1.4.3. DEFINITION. *The  $n$ -fold Massey product  $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$  is defined when all of the lower products  $\langle \alpha_i, \dots, \alpha_j \rangle$  contain zero for  $i < j$  and  $j - i < n - 1$ . It is strictly defined when these lower products also have trivial indeterminacy, e.g., all triple products are strictly defined. In either case the matrices  $a_{i,j}$  chosen above*



for  $0 < i \leq j \leq n$  and  $j - i < n$  constitute a defining system for the product in question, which is, modulo indeterminacy (to be described below), the class represented by the cocycle

$$\sum_{0 < i < n} \bar{a}_{0,i} a_{i,n}.$$

Note that if  $\alpha_i \in H^{s_i}(C)$ , then  $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^{s+2-n}(C)$  where  $s = \sum s_i$ .

In 1.5 of May [3] it is shown that this product is natural with respect to DGA maps  $f$  in the sense that  $\langle f_*(\alpha_1), \dots, f_*(\alpha_n) \rangle$  is defined and contains  $f_*(\langle \alpha_1, \dots, \alpha_n \rangle)$ .

The indeterminacy for  $n \geq 4$  is problematic in that without additional technical assumptions it need not even be a subgroup. Upper bounds on it are given by the following result, which is part of 2.3, 2.4, and 2.7 of May [3]. It expresses the indeterminacy of  $n$ -fold products in terms of  $(n - 1)$ -fold products, which is to be expected since that of a triple product is a certain matrix double product.

**A1.4.4. INDETERMINACY THEOREM.** *Let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be defined. For  $1 \leq k \leq n - 1$  let the degree of  $x_k$  be one less than that of  $\alpha_k \alpha_{k+1}$ .*

(a) *Define matrices  $W_k$  by*

$$W_1 = (\alpha_1 \ x_1),$$

$$W_k = \begin{pmatrix} \alpha_k & x_k \\ 0 & \alpha_{k+1} \end{pmatrix} \quad \text{for } 2 \leq k \leq n - 2$$

and

$$W_{n-1} = \begin{pmatrix} x_{n-1} \\ \alpha_n \end{pmatrix}.$$

Then  $\text{In}\langle \alpha_1, \dots, \alpha_n \rangle \subset \bigcup \langle W_1, \dots, W_n \rangle$  where the union is over all  $x_k$  for which  $\langle W_1, \dots, W_n \rangle$  is defined.

(b) *Let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be strictly defined. Then for  $1 \leq k \leq n - 1$   $\langle \alpha_1, \dots, \alpha_{k-1}, x_k, \alpha_{k+2}, \dots, \alpha_n \rangle$  is strictly defined and*

$$\text{In}\langle \alpha_1, \dots, \alpha_n \rangle \subset \bigcup_{k=1}^{n-1} \sum_{k=1}^{n-1} \langle \alpha_1, \dots, \alpha_{k-1}, x_k, \alpha_{k+2}, \alpha_n \rangle$$

where the union is over all possible  $x_k$ . Equality holds when  $n = 4$ .

(c) *If  $\alpha_k = \alpha'_k + \alpha''_k$  and  $\langle \alpha_1, \dots, \alpha'_k, \dots, \alpha_n \rangle$  is strictly defined, then*

$$\langle \alpha_1, \dots, \alpha_n \rangle \subset \langle \alpha_1, \dots, \alpha'_k, \dots, \alpha_n \rangle + \langle \alpha_1, \dots, \alpha''_k, \dots, \alpha_n \rangle. \quad \square$$

There is a more general formula for the sum of two products, which generalizes the equation

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 = \left\langle (\alpha_1 \alpha_2), \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\rangle$$

and is part of 2.9 of May [3].

**A1.4.5. ADDITION THEOREM.** *Let  $\langle \alpha_1, \dots, \alpha_n \rangle$  and  $\langle \beta_1, \dots, \beta_n \rangle$  be defined. Then so is  $\langle \gamma_1, \dots, \gamma_n \rangle$  where*

$$\gamma_1 = (\alpha_1, \beta_1), \quad \gamma_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix} \quad \text{for } 1 < k < n, \quad \text{and} \quad \gamma_n = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

Moreover  $\langle \alpha_1, \dots, \alpha_n \rangle + \langle \beta_1, \dots, \beta_n \rangle \subset \langle \gamma_1, \dots, \gamma_n \rangle$ .

In Section 3 of May [3] certain associativity formulas are proved, the most useful of which (3.2 and 3.4) relate Massey products and ordinary products and are listed below. The manipulations allowed by this result are commonly known as *juggling*.

A1.4.6. FIRST JUGGLING THEOREM. (a) If  $\langle \alpha_2, \dots, \alpha_n \rangle$  is defined, then so is  $\langle \bar{\alpha}_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle$  and

$$\alpha_1 \langle \alpha_2, \dots, \alpha_n \rangle \subset -\langle \bar{\alpha}_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle.$$

(b) If  $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$  is defined, then so is  $\langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1} \alpha_n \rangle$  and

$$\langle \alpha_1, \dots, \alpha_{n-1} \rangle \alpha_n \subset \langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1} \alpha_n \rangle.$$

(c) If  $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$  and  $\langle \alpha_2, \dots, \alpha_n \rangle$  are strictly defined, then

$$\alpha_1 \langle \alpha_2, \dots, \alpha_n \rangle = \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-1} \rangle \alpha_n.$$

(d) If  $\langle \alpha_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle$  is defined, then so is  $\langle \alpha_1, \bar{\alpha}_2 \alpha_3, \alpha_4, \dots, \alpha_n \rangle$  and

$$\langle \alpha_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle \subset -\langle \alpha_1, \bar{\alpha}_2 \alpha_3, \alpha_4, \dots, \alpha_n \rangle.$$

(e) If  $\langle \alpha_1, \dots, \alpha_{n-2}, \bar{\alpha}_{n-1} \alpha_n \rangle$  is defined, then so is  $\langle \alpha_1, \dots, \alpha_{n-3}, \alpha_{n-1}, \alpha_n \rangle$  and

$$\langle \alpha_1, \dots, \alpha_{n-2}, \bar{\alpha}_{n-1} \alpha_n \rangle \subset -\langle \alpha_1, \dots, \alpha_{n-3}, \alpha_{n-2} \alpha_{n-1}, \alpha_n \rangle.$$

(f) If  $\langle \alpha_1, \dots, \alpha_{k-1}, \alpha_k \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n \rangle$  and  $\langle \alpha_1, \dots, \alpha_k, \bar{\alpha}_{k+1} \alpha_{k+2}, \alpha_{k+3}, \dots, \alpha_n \rangle$  are strictly defined, then the intersection of the former with minus the latter is nonempty.  $\square$

Now we come to some commutativity formulas. For these the DGA  $C$  must satisfy certain conditions (e.g., the cup product must be commutative) which always hold in the cobar complex. We must assume (if  $2 \neq 0$  in  $K$ ) that in each matrix  $\alpha_i$  the degrees of the entries all have the same parity  $\varepsilon_i$ ; i.e.,  $\varepsilon_i$  is 0 if the degrees are all even and 1 if they are all odd. Then we define

$$(A1.4.7) \quad s(i, j) = j - i + \sum_{i \leq k \leq m \leq j} (1 + \varepsilon_k)(1 + \varepsilon_m)$$

and

$$t(k) = (1 + \varepsilon_1) \sum_{j=2}^k (1 + \varepsilon_j).$$

The transpose of a matrix  $\alpha$  will be denoted by  $\alpha'$ . The following result is 3.7 of May [3].

A1.4.8. SECOND JUGGLING THEOREM. Let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be defined and assume that either  $2 = 0$  in  $K$  or the degrees of all of the entries of each  $\alpha_i$  have the same parity  $\varepsilon_i$ . Then  $\langle \alpha'_n, \dots, \alpha'_1 \rangle$  is also defined and

$$\langle \alpha_1, \dots, \alpha_n \rangle' = (-1)^{s(1, n)} \langle \alpha'_n, \dots, \alpha'_1 \rangle.$$

(For the sign see A1.4.7)  $\square$

The next result involves more complicated permutations of the factors. In order to ensure that the permuted products make sense we must assume that we have ordinary, as opposed to matric, Massey products. The following result is 3.8 and 3.9 of May [3].

A1.4.9. THIRD JUGGLING THEOREM. Let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be defined as an ordinary Massey product.

(a) If  $\langle \alpha_{k+1}, \dots, \alpha_n, \alpha_1, \dots, \alpha_k \rangle$  is strictly defined for  $1 \leq k < n$ , then

$$(-1)^{s(1,n)} \langle \alpha_1, \dots, \alpha_n \rangle \subset \sum_{k=1}^{n-1} (-1)^{s(1,k)+s(k+1,n)} \langle \alpha_{k+1}, \dots, \alpha_n, \dots, \alpha_k \rangle.$$

(b) If  $\langle \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n \rangle$  is strictly defined for  $1 \leq k \leq n$  then

$$\langle \alpha_1, \dots, \alpha_n \rangle \subset - \sum_{k=2}^n (-1)^{t(k)} \langle \alpha_2, \dots, \alpha_k, \alpha_1, \alpha_{k+1}, \dots, \alpha_n \rangle.$$

(For the signs see A1.4.7)

Now we consider the behavior of Massey products in spectral sequences. In the previous section we considered essentially three types: the one associated with a resolution (A1.3.2), the one associated with a filtration (decreasing or increasing) of the Hopf algebroid  $\Gamma$  (A1.3.9), and the Cartan–Eilenberg spectral sequence associated with an extension (A1.3.14). In each case the spectral sequence arises from a filtration of a suitable complex. In the latter two cases this complex is the cobar complex of A1.2.11 (in the case of the Cartan–Eilenberg spectral sequence this result is A1.3.16), which is known to be a DGA (A1.2.14) that satisfies the additional hypotheses (not specified here) needed for the commutativity formulas A1.4.8 and A1.4.9. Hence all of the machinery of this section is applicable to those two spectral sequences; its applicability to the resolution spectral sequence of A1.3.2 will be discussed as needed in specific cases.

To fix notation, suppose that our DGA  $C$  is equipped with a decreasing filtration  $\{F^p C\}$  which respects the differential and the product. We do not require  $F^0 C = C$ , but only that  $\lim_{p \rightarrow \infty} F^p C = C$  and  $\lim_{p \rightarrow \infty} F^p C = 0$ . Hence we can have an increasing filtration  $\{F_p C\}$  by defining  $F_p C = F^{-p} C$ . Then we get a spectral sequence with

$$\begin{aligned} E_0^{p,q} &= F^p C^{p+q} / F^{p+1} C^{p+q}, \\ E_1^{p,q} &= H^{p+q}(F^p / F^{p+1}), \\ d_r: E_r^{p,q} &\rightarrow E_r^{p+r, q-r+1}, \end{aligned}$$

and

$$E_\infty^{p,q} = F^p H^{p+q} / F^{p+1} H^{p+q}.$$

We let  $E_{r,\infty}^{p,q} \subset E_r^{p,q}$  denote the permanent cycles and  $i: E_{r,\infty}^{p,q} \rightarrow E_\infty^{p,q}$  and  $\pi: F^p C^{p+q} \rightarrow E_0^{p,q}$  the natural surjections. If  $x \in E_{r,\infty}^{p,q}$  and  $y \in F^p H^{p+q}$  projects to  $i(x) \in E_\infty^{p,q}$  we say that  $x$  converges to  $y$ . If the entries of a matrix  $B \in MC$  are all known to survive to  $E_r$ , we indicate this by writing  $\pi(B) \in ME_r$ . In the following discussions  $\alpha_i$  will denote an element in  $ME_r$  represented by  $a_i \in MC$ . If  $\alpha_i \in ME_{r,\infty}$ ,  $\beta_i \in MH^*(C)$  will denote an element to which it converges.

Each  $E_r$  is a DGA in whose cohomology,  $E_{r+1}$ , Massey products can be defined. Suppose  $\langle \alpha_1, \dots, \alpha_n \rangle$  is defined in  $E_{r+1}$  and that the total bidegree of the  $\alpha_i$  is  $(s, t)$ , i.e., that the ordinary product  $\alpha_1 \alpha_2 \dots \alpha_n$  (which is of course zero if  $n \geq 3$ ) lies in  $E_{r+1}^{s,t}$ . Then the indexing of  $d_r$  implies that the Massey product is a subset of  $E_{r+1}^{s-r(n-2), t+(r-1)(n-2)}$ .

May's first spectral sequence result concerns convergence of Massey products. Suppose that the ordinary triple product  $\langle \beta_1, \beta_2, \beta_3 \rangle \subset H^*(C)$  is defined and that  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is defined in  $E_{r+1}$ . Then one can ask if an element in the latter product is a permanent cycle converging to an element of the former product. Unfortunately, the answer is not always yes. To see how counterexamples can occur, let  $\hat{u}_i \in E_r$  be such that  $d_r(\hat{u}_i) = \alpha_i \alpha_{i+1}$ . Let  $(p, q)$  be the bidegree of one of the  $\hat{u}_i$ . Since  $\langle \beta_1, \beta_2, \beta_3 \rangle$  is defined we have as before  $u_i \in C$  such that  $d(u_i) = a_i a_{i+1}$ . The difficulty is that  $\bar{a}_i a_{i+1}$  need *not* be a coboundary in  $F^p C$ ; i.e., it may not be possible to find a  $u_i \in F^p C$ . Equivalently, the best possible representative  $\tilde{u}_i \in F^p C$  of  $\hat{u}_i$ , may have coboundary  $\bar{a}_i a_{i+1} - e_i$  with  $0 \neq \pi(e_i) \in E_t^{p+t, q-t+1}$  for some  $t > r$ . Then we have  $d(u_i - \tilde{u}_i) = e_i$ . and  $\pi(u_i - \tilde{u}_i) = \pi(u_i) \in E_{m+t}^{p-m, q+m}$  for some  $m > 0$ , so  $d_{m+t}(\pi(u_i)) = \pi(e_i)$ . In other words, the failure of the Massey product in  $E_{r+1}$  to converge as desired is reflected in the presence of a certain higher differential. Thus we can ensure convergence by hypothesizing that all elements in  $E_{m+r+1}^{p-m, q+m}$  for  $m \geq 0$  are permanent cycles.

The case  $m = 0$  is included for the following reason (we had  $m > 0$  in the discussion above). We may be able to find a  $u_i \in F^p C$  with  $d(u_i) = \bar{a}_i a_{i+1}$  but with  $\pi(u_i) \neq u_i$ , so  $d_t(\pi(u - \tilde{u}_i)) = \pi(e_i) \neq 0$ . In this case we can find a convergent element in the Massey product in  $E_{r+1}$ , but it would not be the one we started with.

The general convergence result, which is 4.1 and 4.2 of May [3], is

**A1.4.10. CONVERGENCE THEOREM.** (a) *With notation as above let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be defined in  $E_{r+1}$ . Assume that  $\alpha_i \in ME_{r+1, \infty}$  and  $\alpha_i$  converges to  $\beta_i$ , where  $\langle \beta_1, \dots, \beta_n \rangle$  is strictly defined in  $H^*(C)$ . Assume further that if  $(p, q)$  is the bidegree of an entry of some  $a_{i,j}$  (for  $1 < j-1 < n$ ) in a defining system for  $\langle \alpha_1, \dots, \alpha_n \rangle$  then each element in  $E_{r+m+1}^{p-m, q+m}$  for all  $m \geq 0$  is a permanent cycle. Then each element of  $\langle \alpha_1, \dots, \alpha_n \rangle$  is a permanent cycle converging to an element of  $\langle \beta_1, \dots, \beta_n \rangle$ .*

(b) *Suppose all of the above conditions are met except that  $\langle \alpha_1, \dots, \alpha_n \rangle$  is not known to be defined in  $E_{r+1}$ . If for  $(p, q)$  as above every element of  $E_{r+m}^{p-m, q+m}$  for  $m \geq 1$  is a permanent cycle then  $\langle \alpha_1, \dots, \alpha_n \rangle$  is strictly defined so the conclusion above is valid.  $\square$*

The above result does not prevent the product in question from being hit by a higher differential. In this case  $\langle \beta_1, \dots, \beta_n \rangle$  projects to a higher filtration.

May's next result is a generalized Leibnitz formula which computes the differential on a Massey product in terms of differentials on its factors. The statement is complicated so we first describe the simplest nontrivial situation to which it applies. For this discussion we assume that we are in characteristic 2 so we can ignore signs. Suppose  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is defined in  $E_{r+1}$  but that the factors are not necessarily permanent cycles. We wish to compute  $d_{r+1}$  of this product. Let  $\alpha_i$  have bidegree  $(p_i, q_i)$ . Then we have  $u_i \in F^{p_i+p_{i+1}-2} C$  with  $d(u_i) = a_i a_{i+1} \bmod F^{p_i+p_{i+1}+1} C$ . The product is represented by  $u_1 a_3 + a_1 u_2$ . Now let  $d(a_i) = a'_i$  and  $d(u_i) = a_i a_{i+1} + u'_i$ . Then we have  $d(u_1 a_3 + a_1 u_2) = u'_1 a_3 + u_1 a'_3 + a'_1 u_2 + a_2 u'_2$ . This expression projects to a permanent cycle which we want to describe as a Massey product in  $E_{r+1}$ . Consider

$$\left\langle (d_{r+1}(\alpha_1) \quad \alpha_1), \left( \begin{array}{cc} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{array} \right), \left( \begin{array}{c} \alpha_3 \\ d_{r+1}(\alpha_3) \end{array} \right) \right\rangle.$$

Since  $d(u_i) = \alpha_i \alpha_{i+1} + u'_i$  is a cycle, we have  $d(u'_i) = d(a_i a_{i+1}) = a'_i a_{i+1} + a_i a'_{i+1}$ , so  $d_r(\pi(u'_i)) = d_{r+1}(\alpha_i) \alpha_{i+1} + \alpha_i d_{r+1}(\alpha_{i+1})$ . It follows that the above product contains  $\pi(u'_1 a_3 + u_1 a'_3 + a'_1 u_2 + a_2 u'_2) \in E_{r+1}$ .

Hence we have shown that

$$d_{r+1}(\langle \alpha_1, \alpha_2, \alpha_3 \rangle) \subset \left\langle (d_{r+1}(\alpha_1) \quad \alpha_1), \begin{pmatrix} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ d_{r+1}(\alpha_3) \end{pmatrix} \right\rangle.$$

We would like to show more generally that for  $s > r$  with  $d_t(\alpha_i) = 0$  for  $r < t < s$ , the product is a  $d_t$ -cycle and  $d_s$  on it is given by a similar formula. As in A1.4.10, there are potential obstacles which must be excluded by appropriate technical hypotheses which are vacuous when  $s = r + 1$ . Let  $(p, q)$  be the bidegree of some  $u_i$ . By assumption  $u'_i \in F^{p+r+1}C$  and  $d(u'_i) = a'_i a_{i+1} + a_i a'_{i+1}$ . Hence  $\pi(a_i a'_{i+1} + a'_i a_{i+1}) \in E_*^{p+r+s, q-r-s+2}$  is killed by a  $d_{r+s-t}$  for  $r < t \leq s$ . If the new product is to be defined this class must in fact be hit by a  $d_r$  and we can ensure this by requiring  $E_{r+s-t}^{p+t, q-t+1} = 0$  for  $r < t < s$ . We also need to know that the original product is a  $d_t$ -cycle for  $r < t < s$ . This may not be the case if  $\pi(u'_i) \neq 0 \in E_t^{p+t, q-t+1}$  for  $r < t < s$ , because then we could not get rid of  $\pi(u'_i)$  by adding to  $u_i$  an element in  $F^{p+1}C$  with coboundary in  $F^{p+r+1}C$  (such a modification of  $u_i$  would not alter the original Massey product) and the expression for the Massey product's coboundary could have lower filtration than needed. Hence we also require  $E_t^{p+t, q-t+1} = 0$  for  $r < t < s$ .

We are now ready to state the general result, which is 4.3 and 4.4 of May [3].

A1.4.11. THEOREM (LEIBNITZ FORMULA). (a) *With notation as above let  $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$  be defined in  $E_{r+1}$  and let  $s > r$  be given with  $d_t(\alpha_i) = 0$  for all  $t < s$  and  $1 \leq i \leq n$ . Assume further that for  $(p, q)$  as in A1.4.10 and for each  $t$  with  $r < t < s$ ,*

$$E_t^{p+t, q-t+1} = 0 \quad \text{and} \quad E_{r+s-t}^{p+t, q-t+1} = 0$$

*(for each  $t$  one of these implies the other). Then each element  $\alpha$  of the product is a  $d_t$ -cycle for  $r < t < s$  and there are permanent cycles  $\alpha'_i \in ME_{r+1, \infty}$  which survive to  $d_s(\alpha_i)$  such that  $\langle \gamma_1, \dots, \gamma_n \rangle$  is defined in  $E_{r+1}$  and contains an element  $\gamma$  which survives to  $-d_s(\alpha)$ , where*

$$\gamma_1 = (\alpha'_1 \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_i & 0 \\ \alpha_i & \bar{\alpha}_i \end{pmatrix} \quad \text{for } 1 < i < n,$$

and

$$\gamma_n = \begin{pmatrix} \alpha_n \\ \alpha'_n \end{pmatrix}.$$

(b) *Suppose further that each  $\alpha'_i$  is unique, that each  $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \alpha_i, \alpha_{i+1}, \dots, \alpha_n \rangle$  is strictly defined, and that all products in sight have zero indeterminacy. Then*

$$d_s(\langle \alpha_1, \dots, \alpha_s \rangle) = - \sum_{i=1}^n \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \alpha'_i, \alpha_{i+1}, \dots, \alpha_n \rangle. \quad \square$$

The last result of May [3] concerns the case when  $\langle \alpha_1, \dots, \alpha_n \rangle$  is defined in  $E_{r+1}$ , the  $\alpha_i$  are all permanent cycles, but the corresponding product in  $H^*(C)$  is not defined, so the product in  $E_{r+1}$  supports some nontrivial higher differential. One could ask for a more general result; one could assume  $d_t(\alpha_i) = 0$  for  $t < s$  and, without the vanishing hypotheses of the previous theorem, show that the product supports a nontrivial  $d_t$ . In many specific cases it may be possible to derive such

a result from the one below by passing from the DGA  $C$  to a suitable quotient in which the  $\alpha_i$  are permanent cycles.

As usual we begin by discussing the situation for ordinary triple products, ignoring signs, and using the notation of the previous discussion. If  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is defined in  $E_{r+1}$  and the  $a_i$  are cocycles in  $C$  but the corresponding product in  $H^*(C)$  is not defined, it is because the  $a_i a_{i+1}$  are not both coboundaries; i.e., at least one of the  $u'_i = d(u_i) + a_i a_{i+1}$  is nonzero. Suppose  $\pi(u'_i)$  is nontrivial in  $E_{r+1}^{p+r+1, q-r}$ . As before, the product is represented by  $u_1 a_3 + a_1 u_2$  and its coboundary is  $u'_1 a_3 + a_1 u'_2$ , so  $d_{r+1}(\langle \alpha_1, \alpha_2, \alpha_3 \rangle) = \pi(u'_1 a_3 + a_1 u'_2)$ . Here  $u'_i$  represents the product  $\beta_i \beta_{i+1} \in H^*(C)$ , where  $\beta_i \in H^*(C)$  is the class represented by  $a_i$ . The product  $\beta_i \beta_{i+1}$  has filtration greater than the sum of those of  $\beta_i$  and  $\beta_{i+1}$ , and the target of the differential represents the associator  $(\beta_1 \beta_2) \beta_3 + \beta_1 (\beta_2 \beta_3)$ .

Next we generalize by replacing  $r + 1$  by some  $s > r$ ; i.e., we assume that the filtration of  $\beta_i \beta_{i+1}$  exceeds the sum of those of  $\beta_i$ , and  $\beta_{i+1}$  by  $s - r$ . As in the previous result we need to assume

$$E_t^{p+t, q-t+1} = 0 \quad \text{for } r < t < s;$$

this condition ensures that the triple product is a  $d_t$ -cycle.

The general theorem has some hypotheses which are vacuous for triple products, so in order to illustrate them we must discuss quadruple products, again ignoring signs. Recall the notation used in definition A1.4.3. The elements in the defining system for the product in  $E_{r+1}$  have cochain representatives corresponding to the defining system the product would have if it were defined in  $H^*(C)$ . As above, we denote  $a_{i-1, i}$  by  $a_i$ ,  $a_{i-1, i+1}$  by  $u_i$ , and also  $a_{i-1, i+2}$  by  $v_i$ . Hence we have  $d(a_i) = 0$ ,  $d(u_i) = a_i a_{i+1} + u'_i$ ,  $d(v_i) = a_i u_{i+1} + u_i a_{i+2} + v'_i$ , and the product contains an element  $\alpha$  represented by  $m = a_1 v_2 + u_1 u_3 + v_1 a_4$ , so  $d(m) = a_1 v'_2 + u'_1 u_3 + u_1 u'_3 + v'_1 a_4$ . We also have  $d(u'_i) = 0$  and  $d(v'_i) = u'_i a_{i+2} + a_i u'_{i+1}$ .

We are assuming that  $\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle$  is not defined. There are two possible reasons for this. First, the double products  $\beta_i \beta_{i+1}$  may not all vanish. Second, the double products all vanish, in which case  $u'_i = 0$ , but the two triple products  $\langle \beta_i, \beta_{i+1}, \beta_{i+2} \rangle$  must not both contain zero, so  $v'_i \neq 0$ . More generally there are  $n - 2$  reasons why an  $n$ -fold product may fail to be defined. The theorem will express the differential of the  $n$ -fold product in  $E_{r+1}$  in terms of the highest order subproducts which are defined in  $H^*(C)$ . We will treat these two cases separately.

Let  $(p_i, q_i)$  be the bidegree of  $\alpha_i$ . Then the filtrations of  $u_i, v_i$ , and  $m$  are, respectively,  $p_i + p_{i+1} - r$ ,  $p_i + p_{i+1} + p_{i+2} - 2r$ , and  $p_1 + p_2 + p_3 + p_4 - 2r$ .

Suppose the double products do not all vanish. Let  $s > r$  be the largest integer such that each  $u'_i$  has filtration  $\geq s - r + p_i + p_{i+1}$ . We want to give conditions which will ensure that  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$  is a  $d_t$ -cycle for  $r < t < s$  and that the triple product

$$\left\langle (\pi(u'_1) \alpha_1), \begin{pmatrix} \alpha_3 & 0 \\ \pi(u'_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_4 \\ \pi(u'_3) \end{pmatrix} \right\rangle$$

is defined in  $E_{r+1}$  and contains an element which survives to  $d_s(\alpha)$ ; note that if all goes well this triple product contains an element represented by  $d(m)$ . These conditions will be similar to those of the Leibnitz formula A1.4.11. Let  $(p, q)$  be the bidegree of some  $v_i$ . As before, we ensure that  $d_t(\alpha) = 0$  by requiring  $E_t^{p+t, q-t+1}$ , and that the triple product is defined in  $E_{r+1}$  by requiring  $E_{r+s-t}^{p+t, q-t+1} = 0$ . The

former condition is the same one we made above while discussing the theorem for triple products, but the latter condition is new.

Now we treat the case when the double products vanish but the triple products do not. First consider what would happen if the above discussion were applied here. We would have  $s = \infty$  and  $\alpha$  would be a permanent cycle provided that  $E_t^{p+t, q-t+1} = 0$  for all  $t > r$ . However, this condition implies that  $v$  can be chosen so that  $v' = 0$ , i.e., that the triple products vanish. Hence the above discussion is not relevant here.

Since  $u'_i = 0$ , the coboundary of the Massey product  $m$  is  $a_1 v'_2 + v'_1 a_4$ . Since  $d(v_i) = a_i u_{i+1} + u_i a_{i+2} + v'_i$ ,  $v'_i$  is a cocycle representing an element of  $\langle \beta_i, \beta_{i+1}, \beta_{i+2} \rangle$ . Hence if all goes well we will have  $d_s(\alpha) = \alpha_1 \pi(v'_2) + \pi(v'_1) \alpha_4$ , where  $s > r$  is the largest integer such that each  $v'_i$  has filtration at least  $p_i + p_{i+1} + p_{i+2} + s - 2r$ . To ensure that  $d_t(\alpha) = 0$  for  $t < s$ , we require  $E_t^{p+t, q-t+1} = 0$  for  $r < t < s$  as before, where  $(p, q)$  is the degree of  $v_i$ . We also need to know that  $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2} \rangle$  converges to  $\langle \beta_i, \beta_{i+1}, \beta_{i+2} \rangle$ ; since the former contains zero, this means that the latter has filtration greater than  $p_i + p_{i+1} + p_{i+2} - r$ . We get this convergence from A1.4.10, so we must require that if  $(p, q)$  is the bidegree of  $\pi(u_i)$ , then each element of  $E_{r+m+1}^{p-m, q+m+1}$  for all  $m \geq 0$  is a permanent cycle.

Now we state the general result, which is 4.5 and 4.6 of May [3].

A1.4.12. DIFFERENTIAL AND EXTENSION THEOREM. (a) *With notation as above, let  $\langle \alpha_1, \dots, \alpha_n \rangle$  be defined in  $E_{r+1}$  where each  $\alpha_i$  is a permanent cycle converging to  $\beta_i \in H^*(C)$ . Let  $k$  with  $1 \leq k \leq n - 2$  be such that each  $\langle \beta_i, \dots, \beta_{i+k} \rangle$  is strictly defined in  $H^*(C)$  and such that if  $(p, q)$  is the bidegree of an entry of some  $a_{i,j}$  for  $1 < j - i \leq k$  in a defining system for  $\langle \alpha_1, \dots, \alpha_n \rangle$  then each element of  $E_{r+m+1}^{p-m, q+m}$  for all  $m \geq 0$  is a permanent cycle. Furthermore, let  $s > r$  be such that for each  $(p, q)$  as above with  $k < j - i < n$  and each  $t$  with  $r < t < s$ ,  $E_t^{p+t, q-t+1} = 0$ , and, if  $j - i > k + 1$ ,  $E_{r+s-t}^{p+t, q-t+1} = 0$ .*

*Then for each  $\alpha \in \langle \alpha_1, \dots, \alpha_n \rangle$ ,  $d_t(\alpha) = 0$  for  $r < t < s$ , and there are permanent cycles  $\delta_i \in ME_{r+1, \infty}$  for  $1 \leq i \leq n - k$  which converge to elements of  $\langle \beta_i, \dots, \beta_{i+k} \rangle \subset H^*(C)$  such that  $\langle \gamma_1, \dots, \gamma_{n-k} \rangle$  is defined in  $E_{r+1}$  and contains an element  $\gamma$  which survives to  $-d_s(\alpha)$ , where*

$$\gamma_1 = (\delta_1 \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_{i+k} & 0 \\ \delta_i & \bar{\alpha}_i \end{pmatrix} \quad \text{for } i < n - k,$$

and

$$\gamma_{n-k} = \begin{pmatrix} \alpha_n \\ \delta_{n-k} \end{pmatrix}.$$

(b) *Suppose in addition to the above that each  $\delta_i$  is unique, that each  $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \dots, \alpha_n \rangle$  is strictly defined in  $E_{r+1}$  and that all Massey products in sight (except possibly  $\langle \beta_i, \dots, \beta_{i+k} \rangle$ ) have zero indeterminacy. Then*

$$d_s(\langle \alpha_1, \dots, \alpha_n \rangle) = \sum_{i=1}^{n-k} \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \dots, \alpha_n \rangle. \quad \square$$

Note that in (b) the uniqueness of  $\delta_i$  does not make  $\langle \beta_i, \dots, \beta_{i+k} \rangle$  have zero indeterminacy, but merely indeterminacy in a higher filtration. The theorem does not prevent  $\delta_i$  from being killed by a higher differential. The requirement that  $E_{r+m+1}^{p-m, q+m} \subset E_{r+m+1, \infty}$  is vacuous for  $k = 1$ , e.g., if  $n = 3$ . The condition

$E_{r+s-t}^{p+t, q-t+1} = 0$  is vacuous when  $k = n - 2$ ; both it and  $E_t^{p+t, q-t+1} = 0$  are vacuous when  $s = r + 1$ .

A1.4.13. REMARK. The above result relates differentials to nontrivial extensions in the multiplicative structure (where this is understood to include Massey product structure) since  $\delta_i$  represents  $\langle \beta_i, \dots, \beta_{i+k} \rangle$  but has filtration greater than that of  $\langle \alpha_i, \dots, \alpha_{i+k} \rangle$ . The theorem can be used not only to compute differentials given knowledge of multiplicative extensions, but also vice versa. If  $d_s(\alpha)$  is known, the hypotheses are met, and there are unique  $\delta_i$  which fit into the expression for  $\gamma$ , then these  $\delta_i$  necessarily converge to  $\langle \beta_i, \dots, \beta_{i+k} \rangle$ .

### 5. Algebraic Steenrod Operations

In this section we describe operations defined in  $\text{Cotor}_\Gamma(M, N)$ , where  $\Gamma$  is a Hopf algebroid over  $\mathbf{Z}/(p)$  for  $p$  prime and  $M$  and  $N$  are right and left comodule algebras (A1.1.2) over  $\Gamma$ . These operations were first introduced by Liulevicius [2], although some of the ideas were implicit in Adams [12]. The most thorough account is in May [5], to which we will refer for most of the proofs. Much of the material presented here will also be found in Bruner *et al.* [1]; we are grateful to its authors for sending us the relevant portion of their manuscript. The construction of these operations is a generalization of Steenrod’s original construction (see Steenrod [1]) of his operations in the mod  $(p)$  cohomology of a topological space  $X$ . We recall his method briefly. Let  $G = \mathbf{Z}/(p)$  and let  $E$  be a contractible space on which  $G$  acts freely with orbit space  $B$ .  $X^p$  denotes the  $p$ -fold Cartesian product of  $X$  and  $X^p \times_G E$  denotes the orbit space of  $X^p \times E$  where  $G$  acts canonically on  $E$  and on  $X^p$  by cyclic permutation of coordinates. Choosing a base point in  $E$  gives maps  $X \rightarrow X \times B$  and  $X^p \rightarrow X^p \times_G E$ . Let  $\Delta: X \rightarrow X^p$  be the diagonal embedding. Then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X^p \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & X^p \times_G E \end{array}$$

Given  $x \in H^*(X)$  [all  $H^*$  groups are understood to have coefficients in  $\mathbf{Z}/(p)$ ] it can be shown that  $x \otimes x \otimes \dots \otimes x \in H^*(X^p)$  pulls back canonically to a class  $Px \in H^*(X^p \times_G E)$ . We have  $H^i(B) = \mathbf{Z}/(p)$  generated by  $e_i$  for each  $i \geq 0$ . Hence the image of  $Px$  in  $H^*(X \times B)$  has the form  $\sum_{i \geq 0} x_i \otimes e_i$  with  $x_i \in H^*(X)$  and  $x_0 = x^p$ . These  $x_i$  are certain scalar multiples of various Steenrod operations on  $x$ .

If  $C$  is a suitable DGA whose cohomology is  $H^*(X)$  and  $W$  is a free  $R$ -resolution (where  $R = \mathbf{Z}/(p)[G]$ ) of  $\mathbf{Z}/(p)$ , then we get a diagram

$$\begin{array}{ccc} C & \longleftarrow & C_p \\ \downarrow & & \downarrow \\ C \otimes_R W & \longleftarrow & C_p \otimes_R W \end{array}$$

where  $C_p$  is the  $p$ -fold tensor power of  $C$ ,  $R$  acts trivially on  $C$  and by cyclic permutation on  $C_p$ , and the top map is the iterated product in  $C$ . It is this diagram (with suitable properties) that is essential to defining the operations. The



fact that  $C$  is associated with a space  $X$  is not essential. Any DGA  $C$  which admits such a diagram has Steenrod operations in its cohomology. The existence of such a diagram is a strong condition on  $C$ ; it requires the product to be homotopy commutative in a very strong sense. If the product is strictly commutative the diagram exists but gives trivial operations.

In 11.3 of May [5] it is shown that the cobar complex (A1.2.11)  $C_\Gamma(M, N)$ , for  $M, N$  as above and  $\Gamma$  a Hopf algebra, has the requisite properties. The generalization to Hopf algebroids is not obvious so we give a partial proof of it here, referring to Bruner *et al.* [1] for certain details.

We need some notation to state the result. Let  $C = C_\Gamma(M, N)$  for  $\Gamma$  a Hopf algebroid over  $K$  (which need not have characteristic  $p$ ) and  $M, N$  comodule algebras. Let  $C_r$  denote the  $r$ -fold tensor product of  $C$  over  $K$ . Let  $\pi$  be a subgroup of the  $r$ -fold symmetric group  $\Sigma_r$  and let  $W$  be a negatively graded  $K[\pi]$ -free resolution of  $K$ . Let  $\pi$  act on  $C_r$  by permuting the factors. We will define a map of complexes

$$\theta: W \otimes_{K[\pi]} C_r \rightarrow C$$

with certain properties.

We define  $\theta$  by reducing to the case  $M = \Gamma$ , which is easier to handle because the complex  $d = C_\Gamma(\Gamma, N)$  is a  $\Gamma$ -comodule with a contracting homotopy. We have  $C = M \square_\Gamma D$  and an obvious map

$$j: W \otimes_{K[\pi]} C_r \rightarrow M_r \square_\Gamma (W \otimes_{K[\pi]} D_r),$$

where the comodule structure on  $W \otimes_{K[\pi]} D_r$  is defined by

$$\psi(w \otimes d_1 \cdots \otimes d_r) = d'_1 d'_2 \cdots d'_r \otimes w \otimes d''_1 \otimes \cdots \otimes d''_r$$

for  $w \in W$ ,  $d_i \in D$ , and  $C(d_i) = d'_i \otimes d''_i$ , and the comodule structure on  $M_r$  is defined similarly. Given a suitable map

$$\tilde{\theta}: W \otimes_{K[\pi]} D_r \rightarrow D,$$

we define  $\theta$  to be the composite  $(\mu \square \tilde{\theta})j$ , where  $\mu: M_r \rightarrow M$  is the product.

**A1.5.1. THEOREM.** *With notation as above assume  $W_0 = K[\pi]$  with generator  $e_0$ . Then there are maps  $\theta, \tilde{\theta}$  as above with the following properties.*

- (i) *The restriction of  $\theta$  to  $e_0 \otimes C_r$  is the iterated product (A1.2.15)  $C_r \rightarrow C$ .*
- (ii)  *$\theta$  is natural in  $M, N$ , and  $\Gamma$  up to chain homotopy.*
- (iii) *The analogs of (i) and (ii) for  $\tilde{\theta}$  characterize it up to chain homotopy.*
- (iv) *Let  $\Delta: W \rightarrow W \otimes W$  be a coassociative differential coproduct on  $W$  which is a  $K[\pi]$ -map (where  $K[\pi]$  acts diagonally on  $W \otimes W$ , i.e., given  $\alpha \in \pi$ , and  $w_1, w_2 \in W$ ,  $\alpha \in (w_1 \otimes w_2) = \alpha(w_1) \otimes \alpha(w_2)$ ); such coproducts are known to exist. Let  $\mu: C \otimes C \rightarrow C$  be the product of A1.2.15. Then the following diagram commutes*

up to natural chain homotopy.

$$\begin{array}{ccc}
 W \otimes_{K[\pi]} (C \otimes C)_r & \xrightarrow{W \otimes \mu_r} & W \otimes_{K[\pi]} C_r \\
 \Delta \otimes (C \otimes C)_r \downarrow & & \downarrow \theta \\
 W \otimes W \otimes_{K[\pi]} (C \otimes C)_r & & \\
 W \otimes T \downarrow & & \\
 W \otimes_{K[\pi]} C_r \otimes W \otimes_{K[\pi]} C_r & & \\
 \theta \otimes \theta \downarrow & & \\
 C \otimes C & \xrightarrow{\mu} & C
 \end{array}$$

where  $T$  is the evident shuffle map.

(v) Let  $\pi = \nu = \mathbf{Z}/(p)$ ,  $\sigma = \Sigma_{p^2}$  and let  $\tau$  be the split extensions of  $\nu^p$  by  $\pi$  in which  $\pi$  permutes the factors of  $\nu^p$ . Let  $W, V$ , and  $Y$  be resolutions of  $K$  over  $K[\pi]$ ,  $K[\nu]$ , and  $K[\sigma]$ , respectively. Let  $j: \tau \rightarrow \sigma$  ( $\tau$  is a  $p$ -Sylow subgroup of  $K$ ) induce a map  $j: W \otimes V_p \rightarrow Y$  ( $W \otimes V_p$  is a free  $K[\tau]$  resolution of  $K$ ). Then there is a map  $\omega: Y \otimes_{K[\sigma]} C_{p^2} \rightarrow C$  such that the following diagram commutes up to natural homotopy

$$\begin{array}{ccc}
 (W \otimes V_p) \otimes_{K[\tau]} C_{p^2} & \xrightarrow{j \otimes C_p^2} & Y \otimes_{K[\sigma]} C_{p^2} \\
 \downarrow U & & \searrow \omega \\
 & & C \\
 & & \nearrow \theta \\
 W \otimes_{K[\pi]} (V \otimes_{K[\nu]} C_p)_p & \xrightarrow{W \otimes \theta_p} & W \otimes_{K[\pi]} C_p
 \end{array}$$

where  $U$  is the evident shuffle.

PROOF. The map  $\tilde{\theta}$  satisfying (i), (ii), and (iii) is constructed in Lemma 2.3 of Bruner's chapter in Bruner *et al.* [1]. In his notation let  $M = N$  and  $K = L = C(A, N)$ , which is our  $D$ . Thus his map  $\Phi$  is our  $\tilde{\theta}$ . Since  $\tilde{\theta}$  extends the product on  $N$  it satisfies (i). For (ii), naturality in  $M$  is obvious since cotensor products are natural and everything in sight is natural in  $\Gamma$ . For naturality in  $N$  consider the (not necessarily commutative) diagram

$$\begin{array}{ccc}
 W \otimes_{K[\pi]} C_\Gamma(\Gamma, N)_r & \longrightarrow & W \otimes_{K[\pi]} C_\Gamma(\Gamma, N')_r \\
 \downarrow \tilde{\theta} & & \downarrow \tilde{\theta}' \\
 C_\Gamma(\Gamma, N) & \longrightarrow & C_\Gamma(\Gamma, N').
 \end{array}$$

Bruner's result gives a map

$$W \otimes_{K[\pi]} C_\Gamma(\Gamma, N)_r \rightarrow C_\Gamma(\Gamma, N')$$

extending the map  $N_r \rightarrow N'$ . Both the composites in the diagram have the appropriate properties so they are chain homotopic and  $\tilde{\theta}$  is natural in  $\tilde{N}$  up to the chain homotopy.

For (iv) note that  $\pi$  acts on  $(C \otimes C)_r = C_{2r}$  by permutation, so  $\pi$  is a subgroup of  $\Sigma_{2r}$ . The two composites in the diagram satisfy (i) and (ii) as maps from  $W \otimes_{K[\pi]} C_{2r}$  to  $C$ , so they are naturally homotopic by (iii).

To prove (v), construct  $\omega$  for the group  $\sigma$  in the same way we constructed  $\theta$  for the group  $\pi$ . Then the compositions  $\omega(j \otimes C_{p^2})$  and  $(W \otimes \theta_p)U$  both satisfy (i) and (ii) for the group  $\tau$ , so they are naturally homotopic by (iii).  $\square$

With the above result in hand the machinery of May [5] applies to  $C_\Gamma(M, N)$  and we get Steenrod operations in  $\text{Cotor}_\Gamma(M, N)$  when  $K = \mathbf{Z}/(p)$ . Parts (i), (ii), and (iii) guarantee the existence, naturality, and uniqueness of the operations, while (iv) and (v) give the Cartan formula and Adem relations. These operations have properties similar to those of the topological Steenrod operations with the following three exceptions. First, there is in general no Bockstein operation  $\beta$ . There are operations  $\beta P^i$ , but they need not be decomposable. Recall that in the classical case  $\beta$  was the connecting homomorphism for the short exact sequence

$$0 \rightarrow C \rightarrow \tilde{C} \otimes \mathbf{Z}/(p^2) \rightarrow C \rightarrow 0,$$

where  $\tilde{C}$  is a DGA which is free over  $\mathbf{Z}$ , whose cohomology is the integral cohomology of  $X$  and which is such that  $\tilde{C} \otimes \mathbf{Z}/(p) = C$ . If  $C$  is a cobar complex as above then such a  $\tilde{C}$  may not exist. For example, it does not exist if  $C = C_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$  where  $A_*$  is the dual Steenrod algebra, but if  $C = C_{BP_*(BP)/(p)}(BP_*/(p))$  we have  $\tilde{C} = C_{BP_*(BP)}(BP_*)$ .

Second, when dealing with bigraded complexes there are at least two possible ways to index the operations; these two coincide in the classical singly graded case. In May [5] one has  $P^i: \text{Cotor}^{s,t} \rightarrow \text{Cotor}^{s+(2i-t)(p-1),pt}$ , which means that  $P^i = 0$  if either  $2i < t$  or  $2i > s + t$ . (Classically one would always have  $t = 0$ .) We prefer to index our  $P^i$  so that they raise cohomological degree by  $2i(p-1)$  and are trivial if  $i < 0$  or  $2i > s$  (in May [5] such operations are denoted by  $\tilde{P}^i$ ). This means that we must allow  $i$  to be a half-integer with  $P^i$  nontrivial only if  $2i \equiv t \pmod{2}$ . (This is not a serious inconvenience because in most of our applications for  $p > 2$  the complex  $C^{**}$  will be trivial for odd  $t$ .) The Cartan formula and Adem relations below must be read with this in mind.

Finally,  $P^0: \text{Cotor}^{s,2t} \rightarrow \text{Cotor}^{s,2pt}$  is not the identity as in the classical case. The following is a reindexed form of 11.8 of May [5].

**A1.5.2. STEENROD OPERATIONS THEOREM.** *Let  $\Gamma$  be a Hopf algebroid over  $\mathbf{Z}/(p)$  and  $M$  and  $N$  right and left  $\Gamma$ -comodule algebras. Denote  $\text{Cotor}_\Gamma^{s,t}(M, N)$  by  $H^{s,t}$ . Then there exist natural homomorphisms*

$$\begin{aligned} Sq^i: H^{s,t} &\rightarrow H^{s+i,2t} \quad \text{for } p = 2, \\ P^{i/2}: H^{s,t} &\rightarrow H^{qi/2+s,pt} \end{aligned}$$

and

$$\beta P^{i/2}: H^{s,t} \rightarrow H^{qi/2+s+1,pt} \quad \text{for } p > 2 \text{ and } q = 2p - 2,$$

all with  $i \geq 0$ , having the following properties.

(a) For  $p = 2$ ,  $Sq^i(x) = 0$  if  $i > s$ . For  $p > 2$ ,  $P^{i/2}(x) = 0$  and  $\beta P^{i/2}(x) = 0$  if  $i > s$  or  $2i \not\equiv t \pmod{2}$ .

(b) For  $p = 2$ ,  $Sq^i(x) = x^2$  if  $i = s$ . For  $p > 2$  and  $s + t$  even,  $P^i(x) = x^p$  if  $2i = s$ .

(c) If there exists a Hopf algebroid  $\tilde{\Gamma}$  and  $\tilde{\Gamma}$ -comodule algebras  $\tilde{M}$  and  $\tilde{N}$  all flat over  $\mathbf{Z}/(p)$  with  $\Gamma = \tilde{\Gamma} \otimes \mathbf{Z}/(p)$ ,  $M = \tilde{M} \otimes \mathbf{Z}/(p)$ , and  $N = \tilde{N} \otimes \mathbf{Z}/(p)$ , then  $\beta Sq^i = (i+1)Sq^{i+1}$  for  $p = 2$  and for  $p > 2$   $\beta P^i$  is the composition of  $\beta$  and  $P^i$ , where  $\beta: H^{i,t} \rightarrow H^{s+1,t}$  is the connecting homomorphism for the short exact sequence  $0 \rightarrow N \rightarrow \tilde{N} \otimes \mathbf{Z}/(p^2) \rightarrow N \rightarrow 0$ .

(d)

$$Sq^i(xy) = \sum_{0 \leq j \leq i} Sq^j(x)Sq^{i-j}(y) \quad \text{for } p = 2.$$

For  $p > 2$

$$P^{i/2}(xy) = \sum_{0 \leq j \leq i} P^{j/2}(x)P^{(i-j)/2}(y)$$

and

$$\beta P^{i/2}(xy) = \sum_{0 \leq j \leq i} \beta P^{j/2}(x)P^{(i-j)/2}(y) + P^{i/2}(x)\beta P^{(i-j)/2}(y).$$

Similar external Cartan formulas hold.

(e) The following Adem relations hold. For  $p = 2$  and  $a < 2b$ ,

$$Sq^a Sq^b = \sum_{i \geq 0} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i.$$

For  $p > 2$ ,  $a < pb$ , and  $\varepsilon = 0$  or  $1$  (and, by abuse of notation,  $\beta^0 P^i = P^i$  and  $\beta^1 P^i = \beta P^i$ ),

$$\beta^\varepsilon P^{a/2} P^{b/2} = \sum_{i \geq 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2}$$

and

$$\begin{aligned} \beta^\varepsilon P^{a/2} \beta P^{b/2} &= (1-\varepsilon) \sum_{i \geq 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta P^{(a+b-i)/2} P^{i/2} \\ &\quad - \sum_{i \geq 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2 - 1} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2}, \end{aligned}$$

where, in view of (a), one only considers terms in which  $a$ ,  $b$ , and  $i$  all have the same parity (so the signs and binomial coefficients all make sense).  $\square$

To compute  $Sq^0$  or  $P^0$  we have the following, which is 11.10 of May [5].

**A1.5.3. PROPOSITION.** *With notation as above, let  $x \in H^{s,t}$ , where  $t$  is even if  $p > 2$ , be represented by a cochain which is a sum of elements of the form  $m\gamma_1 | \cdots | \gamma_s n$ . Then  $Sq^0(x)$  or  $P^0(x)$  is represented by a similar sum of elements of the form  $m^p \gamma_1^p | \cdots | \gamma_s^p n^p$ .  $\square$*

The operations also satisfy a certain suspension axiom. Consider the category  $\mathbf{C}$  of triples  $(M, \Gamma, N)$  with  $M, \Gamma, N$  as above. A morphism in  $\mathbf{C}$  consists of maps  $M \rightarrow M'$ ,  $\Gamma \rightarrow \Gamma'$ , and  $N \rightarrow N'$  which respect all the structure in sight. Let  $C_i$ ,  $i = 1, 2, 3$ , be the cobar complexes for three objects in  $\mathbf{C}$  and suppose there are morphisms which induce maps

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

such that the composite  $gf$  is trivial in positive cohomological degree. Let  $H^{**}$ ,  $i = 1, 2, 3$ , denote the corresponding Cotor groups. Define a homomorphism  $\sigma$  (the *suspension*) from  $\ker f^* \subset H_1^{s+t,t}$  to  $H_3^{s,t}/\text{im } g^*$  as follows. Given  $x \in \ker f^*$ , choose a cocycle  $a \in C_1$  representing  $x$  and a cochain  $b \in C_2$  such that  $d(b) = f(a)$ . Then  $g(b)$  is a cocycle representing  $\sigma(x)$ . It is routine to verify that  $\sigma(x)$  is well defined.

**A1.5.4. SUSPENSION LEMMA.** *Let  $\sigma$  be as above. Then for  $p > 2$ ,  $\sigma(P^i(x)) = P^i(\sigma(x))$  and  $\sigma(\beta P^i(x)) = \beta P^i(\sigma(x))$  and similarly for  $p = 2$ .*

**PROOF.** We show how this statement can be derived from ones proved in May [5]. Let  $\bar{C}_1 \subset C_1$  be the subcomplex of elements of positive cohomological degree. It has the structure necessary for defining Steenrod operations in its cohomology since  $C_1$  does. Then May's theorem 3.3 applies to

$$\bar{C}_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

and shows that suspension commutes with the operations in  $\ker f^* \subset H^*(C_1)$ . We have  $H^s(\bar{C}_1) = H^s(C_1)$  for  $s > 1$  and a four-term exact sequence

$$0 \rightarrow M_1 \square_{\Gamma_1} N_1 \rightarrow M_1 \otimes_{A_1} M_1 \rightarrow H^1(\bar{C}_1) \rightarrow H^1(C_1) \rightarrow 0$$

so the result follows. □

**A1.5.5. COROLLARY.** *Let  $\delta$  be the connecting homomorphism associated with an short exact sequence of commutative associative  $\Gamma$ -comodule algebras. Then  $P^i\delta = \delta P^i$  and  $\beta P^i\delta = -\delta\beta P^i$  for  $p > 2$  and similarly for  $p = 2$ . (In this situation the subcomodule algebra must fail to have a unit.)*

**PROOF.** Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be such a short exact sequence. Then set  $N_i = N$  and  $\Gamma_i = \Gamma$  in the previous lemma. Then  $\delta$  is the inverse of  $\sigma$  so the result follows. □

We need a transgression theorem.

**A1.5.6. COROLLARY.** *Let  $(D, \Phi) \xrightarrow{i} (A, \Gamma) \xrightarrow{f} (A, \Sigma)$  be an extension of Hopf algebroids over  $\mathbf{Z}/(p)$  (A1.1.15); let  $M$  be a right  $\Phi$ -comodule algebra and  $N$  a left  $\Gamma$ -comodule algebra, both commutative and associative. Then there is a suspension map  $\sigma$  from  $\ker i^* \subset \text{Cotor}_{\Phi}^{s+1,t}(M, A \square_{\Sigma} N)$  to  $\text{Cotor}_{\Sigma}^{s,t}(M \otimes_D A, N)/\text{im } f^*$  which commutes with Steenrod operations as in A1.5.4.*

**PROOF.**  $A \square_{\Sigma} N$  is a left  $\Phi$ -comodule algebra by A1.3.14(a). We claim the composite  $\bar{\Phi} \xrightarrow{i} \bar{\Gamma} \xrightarrow{f} \bar{\Sigma}$  is zero; since  $\Phi = A \square_{\Sigma} \Gamma \square_{\Sigma} A$ ,  $fi(\Phi) = A \square_{\Sigma} \Sigma \square_{\Sigma} A = A \square_{\Sigma} A = D$ , so  $fi(\bar{\Phi}) = 0$ . Hence  $C_{\Phi}(M, A \square_{\Sigma} N) \rightarrow C_{\Gamma}(M \otimes_D A, N) \rightarrow C_{\Sigma}(M \otimes_D A, N)$  is zero in positive cohomological degree. Hence the result follows from A1.5.4. □

The following is a reformulation of theorem 3.4 of May[5].

**A1.5.7. KUDO TRANSGRESSION THEOREM.** *Let  $\Phi \rightarrow \Gamma \rightarrow \Sigma$  be a cocentral extension (A1.1.15) of Hopf algebras over a field  $K$  of characteristic  $p$ . In the Cartan–Eilenberg spectral sequence (A1.3.14) for  $\text{Ext}_{\Gamma}(K, K)$  we have  $E_2^{s,t} = \text{Ext}_{\Phi}^s(K, K) \otimes \text{Ext}_{\Sigma}^t(K, K)$  with  $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$ . Then the transgression  $d_r: E_r^{0,r-1} \rightarrow E_r^{r,0}$  commutes with Steenrod operations up to sign as in A1.5.4; e.g., if  $d_r(x) = y$  then  $d_{r+2s(p+1)}(P^s(x)) = P^s(y)$ . Moreover for  $p > 2$  and  $r - 1$  even we have*

$$d_{(p-1)(r-1)+1}(x^{p-1}y) = -\beta P^{(r-1)/2}(y). \quad \square$$



## Formal Group Laws

In this appendix we will give a self-contained account of the relevant aspects of the theory of commutative one-dimensional formal group laws. This theory was developed by various algebraists for reasons having nothing to do with algebraic topology. The bridge between the two subjects is the famous result of Quillen [2] (4.1.6) which asserts that the Lazard ring  $L$  (A2.1.8) over which the universal formal group law is defined is naturally isomorphic to the complex cobordism ring. A most thorough and helpful treatment of this subject is given in Hazewinkel [1]. An account of the Lazard ring is also given in Adams [5], while the classification in characteristic  $p$  can also be found in Fröhlich [1].

We now outline the main results of Section 1. We define formal group laws (A2.1.1) and homomorphisms between them (A2.1.5) and show that over a field of characteristic 0 every formal group law is isomorphic to the additive one (A2.1.6). The universal formal group law is constructed (A2.1.8) and the structure of the ring  $L$  over which it is defined is determined (A2.1.10). This result is originally due to Lazard [1]. Its proof depends on a difficult lemma (A2.1.12) whose proof is postponed to the end of the section.

Then we define  $p$ -typical formal group laws (A2.1.17 and A2.1.22) and determine the structure of the  $p$ -typical analog of the Lazard ring,  $V$  (A2.1.24). This result is due to Carrier [1]; Quillen [2] showed that  $V$  is naturally isomorphic to  $\pi^*(BP)$  (4.1.12). Using a point of view due to Landweber [1], we determine the structure of algebraic objects  $LB$  (A2.1.16) and  $VT$  (A2.1.26), which turn out to be isomorphic to  $MU_*(MU)$  (4.1.11) and  $BP_*(BP)$  (4.1.19), respectively.

All of the results of this section can be found in Adams [5], although our treatment of it differs from his.

In Section 2 we give the explicit generators of  $V$  [i.e., of  $\pi_*(BP)$ ] given by Hazewinkel [2] (A2.2.1) and Araki [1] (A2.2.2) and determine the behavior of the right unit  $\eta_R$  on Araki's generators (A2.2.6).

For the Morava theory of Chapter 6 we will need the classification of formal group laws over separably closed fields of characteristic  $p > 0$  (A2.2.12) originally due to Lazard [2], and a description of the relevant endomorphism rings (A2.2.18 and A2.2.19) originally due to Dieudonné [1] and Lubin [1].

For a scheme theoretic approach to this subject, see Strickland [1].

### 1. Universal Formal Group Laws and Strict Isomorphisms

A2.1.1. DEFINITION. *Let  $R$  be a commutative ring with unit. A formal group law over  $R$  is a power series  $F(x, y) \in R[[x, y]]$  satisfying*

- (i)  $F(x, 0) = F(0, x) = x$ ,
- (ii)  $F(x, y) = F(y, x)$ , and
- (iii)  $F(x, F(y, z)) = F(F(x, y), z)$ .

Strictly speaking, such an object should be called a commutative one-dimensional formal group law; we omit the first two adjectives as this is the only type of formal group law we will consider. It is known (Lazard [3]) that (ii) is redundant if  $R$  has no nilpotent elements.

The reason for this terminology is as follows. Suppose  $G$  is a one-dimensional commutative Lie group and  $g: \mathbf{R} \rightarrow U \subset G$  is a homomorphism to a neighborhood  $U$  of the identity which sends 0 to the identity. Then the group operation  $G \times G \rightarrow G$  can be described locally by a real-valued function of two real variables. If the group is analytic then this function has a power series expansion about the origin that satisfies (i)–(iii). These three conditions correspond, respectively, to the identity, commutativity, and associativity axioms of the group. In terms of the power series, the existence of an inverse is automatic, i.e.,

**A2.1.2. PROPOSITION.** *If  $F$  is a formal group law over  $R$  then there is a power series  $i(x) \in R[[x]]$  (called the formal inverse) such that  $F(x, i(x)) = 0$ .*

In the Lie group case this power series must of course converge, but in the formal theory convergence does not concern us. Formal group laws arise in more algebraic situations; e.g., one can extract a formal group law from an elliptic curve defined over  $R$ ; see Chapter 7 of Silverman [1]. One can also reverse the procedure and get a group out of a formal group law; if  $R$  is a complete local ring then  $F(x, y)$  will converge whenever  $x$  and  $y$  are in the maximal ideal, so a group structure is defined on the latter which may differ from the usual additive one.

Before proceeding further note that A2.1.1(i) implies

**A2.1.3. PROPOSITION.** *If  $F$  is a formal group law then*

$$F(x, y) \equiv x + y \pmod{(x, y)^2}. \quad \square$$

**A2.1.4. EXAMPLES OF FORMAL GROUP LAWS.** (a)  $F_a(x, y) = x + y$ , the additive formal group law.

(b)  $F(x, y) = x + y + uxy$  (where  $u$  is a unit in  $R$ ), the multiplicative formal group law, so named because  $1 + uF = (1 + ux)(1 + uy)$ .

(c)  $F(x, y) = (x + y)/(1 + xy)$ .

(d)  $F(x, y) = (x\sqrt{1 - y^4} + y\sqrt{1 - x^4})/(1 + x^2y^2)$ , a formal group law over  $\mathbf{Z}[1/2]$ .

The last example is due to Euler and is the addition formula for the elliptic integral

$$\int_0^x \frac{dt}{\sqrt{1 - t^4}}$$

(see Siegel [1, pp. 1-9]). These examples will be studied further below (A2.2.10).

The astute reader will recognize (c) as the addition formula for the hyperbolic tangent function; i.e., if  $x = \tanh(u)$  and  $y = \tanh(v)$  then  $F(x, y) = \tanh(u + v)$ . Hence we have

$$\tanh^{-1}(F(x, y)) = \tanh^{-1}(x) + \tanh^{-1}(y)$$

or

$$F(x, y) = \tanh(\tanh^{-1}(x) + \tanh^{-1}(y)),$$

where  $\tanh^{-1}(x) = \sum_{i \geq 0} x^{2i+1}/(2i+1) \in R \otimes \mathbf{Q}[[x]]$ .

We have a similar situation in (b), i.e.,

$$\log(1 + uF) = \log(1 + ux) + \log(1 + uy),$$



where  $\log(l + ux) = \sum_{i>0} (-1)^{i+1} (ux)^i / i \in R \otimes \mathbf{Q}[[x]]$ .

This means that the formal group laws of (b) and (c) are isomorphic over  $\mathbf{Q}$  to the additive formal group law (a) in the following sense.

**A2.1.5. DEFINITION.** *Let  $F$  and  $G$  be formal group laws. A homomorphism from  $F$  to  $G$  is a power series  $f(x) \in R[[x]]$  with constant term 0 such that  $f(F(x, y)) = G(f(x), f(y))$ . It is an isomorphism if it is invertible, i.e., if  $f'(0)$  (the coefficient of  $x$ ) is a unit in  $R$ , and a strict isomorphism if  $f'(0) = 1$ . A strict isomorphism from  $F$  to the addition formal group law  $x + y$  is a logarithm for  $F$ , denoted by  $\log_F(x)$ .*

Hence the logarithms for [A2.1.4](#)(b) and (c) are

$$\sum_{i>0} \frac{(-u)^{i-1} x^i}{i} \quad \text{and} \quad \tanh^{-1}(x)$$

respectively.

On the other hand, these formal group laws are not isomorphic to the additive one over  $\mathbf{Z}$ . To see this for (b), set  $u = 1$ . Then  $F(x, x) = 2x + x^2 \equiv x^2 \pmod{2}$ , while  $F_a(x, x) = 2x \equiv 0 \pmod{2}$ , so the two formal group laws are not isomorphic over  $\mathbf{Z}/(2)$ . The formal group law of (c) is isomorphic to  $F_a$  over  $\mathbf{Z}/(2)$ , since its logarithm  $\tanh^{-1}x$  has coefficients in  $\mathbf{Z}/(2)$ , but we have  $F(F(x, x), x) = (3x + x^3)/(1 + 3x^2) \equiv x^3 \pmod{3}$  while  $F_a(F_a(x, x), x) = 3x \equiv 0 \pmod{3}$ . Similarly, it can be shown that  $F$  and  $F_a$  are distinct at every odd prime (see [A2.2.10](#)).

**A2.1.6. THEOREM.** *Let  $F$  be a formal group law and let  $f(x) \in R \otimes \mathbf{Q}[[x]]$  be given by*

$$f(x) = \int_0^x \frac{dt}{F_2(t, 0)}$$

where  $F_2(x, y) = \partial F / \partial y$ . Then  $f$  is a logarithm for  $F$ , i.e.,  $F(x, y) = f^{-1}(f(x) + f(y))$ , and  $F$  is isomorphic over  $R \otimes \mathbf{Q}$  to the additive formal group law.

**PROOF.** Let  $w = f(F(x, y)) - f(x) - f(y)$ . We wish to show  $w = 0$ . We have  $F(F(x, y), z) = F(x, F(y, z))$ . Differentiating with respect to  $z$  and setting  $z = 0$  we get

$$(A2.1.7) \quad F_2(F(x, y), 0) = F_2(x, y)F_2(y, 0).$$

On the other hand, we have  $\partial w / \partial y = f'(F(x, y))F_2(y, 0) - f'(y)$ , which by the definition of  $f$  becomes

$$\frac{\partial w}{\partial y} = \frac{F_2(x, y)}{F_2(F(x, y), 0)} - \frac{1}{F_2(y, 0)} = 0 \quad \text{by } A2.1.7.$$

By symmetry we also have  $\partial w / \partial x = 0$ , so  $w$  is a constant. But  $f$  and  $F$  both have trivial constant terms, so  $w = 0$ .  $\square$

Now we wish to consider the universal formal group law. Its construction is easy.

**A2.1.8. THEOREM.** *There is a ring  $L$  (called the Lazard ring) and a formal group law*

$$F(x, y) = \sum a_{i,j} x^i y^j$$

defined over it such that for any formal group law  $G$  over any commutative ring with unit  $R$  there is a unique ring homomorphism  $\theta: L \rightarrow R$  such that  $G(x, y) = \sum \theta(a_{i,j})x^i y^j$ .

PROOF. Simply set  $L = \mathbf{Z}[a_{i,j}]/I$ , where  $I$  is the ideal generated by the relations among the  $a_{i,j}$  required by the definition A2.1.1, i.e., by  $a_{1,0} - 1$ ,  $a_{0,1} - 1$ ,  $a_{i,0}$ , and  $a_{0,i}$  for (i),  $a_{i,j} - a_{j,i}$  for (ii), and  $b_{ijk}$  for (iii), where

$$F(F(x, y), z) - F(x, F(y, z)) = \sum b_{ijk}x^i y^j z^k.$$

Then  $\theta$  can be defined by the equation it is supposed to satisfy.  $\square$

Determining the structure of  $L$  explicitly is more difficult. At this point it is convenient to introduce a grading on  $L$  by setting  $|a_{i,j}| = 2(i + j - 1)$ . Note that if we have  $|x| = |y| = -2$  then  $F(x, y)$  is a homogeneous expression of degree  $-2$ .

A2.1.9. LEMMA. (a)  $L \otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2, \dots]$  with  $|m_i| = 2i$  and  $F(x, y) = f^{-1}(f(x) + f(y))$  where  $f(x) = x + \sum_{i>0} m_i x^{i+1}$ .

(b) Let  $M \subset L \otimes \mathbf{Q}$  be  $\mathbf{Z}[m_1, m_2, \dots]$ . Then  $\text{im } L \subset M$ .

PROOF. (a) By A2.1.6 every formal group law  $G$  over a  $\mathbf{Q}$ -algebra  $R$  has a logarithm  $g(x)$  so there is a unique  $\phi: \mathbf{Q}[m_1, m_2, \dots] \rightarrow R$  such that  $\phi(f(x)) = g(x)$ . In particular we have  $\phi: \mathbf{Q}[m_1, m_2, \dots] \rightarrow L \otimes \mathbf{Q}$  as well as  $\theta: L \otimes \mathbf{Q} \rightarrow \mathbf{Q}[m_1, m_2, \dots]$  with  $\theta\phi$  and  $\phi\theta$  being identity maps, so  $\theta$  and  $\phi$  are isomorphisms.

(b)  $F(x, y)$  is a power series with coefficients in  $M$ , so the map from  $L$  to  $L \otimes \mathbf{Q}$  factors through  $M$ .  $\square$

Now recall that if  $R$  is a graded connected ring (e.g.,  $L \otimes \mathbf{Q}$ ) the group of indecomposables  $QR$  is  $I/I^2$  where  $I \subset R$  is the ideal of elements of positive degree.

A2.1.10. THEOREM (Lazard [1]). (a)  $L = \mathbf{Z}[x_1, x_2, \dots]$  with  $|x_i| = 2i$  for  $i > 0$ .

(b)  $x_i$  can be chosen so that its image in  $QL \otimes \mathbf{Q}$  is

$$\begin{cases} pm_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ m_i & \text{otherwise.} \end{cases} \quad \square$$

(c)  $L$  is a subring of  $M$  [A2.1.9(b)].

The proof of this is not easy and we will postpone the hardest part of it (A2.1.12) to the end of this section. The difficulty is in effect showing that  $L$  is torsion-free. Without proving A2.1.12 we can determine  $L/\text{torsion}$  with relative ease. We will not give  $F$  in terms of the  $x_i$ , nor will the latter be given explicitly. Such formulas can be found, however, in Hazewinkel [3] and in Section 5 of Hazewinkel [1].

Before stating the hard lemma we need the following exercise in binomial coefficients.

A2.1.11. PROPOSITION. Let  $u_n$  be the greatest common divisor of the numbers  $\binom{n}{i}$  for  $0 < i < n$ . Then

$$u_n = \begin{cases} p & \text{if } n = p^k \text{ for some prime } p \\ 1 & \text{otherwise.} \end{cases} \quad \square$$

Now we are ready for the hard lemma. Define homogeneous symmetric polynomials  $B_n(x, y)$  and  $C_n(x, y)$  of degree  $n$  for all  $n > 0$  by

$$B_n(x, y) = (x + y)^n - x^n - y^n$$

$$C_n(x, y) = \begin{cases} B_n/p & \text{if } n = p^k \text{ for some prime } p \\ B_n & \text{otherwise.} \end{cases}$$

It follows from [A2.1.11](#) that  $C_n(x, y)$  is integral and that it is not divisible by any integer greater than one.

**A2.1.12. COMPARISON LEMMA** (Lazard [[1](#)]). *Let  $F$  and  $G$  be two formal group laws over  $R$  such that  $F \equiv G \pmod{(x, y)^n}$ . Then  $F \equiv G + aC_n \pmod{(x, y)^{n+1}}$  for some  $a \in R$ .  $\square$*

The proof for general  $R$  will be given at the end of this section. For now we give a proof for torsion-free  $R$ .

In this case we lose no information by passing to  $R \otimes \mathbf{Q}$ , where we know ([A2.1.6](#)) that both formal group laws have logarithms, say  $f(x)$  and  $g(x)$ , respectively. Computing  $\pmod{(x, y)^{n+1}}$  we have

$$f(x) \equiv g(x) + bx^n \quad \text{for some } b \in R \otimes \mathbf{Q} \quad \text{so} \quad f^{-1}(x) = g^{-1}(x) - bx^n$$

and

$$\begin{aligned} F - G &= f^{-1}(f(x) + f(y)) - g^{-1}(g(x) + g(y)) \\ &\equiv g^{-1}(g(x) + g(y) + b(x^n + y^n)) - b(x + y)^n - g^{-1}(g(x) + g(y)) \\ &\equiv g^{-1}(g(x) + g(y)) + b(x^n + y^n) - b(x + y)^n - g^{-1}(g(x) + g(y)) \\ &\equiv -bB_n(x, y). \end{aligned}$$

Since this must lie in  $R$  it must have the form  $aC_n(x, y)$ , completing the proof for torsion-free  $R$ .

**A2.1.13. LEMMA.** (a) *In  $QL \otimes \mathbf{Q}$ ,  $a_{i,j} = -\binom{i+j}{j}m_{i+j-1}$ .*  
 (b)  *$QL$  is torsion-free.*

**PROOF.** (a) Over  $L \otimes \mathbf{Q}$  we have  $\sum m_{n-1}(\sum a_{i,j}x^i y^j)^n = \sum m_{n-1}(x^n + y^n)$ . Using [A2.1.3](#) to pass to  $QL \otimes \mathbf{Q}$  we get

$$\sum a_{i,j}x^i y^j + \sum_{n>1} m_{n-1}(x + y)^n = \sum_{n>0} m_{n-1}(x^n + y^n),$$

which gives the desired formula.

(b) Let  $Q_{2n}L$  denote the component of  $QL$  in degree  $2n$ , and let  $R$  be the graded ring  $\mathbf{Z} \oplus Q_{2n}L$ . Let  $F$  be the formal group law over  $R$  induced by the obvious map  $\theta: L \rightarrow R$ , and let  $G$  be the additive formal group law over  $R$ . Then by [A2.1.12](#),  $F(x, y) \equiv x + y + aC_{n+1}(x, y)$  for  $a \in Q_{2n}L$ . It follows that  $Q_{2n}L$  is a cyclic group generated by  $a$ . By (a)  $Q_{2n}L \otimes \mathbf{Q} = \mathbf{Q}$ , so  $Q_{2n}L = \mathbf{Z}$  and  $QL$  is torsion-free.  $\square$

It follows from the above that  $L$  is generated by elements  $x_i$  whose images in  $QL \otimes \mathbf{Q}$  are  $u_i m_i$ , where  $u_i$  is as in [A2.1.11](#), i.e., that  $L$  is a quotient of  $\mathbf{Z}[x_i]$ . By [A2.1.9](#) it is the quotient by the trivial ideal, so [A2.1.10](#) is proved.

Note that having [A2.1.12](#) for torsion-free  $R$  implies that  $L/\text{torsion}$  is as claimed.

The reader familiar with Quillen’s theorem (4.1.6) will recognize  $L$  as  $\pi_*(MU) = MU_*$ . We will now define an object which is canonically isomorphic to  $\pi_*(MU \wedge MU) = MU_*(MU)$ . This description of the latter is due to Landweber [1].

A2.1.14. DEFINITION. *Let  $R$  be a commutative ring with unit. Then  $FGL(R)$  is the set of formal group laws over  $R$  (A2.1.1) and  $SI(R)$  is the set of triples  $(F, f, G)$  where  $F, G \in FGL(R)$  and  $f: F \rightarrow G$  is a strict isomorphism (A2.1.5), i.e.,  $f(x) \in R[[x]]$  with  $f(0) = 0$ ,  $f'(0) = 1$ , and  $f(F(x, y)) = G(f(x), f(y))$ . We call such a triple a matched pair*

A2.1.15. PROPOSITION.  *$FGL(-)$  and  $SI(-)$  are covariant functors on the category of commutative rings with unit.  $FGL(-)$  is represented by the Lazard ring  $L$  and  $SI(-)$  is represented by the ring  $LB = L \otimes \mathbf{Z}[b_1, b_2, \dots]$ . In the grading introduced above,  $|b_i| = 2i$ .*

PROOF. All but the last statement are obvious. Note that a matched pair  $(F, f, G)$  is determined by  $F$  and  $f$  and that  $f$  can be any power series of the form  $f(x) = x + \sum_{i>0} f_i x^{i+1}$ . Hence such objects are in 1-1 correspondence with ring homomorphisms  $\theta: LB \rightarrow R$  with  $\theta(b_i) = f_i$ . □

Now  $LB$  has some additional structure which we wish to describe. Note that  $FGL(R)$  and  $SI(R)$  are the sets of objects and morphisms, respectively, of a groupoid, i.e., a small category in which every morphism is an equivalence. Hence these functors come equipped with certain natural transformations reflecting this structure. The most complicated is the one corresponding to composition of morphisms, which gives a natural (in  $R$ ) map from a certain subset of  $SI(R) \times SI(R)$  to  $SI(R)$ . This structure also endows  $(L, LB)$  with the structure of a Hopf algebroid (A1.1.1). Indeed that term was invented by Haynes Miller with this example in mind. We now describe this structure.

A2.1.16. THEOREM. *In the Hopf algebroid  $(L, LB)$  defined above  $\varepsilon: LB \rightarrow L$  is defined by  $\varepsilon(b_i) = 0$ ;  $\eta_L: L \rightarrow LB$  is the standard inclusion while  $\eta_R: L \otimes \mathbf{Q} \rightarrow LB \otimes \mathbf{Q}$  is given by*

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left( \sum_{j \geq 0} c(b_j) \right)^{i+1},$$

where  $m_0 = b_0 = 1$ ;  $\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} (\sum_{i \geq 0} b_i)^{j+1} \otimes b_j$ ; and  $c: LB \rightarrow LB$  is determined by  $c(m_i) = \eta_R(m_i)$  and  $\sum_{i \geq 0} c(b_i) \left( \sum_{j \geq 0} b_j \right)^{i+1} = 1$ .

These are the structure formulas for  $MU_*(MU)$  (4.1.11).

PROOF.  $\varepsilon$  and  $\eta_L$  are obvious. For  $c$ , if  $f(x) = \sum b_i x^{i+1}$  then  $f^{-1}(x) = \sum c(b_i) x^{i+1}$ . Expanding  $f^{-1}(f(1)) = 1$  gives the formula for  $c(b_i)$ . For  $\eta_R$ , let  $\log x = \sum m_i x^{i+1}$  and  $\text{mog } x = \sum \eta_R(m_i) x^{i+1}$  be the logarithms for  $F$  and  $G$ , respectively. Then we have

$$f^{-1}(G(x, y)) = F(f^{-1}(x), f^{-1}(y))$$

so

$$\log(f^{-1}(G(x, y))) = \log(f^{-1}(x)) + \log(f^{-1}(y)).$$

We also have

$$\text{mog}(G(x, y)) = \text{mog}(x) + \text{mog}(y)$$

for which we deduce

$$\text{mog}(x) = \log f^{-1}(x).$$

Setting  $x = 1$  gives the formula for  $\eta_R$ . For  $\Delta$  let  $f_1(x) = b'_i x^{i+1}$ ,  $f_2(x) = \sum b'_i x^{i+1}$ , and  $f(x) = f_2(f_1(x))$ . Then expanding and setting  $x = 1$  gives  $\sum b_i = \sum b'_i (\sum b'_j)^{i+1}$ . Since  $f_2$  follows  $f_1$  this gives the formula for  $\Delta$ .  $\square$

Note that  $(L, LB)$  is split (A1.1.22) since  $\Delta$  defines a Hopf algebra structure on  $B = \mathbf{Z}[b_i]$ .

Next we will show how the theory simplifies when we localize at a prime  $p$ , and this will lead us to  $BP_*$  and  $BP_*(BP)$ .

A2.1.17. DEFINITION. A formal group law over a torsion-free  $\mathbf{Z}_{(p)}$ -algebra is  $p$ -typical if its logarithm has the form  $\sum_{i \geq 0} \ell_i x^{p^i}$  with  $\ell_0 = 1$ .

Later (A2.1.22) we will give a form of this definition which works even when the  $\mathbf{Z}_{(p)}$ -algebra  $R$  has torsion. Assuming this can be done, we have

A2.1.18. THEOREM (Cartier [1]). Every formal group law over a  $\mathbf{Z}_{(p)}$ -algebra is canonically strictly isomorphic to a  $p$ -typical one.

Actually A2.1.17 is adequate for proving the theorem because it suffices to show that the universal formal group law is isomorphic over  $L \otimes \mathbf{Z}_{(p)}$  to a  $p$ -typical one.

The following notation will be used repeatedly.

A2.1.19. DEFINITION. Let  $F$  be a formal group law over  $R$ . If  $x$  and  $y$  are elements in an  $R$ -algebra  $A$  which also contains the power series  $F(x, y)$ , let

$$x +_F y = F(x, y).$$

This notation may be iterated, e.g.,  $x +_F y +_F z = F(F(x, y), z)$ . Similarly,  $x -_F y = F(x, i(y))$  (A2.1.2). For nonnegative integers  $n$ ,  $[n]_F(x) = F(x, [n-1]_F(x))$  with  $[0]_F(x) = 0$ . (The subscript  $F$  will be omitted whenever possible.)  $\sum^F ( )$  will denote the formal sum of the indicated elements.

A2.1.20. PROPOSITION. If the formal group law  $F$  above is defined over a  $K$ -algebra  $R$  where  $K$  is a subring of  $\mathbf{Q}$ , then for each  $r \in K$  there is a unique power series  $[r]_F(x)$  such that

- (a) if  $r$  is a nonnegative integer,  $[r]_F(x)$  is the power series defined above,
- (b)  $[r_1 + r_2]_F(x) = F([r_1]_F(x), [r_2]_F(x))$ ,
- (c)  $[r_1 r_2]_F(x) = [r_1]_F([r_2]_F(x))$ .

PROOF. Let  $[-1]_F(x) = i(x)$  (A2.1.2), so  $[r]_F(x)$  is defined by (b) for all  $r \in \mathbf{Z}$ . We have  $[r]_F(x) \equiv rx \pmod{x^2}$ , so if  $d \in \mathbf{Z}$  is invertible in  $K$ , the power series  $[d]_F(x)$  is invertible and we can define  $[d^{-1}]_F(x) = [d]_F^{-1}(x)$ .  $\square$

Now we suppose  $q$  is a natural number which is invertible in  $R$ . Let

$$(A2.1.21) \quad f_q(x) = [1/q] \left( \sum_{i=1}^q {}^F \zeta^i x \right)$$

where  $\zeta$  is a primitive  $q$ th root of unity. A priori this is a power series over  $R[\zeta]$ , but since it is symmetric in the  $\zeta^i$  it is actually defined over  $R$ .

If  $R$  is torsion-free and  $\log(x) = \sum_{i \geq 0} m_i x^{i+1}$ , we have

$$\begin{aligned} \log(f_q(x)) &= \frac{1}{q} \sum_{i=1}^q \log(\zeta^i x) \\ &= \frac{1}{q} \sum_{i=1}^q \sum_{j \geq 0} m_j x^{j+1} \zeta^{i(j+1)} \\ &= \frac{1}{q} \sum_{j \geq 0} m_j x^{j+1} \sum_{i=1}^q \zeta^{i(j+1)}. \end{aligned}$$

The expression  $\sum_{i=1}^q \zeta^{i(j+1)}$  vanishes unless  $(j+1)$  is divisible by  $q$ , in which case its value is  $q$ . Hence, we have

$$\log(f_q(x)) = \sum_{j > 0} m_{qj-1} x^{qj}.$$

If  $F$  is  $p$ -typical for  $p \neq q$ , this expression vanishes, so we make

A2.1.22. DEFINITION. A formal group law  $F$  over a  $\mathbf{Z}_{(p)}$ -algebra is  $p$ -typical if  $f_q(x) = 0$  for all primes  $q \neq p$ .

Clearly this is equivalent to our earlier definition A2.1.17 for torsion-free  $R$ .

To prove Cartier's theorem (A2.1.18) we claim that it suffices to construct a strict isomorphism  $f(x) = \sum f_i x^i \in L \otimes \mathbf{Z}_{(p)}[[x]]$  from the image of  $F$  over  $L \otimes \mathbf{Z}_{(p)}$  to a  $p$ -typical formal group law  $F'$ . Then if  $G$  is a formal group law over a  $\mathbf{Z}_{(p)}$ -algebra  $R$  induced by a homomorphism  $\theta: L \otimes \mathbf{Z}_{(p)} \rightarrow R$ ,  $g(x) = \sum \theta(f_i) x^i \in R[[x]]$  is a strict isomorphism from  $G$  to a  $p$ -typical formal group law  $G'$ .

Recall that if  $\text{mog}(x)$  is the logarithm for  $F'$  then

$$\text{mog}(x) = \log(f^{-1}(x)).$$

We want to use the  $f_q(x)$  for various primes  $q \neq p$  to concoct an  $f^{-1}(x)$  such that

$$\log(f^{-1}(x)) = \sum_{i \geq 0} m_{p^i-1} x^{p^i}.$$

It would not do to set

$$f^{-1}(x) = x -_F \sum_{q \neq p}^F f_q(x)$$

because if  $n$  is a product of two or more primes  $\neq p$  then a negative multiple of  $M_{n-1} x^n$  would appear in  $\log f^{-1}(x)$ . What we need is the Möbius function  $\mu(n)$  defined on natural numbers  $n$  by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

Note that  $\mu(1) = 1$  and  $\mu(q) = -1$  if  $q$  is prime. Then we define  $f(x)$  by

$$(A2.1.23) \quad f^{-1}(x) = \sum_{p \nmid q}^F [\mu(q)]_F (f_q(x)).$$

[Note also that  $f_1(x) = x$ .] The sum is over all natural numbers  $q$  not divisible by  $p$ . This infinite formal sum is well defined because  $f_q(x) \equiv 0 \pmod{(x^q)}$ .

Now

$$\log(f^{-1}(x)) = \sum_{p \nmid q} \mu(q) \sum_{j>0} m_{qj-1} x^{qj} = \sum_{n>0} \left( \sum_{\substack{p \nmid q \\ q|n}} \mu(q) \right) m_{n-1} x^n.$$

It is elementary to verify that

$$\sum_{\substack{p \nmid q \\ q|n}} \mu(q) = \begin{cases} 1 & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that  $F'$  has logarithm

$$(A2.1.24) \quad \text{mog}(x) = \sum_{i \geq 0} m_{p^i-1} x^{p^i},$$

so  $F'$  is  $p$ -typical. This completes the proof of [A2.1.18](#).

Now we will construct the universal  $p$ -typical formal group law.

**A2.1.25. THEOREM.** *Let  $V = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  with  $|v_n| = 2(p^n - 1)$ . Then there is a universal  $p$ -typical formal group law  $F$  defined over  $V$ ; i.e., for any  $p$ -typical formal group law  $G$  over a commutative  $\mathbf{Z}_{(p)}$ -algebra  $R$ , there is a unique ring homomorphism  $\theta: V \rightarrow R$  such that  $G(x, y) = \theta(F(x, y))$ . Moreover the homomorphism from  $L \otimes \mathbf{Z}_{(p)}$  to  $V$  corresponding ([A2.1.8](#)) to this formal group law is surjective, i.e.,  $V$  is isomorphic to a direct summand  $L \otimes \mathbf{Z}_{(p)}$ .  $\square$*

We will give an explicit formula for the  $v_n$ 's in terms of the log coefficients  $m_{p^n-1}$  below ([A2.2.2](#)). In [4.1.12](#) it is shown that  $V$  is canonically isomorphic to  $\pi_*(BP)$ .

**PROOF.** Recall that the canonical isomorphism  $f$  above corresponds to an endomorphism  $\phi$  of  $L \otimes \mathbf{Z}_{(p)}$  given by

$$\phi(m_i) = \begin{cases} m_i & \text{if } i = p^k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This  $\phi$  is idempotent, i.e.,  $\phi^2 = \phi$  and its image is a subring  $V \subset L \otimes \mathbf{Z}_{(p)}$  over which the universal  $p$ -typical formal group law is defined. An argument similar to the proof of Lazard's theorem [A2.1.9](#) shows that  $V$  has the indicated structure.  $\square$

Now we will construct a ring  $VT$  canonically isomorphic to  $BP_*(BP)$  and representing the set of  $p$ -typical matched pairs  $(F, f, G)$  ([A2.1.14](#)), i.e., matched pairs with  $F$  and  $G$   $p$ -typical. The power series  $f$  must be chosen carefully to ensure that  $G$  is  $p$ -typical, and this choice depends on  $F$ . There is no such thing as a " $p$ -typical power series," i.e., one that sends any  $p$ -typical  $F$  to a  $p$ -typical  $G$ . To characterize the appropriate  $f$  we have

**A2.1.26. LEMMA.** *Let  $F$  be a  $p$ -typical formal group law over a  $\mathbf{Z}_{(p)}$ -algebra  $R$ . Let  $f(x)$  be an isomorphism ([A2.1.5](#)) from  $F$  to a formal group law  $G$ . Then  $G$  is  $p$ -typical if*

$$f^{-1}(x) = \sum_{i \geq 0}^F t_i x^{p^i}$$

for  $t_i \in R$  with  $t_0$  a unit in  $R$ .

PROOF. For a prime number  $\neq p$  let

$$h_q(x) = [q^{-1}]_G \left( \sum_{i=1}^q \zeta^i x \right)$$

where  $\zeta$  is a primitive  $p$ th root of unity. By A2.1.22 we need to show that  $h_q(x) = 0$  for all  $q \neq p$  iff  $f$  is as specified. From the relation

$$G(x, y) = f(F(f^{-1}(x), f^{-1}(y)))$$

we deduce

$$f^{-1}(h_q(x)) = [q^{-1}]_F \left( \sum_{j=1}^q f^{-1}(\zeta^j x) \right).$$

Now for isomorphism  $f(x)$  there are unique  $c_i \in R$  such that

$$f^{-1}(x) = \sum_{i>0}^F c_i x^i$$

with  $c_1$  a unit in  $R$ . Hence we have

$$\begin{aligned} f^{-1}(h_q(x)) &= [q^{-1}]_F \left( \sum_{i,j}^F c_i \zeta^{ij} x^i \right) \\ &= [q^{-1}]_F \left( \sum_{q^i}^F \sum_j^F \zeta^j c_i x^i \right) +_F [q^{-1}] +_F \left( \sum_i^F [q]_F(c_{qi} x^{qi}) \right) \\ &= \sum_{q^i}^F f_q(c_i x^i) +_F \sum_i^F c_{qi} x^{qi} = \sum_{i>0} c_{qi} x^{qi}. \end{aligned}$$

This expression vanishes for all  $q \neq p$  iff  $c_{qi} = 0$  for all  $i > 0$  and  $q \neq p$ , i.e., iff  $f$  is as specified.  $\square$

It follows immediately that  $VT = V \otimes \mathbf{Z}_{(p)}[t_1, t_2, \dots]$  as a ring since for a strict isomorphism  $t_0 = 1$ . The rings  $V$  and  $VT$  represent the sets of objects and morphisms in the groupoid of strict isomorphisms of  $p$ -typical formal group laws over a  $\mathbf{Z}_{(p)}$ -algebra. Hence  $(V, VT)$ , like  $(L, LB)$ , is a Hopf algebroid (A1.1.1) and it is isomorphic to  $(BP_*, BP_*(BP))$ . Its structure is as follows.

A2.1.27. THEOREM. *In the Hopf algebroid  $(V, VT)$  (see A1.1.1)*

- (a)  $V = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  with  $|v_n| = 2(p^n - 1)$ ,
- (b)  $VT = V \otimes \mathbf{Z}_{(p)}[t_1, t_2, \dots]$  with  $|t_n| = 2(p^n - 1)$ , and
- (c)  $\eta_L: V \rightarrow VT$  is the standard inclusion and  $\varepsilon: VT \rightarrow V$  is defined by  $\varepsilon(t_i) = 0$ ,  $\varepsilon(v_i) = v_i$ .

Let  $\ell_i \in V \otimes \mathbf{Q}$  denote the image of  $m_{p^i-1} \in L \otimes \mathbf{Q}$  (see A2.1.9). Then

- (d)  $\eta_R: V \rightarrow VT$  is determined by  $\eta_R(\ell_n) = \sum_{0 \leq i \leq n} \ell_i t_{n-i}^{p^i}$  where  $\ell_0 = t_0 = 1$ ,
- (e)  $\Delta$  is determined by  $\sum_{i,j \geq 0} \ell_i \Delta(t_j)^{p^i} = \sum_{i,k,j \geq 0} \ell_i t_j^{p^i} \otimes t_k^{p^{i+j}}$ , and
- (f)  $c$  is determined by  $\sum_{i,j,k \geq 0} \ell_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i \geq 0} \ell_i$ .
- (g) The forgetful functor from  $p$ -typical formal group laws to formal group laws induces a surjection of Hopf algebroids (A1.1.19)  $(L \otimes \mathbf{Z}_{(p)}, LB \otimes \mathbf{Z}_{(p)}) \rightarrow (V, VT)$ .



Note that (e) and (f) are equivalent to

$$\sum_{i \geq 0}^F \Delta(t_i) = \sum_{i, j \geq 0}^F t_i \otimes t_j^{p^i} \quad \text{and} \quad \sum_{i, j \geq 0}^F t_i c(t_j)^{p^i} = 1,$$

respectively.

It can be shown that unlike  $(L, LB)$  (A2.1.16),  $(V, VT)$  is not split (A1.1.22).

PROOF. Part (a) was proved in A2.1.23, (b) follows from A2.1.23, and (c) is obvious, as is (g).

For (d) let  $f$  be a strict isomorphism between  $p$ -typical formal group law  $F$  and  $G$  with logarithms  $\log(x)$  and  $\text{mog}(x)$ , respectively. If  $f(x)$  satisfies

$$f^{-1}(x) = \sum_{i \geq 0}^F t_i x^{p^i}$$

and

$$\log(x) = \sum_{i \geq 0} \ell_i x^{p^i}$$

then by definition of  $\eta_R$

$$\text{mog}(x) = \sum_{i \geq 0} \eta_R(\ell_i) x^{p^i}.$$

We have (see the proof of A2.1.16)

$$\begin{aligned} \text{mog}(x) &= \log(f^{-1}(x)) = \log\left(\sum_{i \geq 0}^F t_i x^{p^i}\right) \\ &= \sum_{i \geq 0} \log(t_i x^{p^i}) = \sum_{i, j \geq 0} \ell_i t_j^{p^i} x^{p^{i+j}} \end{aligned}$$

and (d) follows.

For (e) let  $F \xrightarrow{f_1} G \xrightarrow{f_2} H$  be strict isomorphisms of  $p$ -typical formal group laws with

$$f_1^{-1}(x) = \sum_{i \geq 0}^F t'_i x^{p^i} \quad \text{and} \quad f_2^{-1}(x) = \sum_{j \geq 0}^G t''_j x^{p^j}.$$

If we set  $f = f_2 \circ f_1$ , with

$$f^{-1}(x) = \sum_{i \geq 0}^F t_i x^{p^i}$$

then a formula for  $t_i$  in terms of  $t'_i$  and  $t''_i$  will translate to a formula for  $\Delta(t_i)$ .

We have

$$\begin{aligned} f^{-1}(x) &= f_1^{-1}(f_2^{-1}(x)) = f_1^{-1}\left(\sum_{j \geq 0}^G t''_j x^{p^j}\right) \\ &= \sum_j^F f_1^{-1}(t''_j x^{p^j}) = \sum_{i, j}^F t'_i (t''_j x^{p^j})^{p^i}. \end{aligned}$$

This gives

$$\sum_i^F \Delta(t_i) = \sum_{i, j}^F t_i \otimes t_j^{p^i}$$

as claimed.

For (f) let  $f: F \rightarrow G$  be as above. Then

$$f(x) = \sum^G c(t_j)x^{p^j}$$

so

$$\begin{aligned} x &= f^{-1}(f(x)) = f^{-1}\left(\sum_j^G c(t_j)x^{p^j}\right) \\ &= \sum_j^F f^{-1}(c(t_j)x^{p^j}) = \sum_{i,j}^F t_i(c(t_j)x^{p^j})^{p^i} \end{aligned}$$

setting  $x = 1$  gives (f).  $\square$

Our only remaining task is to prove Lazard's comparison lemma [A2.1.12](#). The proof below is due to Fröhlich [\[1\]](#). The lemma states that if  $F$  and  $G$  are formal group laws with  $F \equiv G \pmod{(x, y)^n}$  then

$$F \equiv G + aC_n(x, y) \pmod{(x, y)^{n+1}},$$

where

$$C_n(x, y) = \begin{cases} \frac{(x+y)^n - x^n - y^n}{p} & \text{if } n = p^k \text{ for some prime } p \\ (x+y)^n - x^n - y^n & \text{otherwise.} \end{cases}$$

Let  $\Gamma(x, y)$  be the degree  $n$  component of  $F - G$ .

**A2.1.28. LEMMA.**  $\Gamma(x, y)$  above is a homogeneous polynomial satisfying

- (i)  $\Gamma(x, y) = \Gamma(y, x)$ ,
- (ii)  $\Gamma(x, 0) = \Gamma(0, x) = 0$ ,
- (iii)  $\Gamma(x, y) + \Gamma(x + y, z) = \Gamma(x, y + z) + \Gamma(y, z)$ .

**PROOF.** Parts (i) and (ii) follow immediately [A2.1.1\(ii\)](#) and (i), respectively. For (iii) let  $G(x, y) = x + y + G'(x, y)$ . Then  $\pmod{(x, y, z)^{n+1}}$  we have

$$\begin{aligned} F(F(x, y), z) &\equiv G(F(x, y), z) + \Gamma(F(x, y), z) \\ &\equiv F(x, y) + z + G'(F(x, y), z) + \Gamma(x + y, z) \\ &\equiv G(x, y) + \Gamma(x, y) + z + G'(G(x, y), z) + \Gamma(x + y, z) \\ &\equiv G(G(x, y), z) + \Gamma(x, y) + \Gamma(x + y, z). \end{aligned}$$

Similarly,

$$F(x, F(y, z)) = G(x, G(y, z)) + \Gamma(x, y + z) + \Gamma(y, z)$$

from which (iii) follows.  $\square$

It suffices to show that any such  $\Gamma$  must be a multiple of  $C_n$ .

**A2.1.29. LEMMA.** Let  $R$  be a field of characteristic  $p > 0$ . Then any  $\Gamma(x, y)$  over  $R$  as above is a multiple of  $C_n(x, y)$ .

**PROOF.** It is easy to verify that  $C_n$  satisfies the conditions of [A2.1.28](#), so it suffices to show that the set of all such  $\Gamma$  is one-dimensional vector space. Let  $\Gamma(x, y) = \sum a_i x^i y^{n-i}$ . Then from [A2.1.28](#) we have

$$a_0 = a_n = 0, \quad a_i = a_{n-i},$$

and

$$(A2.1.30) \quad a_i \binom{n-i}{j} = a_{i+j} \binom{i+j}{j} \quad \text{for } 0 < i, i+j < n.$$

The case  $n = 1$  is trivial so we write  $n = sp^k$  with either  $s = p$  or  $s > 1$  and  $s \not\equiv 0 \pmod p$ . We will prove the lemma by showing  $a_i = 0$  if  $i \not\equiv 0 \pmod{p^k}$  and that  $a_{cp^k}$  is a fixed multiple of  $a_{p^k}$ .

If  $i \not\equiv 0 \pmod{p^k}$  we can assume by symmetry that  $i < (s - 1)p^k$  and write  $i = cp^k - j$  with  $0 < c < s$  and  $0 < j < p^k$ . Then [A2.1.30](#) gives

$$a_i \binom{(s - c)p^k + j}{j} = a_{cp^k} \binom{cp^k}{j},$$

i.e.,  $a_i = 0$ .

To show  $a_{cp^k}$  is determined by  $a_{p^k}$  for  $c < s$  let  $i = p^k$  and  $j = (c - 1)p^k$ . Then [A2.1.30](#) gives

$$a_{p^k} \binom{(s - 1)p^k}{(c - 1)p^k} = a_{cp^k} \binom{cp^k}{(c - 1)p^k},$$

i.e.,

$$a_{p^k} \binom{s - 1}{c - 1} = a_{cp^k} c.$$

This determines  $a_{cp^k}$  provided  $c \not\equiv 0 \pmod p$ . Since  $c < s$  we are done for the case  $s = p$ . Otherwise  $a_{cp^k} = a_{(s-c)p^k}$  by symmetry and since  $s \not\equiv 0 \pmod p$  either  $c$  or  $s - c$  is  $\not\equiv 0 \pmod p$ .  $\square$

Note that [A2.1.29](#) is also true for fields of characteristic 0; this can be deduced immediately from [A2.1.30](#). Alternatively, we have already proved [A2.1.12](#), which is equivalent to [A2.1.29](#), for torsion-free rings.

The proof of [A2.1.29](#) is the last hard computation we have to do. Now we will prove the analogous statement for  $R = \mathbf{Z}/(p^m)$  by induction on  $m$ . We have

$$\Gamma(x, y) = aC_n(x, y) + p^{m-1}\Gamma'(x, y),$$

where  $\Gamma'$  satisfies [A2.1.28](#) mod  $p$ . Hence by [A2.1.29](#)  $\Gamma'(x, y) = bC_n(x, y)$  so

$$\Gamma(x, y) = (a + bp^{m-1})C_n(x, y)$$

as claimed.

To prove [A2.1.29](#) (and hence [A2.1.12](#)) for general  $R$  note that the key ingredient [A2.1.30](#) involves only the additive structure of  $R$ ; i.e., we only have to compute in a finitely generated abelian group  $A$  containing the coefficient of  $\Gamma$ . We have to show that symmetry and [A2.1.30](#) imply that the coefficients  $a_i$  are fixed in relation to each other as are the coefficients of  $C_n$ . We have shown that this is true for  $A = \mathbf{Z}$  (from the case  $R = \mathbf{Q}$ ) and  $A = \mathbf{Z}/(p^m)$ . It is clear that if it is true for groups  $A_1$  and  $A_2$  then it is true for  $A_1 \oplus A_2$ , so it is true for all finitely generated abelian groups  $A$ . This completes the proof of [A2.1.12](#).

## 2. Classification and Endomorphism Rings

In order to proceed further we need an explicit choice of the generators  $v_n$ . The first such choice was given by Hazewinkel [\[2\]](#), which was circulating in preprint form six years before it was published. The same generators for  $p = 2$  were defined earlier still by Liulevicius [\[3\]](#). A second choice, which we will use, was given by Araki [\[1\]](#).

Hazewinkel's generators are defined by

$$(A2.2.1) \quad p\ell_n = \sum_{0 \leq i < n} \ell_i v_{n-i}^p$$

which gives, for example,

$$\begin{aligned}\ell_1 &= \frac{v_1}{p}, & \ell_2 &= \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2}, \\ \ell_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p + v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}.\end{aligned}$$

Of course, it is nontrivial to prove that these  $v_n$  are contained in and generate  $V$ . Araki's formula is nearly identical,

$$(A2.2.2) \quad p\ell_n = \sum_{0 \leq i \leq n} \ell_i v_{n-i}^{p^i}$$

where  $v_0 = p$ . These  $v_n$  can be shown to agree with Hazewinkel's mod  $(p)$ . They give messier formulas for  $\ell_n$ , e.g.,

$$\begin{aligned}\ell_1 &= \frac{v_1}{p - p^p}, & (p - p^{p^2})\ell_2 &= v_2 + \frac{v_1^{1+p}}{p - p^p}, \\ (p - p^{p^3})\ell_3 &= v_3 + \frac{v_1 v_2^p}{p - p^p} + \frac{v_2 v_1^{p^2}}{p - p^{p^2}} + \frac{v_1^{1+p+p^2}}{(p - p^p)(p - p^{p^2})},\end{aligned}$$

but a nicer formula (A2.2.6) for  $\eta_R$ .

**A2.2.3. THEOREM** (Hazewinkel [2], Araki [1]). *The sets of elements defined by A2.2.1 and A2.2.2 are contained in and generate  $V$  as a ring, and they are congruent mod  $(p)$ .*

**PROOF.** We first show that Araki's elements generate  $V$ . Equation A2.2.2 yields

$$\sum_{i \geq 0} p\ell_i x^{p^i} = \sum_{i, j \geq 0} \ell_i v_j^{p^i} x^{p^i + j}.$$

Applying  $\exp$  (the inverse of  $\log$ ) to both sides gives

$$(A2.2.4) \quad [p]_F(x) = \sum_{i \geq 0}^F v_i x^{p^i},$$

which proves the integrality of the  $v_n$ , i.e., that  $v_n \in V$ . To show that they generate  $V$  it suffices by A2.1.10 to show  $v_n = pu_n \ell_n$  in  $QV \otimes \mathbf{Q}$ , where  $u_n$  is a unit in  $\mathbf{Z}_{(p)}$ . Reducing A2.2.2 modulo decomposables gives

$$p\ell_n = v_n + \ell_n p^{p^n}$$

so the result follows.

We now denote Hazewinkel's generators of (A2.2.1) by  $w_i$ . Then A2.2.1 gives

$$p \log x - px = \sum_{i > 0} \log w_i x^{p^i}$$

or

$$px = p \log x - \sum_{i > 0} \log w_i x^{p^i}.$$

Exponentiating both sides gives

$$\begin{aligned} \exp px &= [p](x) -_F \sum_{i>0}^F w_i x^{p^i} \\ &= px +_F \sum_{i>0}^F v_i x^{p^i} -_F \sum_{i>0}^F w_i x^{p^i} \text{ by A2.2.4.} \end{aligned}$$

If we can show that  $(\exp px)/p$  is integral then the above equation will give

$$\sum_{i>0}^F v_i x^{p^i} \equiv \sum_{i>0}^F w_i x^{p^i} \pmod{p}$$

and hence  $v_i \equiv w_i \pmod{p}$  as desired.

To show that  $(\exp px)/p$  is integral simply note that its formal inverse is  $(\log px)/p = \sum \ell_i p^{i-1} x^{p^i}$ , which is integral since  $p^i \ell_i$  is.  $\square$

A2.2.5. LEMMA. *Let  $F$  be a  $p$ -typical formal group law over a torsion-free  $\mathbf{Z}_{(p)}$ -algebra. Then for  $p > 2$ ,  $[-1]_F(x) = -x$ , and for  $p = 2$  it is*

$$[-1]_F(x) = -x -_F \sum_{i>0}^F w_i x^{2^i} = x -_F \sum_{i \geq 0}^F v_i x^{2^i}$$

where  $w_i$  is the Hazewinkel generator of (A2.2.1) and  $v_i$  is the Araki generator of (A2.2.2).

PROOF. The series  $[-1]_F(x)$  is characterized by

$$\log([-1]_F(x)) = -\log x.$$

For  $p > 2$ , the  $p$ -typical logarithm is an odd function, so  $\log(-x) = -\log(x)$ . For  $p = 2$  we need to compute the logarithm of the indicated series, first the one given in terms of the Hazewinkel generators. We have

$$\begin{aligned} \log\left(-x -_F \sum_{i>0}^F w_i x^{2^i}\right) &= \log(-x) - \sum_{j>0} \log(w_j x^{2^j}) \\ &= \sum_{i \geq 0} \ell_i (-x)^{2^i} - \sum_{\substack{i \geq 0 \\ j > 0}} \ell_i (w_j x^{2^j})^{2^i} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{\substack{i \geq 0 \\ j > 0}} \ell_i w_j^{2^i} x^{2^{i+j}} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{i>0} \left( \sum_{0 \leq j < i} \ell_j w_{i-j}^{2^j} \right) x^{2^i} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{i>0} 2\ell_i x^{2^i} \\ &= -\log(x). \end{aligned}$$

The result follows.

For the formula in terms of the Araki generators, note that by (A2.2.4), the expression is the same as  $[1]_F x -_F [2]_F(x) = [-1]_F(x)$ .  $\square$

From now on  $v_n$  will denote the Araki generator defined by [A2.2.2](#) or equivalently by [A2.2.4](#). The following formula for  $\eta_R(v_n)$  first appeared in Ravenel [\[1\]](#), where it was stated mod  $(p)$  in terms of the Hazewinkel generators; see also Mor-eira [\[2\]](#).

A2.2.6. THEOREM. *The behavior of  $\eta_R$  on  $v_n$  is defined by*

$$\sum_{i,j \geq 0}^F t_i \eta_R(v_j)^{p^i} = \sum_{i,j \geq 0}^F v_i t_j^{p^i}.$$

PROOF. Applying  $\eta_R$  to [A2.2.2](#) and reindexing we get by [A2.1.27\(d\)](#)

$$\sum p \ell_i t_j^{p^i} = \sum \ell_i t_j^{p^i} \eta_R(v_k)^{p^{i+j}}.$$

Substituting [A2.2.2](#) on the left-hand side and reindexing gives

$$\sum \ell_i v_j^{p^i} t_k^{p^{i+j}} = \sum \ell_i t_j^{p^i} \eta_R(v_k)^{p^{i+j}}.$$

Applying the inverse of log to this gives the desired formula.  $\square$

This formula will be used to prove the classification theorem [A2.2.12](#) below. Computational corollaries of it are given in Section 4.3.

We now turn to the classification in characteristic  $p$ . We will see that formal group laws over a field are characterized up to isomorphism over the separable algebraic closure by an invariant called the height ([A2.2.8](#)). In order to define it we need

A2.2.7. LEMMA. *Let  $F$  be a formal group law over a commutative  $\mathbf{F}_p$ -algebra  $R$  and let  $f(x)$  be a nontrivial endomorphism of  $F$  ([A2.1.5](#)). Then for some  $n$ ,  $f(x) = g(x^{p^n})$  with  $g'(0) \neq 0$ . In particular  $f$  has leading term  $ax^{p^n}$ .*

*When  $R$  is a perfect field  $K$ , we can replace  $g(x^{p^n})$  by  $h(x)^{p^n}$  with  $h'(0) \neq 0$ .*

For our immediate purpose we only need the statement about the leading term, which is easier to prove. The additional strength of the lemma will be needed below ([A2.2.19](#)). The argument we use can be adapted to prove a similar statement about a homomorphism to another formal group law  $G$ .

PROOF. Suppose inductively we have shown that  $f(x) = f_i(x^{p^i})$ , this being trivial for  $i = 0$ , and suppose  $f'_i(0) = 0$ , as otherwise we are done. Define  $F^{(i)}(x, y)$

$$F(x, y)^{p^i} = F^{(i)}(x^{p^i}, y^{p^i}).$$

It is straightforward to show that  $F^{(i)}$  is also a formal group law. Then we have

$$\begin{aligned} f_i(F^{(i)}(x^{p^i}, y^{p^i})) &= f_i(F(x, y)^{p^i}) = f(F(x, y)) \\ &= F(f(x), f(y)) = F(f_i(x^{p^i}), f_i(y^{p^i})) \end{aligned}$$

so

$$f_i(F^{(i)}(x, y)) = F(f_i(x), f_i(y)).$$

Differentiating with respect to  $y$  and setting  $y = 0$  we get

$$f'_i(F^{(i)}(x, 0))F_2^{(i)}(x, 0) = F_2(f_i(x), f_i(0))f'_i(0).$$

Since  $f'_i(0) = 0$ ,  $F_2^{(i)}(x, 0) \neq 0$ , and  $F^{(i)}(x, 0) = x$ , this gives us

$$f'_i(x) = 0 \quad \text{so} \quad f_i(x) = f_{i+1}(x^p).$$

We repeat this process until we get an  $f_n(x)$  with  $f'_n(0) \neq 0$  and set  $g = f_n$ .

The statement about the perfect field case follows from the fact that each coefficient of  $g$  has a  $p^n$ th root.  $\square$

A2.2.8. DEFINITION. A formal group law  $F$  over a commutative  $\mathbf{F}_p$ -algebra  $R$  has height  $n$  if  $[p]_F(x)$  has leading term  $ax^{p^n}$ . If  $[p]_F(x) = 0$  then  $F$  has height  $\infty$ .

A2.2.9. LEMMA. The height of a formal group law is an isomorphism invariant.

PROOF. Let  $f$  be an isomorphism from  $F$  to  $G$ . Then

$$f([p]_F(x)) = [p]_G(f(x));$$

since  $f(x)$  has leading term  $ux$  for  $u$  a unit in  $R$  and the result follows.  $\square$

A2.2.10. EXAMPLES. Just for fun we will compute the heights of the mod  $(p)$  reductions of the formal group laws in A2.1.4.

(a)  $[p]_F(x) = 0$  for all  $p$  so  $F$  has height  $\infty$ .

(b)  $[p]_F(x) = u_{p-1}x^p$  so  $F$  has height 1.

(c) As remarked earlier,  $F$  is isomorphic over  $\mathbf{Z}_{(2)}$  to the additive formal group law, so its height at  $p = 2$  is  $\infty$ . Its logarithm is

$$\sum_{i \geq 0} \frac{x^{2i+1}}{2i+1}$$

so for each odd prime  $p$  we have  $\ell_1 = m_{p-1} = 1/p$ , so  $v_1 \neq 0 \pmod p$  by A2.2.2, so the height is 1 by A2.2.4 and A2.2.8.

(d) Since  $F$  is not defined over  $\mathbf{Z}_{(2)}$  (as can be seen by expanding it through degree 5) it does not have a mod 2 reduction. To compute its logarithm we have

$$F_2(x, 0) = \sqrt{1 - x^4}$$

so by A2.1.6

$$\begin{aligned} \log(x) &= \int_0^x \frac{dt}{\sqrt{1-t^4}} \\ &= \sum_{i \geq 0} \binom{-1/2}{i} \frac{(-1)^i x^{4i+1}}{4i+1} \\ &= \sum_{i \geq 0} \binom{(2i-1)/2}{i} \frac{x^{4i+1}}{4i+1} \\ &= \sum_{i \geq 0} \frac{1 \cdot 3 \cdot 5 \cdots (2i-1)x^{4i+1}}{2^i i! (4i+1)} \\ &= \sum_{i \geq 0} \frac{(2i)! x^{4i+1}}{2^{2i} (i!)^2 (4i+1)}. \end{aligned}$$

Now if  $p \equiv 1 \pmod 4$ , we find that  $\ell_1 = m_{p-1}$  is a unit (in  $\mathbf{Z}_{(p)}$ ) multiple of  $1/p$ , so as in (c) the height is 1. However, if  $p \equiv -1 \pmod 4$ ,  $v_1 = \ell_1 = 0$  so the height is at least 2. We have

$$\ell_2 = m_{p^2-1} = \frac{(2i)!}{4(i!)^2 p^2} \quad \text{where } i = \frac{p^2-1}{4}.$$

Since

$$\frac{p^2-1}{2} = \frac{p(p-1)}{2} + \frac{p-1}{2},$$

$(2i)!$  is a unit multiple of  $p^{(p-1)/2}$ ; since

$$\frac{p^2 - 1}{4} = p \left( \frac{p-3}{4} \right) + \frac{3p-1}{4}$$

$(i)!$  is a unit multiple of  $p^{(p-3)/4}$ . It follows that  $\ell_2$  is a unit multiple of  $1/p$ , so  $v_2 \not\equiv 0 \pmod{p}$  and the height is 2.

It is known that the formal group law attached to a nonsingular elliptic curve always has height 1 or 2. (See Corollary 7.5 of Silverman [1]).

Now we will specify a formal group law of height  $n$  for each  $n$ .

A2.2.11. DEFINITION.  $F_\infty(x, y) = x + y$ . For a natural number  $h$  let  $F_n$  be the  $p$ -typical formal group law (of height  $n$ ) induced by the homomorphism  $\theta: V \rightarrow R$  (A2.1.25) defined by  $\theta(v_n) = 1$  and  $\theta(v_i) = 0$  for  $i \neq n$ .

A2.2.12. THEOREM (Lazard [2]). Let  $K$  be a separably closed field of characteristic  $p > 0$ . A formal group law  $G$  over  $K$  of height  $n$  is isomorphic to  $F_n$ .

PROOF. By Cartier's theorem (A2.1.18) we can assume  $G$  is  $p$ -typical (A2.1.22) and hence induced by a homomorphism  $\theta: V \rightarrow K$  (A2.1.24). If  $n = \infty$  then by A2.2.4  $\theta(v_n) = 0$  for all  $n$  and  $G = F_\infty$ . For  $n$  finite we have  $\theta(v_i) = 0$  for  $i < n$  and  $\theta(v_n) \neq 0$ . Let  $F = F_n$ . We want to construct an isomorphism  $f: F \rightarrow G$  with  $f^{-1}(x) = \sum_{i \geq 0}^F t_i x^{p^i}$ . It follows from A2.2.6 that these  $t_i$  must satisfy

$$(A2.2.13) \quad \sum_{i,j}^F t_i \theta(v_j)^{p^i} x^{p^{i+j}} = \sum_j^F t_j^{p^n} x^{p^{n+j}}$$

since the homomorphism from  $V$  inducing  $F$  is given in A2.2.11, and the  $\eta_R(v_j)$  in A2.2.6 correspond to  $\theta(v_j)$ . Here we are not assuming  $t_0 = 1$ ; the proof of A2.2.6 is still valid if  $t_0 \neq 1$ .

Equating the coefficient of  $x^{p^n}$  in A2.2.13, we get  $t_0 \theta(v_n) = t_0^{p^n}$ , which we can solve for  $t_0$  since  $K$  is separably closed. Now assume inductively that we have solved A2.2.13 for  $t_0, t_1, \dots, t_{i-1}$ . Then equating coefficients of  $x^{p^{i+n}}$  gives

$$t_i \theta(v_n)^{p^i} + c = t_i^{p^n}$$

for some  $c \in K$ . This can also be solved for  $t_i$ , completing the proof.  $\square$

Our last objective in this section is to describe the endomorphism rings of the formal group laws  $F_n$  of A2.2.11.

A2.2.14. LEMMA. Let  $F$  be a formal group law over a field  $K$  of characteristic  $p > 0$  and let  $E$  be the set of endomorphisms of  $F$ .

(a)  $E$  is a ring under composition and formal sum, i.e., the sum of two endomorphisms  $f(x)$  and  $g(x)$  is  $f(x) +_F g(x)$ .

(b)  $E$  is a domain.

(c)  $E$  is a  $\mathbf{Z}_p$ -algebra (where  $\mathbf{Z}_p$  denotes the  $p$ -adic integers) which is a free  $\mathbf{Z}_p$ -module if  $F$  has finite height, and an  $\mathbf{F}_p$ -vector space if  $F$  has infinite height.

PROOF.

(a) We need to verify the distributive law for these two operations. Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be endomorphisms. Then

$$f(g(x) +_F f(x)) = f(g(x)) +_F f(h(x))$$



so

$$f(g + h) = (fg) + (fh) \quad \text{in } E.$$

Similarly,

$$(g +_F h)(f(x)) = g(f(x)) +_F h(f(x))$$

so

$$(g + h)f = (gf) + (hf) \quad \text{in } E.$$

(b) Suppose  $f(x)$  and  $g(x)$  having leading terms  $ax^{p^n}$  and  $bx^{p^n}$ , respectively, with  $a, b \neq 0$  (A2.2.6). Then  $f(g(x))$  has leading term  $ab^{p^m} x^{p^{m+n}}$ , so  $fg \neq 0$  in  $E$ .

(c) We need to show that  $[a]_F(x)$  is defined for  $a \in \mathbf{Z}_p$ . We can write  $a = \sum a_i p^i$  with  $a_i \in \mathbf{Z}$ . Then we can define

$$[a]_F(x) = \sum^F [a_i]_F([p^i]_F(x))$$

because the infinite formal sum on the right is in  $K[[x]]$  since  $[p^i]_F(x) \equiv 0$  modulo  $x^{p^i}$ . If  $h < \infty$  then  $[a]_F(x) \neq 0$  for all  $0 \neq a \in \mathbf{Z}_p$ , so  $E$  is torsion-free by (b). If  $h = \infty$  then  $[p]_F(x) = 0$  so  $E$  is an  $\mathbf{F}_p$ -vector space.  $\square$

Before describing our endomorphism rings we need to recall some algebra.

A2.2.15. LEMMA. *Let  $p$  be a prime and  $q = p^i$  for some  $i > 0$ .*

(a) *There is a unique field  $\mathbf{F}_p$  with  $q$  elements.*

(b) *Each  $x \in \mathbf{F}_q$  satisfies  $x^q - x = 0$ .*

(c)  *$\mathbf{F}_{p^m}$  is a subfield of  $\mathbf{F}_{p^n}$  iff  $m \mid n$ . The extension is Galois with Galois group  $\mathbf{Z}/(m/n)$  generated by the Frobenius automorphism  $x \mapsto x^{p^m}$ .*

(d)  *$\overline{\mathbf{F}}_p$  the algebraic closure of  $\mathbf{F}_p$  and of each  $\mathbf{F}_q$ , is the union of all the  $\mathbf{F}_q$ . Its Galois group is  $\widehat{\mathbf{Z}} = \varprojlim \mathbf{Z}/(m)$ , the profinite integers, generated topologically by the Frobenius automorphism  $x \mapsto x^p$ . The subgroup  $m\mathbf{Z}$  of index  $m$  is generated topologically by  $x \mapsto x^{p^m}$  and fixes the field  $\mathbf{F}_{p^m}$ .*  $\square$

A proof can be found, for example, in Lang [1, Section VII.5]

Now we need to consider the Witt rings  $W(\mathbf{F}_q)$ , which can be obtained as follows. Over  $\mathbf{F}_p$  the polynomial  $x^q - x$  is the product of irreducible factors of degrees at most  $n$  (where  $q = p^n$ ) since it splits over  $\mathbf{F}_q$ , which is a degree  $n$  extension of  $\mathbf{F}_p$ . Let  $h(x) \in \mathbf{Z}_p[x]$  be a lifting of an irreducible factor of degree  $n$  of  $x^q - x$ . Then let  $W(\mathbf{F}_q) = \mathbf{Z}_p[x]/(h(x))$ . It is known to be independent of the choices made and to have the following properties.

A2.2.16. LEMMA. (a)  *$W(\mathbf{F}_q)$  is a  $\mathbf{Z}_p$ -algebra and a free  $\mathbf{Z}_p$ -module of rank  $n$ , where  $q = p^n$  [e.g.,  $W(\mathbf{F}_p) = \mathbf{Z}_p$ ].*

(b)  *$W(\mathbf{F}_q)$  is a complete local ring with maximal ideal  $(p)$  and residue field  $\mathbf{F}_q$ .*

(c) *Each  $w \in W(\mathbf{F}_q)$  can be written uniquely as  $w = \sum_{i \geq 0} w_i p^i$  with  $w_i^q - w_i = 0$  for each  $i$ .*

(d) *The Frobenius automorphism of  $\mathbf{F}_q$  lifts to an automorphism  $\sigma$  of  $W(\mathbf{F}_q)$  defined by*

$$w^\sigma = \sum_{i \geq 0} w_i^p p^i.$$

$\sigma$  generates the Galois group  $\mathbf{Z}/(n)$  of  $W(\mathbf{F}_q)$  over  $\mathbf{Z}_p$ .

- (e)  $W(\mathbf{F}_q) = \varprojlim W(\mathbf{F}_q)/(p^i)$ , so it is a compact topological ring.
- (f) The group of units  $W(\mathbf{F}_q)^\times$  is isomorphic to  $W(\mathbf{F}_q) \oplus \mathbf{F}_q^\times$ , where  $\mathbf{F}_q^\times \cong \mathbf{Z}/(q-1)$ , for  $p > 2$ , and to  $W(\mathbf{F}_q) \oplus \mathbf{F}_q^\times \oplus \mathbf{Z}/(2)$  for  $p = 2$ , the extra summand being generated by  $-1$ .
- (g)  $W(\mathbf{F}_q) \otimes \mathbf{Q} = \mathbf{Q}_p[x]/(h(x))$ , the unramified degree  $n$  extension of  $\mathbf{Q}_p$ , the field of  $p$ -adic numbers.

A proof can be found in Mumford [1, Lecture 26] and in Serre [1, Section 11.5.6]. We will sketch the proof of (f). For  $p > 2$  there is a short exact sequence

$$1 \rightarrow W(\mathbf{F}_q) \xrightarrow{i} W(\mathbf{F}_q)^\times \xrightarrow{j} \mathbf{F}_q^\times \rightarrow 1$$

where  $j$  is mod  $(p)$  reduction and  $i(w) = \exp pw = \sum_{i \geq 0} (pw)^i / i!$  [this power series converges in  $W(\mathbf{F}_q)$ ]. To get a splitting  $\mathbf{F}_q^\times \rightarrow W(\mathbf{F}_q)^\times$  we need to produce  $(q-1)$ th roots of unity in  $W(\mathbf{F}_q)$ , i.e., roots of the equation  $x^q - x = 0$ . [This construction is also relevant to (c).]

These roots can be produced by a device known as the Teichmüller construction. Choose a lifting  $u$  of a given element in  $\mathbf{F}_q$ , and consider the sequence  $\{u, u^q, u^{q^2}, \dots\}$ . It can be shown that it converges to a root of  $x^q - x = 0$  which is independent of the choice of  $u$ .

For  $p = 2$  the power series  $\exp 2w$  need not converge, so we consider instead the short exact sequence

$$1 \rightarrow W(\mathbf{F}_q) \xrightarrow{i} W(\mathbf{F}_q)^\times \xrightarrow{j} W(\mathbf{F}_q)/(4)^\times \rightarrow 1,$$

where  $j$  is reduction mod  $(4)$  and  $i(w) = \exp 4w$ , which always converges. This sequence does not split. We have  $W(\mathbf{F}_q)/(4)^\times \cong \mathbf{F}_q \oplus \mathbf{F}_q^\times$ . Since  $W(\mathbf{F}_q) \otimes \mathbf{Q}$  is a field,  $W(\mathbf{F}_q)^\times$  can have no elements of order 2 other than  $\pm 1$ , so the other elements of order 2 in  $W(\mathbf{F}_q)/(4)^\times$  lift to elements in  $W(\mathbf{F}_q)^\times$  with nontrivial squares.

Next we describe the noncommutative  $\mathbf{Z}_p$ -algebra  $E_n$ , which we will show to be isomorphic to the endomorphism ring of  $F_n$ , for finite  $n$ .

A2.2.17. LEMMA. Let  $E_n$  be the algebra obtained from  $W(\mathbf{F}_q)$  by adjoining an indeterminate  $S$  and setting  $S^n = p$  and  $Sw = w^\sigma S$  for  $w \in W(\mathbf{F}_q)$ . Then

- (a)  $E_n$  is a free  $\mathbf{Z}_p$  module of rank  $n^2$ .
- (b) Each element  $e \in E_n$  can be expressed uniquely as  $\sum_{i \geq 0} e_i S^i$  with  $e_i^q - e_i = 0$ .
- (c)  $E_n$  is generated as a  $\mathbf{Z}_p$ -algebra by  $S$  and a primitive  $(q-1)$ th root of unity  $\omega$  with relations  $S^n - p = 0$ ,  $S\omega = \omega^p S = 0$ , and  $h(\omega) = 0$ , where  $h(x)$  is an irreducible degree  $n$  factor of  $x^q - x$  over  $\mathbf{Z}_p$ .
- (d)  $E_n$  is the maximal order in  $D_n = E_n \otimes \mathbf{Q}$  which is a division algebra with center  $\mathbf{Q}_p$  and invariant  $1/n$ .

The proofs of (a), (b), and (c) are elementary. To see that  $D_n$  is a division algebra, note that any element in  $D_n$  can be multiplied by some power of  $S$  to give an element in  $E_n$  which is nonzero mod  $(S)$ . It is elementary to show that such an element is invertible.

The invariant referred to in (d) is an element in  $\mathbf{Q}/\mathbf{Z}$  which classifies division algebras over  $\mathbf{Q}_p$ . Accounts of this theory are given in Serre [1, Chapters XII and XIII] Cassels and Fröhlich [1, pp. 137–139], Hazewinkel [1, Sections 20.2.16 and 23.1.4]. We remark that for  $0 < i < n$  and  $i$  prime to  $n$  a division algebra with

invariant  $i/n$  has a description similar to that of  $D_n$  except that  $S^n$  is  $p^i$  instead of  $p$ .

Our main results on endomorphism rings are as follows.

A2.2.18. THEOREM (Dieudonné [1] and Lubin [1]). *Let  $K$  be a field of characteristic  $p$  containing  $\mathbf{F}_q$ , with  $q = p^n$ . Then the endomorphism ring of the formal group law  $F_n$  (A2.2.11) over  $K$  is isomorphic to  $E_n$ . The generators  $\omega$  and  $S$  [A2.2.17(c)] correspond to endomorphisms  $\bar{\omega}x$  and  $x^p$ , respectively.*

A2.2.19. THEOREM. *Let  $R$  be a commutative  $\mathbf{F}_p$ -algebra. Then the endomorphism ring of the additive formal group law  $F_\infty$  over  $R$  is the noncommutative power series ring  $R\langle\langle S \rangle\rangle$  in which  $Sa = a^pS$  for  $a \in R$ . The elements  $a$  and  $S$  correspond to the endomorphisms  $ax$  and  $x^p$ , respectively.*

PROOF OF A2.2.19. An endomorphism  $f(x)$  of  $F_\infty$  must satisfy  $f(x + y) = f(x) + f(y)$ . This is equivalent to  $f(x) = \sum_{i \geq 0} a_i x^{p^i}$  for  $a_i \in R$ . The relation  $Sa = a^pS$  corresponds to  $(ax)^p = a^p x^p$ .  $\square$

There is an amusing connection between this endomorphism ring and the Steenrod algebra. Theorem A2.2.19 implies that the functor which assigns to each commutative  $\mathbf{F}_p$ -algebra  $R$  the strict automorphism group of the additive formal group law is represented by the ring

$$P = \mathbf{F}_p[a_1, a_1, \dots]$$

since  $a_0 = 1$  in this case. The group operation is represented by a coproduct  $\Delta: P \rightarrow P \otimes P$ . To compute  $\Delta a_n$  let  $f_1(x) = \sum a'_j x^{p^k}$ ,  $f_2(x) = \sum a''_k x^{p^k}$ , and  $f(x) = f_2(f_1(x)) = \sum a_i x^{p^i}$  with  $a'_0 = a''_0 = a_0 = 1$ . Then we have

$$f(x) = \sum a''_k \left( \sum a'_j x^{p^j} \right)^{p^k} = \sum a''_k (a'_j)^{p^j} x^{p^{j+k}}.$$

It follows that

$$\Delta a_n = \sum_{0 \leq i \leq n} a_{n-i}^{p^i} \otimes a_i \quad \text{with } a_0 = 1,$$

i.e.,  $P$  is isomorphic to the dual of the algebra of Steenrod reduced powers.

Before proving A2.2.18 we need an improvement of A2.2.7. I am grateful to Gerd Laures for finding an error in an earlier version of the following.

A2.2.20. LEMMA. *Let  $F$  be a  $p$ -typical formal group law over a perfect field  $K$  of characteristic  $p > 0$ , and let  $f(x)$  be an endomorphism of  $F$ . Then*

$$f(x) = \sum_{i \geq 0}^F a_i x^{p^i}$$

for some  $a_i \in K$ .

PROOF. As in the proof of A2.2.7, we define for each integer  $n$  a power series  $F^{(n)} \in K[[x, y]]$  by replacing each coefficient by its  $p^n$ th power. It is easily seen that it is a  $p$ -typical formal group law when  $F$  is.

Then we have

$$\begin{aligned} f(F(x, y)) &= h(F(x, y))^{p^n} \\ F(f(x), f(y)) &= F(h(x)^{p^n}, h(y)^{p^n}) = \left( F^{(-n)}(h(x), h(y)) \right)^{p^n}. \end{aligned}$$

These two expressions are equal because  $f$  is an endomorphism, so

$$h(F(x, y)) = F^{(-n)}(h(x), h(y))$$

and  $h$  (and hence  $h^{-1}$ ) is an isomorphism between two  $p$ -typical formal group laws. The result then follows from [A2.1.26](#).  $\square$

[A2.2.18](#) will follow easily from the following.

**A2.2.21. LEMMA.** *Let  $E(F_n)$  be the endomorphism ring of  $F_n$  ([A2.2.11](#)) over a perfect field  $K$  containing  $\mathbf{F}_q$  where  $q = p^n$ . Then*

- (a) *if  $f(x) = \sum^{F_n} a_i x^{p^i}$  is in  $E(F_n)$ , then each  $a_i \in \mathbf{F}_q$ ;*
- (b) *for  $a \in \mathbf{F}_q$ ,  $ax \in E(F_n)$ ;*
- (c)  *$x^p \in E(F_n)$ ; and*
- (d)  *$E(F_n)/(p) = E_n/(p) = \mathbf{F}_q\langle S \rangle / (S^n)$  with  $Sa = a^p S$ .*

**PROOF.** (a) By the definition of  $F_n$  ([A2.2.11](#)) and [A2.2.15](#) we have

$$(A2.2.22) \quad [p](x) = x^{p^n}.$$

Any endomorphism  $f$  commutes with  $[p]$  so by [A2.2.20](#) we have

$$[p](f(x)) = [p]\left(\sum^{F_n} a_i x^{p^i}\right) = \sum^{F_n} [p](a_i x^{p^i}) = \sum^{F_n} a_i^{p^n} x^{p^{i+n}}.$$

This must equal

$$f([p](x)) = \sum^{F_n} a_i ([p](x))^{p^i} = \sum^{F_n} a_i x^{p^{i+n}}.$$

Hence  $a_i^{p^n} = a_i$  for all  $i$  and  $a_i \in \mathbf{F}_q$ .

(b) It suffices to prove this for  $K = \mathbf{F}_q$ .  $\mathbf{F}_n$  can be lifted to a formal group law  $\tilde{F}_n$  over  $W(\mathbf{F}_q)$  ([A2.2.16](#)) by the obvious lifting of  $\theta: V \rightarrow \mathbf{F}_q$ , to  $W(\mathbf{F}_q)$ . It suffices to show that  $\omega x$  is an endomorphism of  $\tilde{F}_n$  if  $\omega^q - \omega = 0$ . By [A2.2.2](#)  $\tilde{F}_n$  has a logarithm of the form

$$\log(x) = \sum a_i x^{q^i}$$

so  $\log(\omega x) = \omega \log(x)$  and  $\omega x$  is an endomorphism.

(c) This follows from the fact that  $F_n$  is defined over  $\mathbf{F}_p$ , so  $F_n(x^p, y^p) = F_n(x, y)^p$ .

(d) By [A2.2.22](#), (b) and (c),  $f(x) \in pE(F_n)$  iff  $a_i = 0$  for  $i < n$ . It follows that for  $f(x), g(x) \in E(F_n)$ ,  $f \equiv g \pmod{(p)E(F_n)}$  iff  $f(x) \equiv g(x) \pmod{(x^q)}$ . Now our lifting  $\tilde{F}_n$  of  $F_n$  above has  $\log x \equiv x \pmod{(x^q)}$ , so  $F_n(x, y) \equiv x + y \pmod{(x, y)^q}$ . It follows that  $E(F_n)/(p)$  is isomorphic to the corresponding quotient of  $E(F_\infty)$  over  $\mathbf{F}_q$ , which is as claimed by [A2.2.18](#).  $\square$

**PROOF OF [A2.2.18](#).** . By [A2.2.17](#)(c)  $E_n$  is generated by  $\omega$  and  $S$ . The corresponding elements are in  $E(F_n)$  by [A2.2.21](#)(b) and (c). The relation  $S\omega = \omega^p S$  corresponds as before to the fact that  $(\bar{\omega}x)^p = \bar{\omega}^p x^p$ , where  $\bar{\omega}$  is mod  $(p)$  reduction of  $\omega$ . Hence we have a homomorphism  $\lambda: E_n \rightarrow E(F_n)$  which is onto by [A2.2.20](#). We know [[A2.2.14](#)(c)] that  $E(F_n)$  is a free  $\mathbf{Z}_p$ -module. It has rank  $n^2$  by [A2.2.21](#)(d), so  $\lambda$  is 1-1 by [A2.2.17](#)(a).  $\square$

## Tables of Homotopy Groups of Spheres

In this appendix we collect most of the known values of the stable homotopy groups of spheres for the primes 2, 3, and 5. Online graphic displays of these are given by Hatcher [1]. The results of Toda [6] on unstable homotopy groups are shown in Table A3.6. A table of unstable 3-primary homotopy groups up to dimension 80 can be found in Toda [8].

Extensive online charts of various Ext groups over the Steenrod algebra have been provided by Nassau [1] and Bruner [3].

In FIGS. A3.1a–c we display the classical Adams  $E_2$ -term for  $p = 2$ ,

$$\text{Ext}_A^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

for  $t - s \leq 61$ , along the differentials and group extensions. The main reference for the calculation of Ext is Tangora [1], which includes a table showing the answer for  $t - s \leq 70$ . We use his notation for the many generators shown in Ext. His table is preceded by a dictionary (not included here) relating this notation to that of the May spectral sequence, which is his main computational tool.

In our table each basis element is indicated by a small circle. Multiplication by the elements  $h_0$ ,  $h_1$ , and  $h_2$  is indicated, respectively, by vertical lines and lines with slopes 1 and  $\frac{1}{3}$ . Most multiplicative generators are labeled, but there are a few unlabeled generators due to limitations of space. In each case the unlabeled generator is in the image of the periodicity operator  $P$  (denoted by  $\Pi$  in Section 3.4), which sends an element  $x \in \text{Ext}^{s,t}$  to the Massey product (Section A1.4)

$$\langle x, h_0^4, h_3 \rangle \in \text{Ext}^{s+4, t+12}.$$

Differentials are indicated by lines with negative slope. For  $t - s \leq 20$  these can be derived by combining the calculation of Ext in this range due to May [1] with the calculation of the corresponding homotopy groups by Toda [6]. For  $21 \leq t - s \leq 45$  the results can be found in various papers by Barratt, Mahowald, Milgram, and Tangora and most recently in Bruner [2], where precise references to the earlier work can be found.

Differentials in the range  $46 \leq t - s \leq 61$  have been computed (tentatively in some cases) by Mahowald (unpublished) and are included here with his kind permission.

Exotic group extensions and some exotic multiplications by  $h_1$  and  $h_2$  are indicated by broken lines with nonnegative slope.

In FIG. A3.2 we display the Adams–Novikov  $E_2$ -term for  $p = 2$  in the range  $t - s \leq 39$ . The method used is that of Section 4.4, where the calculation is described in detail through dimension 25. The small circles in the chart indicate summands of order 2. Larger cyclic summands are indicated by squares. All such summands in

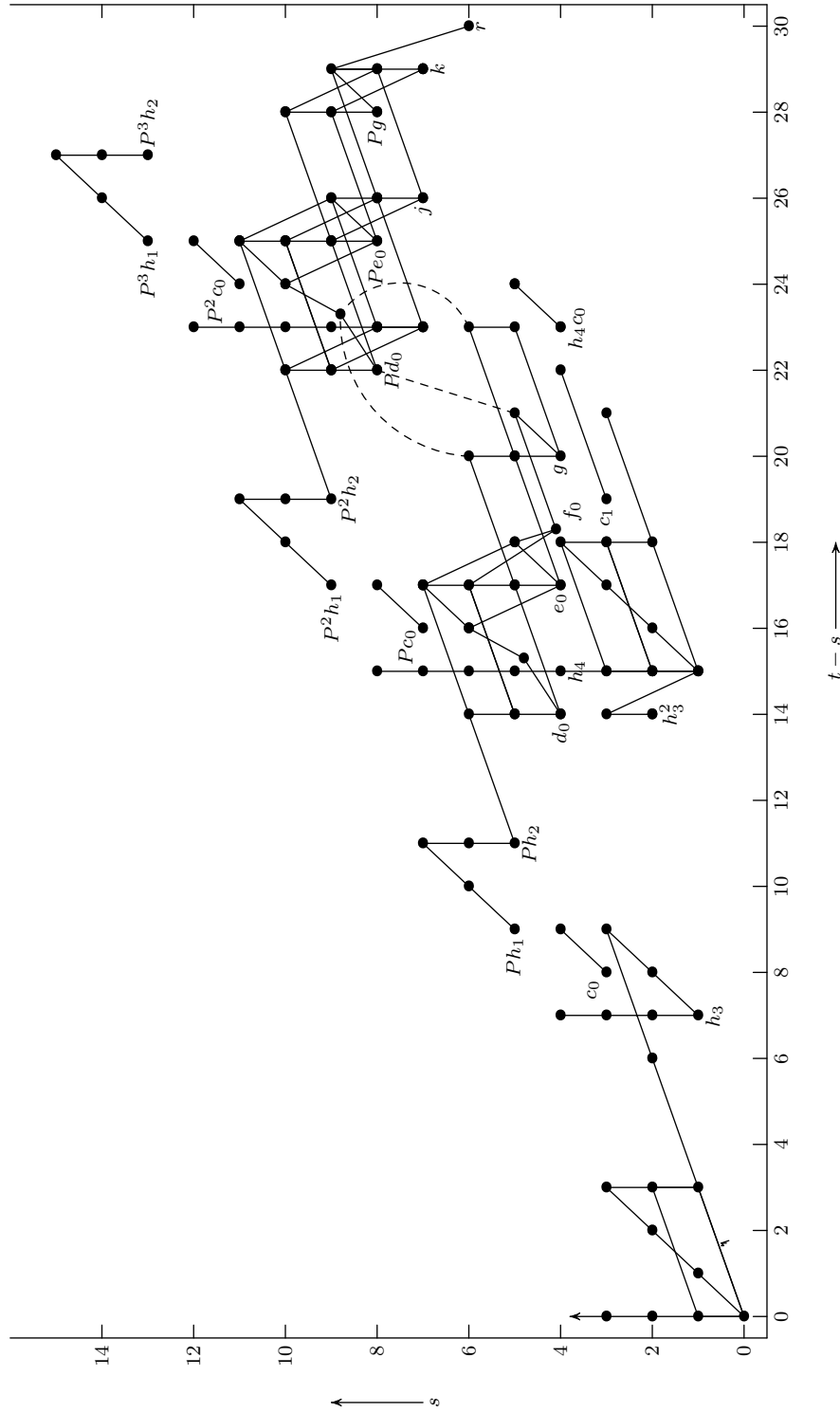


FIGURE A3.1a. The Adams spectral sequence for  $p = 2$ ,  $t - s \leq 29$ .







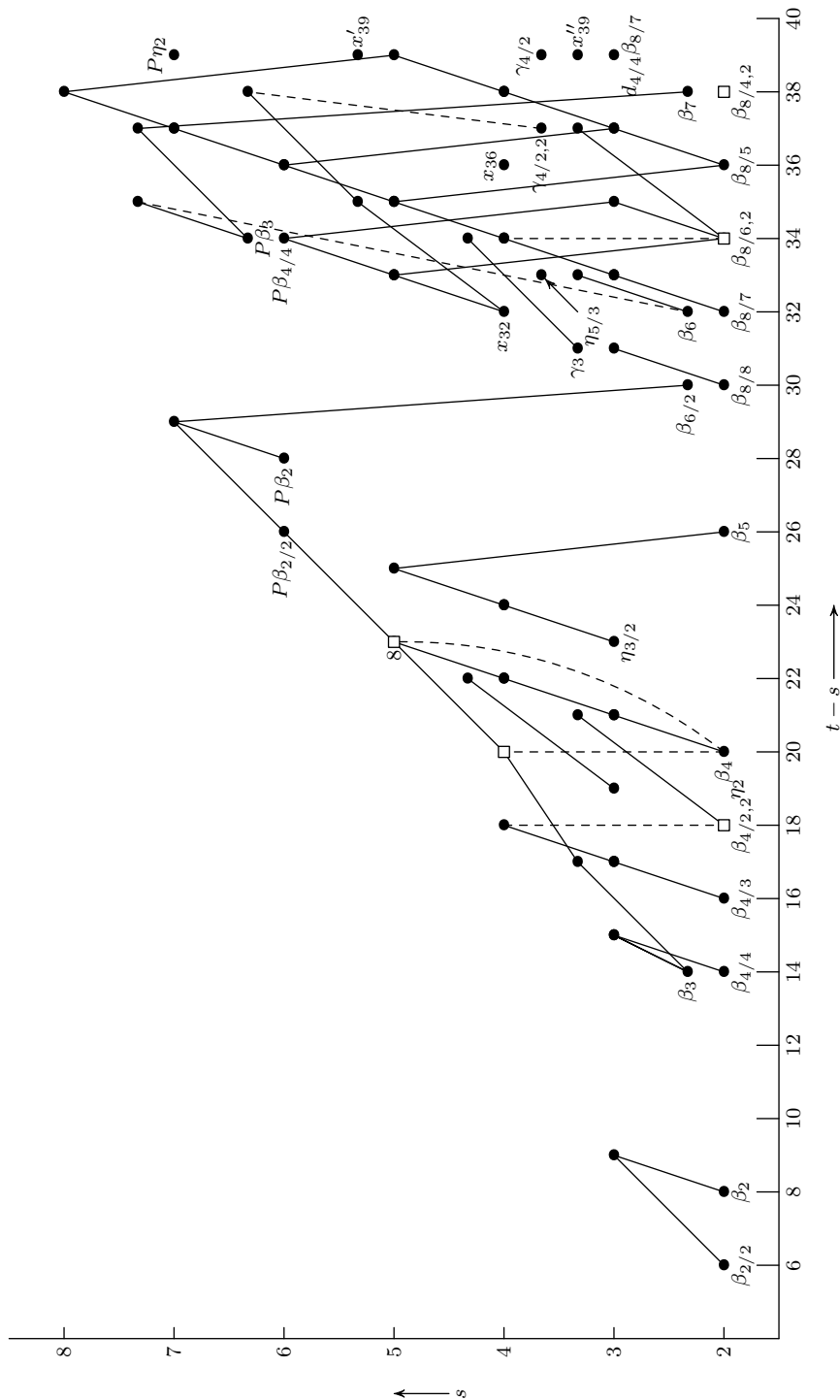


FIGURE A3.2. The Adams–Novikov spectral sequence for  $p = 2$ ,  $t - s \leq 39$ .  $v_1$ -periodic elements omitted. Computations for  $t - s \geq 30$  are tentative. As of February, 2016, this chart is known to contain some errors. A much better chart through a higher range (59) can be found in Isaksen [1].

this range have order 4 except the one in  $\text{Ext}^{5,28}$ , which has order 8. The solid and broken lines in this figure means the same thing as in Figs. A3.1a–c as described above. This figure does *not* include the  $v_1$ -periodic elements described in 5.3.7, i.e., the elements in the image of the  $J$ -homomorphism and the elements constructed in Adams [1].

In TABLE A3.3 we list the values of the 2-component of the stable stems  $\pi_k^S$  for  $k \leq 45$ , showing the name of each element given by Toda [7] (where applicable), by Tangora [1] in the Adams spectral sequence, and by us in the Adams–Novikov spectral sequence. Again we omit the  $v_1$ -periodic elements described in 5.3.7. These omitted summands are as follows.

- $\mathbf{Z}$  for  $k = 0$ ,
- $\mathbf{Z}/(2)$  for  $k = 1$  or  $2$ ,
- $\mathbf{Z}/(4)$  for  $k = 3$ ,
- $\mathbf{Z}/(2^{m+4})$  for  $k = 8t - 1$ , where  $t$  is an odd multiple of  $2^m$ ,
- $\mathbf{Z}/(2)$  for  $k \equiv 0$  or  $2 \pmod{8}$  and  $k > 7$ ,
- $(\mathbf{Z}/(2))^2$  for  $k \equiv 1 \pmod{8}$  and  $k > 7$ , and,
- $\mathbf{Z}/(8)$  for  $k \equiv 3 \pmod{8}$  and  $k > 7$ .

In TABLES A3.4 and A3.5 we do the same for the primes 3 and 5, recapitulating the results obtained in Sections 7.4 and 7.5, respectively. Again we omit the  $v_1$ -periodic elements described in 5.3.7, which in these cases are (in positive dimensions) precisely  $\text{im } J$ , i.e.,

- $\mathbf{Z}$  for  $k = 0$  and
- $\mathbf{Z}/(p^{m+1})$  for  $k = (2p - 2)t - 1$ ,

where  $t = sp^m$  and  $s$  is prime to  $p$ .

In FIG. A3.6 we reproduce the table of unstable homotopy groups of spheres through the 19-stem, given in Toda [6].

TABLE A3.3.  $\pi_*^S$  at  $p = 2^a$

Stem	Toda's name	Tangora's name	Adams–Novikov name
6	$\nu^2$	$h_2^2$	$\beta_{2/2}$
8	$\varepsilon = \langle \nu^2, 2, \eta \rangle$	$c_0$	$\beta_2$
9	$\nu^3$	$h_1^2 h_3$	$\alpha_1 \beta_2$
14	$\sigma^2$	$h_3^2$	$\beta_{4/4}$
	$\kappa$	$d_0$	$\beta_3$
15	$\eta \kappa$	$h_1 d_0$	$\alpha_1 \beta_3 = \alpha_1 \beta_{4/4}$
16	$\eta^* \langle \sigma, 2\sigma, \eta \rangle$	$h_1 h_4$	$\beta_{4/3}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
17	$\eta\eta^*$	$h_1^2h_4$	$\alpha_1\beta_{4/3}$
	$\nu\kappa$	$h_2d_0$	$\alpha_{2/2}\beta_3 = \alpha_{2/2}\beta_{4/4}$
18	$\nu^*\langle\sigma, 2\sigma, \nu\rangle$	$h_2h_4, h_0h_2h_4$ $h_0^2h_2h_4 = h_1^3h_4$	$\beta_{4/2,2}$ $4\beta_{4/2,2} = \alpha_1^2\beta_{4/3}$
19	$\bar{\sigma} = \langle\sigma^2 + \kappa, \eta, \nu\rangle$	$c_1$	$\eta_2$
20	$\bar{\kappa}$	$g$	$\beta_4$
	$2\bar{\kappa}$	$h_0g$	$2\beta_4 = x_{20} = \langle 2, \alpha_1^3, \beta_{4/3} \rangle$
	$4\bar{\kappa}$	$h_0^2g$	$2x_{20} = \alpha_{2/2}^2\beta_3$
21	$\sigma^3$	$h_2^2h_4$	$\alpha_{2/2}\beta_{4/2,2}$
	$\eta\bar{\kappa}$	$h_1g$	$\alpha_1\beta_4$
22	$\nu\bar{\sigma}$	$h_2c_1$	$\alpha_{2/2}\eta_2$
	$\eta^2\bar{\kappa}$	$Pd_0$	$\alpha_1^2\beta_4$
23	$\mathbf{Z}/2 \otimes \mathbf{Z}/8$	$h_4c_0$	$\eta_{3/2}$
		$h_2g$	$x_{23} = \langle \alpha_{2/2}, \alpha_1^3, \beta_{4/3} \rangle$
		$h_0h_2g$	$2x_{23}$
		$Ph_1d_0$	$4x_{23}$
24	$\mathbf{Z}/2$	$h_1h_4c_0$	$\alpha_1\eta_{3/2}$
26	$\mathbf{Z}/2$	$h_2^2g$	$\alpha_{2/2}x_{23}$
28	$\mathbf{Z}/2$	$Pg = d_0^2$	$x_{28} = \langle \beta_2, \alpha_1^3, \beta_{4/3} \rangle$
30	$\mathbf{Z}/2$	$h_4^2$	$\beta_{8/8}$
31	$(\mathbf{Z}/2)^2$	$h_1^2h_4$	$\alpha_1\beta_{8/8}$
		$n$	$\gamma_3$
32	$(\mathbf{Z}/2)^3$	$h_1h_5$	$\beta_{8/7}$
		$d_1$	$x_{32} = \langle \alpha_1, \beta_{4/4} + \beta_3, \alpha_1, \beta_{4/4} + \beta_3 \rangle$
		$q$	$\beta_6$
33	$(\mathbf{Z}/2)^3$	$h_1^2h_5$	$\alpha_1\beta_{8/7}$
		$p$	$\eta_{5/6}$
		$h_1q$	$\alpha_1\beta_3$
34	$\mathbf{Z}/4 \otimes (\mathbf{Z}/2)^2$	$h_0h_2h_5$	$\beta_{8/6,2}$
		$h_0^2h_2h_5 = h_1^3h_5$	$\alpha_1^2\beta_{8/7}$
		$e_0^2$	$\alpha_{2/2}\gamma_3$
		$C_0^2$	$x_{34} = \langle \beta_3, \alpha_1^3, \beta_{4/3} \rangle = P\beta_3$
35	$(\mathbf{Z}/2)^2$	$h_2d_1$	$\alpha_{2/2}x_{32}$
		$h_1e_0^2$	$\alpha_1x_{34}$
36	$\mathbf{Z}/2$	$t$	$x_{36} = ?$
37	$(\mathbf{Z}/2)^2$	$h_2^2h_5$	$\alpha_{2/2}\beta_{8/6,2}$
		$x$	$\gamma_{4/2,2}$
38	$(\mathbf{Z}/4) \oplus \mathbf{Z}/2$	$h_0^2h_3h_5, h_0^3h_3h_5$	$\beta_{8/4,2}$
		$h_1x$	$\alpha_1\gamma_{4/2,2}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
39	$(\mathbf{Z}/2)^5$	$h_1h_3h_5$	$\alpha_{4/4}\beta_{8/7}$
		$h_5c_0$	$\gamma_{4/2}$
		$h_1c_1$	$x_{39} = \langle \alpha_1, \beta_{2/2}, \gamma_3 \rangle$
		$c_1g$	$x'_{39} = \langle \eta_2, \alpha_1^3, \beta_{4/3} \rangle = P\eta_2$
		$u$	$x''_{39} = \langle \beta_{8/6}, a_1, \alpha_{2/2} \rangle$
40	$\mathbf{Z}/4 + (\mathbf{Z}/2)^4$	$h_1^2h_3h_5$	$\alpha_1\alpha_{4/4}\beta_{8/7}$
		$f_1$	$x_{40}?$
		$h_1h_5c_0$	$\beta_2\beta_{8/7} = \gamma_{4/2}\alpha_1$
		$Ph_1h_5$	$\beta_{8/3}$
		$g^2$	?
		$h_1n$	$\alpha_1x''_{39}$
41	$(\mathbf{Z}/2)^3$	$h_1f_1$	$\alpha_1x_{40}$
		$Ph_1^2h_5$	$\alpha_1x_{40}$
		$z$	?
42	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$	$Ph_2h_5, Ph_0h_2h_5$	$\beta_{\alpha_2, 2}$
		$Ph_0^2h_2h_5 = Ph_1^3h_5$	$4\beta_{8/2, 2} = \alpha_1^2\beta_{8/3}$
44	$\mathbf{Z}/8$	$Pe_0^2$	?
		$g_2$	$\beta_8?$
		$h_0g_2$	
45	$(\mathbf{Z}/16) \oplus (\mathbf{Z}/2)^3$	$h_0^2g_2$	
		$h_4^3$	$\gamma_4?$
		$h_0h_4^3$	
		$hg_2$	$\alpha_1\beta_8?$
		$h_5d_0$	
		$h_0h_5d_0$	
		$h_0^2h_5d_0$	
		$w$	

<sup>a</sup> All element have order 2 unless otherwise indicated. (im  $J$  and  $\mu_{8k+1}, \mu_{8k+2}$  omitted.)

TABLE A3.4. 3-Primary Stable Homotopy Excluding  $\text{im } J^a$ 

Stem	Element	Stem	Element
10	$\beta_1$	81	$\gamma_2$
13	$\alpha_1\beta_1$		$x_{81} = \langle \alpha_1, \alpha_1, \beta_5 \rangle$
20	$\beta_1^2$	82	$\beta_{6/3}$
23	$\alpha_1\beta_1^2$	84	$\alpha_1\gamma_2$
26	$\beta_2$		$\beta_1\beta_5 = \alpha_1x_{81}$
29	$\alpha_1\beta_2$	85	$\langle \alpha_1, \alpha_1, \beta_2^3 \rangle = \beta_1\mu$
30	$\beta_1^3 = \langle \beta_2, 3, \alpha_1 \rangle$		$\alpha_1\beta_{6/3}$
36	$\beta_1\beta_2$	86	$\beta_{6/2}$
37	$\langle \alpha_1, \alpha_1, \beta_1^3 \rangle = \langle \beta_1, 3, \beta_2 \rangle$	90	$\beta_6$
38	$\beta_{3/2} = \langle \alpha_1, \beta_1^3, 3, \alpha_1 \rangle$	91	$\beta_1\gamma_2$
39	$\alpha_1\beta_1\beta_2$		$\beta_1x_{81}$
40	$\beta_1^4$	92	$\beta_1\beta_{6/3}$
42	$\beta_3$		$x_{92} = \langle \beta_1, 3, \gamma_2 \rangle$
45	$x_{45} = \langle \alpha_1, \alpha_1, \beta_{3/2} \rangle$ with $3x_{45} = \alpha_1\beta_3$	93	$x_{93} = \langle \alpha_1, \alpha_1, \beta_{6/2} \rangle$ with $3x_{93} = \alpha_1\beta_6$
46	$\beta_1^2\beta_2$	94	$\alpha_1\beta_1\gamma_2$
47	$\langle \alpha_1, \alpha_1, \beta_1^4 \rangle$		$\beta_1^2\beta_5$
49	$\alpha_1\beta_1^2\beta_2$	95	$\alpha_1\beta_1\beta_{6/3}$
50	$\beta_1^5$	99	$\langle \alpha_1, \alpha_1, x_{92} \rangle$
52	$\beta_2^2 = \langle \alpha_1, \alpha_1, x_{45} \rangle$	100	$\beta_2\beta_5$
55	$\alpha_1\beta_2^2$	101	$\beta_1^2\gamma_2$
62	$\beta_1\beta_2^2$		$\beta_1^2x_{81}$
65	$\alpha_1\beta_1\beta_2^2$	102	$\beta_1^2\beta_{6/3}$
68	$x_{68} = \langle \alpha_1, \beta_{3/2}, \beta_2 \rangle$		$\beta_1x_{92}$
72	$\beta_1^2\beta_2^2 = \langle \alpha_1, 3, x_{68} \rangle$	104	$\alpha_1\beta_1^2\gamma_2$
74	$\beta_5$	106	$x_{106} = \beta_7 \pm \beta_{9/9}$
75	$x_{75} = \langle \alpha_1, \alpha_1, x_{68} \rangle = \langle \beta_1, \beta_{3/2}, \beta_2 \rangle$ with $3x_{75} = \alpha_1\beta_1^2\beta_2^2$	107	$\gamma_2\beta_2$
78	$\beta_2^3 = \beta_1x_{68}$		$\beta_2x_{81}$
		108	$\beta_2\beta_{6/3} = \langle \alpha_1, \alpha_1, \beta_1^2x_{81} \rangle$

<sup>a</sup> (See 7.5.3 and subsequent discussion.) All elements have order 3 unless otherwise indicated.

TABLE A3.5. 5-Primary Stable Homotopy Excluding  $\text{im } J$ 

Stem	Element	Stem	Element
38	$\beta_1$	255	$\alpha_1\beta_1^3\beta_3$
45	$\alpha_1\beta_1$	258	$\beta_1^2\beta_4$
76	$\beta_1^2$	265	$\alpha_1\beta_1^2\beta_4$
83	$\alpha_1\beta_1^2$	266	$\beta_1^7$
86	$\beta_2$	268	$\beta_2\beta_4$ with $\beta_1\beta_5 = 0$
93	$\alpha_1\beta_2$	275	$\alpha_1\beta_2\beta_4$
114	$\beta_1^3$	278	$\beta_6$
121	$\alpha_1\beta_1^3$	281	$\underline{2}\beta_1^7$
124	$\beta_1\beta_2$	285	$\alpha_1\beta_6$
131	$\alpha_1\beta_1\beta_2$	286	$\beta_1^4\beta_3$
134	$\beta_3$	293	$\alpha_1\beta_1^4\beta_3$
141	$\alpha_1\beta_3$	296	$\beta_1^3\beta_4$
152	$\beta_1^4$	303	$\alpha_1\beta_1^3\beta_4$
159	$\alpha_1\beta_1^4$	304	$\beta_1^8$
162	$\beta_1^2\beta_2$	306	$\beta_1\beta_2\beta_4$
169	$\alpha_1\beta_1^2\beta_2$	313	$\alpha_1\beta_1\beta_2\beta_4$
172	$\beta_1\beta_3$	316	$\beta_1\beta_6$
179	$\alpha_1\beta_1\beta_3$	319	$\underline{2}\beta_1^8$
182	$\beta_4$	326	$\beta_7$
189	$\alpha_1\beta_4 = \gamma_1$	331	$\underline{2}\beta_1\beta_6$
190	$\beta_1^5$	333	$\alpha_1\beta_7$
200	$\beta_1^3\beta_2$	334	$\beta_1^4\beta_4$
205	$\underline{2}\beta_1^5 = \langle \alpha_1, \alpha_1, \beta_1^5 \rangle$	341	$\alpha_1\beta_1^4\beta_4$
206	$\beta_{5/4} = \langle \alpha_1, \beta_1^5, 5, \alpha_1 \rangle$	342	$\beta_1^9$
207	$\alpha_1\beta_1^3\beta_2$	344	$\beta_1^2\beta_2\beta_4$
210	$\beta_1^2\beta_3$	351	$\alpha_1\beta_1^2\beta_2\beta_4$
213	$\alpha_1\beta_{5/4}$	354	$\beta_1^2\beta_6$
214	$\beta_{5/3}$	357	$\underline{2}\beta_1^9$
217	$\alpha_1\beta_1^2\beta_3$	364	$\beta_1\beta_7$
220	$\beta_1\beta_4$	369	$\underline{2}\beta_1^2\beta_6$
221	$\alpha_1\beta_{5/3}$	374	$\beta_8$
222	$\beta_{5/2}$	379	$\beta_1\beta_7$
227	$\alpha_1\beta_1\beta_4$	380	$\beta_1^{10}$
228	$\beta_1^6$	381	$\alpha_1\beta_8$
230	$\beta_5$	382	$\beta_1^3\beta_2\beta_4$
237	$\underline{2}\beta_{5/2}$ with $5(\underline{2}\beta_{5/2}) = \alpha_1\beta_5$	389	$\alpha_1\beta_1^3\beta_2\beta_4$
238	$\beta_1^4\beta_2$	392	$\beta_1^3\beta_6$
243	$\underline{2}\beta_1^6$	402	$\beta_1^2\beta_7$
245	$\alpha_1\beta_1^4\beta_2$	403	$\underline{3}\beta_1^{10}$
248	$\beta_1^3\beta_3$	404	$x_{404} = \langle \alpha_1\beta_1^4, \beta_1, \beta_{5/4} \rangle$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
407	$2\beta_1^3\beta_6$	491	$2\bar{5}\gamma_2 = \langle \beta_1\alpha_2, \gamma_2 \rangle$
411	$\alpha_1x_{404} = \beta_{5/4}2\beta_1^5$		$= \langle \alpha_1\beta_1, \alpha_1, 5, \gamma_2 \rangle$
412	$\beta_1\beta_8$	493	$2\beta_1^4\beta_7$
	$x_{412} = \beta_1\beta_8 + \beta_{5/4}^2$	494	$\beta_1^{13}$
417	$2\beta_1^2\beta_7$	498	$\beta_1^2\beta_9$
418	$\beta_1^{11}$	503	$2\beta_1^3\beta_8$
419	$\alpha_1\beta_1\beta_8$ with $\alpha_1x_{412} = 0$	508	$\beta_2\beta_9$ with $\beta_1\beta_{10} = 0$
420	$\beta_1^4\beta_2\beta_4 = \langle \alpha_1, 5, x_{412} \rangle$	513	$\beta_1^2\gamma_2$
422	$\beta_9$		$2\beta_1^2\beta_9$
427	$2x_{412}$ with	514	$\beta_1^2\beta_{10/5}$
	$5(2x_{412}) = \alpha_1\beta_1^4\beta_2\beta_4$	517	$3\beta_1^{13}$
430	$\beta_1^4\beta_6$	518	$\beta_{11}$
437	$2\beta_9$	520	$\alpha_1\beta_1^2\gamma_2$
	$\gamma_2$	523	$2\beta_2\beta_9$
438	$\beta_{10/5}$		$\beta_1\gamma_2$
440	$\beta_1^3\beta_7$	524	$\beta_2\beta_{10/5}$
441	$3\beta_1^{11}$	525	$\alpha_1\beta_{11}$
444	$\alpha_1\gamma_2$	526	$\beta_1^4\beta_8$
445	$\alpha_1\beta_{10/5}$	529	$2\bar{5}\beta_1\gamma_2$
	$2\beta_1^4\beta_6$	530	$\alpha_1\beta_2\gamma_2$
446	$\beta_{10/4}$	531	$\beta_72\beta_1^5 = \alpha_1\beta_2\beta_{10/5}$
450	$\beta_1^2\beta_8$ with $\beta_1\beta_{5/4}^2 = 0$	532	$\beta_1^{14}$
453	$\alpha_1\beta_{10/4}$	536	$\beta_1^3\beta_9$
454	$\beta_{10/3}$	541	$2\beta_1^4\beta_8$
455	$2\beta_1^3\beta_7$	546	$\beta_1\beta_2\beta_9$
456	$\beta_1^{12}$	551	$\beta_1^3\gamma_2$
460	$\beta_1\beta_9$		$2\beta_1^3\beta_9$
461	$\alpha_1\beta_{10/3}$	552	$\beta_1^2\bar{5}\gamma_2$
462	$\beta_{10/2}$	555	$3\beta_1^{14}$
465	$2\beta_1^2\beta_8$	556	$\beta_1\beta_{11}$
470	$\beta_{10}$	558	$\alpha_1\beta_1^3\gamma_2$
475	$\beta_1\gamma_2$	561	$\beta_12\beta_2\beta_9$ with $\beta_1\beta_2\gamma_2 = ?\beta_12\beta_2\beta_9$
476	$\beta_1\beta_{10/5} = \langle \alpha_1, \beta_1\beta_6, \beta_1^4 \rangle$	566	$\beta_{12}$
	$\bar{5}\gamma_2 = \langle \beta_1, 5, \gamma_2 \rangle$	567	$\beta_1^2\bar{5}\gamma_2$
477	$2\beta_{10/2}$ with $5(2\beta_{10/2}) = \alpha_1\beta_{10}$	570	$\beta_1^{15}$
478	$\beta_1^4\beta_7$	571	$2\beta_1\beta_{11}$
479	$3\beta_1^{12}$	572	$\beta_3\beta_{10/5}$
482	$\alpha_1\beta_1\gamma_2$	573	$\alpha_1\beta_{12}$
483	$\alpha_1\beta_1\beta_{10/5}$	574	$\beta_1^4\beta_9$
488	$\beta_1^3\beta_8$	579	$\beta_92\beta_1^5 = \alpha_1\beta_3\beta_{10/5}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
583	$\beta_1^2\beta_2\beta_9$	659	$\alpha_1\beta_1\beta_{13}$
589	$\beta_1^4\gamma_2$		with $\alpha_1x_{652} = 0$
	$\underline{2}\beta_1^4\beta_9$	660	$\beta_1^4\beta_2\beta_9$
590	$\beta_1^3\underline{5}\gamma_2$	662	$\beta_{14}$
594	$\beta_1^2\beta_{11}$	665	$\underline{3}\beta_1^2\beta_{12}$
596	$\alpha_1\beta_1^4\gamma_2$		$\beta_1^6\gamma_2$
599	$\beta_1^2\underline{2}\beta_2\beta_9$	666	$\beta_1^3\underline{5}\gamma_2$
601	$\underline{4}\beta_1^{15}$	667	$\underline{2}x_{652}$
602	$x_{602} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4} \rangle$	670	$\beta_1^4\beta_{11}$
604	$\beta_1\beta_{12}$	675	$\underline{2}\beta_1^4\beta_2\beta_9$
605	$\beta_1^3\underline{2}\underline{5}\gamma_2$	677	$\underline{2}\beta_{14}$
608	$\beta_1^{16}$		$\underline{4}\beta_1^{17}$
609	$\alpha_1x_{602}$ with $\underline{2}\beta_1^2\beta_{11} = ?\alpha_1x_{602}$	678	$\beta_{15/5}$
610	$\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, \underline{5}, x_{602} \rangle$	680	$\beta_1^3\beta_{12}$
614	$\beta_{13}$		$\underline{2}\beta_1^6\underline{5}\gamma_2$
617	$x_{617} = \langle \alpha_1, (\alpha_1\underline{2}\beta_2\beta_6), (\beta_6^{x_{602}}) \rangle$	685	$\alpha_1\beta_{15/5}$
	with	686	$\beta_{15/4}$
	$5x_{617} = \alpha_1\beta_1\beta_3\beta_{10/5}$	689	$\underline{3}\beta_1^5\underline{5}\gamma_2$
620	$\beta_4\beta_{10/5}$	690	$\beta_1^2\beta_{13}$
621	$\alpha_1\beta_{13}$	692	$x_{692} = \langle \alpha_1, \beta_1^5, \beta_1^{13} \rangle$
622	$\beta_1^3\beta_2\beta_9$	693	$\alpha_1\beta_{15/4}$
627	$\underline{3}\beta_1\beta_{12}$		$\underline{3}\beta_1^4\beta_{11}$
	$\beta_1^5\gamma_2$	694	$\beta_{15/3}$
628	$\beta_1^4\underline{5}\gamma_2$	700	$\beta_1\beta_{14}$
632	$\beta_1^3\beta_{11}$	701	$\alpha_1\beta_{15/3}$
635	$\underline{2}\beta_4\beta_{10/5}$	702	$\beta_{15/2}$
636	$x_{636} = \langle \beta_1^3, \alpha_1\beta_1^2, \beta_{10/5} \rangle$	703	$\underline{3}\beta_1^3\beta_{12}$
	$= \langle \beta_{5/4}, \beta_9, \alpha_1 \rangle$		$\beta_1^7\gamma_2$
637	$\underline{2}\beta_1^3\beta_2\beta_9$	704	$\beta_1^6\underline{5}\gamma_2$
639	$\underline{4}\beta_1^{16}$	710	$\beta_{15}$
642	$\beta_1^2\beta_{12}$	713	$\beta_1^2\beta_{13}$
	$\underline{2}\beta_1^5\gamma_2$	714	$x_{714} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9 \rangle$
643	$\underline{2}\underline{5}\beta_1^4\gamma_2 = \beta_{5/4}\gamma_2$	715	$\underline{2}\beta_1\beta_{14}$
	$\alpha_1x_{636}$		$\underline{3}x_{692}$
644	$\beta_{5/4}\beta_{10/5}$	716	$\beta_1\beta_{15/5}$
646	$\beta_1^{17}$	717	$\underline{2}\beta_{15/2}$ with
651	$\alpha_1\beta_{5/4}\beta_{10/5}$		$5(\underline{2}\beta_{15/2}) = \alpha_1\beta_{15}$
652	$\beta_1\beta_{13}$	718	$\beta_1^4\beta_{12}$
	$\beta_{5/3}\beta_{10/5} + \beta_1\beta_{13} = x_{652}$		$\underline{2}\beta_1^7\gamma_2$
655	$\underline{3}\beta_1^3\beta_{11}$	721	$\alpha_1x_{714}$



TABLE A3.5 (continued)

Stem	Element	Stem	Element
723	$\alpha_1\beta_1\beta_{15/5}$	786	$\beta_1\beta_2\beta_{14}$
724	$x_{724} = \langle \beta_1, 5, \beta_1, \beta_1^{17} \rangle$ $x'_{724} = \langle \beta_1^2, \beta_1^2\beta_{11}, \alpha_1\alpha_1 \rangle$ with $\beta_1\beta_{15/4} = 0$	789	$\underline{3}\beta_1^4\beta_{13}$
727	$\underline{3}\beta_1^6\underline{5}\gamma_2$	794	$\underline{2}\beta_1^9\gamma_2$
728	$\beta_1^3\beta_{13}$	796	$\beta_1\beta_{16}$
730	$\beta_1x_{692}$	799	$\underline{3}\beta_1^3\beta_{14}$ $\beta_1x_{761}$ with $\underline{4}\beta_1^2x_{692} = ?$
731	$\alpha_1x'_{724}$ with $\alpha_1x_{724} = 0$	800	$\beta_1^2x_{724}$ $\beta_1^2x'_{724}$
738	$\beta_1^2\beta_{14}$	803	$\underline{3}\beta_1^8\underline{5}\gamma_2$
739	$\underline{2}x_{724}$	806	$\beta_{17}$ $\alpha_1\beta_1x_{761} = \beta_1^3x_{692}$
741	$\underline{3}\beta_1^4\beta_{12}$ $\beta_1^8\gamma_2$	807	$\alpha_1\beta_1^2x'_{724}$
742	$\beta_1^7\underline{5}\gamma_2$	809	$\beta_1x_{771}$
748	$\beta_2\beta_{14}$ with $\beta_1\beta_{15} = 0$	810	$\beta_2x'_{724}$
751	$\underline{3}\beta_1^3\beta_{13}$	811	$\underline{2}\beta_1\beta_{16}$
753	$\underline{3}\beta_1x_{692}$	812	$\beta_3\beta_{15/5}$
756	$\underline{2}\beta_1^8\gamma_2$ with $\beta_1^5\beta_{12} = 0$	813	$\alpha_1\beta_{17}$
758	$\beta_{16}$	814	$\beta_1^4\beta_{14}$
761	$x_{761} = \langle \beta_3, \gamma_1, \gamma_2 \rangle$ $\underline{3}\beta_1\beta_{14}$	815	$\underline{2}\beta_1^2x_{724}$
762	$\beta_1x_{724}$ $\beta_1x'_{724}$	816	$\alpha_1\beta_1x_{771}$
763	$\underline{2}\beta_2\beta_{14}$	817	$\beta_1^{10}\gamma_2$ $\underline{4}\beta_1\beta_2\beta_{14}$ with $\underline{2}\beta_2\beta_{15/5} = 0$
764	$\beta_2\beta_{15/5}$	818	$\beta_1^{10}\beta_{10/5}$
765	$\alpha_1\beta_{16}$ $\underline{3}\beta_1^7\underline{5}\gamma_2$	824	$\beta_1^2\beta_1\beta_{14}$
766	$\beta_1^4\beta_{13}$	825	$\underline{2}\beta_2x'_{724}$
768	$\alpha_1x_{761} = \gamma_2\underline{2}\beta_1\beta_6$ $\beta_1^2x_{692}$	826	$x_{826} = \langle \alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5} \rangle$
769	$\alpha_1\beta_1x'_{724}$	827	$\underline{2}\beta_3\beta_{15/5}$
771	$x_{771} = \langle \beta_2, \beta_1^5, \beta_1^{13} \rangle$	833	$\alpha_1x_{826}$
776	$\beta_1^3\beta_{14}$	834	$\beta_1^2\beta_{16}$ $x_{834} = \langle \beta_1^4, \underline{2}\beta_1^6, \beta_{10/5} \rangle$
777	$\underline{2}\beta_1x_{724}$	837	$\underline{3}\beta_1^4\beta_{14}$ $\beta_1^2x_{761}$
778	$\alpha_1x_{771}$	838	$\beta_1^3x_{724}$ with $\beta_1^3x'_{724} = 0$
779	$\beta_1^9\gamma_2$ $\underline{2}\beta_2\beta_{15/5}$	840	$\underline{3}\beta_1^{10}\gamma_2$
780	$\beta_1^8\underline{5}\gamma_2$	841	$\alpha_1x_{834}$ $\underline{3}\beta_1^{10}\beta_{10/5}$
		842	$x_{842} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{10/4} \rangle$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
844	$\alpha_1\beta_1^2x_{761}$	894	$\beta_1^{12}\beta_{10/9}$
	$\beta_1\beta_{17}$	899	$\alpha_1\beta_1\beta_{18}$ with
847	$\beta_1^2x_{771}$		$\alpha_1x_{892} = 0$
849	$\alpha_1x_{842} = \underline{2}\beta_1^2\beta_{16}$	900	$\beta_1^4\beta_2\beta_{14}$
850	$\beta_1\beta_3\beta_{15/5}$	902	$\beta_{19}$
853	$\underline{2}\beta_1^3x_{724}$	903	$\underline{4}\beta_1^3\beta_{16}$
854	$\beta_{18}$	905	$\underline{3}\beta_1^2\beta_{17}$
	$\alpha_1\beta_1^2x_{771}$	906	$\beta_1x_{868}$
855	$\beta_1^{11}\gamma_2$	907	$\underline{2}x_{892}$
856	$\beta_1^{11}\beta_{10/5}$ with	910	$\beta_1^4\beta_{16}$
	$\underline{4}\underline{2}x_{810} = 0$	913	$\alpha_1\beta_1x_{868}$
	$\underline{4}\beta_1^2\beta_1\beta_{14}$		$\beta_1^4x_{761}$ with
857	$x_{857} = \langle \alpha_1(\alpha_1\underline{2}\beta_1\beta_6), (x_{842})_{\beta_{11}} \rangle$		$\underline{2}\beta_1\beta_4\beta_{15/5} = 0$
860	$\beta_4\beta_{15/5}$	914	$\beta_1^5x_{724}$ with
861	$\alpha_1\beta_{18}$		$\beta_1\beta_{10/5}^2 = 0$
862	$\beta_1^3\beta_2\beta_{14}$	916	$\beta_1^{12}\gamma_2$
865	$\underline{2}\beta_1\beta_3\beta_{15/5}$	917	$\underline{2}\beta_{19}$
867	$\underline{3}\beta_1\beta_{17}$	918	$\beta_{20/5}$
868	$x_{868}$ (see 7.6.5)	920	$\beta_1^3\beta_{17}$
872	$\beta_1^3\beta_{16}$ with $\beta_1x_{834} = 0$	923	$\beta_1^4x_{771}$
875	$\alpha_1x_{868}$	925	$\alpha_1\beta_{20/5}$
	$\underline{2}\beta_4\beta_{15/5}$		$\underline{4}\beta_1^{12}\beta_{10/5}$
	$\beta_1^3x_{761} = \beta_{10/5}\gamma_2$	926	$\beta_{20/4}$
876	$\beta_{10/5}^2$	928	$\underline{2}\beta_1^4x_{761}$
	$\beta_1^4x_{724}$	930	$\alpha_1\beta_1^4x_{771}$
878	$\underline{3}\beta_1^{11}\gamma_2$		$\beta_1^2\beta_{18}$ with
882	$\beta_1^2\beta_{17}$ with		$\beta_1x_{892} = 0$
	$\alpha_1\beta_1^3x_{761} = 0$	931	$\beta_1^{13}\gamma_2$
883	$\alpha_1\beta_{10/5}^2$		$\underline{4}\beta_1^4\beta_2\beta_{14}$
884	$\beta_{5/4}\beta_{15/5}$	932	$\beta_1^{13}\beta_{10/5}?$
885	$\beta_1^3x_{771}$	933	$\gamma_4?$
887	$\underline{4}\beta_1^{11}\beta_{10/5}$		$\alpha_1\beta_{20/4}$
890	$\underline{2}\beta_1^3x_{761}$	934	$\beta_{20/3}$
891	$\underline{2}\beta_1^4x_{724}$	937	$\underline{3}\beta_1^5x_{724}$ with
	$\alpha_1\beta_{5/4}\beta_{15/3}$		$\alpha_1\beta_1\beta_{10/4}^2 = 0$
892	$\beta_1\beta_{18}$	940	$\alpha_1\gamma_4$
	$\alpha_1\beta_1^3x_{771}$		$\beta_1\beta_{19}$
	$x_{892} = \beta_{10/4}^2 + \beta_1\beta_{18}$	941	$\alpha_1\beta_{20/3}$
893	$\beta_1^{12}\gamma_2$		$\underline{4}\beta_1^4\beta_{16}$
	$\underline{4}\beta_1^3\beta_2\beta_{14}$	942	$\beta_{20/2}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
950	$\beta_{20}$	969	$\beta_1^{14}\gamma_2$
951	$\beta_1^5 x_{761}$ $\underline{4}\beta_1^3\beta_{17}$ with $\alpha_1\beta_1^2 x_{868} = 0$		$\underline{2}x_{954}$
952	$\beta_1^6 x_{724}$ $x_{952} = \langle \beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2 \rangle$	970	$\beta_1^{14}\beta_{10/5}?$
953	$\underline{3}\beta_1^2\beta_{18}$	971	$\beta_1\gamma_4?$ with $\alpha_1\beta_1\beta_{20/4} = 0$
954	$\underline{3}\beta_1^{13}\gamma_2$ $x_{954} = \langle \alpha_1, \alpha_1, \beta_1\beta_2\beta_{14}, \beta_1^4 \rangle$	972	$\underline{5}\gamma_4$ with $\beta_1\beta_{20/3} = 0$
955	$\underline{2}\beta_1\beta_{19}$	975	$\underline{3}\beta_1^6 x_{724}$
956	$\beta_1\beta_{20/5}$	978	$\alpha_1\beta_1\gamma_4$ $\beta_1^2\beta_{19}$
957	$\underline{2}\beta_{20/2}$ with $5(\underline{2}\beta_{20/2}) = \alpha_1\beta_{20}$	979	$\underline{2}x_{964}$ with $\alpha_1\underline{5}\gamma_4 = 0$ and $\alpha_1/\beta_1\beta_{20/3} = 0$
958	$\beta_1^4\beta_{17}$	987	$\underline{2}\underline{5}\gamma_4$
959	$\alpha_1 x_{952}$	988	$\beta_2\beta_{19}$ with $\beta_1\beta_{20} = 0$
963	$\alpha_1\beta_1\beta_{20/5}$ $\underline{4}\beta_1^{13}\beta_{10/5}$	989	$\beta_1^6 x_{761}$
964	$x_{964} = \underline{3}\underline{4}\beta_1^4\beta_{16}$ with $\beta_1\beta_{20/4} = 0$	990	$\beta_1^7 x_{724}$ $\beta_1 x_{952}$
966	$\underline{2}\beta_1^5 x_{761}$	992	$\underline{3}\beta_1^{14}\gamma_2$ with $\beta_1 x_{954} = 0$
968	$\beta_1^3\beta_{18}$ with $\alpha_1\beta_1^5 x_{771} = 0$	998	$\beta_{21}$
		999	$x_{999} = \langle \beta_1\beta_2, \gamma_2, \gamma_2 \rangle$ with $\underline{4}\beta_1^3\beta_{18} = 0$

$k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19		
1	$\infty$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	$\infty$	2	2	12	2	2	3	15	2	2 <sup>2</sup>	2 <sup>2</sup>	12.2	84.2 <sup>2</sup>	2 <sup>2</sup>	6	30	30	6.2	12.2 <sup>2</sup>	6.2	6.2	
3	2	2	12	2	2	3	15	2	2 <sup>2</sup>	12.2	12.2	84.2 <sup>2</sup>	2 <sup>2</sup>	6	30	30	6.2	12.2 <sup>2</sup>	12.2 <sup>2</sup>	132.2	132.2 <sup>5</sup>	
4	2	2	$\infty$ .12	2 <sup>2</sup>	2 <sup>2</sup>	2 <sup>2</sup>	24.3	15	2	2 <sup>3</sup>	120.12.2	84.2 <sup>5</sup>	2 <sup>6</sup>	24.6.2	2520.6.2	30	6 <sup>2</sup> .2	24.12.4.2 <sup>2</sup>	120.12.2 <sup>5</sup>	132.2 <sup>5</sup>	132.2 <sup>5</sup>	
5	24	2	2	2	2	2	30	2	2	2 <sup>3</sup>	72.2	504.2 <sup>2</sup>	2 <sup>3</sup>	6.2	6.2	30.2	2 <sup>2</sup>	4.2 <sup>2</sup>	24.2 <sup>2</sup>	264.2	264.2	
6	0	$\infty$	0	$\infty$	2	2	60	24.2	2 <sup>3</sup>	72.2	504.4	240	240	6	12.2	60.6	504.2 <sup>2</sup>	2 <sup>4</sup>	24.6.2	1056.8	1056.8	
7	0	0	2	120	2 <sup>3</sup>	2 <sup>4</sup>	24.2	24.2	2 <sup>4</sup>	24.2	504.2	0	0	6	24.4	120.2 <sup>3</sup>	2 <sup>4</sup>	2 <sup>4</sup>	24.2	264.2	264.2	
8	2	2	$\infty$ .120	2 <sup>4</sup>	2 <sup>5</sup>	24 <sup>2</sup> .2	504.2	0	6.2	240.24.4	120.2 <sup>5</sup>	2 <sup>7</sup>	2 <sup>7</sup>	6.2	540.24.2	264.2	264.2	264.2	264.2	264.2	264.2	264.2
9	240	2 <sup>3</sup>	2 <sup>4</sup>	24.2	504.2	0	6	16.4	240.2 <sup>3</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	6	16.4	240.2 <sup>3</sup>	2 <sup>4</sup>	2 <sup>4</sup>	24.2	264.2	264.2	
10	2 <sup>2</sup>	$\infty$ .2 <sup>3</sup>	12.2	504	12	504	12	6	16.2	240.2 <sup>2</sup>	240.2	240.2	240.2	6	16.2	240.2 <sup>2</sup>	240.2	2 <sup>3</sup>	24.2 <sup>2</sup>	264.6	264.6	
11	2 <sup>3</sup>	2 <sup>3</sup>	6.2	504	2	504	2	6.2	16.2	240.2	2	2	2	6.2	16.2	240.2	2	2 <sup>3</sup>	8.4.2	264.2 <sup>3</sup>	264.2 <sup>3</sup>	
12	6	6	6	$\infty$ .504	2 <sup>2</sup>	504	2	6.2	48.4.2	240.2	2	2	2	6.2	48.4.2	240.2	2	2 <sup>4</sup>	480.4 <sup>2</sup> .2	264.2 <sup>5</sup>	264.2 <sup>5</sup>	
13	504	2	2	504	2	504	2	6	16.2	480.2	2	2	2	6	16.2	480.2	2	2 <sup>4</sup>	8 <sup>2</sup> .2	264.2 <sup>3</sup>	264.2 <sup>3</sup>	
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2	8.2
16	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>	2 <sup>4</sup>
17	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>	2 <sup>5</sup>
18	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>	2 <sup>6</sup>
19	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>	2 <sup>7</sup>
20	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>	2 <sup>8</sup>
21	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>	2 <sup>9</sup>

TABLE A3.6. Toda's calculation of unstable homotopy groups  $\pi_{n+k}(S^n)$  for  $n \leq k+2$  and  $k \leq 19$ .

For  $n > k+1$  the group is isomorphic to the one for  $n = k+1$ . The notation  $a.b.c\dots$  denotes the direct sum of cyclic groups of order  $a, b, c$ , etc. The notation  $a^j$  denotes the direct sum of  $j$  cyclic groups, each having order  $a$ . (After Toda [6].)

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