Stable homotopy theory

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1 Smooth bordism

1.1 Transversality

Spheres $S^n \cong D^n/\partial D^n \cong \mathbb{R}^n \cup \{\infty\}$. Deform any map $f\colon S^{n+k} \to S^n$ to be transverse to $0 \in \mathbb{R}^n \subset S^n$. Preimage $M^k = f^{-1}(0)$ closed (always smooth) k-manifold. Given f_0, f_1 transverse to 0, any homotopy $F\colon S^{n+k} \times I \to S^n$ from f_0 to f_1 can be deformed relative to $S^{n+k} \times \{0, 1\}$ to be transverse to 0. Preimage $W^{k+1} = F^{-1}(0)$ is compact $(k+1)$-manifold with $\partial W \cong M_0 \sqcup M_1$. Call W a bordism from $M_0 = f_0^{-1}(0)$ to $M_1 = f_1^{-1}(0)$. Say that M_0 and M_1 are (co-)bordant. Let $\mathscr{N}_k = \Omega_k^O$ be the set of bordism classes of closed k-manifolds. Get function $\pi_{n+k}(S^n) \to \mathscr{N}_k$ mapping homotopy class $[f]$ to bordism class $[M]$. Suspension $\Sigma f\colon S^{n+k+1} \to S^{n+1}$ gives same preimage, so $\pi_k^s = \pi_k(S) = \operatorname{colim}_n \pi_{k+n}(S^n) \to \mathscr{N}_k$.

Compatible with sum and product: $f, g\colon S^{n+k} \to S^n$ transverse to 0 then $f + g = (f \vee g)\nabla\colon S^{n+k} \to S^{n+k} \vee S^{n+k} \to S^n$ is transverse to 0, with $(f + g)^{-1}(0) \cong f^{-1}(0) \sqcup g^{-1}(0)$, so sum in π_k^s corresponds to sum in $\mathscr{N}_k$ induced by disjoint union. If $f\colon S^{n+k} \to S^n$ and $g\colon S^{m+\ell} \to S^m$ are transverse to 0 then $f \cdot g = (f \wedge g)\chi\colon S^{n+m+k+\ell} \cong S^{n+k} \wedge S^{m+\ell} \to S^n \wedge S^m = S^{n+m}$ is transverse to 0 and $(f \cdot g)^{-1}(0) \cong f^{-1}(0) \times g^{-1}(0)$, so that smash product pairing $\pi_k^s \times \pi_\ell(S) \to \pi_{k+\ell}(S)$ corresponds to pairing $\mathscr{N}_k \times \mathscr{N}_\ell \to \mathscr{N}_{k+\ell}$ induced by Cartesian product. Get a homomorphism $\pi_*(S) \to \mathscr{N}_*$ of graded (commutative) rings.

1.2 Framed bordism

More structure on manifolds: Tangent bundle $\tau\colon TM \to M$ embeds in trivial bundle $\epsilon^{n+k}\colon M \times \mathbb{R}^{n+k} \to M$, with normal complement $\nu\colon NM \to M$. For each $x \in M$, $N_xM \subset \mathbb{R}^{n+k}$ is the orthogonal complement of $T_xM \subset \mathbb{R}^{n+k}$, mapping isomorphically to the quotient $\mathbb{R}^{n+k}/T_xM$. Derivative of f along M induces bundle isomorphism $\theta\colon NM \to M \times \mathbb{R}^n$. A trivialization, or framing, of the normal bundle of M. The normal bundle of $M \subset \mathbb{R}^{n+k+1}$ is $\nu \oplus \epsilon^1\colon NM \times \mathbb{R} \to M$. The trivialization $\theta \oplus \epsilon^1$ defines the same stable framing as θ. Let Ω_k^{fr} be the set of stably framed bordism classes of stably framed closed k-manifolds. Get ring homomorphism $\iota^{fr}\colon \pi_*(S) \to \Omega_*^{fr}$.

Theorem (Pontryagin–Thom): ϕ^{fr} is an isomorphism.

Inverse construction: Given a closed k-manifold M choose embedding $E(\nu) = NM \to \mathbb{R}^{n+k}$. Let the Thom complex $Th(\nu) = E(\nu) \cup \{\infty\}$ be the one-point compactification, so that $Th(\nu) \cong D(\nu)/S(\nu)$ where $S(\nu) \subset D(\nu) \subset E(\nu)$ are the unit sphere and unit disc subbundles. Define Pontryagin–Thom collapse map $\nu^*\colon S^{n+k} \to S^{n+k}/(S^{n+k} \setminus E(\nu)) \cong Th(\nu)$. Alternatively, $S^{n+k} \to S^{n+k}/(S^{n+k} \setminus intD(\nu)) \cong$

$D(\nu)/S(\nu) \cong Th(\nu)$. Suppose M is (stably) framed, so that trivialization θ gives map $E(\nu) \to \mathbb{R}^n$ and $\theta_*: Th(\nu) \to S^n$. The composite $\theta_*\nu^*: S^{n+k} \to Th(\nu) \to S^n$ defines a homotopy class in $\pi_{n+k}(S^n) \to \pi_k^s$.

Homology theories: X any space. Let $\pi_k^s(X_+) = \mathrm{colim}_n\, \pi_{k+n}(X_+ \wedge S^n)$. A map $f: S^{n+k} \to X_+ \wedge S^n = X \times S^n / X \times \{\infty\}$ can be deformed to be transversal to $X \times \{0\} \subset X \times \mathbb{R}^n \subset X_+ \wedge S^n$. (Project to $\{0\} \subset \mathbb{R}^n$ to specify transversality.) Then $M = f^{-1}(X \times \{0\})$ is a framed closed k-manifold. The restriction of f specifies a map $\alpha: M \to X$, so M is a k-manifold over X. A homotopy $F: S^{n+k} \times I \to X_+ \wedge S^n$ from f_0 to f_1 can also be taken to be transverse to $X \times \{0\}$, with preimage $W = F^{-1}(X \times \{0\})$ a stably framed bordism from M_0 to M_1. Restriction of F gives a map $\beta: W \to X$ that restricts to $\alpha_0 \sqcup \alpha_1$ on $\partial W \cong M_0 \sqcup M_1$, so F is a bordism over X. Let $\Omega_k^{fr}(X)$ be the set of bordism classes of stably framed closed k-manifolds over X, up to stably framed bordism over X.

Theorem: $\phi^{fr}: \pi_*^s(X_+) \to \Omega_*^{fr}(X)$ is an isomorphism of generalized homology theories in X.

1.3 Unoriented bordism

Homotopical analogue of forgetting framings: Given a closed k-manifold M choose embedding $M \subset \mathbb{R}^{k+n} \subset S^{k+n}$ with normal bundle $\nu: NM \to M$. May embed NM as a tubular neighborhood $NM \subset \mathbb{R}^{k+n}$. For each $x \in M$ the normal space $N_xM \subset \mathbb{R}^{k+n}$ gives a point $g(x) \in Gr_n(\mathbb{R}^{k+n})$ in the Grassmannian manifold of n-planes in $\mathbb{R}^{k+n}$. Get a Gauss map $g: M \to Gr_n(\mathbb{R}^{k+n})$ covered by a bundle map $\nu \to \gamma^n(\mathbb{R}^{n+k})$, where $\gamma^n(\mathbb{R}^{n+k})$ is the canonical n-bundle over $Gr_n(\mathbb{R}^{k+n})$. Include $\mathbb{R}^{n+k} \subset \mathbb{R}^\infty$ to map to the infinite Grassmannian $Gr_n(\mathbb{R}^\infty)$. There is a fiber bundle $O(n) \to V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ where the Stiefel variety $V_n(\mathbb{R}^\infty) \simeq EO(n)$ of orthonormal n-frames in $\mathbb{R}^\infty$ is contractible, so $Gr_n(\mathbb{R}^\infty) \simeq BO(n)$ is a classifying space for principal $O(n)$-bundles, and the canonical n-bundle γ^n over $BO(n)$ has total space $E(\gamma^n) = EO(n) \times_{O(n)} \mathbb{R}^n$. Its Thom complex is $Th(\gamma^n) = EO(n) \times_{O(n)} S^n / EO(n) \times_{O(n)} \{\infty\} \cong EO(n)_+ \wedge_{O(n)} S^n = MO(n)$. Get a Gauss map $g: M \to BO(n)$ covered by a bundle map $\nu \to \gamma^n$, or $NM \to E(\gamma^n)$, with induced map of Thom complexes $g_*: Th(\nu) \to Th(\gamma^n) = MO(n)$.

Compose with the Pontryagin–Thom collapse map $\nu^*: S^{n+k} \to Th(\nu)$ to get a map $g_*\nu^*: S^{n+k} \to MO(n)$, with homotopy class in $\pi_{n+k}MO(k)$. Different choices of embeddings become isotopic, and induce homotopic maps, after replacing $\mathbb{R}^{k+n}$ with $\mathbb{R}^{k+n+1}$ sufficiently often. This replaces ν with $\nu \oplus \epsilon^1$, finitely often, which corresponds to replacing $\nu^*: S^{n+k} \to Th(\nu)$ with its suspension $S^{n+k+1} \to Th(\nu) \wedge S^1 \cong Th(\nu \oplus \epsilon^1)$, and replacing g_* with the composite of its suspension $Th(\nu) \wedge S^1 \to Th(\gamma^n) \wedge S^1$ and a map $\sigma: Th(\gamma^n) \wedge S^1 \to Th(\gamma^{n+1})$. The latter is induced by the bundle map $\gamma^n \oplus \epsilon^1 \to \gamma^{n+1}$ covering the inclusion $BO(n) \to BO(n+1)$. Get homomorphism $\psi: \mathscr{N}_k \to \mathrm{colim}_n\, \pi_{k+n}MO(n) = \pi_k(MO)$.

Inverse construction: A map $f: S^{n+k} \to MO(n) = Th(\gamma^n)$ can be deformed to be transversal to 0-section $s_0: BO(n) \to E(\gamma^n)$. Then $M = f^{-1}(BO(n)) \subset S^{n+k}$ is a closed k-manifold, with a map $g: M \to BO(n)$ covered by a bundle map $g: \nu \to \gamma^n$. (Here γ^n is also the normal bundle of the 0-section.) Stabilizing f by increasing n does not alter M. Get a homomorphism $\phi: \pi_k(MO) \to \mathscr{N}_k$, inverse to ϕ.

Theorem: $\phi: \pi_*(MO) \to \mathscr{N}_*$ is an isomorphism.

For each space X let $MO_k(X) = \mathrm{colim}_n\, \pi_{k+n}(MO(n) \wedge X_+)$. Let $\mathscr{N}_k(X)$ be the set of bordism classes of closed k-manifolds (M, α) over X, up to bordism over X. (No framings.)

Theorem: $\phi: MO_*(X) \to \mathscr{N}_*(X)$ is an isomorphism of generalized homology theories in X.

Steenrod problem: For a k-manifold (M, α) over X, the fundamental class $[M] \in H_k(M; \mathbb{Z}/2)$ maps to a class $\alpha_*[M] \in H_k(X; \mathbb{Z}/2)$. (No framing is needed to have this homology class.) Cobordant manifolds over X give the same class in $H_k(X; \mathbb{Z}/2)$, so $\mathscr{N}_k(X) \to H_k(X; \mathbb{Z}/2)$. What homology classes arise in this way? Thom: All!

[[Prove that there is a split surjection $MO \to H\mathbb{Z}/2$. In fact, $MO \simeq \bigvee \Sigma^? H\mathbb{Z}/2$ is a wedge sum of suspensions of copies of $H\mathbb{Z}/2$, and $\mathscr{N}_* \cong \pi_*(MO) \cong \mathbb{F}_2[x_i \mid 1 \le i \ne 2^j - 1]$, with $|x_i| = i$. The equivalence realizes an isomorphism $\bigoplus \Sigma^? \mathscr{A} \cong H^*(MO; \mathbb{Z}/2)$ of free $\mathscr{A}$-modules, where $\mathscr{A} = H^*(H\mathbb{Z}/2; \mathbb{Z}/2)$ is the mod 2 Steenrod algebra. Here $H^*(MO; \mathbb{Z}/2) \cong H^*(BO; \mathbb{Z}/2)$ by Thom isomorphisms, $H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_i \mid i \ge 1]$ with $|w_i| = i$, and $\mathscr{A}$ is dual to $\mathscr{A}_* = \mathbb{Z}/2[\xi_j \mid j \ge 1]$ with $|\xi_j| = 2^j - 1$. In low dimensions, $\mathscr{N}_* = (\mathbb{F}_2\{1\}, 0, \mathbb{F}_2\{x_2\}, 0, \mathbb{F}_2\{x_2^2, x_4\}, \dots)$ with x_2 represented by $\mathbb{R}P^2$.]]

[[$\pi_*(MO) \to H_*(MO; \mathbb{Z}/2)$ injective, so k-dimensional bordism classes are detected by the Stiefel–Whitney characteristic numbers $\langle w(\tau_M), [M] \rangle$ where $w = w_1^{i_1} \dots w_k^{i_k}$ ranges over a basis for $H^k(BO)$.]]

1.4 Oriented bordism

Let $\Omega_k = \Omega_k^{SO}$ be the group of oriented bordism classes of oriented k-manifolds. Let $\widetilde{Gr}_n(\mathbb{R}^{n+k})$ be the oriented Grassmannian of oriented n-planes in $\mathbb{R}^{n+k}$. Then $\widetilde{Gr}_n(\mathbb{R}^\infty) \simeq BSO(n)$, and the Thom complex of the canonical oriented bundle is $Th(\tilde{\gamma}^n) \simeq MSO(n) = ESO(n)_+ \wedge_{SO(n)} S^n$. Get an isomorphism $\tilde{\phi}\colon \pi_*(MSO) \to \Omega_*$, where $\pi_k(MSO) = \operatorname{colim}_n \pi_{k+n} MSO(n)$. More generally, $MSO_*(X) \cong \Omega_*(X)$, where $MSO_k(X) = \operatorname{colim}_n \pi_{k+n}(MSO(n) \wedge X_+)$. For an oriented k-manifold $\alpha\colon M \to X$ over X, the integral fundamental class $[M] \in H_k(M)$ maps to a class $\alpha_*[M] \in H_k(X)$ that only depends on the oriented bordism class of M over X. What homology classes are in the image of this homomorphism $\Omega_*(X) \to H_*(X)$? Thom: Not all! For $k = 7$, some homology classes are not represented by (smooth) orientable manifolds. Led to Sullivan–Baas's bordism with singularities.

$$\begin{tikzcd} S \arrow[r] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\ & H & H\mathbb{Z}/2 \end{tikzcd}$$

represents

$$\begin{tikzcd} \Omega_*^{fr}(X) \arrow[r] & \Omega_*(X) \arrow[r] \arrow[d] & \mathscr{N}_*(X) \arrow[d] \\ & H_*(X) & H_*(X;\mathbb{Z}/2) \end{tikzcd}$$

2 Topological K-theory

Principal G-bundle $P \to X$, associated fiber bundle $E = P \times_G F \to X$. Example: Vector bundle $P \times_{O(n)} \mathbb{R}^n \to X$ with structure group $O(n)$. Classification of principal G-bundles over X, or fiber bundles over X with structure group G, by homotopy classes of maps $X \to BG$, where $EG \to BG$ is a principal G-bundle with EG contractible. Note that $G \simeq \Omega(BG)$.

Real vector bundles over X classified by $\coprod_{n\ge 0} BO(n)$. Topological K-theory $KO(X) = K(Vect(X))$ is group completion of $[X, \coprod_{n\ge 0} BO(n)]$. Initial homomorphism $Vect(X) \to KO(X)$ to a group. Over compact Hausdorff spaces/finite CW complexes, each bundle admits a stable inverse, so group completion is the localization inverting $\xi \mapsto \xi \oplus \epsilon^1$. Represented by group completion $\mathbb{Z}\times BO$: $KO(X) = [X, \mathbb{Z}\times BO]$.

Bott periodicity: $\mathbb{Z}\times BO \simeq \Omega^8(\mathbb{Z}\times BO)$ and $\mathbb{Z}\times BU \simeq \Omega^2(\mathbb{Z}\times BU)$. More precisely, the loop spaces $\Omega^i(\mathbb{Z}\times BO)$ for $0 \le i \le 8$ are homotopy equivalent to $\mathbb{Z}\times BO$, O, O/U, U/Sp, $\mathbb{Z}\times BSp$, Sp, Sp/U, U/O and $\mathbb{Z}\times BO$. The homotopy groups $\pi_i(\mathbb{Z}\times BO)$ for $i \ge 0$ begin $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 and repeat 8-periodically. The loop spaces $\Omega^i(\mathbb{Z}\times BU)$ for $0 \le i \le 2$ are homotopy equivalent to $\mathbb{Z}\times BU$, U and $\mathbb{Z}\times BU$. The homotopy groups $\pi_i(\mathbb{Z}\times BU)$ for $i \ge 0$ begin $\mathbb{Z}$, 0 and repeat 2-periodically.

Morse theory proof: The space of minimal (shortest) geodesics from I to $-I$ in $SU(2n)$ is $Gr_n(\mathbb{C}^{2n}) = U(2n)/U(n)\times U(n)$, and the inclusion $Gr_n(\mathbb{C}^{2n}) \to \Omega SU(2n)$ is $(2n+1)$-connected. Hence $BU \to \Omega SU$ is an equivalence.

Atiyah–Hirzebruch: Define $\widetilde{KO}^j(X) = [X, \Omega^i(\mathbb{Z}\times BO)]$ where $i + j \equiv 0 \mod 8$, $0 \le i < 8$, and $\widetilde{KU}^j(X) = [X, \Omega^i(\mathbb{Z}\times BU)]$ where $i + j \equiv 0 \mod 2$, $0 \le i < 2$. These give generalized cohomology theories $KO^*(X)$ and $KU^*(X)$, with coefficients $KO^j = \pi_i(\mathbb{Z}\times BO)$ and $KU^j = \pi_i(\mathbb{Z}\times BU)$, with i and j as above. Note that $\widetilde{KO}^*(X) \cong \widetilde{KO}^{*+8}(X)$ and $\widetilde{KU}^*(X) \cong \widetilde{KU}^{*+2}(X)$.

3 The stable homotopy category

3.1 Compactly generated spaces

Steenrod, Moore, McCord: Let $\mathscr{U}$ be the category of compactly generated weak Hausdorff spaces. A space X is compactly generated if the closed subsets A of X are precisely those for which preimage $u^{-1}(A)$ is closed in K, for any map $u\colon K \to X$ with K compact Hausdorff. The space X is weak Hausdorff if the image $u(K)$ is closed in X for any map $u\colon K \to X$ with K compact Hausdorff.

Every metric space, and every locally compact Hausdorff space is compactly generated weak Hausdorff.

Let X, $Y \in \mathscr{U}$. The traditional product topology on the set $X \times Y$ of pairs(x, y) might not be compactly generated, but by declaring the subsets $A \subset X \times Y$ for which the preimage $u^{-1}(A)$ is closed in K, for any map $u\colon K \to X \times Y$ with K compact Hausdorff, to be closed, gives a potentially finer topology that is compactly generated weak Hausdorff. The traditional compact-open topology on the set $Y^X = Map(X, Y)$ of maps $f\colon X \to Y$ might also not be compactly generated, but can be refined to a compactly generated weak Hausdorff topology by the same method.

Proposition: $\mathscr{U}$ has all (small) limits and colimits, hence is complete and cocomplete.

Theorem: $\mathscr{U}$ is Cartesian closed, in the sense that there is a natural homeomorphism

$$Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$$

given by sending $f\colon X \times Y \to Z$ to $g\colon X \to Map(Y, Z)$ given by $g(x)(y) = f(x, y)$.

Let $i\colon A \to X$. The problem of extending a map $f\colon A \to E$ over i to a map $h\colon X \to E$ with $hi = f$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \\ X \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

is homotopy invariant, i.e., does only depend on the homotopy class of f, if i has the homotopy extension property (HEP). If i has the homotopy extension property for all E then i is a (Hurewicz) cofibration. For a CW pair (X, A), the inclusion $A \to X$ is a cofibration.

Let $p\colon E \to B$. The problem of lifting a map $g\colon X \to B$ over p to a map $h\colon X \to E$ with $ph = g$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "h"] \arrow[r, "g"'] & B \end{tikzcd}$$

is homotopy invariant, i.e., does only depend on the homotopy class of g, if p has the homotopy lifting property (HLP). If p has the homotopy lifting property for all X then p is a (Hurewicz) fibration. For a fiber bundle $p\colon E \to B$ over a paracompact Hausdorff base space, the projection is a fibration.

Proposition (Strøm): If $i\colon A \to X$ is a cofibration, $p\colon E \to B$ a fibration, and i or p a homotopy equivalence, then in any commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "h"] \arrow[r, "g"'] & B \end{tikzcd}$$

there exists a map $h\colon X \to E$ making both triangles commute.

Earlier, Quillen proved a similar result with retracts of relative CW complexes (in place of Hurewicz cofibrations), Serre fibrations (in place of Hurewicz fibrations) and weak homotopy equivalences (in place of homotopy equivalences). Give different models for homotopy theory, i.e., model categories.

Alternative: Use simplicial sets, degreewise monomorphisms, weak homotopy equivalences and Kan fibrations.

3.2 Based spaces

Let $\mathscr{T}$ be the category of based compactly generated weak Hausdorff spaces with a chosen base point, and basepoint-preserving maps.

Proposition: $\mathscr{T}$ has all (small) limits and colimits, hence is complete and cocomplete.

Recall the wedge sum

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$$

and the smash product

$$X \wedge Y = X \times Y / X \vee Y\,.$$

Let $F(X,Y)$ denote the space of basepoint-preserving maps $X \to Y$.

$(\mathscr{T}, \wedge, S^0)$ is monoidal, in the sense that the smash product is a bifunctor $\wedge\colon \mathscr{T} \times \mathscr{T} \to \mathscr{T}$, mapping (X,Y) to $X \wedge Y$, which is associative up to a natural homeomorphism

$$\alpha\colon X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$$

and unital up to natural homeomorphisms $\lambda\colon S^0 \wedge Y \cong Y$ and $\rho\colon Y \wedge S^0 \cong Y$, subject to the coherence conditions that the diagrams

$$\begin{tikzcd}
 & (X \wedge Y) \wedge (Z \wedge W) \arrow[dr, "\alpha"] & \\
X \wedge (Y \wedge (Z \wedge W)) \arrow[ur, "\alpha"] \arrow[d, "\mathbb{1}\wedge\alpha"'] & & ((X \wedge Y) \wedge Z) \wedge W \arrow[d, "\alpha\wedge\mathbb{1}"] \\
X \wedge ((Y \wedge Z) \wedge W) \arrow[rr, "\alpha"] & & (X \wedge (Y \wedge Z)) \wedge W
\end{tikzcd}$$

(Mac Lane pentagon)

$$\begin{tikzcd}
X \wedge (S^0 \wedge Z) \arrow[rr, "\alpha"] \arrow[dr, "\mathbb{1}\wedge\lambda"'] & & (X \wedge S^0) \wedge Z \arrow[dl, "\rho\wedge\mathbb{1}"] \\
 & X \wedge Z &
\end{tikzcd}$$

and

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[d, bend right, "\lambda"'] \arrow[d, bend left, "\rho"] \\
S^0
\end{tikzcd}$$

commute.

Moreover, $(\mathscr{T}, \wedge, S^0, \gamma)$ is symmetric monoidal, in the sense that there is a natural homeomorphism

$$\gamma\colon X \wedge Y \cong Y \wedge X$$

subject to the coherence conditions that the diagrams

$$\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dr, "\gamma"] & \\
X \wedge (Y \wedge Z) \arrow[ur, "\alpha"] \arrow[d, "\mathbb{1}\wedge\gamma"'] & & Z \wedge (X \wedge Y) \arrow[d, "\alpha"] \\
X \wedge (Z \wedge Y) \arrow[dr, "\alpha"'] & & (Z \wedge X) \wedge Y \\
 & (X \wedge Z) \wedge Y \arrow[ur, "\gamma\wedge\mathbb{1}"'] &
\end{tikzcd}$$

(Mac Lane hexagon)

$$\begin{tikzcd}
X \wedge Y \arrow[d, bend right, "\gamma"'] \\
Y \wedge X \arrow[u, bend right, "\gamma"']
\end{tikzcd}$$

and

$$\begin{tikzcd} Y \wedge S^0 \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & S^0 \wedge Y \arrow[dl, "\lambda"] \\ & Y & \end{tikzcd}$$

commute. (The last triangle makes ρ, or λ, superfluous.)

Mac Lane proved a coherence theorem saying that all diagrams involving these structures that can reasonably expected to commute do indeed commute. An diagram than cannot be expected to commute is that given by the two maps $\mathbb{1}, \gamma \colon X \wedge X \to X \wedge X$.

A monoidal structure allows us to define what we mean by a monoid in $\mathscr{T}$: A space X with a multiplication $\mu \colon X \wedge X \to X$ and a unit $\eta \colon S^0 \to X$ such that the diagrams

$$\begin{tikzcd} X \wedge (X \wedge X) \arrow[rr, "\alpha"] \arrow[d, "\mathbb{1}\wedge\mu"'] & & (X \wedge X) \wedge X \arrow[d, "\mu\wedge\mathbb{1}"] \\ X \wedge X \arrow[r, "\mu"] & X & X \wedge X \arrow[l, "\mu"'] \end{tikzcd}$$

and

$$\begin{tikzcd} S^0 \wedge X \arrow[r, "\eta\wedge\mathbb{1}"] \arrow[dr, "\lambda"'] & X \arrow[d, "\mu"] & X \wedge S \arrow[l, "\mathbb{1}\wedge\eta"'] \arrow[dl, "\rho"] \\ & X & \end{tikzcd}$$

commute. A monoid map from (X, μ, η) to (Y, μ, η) is a map $f \colon X \to Y$ that makes the diagrams

$$\begin{tikzcd} X \wedge X \arrow[r, "f\wedge f"] \arrow[d, "\mu"'] & Y \wedge Y \arrow[d, "\mu"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and

$$\begin{tikzcd} & S^0 \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

commute. The monoids in $(\mathscr{T}, \wedge, S^0)$ thus form a category. In a monoidal category we can also define what we mean by a module over a monoid. (Details given later, in the context of S-modules.)

A symmetric monoidal structure allows us to specify when a monoid (X, μ, η) is commutative. This means that that diagram

$$\begin{tikzcd} X \wedge X \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & X \wedge X \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes. The commutative monoids (in a symmetric monoidal category) form a full subcategory of the monoids (in the underlying monoidal category).

Finally, the mapping space F makes $\mathscr{T}$ a closed symmetric monoidal category, in the sense that there is a natural homeomorphism

$$\theta \colon F(X \wedge Y, Z) \cong F(X, F(Y, Z)) .$$

For each Y, we say that the functors $L \colon X \mapsto X \wedge Y$ and $R \colon Z \mapsto F(Y, Z)$ are adjoint, with L the left adjoint, and R the right adjoint. The natural homeomorphism

$$\theta \colon F(L(X), Z) \cong F(X, R(Z))$$

is called an adjunction. In the case $Z = L(X)$, the identity map on the left hand side corresponds to a natural map

$$\eta \colon X \to R(L(X))$$

called the adjunction unit. In the case $X = R(Z)$, the identity map on the right hand side corresponds to a natural map

$$\epsilon \colon L(R(Z)) \to Z$$

called the adjunction counit.

For example, with $Y = S^1$, $X \wedge S^1 = \Sigma X$ is the suspension and $F(S^1, Z) = \Omega Z$ is the loop space, and the adjunction $F(\Sigma X, Z) \cong F(X, \Omega Z)$ takes a map $f \colon \Sigma X \to Z$ to the map $g \colon X \to \Omega Z$ with $g(x)(s) = f(x, s)$ with $s \in S^1$. The adjunction unit $\eta \colon X \to \Omega\Sigma X$ and counit $\epsilon \colon \Sigma\Omega Z \to Z$ are the natural maps given by $\eta(x)(s) = (x, s)$ and $\epsilon(\omega, s) = \omega(s)$, respectively,

More generally, with $Y = S^n$ we call $X \wedge S^n = \Sigma^n X$ the n-fold suspension and $\Omega^n Z = F(S^n, Z)$ the n-fold loop space, and $F(\Sigma^n X, Z) \cong F(X, \Omega^n Z)$. The adjunction units are $\eta \colon X \to \Omega^n \Sigma^n X$ and $\epsilon \colon \Sigma^n \Omega^n Z \to Z$.

A space (X, x_0) is nondegenerately based (= well-based) if the inclusion $\{x_0\} \to X$ is a Hurewicz cofibration, i.e., if $(X, \{x_0\})$ is an NDR-pair.

3.3 The homotopy category of spaces

Homotopy functors defined on $\mathscr{T}$ take homotopy equivalences to isomorphisms. For the purpose of algebraic topology, we focus on weak homotopy functors such as H_* and π_*, which take all weak homotopy equivalences to isomorphisms. Let $\mathscr{W} \subset \mathscr{T}$ be the subcategory of weak homotopy equivalences. The localization functor

$$\mathscr{T} \to \mathrm{Ho}(\mathscr{T}) = \mathscr{T}[\mathscr{W}^{-1}]$$

is the initial functor from $\mathscr{T}$ that takes each weak equivalence $f \colon X \simeq_w Y$ to an isomorphism. It can be constructed with the same objects as $\mathscr{T}$, and with morphisms

$$\mathrm{Ho}(\mathscr{T})(X, Y) = [X, Y] = \{\Gamma X \to Y\}/\simeq$$

the set of homotopy classes of maps $\Gamma X \to Y$ from a CW approximation $\Gamma X \simeq_w X$ of X to Y. (Uses that $\emptyset \to \Gamma X$ is a cofibration and $Y \to *$ is a fibration in Quillen's model structure.) When X is (homotopy equivalent to) a CW complex, this is the same as the set of homotopy classes of maps $X \to Y$. Weak homotopy functors from $\mathscr{T}$ thus factor uniquely through $\mathrm{Ho}(\mathscr{T})$.

We review Puppe's theory of homotopy cofiber and fiber sequences, using May's notation (Lewis–May–Steinberger, May–Ponto).

Given $f \colon X \to Y$ in $\mathscr{T}$ let the cone of X be $CX = I \wedge X$ with $I = [0, 1]$ based at 1, and let the mapping cone $Cf = Y \cup_f CX$ be the pushout of f and $i_0 \colon X \to CX$. The map f and the inclusion $i \colon Y \to Cf$ induce an exact sequence

$$[X, T] \xleftarrow{f^*} [Y, T] \xleftarrow{i^*} [Cf, T]$$

for any space T, in the sense that the image of i^* is equal to the kernel of f^*, i.e., the preimage of the class 0 of the constant map. Any other diagram $X' \to Y' \to Z'$ that is homotopy equivalent to $X \to Y \to Cf$ will also induce such exact sequences, and is called a homotopy cofiber sequence. We often call Cf the homotopy cofiber of f.

The collapse map $Cf \to Y/X$ is a homotopy equivalence if f is a cofibration. In particular, i is a cofibration, so the collapse map $Ci \to Cf/Y$ is a homotopy equivalence. Here $Cf/Y \cong I/\partial I \wedge X \cong S^1 \wedge X \cong \Sigma X$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "j"] \arrow[dr, "\pi"'] & Ci \arrow[r, "k"] \arrow[d, "\simeq"] & Cj \arrow[d, "\simeq"] \\
& & & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

(The square commutes up to homotopy.) Hence $Y \to Cf \to \Sigma X$ and $Cf \to \Sigma X \to \Sigma Y$ are also homotopy cofiber sequences. The exact sequence extends without bound to the right:

$$[X,T] \xleftarrow{f^*} [Y,T] \xleftarrow{i^*} [Cf,T] \xleftarrow{\pi^*} [\Sigma X,T] \xleftarrow{-\Sigma f^*} [\Sigma Y,T] \xleftarrow{-\Sigma i^*} [\Sigma Cf,T] \xleftarrow{-\Sigma\pi^*} [\Sigma^2 X,T] \xleftarrow{\Sigma^2 f^*} \dots$$

Starting at $[\Sigma X,T]$ the pinch map $S^1 \to S^1 \vee S^1$ and the reflection $S^1 \to S^1$ induce a group structure, which is abelian from $[\Sigma^2 X,T]$ and onwards.

For example, with $T = \mathbb{Z} \times BO$, this gives half of the long exact sequence in topological K-theory:

$$\widetilde{KO}^0(X) \xleftarrow{f^*} \widetilde{KO}^0(Y) \xleftarrow{i^*} \widetilde{KO}^0(Cf) \xleftarrow{\pi^*} \widetilde{KO}^{-1}(X) \xleftarrow{-f^*} \widetilde{KO}^{-1}(Y) \xleftarrow{-i^*} \dots$$

which can be extended to the left by Bott periodicity.

Given $f\colon X \to Y$ let the path space of Y be $PY = F(I,Y)$ with $I = [0,1]$ based at 0, and let the homotopy fiber $Ff = X \times_Y PY$ be the pullback of f and $p_1\colon PY \to Y$. The projection $p\colon Ff \to X$ and $f\colon X \to Y$ induce an exact sequence

$$[T,Ff] \xrightarrow{p_*} [T,X] \xrightarrow{f_*} [T,Y]$$

for any space T, in the sense that the image of p_* is the kernel of f_*. Any other diagram $W' \to X' \to Y'$ that is homotopy equivalent to $Ff \to X \to Y$ will also induce such exact sequences, and is called a homotopy fiber sequence.

The inclusion $f^{-1}(y_0) \to Ff$ is a homotopy equivalence if f is a fibration. In particular, p is a fibration, so the inclusion $p^{-1}(y_0) \to Fp$ is a homotopy equivalence. Here $p^{-1}(y_0) \cong F(I/\partial I, Y) \cong \Omega Y$.

$$\begin{tikzcd}
\Omega X \arrow[r, "-\Omega f"] \arrow[d, "\simeq"'] & \Omega Y \arrow[d, "\simeq"'] \arrow[dr, "\iota"] & & & \\
Fq \arrow[r, "r"'] & Fp \arrow[r, "q"'] & Ff \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

(The square commutes up to homotopy.) Hence $\Omega Y \to Ff \to X$ and $\Omega X \to \Omega Y \to Ff$ are also homotopy fiber sequences. The exact sequence extends without bound to the left:

$$\dots \xrightarrow{\Omega^2 f_*} [T,\Omega^2 Y] \xrightarrow{-\Omega^2\iota_*} [T,\Omega Ff] \xrightarrow{-\Omega p_*} [T,\Omega X] \xrightarrow{-\Omega f_*} [T,\Omega Y] \xrightarrow{\iota_*} [T,Ff] \xrightarrow{p_*} [T,X] \xrightarrow{f_*} [T,Y]$$

For example, with $T = S^k$ this gives half of the long exact sequence of (higher) homotopy groups

$$\dots \xrightarrow{-p_*} \pi_{k+1}(X) \xrightarrow{-f_*} \pi_{k+1}(Y) \xrightarrow{\iota_*} \pi_k(Ff) \xrightarrow{p_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y)$$

These exact sequences are compatible up to one change of sign.

Let $\eta\colon X \to \Omega\Sigma X$ be the suspension-loop adjunction unit, left adjoint to the identity map on ΣX. Hence $\eta(x)(t) = (x,t)$ for $x \in X$ and $t \in I$. Let $\epsilon\colon \Sigma\Omega Y \to Y$ be the adjunction counit, right adjoint to the identity map on ΩY. Hence $\epsilon(\omega,t) = \omega(t)$.

Let $\eta\colon Ff \to \Omega Cf$ and $\epsilon\colon \Sigma Ff \to Cf$ be the adjoint pair of maps given by

$$\eta(x,\omega)(t) = \epsilon(x,\omega,t) = \begin{cases} \omega(2t) & \text{for } 0 \le t \le 1/2 \\ (x, 2t-1) & \text{for } 1/2 \le t \le 1 \end{cases}$$

where $x \in X$, $\omega \in PY$ and $t \in I$, with $f(x) = \omega(1)$.

Lemma 3.1. *The diagrams*

$$\begin{tikzcd}
\Omega Y \arrow[r, "\iota"] \arrow[d, "="] & Ff \arrow[r, "p"] \arrow[d, "\eta"] & X \arrow[r, "f"] \arrow[d, "\eta"] & Y \arrow[d, "\eta"] \\
\Omega Y \arrow[r, "\Omega i"] & \Omega Cf \arrow[r, "\Omega\pi"] & \Omega\Sigma X \arrow[r, "\Omega\Sigma f"] & \Omega\Sigma Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\epsilon"] & \Sigma\Omega Y \arrow[r, "\Sigma\iota"] \arrow[d, "\epsilon"] & \Sigma F f \arrow[r, "\Sigma p"] \arrow[d, "\epsilon"] & \Sigma X \arrow[d, "="] \\
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "\pi"] & \Sigma X
\end{tikzcd}$$

commute up to homotopy.

3.4 The Spanier–Whitehead category

(Margolis)

Homotopy excision theorem: If X, Y and Cf are c-connected and T is a CW complex of dimension $\le 2c$, then f and i induce an exact sequence

$$[T, X] \xrightarrow{f_*} [T, Y] \xrightarrow{i_*} [T, Cf]$$

Freudenthal suspension theorem: If X is c-connected, and T is a CW complex of dimension $\le 2c$, then the suspension homomorphism

$$\Sigma\colon [T, X] \to [\Sigma T, \Sigma X]$$

is an isomorphism.

Notice that if $d = \dim(T) + 2$, the dimension condition on T can be achieved by replacing all spaces and maps in sight by their d-fold suspensions.

Spanier and J.H.C. Whitehead define the stable homotopy classes of maps

$$\{X, Y\} = \operatorname*{colim}_n [X \wedge S^n, Y \wedge S^n] = \operatorname*{colim}_n [\Sigma^n X, \Sigma^n Y] \cong \operatorname*{colim}_n [X, \Omega^n \Sigma^n Y] \,.$$

This a colimit of groups for $n \ge 1$, which are abelian for $n \ge 2$, so $\{X, Y\}$ is naturally an abelian group. The diagram

$$\{X, T\} \xleftarrow{f^*} \{Y, T\} \xleftarrow{i^*} \{Cf, T\}$$

remains exact for any cofiber sequence $X \to Y \to Cf$, since this is the sequential colimit over n of the exact sequences $[\Sigma^n X, \Sigma^n T] \leftarrow [\Sigma^n Y, \Sigma^n T] \leftarrow [\Sigma^n Cf, \Sigma^n T]$. More interesting is that

$$\{T, X\} \xrightarrow{f_*} \{T, Y\} \xrightarrow{i^*} \{T, Cf\}$$

is also exact, at least for finite-dimensional T, because this is the sequential colimit over n of the sequences $[\Sigma^n T, \Sigma^n X] \to [\Sigma^n T, \Sigma^n Y] \to [\Sigma^n T, \Sigma^n Cf]$, which are exact for $\dim(T) + n \le 2(n-1)$, i.e., for $n \ge \dim(T) + 2$. (Discuss passage to limit to account for infinite-dimensional T?) Likewise, the suspension homomorphism

$$\Sigma\colon \{T, X\} \to \{\Sigma T, \Sigma X\}$$

is also an isomorphism, more-or-less by construction.

Hence the category with objects $\mathscr{T}$ and morphisms from X to Y given by the abelian group $\{X, Y\}$ is almost a localization of $\mathrm{Ho}(\mathscr{T})$ with respect to Σ, in the sense that Σ becomes an equivalence of categories. However, Σ is not essentially surjective on objects. To achieve this, one can introduce formal desuspensions of spaces. The resulting Spanier–Whitehead category $\mathscr{SW}$ has objects pairs (X, k) with X in $\mathscr{T}$ (or X a CW complex) and k an integer. We think of (X, k) as a model for $\Sigma^k X$, also when k is negative. The morphisms from (X, k) to (Y, ℓ) are

$$\mathscr{SW}((X, k), (Y, \ell)) = \operatorname*{colim}_n [\Sigma^{k+n} X, \Sigma^{\ell+n} Y] \,.$$

The functor

$$\Sigma^\infty\colon \mathrm{Ho}(\mathscr{T}) \to \mathscr{SW} = \mathrm{Ho}(\mathscr{T})[\Sigma^{-1}]$$

given by $X \mapsto (X, 0)$ takes the suspension functor $\Sigma\colon \mathrm{Ho}(\mathscr{T}) \to \mathrm{Ho}(\mathscr{T})$ to the shift equivalence $(X, k) \mapsto (X, k+1)$, up to a natural isomorphism $(\Sigma X, 0) \cong (X, 1)$. (Universal property: See Margolis (1983), p. 7.)

Spanier–Whitehead duality: Finite CW complexes (and their desuspensions) are dualizable in $\mathscr{SW}$, in the sense that to each such object Y there is a dual object DY and a natural isomorphism

$$\{X \wedge Y, Z\} \cong \{X, DY \wedge Z\}\,.$$

The dual of S^n is S^{-n}, and to each homotopy cofiber sequence $Y' \to Y \to Y''$ there is a dual homotopy (co-)fiber sequence $D(Y'') \to DY \to D(Y')$. For each dualizable Y the functor $L(X) = X \wedge Y$ is left adjoint to the functor $R(Z) = DY \wedge Z$, with adjunction unit $\eta\colon S \to DY \wedge Y$ and counit $\epsilon\colon DY \wedge Y \to S$.

3.5 Triangulated categories

The Spanier–Whitehead category is an Ab-category, in the sense that each morphism set $\{X, Y\}$ is an abelian group and each composition pairing $\{Y, Z\} \times \{X, Y\} \to \{X, Z\}$ is bilinear. (This is a case of an enriched category: instead of being formed in the symmetric monoidal category of sets, with pairing the Cartesian product $\times$ and with unit the singleton set $*$, the notion of an Ab-category is formed in the symmetric monoidal category Ab of abelian groups, with pairing the tensor product $\otimes$ and unit $\mathbb{Z}$.)

An additive category is an Ab-category with all finite coproducts, i.e., an initial object give by the empty coproduct, and a binary coproduct. It will then also have all finite products, the initial object is also terminal, and the canonical map from a binary coproduct to a binary product is an isomorphism. (A simultaneously initial and terminal object is called a zero object. A simultaneous coproduct and product is called a biproduct.)

The Spanier–Whitehead category is an additive category, with initial object $*$ and coproduct $X \vee Y$. The canonical map $X \vee Y \to X \times Y$ is an isomorphism in $\mathscr{SW}$. Another example of an additive category is the category of (right) R-modules for a fixed ring R. Each morphism set $\mathrm{Hom}_R(M, N)$ is naturally an abelian group, and composition is bilinear. The initial and terminal object is 0, and the canonical map $M \oplus N \to M \times N$ is an isomorphism.

An abelian category is an additive category such that every morphism has a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel. Such a category is a convenient setting for homological algebra. The category of R-modules is abelian, but the Spanier–Whitehead category is not abelian. For instance, a cokernel C of a morphism $f \in \{X, Y\}$ would be a coequalizer of f and the zero morphism 0 (which is the unique map that factors through the zero object):

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \arrow[r, "i"] & C \end{tikzcd}$$

The mapping cone Cf with the canonical inclusion $i\colon Y \to Cf$ fits in such a diagram, with $if = i0$, hence is a weak cokernel. However, it does not in general satisfy the uniqueness condition required of a coequalizer. Given $g\colon Y \to T$ with $gf = g0$ there exists a map $h\colon Cf \to T$ with $hi = g$, but h is only determined up to addition of a class in $j^*\{\Sigma X, T\}$:

$$\begin{tikzcd} \{X, T\} & \{Y, T\} \arrow[l, shift right, "f^*"'] \arrow[l, shift left, "0"] & \{Cf, T\} \arrow[l, "i^*"'] & \{\Sigma X, T\} \arrow[l, "j^*"'] \end{tikzcd}$$

Since $j^*\{\Sigma X, T\}$ is in general nonzero, Cf is not a cokernel in the sense of abelian categories. Instead, the Spanier–Whitehead category is a triangulated category, somewhat intermediate between abelian categories and the long exact sequences arising from homological algebra.

Triangulated categories were defined by Puppe (1962/1967), with an additional axiom added by Verdier (1966/1971). (Beilinson–Bernstein–Deligne (1982), Margolis (1983), Hovey–Palmieri–Strickland (1997)). We follow May (2001):

Definition 3.2. A triangulation on an additive category $\mathscr{C}$ is an additive self-equivalence $\Sigma\colon \mathscr{C} \to \mathscr{C}$ together with a collection of triangles, i.e., diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $\mathscr{C}$, called the distinguished triangles, such that the following axioms hold.

(T1) Let T be any object and $f\colon X \to Y$ be any map in $\mathscr{C}$.

- The triangle $T \xrightarrow{\mathbb{1}} T \longrightarrow * \longrightarrow \Sigma T$ is distinguished.
- The map f is part of a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.
- Any triangle isomorphic to a distinguished triangle is distinguished.

(T2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished, then so is $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.

(T3) Consider the following braid diagram.

$$\begin{tikzcd}
X \arrow[dr, "f"'] \arrow[rr, bend left, "h"] & & Z \arrow[dr, dashed, "h'"] \arrow[rr, bend left, "g'"] & & W \arrow[dr, "g''"] \arrow[rr, bend left, "j''"] & & \Sigma U \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"'] & & V \arrow[ur, dashed, "j'"] \arrow[dr, dashed, "h''"] & & \Sigma Y \arrow[ur, "\Sigma f'"'] & \\
& & U \arrow[ur, dashed, "j"] \arrow[rr, bend right, "f''"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

Assume that $h = g \circ f$ and $j'' = \Sigma f' \circ g''$, and that (f, f', f'') and (g, g', g'') are distinguished. If h' and h'' are such that (h, h', h'') is distinguished, then there are maps j and j' such that the diagram commutes and (j, j', j'') is distinguished.

Axiom (T3) is Verdier's octahedral axiom. May shows that the axioms (T1), (T2) and (T3) imply the following lemma, even though Verdier assumes it as another axiom.

Lemma 3.3. *If the rows are distinguished and the left hand square commutes in the following diagram, then there is a map k that makes the remaining squares commute.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

We call k a fill-in map.

Proposition 3.4. *For (f, g, h) distinguished and T any object, the sequences*

$$\dots \longrightarrow \mathscr{C}(T, X) \xrightarrow{f_*} \mathscr{C}(T, Y) \xrightarrow{g_*} \mathscr{C}(T, Z) \xrightarrow{h_*} \mathscr{C}(T, \Sigma X) \longrightarrow \dots$$

and

$$\dots \longleftarrow \mathscr{C}(X, T) \xleftarrow{f^*} \mathscr{C}(Y, T) \xleftarrow{g^*} \mathscr{C}(Z, T) \xleftarrow{h^*} \mathscr{C}(\Sigma X, T) \longleftarrow \dots$$

are exact.

Proof. We show that $\mathrm{im}(f_*) = \ker(g_*)$. Given $i\colon T \to X$ in $\mathscr{C}(T, X)$ we have

$$\begin{tikzcd}
T \arrow[r, "\mathbb{1}"] \arrow[d, "i"'] & T \arrow[r] \arrow[d, "j"] & * \arrow[r] \arrow[d, dashed, "k"] & \Sigma T \arrow[d, "\Sigma i"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

with $j = f \circ i$, and there is a fill-in map k. Hence $gfi = 0$, so $\mathrm{im}(f_*) \subset \ker(g_*)$.

Conversely, given $j\colon T \to Y$ in $\mathscr{C}(T,Y)$ with $g \circ j = 0$ we have

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "j"] & * \arrow[r] \arrow[d] & \Sigma T \arrow[r, "-\Sigma\mathbb{1}"] \arrow[d, dashed, "\Sigma i"] & \Sigma T \arrow[d, "\Sigma j"] \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

and there is a fill-in map Σi. Hence $\Sigma j = \Sigma(f \circ i)$, so $j = f \circ i$, and $\ker(g_*) \subset \operatorname{im}(f_*)$. $\square$

Lemma 3.5 (The 3×3 lemma)**.** *Assume that $j \circ f = f' \circ i$ and the two top rows and two left columns are distinguished in the following diagram.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"'] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] \arrow[d, dashed, "k'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[r, dashed, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, dashed, "g''"] \arrow[d, "j''"] & Z'' \arrow[r, dashed, "h''"] \arrow[d, dashed, "k''"] & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

Then there is an object Z'' and maps f'', g'', h'', k, k' and k'' such that the diagram is commutative, except for its bottom right hand square, which commutes up to the sign -1, and all four rows and columns are distinguished.

3.6 Boardman's stable homotopy category

The Spanier–Whitehead category is not large enough to represent ordinary cohomology theories, and does not admit arbitrary coproducts. For example, given spaces X_n and maps $\sigma\colon \Sigma X_n \to X_{n+1}$ for each $n \ge 0$, the sequence

$$(X_0, 0) \to (X_1, -1) \to \dots \to (X_n, -n) \to \dots$$

might not have a colimit in $\mathscr{SW}$. Boardman (1965 and later) and Adams (1971 or earlier) construct a larger category $\mathscr{B}$ with better formal properties. It satisfies Margolis' axioms:

Definition 3.6. A stable homotopy category is a category $\mathscr{S}$ with objects called spectra and with morphisms $\mathscr{S}(X,Y) = [X,Y]$ which satisfies the following axioms:

Axiom 1: $\mathscr{S}$ has arbitrary coproducts $\coprod_\alpha X_\alpha$.

There is a suspension functor $\Sigma\colon \mathscr{S} \to \mathscr{S}$ and a collection Δ of distinguished triangles of the form $X \to Y \to Z \to \Sigma X$.

Axiom 2: $(\mathscr{S}, \Sigma, \Delta)$ is a triangulated category.

Axiom 3: There is an additive functor $\wedge\colon \mathscr{S} \times \mathscr{S} \to \mathscr{S}$, called the smash product, satisfying

(a) $(\mathscr{S}, \wedge)$ is a symmetric monoidal category with unit S;

(b) There is a natural isomorphism $\Sigma(X) \wedge W \cong \Sigma(X \wedge W)$ [[ETC]];

(c) For $X \to Y \to Z \to \Sigma X$ distinguished and any W, the diagram $X \wedge W \to Y \wedge W \to Z \wedge W \to \Sigma(X \wedge W)$ is distinguished;

(d) The natural map $\coprod_\alpha (X \wedge Y_\alpha) \to X \wedge \coprod_\alpha Y_\alpha$ is an isomorphism.

We define $\pi_k(X) = [S,X]_k$, which is $[\Sigma^k S, X]$ for $k \ge 0$ and $[S, \Sigma^{-k} X]$ for $k \le 0$.

Axiom 4: S is a small graded weak generator, i.e., the natural map $\bigoplus_\alpha \pi_*(X_\alpha) \to \pi_*(\coprod_\alpha X_\alpha)$ is an isomorphism, and $f\colon X \to Y$ is an isomorphism if (and only if) $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Let $\mathscr{F}$ be the subcategory of finite spectra in $\mathscr{S}$, i.e., the minimal full subcategory containing S and closed under the formation of fibers and cofibers, in the sense that X is in $\mathscr{F}$ if $X \to Y \to Z \to \Sigma(X)$ is distinguished and Y and Z are in $\mathscr{F}$.

Axiom 5: The subcategory $\mathscr{F}$ of finite spectra in $\mathscr{S}$ is equivalent to the Spanier–Whitehead category of finite CW spectra [ETC].

Margolis (1983) conjectured that these axioms characterize Boardman's stable homotopy category $\mathscr{B}$. Schwede (Annals of Math, 2007) proved that: "if $\mathscr{C}$ is a stable model category whose homotopy category is compactly generated, and the full subcategory of compact objects in the homotopy category of $\mathscr{C}$ is equivalent as a triangulated category to the usual homotopy category of finite spectra, then $\mathscr{C}$ is Quillen equivalent as a model category to any of the standard model categories of spectra. That is, given the primary homotopy theory of finite spectra (cofiber sequences and suspensions), the secondary homotopy theory of all spectra (such as Toda brackets and function spaces) is determined." (This is quoted from Hovey's Math Review.)

Freyd's generating hypothesis (1966): If $f\colon X \to Y$ is a map of finite spectra and $\pi_*(f) = 0$ then $f = 0$. This is an open problem.

3.7 Representation of homology and cohomology theories

Brown representability (1962): Let F be a contravariant homotopy functor from the category of pointed connected CW complexes to pointed sets, such that (product axiom)

$$F(\bigvee_\alpha X_\alpha) \cong \prod_\alpha F(X_\alpha)$$

for any collection $\{X_\alpha\}_\alpha$, and (Mayer–Vietoris)

$$F(X) \longrightarrow F(A) \times F(B) \longrightarrow F(A \cap B)$$

is exact at $F(A) \times F(B)$, whenever $X = A \cup B$ is the union of two subcomplexes. Then there exists a (pointed connected) CW complex Y and a natural isomorphism $[X, Y] \cong F(X)$. The converse is immediate.

This applies in each degree for a generalized (reduced) cohomology theory $X \mapsto h^*(X)$. Let the space E_n represent the functor $X \mapsto h^n(X)$, for each $n \ge 0$. Then $[X, E_n] \cong h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}]$, so there is a (weak) homotopy equivalence $E_n \simeq \Omega_0 E_{n+1}$. (Here $\Omega_0 Y$ denotes the path component of the base point in ΩY.) Replacing E_n with ΩE_{n+1} we have (weak) homotopy equivalences $\tilde\sigma\colon E_n \simeq \Omega E_{n+1}$. Let $\sigma\colon \Sigma E_n \to E_{n+1}$ be the left adjoint map. Then

$$h^n(X) = \begin{cases} [X, E_n] & \text{for } n \ge 0 \\ [\Sigma^{-n} X, E_0] & \text{for } n \le 0 \end{cases}$$

for all CW complexes X.

For example, with $h^*(X) = \widetilde{KU}^*(X)$ given by reduced complex topological K-theory we representing spaces $\underline{KU}_n \simeq \mathbb{Z} \times BU$ for each even n and $\underline{KU}_n \simeq U$ for each odd n. The adjoint structure maps are standard equivalence $U \simeq \Omega(\mathbb{Z} \times BU)$ for n odd, and the Bott equivalence $\mathbb{Z} \times BU \simeq \Omega U$ for n even.

Recall also the formula

$$\widetilde{MO}_k(X) = \operatorname*{colim}_n \pi_{k+n}(MO(n) \wedge X)$$

for the reduced bordism group of X. Here $\sigma\colon \Sigma MO(n) \to MO(n+1)$ was defined as the map of Thom complexes $\Sigma MO(n) = Th(\gamma^n \oplus \epsilon^1) \to Th(\gamma^{n+1}) = MO(n+1)$ induced by the map $\gamma^n \oplus \epsilon^1 \to \gamma^{n+1}$ covering the inclusion $BO(n) \to BO(n+1)$. Alternatively, it is the map

$$EO(n+1)_+ \wedge_{O(n)} S^n \wedge S^1 \longrightarrow EO(n+1)_+ \wedge_{O(n+1)} S^{n+1}$$

associated to the inclusion $O(n) \cong O(n) \times 1 \subset O(n+1)$.

Lima–Whitehead (1959/1962): A (sequential) spectrum E is a sequence of based spaces E_n and structure maps $\Sigma E_n \to E_{n+1}$, for $n \ge 0$. It is an Ω-spectrum if each adjoint structure map $\tilde\sigma\colon E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence.

The complex K-theory spectrum $KU = \{n \mapsto \underline{KU}_n\}$ is an Ω-spectrum; the bordism spectrum $MO = \{n \mapsto MO(n)\}$ is not.

The homology theory $X \mapsto E_*(X)$ associated to E is the covariant functor defined by

$$E_k(X) = \operatorname*{colim}_n \pi_{k+n}(E_n \wedge X)$$

where the colimit is formed for $k + n \geq 0$ over the composites

$$\pi_{k+n}(E_n \wedge X) \xrightarrow{\Sigma} \pi_{k+n+1}\Sigma(E_n \wedge X) \xrightarrow{\cong} \pi_{k+n+1}(\Sigma E_n) \wedge X \xrightarrow{\sigma_*} \pi_{k+n+1}(E_{n+1} \wedge X)\,.$$

If E is an Ω-spectrum, the associated cohomology theory $X \mapsto E^*(X)$ is the contravariant functor defined by

$$E^k(X) = \begin{cases} [X, E_k] & \text{for } k \geq 0 \\ [\Sigma^{-k} X, E_0] & \text{for } k \leq 0. \end{cases}$$

If E is not an Ω-spectrum, the cohomology theory is given for X of the homotopy type of a finite CW complex by the colimit $E^k(X) = \operatorname{colim}_n [\Sigma^n X, E_{k+n}]$, where $k + n \geq 0$. (Forward reference?)

In the stable homotopy category of spectra, these formulas can be rewritten as natural isomorphisms

$$E_k(X) \cong [S^k, E \wedge X] \qquad \text{and} \qquad E^k(X) \cong [X, \Sigma^k E]\,,$$

for all integers k.

We define the stable homotopy groups of E for all integers k by

$$\pi_k(E) = \operatorname*{colim}_n \pi_{k+n}(E_n)$$

where the colimit is formed for $k + n \geq 0$ over the composites

$$\pi_{k+n}(E_n) \xrightarrow{\Sigma} \pi_{k+n+1}(\Sigma E_n) \xrightarrow{\sigma_*} \pi_{k+n+1}(E_{n+1})\,.$$

A map $f\colon E \to E'$ of spectra is a sequence of basepoint-preserving maps $f_n\colon E_n \to E'_n$ commuting with the structure maps, in the sense that $f_{n+1} \circ \sigma = \sigma \circ \Sigma f_n\colon \Sigma E_n \to E'_{n+1}$, for each $n \geq 0$.

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\sigma"] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma E'_n \arrow[r, "\sigma"] & E'_{n+1} \end{tikzcd}$$

The map $f\colon E \to E'$ induces homomorphisms $f_*\colon \pi_k(E) \to \pi_k(E')$, and is called a stable equivalence, or a π_*-isomorphism, if f_* is an isomorphism for each integer k.

The stable homotopy category can be constructed as the localization $\mathrm{Sp}[\mathscr{W}^{-1}]$ of this category Sp of spectra with respect to the class $\mathscr{W}$ of stable equivalences, i.e., the category that results by making each stable equivalence into an isomorphism. It is not evident that such a category exists, nor how one can calculate its morphism sets. (How can one see that $\pi_k(X) \cong [S^k, X]$ in $\mathrm{Sp}[\mathscr{W}^{-1}]$?)

3.8 The problem of representing products

The smash product in (integral, singular) cohomology

$$\wedge\colon \tilde{H}^k(X) \otimes \tilde{H}^\ell(Y) \longrightarrow \tilde{H}^{k+\ell}(X \wedge Y)$$

can be represented by a map of representing spaces

$$\phi_{k,\ell}\colon H_k \wedge H_\ell \longrightarrow H_{k+\ell}\,,$$

where in this case $H_k = K(\mathbb{Z}, k)$ is the Eilenberg–Mac Lane space. If $x \in \tilde{H}^k(X)$ is represented by $f\colon X \to H_k$ and $y \in \tilde{H}^\ell(Y)$ is represented by $g\colon Y \to H_\ell$, the smash product $x \wedge y$ is represented by the composite

$$X \wedge Y \xrightarrow{f \wedge g} H_k \wedge H_\ell \xrightarrow{\phi_{k,\ell}} H_{k+\ell}\,.$$

Is there a spectrum $H \wedge H$, and natural maps $\iota_{k,\ell}\colon H_k \wedge H_\ell \to (H \wedge H)_{k+\ell}$, so that the maps $\phi_{k,\ell}$ arise from a spectrum map $\mu\colon H \wedge H \to H$ with components $\mu_n\colon (H \wedge H)_n \to H_n$ as the composites

$$H_k \wedge H_\ell \xrightarrow{\iota_{k,\ell}} (H \wedge H)_{k+\ell} \xrightarrow{\mu_{k+\ell}} H_{k+\ell}\,?$$

Recall that the cup product in cohomology is graded commutative. The smash product $\wedge\colon \tilde{H}^k(X) \wedge \tilde{H}^\ell(Y) \to \tilde{H}^{k+\ell}(X \wedge Y)$ satisfies $\gamma_*(x \wedge y) = (-1)^{k\ell} y \wedge x$, where $\gamma\colon X \wedge Y \to Y \wedge X$ is the symmetry homeomorphism. With the convention $\Sigma X = X \wedge S^1$ we can write the suspension isomorphism $\Sigma\colon \tilde{H}^k(X) \to \tilde{H}^{k+1}(\Sigma X) = \tilde{H}^{k+1}(X \wedge S^1)$ as $\Sigma(x) = x \wedge s$, where $s \in \tilde{H}^1(S^1)$ is the preferred generator. Then $x \wedge y \wedge s = (-1)^\ell x \wedge s \wedge y$, so

$$\Sigma(x \wedge y) = x \wedge \Sigma y = (-1)^\ell \Sigma x \wedge y\,.$$

Hence, at the level of representing spaces the diagram

$$\begin{tikzcd}
H_k \wedge H_\ell \wedge S^1 \arrow[r, "1\wedge\sigma"] \arrow[d, "\phi_{k,\ell}\wedge 1"'] & H_k \wedge H_{\ell+1} \arrow[d, "\phi_{k,\ell+1}"] \\
H_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & H_{k+\ell+1}
\end{tikzcd}$$

will commute up to homotopy, whereas the diagram

$$\begin{tikzcd}
H_k \wedge H_\ell \wedge S^1 \arrow[r, "1\wedge\gamma", "\cong"'] \arrow[d, "\phi_{k,\ell}\wedge 1"'] & H_k \wedge S^1 \wedge H_\ell \arrow[r, "\sigma\wedge 1"] & H_{k+1} \wedge H_\ell \arrow[d, "\phi_{k+1,\ell}"] \\
H_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & H_{k+\ell+1} \arrow[r, "="] & H_{k+1+\ell}
\end{tikzcd}$$

will only commute up to the sign $(-1)^\ell$.

The structure maps $\iota_{k,\ell}\colon H_k \wedge H_\ell \to (H \wedge H)_{k+\ell}$ must therefore make

$$\begin{tikzcd}
H_k \wedge H_\ell \wedge S^1 \arrow[r, "1\wedge\sigma"] \arrow[d, "\iota_{k,\ell}\wedge 1"'] & H_k \wedge H_{\ell+1} \arrow[d, "\iota_{k,\ell+1}"] \\
(H \wedge H)_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & (H \wedge H)_{k+\ell+1}
\end{tikzcd}$$

commute up to homotopy, and

$$\begin{tikzcd}
H_k \wedge H_\ell \wedge S^1 \arrow[r, "1\wedge\gamma", "\cong"'] \arrow[d, "\iota_{k,\ell}\wedge 1"'] & H_k \wedge S^1 \wedge H_\ell \arrow[r, "\sigma\wedge 1"] & H_{k+1} \wedge H_\ell \arrow[d, "\iota_{k+1,\ell}"] \\
(H \wedge H)_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & (H \wedge H)_{k+\ell+1} \arrow[r, "="] & (H \wedge H)_{k+1+\ell}
\end{tikzcd}$$

commute up to the sign $(-1)^\ell$. More generally, given spectra D, E and F, if a pairing of reduced cohomology theories $D^k(X) \otimes E^\ell(Y) \longrightarrow F^{k+\ell}(X \wedge Y)$ is to be represented by an external pairing

$$\wedge\colon D^k(X) \otimes E^\ell(Y) \longrightarrow (D \wedge E)^{k+\ell}(X \wedge Y)$$

followed the homomorphism $\mu_*\colon (D \wedge E)^{k+\ell}(X \wedge Y) \longrightarrow F^{k+\ell}(X \wedge Y)$ induced by a map of spectra $\mu\colon D \wedge E \to F$, then the structure maps $\sigma\colon \Sigma(D \wedge E)_n \to (D \wedge E)_{n+1}$ should apparently make

$$\begin{tikzcd}
D_k \wedge E_\ell \wedge S^1 \arrow[r, "1\wedge\sigma"] \arrow[d, "\iota_{k,\ell}\wedge 1"'] & D_k \wedge E_{\ell+1} \arrow[d, "\iota_{k,\ell+1}"] \\
(D \wedge E)_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & (D \wedge E)_{k+\ell+1}
\end{tikzcd}$$

commute up to homotopy, and

$$\begin{tikzcd} D_k \wedge E_\ell \wedge S^1 \arrow[r, "1\wedge\gamma", "\cong"'] \arrow[d, "\iota_{k,\ell}\wedge 1"'] & D_k \wedge S^1 \wedge E_\ell \arrow[r, "\sigma\wedge 1"] & D_{k+1} \wedge E_\ell \arrow[d, "\iota_{k+1,\ell}"] \\ (D\wedge E)_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & (D\wedge E)_{k+\ell+1} \arrow[r, "="] & (D\wedge E)_{k+1+\ell} \end{tikzcd}$$

commute up to the sign $(-1)^\ell$. Adams achieves something like this by letting

$$(D \wedge E)_n = \operatorname*{hocolim}_{k+\ell\le n} (D_k \wedge E_\ell \wedge S^{n-k-\ell})$$

be carefully gluing together from the spaces $D_k \wedge E_\ell$ with $k + \ell = n$, together with several other spaces. (I am not sure if the definition can be formalized as a homotopy colimit, but it is close.)

We might reduce to the case $k = \ell = n \ge 0$ by representing $\wedge\colon D^k(X)\otimes E^\ell(Y) \to (D\wedge E)^{k+\ell}(X\wedge Y)$ as the composite

$$D^k(X)\otimes E^\ell(Y) \cong D^n(\Sigma^{n-k}X)\otimes E^n(\Sigma^{n-\ell}Y) \longrightarrow (D\wedge E)^{2n}(\Sigma^{n-k}X \wedge \Sigma^{n-\ell}Y) \cong (D\wedge E)^{k+\ell}(X\wedge Y)\,,$$

for n sufficiently large. We could then try define $(D\wedge E)_{2n} = D_n \wedge E_n$ and $(D\wedge E)_{2n+1} = D_n \wedge E_n \wedge S^1$, with structure map

$$\sigma\colon (D\wedge E)_{2n} \wedge S^1 \wedge S^1 \longrightarrow (D\wedge E)_{2n+2}$$

given by the composite

$$D_n \wedge E_n \wedge S^1 \wedge S^1 \xrightarrow{1\wedge\gamma\wedge 1} D_n \wedge S^1 \wedge E_n \wedge S^1 \xrightarrow{\sigma\wedge\sigma} D_{n+1} \wedge E_{n+1}\,.$$

This pairing is not quite associative, since $((D\wedge E)\wedge F)_{4n} = D_n \wedge E_n \wedge F_{2n}$, while $(D\wedge(E\wedge F))_{4n} = D_{2n}\wedge E_n \wedge F_n$. It is also not quite left or right unital, since $(S\wedge E)_{2n} = S^n \wedge E_n$ and $(E\wedge S)_{2n} = E_n \wedge S^n$, where $S = \{n \mapsto S^n\}$ is the sphere spectrum. Commutativity also fails, since the sequence of maps $\gamma_{2n}\colon (D\wedge E)_{2n} \to (E\wedge D)_{2n}$ given by $\gamma\colon D_n \wedge E_n \to E_n \wedge D_n$ only commutes up to the sign (-1) with the structure maps, hence does not induce a (strict) spectrum map $\gamma\colon D\wedge E \to E\wedge D$. The diagram

$$\begin{tikzcd} D_n \wedge E_n \wedge S^1 \wedge S^1 \arrow[r, "1\wedge\gamma\wedge 1"] \arrow[d, "\gamma\wedge 1\wedge 1"'] & D_n \wedge S^1 \wedge E_n \wedge S^1 \arrow[r, "\sigma\wedge\sigma"] & D_{n+1} \wedge E_{n+1} \arrow[d, "\gamma"] \\ E_n \wedge D_n \wedge S^1 \wedge S^1 \arrow[r, "1\wedge\gamma\wedge 1"] & E_n \wedge S^1 \wedge D_n \wedge S^1 \arrow[r, "\sigma\wedge\sigma"] & E_{n+1} \wedge D_{n+1} \end{tikzcd}$$

only becomes (homotopy) commutative upon replacing the left hand vertical map with $\gamma \wedge \gamma$, where $\gamma\colon S^1 \wedge S^1 \to S^1 \wedge S^1$ has degree (-1). Early attempts relaxed the notion of a spectrum map to only ask for compatibility up to homotopy with the structure maps, and restricting attention to even integers n in the notation above, since $\gamma\colon S^2 \wedge S^2 \to S^2 \wedge S^2$ has degree $+1$, even if the map is not equal to the identity.

To summarize, it is possible to define a smash product $D \wedge E$ of two sequential spectra, but it does not make $(\mathrm{Sp}, \wedge, S)$ symmetric monoidal. It only achieves this after passage to the homotopy category, so that $(\mathrm{Sp}[\mathscr{W}^{-1}], \wedge, S)$ is symmetric monoidal.

4 Diagram spaces and diagram spectra

Building on an idea due to Jeff Smith, and realized in the case of symmetric spectra in Hovey–Shipley–Smith (1999), we follow Mandell–May–Schwede–Shipley (2001) and define an orthogonal spectrum X to be a sequence of based $O(n)$-spaces X_n, for each $n \ge 0$, equipped with structure maps $\sigma\colon X_n \wedge S^1 \to X_{n+1}$, such that the ℓ-fold composite

$$\sigma^\ell\colon X_k \wedge S^\ell \longrightarrow X_{k+\ell}$$

is $O(k)\times O(\ell)$-equivariant, for each $k, \ell \ge 0$. To put this definition in a context, we shall interpret it in terms of an underlying category of orthogonal sequences.

((Is it better to say "orthogonal space" than "orthogonal sequence"?))

The category of sequential spectra can be considered as the category of (right) S-modules, for a monoid S in the category of sequences of based spaces. This is a symmetric monoidal category, but the monoid S is not commutative, so the monoidal pairing does not induce a pairing of S-modules. By analogy, a ring R is a monoid in the category of abelian groups, but the tensor product $\otimes$ of abelian groups only induces a tensor product $\otimes_R$ of right R-modules if R is commutative:

$$M \otimes R \otimes N \underset{1\otimes\sigma'}{\overset{\sigma\otimes 1}{\rightrightarrows}} M \otimes N \xrightarrow{\pi} M \otimes_R N$$

Here $\sigma\colon M \otimes R \to M$ is the right R-module structure map of M, $\sigma' = \sigma\gamma\colon R \otimes N \cong N \otimes R \to N$ is the right R-action on N turned into a left R-action (which works when R is commutative), and π is the coequalizer of $\sigma \otimes 1$ and $1 \otimes \sigma'$. This explains why there is no easy definition of a smash product pairing of sequential spectra.

The category of orthogonal spectra can be considered as the category of (right) S-modules in a category of orthogonal sequences, i.e., sequences of based spaces $\{n \mapsto X_n\}$, where X_n comes equipped with a continuous, basepoint-preserving $O(n)$-action, for each $n \ge 0$. The latter category has a symmetric monoidal structure for which the sequence $S = \{n \mapsto S^n\}$ is a commutative monoid. Hence the symmetric monoidal pairing of orthogonal sequences induces a symmetric monoidal pairing of orthogonal spectra. This is the smash product pairing of orthogonal spectra.

4.1 Sequences of spaces

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of non-negative integers, viewed as a category with only identity morphisms. The addition $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ sending (k, ℓ) to $k + \ell$ and the zero object $0 \in \mathbb{N}$ define a symmetric monoidal structure on $\mathbb{N}$. The coherent isomorphisms $\alpha\colon k + (\ell + m) \cong (k + \ell) + m$, $\lambda\colon 0 + \ell \cong \ell$, $\rho\colon \ell + 0 \cong \ell$ and $\gamma\colon k + \ell \cong \ell + k$ are all identity morphisms.

By an $\mathbb{N}$-space, or a sequence of spaces, we mean a functor $X\colon \mathbb{N} \to \mathscr{T}$. Writing $X_n = X(n)$, this is just a sequence $\{n \mapsto X_n\}$ of based spaces X_n for $n \ge 0$. By a map $f\colon X \to Y$ of $\mathbb{N}$-spaces, we mean a natural transformation of functors, i.e., a sequence of basepoint-preserving maps $f_n\colon X_n \to Y_n$. Let $\mathscr{T}^{\mathbb{N}}$ denote the topological category of $\mathbb{N}$-spaces. Each morphism space $\mathscr{T}^{\mathbb{N}}(X, Y)$ is based at the constant map $X \to Y$.

The category of $\mathbb{N}$-spaces has all small colimits and limits, created levelwise: For each small diagram $\alpha \mapsto X_\alpha$ of $\mathbb{N}$-spaces, we have

$$(\operatorname*{colim}_\alpha X_\alpha)_n = \operatorname*{colim}_\alpha (X_\alpha)_n \quad\text{and}\quad (\lim_\alpha X_\alpha)_n = \lim_\alpha (X_\alpha)_n$$

for each $n \ge 0$.

It is tensored and cotensored over $\mathscr{T}$, and these structures are again created levelwise: For each based space T and $\mathbb{N}$-spaces X and Y the $\mathbb{N}$-spaces $X \wedge T$ and $F(T, Y)$ are defined by

$$(X \wedge T) = X_n \wedge T \quad\text{and}\quad F(T, Y)_n = F(T, Y_n)$$

for each $n \ge 0$. There are natural homeomorphisms

$$\mathscr{T}(T, \mathscr{T}^{\mathbb{N}}(X, Y)) \cong \mathscr{T}^{\mathbb{N}}(X \wedge T, Y) \cong \mathscr{T}^{\mathbb{N}}(X, F(T, Y))\,.$$

The smash product $\wedge\colon \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ and the sum $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ give rise to a pairing

$$\otimes\colon \mathscr{T}^{\mathbb{N}} \times \mathscr{T}^{\mathbb{N}} \longrightarrow \mathscr{T}^{\mathbb{N}}$$

of $\mathbb{N}$-spaces, mapping X and Y to the $\mathbb{N}$-space $X \otimes Y$ with

$$(X \otimes Y)_n = \bigvee_{k+\ell=n} X_k \wedge Y_\ell\,.$$

Maps $X \otimes Y \to Z$ in $\mathscr{T}^{\mathbb{N}}$ are in one-to-one correspondence with collections of basepoint-preserving maps

$$X_k \wedge Y_\ell \longrightarrow Z_{k+\ell} \tag{1}$$

for all $k,\ell \geq 0$. This is an instance of the Day convolution product, created by a left Kan extension along $+\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$.

Let U be the (unit) sequence with $U_0 = S^0$ and $U_n = *$ for $n>0$. Then there are evident isomorphisms $\alpha\colon X\otimes(Y\otimes Z)\cong(X\otimes Y)\otimes Z$, $\lambda\colon U\otimes Y\cong Y$, $\rho\colon Y\otimes U\cong Y$ and $\gamma\colon X\otimes Y\cong Y\otimes X$. In the last case,

$$\gamma_n\colon \bigvee_{k+\ell=n} X_k\wedge Y_\ell \longrightarrow \bigvee_{k+\ell=n} Y_k\wedge X_\ell$$

maps $(x,y)\in X_k\wedge Y_\ell$ to $(y,x)\in Y_\ell\wedge X_k$, which is admissible since $\ell+k=k+\ell$. These isomorphism are coherent, so $(\mathscr{T}^{\mathbb{N}},\otimes,U)$ is a symmetric monoidal category.

The category of $\mathbb{N}$-spaces is also closed. The internal Hom object functor

$$\mathrm{Hom}\colon (\mathscr{T}^{\mathbb{N}})^{op}\times\mathscr{T}^{\mathbb{N}}\longrightarrow\mathscr{T}^{\mathbb{N}}$$

takes Y and Z to the $\mathbb{N}$-space $\mathrm{Hom}(Y,Z)$ with

$$\mathrm{Hom}(Y,Z)_k = \prod_{k+\ell=n}\mathscr{T}(Y_\ell,Z_n) = \mathscr{T}^{\mathbb{N}}(Y,\mathrm{sh}^k Z)\,.$$

Here $\mathrm{sh}^k Z$ is the (left) k-shifted $\mathbb{N}$-space given by $(\mathrm{sh}^k Z)_\ell = Z_{k+\ell}$. The functor $\mathrm{Hom}(Y,-)$ is right adjoint to the functor $(-)\otimes Y$:

$$\mathscr{T}^{\mathbb{N}}(X\otimes Y,Z)\cong\mathscr{T}^{\mathbb{N}}(X,\mathrm{Hom}(Y,Z))\,,$$

with maps $X\otimes Y\to Z$ and $X\to\mathrm{Hom}(Y,Z)$ corresponding to collections of maps as in (1) above.

4.2 Sequential spectra as right S-modules

Consider the sphere sequence $S=\{n\mapsto S^n\}$. Here we may assume that $S^n = S^1\wedge\cdots\wedge S^1$, with $n\geq 0$ copies of S^1, so that there are compatible preferred homeomorphisms $S^k\wedge S^\ell\cong S^{k+\ell}$. Let $\mu\colon S\otimes S\to S$ be the map of $\mathbb{N}$-spaces given in degree n by the wedge sum

$$\mu_n\colon \bigvee_{k+\ell=n} S^k\wedge S^\ell\longrightarrow S^n$$

of these homeomorphisms. Let $\eta\colon U\to S$ be the map of $\mathbb{N}$-space given by the identity $\eta_0\colon S^0\to S^0$ in degree 0, and by the inclusion of the base point $\eta_n\colon *\to S^n$ for each $n>0$. Then (S,μ,η) is an associative monoid in $(\mathscr{T}^{\mathbb{N}},\otimes,U)$. However, it is not commutative, since the diagram

$$\begin{tikzcd}
\bigvee_{k+\ell=n} S^k\wedge S^\ell \arrow[rr, "\gamma_n"] \arrow[dr, "\mu_n"'] & & \bigvee_{k+\ell=n} S^k\wedge S^\ell \arrow[dl, "\mu_n"] \\
& S^n &
\end{tikzcd}$$

does not commute for $n\geq 2$. For instance, when $k=\ell=1$, a point $(s,t)\in S^1\wedge S^1$ is mapped by μ_2 to $(s,t)\in S^2$, but by $\mu_2\gamma_2$ to $(t,s)\in S^2$. This is precisely the source of the difficulties discussed for pairings of spectra.

What is a right S-module in $(\mathscr{T}^{\mathbb{N}},\otimes,U)$? It is an $\mathbb{N}$-space X with a map $\sigma\colon X\otimes S\to X$ of $\mathbb{N}$-spaces, such that the diagrams

$$\begin{tikzcd}
X\otimes S\otimes S \arrow[r, "\sigma\otimes 1"] \arrow[d, "1\otimes\mu"'] & X\otimes S \arrow[d, "\sigma"] \\
X\otimes S \arrow[r, "\sigma"] & X
\end{tikzcd}$$

(where we have suppressed the isomorphism α) and

$$\begin{tikzcd}
X\otimes S \arrow[d, "\sigma"'] & X\otimes U \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\
X &
\end{tikzcd}$$

commute. In other words, it is a sequence of spaces $\{n \mapsto X_n\}$ and maps

$$\sigma_n \colon \bigvee_{k+\ell=n} X_k \wedge S^\ell \longrightarrow X_n$$

such that the two maps

$$\sigma_n \circ (1 \otimes \mu)_n \ , \ \sigma_n \circ (\sigma \otimes 1)_n \colon \bigvee_{k+\ell+m=n} X_k \wedge S^\ell \wedge S^m \longrightarrow X_n$$

are equal, and the component $X_n \wedge S^0 \to X_n$ of σ_n is the standard identification. It follows that the component $X_k \wedge S^\ell \to X_n$ of σ_n is the composite

$$X_k \wedge S^\ell \xrightarrow{\sigma\wedge 1} X_{k+1} \wedge S^{\ell-1} \xrightarrow{\sigma\wedge 1} \dots \xrightarrow{\sigma} X_{k+\ell} \wedge S^0 = X_n$$

of ℓ suspended copies of the components $\sigma \colon X_m \wedge S^1 \to X_{m+1}$ of σ_m, for $k \le m < n$. Hence the sequence of spaces X_n and the sequence of maps $\sigma \colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$, for $n \ge 0$, define a sequential spectrum.

Conversely, from a sequential spectrum X with structure maps σ, we can recover the right S-module action $\sigma \colon X \otimes S \to X$ by the same formula. In degree n the component $X_k \wedge S^\ell \to X_n$ where $k + \ell = n$ is the ℓ-fold composite

$$\sigma^\ell = \sigma \circ \dots \circ \Sigma^{\ell-2}(\sigma) \circ \Sigma^{\ell-1}(\sigma) \colon X_k \wedge S^\ell \longrightarrow X_n \, .$$

A map $f \colon X \to Y$ of right S-modules corresponds precisely to a map $f \colon X \to Y$ of sequential spectra. Hence the category of right S-modules in $\mathscr{T}^{\mathbb{N}}$ is equivalent to the category $\mathrm{Sp}^{\mathbb{N}}$ of sequential spectra.

4.3 Orthogonal sequences

Let $\mathbb{O}$ be the topological category with objects non-negative integers $\{0, 1, 2, \dots\}$, and with morphism spaces given by $\mathbb{O}(n, n) = O(n)$ and $\mathbb{O}(m, n) = \emptyset$ for $m \ne n$. Here $O(n)$ denotes the topological group of orthogonal $n \times n$ matrices. The continuous composition in $\mathbb{O}$ is given by matrix multiplication: $A \circ B = AB$ for $A, B \in O(n)$. The identity morphism of n is given by the identity matrix $I = I_n$. We may think of the object n in $\mathbb{O}$ as a label for the real inner-product space $\mathbb{R}^n$, with the standard Euclidean dot product, in which case the morphisms $A \in \mathbb{O}(n, n) = O(n)$ are thought of as the isometries $A \colon \mathbb{R}^n \to \mathbb{R}^n$, mapping $v \in \mathbb{R}^n$ to $Av \in \mathbb{R}^n$.

The sum $+ \colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ sending the pair of objects (k, ℓ) to the object $k + \ell$, and sending a pair of morphisms $(A, B) \in O(k) \otimes O(\ell)$ to the block sum

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in O(k + \ell) \, ,$$

is part of a symmetric monoidal structure on $\mathbb{O}$. The zero object is 0. The coherent natural isomorphisms $\alpha \colon k+(\ell+m) \cong (k+\ell)+m$, $\lambda \colon 0+\ell \cong \ell$ and $\rho \colon \ell+0 \cong \ell$ are given by the identity matrices in $O(k+\ell+m)$, $O(\ell)$ and $O(\ell)$, respectively. These correspond to the standard identifications $\mathbb{R}^k \oplus (\mathbb{R}^\ell \oplus \mathbb{R}^m) \cong (\mathbb{R}^k \oplus \mathbb{R}^\ell) \oplus \mathbb{R}^m$, $\mathbb{R}^0 \oplus \mathbb{R}^\ell \cong \mathbb{R}^\ell$ and $\mathbb{R}^\ell \oplus \mathbb{R}^0 \cong \mathbb{R}^\ell$.

However, the symmetry isomorphism $\gamma \colon k+\ell \cong \ell+k$ is not the identity. It is given by the permutation matrix

$$\chi_{k,\ell} = \begin{pmatrix} 0 & I_\ell \\ I_k & 0 \end{pmatrix} \in O(k + \ell) \, ,$$

corresponding to the twist isomorphism $\gamma \colon \mathbb{R}^k \oplus \mathbb{R}^\ell \cong \mathbb{R}^\ell \oplus \mathbb{R}^k$. It is natural as a transformation from the functor $(k, \ell) \mapsto k + \ell$ to the functor $(k, \ell) \mapsto \ell + k$, because of the relation

$$\begin{pmatrix} 0 & I_\ell \\ I_k & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & I_\ell \\ I_k & 0 \end{pmatrix} \, .$$

The inclusion functor $\mathbb{N} \to \mathbb{O}$ is strong monoidal (to be defined when we need it), but is not symmetric monoidal, since the symmetry isomorphism in $\mathbb{N}$ is not compatible under ι with the symmetry isomorphism in $\mathbb{O}$. (This seems to be incorrectly stated near equation (8.1) in Mandell–May–Schwede–Shipley (2001).)

By an $\mathbb{O}$-space, or an orthogonal sequence of spaces, we mean a continuous functor $X\colon \mathbb{O} \to \mathscr{T}$. Writing $X_n = X(n)$, this is a sequence $\{n \mapsto X_n\}$ of based spaces X_n, equipped with a continuous, basepoint-preserving left $O(n)$-action $O(n)_+ \wedge X_n \to X_n$, for each $n \geq 0$. Writing Ax for the action of $A \in O(n)$ on $x \in X_n$, we require that $A(Bx) = (AB)x$ and $Ix = x$.

By a map $f\colon X \to Y$ of orthogonal sequences (of spaces), we mean a natural transformation of continuous functors, i.e., a sequence of basepoint-preserving $O(n)$-equivariant maps $f_n\colon X_n \to Y_n$, for each $n \geq 0$. Hence $Af_n(x) = f_n(Ax)$ for each $x \in X_n$.

Let $\mathscr{T}^{\mathbb{O}}$ denote the topological category of orthogonal sequences. Each morphism space $\mathscr{T}^{\mathbb{O}}(X, Y)$ is based at the constant map.

The category of orthogonal sequences has all small colimits and limits, created levelwise: For each small diagram $\alpha \mapsto X_\alpha$ of orthogonal sequences, we have

$$(\operatorname*{colim}_\alpha X_\alpha)_n = \operatorname*{colim}_\alpha (X_\alpha)_n \quad \text{and} \quad (\lim_\alpha X_\alpha)_n = \lim_\alpha (X_\alpha)_n$$

for each $n \geq 0$. The right hand sides are both formed in the category of based $O(n)$-spaces.

The category $\mathscr{T}^{\mathbb{O}}$ is tensored and cotensored over $\mathscr{T}$, and these structures are again created levelwise: For each based space T and orthogonal sequences X and Y the orthogonal sequences $X \wedge T$ and $F(T, Y)$ are defined by

$$(X \wedge T)_n = X_n \wedge T \quad \text{and} \quad F(T, Y)_n = F(T, Y_n)$$

for each $n \geq 0$. The right hand sides are both formed in the category of based $O(n)$-spaces. For $A \in O(n)$, $x \in X_n$ and $t \in T$ we set $A(x \wedge t) = Ax \wedge t$. For $f\colon T \to Y_n$ we set $(Af)(t) = A(f(t))$. There are natural homeomorphisms

$$\mathscr{T}(T, \mathscr{T}^{\mathbb{O}}(X, Y)) \cong \mathscr{T}^{\mathbb{O}}(X \wedge T, Y) \cong \mathscr{T}^{\mathbb{O}}(X, F(T, Y)) .$$

We can also define an orthogonal sequence $T \wedge X$ with $(T \wedge X)_n = T \wedge X_n$, and there is a natural isomorphism $\gamma\colon T \wedge X \to X \wedge T$ given by $\gamma\colon T \wedge X_n \cong X_n \wedge T$ at each level n.

Remark 4.1. Restriction along $\mathbb{N} \to \mathbb{O}$ defines a forgetful functor $\mathbb{U}\colon \mathscr{T}^{\mathbb{O}} \to \mathscr{T}^{\mathbb{N}}$ from orthogonal sequences to sequences of spaces. It preserves small colimits and limits, as well as tensors and cotensors with based spaces. It does not preserve the symmetric monoidal pairing $\otimes$ and closed structure Hom that we are about to define.

The smash product $\wedge\colon \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ and the sum $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ give rise to a pairing

$$\otimes\colon \mathscr{T}^{\mathbb{O}} \times \mathscr{T}^{\mathbb{O}} \longrightarrow \mathscr{T}^{\mathbb{O}}$$

of orthogonal sequences, mapping X and Y to the orthogonal sequence $X \otimes Y$ with

$$(X \otimes Y)_n = \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell .$$

Here the balanced product over $O(k) \times O(\ell)$ is the orbit space of $O(n)_+ \wedge X_k \wedge Y_\ell$ where

$$A \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \wedge x \wedge y$$

is identified with

$$A \wedge Bx \wedge Cy$$

for $A \in O(n)$, $B \in O(k)$, $C \in O(\ell)$, $n = k + \ell$, $x \in X_k$ and $y \in Y_\ell$. The $O(n)$-action on $(X \otimes Y)_n$ is from the left on the copy of $O(n)$: We set $A(B \wedge x \wedge y) = AB \wedge x \wedge y$ for $A, B \in O(n)$, $x \in X_k$ and $y \in Y_\ell$. In other words,

$$O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell$$

is the based $O(n)$-space obtained by inducing up the based $O(k) \times O(\ell)$-space structure on $X_k \wedge Y_\ell$ to an $O(n)$-space structure along the direct sum embedding $O(k) \times O(\ell) \to O(k + \ell) = O(n)$.

Maps $X \otimes Y \to Z$ in $\mathscr{T}^{\mathbb{O}}$ are in one-to-one correspondence with collections of $O(k) \times O(\ell)$-equivariant basepoint-preserving maps

$$X_k \wedge Y_\ell \longrightarrow Z_{k+\ell} \tag{2}$$

for all $k, \ell \geq 0$. Here $X_k \wedge Y_\ell$ is the $O(k) \times O(\ell)$-space given as an external product of the $O(k)$-space X_k and the $O(\ell)$-space Y_ℓ, while the $O(k+\ell)$-space $Z_{k+\ell}$ is treated as an $O(k) \times O(\ell)$-space through the homomorphism $h\colon O(k) \times O(\ell) \to O(k+\ell)$. Alternatively we might say that $X_k \wedge Y_\ell \to Z_{k+\ell}$ is h-equivariant, or $O(k) \times O(\ell) \to O(k+\ell)$-equivariant.

The tensor product $X \otimes Y$ is another instance of Day's convolution product, and can be viewed as the left Kan extension of the external product $\wedge \circ (X \times Y)\colon \mathbb{O} \times \mathbb{O} \to \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ along $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$:

$$\begin{tikzcd} \mathbb{O} \times \mathbb{O} \arrow[r, "X \times Y"] \arrow[d, "+"'] & \mathscr{T} \times \mathscr{T} \arrow[d, "\wedge"] \\ \mathbb{O} \arrow[r, "\otimes"'] & \mathscr{T} \end{tikzcd}$$

This point of view can be expressed by the formula

$$(X \otimes Y)_n = \operatorname*{colim}_{k,\ell,k+\ell \to n} X_k \wedge Y_\ell$$

where $(k, \ell, k+\ell \to n)$ ranges over the left fiber category $+/n$ of $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ at n in $\mathbb{O}$.

Let U be the (unit) orthogonal sequence with $U_0 = S^0$ and $U_n = *$ for $n > 0$. There is only one choice of $O(n)$-actions. There is a natural associativity isomorphism $\alpha\colon X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$, given by the $O(n)$-equivariant homeomorphism

$$\begin{aligned} \alpha_n\colon \bigvee_{k+q=n} O(n)_+ \underset{O(k)\times O(q)}{\wedge} X_k \wedge \Big(\bigvee_{\ell+m=q} O(q)_+ \underset{O(\ell)\times O(m)}{\wedge} Y_\ell \wedge Z_m \Big) \\ \cong \bigvee_{k+\ell+m=n} O(n)_+ \underset{O(k)\times O(\ell)\times O(m)}{\wedge} X_k \wedge Y_\ell \wedge Z_k \\ \cong \bigvee_{p+m=n} O(n)_+ \underset{O(p)\times O(m)}{\wedge} \Big(\bigvee_{k+\ell=p} O(p)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell \Big) \wedge Z_m \,, \end{aligned}$$

for each $n \geq 0$. There are natural left and right unitality isomorphisms $\lambda\colon U \otimes Y \cong Y$ and $\rho\colon Y \otimes U \cong Y$ given by

$$\lambda_n\colon O(n)_+ \underset{O(0)\times O(n)}{\wedge} S^0 \wedge Y_n \cong Y_n \quad \text{and} \quad \rho_n\colon O(n)_+ \underset{O(n)\times O(0)}{\wedge} Y_n \wedge S^0 \cong Y_n \,,$$

respectively. Here $O(0)$ is the trivial group. Less obviously, there is a natural symmetry isomorphism $\gamma\colon X \wedge Y \cong Y \wedge X$ given by the $O(n)$-equivariant homeomorphism

$$\gamma_n\colon \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell \xrightarrow{\cong} \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} Y_k \wedge X_\ell$$

that maps

$$A\chi_{k,\ell} \wedge x \wedge y \in O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell$$

at the left hand side to

$$A \wedge y \wedge x \in O(n)_+ \underset{O(\ell)\times O(k)}{\wedge} Y_\ell \wedge X_k$$

at the right hand side. Recall that $\chi_{k,\ell} \in O(n)$ is the block permutation matrix that corresponds to $\gamma\colon \mathbb{R}^k \oplus \mathbb{R}^\ell \cong \mathbb{R}^\ell \oplus \mathbb{R}^k$. This gives a well-defined map, because

$$A \begin{pmatrix} C & 0 \\ 0 & B \end{pmatrix} \chi_{k,\ell} \wedge x \wedge y = A\chi_{k,\ell} \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \wedge x \wedge y \equiv A\chi_{k,\ell} \wedge Bx \wedge Cy$$

is mapped to

$$A \begin{pmatrix} C & 0 \\ 0 & B \end{pmatrix} \wedge y \wedge x \equiv A \wedge Cy \wedge Bx \,.$$

Here $x \in X_k$, $y \in Y_\ell$, $A \in O(n)$, $B \in O(k)$, $C \in O(\ell)$ and $k + \ell = n$. Equivalently, γ_n maps

$$A \wedge x \wedge y \in O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell$$

at the left hand side to

$$A\chi_{\ell,k} \wedge y \wedge x \in O(n)_+ \underset{O(\ell)\times O(k)}{\wedge} Y_\ell \wedge X_k$$

at the right hand side, since $\chi_{\ell,k} = \chi_{k,\ell}^{-1}$.

Remark 4.2. Sending $A \wedge x \wedge y$ to $A \wedge y \wedge x$, without the factor $\chi_{k,\ell}$, would not give a well-defined map of orthogonal sequences.

These isomorphisms are coherent, so $(\mathscr{T}^{\mathbb{O}}, \otimes, U)$ is a symmetric monoidal category.

The category of orthogonal sequences is also closed. The internal Hom object functor

$$\mathrm{Hom}\colon (\mathscr{T}^{\mathbb{O}})^{op} \times \mathscr{T}^{\mathbb{O}} \longrightarrow \mathscr{T}^{\mathbb{O}}$$

takes a pair of orthogonal sequences Y and Z to the orthogonal sequence $\mathrm{Hom}(Y, Z)$ with

$$\mathrm{Hom}(Y, Z)_k = \prod_{k+\ell=n} \mathscr{T}(Y_\ell, Z_n)^{O(\ell)} = \mathscr{T}^{\mathbb{O}}(Y, \mathrm{sh}^k Z)\,.$$

Here $\mathscr{T}(Y_\ell, Z_n)^{O(\ell)}$ is the space of $O(\ell)$-equivariant maps $Y_\ell \to Z_n$, where $O(\ell)$ acts on Z_n through the (right) inclusion $O(\ell) \to O(n)$ mapping $B \in O(\ell)$ to $\begin{pmatrix} I_k & 0 \\ 0 & B \end{pmatrix} \in O(n)$, and $\mathrm{sh}^k Z$ is the (left) k-shifted $\mathbb{O}$-space given by $(\mathrm{sh}^k Z)_\ell = Z_{k+\ell}$ for each $\ell \geq 0$, with the $O(\ell)$-action just indicated. The group $O(k)$ acts on each factor $\mathscr{T}(Y_\ell, Z_n)^{O(\ell)}$ through its action on Z_n, given by the (left) inclusion $O(k) \to O(n)$ mapping A to $\begin{pmatrix} A & 0 \\ 0 & I_\ell \end{pmatrix}$.

The functor $\mathrm{Hom}(Y, -)$ is right adjoint to the functor $(-) \otimes Y$:

$$\mathscr{T}^{\mathbb{O}}(X \otimes Y, Z) \cong \mathscr{T}^{\mathbb{O}}(X, \mathrm{Hom}(Y, Z))\,,$$

with maps $X \otimes Y \to Z$ and $X \to \mathrm{Hom}(Y, Z)$ corresponding to collections of $O(k) \times O(\ell)$-equivariant maps as in (2) above. Furthermore, this adjunction lifts to an isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

of orthogonal sequences. (Give proof?)

Examples: For each $n \geq 0$, let $\mathrm{Ev}_n\colon \mathscr{T}^{\mathbb{O}} \to \mathscr{T}$ be the (continuous) evaluation functor taking X to $\mathrm{Ev}_n X = X_n$, forgetting the spaces X_m for $m \neq n$, and forgetting the $O(n)$-action on X_n. It has a left adjoint, the free orthogonal sequence functor $G_n\colon \mathscr{T} \to \mathscr{T}^{\mathbb{O}}$, taking T to $G_n T$ with

$$(G_n T)_n = O(n)_+ \wedge T$$

with $O(n)$ acting by $A(B \wedge t) = AB \wedge t$, and $(G_n T)_m = *$ for $m \neq n$. There is a natural homeomorphism

$$\mathscr{T}^{\mathbb{O}}(G_n T, X) \cong \mathscr{T}(T, \mathrm{Ev}_n X)\,.$$

The free functors interact with the closed symmetric monoidal structure as follows: There are natural isomorphisms

$$G_k T \otimes G_\ell V \cong G_{k+\ell}(T \wedge V) \cong (G_{k+\ell} T) \wedge V$$

and

$$\mathrm{Hom}(G_\ell V, Z) \cong \mathrm{sh}^\ell F(V, Z) \cong F(V, \mathrm{sh}^\ell Z)$$

for $k, \ell \geq 0$, $T, V \in \mathscr{T}$ and $Z \in \mathscr{T}^{\mathbb{O}}$. As a special case, $G_0 S^0 = U$ is the unit orthogonal sequence.

4.4 Orthogonal spectra as S-modules

The orthogonal group $O(n)$ acts on the n-sphere S^n by way of its natural action on $\mathbb{R}^n$, where S^n is viewed as the one-point compactification of $\mathbb{R}^n$. The action is continuous and preserves the base point at infinity. The sphere orthogonal sequence S is the sequence $\{n \mapsto S^n\}$ of n-spheres with these actions, for all $n \geq 0$.

The standard identification $\mathbb{R}^k \oplus \mathbb{R}^\ell \cong \mathbb{R}^{k+\ell}$ extends to a homeomorphism $S^k \wedge S^\ell \cong S^{k+\ell}$, which is $O(k) \times O(\ell) \to O(k+\ell)$-equivariant with respect to the given actions. We write $x \wedge y$ for the image of $x \in S^k$ and $y \in S^\ell$ in $S^{k+\ell}$. Let $\mu\colon S \otimes S \to S$ be the map of orthogonal sequences given by these $O(k) \times O(\ell)$-equivariant homeomorphisms

$$S^k \wedge S^\ell \xrightarrow{\cong} S^{k+\ell},$$

or equivalently, by the sequence of $O(n)$-maps

$$\mu_n\colon \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} S^k \wedge S^\ell \longrightarrow S^n$$

that take $A \wedge x \wedge y$ to $A(x \wedge y)$, for $A \in O(n)$, $x \in S^k$ and $y \in S^\ell$, for $n \geq 0$. Let $\eta\colon U \to S$ be the map of orthogonal sequences that is the identity on S^0 at level $n = 0$, and the constant map $* \to S^n$ for $n > 0$.

Proposition 4.3. *(S, μ, η) is a commutative monoid in the symmetric monoidal category $(\mathscr{T}^{\mathbb{O}}, \otimes, U)$ of orthogonal sequences.*

Proof. It is evident that $\mu(1 \wedge \mu) = \mu(\mu \wedge 1)\colon S \otimes S \otimes S \to S$, and that $\mu(\eta \wedge 1) = id = \mu(1 \wedge \eta)\colon S \to S$. To check that S is commutative, we must verify that the diagram

$$\begin{tikzcd}
\bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} S^k \wedge S^\ell \arrow[rr, "\gamma_n"] \arrow[dr, "\mu_n"'] & & \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} S^k \wedge S^\ell \arrow[dl, "\mu_n"] \\
& S^n &
\end{tikzcd}$$

commutes, for each $n \geq 0$. Here γ_n maps

$$A\chi_{k,\ell} \wedge x \wedge y \in O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} S^k \wedge S^\ell$$

to

$$A \wedge y \wedge x \in O(n)_+ \underset{O(\ell)\times O(k)}{\wedge} S^\ell \wedge S^k,$$

and μ_n maps this to

$$A(y \wedge x) \in S^n.$$

Along the left hand side, μ_n maps $A\chi_{k,\ell} \wedge x \wedge y$ to $A\chi_{k,\ell}(x \wedge y)$. This is also equal to $A(y \wedge x)$, since $\chi_{k,\ell} \in O(n)$ acts on S^n as the twist map $\gamma\colon S^k \wedge S^\ell \to S^\ell \wedge S^k$, taking $x \wedge y$ to $y \wedge x$. $\square$

Definition 4.4. An orthogonal spectrum X is a right S-module in orthogonal sequences. In other words, X is an orthogonal sequence equipped with a map $\sigma\colon X \otimes S \to X$ of orthogonal sequences, such that the diagrams

$$\begin{tikzcd}
X \otimes S \otimes S \arrow[r, "\sigma\otimes 1"] \arrow[d, "1\otimes\mu"'] & X \otimes S \arrow[d, "\sigma"] \\
X \otimes S \arrow[r, "\sigma"] & X
\end{tikzcd}$$

(where we have suppressed the isomorphism α) and

$$\begin{tikzcd}
X \otimes S \arrow[d, "\sigma"'] & X \otimes U \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\
X &
\end{tikzcd}$$

commute. A map $f\colon X \to Y$ of orthogonal spectra is a map of right S-modules, i.e., a map $f\colon X \to Y$ of orthogonal sequences such that the diagram

$$\begin{tikzcd}
X \otimes S \arrow[r, "f\otimes 1"] \arrow[d, "\sigma"'] & Y \otimes S \arrow[d, "\sigma"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes. We write $\mathrm{Sp}^{\mathbb{O}}$ for the topological category of orthogonal spectra.

Equivalently, X is equipped with suitable $O(n)$-equivariant maps

$$\sigma_n \colon \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge S^\ell \longrightarrow X_n$$

for all $n \geq 0$. This is the same as saying that X is equipped with suitable $O(k)\times O(\ell)$-equivariant maps

$$\sigma^\ell \colon X_k \wedge S^\ell \longrightarrow X_{k+\ell}$$

for all $k, \ell \geq 0$. The associativity and unitality conditions amount to saying that in each case σ^ℓ is the composite

$$X_k \wedge S^\ell \xrightarrow{\sigma\wedge 1} X_{k+1} \wedge S^{\ell-1} \xrightarrow{\sigma\wedge 1} \dots \longrightarrow X_{k+\ell-1} \wedge S^1 \xrightarrow{\sigma} X_{k+\ell}$$

of ℓ suspended copies of structure maps

$$\sigma \colon X_m \wedge S^1 \longrightarrow X_{m+1}\,,$$

for $k \leq m < k+\ell$. (Elaborate on the equivalence of these two points of view?)

Hence we have the alternative definition of an orthogonal spectrum given at the outset of this section: It is a sequence of based $O(n)$-spaces X_n for $n \geq 0$, and a sequence of structure maps $\sigma \colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$ for $n \geq 0$, with the property that the ℓ-fold composite

$$\sigma^\ell \colon X_k \wedge S^\ell \longrightarrow X_{k+\ell}$$

is $O(k)\times O(\ell) \to O(k+\ell)$-equivariant for $k, \ell \geq 0$, where $O(\ell)$ acts in the standard way on S^ℓ.

A map $f \colon X \to Y$ of orthogonal spectra is a sequence of basepoint-preserving $O(n)$-maps $f_n \colon X_n \to Y_n$ such that the diagram

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n\wedge 1"] \arrow[d, "\sigma"'] & Y_n \wedge S^1 \arrow[d, "\sigma"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes for each $n \geq 0$.

4.5 The closed symmetric monoidal category of orthogonal spectra

The category of orthogonal spectra has all small colimits and limits, created at the level of underlying orthogonal sequences. For each small diagram $\alpha \mapsto X_\alpha$ of orthogonal spectra, the colimit $\operatorname{colim}_\alpha X_\alpha$ of underlying orthogonal sequences has the right S-module structure given by a composite

$$(\operatorname*{colim}_\alpha X_\alpha) \otimes S \xleftarrow{\cong} \operatorname*{colim}_\alpha (X_\alpha \otimes S) \xrightarrow{\operatorname{colim}\sigma} \operatorname*{colim}_\alpha X_\alpha\,.$$

Here $-\otimes S$ is a left adjoint, hence preserves colimits. The limit $\lim_\alpha X_\alpha$ of underlying orthogonal sequences has the right S-module structure given by a composite

$$(\lim_\alpha X_\alpha) \otimes S \longrightarrow \lim_\alpha (X_\alpha \otimes S) \xrightarrow{\lim\sigma} \lim_\alpha X_\alpha\,.$$

Equivalently, it is given by the composite

$$\lim_\alpha X_\alpha \xrightarrow{\lim\tilde\sigma} \lim_\alpha \operatorname{Hom}(S, X_\alpha) \xleftarrow{\cong} \operatorname{Hom}(S, \lim_\alpha X_\alpha)\,.$$

Here $\operatorname{Hom}(S, -)$ is a right adjoint, hence preserves limits. More explicitly, these are given by composites

$$\sigma \colon (\operatorname*{colim}_\alpha (X_\alpha)_n) \wedge S^1 \xrightarrow{\cong} \operatorname*{colim}_\alpha ((X_\alpha)_n \wedge S^1) \longrightarrow \operatorname*{colim}_\alpha (X_\alpha)_{n+1}$$

and

$$\sigma \colon (\lim_\alpha (X_\alpha)_n) \wedge S^1 \xrightarrow{\nu} \lim_\alpha ((X_\alpha)_n \wedge S^1) \longrightarrow \lim_\alpha (X_\alpha)_{n+1}\,.$$

(Explain ν?)

The category of orthogonal spectra is tensored and cotensored over based topological spaces, and these structures are created at the level of orthogonal sequences. For each based space T and orthogonal spectra X and Y, the orthogonal sequences $X \wedge T$ and $F(T, Y)$ have right S-module structures given by composites

$$(X \wedge T) \otimes S \xrightarrow{\cong} (X \otimes S) \wedge T \xrightarrow{\sigma \wedge 1} X \wedge T$$

and

$$F(T, Y) \otimes S \longrightarrow F(T, Y \otimes S) \xrightarrow{F(1,\sigma)} F(T, Y)\,.$$

More explicitly, these are given by composites

$$\sigma\colon X_n \wedge T \wedge S^1 \xrightarrow{1\wedge\gamma} X_n \wedge S^1 \wedge T \xrightarrow{\sigma\wedge 1} X_{n+1} \wedge T$$

and

$$\sigma\colon F(T, Y_n) \wedge S^1 \xrightarrow{\nu} F(T, Y_n \wedge S^1) \xrightarrow{F(1,\sigma)} F(T, Y_{n+1})\,.$$

(Explain ν?) There are natural homeomorphisms

$$\mathscr{T}(T, \mathrm{Sp}^{\mathbb{O}}(X, Y)) \cong \mathrm{Sp}^{\mathbb{O}}(X \wedge T, Y) \cong \mathrm{Sp}^{\mathbb{O}}(X, F(T, Y))\,.$$

We write $\Sigma X = X \wedge S^1$ and $\Omega X = F(S^1, X)$ for the sequential (or orthogonal) spectra obtained from the tensored and cotensored structure, in the case $T = S^1$. The adjunction unit $\eta\colon X \to F(T, X \wedge T)$ and counit $\epsilon\colon F(T, X) \wedge T \to X$ specialize to natural maps $\eta\colon X \to \Omega\Sigma X$ and $\epsilon\colon \Sigma\Omega X \to X$ of orthogonal spectra.

We can also define an orthogonal spectrum $T \wedge X$ as the orthogonal sequence $T \wedge X$ with right S-module structure given by the composite

$$(T \wedge X) \otimes S \cong T \wedge (X \otimes S) \xrightarrow{1\wedge\sigma} T \wedge X\,.$$

More explicitly, $(T \wedge X)_n = T \wedge X_n$, and $\sigma\colon (T \wedge X)_n \wedge S^1 \to (T \wedge X)_{n+1}$ is $1 \wedge \sigma\colon T \wedge X_n \wedge S^1 \to T \wedge X_{n+1}$. The natural isomorphism $\gamma\colon T \wedge X \to X \wedge T$ of orthogonal sequences is also an isomorphism of orthogonal spectra.

Remark 4.5. Restriction along $\mathbb{N} \to \mathbb{O}$ defines a forgetful functor $\mathbb{U}\colon \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Sp}^{\mathbb{N}}$ from orthogonal spectra to sequential spectra. It preserves small colimits and limits, as well as tensors and cotensors with based spaces. It does not preserve the symmetric monoidal pairing $\wedge$ and closed structure F that we are about to define.

By analogy with the tensor product $M \otimes_R N$ and Hom object $\mathrm{Hom}_R(M, N)$ of R-modules, for a commutative ring R, we can now define internal smash products and function objects for orthogonal spectra.

Definition 4.6. The smash product $X \wedge Y$ of two orthogonal spectra X and Y is the coequalizer

$$X \otimes S \otimes Y \underset{1\otimes\sigma'}{\overset{\sigma\otimes 1}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} X \wedge Y$$

in the category of orthogonal sequences, i.e., the colimit of the two parallel arrows. Here $\sigma' = \sigma \circ \gamma$ is the left S-module action on Y given by the composite

$$\sigma'\colon S \otimes Y \xrightarrow{\gamma} Y \otimes S \xrightarrow{\sigma} Y\,.$$

Hence, $(X \wedge Y)_n$ is the coequalizer

$$\bigvee_{k+\ell+m=n} O(n)_+ \underset{O(k)\times O(\ell)\times O(m)}{\wedge} X_k \wedge S^\ell \wedge Y_m \underset{(1\otimes\sigma')_n}{\overset{(\sigma\otimes 1)_n}{\rightrightarrows}} \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell \xrightarrow{\pi_n} (X \wedge Y)_n$$

in the category of based $O(n)$-spaces, for each $n \ge 0$. (Discuss right S-module structure inherited from X or from Y, and why they are the same.)

The multiplication $\mu\colon S \otimes S \to S$ makes the orthogonal sequence S into a right S-module, i.e., an orthogonal spectrum.

Proposition 4.7. *There are natural isomorphisms $\alpha\colon X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$, $\lambda\colon S \wedge Y \cong Y$, $\rho\colon Y \wedge S \cong Y$ and $\gamma\colon X \wedge Y \cong Y \wedge X$. These make $(\mathrm{Sp}^{\mathbb{O}}, \wedge, S)$ a symmetric monoidal category.*

Proof. In each case the isomorphism is induced from the corresponding isomorphism of orthogonal sequences by passage to a coequalizer. For instance, $X \wedge S$ is defined as the coequalizer

$$X \otimes S \otimes S \underset{1\otimes\mu}{\overset{\sigma\otimes 1}{\rightrightarrows}} X \otimes S \xrightarrow{\pi} X \wedge S$$

which is isomorphic to the coequalizer

$$X \otimes S \otimes S \underset{1\otimes\mu}{\overset{\sigma\otimes 1}{\rightrightarrows}} X \otimes S \xrightarrow{\sigma} X$$

split by $X \cong X \otimes U \overset{1\otimes\eta}{\longrightarrow} X \otimes S$ and $X \otimes S \cong X \otimes S \otimes U \overset{1\otimes 1\otimes\eta}{\longrightarrow} X \otimes S \otimes S$. See Mac Lane (1971/1998), Section VI.6 regarding split coequalizers. □

Recall the equalizer diagram

$$\mathrm{Hom}_R(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \rightrightarrows \mathrm{Hom}(M \otimes R, N)$$

for R-modules M and N, where the two parallel arrows take $f\colon M \to N$ to the homomorphisms $M \otimes R \to N$ given by $m \otimes r \mapsto f(mr)$ and $m \otimes r \mapsto f(m)r$, respectively.

Definition 4.8. The function spectrum $F(Y, Z)$ associated to two orthogonal spectra Y and Z is the equalizer

$$F(Y, Z) \xrightarrow{\iota} \mathrm{Hom}(Y, Z) \underset{\sigma^\vee}{\overset{\sigma^*}{\rightrightarrows}} \mathrm{Hom}(Y \otimes S, Z)$$

in the category of orthogonal spectra, i.e., the limit of the two parallel arrows. Here $\sigma^* = \mathrm{Hom}(\sigma, 1)$ and $\sigma^\vee$ is the right adjoint of the composite

$$\mathrm{Hom}(Y, Z) \otimes Y \otimes S \overset{\epsilon\otimes 1}{\longrightarrow} Z \otimes S \overset{\sigma}{\longrightarrow} Z\,.$$

Here $\epsilon\colon \mathrm{Hom}(Y, Z) \otimes Y \to Z$ is an adjunction counit, left adjoint to the identity on $\mathrm{Hom}(Y, Z)$. Hence, $F(Y, Z)_k$ is the equalizer

$$F(Y, Z)_k \xrightarrow{\iota} \prod_{k+\ell=n} \mathscr{T}(Y_\ell, Z_n)^{O(\ell)} \underset{\sigma_k^\vee}{\overset{\sigma_k^*}{\rightrightarrows}} \prod_{k+\ell+m=n} \mathscr{T}(Y_\ell \wedge S^m, Z_n)^{O(\ell)\times O(m)}$$

in the category of based $O(k)$-spaces, for each $k \geq 0$. (Elaborate on σ_k^*, $\sigma_k^\vee$ and the group actions?) (Discuss right S-module structure inherited from Y or from Z, and why they are the same.)

Proposition 4.9. *There is a natural homeomorphism*

$$\mathrm{Sp}^{\mathbb{O}}(X \wedge Y, Z) \cong \mathrm{Sp}^{\mathbb{O}}(X, F(Y, Z))$$

and a natural isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z))$$

for all orthogonal spectra X, Y and Z. Hence $\mathrm{Sp}^{\mathbb{O}}$ is a closed category.

Proof. (These are induced from the corresponding homeomorphism and isomorphism of orthogonal sequences by passage to an equalizer.) □

Examples: The composite evaluation functor $\mathrm{Ev}_k\colon \mathrm{Sp}^{\mathbb{O}} \to \mathscr{T}^{\mathbb{O}} \to \mathscr{T}$ mapping X to $\mathrm{Ev}_k X = X_k$ has a left adjoint, the free functor $F_k\colon \mathscr{T} \to \mathrm{Sp}^{\mathbb{O}}$ given by $F_k T = G_k T \otimes S$. More explicitly,

$$(F_k T)_n = O(n)_+ \underset{O(\ell)}{\wedge} (T \wedge S^\ell)$$

for $n = k+\ell$, with $(F_k T)_n = *$ for $n < k$. In particular, $F_0 T = \Sigma^\infty T = T \wedge S$ is the suspension spectrum of T, with

$$(F_0 T)_n = (\Sigma^\infty T)_n = T \wedge S^n$$

for each $n \geq 0$, where $O(n)$ acts as usual on S^n.

There is a natural homeomorphism

$$\mathrm{Sp}^{\mathbb{O}}(F_k T, X) \cong \mathscr{T}(T, \mathrm{Ev}_k X)$$

and natural isomorphisms

$$F_k T \wedge F_\ell V \cong F_{k+\ell}(T \wedge V) \cong F_{k+\ell}(T) \wedge V$$

and

$$F(F_\ell V, Z) \cong \mathrm{sh}^\ell F(V, Z) \cong F(V, \mathrm{sh}^\ell Z)$$

for $k, \ell \geq 0$, $T, V \in \mathscr{T}$ and $Z \in \mathrm{Sp}^{\mathbb{O}}$. (Proof?)

Note that $F_k T = T \wedge F_k S^0$, and $(F_k S^0)_n = O(n)_+ \wedge_{O(\ell)} S^\ell = Th(\gamma^\perp)$ is the Thom complex of the orthogonal complement $\gamma^\perp$ in ϵ^n to the canonical k-bundle γ over the Stiefel manifold $V_k(\mathbb{R}^n) = O(n)/O(\ell)$ of orthonormal k-frames in $\mathbb{R}^n$. This equals the space $\mathrm{Sp}^{\mathbb{O}}(F_n S^0, F_k S^0)$ of orthogonal spectrum maps $F_n S^0 \to F_k S^0$.

5 Homotopy groups of spectra

We turn to §7 of Mandell–May–Schwede–Shipley.

5.1 π_*-isomorphisms

Definition 5.1. The homotopy groups of a sequential spectrum X are defined by

$$\pi_k(X) = \operatorname*{colim}_n \pi_{k+n}(X_n)$$

for each integer k. The colimit is formed over the composite homomorphisms

$$\pi_{k+n}(X_n) \xrightarrow{-\wedge S^1} \pi_{k+n+1}(X_n \wedge S_1) \xrightarrow{\sigma_*} \pi_{k+n+1}(X_{n+1}),$$

mapping the homotopy class of $f\colon S^{k+n} \to X_n$ to the homotopy class of $\sigma \circ (f \wedge S^1)\colon S^{k+n+1} \cong S^{k+n} \wedge S^1 \to X_n \wedge S^1 \to X_{n+1}$. When $k+n \geq 2$ these are homomorphisms of abelian groups, so each $\pi_k(X)$ is an abelian group. We write $\pi_*(X)$ for the graded abelian group with $\pi_k(X)$ in degree k.

Equivalently, the colimit is formed over the composite homomorphisms

$$\pi_{k+n}(X_n) \xrightarrow{\tilde{\sigma}_*} \pi_{k+n}(\Omega X_{n+1}) \cong \pi_{k+n+1}(X_{n+1}),$$

where $\tilde{\sigma}\colon X_n \to \Omega X_{n+1}$ is the right adjoint of the structure map. If X is an Ω-spectrum, then $\tilde{\sigma}_*$ is an isomorphism for each $k+n \geq 0$. Hence in these cases

$$\pi_k(X) \cong \begin{cases} \pi_k(X_0) & \text{for } k \geq 0, \\ \pi_0(X_{-k}) & \text{for } k \leq 0. \end{cases}$$

Each map $f\colon X \to Y$ of sequential spectra induces a commuting diagram

$$\begin{tikzcd}
\pi_{k+n}(X_n) \arrow[r, "-\wedge S^1"] \arrow[d, "f_{n*}"'] & \pi_{k+n+1}(X_n \wedge S^1) \arrow[r, "\sigma_*"] \arrow[d, "(f_n \wedge 1)_*"'] & \pi_{k+n+1}(X_{n+1}) \arrow[d, "f_{n+1*}"] \\
\pi_{k+n}(Y_n) \arrow[r, "-\wedge S^1"] & \pi_{k+n+1}(Y_n \wedge S^1) \arrow[r, "\sigma_*"] & \pi_{k+n+1}(Y_{n+1})
\end{tikzcd}$$

hence also a homomorphism

$$\pi_k(f) = \operatorname*{colim}_n \pi_{k+n}(f_n) \colon \pi_k(X) \to \pi_k(Y)$$

and a homomorphism of graded abelian groups $\pi_*(f) \colon \pi_*(X) \to \pi_*(Y)$. We often write f_* for $\pi_k(f)$ or $\pi_*(f)$. It is easy to check that $\pi_* \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{grAb}$ defines a functor from sequential spectra to graded abelian groups.

Remark 5.2. The homotopy groups of an orthogonal spectrum X are defined as the homotopy groups of the underlying sequential spectrum $\mathbb{U}X$, where $\mathbb{U} \colon \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Sp}^{\mathbb{N}}$ forgets the actions by orthogonal groups.

Definition 5.3. A homotopy of maps $X \to Y$ is a map $X \wedge I_+ \to Y$, where $I = [0, 1]$. Homotopic maps induce the same homomorphism of homotopy groups. A homotopy equivalence is a map $f \colon X \to Y$ that admits a homotopy inverse, i.e., a map $g \colon Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$.

A map $f \colon X \to Y$ is called a level equivalence if $f_n \colon X_n \to Y_n$ is a weak equivalence, for each $n \geq 0$.

A map $f \colon X \to Y$ is called a π_*-isomorphism if the induced homomorphism $f_* = \pi_*(f) \colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Lemma 5.4. *A homotopy equivalence is a level equivalence. A level equivalence is a π_*-isomorphism. A π_*-isomorphism between Ω-spectra is a level equivalence.*

Proof. The first two claims are clear. If $f \colon X \to Y$ is a π_*-isomorphism between Ω-spectra, then for each $n \geq 0$ and each $i \geq 0$ the homomorphism $\pi_i(f_n) \colon \pi_i(X_n) \to \pi_i(Y_n)$ is identified with $\pi_{i-n}(f) \colon \pi_{i-n}(X) \to \pi_{i-n}(Y)$, hence is an isomorphism. □

Remark 5.5. The stable homotopy category can be defined as the localization of the category $\mathrm{Sp}^{\mathbb{N}}$ where the subcategory $\mathscr{W}$ of π_*-isomorphisms have been turned into isomorphisms:

$$\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) = \mathrm{Sp}^{\mathbb{N}}[\mathscr{W}^{-1}]\,.$$

It is equivalent to the localization of the full subcategory category of Ω-spectra where the level equivalences have been turned into isomorphisms, since every spectrum is π_*-isomorphic to an Ω-spectrum. Replacing $\mathrm{Sp}^{\mathbb{N}}$ with $\mathrm{Sp}^{\mathbb{O}}$ gives equivalent localized categories:

$$\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \simeq \mathrm{Ho}(\mathrm{Sp}^{\mathbb{O}})\,.$$

Example 5.6. Let S^c be the orthogonal spectrum with $S^c_0 = *$ and $S^c_n = S^n$ for $n > 0$, with the usual $O(n)$-action. The inclusion $S^c \to S$ is a π_*-isomorphism. (S^c is a positive cofibrant replacement of S.)

Let S^f be the orthogonal spectrum with $S^f_n = Q(S^n) = \mathrm{colim}_k\, \Omega^k(S^k \wedge S^n)$, with the induced $O(n)$-action. The inclusion $S \to S^f$ is a π_*-isomorphism, and S^f is an Ω-spectrum. (S^f is a fibrant replacement of S.)

Proposition 5.7. *There is a natural isomorphism*

$$S^1 \wedge - \colon \pi_k(X) \longrightarrow \pi_{1+k}(S^1 \wedge X)\,.$$

Proof. For $k + n \geq 0$ there is a natural map $S^1 \wedge - \colon \pi_{k+n}(X_n) \to \pi_{1+k+n}(S^1 \wedge X_n)$ that takes the homotopy class of $f \colon S^{k+n} \to X_n$ to the homotopy class of $S^1 \wedge f \colon S^{1+k+n} \cong S^1 \wedge S^{k+n} \to S^1 \wedge X_n$. The diagram

$$\begin{tikzcd}
\pi_{k+n}(X_n) \arrow[r, "-\wedge S^1"] \arrow[d, "S^1\wedge -"'] & \pi_{k+n+1}(X_n \wedge S^1) \arrow[r, "\sigma"] \arrow[d, "S^1\wedge -"'] & \pi_{k+n+1}(X_{n+1}) \arrow[d, "S^1\wedge -"'] \\
\pi_{1+k+n}(S^1 \wedge X_n) \arrow[r, "-\wedge S^1"] & \pi_{1+k+n+1}(S^1 \wedge X_n \wedge S^1) \arrow[r, "\sigma"] & \pi_{1+k+n+1}(S^1 \wedge X_{n+1})
\end{tikzcd}$$

commutes, and can be repeated indefinitely to the right, so $S^1 \wedge -$ induces a homomorphism $S^1 \wedge - \colon \pi_k(X) \to \pi_{1+k}(S^1 \wedge X)$ of horizontal colimits.

We claim that $\ker(-\wedge S^1) = \ker(S^1 \wedge -)$ in $\pi_{k+n}(X_n)$, and $\mathrm{im}(-\wedge S^1) = \mathrm{im}(S^1 \wedge -)$ in $\pi_{1+k+n+1}(S^1 \wedge X_n \wedge S^1)$.

The first claim implies that $S^1 \wedge -\colon \pi_k(X) \to \pi_{1+k}(S^1 \wedge X)$ is injective: If $f\colon S^{k+n} \to X_n$ represents a class in $\pi_k(X)$ that maps to zero in $\pi_{1+k}(S^1 \wedge X)$, then $S^1 \wedge f$ maps to zero in $\pi_{1+k+m}(S^1 \wedge X_m)$ for some $m \ge n$. The image of f in $\pi_{k+m}(X_m)$ is then in $\ker(S^1 \wedge -)$. By the claim, this equals $\ker(- \wedge S^1)$, so the images of f in $\pi_{k+m+1}(X_m \wedge S^1)$ and $\pi_{k+m+1}(X_{m+1})$ are zero. Hence the class of f in $\pi_k(X)$ is also zero.

The second claim implies that $S^1 \wedge -\colon \pi_k(X) \to \pi_{1+k}(S^1 \wedge X)$ is surjective: If $g\colon S^{1+k+n} \to S^1 \wedge X_n$ represents a given class in $\pi_{1+k}(S^1 \wedge X)$, then the image of g in $\pi_{1+k+n+1}(S^1 \wedge X_n \wedge S^1)$ is in $\operatorname{im}(- \wedge S^1)$. By the claim, this equals $\operatorname{im}(S^1 \wedge -)$, so there is a map $f\colon S^{k+n+1} \to X_n \wedge S^1$ such that $S^1 \wedge f$ is homotopic to $g \wedge S^1$. The image of f in $\pi_{k+n+1}(X_{n+1})$ then maps to the image of g in $\pi_{1+k+n+1}(S^1 \wedge X_{n+1})$, so the image of f in $\pi_k(X)$ maps to the given class in $\pi_{1+k}(S^1 \wedge X)$.

Hence the two claims imply that $S^1 \wedge -$ is an isomorphism.

To prove the first claim, consider a map $f\colon S^{k+n} \to X_n$. There is a commutative diagram

$$\begin{tikzcd}
S^1 \wedge S^{k+n} \arrow[r, "S^1 \wedge f"] \arrow[d, "\gamma"'] & S^1 \wedge X_n \arrow[d, "\gamma"] \\
S^{k+n} \wedge S^1 \arrow[r, "f \wedge S^1"] & X_n \wedge S^1
\end{tikzcd}$$

with vertical homeomorphisms, showing that $f \wedge S^1 = \gamma \circ (S^1 \wedge f) \circ \gamma^{-1}$. Hence, if $S^1 \wedge f$ is null-homotopic then $f \wedge S^1$ is null-homotopic. By symmetry, the opposite implication also holds.

To prove the second claim, consider a map $g\colon S^{1+k+n} \cong S^1 \wedge S^{k+n} \to S^1 \wedge X_n$. There is a commutative diagram

$$\begin{tikzcd}
S^1 \wedge S^{k+n} \wedge S^1 \arrow[r, "S^1 \wedge h"] \arrow[d, "(13)"'] & S^1 \wedge X_n \wedge S^1 \arrow[d, "(13)"] \\
S^1 \wedge S^{k+n} \wedge S^1 \arrow[r, "g \wedge S^1"] & S^1 \wedge X_n \wedge S^1
\end{tikzcd}$$

where (13) denotes the transposition of the first and third smash factors, and $h = \gamma \circ g \circ \gamma^{-1}\colon S^{k+n} \wedge S^1 \to X_n \wedge S^1$. A path in $O(2)$ from $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ induces a homotopy from the transposition $\gamma\colon S^1 \wedge S^1 \to S^1 \wedge S^1$ to $r \wedge S^1\colon S^1 \wedge S^1 \to S^1 \wedge S^1$, where $r\colon S^1 \to S^1$ reverses the orientation. Hence the two transpositions (13) are homotopic to $r \wedge S^{k+n} \wedge S^1$ and $r \wedge X_n \wedge S^1$, respectively, and $g \wedge S^1$ is homotopic to

$$(r \wedge X_n \wedge S^1) \circ (S^1 \wedge h) \circ (r \wedge S^{k+n} \wedge S^1)^{-1} = rr^{-1} \wedge h = S^1 \wedge h\,.$$

Thus the image of $- \wedge S^1$ in $\pi_{1+k+n+1}(S^1 \wedge X_n \wedge S^1)$ is contained in the image of $S^1 \wedge -$. By symmetry, the opposite inclusion also holds. $\square$

Corollary 5.8. *A map $f\colon X \to Y$ is a π_*-isomorphism if and only if $S^1 \wedge f\colon S^1 \wedge X \to S^1 \wedge Y$ is one.*

We write $\Sigma X = X \wedge S^1$ and $\Omega X = F(S^1, X)$ for the sequential (or orthogonal) spectra obtained from the tensored and cotensored structure, in the case $T = S^1$.

Proposition 5.9. *The adjunction unit $\eta\colon X \to \Omega\Sigma X$ and counit $\epsilon\colon \Sigma\Omega X \to X$ are π_*-isomorphisms.*

Proof. There are commutative diagrams

$$\begin{tikzcd}
\pi_{k+n}(X_n) \arrow[r, "\eta_{n*}"] \arrow[d, "S^1 \wedge -"'] & \pi_{k+n}(\Omega\Sigma X_n) \arrow[d, "\cong"] \\
\pi_{1+k+n}(S^1 \wedge X_n) \arrow[r, "\cong"] & \pi_{k+n}(\Omega(S^1 \wedge X_n))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_{k+n}(\Omega X_n) \arrow[r, "\cong"] \arrow[d, "S^1 \wedge -"'] & \pi_{1+k+n}(X_n) \\
\pi_{1+k+n}(S^1 \wedge \Omega X_n) \arrow[r, "\cong"] & \pi_{1+k+n}(\Sigma\Omega X_n) \arrow[u, "\epsilon_{n*}"']
\end{tikzcd}$$

for all $k+n \geq 0$. These are compatible for varying n, and induce commutative diagrams

$$\begin{tikzcd}
\pi_k(X) \arrow[r, "\eta_*"] \arrow[d, "S^1\wedge -"'] & \pi_k(\Omega\Sigma X) \arrow[d, "\cong"] \\
\pi_{1+k}(S^1 \wedge X) \arrow[r, "\cong"] & \pi_k(\Omega(S^1 \wedge X))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_k(\Omega X) \arrow[r, "\cong"] \arrow[d, "S^1\wedge -"'] & \pi_{1+k}(X) \\
\pi_{1+k}(S^1 \wedge \Omega X) \arrow[r, "\cong"] & \pi_{1+k}(\Sigma\Omega X) \arrow[u, "\epsilon_*"']
\end{tikzcd}$$

for all integers k. Since $S^1 \wedge -$ is an isomorphism, so are η_* and ϵ_*. □

Corollary 5.10. *A map $f\colon X \to Y$ is a π_*-isomorphism if and only if $\Omega f\colon \Omega X \to \Omega Y$ is one.*

Theorem 5.11. *The functors*

$$\Sigma\colon \mathrm{Sp}^{\mathbb{N}} \longrightarrow \mathrm{Sp}^{\mathbb{N}} \qquad \text{and} \qquad \Omega\colon \mathrm{Sp}^{\mathbb{N}} \longrightarrow \mathrm{Sp}^{\mathbb{N}}$$

preserve π_*-isomorphisms. The induced functors

$$\Sigma\colon \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \longrightarrow \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \qquad \text{and} \qquad \Omega\colon \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \longrightarrow \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$$

are mutually inverse equivalences of categories. The same results apply with $\mathrm{Sp}^{\mathbb{O}}$ in place of $\mathrm{Sp}^{\mathbb{N}}$.

Proof. The first two claims are contained in Corollaries 5.8 and 5.10. Hence Σ and Ω induce endofunctors of the stable homotopy category $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) = \mathrm{Sp}^{\mathbb{N}}[\mathscr{W}^{-1}]$ where the π_*-isomorphisms have been inverted (assuming that this localization exists). The unit $\eta\colon \mathbb{1} \to \Omega\Sigma$ and counit $\epsilon\colon \Sigma\Omega \to \mathbb{1}$ induce natural isomorphisms, by Proposition 5.9, which means that the endofunctors Σ and Ω of $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ are mutually inverse equivalences. □

5.2 Long exact sequences

For a map $f\colon X \to Y$ of sequential or orthogonal spectra we define Cf as the pushout

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & CX \arrow[d] \\
Y \arrow[r, "i"] & Cf
\end{tikzcd}$$

where $CX = I \wedge X$ with $I = [0,1]$ based at 1, and Ff as the pullback

$$\begin{tikzcd}
Ff \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\
PY \arrow[r, "p_1"] & Y
\end{tikzcd}$$

where $PY = F(I, Y)$ with I based at 0. These constructions are compatible under the forgetful functor $\mathbb{U}\colon \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Sp}^{\mathbb{N}}$, since they only involve colimits, limits, tensors and cotensors. As in the category of spaces, we have (iterated) homotopy cofiber and fiber sequences

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} S^1 \wedge X \xrightarrow{-S^1\wedge f} S^1 \wedge Y \longrightarrow \dots$$

and

$$\dots \longrightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{p} X \xrightarrow{f} Y\,.$$

(This uses that the collapse map $Cf \to Y/X = Y \cup_X *$ is a homotopy equivalence when $f\colon X \to Y$ is a Hurewicz cofibration, and that $i\colon Y \to Cf$ is a Hurewicz cofibration, so that $Ci \to Cf/Y \cong S^1 \wedge X$ is a homotopy equivalence, and the dual facts about the inclusion $X \times_Y * = f^{-1}(*) \to Ff$ and $p\colon Ff \to X$.)

Proposition 5.12. *For any map $f\colon X \to Y$ there are natural long exact sequences*

$$\dots \to \pi_{1+k}(Y) \xrightarrow{\iota_*} \pi_k(Ff) \xrightarrow{p_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \to \pi_{-1+k}(Ff) \to \dots$$

and

$$\dots \to \pi_{1+k}(Cf) \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\pi_*} \pi_{-1+k}(X) \to \dots .$$

The natural maps $\eta\colon Ff \to \Omega Cf$ and $\epsilon\colon \Sigma Ff \to Cf$ are π_*-isomorphisms.

Proof. For $k + n \geq 0$ we have long exact sequences

$$\dots \to \pi_{1+k+n}(X_n) \xrightarrow{-f_{n*}} \pi_{1+k+n}(Y_n) \xrightarrow{\iota_{n*}} \pi_{k+n}(Ff_n) \xrightarrow{p_{n*}} \pi_{k+n}(X_n) \xrightarrow{f_{n*}} \pi_{k+n}(Y_n)$$

that are compatible for varying n. Here we have identified $\pi_{k+n}(\Omega Y_n)$ with $\pi_{1+k+n}(Y_n)$, etc. Passing to sequential colimits preserves exactness, so we also get a long exact sequence

$$\dots \to \pi_{1+k}(X) \xrightarrow{-f_*} \pi_{1+k}(Y) \xrightarrow{\iota_*} \pi_k(Ff) \xrightarrow{p_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y)$$

for each integer k. Letting k vary these extend indefinitely to the right, as claimed.

Next, we prove exactness of

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf)$$

at $\pi_k(Y)$. The composite $i \circ f$ is null-homotopic, so $i_* \circ f_* = 0$. Let $g\colon S^{k+n} \to Y_n$ represent a class in $\ker(i_*)$. By increasing n we may assume that $i_n \circ g\colon S^{k+n} \to Cf_n$ is null-homotopic, hence extends over a map $h\colon CS^{k+n} \to Cf_n$.

$$\begin{tikzcd}
S^{k+n} \arrow[r, "\mathbb{1}"] & S^{k+n} \arrow[r, "i_n"] \arrow[d, "g"] & CS^{k+n} \arrow[r, "\pi_n"] \arrow[d, "h"] & S^1 \wedge S^{k+n} \arrow[r, "-\mathbb{1}"] \arrow[d, "j"] & S^1 \wedge S^{k+n} \arrow[d, "S^1 \wedge g"] \\
X_n \arrow[r, "f_n"] & Y_n \arrow[r, "i_n"] & Cf_n \arrow[r, "\pi_n"] & S^1 \wedge X_n \arrow[r, "-(S^1 \wedge f_n)"] & S^1 \wedge Y_n
\end{tikzcd}$$

Let $j\colon S^1 \wedge S^{k+n} \to S^1 \wedge X_n$ be the induced maps of quotients. Then $(S^1 \wedge g) \circ (-\mathbb{1}) \simeq -(S^1 \wedge f_n) \circ j$, so the class of $S^1 \wedge g$ in $\pi_{1+k}(S^1 \wedge Y)$ is in the image of $(S^1 \wedge f)_*$. (Elaborate on the role of signs?) The natural isomorphism $S^1 \wedge -$ now tells us that the class of g in $\pi_k(Y)$ is in the image of f_*, as claimed.

$$\begin{tikzcd}
\pi_k(X) \arrow[r, "S^1 \wedge -", "\cong"'] \arrow[d, "f_*"] & \pi_{1+k}(S^1 \wedge X) \arrow[d, "(S^1 \wedge f)_*"] \\
\pi_k(Y) \arrow[r, "S^1 \wedge -", "\cong"'] & \pi_{1+k}(S^1 \wedge Y)
\end{tikzcd}$$

By iteration, this implies exactness of

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\pi_*} \pi_k(S^1 \wedge X) \xrightarrow{-(S^1 \wedge f)_*} \pi_k(S^1 \wedge Y) \to \dots$$

hence also of

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\pi_*} \pi_{-1+k}(X) \xrightarrow{-f_*} \pi_{-1+k}(Y) \to \dots$$

for all integers k. (This uses the isomorphism $S^1 \wedge -$.) Letting k vary these extend without bound to the left, as claimed.

The homotopy-commutative diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "-\Omega f"] \arrow[d, "="] & \Omega Y \arrow[r, "\iota"] \arrow[d, "="] & Ff \arrow[r, "p"] \arrow[d, "\eta"] & X \arrow[r, "f"] \arrow[d, "\eta"] & Y \arrow[d, "\eta"] \\
\Omega X \arrow[r, "-\Omega f"] & \Omega Y \arrow[r, "\Omega i"] & \Omega Cf \arrow[r, "\Omega \pi"] & \Omega \Sigma X \arrow[r, "\Omega \Sigma f"] & \Omega \Sigma Y
\end{tikzcd}$$

induces a vertical map of five-term exact sequences

$$\begin{tikzcd}
\pi_{1+k}(X) \arrow[r, "-\Omega f_*"] \arrow[d, "="'] & \pi_{1+k}(Y) \arrow[r, "\iota_*"] \arrow[d, "="'] & \pi_k(Ff) \arrow[r, "p_*"] \arrow[d, "\eta_*"'] & \pi_k(X) \arrow[r, "f_*"] \arrow[d, "\eta_*"'] & \pi_k(Y) \arrow[d, "\eta_*"'] \\
\pi_{1+k}(X) \arrow[r, "-\Omega f_*"] & \pi_{1+k}(Y) \arrow[r, "\Omega i_*"] & \pi_k(\Omega Cf) \arrow[r, "\Omega \pi_*"] & \pi_k(\Omega\Sigma X) \arrow[r, "\Omega\Sigma f_*"] & \pi_k(\Omega\Sigma Y)\,.
\end{tikzcd}$$

The five-lemma implies that the middle vertical homomorphism η_* is an isomorphism.

It follows that $\epsilon\colon \Sigma Ff \to Cf$ is a π_*-isomorphism, since it factors as $\Sigma Ff \xrightarrow{\Sigma\eta} \Sigma\Omega Cf \xrightarrow{\epsilon} Cf$. Alternatively one can start from the homotopy-commutative diagram relating $\epsilon\colon \Sigma Ff \to Cf$ to $\epsilon\colon \Sigma\Omega X \to X$ and $\epsilon\colon \Sigma\Omega Y \to Y$. □

For any small diagram $\alpha \mapsto X_\alpha$ of sequential spectra, there are canonical homomorphisms

$$\operatorname*{colim}_\alpha \pi_*(X_\alpha) \longrightarrow \pi_*(\operatorname*{colim}_\alpha X_\alpha)$$

and

$$\pi_*(\lim_\alpha X_\alpha) \longrightarrow \lim_\alpha \pi_*(X_\alpha)\,.$$

Lemma 5.13. *For any finite collection $(X_\alpha)_\alpha$ of spectra the canonical homomorphisms*

$$\bigoplus_\alpha \pi_*(X_\alpha) \xrightarrow{\cong} \pi_*(\bigvee_\alpha X_\alpha)$$

and

$$\pi_*(\prod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha \pi_*(X_\alpha)$$

are isomorphisms. Hence $\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$ is a π_*-isomorphism.

Proof. By induction, it suffices to prove this in the case of two spectra X and Y.

The mapping cone Cf of the inclusion $f\colon X \to X \vee Y$ is homotopy equivalent to Y, so there is a long exact sequence

$$\dots \to \pi_{1+k}(Y) \to \pi_k(X) \xrightarrow{f_*} \pi_k(X \vee Y) \xrightarrow{i_*} \pi_k(Y) \xrightarrow{\pi_*} \dots\,.$$

The inclusion $g\colon Y \to X \vee Y$ defines a right inverse to $i\colon X \vee Y \to Cf \simeq Y$, so i_* is split surjective. Hence the long exact sequence breaks up into short exact sequences, and $f_* + g_*\colon \pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \vee Y)$ is an isomorphism in each degree k.

The homotopy fiber Ff of the projection $f\colon X \times Y \to Y$ is homotopy equivalent to X, so there is a long exact sequence

$$\dots \to \pi_{1+k}(Y) \xrightarrow{\iota_*} \pi_k(X) \xrightarrow{p_*} \pi_k(X \times Y) \xrightarrow{f_*} \pi_k(Y) \to \dots\,.$$

The projection $g\colon X \times Y \to X$ defines a left inverse to $p\colon X \simeq Ff \to X \times Y$, so p_* is split injective. Hence he long exact sequence breaks up into short exact sequences, and $(f_*, g_*)\colon \pi_k(X \times Y) \to \pi_k(X) \times \pi_k(Y)$ is an isomorphism in each degree k.

The canonical homomorphism $\pi_*(X) \oplus \pi_*(Y) \to \pi_*(X) \times \pi_*(Y)$ is an isomorphism, hence $X \vee Y \to X \times Y$ is a π_*-isomorphism. □

Lemma 5.14. *For any collection $(X_\alpha)_\alpha$ of spectra, finite or infinite, the canonical homomorphism*

$$\bigoplus_\alpha \pi_*(X_\alpha) \xrightarrow{\cong} \pi_*(\bigvee_\alpha X_\alpha)$$

is an isomorphism.

Proof. At each level n the wedge sum $\bigvee_\alpha (X_\alpha)_n$ is the colimit over the finite subsets $F \subset \{\alpha\}$ of the subspaces $\bigvee_{\alpha\in F}(X_\alpha)_n$. This colimit is strongly filtered (see Strickland, Lemma 3.6), so that each map $K \to \bigvee_\alpha (X_\alpha)_n$ from a compact space K factors through $\bigvee_{\alpha\in F}(X_\alpha)_n$ for some finite F. Applying this with $K = S^{k+n}$ (for surjectivity) and with $K = S^{k+n} \wedge I_+$ (for injectivity) it follows that

$$\operatorname*{colim}_F \pi_{n+k}(\bigvee_{\alpha\in F}(X_\alpha)_n) \xrightarrow{\cong} \pi_{n+k}(\bigvee_{\alpha\in F}(X_\alpha)_n)$$

is an isomorphism. Passing to colimits over n, and noting that independent colimits commute, we get the isomorphism

$$\operatorname*{colim}_F \pi_k(\bigvee_{\alpha\in F} X_\alpha) \xrightarrow{\cong} \pi_k(\bigvee_{\alpha\in F} X_\alpha)\,.$$

When combined with the previous lemma, this yields the conclusion. □

Corollary 5.15. *A finite product of π_*-isomorphisms is a π_*-isomorphism. An arbitrary wedge sum of π_*-isomorphisms is a π_*-isomorphism.*

5.3 Hurewicz cofibrations

A map $f\colon X \to Y$ of (orthogonal or sequential) spectra is a Hurewicz cofibration (= h-cofibration in [MMSS]) if it has the homotopy extension property (HEP) with respect to every spectrum Z: Given any map $g\colon Y \to Z$ and any homotopy $h\colon X \wedge I_+ \to Z$ with $g \circ f = h \circ i_0$, there exists a homotopy $k\colon Y \wedge I_+ \to Z$ with $k \circ (f \wedge \mathbb{1}) = h$ and $k \circ i_0 = g$.

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & X \wedge I_+ \arrow[d, "f\wedge\mathbb{1}"'] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "i_0"] \arrow[drr, bend right, "g"'] & Y \wedge I_+ \arrow[dr, dashed, "k"] & \\
& & Z
\end{tikzcd}$$

The universal case of this property is given by the mapping cylinder $Mf = Z = Y \cup_X (X \wedge I_+)$, in which case k provides a retraction to the canonical map $Y \cup_X (X \wedge I_+) \to Y \wedge I_+$. Hence f is a Hurewicz cofibration if and only if $Y \cup_X (X \wedge I_+) \to Y \wedge I_+$ admits a left inverse.

A retraction in the category of orthogonal spectra gives a retraction in the category of sequential spectra, so for each Hurewicz cofibration of orthogonal spectra the underlying map of sequential spectra is (also) a Hurewicz cofibration.

A retraction of sequential spectra gives a retraction at each level, so for each Hurewicz cofibration $f\colon X \to Y$ of spectra the map $f_n\colon X_n \to Y_n$ is a Hurewicz cofibration of based spaces, for each $n \ge 0$. In particular, each f_n is a closed embedding (= closed inclusion in [MMSS]), meaning that f_n maps X_n homeomorphically to its image, which is a closed subset of Y_n. (Reference?)

Proposition 5.16 (Cobase change)**.** *Consider a pushout square*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "\bar f"] \\
Y \arrow[r, "\bar g"] & Y \cup_X Z
\end{tikzcd}$$

of spectra, where $f\colon X \to Y$ is a Hurewicz cofibration. If f (resp. g) is a π_*-isomorphism then $\bar f$ (resp. $\bar g$) is a π_*-isomorphism.

Proof. It is formal that the pushout $\bar f$ of f along g is also a Hurewicz cofibration. Hence we have a commutative square

$$\begin{tikzcd}
Cf \arrow[r] \arrow[d, "\simeq"'] & C\bar f \arrow[d, "\simeq"] \\
Y/X \arrow[r, "\cong"] & (Y \cup_X Z)/Z
\end{tikzcd}$$

with vertical homotopy equivalences and one horizontal homeomorphism. It follows that $Cf \to C\bar{f}$ is a homotopy equivalence. We get a map of long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & \pi_k(X) \arrow[r, "f_*"] \arrow[d, "g_*"'] & \pi_k(Y) \arrow[r] \arrow[d, "\bar{g}_*"'] & \pi_k(Cf) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \pi_k(Z) \arrow[r, "\bar{f}_*"] & \pi_k(Y \cup_X Z) \arrow[r] & \pi_k(C\bar{f}) \arrow[r] & \dots
\end{tikzcd}$$

On one hand, if f_* is an isomorphism then $\pi_*(Cf) = 0$, so $\pi_*(C\bar{f}) = 0$ and $\bar{f}_*$ is an isomorphism. On the other hand, if g_* is an isomorphism then by the five-lemma $\bar{g}_*$ is also an isomorphism. □

Proposition 5.17 (Gluing lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
Y \arrow[d] & X \arrow[l, hook', "f"'] \arrow[r] \arrow[d] & Z \arrow[d] \\
Y' & X' \arrow[l, hook', "f'"'] \arrow[r] & Z'
\end{tikzcd}$$

of spectra, where f and f' are Hurewicz cofibrations. If $X \to X'$, $Y \to Y'$ and $Z \to Z'$ are π_*-isomorphisms, then so is the induced map $Y \cup_X Z \to Y' \cup_{X'} Z'$.

Proof. We have maps of long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & \pi_k(X) \arrow[r, "f_*"] \arrow[d] & \pi_k(Y) \arrow[r] \arrow[d] & \pi_k(Y/X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi_k(X') \arrow[r, "f'_*"] & \pi_k(Y') \arrow[r] & \pi_k(Y'/X') \arrow[r] & \dots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\dots \arrow[r] & \pi_k(Z) \arrow[r, "\bar{f}_*"] \arrow[d] & \pi_k(Y \cup_X Z) \arrow[r] \arrow[d] & \pi_k(Y \cup_X Z/Z) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi_k(Z') \arrow[r, "\bar{f}'_*"] & \pi_k(Y' \cup_{X'} Z') \arrow[r] & \pi_k(Y' \cup_{X'} Z'/Z') \arrow[r] & \dots .
\end{tikzcd}$$

By assumption $\pi_*(X) \to \pi_*(X')$ and $\pi_*(Y) \to \pi_*(Y')$ are isomorphisms, so by the five-lemma $\pi_*(Y/X) \to \pi_*(Y'/X')$ is an isomorphism. Hence $\pi_*(Y \cup_X Z/Z) \to \pi_*(Y' \cup_{X'} Z'/Z')$ is an isomorphism. By assumption $\pi_*(Z) \to \pi_*(Z')$ is an isomorphism, so by the five-lemma $\pi_*(Y \cup_X Z) \to \pi_*(Y' \cup_{X'} Z')$ is an isomorphism. □

Proposition 5.18. *If Y is the colimit of a sequence*

$$X_0 \to X_1 \to \dots \to X_\alpha \to X_{\alpha+1} \to \dots$$

of Hurewicz cofibrations, then

$$\operatorname*{colim}_\alpha \pi_*(X_\alpha) \xrightarrow{\cong} \pi_*(Y)$$

is an isomorphism.

Proof. At each level n, the space Y_n is the colimit of the sequence

$$(X_0)_n \to (X_1)_n \to \dots \to (X_\alpha)_n \to (X_{\alpha+1})_n \to \dots$$

of closed embeddings. Such a colimit is strongly filtered (e.g. by Strickland, Lemma 3.6), so each map $K \to Y_n$ from a compact space K factors through $(X_\alpha)_n$ for some finite α. Applying this with $K = S^{k+n}$ and $K = S^{k+n} \wedge I_+$ shows that

$$\operatorname*{colim}_\alpha \pi_{k+n}(X_\alpha)_n \xrightarrow{\cong} \pi_{k+n}(Y_n)$$

is an isomorphism. Passing to colimits over n gives the stated conclusion. □

Corollary 5.19. *If each map $X_\alpha \to X_{\alpha+1}$ is a π_*-isomorphism, then so is the canonical map $X_0 \to \operatorname{colim}_\alpha X_\alpha$.*

The following result shows that the tensor $A \wedge X \cong X \wedge A$ is homotopically meaningful when A is a CW complex, and that the cotensor $F(B, X)$ is homotopically meaningful when B is a finite CW complex.

Theorem 5.20. *Let $f\colon X \to Y$ be a π_*-isomorphism of spectra. If A is a based CW complex, then $A \wedge f\colon A \wedge X \to A \wedge Y$ is a π_*-isomorphism. If B is a finite based CW complex then $F(B, f)\colon F(B, X) \to F(B, Y)$ is a π_*-isomorphism.*

Proof. We have seen that $S^1 \wedge f\colon S^1 \wedge X \to S^1 \wedge Y$ and $F(S^1, f)\colon F(S^1, X) \to F(S^1, Y)$ are π_*-isomorphisms. By induction it follows that $S^m \wedge f$ and $F(S^m, f)$ are π_*-isomorphisms for all $m \geq 0$. Hence

$$\bigvee_\alpha (S^m \wedge f) \cong (\bigvee_\alpha S^m) \wedge f$$

is a π_*-isomorphism for arbitrary indexing sets $\{\alpha\}$, and

$$F(\bigvee_\alpha S^m, f) \cong \prod_\alpha F(S^m, f)$$

is a π_*-isomorphism for finite indexing sets $\{\alpha\}$.

Let A be an $(m+1)$-dimensional (based) CW complex, with m-skeleton A' and attaching map $\phi\colon \bigvee_\alpha S^m \to A'$, so that $C\phi \cong A$. Consider the commutative diagram

$$\begin{tikzcd}
\bigvee_\alpha S^m \wedge X \arrow[r, "\phi \wedge X"] \arrow[d, "\mathbb{1} \wedge f"'] & A' \wedge X \arrow[r, "i \wedge X"] \arrow[d, "A' \wedge f"'] & A \wedge X \arrow[d, "A \wedge f"'] \\
\bigvee_\alpha S^m \wedge Y \arrow[r, "\phi \wedge Y"] & A' \wedge Y \arrow[r, "i \wedge Y"] & A \wedge Y
\end{tikzcd}$$

of horizontal homotopy cofiber sequences. By induction on m we may assume that $A' \wedge f\colon A' \wedge X \to A' \wedge Y$ is a π_*-isomorphism. We have just shown that $\mathbb{1} \wedge f\colon \bigvee_\alpha S^m \wedge X \to \bigvee_\alpha S^m \wedge Y$ is a π_*-isomorphism. By the five-lemma applied to the associated map of long exact sequences of homotopy groups, we deduce that $A \wedge f\colon A \wedge X \to A \wedge Y$ is a π_*-isomorphism.

Let B be a finite $(m+1)$-dimensional (based) CW complex, with finite m-skeleton B' and attaching map $\phi\colon \bigvee_\alpha S^m \to B'$, so that $C\phi \cong B$. Consider the commutative diagram

$$\begin{tikzcd}
F(B, X) \arrow[r, "F(p, X)"] \arrow[d, "F(B, f)"'] & F(B', X) \arrow[r, "F(\phi, X)"] \arrow[d, "F(B', f)"'] & F(\bigvee_\alpha S^m, X) \arrow[d, "F(\mathbb{1}, f)"'] \\
F(B, Y) \arrow[r, "F(p, Y)"] & F(B', Y) \arrow[r, "F(\phi, Y)"] & F(\bigvee_\alpha S^m, Y)
\end{tikzcd}$$

of horizontal homotopy fiber sequences. By induction on m we may assume that $F(B', f)\colon F(B', X) \to F(B', Y)$ is a π_*-isomorphism. We have shown that $F(\mathbb{1}, f)\colon F(\bigvee_\alpha S^m, X) \to F(\bigvee_\alpha S^m, Y)$ is a π_*-isomorphism. By the five-lemma applied to the associated map of long exact sequences of homotopy groups, we deduce that $F(B, f)\colon F(B, X) \to F(B, Y)$ is a π_*-isomorphism.

Finally, let A be an arbitrary based CW complex, with m-skeleton $A^{(m)}$. We have proved that $A^{(m)} \wedge f$ is a π_*-isomorphism for each m. Since

$$\dots \to A^{(m)} \wedge X \to A^{(m+1)} \wedge X \to \dots$$

is a sequence of Hurewicz cofibrations with colimit $A \wedge X$, and likewise with Y in place of X, we deduce that $(A \wedge f)_*$ maps $\pi_*(A \wedge X) \cong \operatorname{colim}_m \pi_*(A^{(m)} \wedge X)$ isomorphically to $\pi_*(A \wedge Y) \cong \operatorname{colim}_m \pi_*(A^{(m)} \wedge Y)$. $\square$

(Discuss $F(B, f)$ for level equivalences $f\colon X \to Y$ and (infinite) based CW complexes B.)

Remark 5.21. Let E be a spectrum, and $f\colon X \to Y$ a map of spaces, with mapping cone $Cf = Y \cup_f CX$. Let $f_\# = E \wedge f\colon E \wedge X \to E \wedge Y$ and $f^\# = F(f, E)\colon F(Y, E) \to F(X, E)$ be the induced maps of spectra. There are natural isomorphisms

$$C(f_\#) \cong E \wedge Cf \qquad \text{and} \qquad F(f^\#) \cong F(Cf, E)$$

and associated long exact sequences

$$\dots \to \pi_k(E \wedge X) \xrightarrow{f_{\#*}} \pi_k(E \wedge Y) \xrightarrow{i_*} \pi_k(E \wedge Cf) \xrightarrow{\pi_*} \pi_{-1+k}(E \wedge X) \to \dots$$

and

$$\dots \to \pi_{1+k} F(X, E) \xrightarrow{\iota_*} \pi_k F(Cf, E) \xrightarrow{p_*} \pi_k F(Y, E) \xrightarrow{f^\#_*} \pi_k F(X, E) \to \dots .$$

When X is a CW complex we let $E_k(X) = \pi_k(E \wedge X)$, so that the first sequence becomes the long exact sequence

$$\dots \to E_k(X) \xrightarrow{f_*} E_k(Y) \longrightarrow E_k(Cf) \xrightarrow{\partial} E_{k-1}(X) \to \dots$$

in reduced E-homology (for f cellular). When X is a finite CW complex, or E is an Ω-spectrum, we let $E^\ell(X) = \pi_{-\ell} F(X, E)$, so that the second sequence becomes the long exact sequence

$$\dots \to E^{\ell-1}(X) \xrightarrow{\delta} E^\ell(Cf) \longrightarrow E^\ell(Y) \xrightarrow{f^*} E^\ell(X) \to \dots$$

in reduced E-cohomology (for f cellular). When f is the inclusion of X as a subcomplex of Y we can replace Cf by Y/X, in view of the homotopy equivalence $Cf \to Y/X$ that we have in this case. By the theorem above, any π_*-isomorphism $D \to E$ induces natural isomorphisms $D_*(X) \cong E_*(X)$ for CW complexes X, and $D^*(X) \cong E^*(X)$ for finite CW complexes X, so in this restricted sense E-homology and E-cohomology only depend on the π_*-isomorphism class of E.

6 Products

6.1 Pairings of spectra

Let X, Y and Z be orthogonal spectra. A pairing of X and Y with values in Z is a map

$$\mu\colon X \wedge Y \longrightarrow Z$$

of orthogonal spectra, i.e., a map of right S-modules $X \wedge Y \to Z$ in orthogonal sequences. Recall that $X \wedge Y$ is the coequalizer of $\sigma \otimes 1$ and $1 \otimes \sigma' = 1 \otimes \sigma\gamma\colon X \otimes S \otimes Y \to X \otimes Y$. The right S-actions on X and Y induce right S-actions $(\sigma \otimes 1)(1 \otimes \gamma)$ and $1 \otimes \sigma$, respectively, on $X \otimes Y$, and these induce the same right S-action on $X \wedge Y$.

$$\begin{tikzcd}
X \otimes S \otimes Y \arrow[r, shift left, "\sigma\otimes 1"] \arrow[r, shift right, "1\otimes\sigma'"'] & X \otimes Y \arrow[r, "\pi"] \arrow[dr, "\phi"'] & X \wedge Y \arrow[d, "\mu"] \\
& & Z
\end{tikzcd}$$

By the universal property of the coequalizer, μ corresponds to a unique map

$$\phi = \mu\pi\colon X \otimes Y \longrightarrow Z$$

of orthogonal sequences, whose composites with $\sigma \otimes 1$ and $1 \otimes \sigma'$ are equal and which takes either one of the right S-actions on $X \otimes Y$ to the right S-action on Z.

$$\begin{tikzcd}
X \otimes S \otimes Y \arrow[rr, leftrightarrow, "1\otimes\gamma"] \arrow[d, "\sigma\otimes 1"'] \arrow[dr, "1\otimes\sigma'"] & & X \otimes Y \otimes S \arrow[dl, "1\otimes\sigma"'] \arrow[d, "\phi\otimes 1"] \\
X \otimes Y \arrow[dr, "\phi"'] & X \otimes Y \arrow[d, "\phi"] & Z \otimes S \arrow[dl, "\sigma"] \\
& Z &
\end{tikzcd}$$

Here $\sigma' = \sigma\gamma\colon S \otimes Y \to Y$ is the map of orthogonal sequences given in degree n by the composite $O(n)$-map

$$\sigma'_n = \sigma_n \circ \gamma_n \colon \bigvee_{k+\ell=n} O(n)_+ \underset{O(k)\times O(\ell)}{\wedge} S^k \wedge Y_\ell \longrightarrow Y_n$$

taking $A\chi_{k,\ell} \wedge s \wedge y$ via $A \wedge y \wedge s$ to $A(\sigma_n(y \wedge s))$, for $A \in O(n)$, $s \in S^k$ and $y \in Y_\ell$. In particular, it takes $I_n \wedge s \wedge y$ to $\chi_{\ell,k}(\sigma_n(y \wedge s))$, where $\chi_{\ell,k} = \chi_{k,\ell}^{-1}$. In other words, σ'_n corresponds to the collection of $O(k) \times O(\ell)$-maps, for $k + \ell = n$, given by the composites

$$S^k \wedge Y_\ell \xrightarrow{\gamma} Y_\ell \wedge S^k \xrightarrow{\sigma^k} Y_{\ell+k} \xrightarrow{\chi_{\ell,k}} Y_{k+\ell}\,.$$

Hence $\phi\colon X \otimes Y \to Z$ corresponds to a collection of $O(k) \times O(\ell)$-equivariant maps

$$\phi_{k,\ell}\colon X_k \wedge Y_\ell \longrightarrow Z_{k+\ell}\,,$$

for $k, \ell \geq 0$, such that the diagrams

$$\begin{tikzcd}
X_k \wedge S^1 \wedge Y_\ell \arrow[r, "1\wedge\gamma"] \arrow[d, "\sigma\wedge 1"'] & X_k \wedge Y_\ell \wedge S^1 \arrow[r, "1\wedge\sigma"] & X_k \wedge Y_{\ell+1} \arrow[r, "1\wedge\chi_{\ell,1}"] & X_k \wedge Y_{1+\ell} \arrow[d, "\phi_{k,1+\ell}"] \\
X_{k+1} \wedge Y_\ell \arrow[rrr, "\phi_{k+1,\ell}"] & & & Z_{k+1+\ell}
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_k \wedge Y_\ell \wedge S^1 \arrow[r, "1\wedge\sigma"] \arrow[d, "\phi_{k,\ell}\wedge 1"'] & X_k \wedge Y_{\ell+1} \arrow[d, "\phi_{k,\ell+1}"] \\
Z_{k+\ell} \wedge S^1 \arrow[r, "\sigma"] & Z_{k+\ell+1}
\end{tikzcd}$$

commute.

Proposition 6.1. *Pairings $\mu\colon X \wedge Y \to Z$ are in bijective correspondence with collections of $O(k)\times O(\ell)$-equivariant maps*

$$\phi_{k,\ell}\colon X_k \wedge Y_\ell \longrightarrow Z_{k+\ell}$$

that make the bilinearity diagram

$$\begin{tikzcd}
X_k \wedge S^1 \wedge Y_\ell \arrow[d, "\sigma\wedge 1"'] & X_k \wedge Y_\ell \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "\phi_{k,\ell}\wedge 1"'] \arrow[dr, "1\wedge\sigma"] & \\
X_{k+1} \wedge Y_\ell \arrow[d, "\phi_{k+1,\ell}"'] & Z_{k+\ell} \wedge S^1 \arrow[d, "\sigma"'] & X_k \wedge Y_{\ell+1} \arrow[dl, "\phi_{k,\ell+1}"] \\
Z_{k+1+\ell} \arrow[r, "I_k\oplus\chi_{1,\ell}"] & Z_{k+\ell+1} &
\end{tikzcd}$$

commute, for each $k, \ell \geq 0$.

Proof. Using the equivariance relation $(I_k \oplus \chi_{\ell,1})\phi_{k,\ell+1} = \phi_{k,1+\ell}(1 \wedge \chi_{\ell,1})$ the two diagrams preceding the proposition can be rewritten and combined into this one diagram, as asserted. $\square$

Remark 6.2. The identity map $X \wedge Y \to X \wedge Y$ corresponds to the canonical map $\pi\colon X \otimes Y \to X \wedge Y$. Its components

$$\iota_{k,\ell}\colon X_k \wedge Y_\ell \subset (X \otimes Y)_{k+\ell} \longrightarrow (X \wedge Y)_{k+\ell}$$

are $O(k) \times O(\ell)$-equivariant and make the bilinearity diagram

$$\begin{tikzcd}
X_k \wedge S^1 \wedge Y_\ell \arrow[d, "\sigma\wedge 1"'] & X_k \wedge Y_\ell \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "\iota_{k,\ell}\wedge 1"'] \arrow[dr, "1\wedge\sigma"] & \\
X_{k+1} \wedge Y_\ell \arrow[d, "\iota_{k+1,\ell}"'] & (X\wedge Y)_{k+\ell} \wedge S^1 \arrow[d, "\sigma"'] & X_k \wedge Y_{\ell+1} \arrow[dl, "\iota_{k,\ell+1}"] \\
(X\wedge Y)_{k+1+\ell} \arrow[r, "I_k \oplus \chi_{1,\ell}"] & (X\wedge Y)_{k+\ell+1} &
\end{tikzcd}$$

commute, for each $k, \ell \geq 0$. This reflects the identification made in the passage from $X \otimes Y$ to $X \wedge Y$.

Lemma 6.3. *If the pairing $\mu\colon X \wedge Y \to Z$ corresponds to the collection of maps $\phi_{k,\ell}\colon X_k \wedge Y_\ell \to Z_{k+\ell}$, then the "opposite" pairing $\mu\gamma\colon Y \wedge X \to X$ corresponds to the collection of maps*

$$\chi_{k,\ell} \circ \phi_{k,\ell} \circ \gamma\colon Y_\ell \wedge X_k \longrightarrow Z_{\ell+k}\,.$$

Proof. The diagram

$$\begin{tikzcd}
Y_\ell \wedge X_k \arrow[rr, "\gamma"] \arrow[d, "\iota_{\ell,k}"'] & & X_k \wedge Y_\ell \arrow[d, "\iota_{k,\ell}"] \\
(Y\wedge X)_{\ell+k} \arrow[r, "\gamma_{\ell+k}"] \arrow[dr, "(\mu\gamma)_{\ell+k}"'] & (X\wedge Y)_{\ell+k} \arrow[r, "\chi_{\ell,k}"] \arrow[d, "\mu_{\ell+k}"] & (X\wedge Y)_{k+\ell} \arrow[d, "\mu_{k+\ell}"] \\
& Z_{\ell+k} \arrow[r, "\chi_{\ell,k}"] & Z_{k+\ell}
\end{tikzcd}$$

commutes. □

6.2 Pairings of homotopy groups

We will define a natural pairing

$$\pi_i(X) \times \pi_j(Y) \overset{\cdot}{\longrightarrow} \pi_{i+j}(X \wedge Y)$$

for orthogonal spectra X and Y, mapping the class in $\pi_i(X)$ of $[f] \in \pi_{i+k}(X_k)$ and the class in $\pi_j(Y)$ of $[g] \in \pi_{j+\ell}(Y_\ell)$ to the class of an element $[f]\cdot[g] \in \pi_{i+j+k+\ell}((X\wedge Y)_{k+\ell})$. It will be bilinear, and hence induce a linear pairing

$$\pi_i(X) \otimes \pi_j(Y) \overset{\cdot}{\longrightarrow} \pi_{i+j}(X \wedge Y)$$

for all i and j. Any pairing $\mu\colon X \wedge Y \to Z$ will thus induce a pairing

$$\mu_*\colon \pi_i(X) \otimes \pi_j(Y) \longrightarrow \pi_{i+j}(Z)\,.$$

Definition 6.4. Let $(\mathscr{C}, \wedge, S)$ and $(\mathscr{D}, \otimes, U)$ be monoidal categories, with coherent isomorphisms α, λ and ρ. A lax monoidal functor F from $(\mathscr{C}, \wedge, S)$ to $(\mathscr{D}, \otimes, Z)$ is a functor $F\colon \mathscr{C} \to \mathscr{D}$, a natural morphism

$$F(X) \otimes F(Y) \longrightarrow F(X \wedge Y)$$

for $X, Y \in \mathscr{C}$ and a morphism

$$U \longrightarrow F(S)$$

such that the diagrams

$$\begin{tikzcd}
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\alpha"] \arrow[d] & (F(X) \otimes F(Y)) \otimes F(Z) \arrow[d] \\
F(X) \otimes F(Y \wedge Z) \arrow[d] & F(X \wedge Y) \otimes F(Z) \arrow[d] \\
F(X \wedge (Y \wedge Z)) \arrow[r, "F(\alpha)"] & F((X \wedge Y) \wedge Z)
\end{tikzcd}$$

and

$$\begin{tikzcd} U \otimes F(Y) \arrow[r, "\lambda"] \arrow[d] & F(Y) \\ F(S) \otimes F(Y) \arrow[r] & F(S \wedge Y) \arrow[u, "F(\lambda)"'] \end{tikzcd} \qquad \begin{tikzcd} F(Y) \otimes U \arrow[r, "\rho"] \arrow[d] & F(Y) \\ F(Y) \otimes F(S) \arrow[r] & F(Y \wedge S) \arrow[u, "F(\rho)"'] \end{tikzcd}$$

commute.

A strong monoidal functor F is a monoidal functor F, as above, such that each morphism $F(X) \otimes F(Y) \to F(X \wedge Y)$ and $U \to F(S)$ is an isomorphism.

A strict monoidal functor F is a monoidal functor F such that each morphism $F(X) \otimes F(Y) \to F(X \wedge Y)$ and $U \to F(S)$ is an identity.

Remark 6.5. A lax monoidal functor maps monoids to monoids. An object X in $\mathscr{C}$ with multiplication $\mu \colon X \wedge X \to X$ and unit $\eta \colon S \to X$ maps to a monoid $F(X)$ in $\mathscr{D}$ with multiplication

$$F(X) \otimes F(X) \longrightarrow F(X \wedge X) \overset{F(\mu)}{\longrightarrow} F(X)$$

and unit

$$U \longrightarrow F(S) \overset{F(\eta)}{\longrightarrow} F(X)\,.$$

A map $f \colon X \to Y$ of monoids in $(\mathscr{C}, \wedge, S)$ induces a map $F(f) \colon F(X) \to F(Y)$ of monoids in $(\mathscr{D}, \otimes, U)$.

Definition 6.6. Let $(\mathscr{C}, \wedge, S)$ and $(\mathscr{D}, \otimes, U)$ be symmetric monoidal categories, with coherent isomorphisms α, λ, ρ and γ. A lax symmetric monoidal functor F from $\mathscr{C}$ to $\mathscr{D}$ is a lax monoidal functor such that the diagram

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\gamma"] \arrow[d] & F(Y) \otimes F(X) \arrow[d] \\ F(X \wedge Y) \arrow[r, "F(\gamma)"] & F(Y \wedge X) \end{tikzcd}$$

commutes.

Remark 6.7. A lax monoidal functor maps monoids to monoids. An object X in $\mathscr{C}$ with multiplication $\mu \colon X \wedge X \to X$ and unit $\eta \colon S \to X$ maps to a monoid $F(X)$ in $\mathscr{D}$ with multiplication

$$F(X) \otimes F(X) \longrightarrow F(X \wedge X) \overset{F(\mu)}{\longrightarrow} F(X)$$

and unit

$$U \longrightarrow F(S) \overset{F(\eta)}{\longrightarrow} F(X)\,.$$

A lax symmetric monoidal functor also maps commutative monoids to commutative monoids, since the diagram

$$\begin{tikzcd} F(X) \otimes F(X) \arrow[rr, "\gamma"] \arrow[d] & & F(X) \otimes F(X) \arrow[d] \\ F(X \wedge X) \arrow[rr, "F(\gamma)"] \arrow[dr, "F(\mu)"'] & & F(X \wedge X) \arrow[dl, "F(\mu)"] \\ & F(X) & \end{tikzcd}$$

commutes. A map $f \colon X \to Y$ of commutative monoids in $(\mathscr{C}, \wedge, S, \gamma)$ induces a map $F(f) \colon F(X) \to F(Y)$ of commutative monoids in $(\mathscr{D}, \otimes, U, \gamma)$.

As usual in algebraic topology, we give the category grAb of graded abelian groups the symmetric monoidal structure where

$$(A_* \otimes B_*)_n = \bigoplus_{i+j=n} A_i \otimes B_j$$

and

$$\gamma \colon A_* \otimes B_* \to B_* \otimes A_*$$

maps $a \otimes b$ to $(-1)^{ij} b \otimes a$, for $a \in A_i$ and $b \in B_j$.

Theorem 6.8. *There is a natural pairing*

$$\cdot\colon \pi_*(X) \otimes \pi_*(Y) \longrightarrow \pi_*(X \wedge Y)$$

and a homomorphism

$$\mathbb{Z} \longrightarrow \pi_*(S)$$

that make make π_* a lax symmetric monoidal functor from $(\mathrm{Sp}^{\mathbb{O}}, \wedge, S)$ *to* $(\mathrm{grAb}, \otimes, \mathbb{Z})$.

Given maps $f\colon S^{i+k} \to X_k$ and $g\colon S^{j+\ell} \to Y_\ell$, we can form the composite

$$f * g = \iota_{k,\ell}(f \wedge g)\colon S^{i+k} \wedge S^{j+\ell} \xrightarrow{f\wedge g} X_k \wedge Y_\ell \xrightarrow{\iota_{k,\ell}} (X \wedge Y)_{k+\ell}\,.$$

The homotopy class $[f * g] \in \pi_{i+k+j+\ell}((X \wedge Y)_{k+\ell})$ only depends on the homotopy classes $[f]$ and $[g]$, so we can let $[f * g] = [f] * [g]$. We must address how to make the pairings $*\colon \pi_{i+k}(X_k) \times \pi_{j+\ell}(Y_\ell) \to \pi_{i+k+j+\ell}((X \wedge Y)_{k+\ell})$ induce a pairing $\cdot\colon \pi_i(X) \times \pi_j(Y) \to \pi_{i+j}(X \wedge Y)$, where $\pi_i(X) = \operatorname{colim}_k \pi_{i+k}(X_k)$ and $\pi_j(Y) = \operatorname{colim}_\ell \pi_{j+\ell}(Y_\ell)$. The class in $\pi_i(X)$ of $[f] \in \pi_{i+k}(X_k)$ is the same as the class of its image $[\sigma(f \wedge 1)] \in \pi_{i+k+1}(X_{k+1})$. Let use write

$$f' = \sigma(f \wedge 1)\colon S^{i+k} \wedge S^1 \xrightarrow{f\wedge 1} X_k \wedge S^1 \xrightarrow{\sigma} X_{k+1}$$

for this composite. By bilinearity, the diagrams

$$\begin{tikzcd}
S^{i+k} \wedge S^1 \wedge S^{j+\ell} \arrow[d, "f\wedge 1\wedge g"'] & S^{i+k} \wedge S^{j+\ell} \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "f\wedge g\wedge 1"'] \\
X_k \wedge S^1 \wedge Y_\ell \arrow[d, "\sigma\wedge 1"'] & X_k \wedge Y_\ell \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "\iota_{k,\ell}\wedge 1"'] \\
X_{k+1} \wedge Y_\ell \arrow[d, "\iota_{k+1,\ell}"'] & (X \wedge Y)_{k+\ell} \wedge S^1 \arrow[d, "\sigma"'] \\
(X \wedge Y)_{k+1+\ell} \arrow[r, "I_k \oplus \chi_{1,\ell}"'] & (X \wedge Y)_{k+\ell+1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
S^{i+k} \wedge S^{j+\ell} \wedge S^1 \arrow[d, "f\wedge g\wedge 1"'] & \\
X_k \wedge Y_\ell \wedge S^1 \arrow[d, "\iota_{k,\ell}\wedge 1"'] \arrow[dr, "1\wedge\sigma"] & \\
(X \wedge Y)_{k+\ell} \wedge S^1 \arrow[d, "\sigma"'] & X_k \wedge Y_{\ell+1} \arrow[dl, "\iota_{k,\ell+1}"] \\
(X \wedge Y)_{k+\ell+1} &
\end{tikzcd}$$

commute, hence so do the diagrams

$$\begin{tikzcd}
S^{i+k} \wedge S^1 \wedge S^{j+\ell} \arrow[d, "f' * g"'] & S^{i+k} \wedge S^{j+\ell} \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "(f*g)'"] \\
(X \wedge Y)_{k+1+\ell} \arrow[r, "I_k \oplus \chi_{1,\ell}"'] & (X \wedge Y)_{k+\ell+1}
\end{tikzcd}$$

and

$$
\begin{tikzcd}
& S^{i+k} \wedge S^{j+\ell} \wedge S^1 \arrow[dl, bend right, "(f*g)'"'] \arrow[dr, bend left, "f*g'"] & \\
& (X \wedge Y)_{k+\ell+1} \, . &
\end{tikzcd}
$$

In formulas,

$$(I_k \oplus \chi_{1,\ell})(f' * g)(1 \wedge \gamma) = (f * g)' = f * g'$$

as maps $S^{i+k} \wedge S^{j+\ell} \wedge S^1 \to (X \wedge Y)_{k+\ell+1}$ Here composition with $I_k \oplus \chi_{1,\ell}$ and $1 \wedge \gamma$ are compatible with multiplication by $(-1)^{\ell}$ and $(-1)^{j+\ell}$, respectively, on $\pi_{i+j}(X \wedge Y)$.

Proposition 6.9. *For each $A \in O(n)$, inducing a map $A\colon X_n \to X_n$, the diagram*

$$
\begin{tikzcd}
\pi_{k+n}(X_n) \arrow[r] \arrow[d, "A_*"'] & \pi_k(X) \arrow[d, "\det(A)"] \\
\pi_{k+n}(X_n) \arrow[r] & \pi_k(X)
\end{tikzcd}
$$

commutes, where the horizontal arrows are the canonical morphisms.

Proof. The matrices $A + (1)$ and $I_n + (\det(A))$ lie in the same path component of $O(n+1)$, hence induce homotopic maps $X_{n+1} \to X_{n+1}$. It therefore suffices to note that the diagram

$$
\begin{tikzcd}
\pi_{k+n+1}(X_n \wedge S^1) \arrow[r] \arrow[d, "(1\wedge\det(A))_*"'] & \pi_k(X) \arrow[d, "\det(A)"] \\
\pi_{k+n+1}(X_n \wedge S^1) \arrow[r] & \pi_k(X)
\end{tikzcd}
$$

commutes. □

In other words, the class of $[f*g]$ and $[f*g']$ in $\pi_{i+j}(X\wedge Y)$ is equal to $(-1)^j$ times the class of $[f'*g]$. To compensate for the sign change by $(-1)^j$ when $f\colon S^{i+k} \to X_k$ is replaced by $f'\colon S^{i+k+1} \to X_{k+1}$, i.e., when k increases by one, we can multiply $[f * g]$ by $(-1)^{jk}$.

Definition 6.10. For $f\colon S^{i+k} \to X_k$ and $g\colon S^{j+\ell} \to Y_\ell$, with $[f] \in \pi_{i+k}(X_k)$ and $[g] \in \pi_{j+\ell}(Y_\ell)$ representing classes in $\pi_i(X)$ and $\pi_j(Y)$, respectively, let

$$[f] \cdot [g] = (-1)^{jk}[f * g]$$

in $\pi_{i+j}(X \wedge Y)$ be $(-1)^{jk}$ times the class of $f * g = \iota_{k,\ell}(f \wedge g)\colon S^{i+k} \wedge S^{j+\ell} \to X_k \wedge Y_\ell \to (X \wedge Y)_{k+\ell}$.

By the discussion above this is well-defined, since $[f'] \cdot [g] = [f] \cdot [g] = [f] \cdot [g']$.

Remark 6.11. The sign $(-1)^{jk}$ can be viewed as arising from the identification of $[f*g] \in \pi_{i+k+j+\ell}((X\wedge Y)_{k+\ell})$ with a class in $\pi_{i+j+k+\ell}((X\wedge Y)_{k+\ell})$, which stabilizes to $\pi_{i+j}(X\wedge Y)$. Note the interchange of j and k from $i+k+j+\ell$ to $i+j+k+\ell$. When $i, j \geq 0$ this interchange can be realized by the symmetry $\gamma\colon S^k \wedge S^j \to S^j \wedge S^k$, so that $[f] \cdot [g]$ is the class of $[f \cdot g]$, where

$$f \cdot g = \iota_{k,\ell}(f \wedge g)(1 \wedge \gamma \wedge 1)$$

is defined to be the composite

$$S^i \wedge S^j \wedge S^k \wedge S^\ell \xrightarrow{1\wedge\gamma\wedge 1} S^i \wedge S^k \wedge S^j \wedge S^\ell \xrightarrow{f\wedge g} X_k \wedge Y_\ell \xrightarrow{\iota_{k,\ell}} (X \wedge Y)_{k+\ell} \, .$$

However, when $i < 0$ or $j < 0$ this does not make sense at the space level, since S^i or S^j does not exist. The use of the algebraic sign $(-1)^{jk}$ works also for negative j. Of course, if $\pi_i(X) = 0$ for $i < 0$ and $\pi_j(Y) = 0$ for $j < 0$, in which case we say that X and Y are connective, then the space level definition $[f] \cdot [g] = [f \cdot g]$ will be satisfactory.

The unit homomorphism $\mathbb{Z} \to \pi_*(S)$ takes $d \in \mathbb{Z}$ to the class in $\pi_0(S)$ of the degree d map $S^1 \to S^1$ in $\pi_1(S^1)$.

(Check bilinearity of $*$ or $\cdot$.)

(Check associativity and unitality diagrams.)

It remains to show that π_* is lax symmetric monoidal. Recall that the isomorphism $\gamma\colon \pi_*(X) \otimes \pi_*(Y) \to \pi_*(Y) \otimes \pi_*(X)$ involves the usual sign $(-1)^{ij}$ that is introduced when two classes of degree i and j are interchanged.

Proposition 6.12. *The diagram*

$$\begin{tikzcd}
\pi_*(X) \otimes \pi_*(Y) \arrow[r, "\gamma"] \arrow[d, "\cdot"'] & \pi_*(Y) \otimes \pi_*(X) \arrow[d, "\cdot"] \\
\pi_*(X \wedge Y) \arrow[r, "\gamma_*"] & \pi_*(Y \wedge X)
\end{tikzcd}$$

commutes. In symbols,

$$\gamma_*(a \cdot b) = (-1)^{ij} b \cdot a$$

for $a \in \pi_i(X)$ and $b \in \pi_j(Y)$.

Proof. The twist isomorphism $\gamma\colon X \wedge Y \to Y \wedge X$ is induced by $\gamma\colon X \otimes Y \to Y \otimes X$, so that the diagram

$$\begin{tikzcd}
X_k \wedge Y_\ell \arrow[r] \arrow[dr, "\iota_{k,\ell}"'] & O(k+\ell)_+ \underset{O(k)\times O(\ell)}{\wedge} X_k \wedge Y_\ell \arrow[r, "\gamma"] \arrow[d] & O(\ell+k)_+ \underset{O(\ell)\times O(k)}{\wedge} Y_\ell \wedge X_k \arrow[d] & Y_\ell \wedge X_k \arrow[l] \arrow[dl, "\iota_{\ell,k}"] \\
& (X \wedge Y)_{k+\ell} \arrow[r, "\gamma"] & (Y \wedge X)_{\ell+k} &
\end{tikzcd}$$

commutes. Here the upper central arrow γ maps $A \wedge x \wedge y$ to $A\chi_{\ell,k} \wedge y \wedge x$. Given maps $f\colon S^{i+k} \to X_k$ and $g\colon S^{j+\ell} \to Y_\ell$, the composite

$$S^{i+k} \wedge S^{j+\ell} \xrightarrow{f\wedge g} X_k \wedge Y_\ell \xrightarrow{\iota_{k,\ell}} (X \wedge Y)_{k+\ell} \xrightarrow{\gamma} (Y \wedge X)_{\ell+k}$$

is equal to the composite

$$S^{i+k} \wedge S^{j+\ell} \xrightarrow{\gamma} S^{j+\ell} \wedge S^{i+k} \xrightarrow{g\wedge f} Y_\ell \wedge X_k \xrightarrow{\iota_{\ell,k}} (Y \wedge X)_{\ell+k} \xrightarrow{\chi_{\ell,k}} (Y \wedge X)_{k+\ell}\,,$$

so that

$$\gamma(f * g) = \chi_{\ell,k}(g * f)\gamma\,.$$

Here the right hand map γ has degree $(-1)^{(i+k)(j+\ell)}$, and $\chi_{\ell,k}$ induced multiplication by $(-1)^{k\ell}$ after stabilization. Recall that $[f]\cdot[g] = (-1)^{jk}[f * g]$, so that $[g]\cdot[f] = (-1)^{i\ell}[g * f]$. Hence

$$\gamma_*([f]\cdot[g]) = (-1)^{jk}[\gamma(f*g)] = (-1)^{jk}(-1)^{k\ell}(-1)^{(i+k)(j+\ell)}(-1)^{i\ell}[g]\cdot[f] = (-1)^{ij}[g]\cdot[f]$$

in $\pi_{i+j}(Y \wedge X)$. □

There is also good compatibility with the closed structure. For graded abelian groups B_* and C_* we let $\mathrm{Hom}(B_*, C_*)$ be the graded abelian group with

$$\mathrm{Hom}(B_*, C_*)_i = \prod_{i+j=n} \mathrm{Hom}(B_j, C_n)\,,$$

so that there is an adjunction isomorphism

$$\mathrm{grAb}(A_* \otimes B_*, C_*) \cong \mathrm{grAb}(A_*, \mathrm{Hom}(B_*, C_*))$$

that can be enriched to a natural isomorphism

$$\mathrm{Hom}(A_* \otimes B_*, C_*) \cong \mathrm{Hom}(A_*, \mathrm{Hom}(B_*, C_*))\,.$$

Theorem 6.13. *There is a natural homomorphism*

$$\pi_* F(Y, Z) \longrightarrow \mathrm{Hom}(\pi_*(Y), \pi_*(Z))$$

that makes π_* a closed functor: If $\mu\colon X \wedge Y \to Z$ in $\mathrm{Sp}^{\mathbb{O}}$ *is left adjoint to $\tilde{\mu}\colon X \to F(Y, Z)$, then the composite*

$$\pi_*(X) \otimes \pi_*(Y) \xrightarrow{\cdot} \pi_*(X \wedge Y) \xrightarrow{\mu_*} \pi_*(Z)$$

in grAb *is left adjoint to the composite*

$$\pi_*(X) \xrightarrow{\tilde{\mu}_*} \pi_* F(Y, Z) \longrightarrow \mathrm{Hom}(\pi_*(Y), \pi_*(Z)) .$$

Proof. The homomorphism is determined by the case $\tilde{\mu} = \mathbb{1}$, with $X = F(Y, Z)$, which is right adjoint to the counit $\mu = \epsilon\colon F(Y, Z) \wedge Y \to Z$, so

$$\pi_* F(Y, Z) \longrightarrow \mathrm{Hom}(\pi_*(Y), \pi_*(Z))$$

is right adjoint to the composite

$$\pi_* F(Y, Z) \otimes \pi_*(Y) \xrightarrow{\cdot} \pi_*(F(Y, X) \wedge Y) \xrightarrow{\epsilon_*} \pi_*(Z) .$$

To make this more explicit, consider maps $f\colon S^{i+k} \to F(Y, Z)_k$ and $g\colon S^{j+\ell} \to Y_\ell$, with homotopy classes $[f] \in \pi_{i+k} F(Y, Z)_k$ and $[g] \in \pi_{j+\ell}(Y_\ell)$ representing elements in $\pi_i F(Y, Z)$ and $\pi_j(Y)$. The class of $[f]$ then maps to the homomorphism $\pi_*(Y) \to \pi_*(Z)$ of degree i that maps $\pi_j(Y)$ to $\pi_{i+j}(Z)$ by sending the class of $[g]$ to the class of

$$\epsilon_*([f] \cdot [g]) = (-1)^{jk} [\epsilon(f * g)] = (-1)^{jk} [\epsilon\iota_{k,\ell}(f \wedge g)] .$$

Here $\epsilon\iota_{k,\ell}$ is given by

$$F(Y, Z)_k \wedge Y_\ell \xrightarrow{\iota_\ell \wedge 1} \mathscr{T}(Y_\ell, Z_{k+\ell})^{O(\ell)} \wedge Y_\ell \xrightarrow{\epsilon} Z_{k+\ell} ,$$

where ι_ℓ is the ℓ-th component of the canonical map ι in the equalizer diagram that characterizes $F(Y, Z)_k$. Hence $\epsilon\iota_{k,\ell}(f \wedge g)$ is the composite map

$$S^{i+k} \wedge S^{j+\ell} \xrightarrow{f \wedge g} F(Y, Z)_k \wedge Y_\ell \xrightarrow{\iota_\ell} \mathscr{T}(Y_\ell, Z_{k+\ell})^{O(\ell)} \wedge Y_\ell \xrightarrow{\epsilon} Z_{k+\ell} .$$

taking $s \in S^{i+k}$ and $t \in S^{j+\ell}$ to $(\iota_\ell f(s))(g(t))$, where $\iota_\ell f(s)\colon Y_\ell \to Z_{k+\ell}$ and $g(t) \in Y_\ell$. $\square$

6.3 Ring, module and algebra spectra

An orthogonal ring spectrum is an orthogonal spectrum R with a multiplication $\mu\colon R \wedge R \to R$ and unit $\eta\colon S \to R$ such that the associativity and unitality diagrams commute. In other words, (R, μ, η) is a monoid in $(\mathrm{Sp}^{\mathbb{O}}, \wedge, S)$. A map $f\colon Q \to R$ of orthogonal ring spectra is a map of monoids, i.e., a map $f\colon Q \to R$ of orthogonal spectra that commute with the multiplication and unit maps.

The homotopy groups $\pi_*(R)$ form a graded ring, with multiplication $\mu_*\colon \pi_*(R) \otimes \pi_*(R) \to \pi_*(R)$ and unit $\eta_*\colon \mathbb{Z} \to \pi_*(R)$. A map $f\colon Q \to R$ of orthogonal ring spectra induces a homomorphism $f_*\colon \pi_*(Q) \to \pi_*(R)$ of graded rings.

A commutative orthogonal ring spectrum is an orthogonal ring spectrum R such that the commutativity diagram

$$\begin{tikzcd} R \wedge R \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & R \wedge R \arrow[dl, "\mu"] \\ & R & \end{tikzcd}$$

commutes. The homotopy groups $\pi_*(R)$ of a commutative orthogonal ring spectrum form a graded commutative ring, so that $ab = (-1)^{ij} ba$ for $a \in \pi_i(R)$ and $b \in \pi_j(R)$. In particular, $2a^2 = 0$ for a in odd degrees.

The sphere spectrum S is the initial orthogonal ring spectrum, and also the initial commutative orthogonal ring spectrum.

Remark 6.14. The term "ring spectrum" usually refers to a monoid in the stable homotopy category, i.e., a spectrum R with morphisms $\mu\colon R \wedge R \to R$ and $\eta\colon S \to R$ such that the associativity and unitality diagrams commute, in the stable homotopy category. An orthogonal ring spectrum determines a ring spectrum in this sense, but the converse does not generally hold: Having commuting diagrams in the stable homotopy category is not generally enough to find representing maps μ and η that make the associativity and unitality diagrams commute in the category of orthogonal spectra. Similarly the term "commutative ring spectrum" traditionally refers to a commutative monoid in the stable homotopy category, and these may or may not come from commutative orthogonal ring spectra.

Let R be an orthogonal ring spectrum. A left R-module is an orthogonal spectrum M with a pairing $\lambda\colon R \wedge M \to M$ such that the diagrams

$$\begin{tikzcd}
R \wedge R \wedge M \arrow[r, "1\wedge\lambda"] \arrow[d, "\mu\wedge 1"'] & R \wedge M \arrow[d, "\lambda"] \\
R \wedge M \arrow[r, "\lambda"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & R \wedge M \arrow[d, "\lambda"] \\
& M
\end{tikzcd}$$

commute. A right R-module is an orthogonal spectrum M with a pairing $\rho\colon M \wedge R \to M$ such that the diagrams

$$\begin{tikzcd}
M \wedge R \wedge R \arrow[r, "\rho\wedge 1"] \arrow[d, "1\wedge\mu"'] & M \wedge R \arrow[d, "\rho"] \\
M \wedge R \arrow[r, "\rho"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \wedge R \arrow[d, "\rho"'] & M \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\cong"] \\
M &
\end{tikzcd}$$

commute. If R is commutative, then left R-modules correspond to right R-modules via $\rho = \lambda\gamma$ and vice versa.

The homotopy groups $\pi_*(M)$ of a left R-module M form a left $\pi_*(R)$-module, with left action

$$\pi_*(R) \otimes \pi_*(M) \xrightarrow{\cdot} \pi_*(R \wedge M) \xrightarrow{\lambda_*} \pi_*(M)\,,$$

and similarly for right R-modules and right $\pi_*(R)$-modules.

Given a right R-module M and a left R-module N, the smash product $M \wedge_R N$ is the orthogonal spectrum defined as the coequalizer

$$\begin{tikzcd}
M \wedge R \wedge N \arrow[r, shift left, "\rho\wedge 1"] \arrow[r, shift right, "1\wedge\lambda"'] & M \wedge N \arrow[r, "\pi"] & M \wedge_R N\,.
\end{tikzcd}$$

The induced diagram

$$\begin{tikzcd}
\pi_*(M) \otimes \pi_*(R) \otimes \pi_*(N) \arrow[r, shift left, "\rho_*\otimes 1"] \arrow[r, shift right, "1\otimes\lambda_*"'] & \pi_*(M) \otimes \pi_*(N) \arrow[r, "\pi_*"] & \pi_*(M \wedge_R N)
\end{tikzcd}$$

commutes, so there is an induced homomorphism

$$\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N) \xrightarrow{\cdot} \pi_*(M \wedge_R N)\,.$$

If R is commutative, then $M \wedge_R N$ is naturally a left (and right) R-module, and the homomorphism above is one of left (and right) $\pi_*(R)$-modules.

In particular, for $R = S$ the canonical map $M \wedge N \to M \wedge_S N$ is an isomorphism, and the natural pairing

$$\pi_*(M) \otimes \pi_*(N) \overset{\cdot}{\longrightarrow} \pi_*(M \wedge N)$$

of graded abelian groups descends to a natural pairing

$$\pi_*(M) \otimes_{\pi_*(S)} \pi_*(N) \overset{\cdot}{\longrightarrow} \pi_*(M \wedge N)$$

of $\pi_*(S)$-modules.

Given two left R-modules M and N, the R-linear function spectrum $F_R(M,N)$ is the orthogonal spectrum defined as the equalizer

$$\begin{tikzcd} F_R(M,N) \arrow[r, "\iota"] & F(M,N) \arrow[r, shift left, "\lambda^*"] \arrow[r, shift right, "\lambda^\vee"'] & F(R \wedge M, N) \end{tikzcd}$$

where $\lambda^* = F(\lambda, 1)$ and $\lambda^\vee$ is adjoint to the composite

$$F(M,N) \wedge R \wedge M \xrightarrow{\gamma\wedge 1} R \wedge F(M,N) \wedge M \xrightarrow{1\wedge\epsilon} R \wedge N \xrightarrow{\lambda} N\,.$$

The induced diagram

$$\begin{tikzcd} \pi_* F_R(M,N) \arrow[r, "\iota_*"] & \mathrm{Hom}(\pi_*(M), \pi_*(N)) \arrow[r, shift left, "(\lambda_*)^*"] \arrow[r, shift right, "(\lambda_*)^\vee"'] & \mathrm{Hom}(\pi_*(R) \otimes \pi_*(M), \pi_*(N)) \end{tikzcd}$$

commutes, so there is an induced homomorphism

$$\pi_* F_R(M,N) \longrightarrow \mathrm{Hom}_{\pi_*(R)}(\pi_*(M), \pi_*(N))$$

to the graded abelian group of left $\pi_*(R)$-module homomorphisms. If R is commutative then $F_R(M,N)$ is naturally a left (and right) R-module, and the homomorphism above is one of left (and right) $\pi_*(R)$-modules.

In the case $R = S$, the canonical map $F_S(M,N) \to F(M,N)$ is an isomorphism, so the natural homomorphism

$$\pi_* F(M,N) \longrightarrow \mathrm{Hom}(\pi_*(M), \pi_*(N))$$

of graded abelian groups factors through a natural homomorphism

$$\pi_* F(M,N) \longrightarrow \mathrm{Hom}_{\pi_*(S)}(\pi_*(M), \pi_*(N))$$

of $\pi_*(S)$-modules.

Let R be a commutative orthogonal ring spectrum. The smash product over R, taking M and N to $M \wedge_R N$, defines a symmetric monoidal structure on the category of left (and right) R-modules, with unit object R.

An R-algebra A is a monoid in this category of R-modules, i.e., an R-module A with a multiplication $\mu\colon A \wedge_R A \to A$ and a unit $\eta\colon R \to A$, such that that associativity and unitality diagrams

$$\begin{tikzcd} A \wedge_R A \wedge_R A \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & A \wedge_R A \arrow[d, "\mu"] \\ A \wedge_R A \arrow[r, "\mu"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} R \wedge_R A \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & A \wedge_R A \arrow[d, "\mu"] & A \wedge_R R \arrow[l, "1\wedge\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

commute, in the category of R-modules. A map $A \to B$ of R-algebras is a monoid map in R-modules.

In view of the definition of $A \wedge_R A$ as a coequalizer, an equivalent definition of an R-algebra A is an orthogonal spectrum A with maps $\phi\colon A \wedge A \to A$ and $\eta\colon R \to A$ such that the composite

$$S \wedge A \longrightarrow R \wedge A \xrightarrow{\eta\wedge 1} A \wedge A \xrightarrow{\phi} A$$

is equal to the canonical isomorphism $S \wedge A \cong A$, and the associativity, multiplicativity and centrality diagrams

$$\begin{tikzcd} A \wedge A \wedge A \arrow[r, "\phi\wedge 1"] \arrow[d, "1\wedge\phi"'] & A \wedge A \arrow[d, "\phi"] \\ A \wedge A \arrow[r, "\phi"] & A \end{tikzcd}$$

$$\begin{tikzcd} R \wedge R \arrow[r, "\eta\wedge\eta"] \arrow[d] & A \wedge A \arrow[d, "\phi"] \\ R \arrow[r, "\eta"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} R \wedge A \arrow[d, "\eta\wedge 1"'] & A \wedge R \arrow[l, "\gamma"'] \arrow[d, "1\wedge\eta"] \\ A \wedge A \arrow[dr, "\phi"'] & A \wedge A \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

commute, in the category of orthogonal spectra. The left R-module structure on A is then given by the composite $\lambda = \phi \circ (\eta \wedge 1)\colon R \wedge A \to A \wedge A \to A$.

A commutative R-algebra A is a commutative monoid in R-modules, i.e., an R-algebra such that the commutativity diagram

$$\begin{tikzcd} A \wedge_R A \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & A \wedge_R A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

commutes. A map $A \to B$ of commutative R-algebras is a commutative monoid map in R-modules.

Equivalently, a commutative R-algebra is an orthogonal spectrum A with maps $\phi\colon A \wedge A \to A$ and $\eta\colon R \to A$, satisfying the conditions for an R-algebra, together with the condition that the commutativity diagram

$$\begin{tikzcd} A \wedge A \arrow[rr, "\gamma"] \arrow[dr, "\phi"'] & & A \wedge A \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

commutes, in orthogonal spectra. Note that in this case the centrality diagram is superfluous.

In the category of commutative R-algebras, the coproduct of A and B is given by the smash product $A \wedge_R B$, with the multiplication

$$(A \wedge_R B) \wedge_R (A \wedge_R B) \overset{1\wedge\gamma\wedge 1}{\longrightarrow} A \wedge_R A \wedge_R B \wedge_R B \overset{\phi\wedge\phi}{\longrightarrow} A \wedge_R B$$

and the unit

$$R \cong R \wedge_R R \overset{\eta\wedge\eta}{\longrightarrow} A \wedge_R B\,.$$

In the special case $R = S$, an S-algebra (in orthogonal spectra) is the same as an orthogonal ring spectrum, and a commutative S-algebra is the same as a commutative orthogonal ring spectrum. The coproduct of two commutative S-algebras, A and B, is the smash product $A \wedge B$.

The homotopy groups $\pi_*(A)$ of an R-algebra A form a graded $\pi_*(R)$-algebra, with multiplication

$$\pi_*(A) \otimes_{\pi_*(R)} \pi_*(A) \xrightarrow{\cdot} \pi_*(A \wedge_R A) \xrightarrow{\mu_*} \pi_*(A)$$

and unit

$$\pi_*(R) \xrightarrow{\eta_*} \pi_*(A)\,,$$

which is graded commutative if A is a commutative R-algebra.

Remark 6.15. For orthogonal spectra D and E, and based CW complexes X and Y, the pairing

$$\begin{aligned}\wedge\colon D_k(X) \otimes E_\ell(Y) = \pi_k(D \wedge X) \otimes \pi_\ell(E \wedge Y) &\xrightarrow{\cdot} \pi_{k+\ell}(D \wedge X \wedge E \wedge Y) \\ &\xrightarrow{1\wedge\gamma\wedge 1} \pi_{k+\ell}(D \wedge E \wedge X \wedge Y) = (D \wedge E)_{k+\ell}(X \wedge Y)\end{aligned}$$

defines an external smash product in homology. If E is an orthogonal ring spectrum, the composite

$$\cdot\colon E_k(X) \otimes E_\ell(Y) \xrightarrow{\wedge} (E \wedge E)_{k+\ell}(X \wedge Y) \xrightarrow{\mu_*} E_{k+\ell}(X \wedge Y)$$

defines an internal smash product. (Get Pontryagin product in the case $X = Y = M_+$ with M an H-space.)

For finite based CW complexes X and Y, the pairing

$$\begin{aligned}\wedge\colon D^k(X) \otimes E^\ell(Y) = \pi_{-k}F(X, D) \otimes \pi_{-\ell}(Y, E) &\xrightarrow{\cdot} \pi_{-k-\ell}(F(X, D) \wedge F(Y, E)) \\ &\xrightarrow{\wedge} \pi_{-k-\ell}(F(X \wedge Y, D \wedge E) = (D \wedge E)^{k+\ell}(X \wedge Y)\end{aligned}$$

defines an external smash product in cohomology. If E is an orthogonal ring spectrum, the composite

$$E^k(X) \otimes E^\ell(Y) \xrightarrow{\wedge} (E \wedge E)^{k+\ell}(X \wedge Y) \xrightarrow{\mu_*} E^{k+\ell}(X \wedge Y)$$

defines an internal smash product. In the case $X = Y$, the diagonal $\Delta\colon X \to X \wedge X$ induces an internal cup product

$$\cup\colon E^k(X) \otimes E^\ell(X) \longrightarrow E^{k+\ell}(X \wedge X) \xrightarrow{\Delta^*} E^{k+\ell}(X)\,.$$

(Get graded ring in the case $X = T_+$, with T a finite CW complex.)

7 Examples

7.1 Suspension spectra

For any based space $T \in \mathscr{T}$, the suspension spectrum

$$\Sigma^\infty T = T \wedge S$$

is given by $(\Sigma^\infty T)_n = \Sigma^n T = T \wedge S^n$, with the standard $O(n)$-action on S^n. The homotopy groups

$$\pi_k(\Sigma^\infty T) = \operatorname*{colim}_n \pi_{k+n}(T \wedge S^n)$$

are the stable homotopy groups of T, often denoted $\pi_k^s(T)$. For instance,

$$\pi_k^s(S^0) = \pi_k(S) = \operatorname*{colim}_n \pi_{k+n}(S^n)$$

is the k-th stable stem. By the Pontryagin–Thom construction and transversality, it is isomorphic to the bordism group of framed k-dimensional (smooth) manifolds. These groups are trivial for $k < 0$, $\pi_0(S) \cong \mathbb{Z}$, and Serre showed that $\pi_k(S)$ is finite for each $k > 0$.

If $U \in \mathscr{U}$ is an unbased space, let $U_+ \in \mathscr{T}$ denote the based space given by adding a disjoint base point to U. Let

$$S[U] = \Sigma^\infty(U_+) = U_+ \wedge S$$

be the "unreduced" suspension spectrum on U.

k	π_k^s	gen.
0	$\mathbb{Z}$	ι
1	$\mathbb{Z}/2$	η
2	$\mathbb{Z}/2$	η^2
3	$\mathbb{Z}/24$	ν
4	0	
5	0	
6	$\mathbb{Z}/2$	ν^2
7	$\mathbb{Z}/240$	σ
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\bar{\nu}, \epsilon$
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\nu^3, \eta\epsilon, \mu$
10	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\eta\mu, \beta_1$
11	$\mathbb{Z}/504$	ζ
12	0	
13	$\mathbb{Z}/3$	$\alpha_1\beta_1$
14	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	σ^2, κ
15	$\mathbb{Z}/480 \oplus \mathbb{Z}/2$	$\rho, \eta\kappa$
16	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta^*, \eta\rho$
17	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta\eta^*, \nu\kappa, \eta^2\rho, \bar{\mu}$
18	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\nu^*, \eta\bar{\mu}$
19	$\mathbb{Z}/2 \oplus \mathbb{Z}/264$	$\bar{\sigma}, \bar{\zeta}$
20	$\mathbb{Z}/8 \oplus \mathbb{Z}/3$	$\bar{\kappa}, \beta_1^2$
21	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\nu\nu^*, \eta\bar{\kappa}$
22	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\nu\bar{\sigma}, \eta^2\bar{\kappa}$
$\vdots$	$\vdots$	$\vdots$

Figure 1: The first twenty-odd stable stems

If M is a topological monoid, with unit element e, the multiplication $M \times M \to M$ induces a pairing

$$\mu\colon S[M] \wedge S[M] \cong S[M \times M] \longrightarrow S[M]$$

and the inclusion $\{e\} \to M$ induces a unit

$$\eta\colon S \cong S[\{e\}] \longrightarrow S[M]$$

making $S[M]$ an orthogonal ring spectrum, called the spherical monoid ring of M. Its homotopy groups

$$\pi_* S[M] = \pi_*^s(M_+)$$

form a graded algebra over $\pi_*(S)$. If M is commutative then $S[M]$ is a commutative orthogonal ring spectrum, and $\pi_* S[M]$ is a graded commutative $\pi_*(S)$-algebra. If G is a topological group, we might call $S[G]$ the spherical group ring of G. The group inverse $\chi\colon g \mapsto g^{-1}$ then induces anti-homomorphisms $\chi\colon S[G] \to S[G]$ and $\chi\colon \pi_* S[G] \to \pi_* S[G]$.

(The diagonal $\Delta\colon M \to M \times M$ induces a cocommutative copairing $\psi\colon S[M] \to S[M] \wedge S[M]$. Discuss when there is an induced graded cocommutative coproduct $\psi\colon \pi_* S[M] \to \pi_* S[M] \otimes_{\pi_*(S)} \pi_* S[M]$, making $\pi_* S[M]$ a graded cocommutative bialgebra over $\pi_*(S)$. Similarly for when $\pi_* S[G]$ becomes a graded cocommutative Hopf algebra over $\pi_*(S)$.)

7.2 Eilenberg–Mac Lane spectra

For each abelian group G and non-negative integer n we can construct a CW-complex $K(G,n)$ with

$$\pi_i K(G,n) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

e.g., by first building a Moore space $M(G,n)$ with n- and $(n+1)$-cells given by generators and relations in a presentation for G, and then attaching m-cells for $m \ge n+2$ to kill π_i for $i > n$. There is then a homotopy equivalence

$$\tilde\sigma\colon K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$$

for each $n \ge 0$, with left adjoint $\sigma\colon \Sigma K(G,n) \to K(G,n+1)$. The sequence of based spaces

$$HG = \{n \mapsto K(G,n)\}$$

with these structure maps defines a sequential spectrum, called the Eilenberg–Mac Lane spectrum of G. It is an Ω-spectrum, since the adjoint structure maps are (weak) homotopy equivalences. It represents ordinary homology and cohomology with coefficients in G, in the sense that there are natural isomorphisms

$$HG_k(X) = \pi_k(HG \wedge X) \cong \tilde H_k(X;G) \qquad \text{and} \qquad HG^k(X) = \pi_{-k} F(X, HG) \cong \tilde H^k(X;G)$$

for all based CW complexes X and integers k.

(The second claim is the natural isomorphism $\pi_0 F(X, K(G,k)) = [X, K(G,k)] \cong \tilde H^k(X;G)$ for $k \ge 0$, and the observation that $K(G,0) \simeq G$ is discrete, so that $\pi_\ell F(X, K(G,0)) = 0$ for $\ell > 0$. The first claim follows by Spanier–Whitehead duality and passage to colimits, or perhaps by a more direct argument.)

At the level of homotopy groups

$$\pi_k(HG) = \begin{cases} G & \text{for } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

so $\pi_*(HG) = G$ concentrated in degree 0.

To promote HG to an orthogonal spectrum, we need $O(n)$-actions on the spaces $HG_n \simeq K(G,n)$ that are compatible with the structure maps. This is not generally possible with the construction above. There is a simplicial construction due to Milgram (and Steenrod?), called the bar construction, that to each topological group G produces a topological space BG, with $G \simeq \Omega BG$. (This assumes that G is well-based, i.e., that the inclusion $\{e\} \to G$ is a cofibration.) When G is abelian, BG is again an abelian topological group. Here

$$BG = B(*, G, *) = \coprod_{q \geq 0} G^q \times \Delta^q/\!\sim$$

where $\Delta^q \subset \mathbb{R}^{q+1}$ is the standard q-simplex, and $\sim$ is generated by the relations

$$(d_i(x), \xi) \sim (x, \delta_i(\xi))$$

for $x = [g_1|\dots|g_q] \in G^q$, $\xi = (t_0, \dots, t_{q-1}) \in \Delta^{q-1}$, $0 \leq i \leq q$,

$$d_i(x) = \begin{cases} [g_2|\dots|g_q] & \text{for } i = 0, \\ [g_1|\dots|g_i \cdot g_{i+1}|\dots|g_q] & \text{for } 0 < i < q, \\ [g_1|\dots|g_{q-1}] & \text{for } i = q \end{cases}$$

and

$$\delta_i(\xi) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1})\,,$$

and the relations

$$(s_j(x), \xi) \sim (x, \sigma_j(\xi))$$

for $x = [g_1|\dots|g_q] \in G^q$, $\xi = (t_0, \dots, t_{q+1}) \in \Delta^{q+1}$, $0 \leq j \leq q$,

$$s_j(x) = [g_1|\dots|g_j|e|g_{j+1}|\dots|g_q]$$

and

$$\sigma_j(\xi) = (t_0, \dots, t_j + t_{j+1}, \dots, t_{q+1})\,.$$

Note that the subspace of BG generated by $G^q \times \Delta^q$ for $0 \leq q \leq 1$ is $G \wedge \Delta^1/\partial\Delta^1 \cong \Sigma G$, so there is a natural inclusion $\Sigma G \to BG$, with adjoint the natural (weak?) equivalence $G \to \Omega BG$.

Starting with a discrete abelian group G and iterating this construction n times, we obtain a space $B^n G = B \cdots BG \simeq K(G, n)$. Permuting the order in which the n bar constructions are performed defines an action by the symmetric group Σ_n on $B^n G$, and this recipe defines $HG = \{n \mapsto B^n G\}$ as a symmetric spectrum. However, this action by Σ_n does not naturally extend to an action by $O(n)$.

A construction by McCord generalizes the bar construction B and the infinite symmetric product of Dold and Thom, and can be used to construct HG as an orthogonal spectrum.

Definition 7.1. For each abelian topological monoid $(G, +, 0)$ and each based space (X, x_0) let $B(G, X)$ be the space of finite sums

$$u = \sum_i g_i x_i$$

with $g_i \in G$, $x_i \in X$ for all i, subject to the relations $gx + g'x = (g + g')x$ and $gx_0 = 0$, for $g, g' \in G$ and $x \in X$. Equivalently, u is a function $u \colon X \to G$ such that $u(x) \neq 0$ for only finitely many $x \in X$, and $u(x_0) = 0$.

For each $n \geq 0$ let $B_n(G, X)$ be the image of the map $(G \times X)^n \to G(X)$ sending $(g_1, x_1, \dots, g_n, x_n)$ to $g_1 x_1 + \dots + g_n x_n$. We give $B_n(G, X)$ the quotient topology from $(G \times X)^n$. Each inclusion $B_n(G, X) \to B_{n+1}(G, X)$ is a closed embedding, and we give $B(G, X) = \bigcup_n B_n(G, X)$ the colimit topology. Hence $U \subset B(G, X)$ is open if and only if its preimage in $(G \times X)^n$ is open, for each $n \geq 0$.

McCord (1969, Theorem 8.8) shows that if G is a discrete abelian group, the map $CS^n \to CS^n/S^n \cong S^{n+1}$ induces a numerable principal $B(G, S^n)$-bundle

$$B(G, CS^n) \to B(G, S^{n+1})$$

with $B(G, CS^n)$ contractible. Hence $B(G, S^n) \simeq \Omega B(G, S^{n+1})$ is an Eilenberg–Mac Lane space of type $K(G, n)$.

Definition 7.2. We define the orthogonal Eilenberg–Mac Lane spectrum HG by

$$HG_n = B(G, S^n)$$

with the $O(n)$-action induced by the $O(n)$-action on S^n, i.e., $A \in O(n)$ maps $\sum_i g_i x_i$ to $\sum_i g_i A x_i$, for $g_i \in G$ and $x_i \in S^n$. The structure map $\sigma\colon HG_n \wedge S^1 \to HG_{n+1}$ is given by the map

$$B(G, S^n) \wedge S^1 \longrightarrow B(G, S^n \wedge S^1)$$

that takes $(\sum_i g_i x_i) \wedge s$ to $\sum_i g_i(x_i \wedge s)$. The iterated structure map

$$\sigma^\ell\colon B(G, S^k) \wedge S^\ell \longrightarrow B(G, S^{k+\ell})$$

is then evidently $O(k) \wedge O(\ell)$-equivariant, so HG is an orthogonal Ω-spectrum.

Proposition 7.3. *The Eilenberg–Mac Lane functor H from abelian groups to orthogonal spectra is lax symmetric monoidal: There is a natural transformation*

$$HG \wedge HG' \longrightarrow H(G \otimes G')$$

and a map

$$S \longrightarrow H\mathbb{Z}$$

making the required diagrams commute.

Proof. The pairing

$$B(G, S^k) \wedge B(G', S^\ell) \xrightarrow{\cdot} B(G \otimes G', S^k \wedge S^\ell)$$

takes $(\sum_i g_i x_i, \sum_j g'_j y_j)$ to

$$\sum_{i,j} (g_i \otimes g'_j)(x_i \wedge y_j)\,.$$

It is $O(k) \times O(\ell)$-equivariant, and makes the bilinearity diagram

$$\begin{tikzcd}
B(G,S^k) \wedge S^1 \wedge B(G',S^\ell) \arrow[d, "\sigma\wedge 1"'] & B(G,S^k) \wedge B(G',S^\ell) \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "\phi_{k,\ell}\wedge 1"'] \arrow[dr, "1\wedge\sigma"] & \\
B(G,S^{k+1}) \wedge B(G',S^\ell) \arrow[d, "\phi_{k+1,\ell}"'] & B(G\otimes G',S^{k+\ell}) \wedge S^1 \arrow[d, "\sigma"'] & B(G,S^k) \wedge B(G',S^{\ell+1}) \arrow[dl, "\phi_{k,\ell+1}"] \\
B(G\otimes G',S^{k+1+\ell}) \arrow[r, "I_k\oplus\chi_{1,\ell}"] & B(G\otimes G',S^{k+\ell+1}) &
\end{tikzcd}$$

commute.

The spectrum level Hurewicz map $h\colon S \to H\mathbb{Z}$ is given at level n by the embedding $h_n\colon S^n \to B(\mathbb{Z}, S^n)$ sending $x \in S^n$ to $1 \cdot x \in B(\mathbb{Z}, S^n)$. It is $O(n)$-equivariant and compatible with the structure maps.

The compatibility diagrams for α, λ and ρ commute. The compatibility diagram

$$\begin{tikzcd}
HG \wedge HG' \arrow[r, "\gamma"] \arrow[d] & HG' \wedge HG \arrow[d] \\
H(G\otimes G') \arrow[r, "H(\gamma)"] & H(G'\otimes G)\,.
\end{tikzcd}$$

commutes, because each diagram

$$\begin{tikzcd}
B(G,S^k) \wedge B(G',S^\ell) \arrow[r, "\gamma"] \arrow[dd, "\cdot"'] & B(G',S^\ell) \wedge B(G,S^k) \arrow[d, "\cdot"] \\
 & B(G'\otimes G, S^\ell \wedge S^k) \arrow[d, "\chi_{\ell,k}"] \\
B(G\otimes G', S^k \wedge S^\ell) \arrow[r, "B(\gamma,1)"] & B(G'\otimes G, S^k \wedge S^\ell)
\end{tikzcd}$$

commutes. □

For each ring R the Eilenberg–Mac Lane spectrum HR is an orthogonal ring spectrum, with multiplication $\mu\colon HR \wedge HR \to H(R \otimes R) \to HR$ and unit $\eta\colon S \to H\mathbb{Z} \to HR$. If R is commutative then HR is a commutative orthogonal ring spectrum. The induced product on $\pi_*(HR) \cong R$ is the ring multiplication in R.

For each left R-module M the Eilenberg–Mac Lane spectrum HM is a left HR-module, with respect to the pairing $\lambda\colon HR \wedge HM \to H(R \otimes M) \to HM$, and similarly for right modules. The induced pairing $\pi_*(HR) \otimes \pi_*(HM) \to \pi_*(HM) \cong M$ is the left R-module action on M.

The functors $H\colon (\mathrm{Ab}, \otimes, \mathbb{Z}) \to (\mathrm{Sp}^{\mathbb{O}}, \wedge, S)$ and $\pi_0\colon (\mathrm{Sp}^{\mathbb{O}}, \wedge, S) \to (\mathrm{Ab}, \otimes, \mathbb{Z})$ are thus both lax symmetric monoidal, and the composite $\pi_0 \circ H$ is naturally isomorphic to the identity. In this way, the algebra of abelian groups is faithfully embedded in that of orthogonal spectra. (Elaborate?)

The Hurewicz map $h\colon S \to H\mathbb{Z}$ induces the stable Hurewicz homomorphism $\pi_k^s(T) = \pi_k(S \wedge T) \to \pi_k(H\mathbb{Z} \wedge T) = \tilde{H}_k(T; \mathbb{Z})$ for any based space T.

7.3 Bordism spectra

The bar construction $BO(n)$ and the infinite Grassmannian $Gr_n(\mathbb{R}^\infty)$ are homotopy equivalent. There are principal $O(n)$-bundles

$$O(n) \longrightarrow EO(n) \longrightarrow BO(n)$$

and

$$O(n) \longrightarrow V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$$

with contractible total spaces (that admit CW structures), so any $O(n)$-equivariant equivalence $EO(n) \simeq V_n(\mathbb{R}^\infty)$ induces an equivalence $BO(n) \simeq Gr_n(\mathbb{R}^\infty)$ of $O(n)$-orbit spaces. Here $EO(n)$ is the case $G = O(n)$ of the construction

$$EG = B(*, G, G) = \coprod_{q \ge 0} (G^q \times G) \times \Delta^q / {\sim}$$

where $\sim$ is generated by the relations $(d_i(x), \xi) \simeq (x, \delta_i(\xi))$ and $(s_j(x), \xi) \simeq (x, \sigma_j(\xi))$, for $x = [g_1|\dots|g_q]g_{q+1} \in G^q \times G$, with

$$d_i(x) = \begin{cases} [g_2|\dots|g_q]g_{q+1} & \text{for } i = 0, \\ [g_1|\dots|g_i g_{i+1}|\dots|g_q]g_{q+1} & \text{for } 0 < i < q, \\ [g_1|\dots|g_{q-1}]g_q g_{q+1} & \text{for } i = q \end{cases}$$

and

$$s_j(x) = [g_1|\dots|g_{j-1}|e|g_j|\dots|g_q]g_{q+1}\,.$$

The group G acts freely from the right on EG, with $EG/G \cong BG$. (Give contraction of EG?) On the other hand, $V_n(\mathbb{R}^\infty)$ is the Stiefel "variety" of orthonormal n-tuples $(v_1, \dots, v_n)$ in $\mathbb{R}^\infty$. It is the colimit over k of the homogeneous spaces $V_n(\mathbb{R}^{n+k}) = O(n+k)/(1 \times O(k))$. Since each map $O(k) \to O(1+k) \to \dots \to O(n+k)$ is $(k-1)$-connected, the space $V_n(\mathbb{R}^\infty)$ is contractible. There is a principal $O(n)$-bundle $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ mapping $(v_1, \dots, v_n)$ to the subspace of $\mathbb{R}^\infty$ spanned by those n vectors.

The $\mathbb{R}^n$-bundle

$$\mathbb{R}^n \longrightarrow EO(n) \times_{O(n)} \mathbb{R}^n \longrightarrow BO(n)$$

and the canonical bundle

$$\mathbb{R}^n \longrightarrow E(\gamma^n) \longrightarrow Gr_n(\mathbb{R}^\infty)$$

correspond under these equivalences, so there is an equivalence

$$MO(n) = EO(n)_+ \wedge_{O(n)} S^n \simeq Th(\gamma^n)$$

of Thom complexes. The inclusion $O(n) \cong O(n) \times 1 \subset O(n+1)$ induces a map $i\colon BO(n) \to BO(n+1)$, and the pullback of the $\mathbb{R}^{n+1}$-bundle

$$EO(n+1) \times_{O(n+1)} \mathbb{R}^{n+1} \to BO(n+1)$$

along i is the product

$$EO(n) \times_{O(n)} \mathbb{R}^n \times \mathbb{R} \to BO(n)$$

of the $\mathbb{R}^n$-bundle over $BO(n)$ with a copy of $\mathbb{R}$. In other words, this is the Whitney sum of the $\mathbb{R}^n$-bundle and the trivial line bundle ϵ^1. Hence its Thom complex

$$EO(n)_+ \wedge_{O(n)} S^{n+1} \cong EO(n)_+ \wedge_{O(n)} S^n \wedge S^1 = MO(n) \wedge S^1$$

maps naturally to the Thom complex

$$MO(n+1) = EO(n+1)_+ \wedge_{O(n+1)} S^{n+1} .$$

The sequence of spaces $\{n \mapsto MO(n)\}$ with the structure maps $\sigma = Th(i)\colon MO(n) \wedge S^1 \to MO(n+1)$ defines the Thom spectrum MO as a sequential spectrum, with $MO_n = MO(n)$.

There is also an inclusion $i\colon Gr_n(\mathbb{R}^\infty) \to Gr_{n+1}(\mathbb{R}^\infty)$, mapping $V \subset \mathbb{R}^\infty$ to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$, and the pullback of γ^{n+1} along i is isomorphic to $\gamma^n \oplus \epsilon^1$, so that i induces a map

$$\sigma = Th(i)\colon Th(\gamma^n) \wedge S^1 \cong Th(\gamma^n \oplus \epsilon^1) \longrightarrow Th(\gamma^{n+1}) .$$

The resulting spectrum $\{n \mapsto Th(\gamma^n)\}$ is level equivalent to MO, as defined above. (Clarify role of isomorphism $\mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$?)

By the Pontryagin–Thom construction and transversality,

$$\pi_k(MO) = \operatorname*{colim}_n \pi_{k+n} MO_n \cong \mathscr{N}_k$$

is the bordism group of closed (smooth, unoriented) k-manifolds. More generally,

$$MO_k(X) = \operatorname*{colim}_n \pi_{k+n}(MO_n \wedge X_+) \cong \mathscr{N}_k(X)$$

is the bordism group of closed k-manifolds over X.

We can promote MO to an orthogonal spectrum, cf. May (1977), p. 75. The group $O(n)$ acts on itself by conjugation, so that $A \in O(n)$ induces the homomorphism $O(n) \to O(n)$ given by $g \mapsto AgA^{-1}$. When combined with the standard action on $S^n = \mathbb{R}^n \cup \{\infty\}$, taking s to As, this induces an action on

$$MO(n) = EO(n)_+ \wedge_{O(n)} S^n$$

taking $([g_1|\dots|g_q]g_{q+1}, \xi, s)$ to $([Ag_1A^{-1}|\dots|Ag_qA^{-1}]Ag_{q+1}A^{-1}, \xi, As)$. The iterated structure map

$$\sigma^\ell\colon MO(k) \wedge S^\ell = EO(k)_+ \wedge_{O(k)} S^k \wedge S^\ell \longrightarrow EO(k+\ell)_+ \wedge_{O(k+\ell)} S^{k+\ell}$$

is $O(k) \times O(\ell)$-equivariant: $(A, B) \in O(k) \times O(\ell)$ acts on $([g_1|\dots|g_q]g_{q+1}, \xi, s, t)$ to give

$$([Ag_1A^{-1}|\dots|Ag_qA^{-1}]Ag_{q+1}A^{-1}, \xi, As, Bt) ,$$

which maps to

$$([Ag_1A^{-1} \oplus I_\ell|\dots|Ag_qA^{-1} \oplus I_\ell]Ag_{q+1}A^{-1} \oplus I_\ell, \xi, As, Bt) .$$

This is equal to the action of $A \oplus B \in O(k+\ell)$ on the image

$$([g_1 \oplus I_\ell|\dots|g_q \oplus I_\ell]g_{q+1} \oplus I_\ell, \xi, s, t) ,$$

since $(A \oplus B)(g \oplus I_\ell)(A \oplus B)^{-1} = AgA^{-1} \oplus I_\ell$, for each $g \in O(k)$. Hence MO is an orthogonal spectrum.

In fact, MO is a commutative orthogonal ring spectrum, i.e., a commutative S-algebra. The multiplication $\mu\colon MO \wedge MO \to MO$ is induced by the $O(k) \times O(\ell)$-equivariant map

$$\phi_{k,\ell}\colon MO(k) \wedge MO(\ell) \longrightarrow MO(k+\ell)$$

given by the composite

$$\begin{aligned} EO(k)_+ \wedge_{O(k)} S^k \wedge EO(\ell)_+ \wedge_{O(\ell)} S^\ell &\xrightarrow{\cong} E(O(k) \times O(\ell))_+ \wedge_{O(k)\times O(\ell)} S^k \wedge S^\ell \\ &\longrightarrow EO(k+\ell)_+ \wedge_{O(k+\ell)} S^{k+\ell} , \end{aligned}$$

for each $k, \ell \geq 0$. The first map uses the natural $(G \times H)$-equivariant homeomorphism $EG \times EH \cong E(G \times H)$, in the case $G = O(k)$ and $H = O(\ell)$. (This relies on Milnor's homeomorphism $|X \times Y| \cong |X| \times |Y|$ for (nice) simplicial spaces X and Y.) The resulting map $\phi\colon MO \otimes MO \to MO$ of orthogonal sequences descends to define a map μ of orthogonal spectra, because the bilinearity diagram

$$\begin{tikzcd}
MO(k) \wedge S^1 \wedge MO(\ell) \arrow[d, "\sigma\wedge 1"'] & MO(k) \wedge MO(\ell) \wedge S^1 \arrow[l, "1\wedge\gamma"'] \arrow[d, "\iota_{k,\ell}\wedge 1"'] \arrow[dr, "1\wedge\sigma"] & \\
MO(k+1) \wedge MO(\ell) \arrow[d, "\iota_{k+1,\ell}"'] & MO(k+\ell) \wedge S^1 \arrow[d, "\sigma"'] & MO(k) \wedge MO(\ell+1) \arrow[dl, "\iota_{k,\ell+1}"] \\
MO(k+1+\ell) \arrow[r, "I_k \oplus \chi_{1,\ell}"] & MO(k+\ell+1) &
\end{tikzcd}$$

commutes. (Spell out in terms of $EO(k)_+ \wedge_{O(k)} S^k$, etc.?) It gives a commutative product, because the square

$$\begin{tikzcd}
MO(k) \wedge MO(\ell) \arrow[r, "\gamma"] \arrow[d, "\phi_{k,\ell}"'] & MO(\ell) \wedge MO(k) \arrow[d, "\phi_{\ell,k}"] \\
MO(k+\ell) \arrow[r, "\chi_{k,\ell}"] & MO(\ell+k)
\end{tikzcd}$$

commutes, so that $\mu\gamma = \mu$. More explicitly, this is the diagram

$$\begin{tikzcd}
EO(k)_+ \wedge_{O(k)} S^k \wedge EO(\ell)_+ \wedge_{O(\ell)} S^\ell \arrow[r, "\gamma"] \arrow[d, "\phi_{k,\ell}"'] & EO(\ell)_+ \wedge_{O(\ell)} S^\ell \wedge EO(k)_+ \wedge_{O(k)} S^k \arrow[d, "\phi_{\ell,k}"] \\
EO(k+\ell)_+ \wedge_{O(k+\ell)} S^{k+\ell} \arrow[r, "\chi_{k,\ell}"] & EO(\ell+k)_+ \wedge_{O(\ell+k)} S^{\ell+k}\,.
\end{tikzcd}$$

The unit $\eta\colon S \to MO$ is given by the inclusion $S^n \to EO(n)_+ \wedge_{O(n)} S^n = MO(n)$ taking s to $([]e, s)$ (in the case $q = 0$). It is $O(n)$-equivariant.

Under the equivalences $MO(n) \simeq Th(\gamma^n)$, the map $\phi_{k,\ell}$ corresponds to the map

$$Th(\gamma^k) \wedge Th(\gamma^\ell) \cong Th(\gamma^k \times \gamma^\ell) \longrightarrow Th(\gamma^{k+\ell})$$

of Thom complexes induced by the bundle map

$$E(\gamma^k) \times E(\gamma^\ell) \longrightarrow E(\gamma^{k+\ell})$$

covering the map

$$Gr_k(\mathbb{R}^\infty) \times Gr_\ell(\mathbb{R}^\infty) \longrightarrow Gr_{k+\ell}(\mathbb{R}^\infty)$$

that sends $V \subset \mathbb{R}^\infty$ and $W \subset \mathbb{R}^\infty$ to the direct sum $V \oplus W \subset \mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$.

The induced graded commutative ring structure on $\pi_*(MO)$ corresponds under the isomorphism

$$\pi_*(MO) \cong \mathscr{N}_*$$

to the graded commutative ring structure on the bordism groups, given by taking the classes of M and N to the class of $M \times N$.

The unit map $S \to MO$ induces a homomorphism $\pi_*(S) \to \pi_*(MO)$ that corresponds to the ring homomorphism $\Omega_*^{fr} \to \mathscr{N}_*$ taking a framed bordism class to the underlying unoriented bordism class. It is not a very interesting homomorphism: The underlying unoriented manifold of any framed k-manifold bounds an unoriented $(k+1)$-manifold, for $k > 0$, so $\pi_k(S) \to \pi_k(MO)$ is zero except for $k = 0$. This is a consequence of Thom's theorem that unoriented bordism classes are detected by Stiefel–Whitney numbers, and these vanish for framed manifolds (whose tangent bundles are stably trivial).

Theorem 7.4 (Thom)**.** *The mod 2 Hurewicz homomorphism*

$$\pi_k(MO) = \operatorname*{colim}_n \pi_{k+n} MO(n) \longrightarrow \operatorname*{colim}_n \tilde{H}_{k+n}(MO(n); \mathbb{Z}/2) \cong H_k(BO; \mathbb{Z}/2)$$

is injective for each integer k, and identifies

$$\pi_*(MO) \cong \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, \dots],$$

with one generator x_i in each positive degree i not of the form $2^j - 1$, for $j \ge 1$, as a polynomial subalgebra of $H_*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, \dots]$ with one generator a_k in each positive degree k. There is a π_*-isomorphism of spectra

$$MO \simeq \bigvee_X \Sigma^{|X|} H\mathbb{Z}/2$$

where $X = x_2^{e_2} x_4^{e_4} x_5^{e_5} \cdot \dots$ ranges over a monomial basis for $\mathbb{Z}/2[x_2, x_4, x_5, \dots]$, and $|X| = 2e_2 + 4e_4 + 5e_5 + \dots$ is the degree of the monomial X.

Remark 7.5. The class $a_k \in H_k(BO; \mathbb{Z}/2)$ is the image of the generator $\alpha_k \in H_k(BO(1); \mathbb{Z}/2)$, where $BO(1) \simeq \mathbb{R}P^\infty$. As a bicommutative Hopf algebra, $H_*(BO; \mathbb{Z}/2)$ is dual to the Hopf algebra $H^*(BO; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \dots]$ generated by the Stiefel–Whitney classes $w_i \in H^i(BO; \mathbb{Z}/2)$.)

Let $SO(n) \subset O(n)$ be the special orthogonal group. The principal $SO(n)$-bundles

$$SO(n) \longrightarrow ESO(n) \longrightarrow BSO(n)$$

and

$$SO(n) \longrightarrow V_n(\mathbb{R}^\infty) \longrightarrow \widetilde{Gr}_n(\mathbb{R}^\infty)$$

are both universal, hence equivalent, where $\widetilde{Gr}_n(\mathbb{R}^\infty)$ is the infinite Grassmannian of oriented n-dimensional subspaces of $\mathbb{R}^\infty$. The oriented $\mathbb{R}^n$-bundles

$$\mathbb{R}^n \longrightarrow ESO(n) \times_{SO(n)} \mathbb{R}^n \longrightarrow BSO(n)$$

and

$$\mathbb{R}^n \longrightarrow E(\tilde{\gamma}^n) \longrightarrow \widetilde{Gr}_n(\mathbb{R}^\infty)$$

are equivalent, where $\tilde{\gamma}^n$ denotes the canonical $\mathbb{R}^n$-bundle over $\widetilde{Gr}_n(\mathbb{R}^\infty)$, so there is an equivalence

$$MSO(n) = ESO(n)_+ \wedge_{SO(n)} S^n \simeq Th(\tilde{\gamma}^n)$$

of Thom complexes. Replacing $O(n)$ with $SO(n)$ in the role of G in the constructions above, we obtain a commutative orthogonal ring spectrum MSO, with n-th space $MSO(n)$, structure maps

$$\sigma\colon MSO(n) \wedge S^1 = ESO(n)_+ \wedge_{SO(n)} S^n \wedge S^1 \longrightarrow ESO(n+1)_+ \wedge_{SO(n+1)} S^{n+1} = MSO(n+1),$$

$O(n)$-action on $MSO(n)$ given by conjugation, since $AgA^{-1} \in SO(n)$ for $A \in O(n)$ and $g \in SO(n)$, product $\mu\colon MSO \wedge MSO \to MSO$ induced by maps

$$\phi_{k,\ell}\colon MSO(k) \wedge MSO(\ell) \longrightarrow MSO(k+\ell)$$

and unit $\eta\colon S \to MSO$ induced by the inclusions $\eta_n\colon S^n \to ESO(n)_+ \wedge_{SO(n)} S^n = MSO(n)$.

Let $Spin(n)$ be the spin group, realizing a double cover of $SO(n)$ for each $n \ge 0$. The Thom complex of the spin $\mathbb{R}^n$-bundle

$$\mathbb{R}^n \longrightarrow ESpin(n) \times_{Spin(n)} \mathbb{R}^n \longrightarrow BSpin(n)$$

is

$$MSpin(n) = ESpin(n)_+ \wedge_{Spin(n)} S^n$$

Replacing $O(n)$ with $Spin(n)$ in the role of G in the constructions above, we obtain a commutative orthogonal ring spectrum $MSpin$, with n-th space $MSpin(n)$, structure maps

$$\sigma\colon MSpin(n) \wedge S^1 = ESpin(n)_+ \wedge_{Spin(n)} S^n \wedge S^1 \longrightarrow ESpin(n+1)_+ \wedge_{Spin(n+1)} S^{n+1} = MSpin(n+1),$$

$O(n)$-action on $MSpin(n)$ given by the unique lift $Spin(n) \to Spin(n)$ of the conjugation homomorphism $g \mapsto AgA^{-1} \colon SO(n) \to SO(n)$, product $\mu \colon MSpin \wedge MSpin \to MSpin$ induced by maps

$$\phi_{k,\ell} \colon MSpin(k) \wedge MSpin(\ell) \longrightarrow MSpin(k+\ell)$$

derived from the homomorphism $Spin(k) \times Spin(\ell) \to Spin(k+\ell)$ lifting the block sum $SO(k) \times SO(\ell) \to SO(k+\ell)$, and unit $\eta \colon S \to MSpin$ induced by the inclusions $\eta_n \colon S^n \to ESpin(n)_+ \wedge_{Spin(n)} S^n = MSpin(n)$.

We obtain maps

$$S \longrightarrow MSpin \longrightarrow MSO \longrightarrow MO$$

of orthogonal ring spectra inducing graded ring homomorphisms

$$\pi_*(S) \longrightarrow \pi_*(MSpin) \longrightarrow \pi_*(MSO) \longrightarrow \pi_*(MO)$$

that are isomorphic to the graded homomorphisms

$$\Omega_*^{fr} \longrightarrow \Omega_*^{Spin} \longrightarrow \Omega_* = \Omega_*^{SO} \longrightarrow \mathscr{N}_* = \Omega_*^{O}$$

of framed, spin, oriented and unoriented bordism rings, respectively.

Theorem 7.6 (Thom)**.** *The rational Hurewicz homomorphism*

$$\pi_k(MSO) \otimes \mathbb{Q} = \operatorname*{colim}_n \pi_{k+n} MSO(n) \otimes \mathbb{Q} \longrightarrow \operatorname*{colim}_n \tilde{H}_{k+n}(MSO(n); \mathbb{Q}) \cong H_k(BSO; \mathbb{Q})$$

is an isomorphism for each integer k, and identifies

$$\pi_*(MSO) \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, x_{12}, \dots]$$

with the polynomial ring on one generator in each positive degree of the form $4i$.

(Discuss generators of $H_*(BSO; \mathbb{Q})$ and duality with $H^*(BSO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, \dots]$, where $p_i \in H^{4i}(BSO; \mathbb{Q})$ is the i-th Pontryagin class. Decide about $\mathbb{Z}$-, $\mathbb{Z}[1/2]$- or $\mathbb{Q}$-coefficients.)

Wall (1960) determined the structure of $\pi_*(MSO) \cong \Omega_*$ completely. Oriented bordism classes are detected by Pontryagin- and Stiefel–Whitney numbers, and all torsion is of exponent 2.

(Splitting of MSO when localized at $p = 2$.)

(String bordism?)

7.4 Complex bordism and formal group laws

Let $U(n)$ denote the unitary group, acting linearly on complex n-space, $\mathbb{C}^n$. There is a universal principal $U(n)$-bundle

$$U(n) \longrightarrow EU(n) \longrightarrow BU(n)$$

with associated $\mathbb{C}^n$-bundle

$$\mathbb{C}^n \longrightarrow EU(n) \times_{U(n)} \mathbb{C}^n \longrightarrow BU(n)$$

and Thom complex

$$MU(n) = EU(n)_+ \wedge_{U(n)} S^{2n} ,$$

where $S^{2n} = \mathbb{C}^n \cup \{\infty\}$. The inclusion $U(n) \cong U(n) \times 1 \subset U(n+1)$ induces a map $i \colon BU(n) \to BU(n+1)$, covered by a $\mathbb{C}^{n+1}$-bundle map

$$EU(n) \times_{U(n)} \mathbb{C}^n \times \mathbb{C} \longrightarrow EU(n+1) \times_{U(n+1)} \mathbb{C}^{n+1}$$

that induces a map

$$Th(i) \colon MU(n) \wedge S^2 \longrightarrow MU(n+1)$$

of Thom complexes. Here S^2 arises as the one-point compactification of $\mathbb{C} \cong \mathbb{R}\{1\} \oplus \mathbb{R}\{i\}$, with $i = \sqrt{-1}$, hence can be decomposed as $S^{\mathbb{R}} \wedge S^{i\mathbb{R}} \cong S^1 \wedge S^1$, i.e., the smash product of one 'real' and one 'imaginary' circle.

We obtain a sequential spectrum, here denoted $M'U$, with $2n$-th space $MU(n)$, $2n+1$-th space $\Sigma MU(n)$, and structure maps that alternate between the identity

$$\mathbb{1}\colon MU(n) \wedge S^1 \longrightarrow \Sigma MU(n)$$

and the map

$$Th(i)\colon \Sigma MU(n) \wedge S^1 \longrightarrow MU(n+1)\,.$$

The homotopy groups

$$\pi_k(M'U) = \operatorname{colim} \pi_{k+2n} MU(n)$$

are the complex bordism groups $U_k = \Omega_k^U$, i.e., the bordism groups of closed (smooth) k-manifolds whose (stable) normal bundle comes equipped with a complex structure.

Theorem 7.7 (Milnor, Novikov)**.** *The integral Hurewicz homomorphism*

$$\pi_k(M'U) = \operatorname*{colim}_n \pi_{k+2n} MU(n) \longrightarrow \operatorname*{colim}_n \tilde{H}_{k+2n}(MU(n)) \cong H_k(BU)$$

is injective for each integer k, and identifies

$$\pi_*(M'U) \cong \mathbb{Z}[x_1, x_2, x_3, \dots]\,,$$

with one generator x_i in each positive even degree $2i$, as a polynomial subring of $H_*(BU) = \mathbb{Z}[b_1, b_2, \dots]$, with one generator b_i in each positive even degree $2i$.

Remark 7.8. The class $b_i \in H_{2i}(BU)$ is the image of a generator $\beta_i \in H_{2i}(BU(1))$, where $BU(1) \simeq \mathbb{C}P^\infty$. As a bicommutative Hopf algebra, $H_*(BU)$ is dual to the Hopf algebra $H^*(BU) = \mathbb{Z}[c_1, c_2, \dots]$ generated by the Chern classes $c_i \in H^{2i}(BU)$.

With the definition above, there is no evident $O(2n)$-action on the $2n$-th space $MU(n)$ in the sequential spectrum $M'U$, so with this definition $M'U$ does not arise as an orthogonal spectrum. There is, however, a natural $U(n)$-action on $MU(n)$, similar to the natural $O(n)$-action on $MO(n)$. This $U(n)$-action restricts to an $O(n)$-action on $MU(n)$ via the complexification homomorphism $c\colon O(n) \to U(n)$.

Following Schwede, we define MU as the orthogonal spectrum given at level n by $MU_n = \Omega^n MU(n)$. More explicitly, we keep the real and imaginary summands of $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$ separate, and set

$$MU_n = F(S^{i\mathbb{R}^n}, MU(n))$$

and write $Th(i)$ as

$$MU(n) \wedge S^{\mathbb{R}} \wedge S^{i\mathbb{R}} \longrightarrow MU(n+1)$$

with right adjoint

$$MU(n) \wedge S^{\mathbb{R}} \longrightarrow F(S^{i\mathbb{R}}, MU(n+1))\,.$$

The structure map σ is then the composite

$$\begin{aligned}\Sigma MU_n = MU(n) \wedge S^{\mathbb{R}} = F(S^{i\mathbb{R}^n}, MU(n)) \wedge S^{\mathbb{R}} &\xrightarrow{\nu} F(S^{i\mathbb{R}^n}, MU(n) \wedge S^{\mathbb{R}}) \\ &\longrightarrow F(S^{i\mathbb{R}^n}, F(S^{i\mathbb{R}}, MU(n+1))) \cong F(S^{i\mathbb{R}^{n+1}}, MU(n+1)) = MU_{n+1}\,.\end{aligned}$$

The group $O(n)$ acts on MU_n by conjugation, with $A \in O(n)$ taking the map $f\colon S^{i\mathbb{R}^n} \to MU(n)$ to the map $Af\colon S^{i\mathbb{R}^n} \to MU(n)$ given by $(Af)(s) = A(f(A^{-1}s))$. The $O(k) \times O(\ell)$-equivariant maps

$$\phi_{k,\ell}\colon MU_k \wedge MU_\ell = \Omega^k MU(k) \wedge \Omega^\ell MU(\ell) \xrightarrow{\wedge} \Omega^{k+\ell}(MU(k) \wedge MU(\ell)) \longrightarrow \Omega^{k+\ell} MU(k+\ell)$$

define a multiplication $\mu\colon MU \wedge MU \to MU$, and the $O(n)$-equivariant maps

$$\eta_n\colon S^n \longrightarrow \Omega^n MU(n)\,,$$

adjoint to the inclusion $S^{2n} \to MU(n)$, define a unit $\eta\colon S \to MU$, making MU a commutative orthogonal ring spectrum.

There are isomorphisms

$$\pi_k(MU) = \operatorname*{colim}_n \pi_{k+n}(MU_n) = \operatorname*{colim}_n \pi_{k+n}(\Omega^n MU(n)) \cong \operatorname*{colim}_n \pi_{k+2n}(MU(n)) = \pi_k(M'U)$$

and more generally isomorphisms

$$\begin{aligned} MU_k(X) &= \operatorname*{colim}_n \pi_{k+n}(MU_n \wedge X) = \operatorname*{colim}_n \pi_{k+n}(\Omega^n MU(n) \wedge X) \\ &\xrightarrow{\nu} \operatorname*{colim}_n \pi_{k+n}(\Omega^n(MU(n) \wedge X)) \cong \operatorname*{colim}_n \pi_{k+2n}(MU(n) \wedge X) = M'U_k(X) \end{aligned}$$

for based CW complexes X. Presumably $M'U$ and MU are π_*-isomorphic as sequential spectra, defining the same homology and cohomology theory (= complex bordism).

Lemma 7.9. *The zero section in the Hopf $\mathbb{C}$-bundle*

$$\eta\colon \mathbb{C} \longrightarrow EU(1) \times_{U(1)} \mathbb{C} \longrightarrow BU(1)$$

induces a homotopy equivalence

$$BU(1) \xrightarrow{\simeq} MU(1)\,.$$

Proof. The unit circle bundle

$$S(\mathbb{C}) \longrightarrow EU(1) \times_{U(1)} S(\mathbb{C}) \longrightarrow BU(1)$$

has total space $S(\eta) \cong EU(1)$ which is contractible, so the quotient map

$$D(\eta) \xrightarrow{\simeq} D(\eta)/S(\eta)$$

is a homotopy equivalence. The zero section in η induces a homotopy equivalence $BU(1) \to D(\eta)$, and there is a standard homotopy equivalence $MU(1) = Th(\eta) \to D(\eta)/S(\eta)$. Hence $BU(1) \to MU(1)$ is a homotopy equivalence. □

Definition 7.10. Let $t \in MU^2(BU(1))$ be the image under

$$\pi_{k+2}F(BU(1), MU(1)) \longrightarrow \operatorname{colim} \pi_{k+2n}F(BU(1), MU(n)) = MU^{-k}(BU(1))$$

of the homotopy class of the map $BU(1) \to MU(1)$, for $k = -2$.

Proposition 7.11. *1. $MU^*(BU(1)) \cong MU_*[[t]]$.*

2. $MU^*(BU(1) \times BU(1)) \cong MU_*[[t_1, t_2]]$, where $t_1 = pr_1^*(t)$ and $t_2 = pr_2^*(t)$.

3. The multiplication

$$m\colon BU(1) \times BU(1) \longrightarrow BU(1)$$

maps $t \in MU^2(BU(1))$ to a class $m^*(t) \in MU^2(BU(1) \times BU(1))$ that corresponds to a formal sum

$$F_{MU}(t_1, t_2) = \sum_{i,j \geq 0} a_{i,j} t_1^i t_2^j \in MU_*[[t_1, t_2]]$$

with $a_{i,j} \in MU_{2(i+j-1)}$ for each $i, j \geq 0$.

4. $F_{MU}(0, t_2) = t_2 = F_{MU}(t_2, 0)$, $F_{MU}(t_1, F_{MU}(t_2, t_3)) = F_{MU}(F_{MU}(t_1, t_2), t_3)$ and $F_{MU}(t_1, t_2) = F_{MU}(t_2, t_1)$.

These results show that F_{MU} is a formal group law over MU_*, homogeneous of degree (-2).

Definition 7.12. Let R be a commutative ring. A (1-dimensional, commutative) formal group law over R is a power series

$$F(t_1, t_2) \in R[[t_1, t_2]]$$

such that $F(0, t_2) = t_2 = F(t_2, 0)$, $F(t_1, F(t_2, t_3)) = F(F(t_1, t_2), t_3)$ and $F(t_1, t_2) = F(t_2, t_1)$.

A formal group law on R specifies a formal group structure on the formal affine line $\hat{\mathbb{A}}^1_R = \operatorname{Spf} R[[t]]$ over $\operatorname{Spec} R$, containing the thickenings $\operatorname{Spec} R[t]/(t^n)$ for all $n \geq 0$. It also specifies a functor from commutative R-algebras A to abelian groups, taking A to the set $N(A)$ of nilpotent elements in A, with the group operation $+_F$ given by

$$n_1 +_F n_2 = \sum_{i,j\geq 0} a_{i,j} n_1^i n_2^j = n_1 + n_2 + \sum_{i,j\geq 1} a_{i,j} n_1^i n_2^j$$

for $n_1, n_2 \in N(A)$. (The sum is finite because n_1 and n_2 are nilpotent in A.)

The additive formal group, $\hat{\mathbb{G}}_a$, has formal group law given by

$$F_a(t_1, t_2) = t_1 + t_2 .$$

It represents the additive group structure on $N(A)$.

The multiplicative formal group, $\hat{\mathbb{G}}_m$, has formal group law given by

$$F_m(t_1, t_2) = t_1 + t_2 + t_1 t_2 .$$

It represents the group structure on $N(A)$ that corresponds to the multiplicative group structure on $1 + N(A) \subset A^\times$. More generally, for any unit $u \in R^\times$ there is a variant multiplicative formal group law given by

$$F_u(t_1, t_2) = t_1 + t_2 - u t_1 t_2 ,$$

with the property that $1 - uF_u(t_1, t_2) = (1 - ut_1)(1 - ut_2)$.

To each 1-dimensional commutative group scheme $\mathbb{G}$ there is an associated formal group $\hat{\mathbb{G}}$. A choice of local parameter near the unit element specifies a formal group law. In the case of elliptic curves, these are called elliptic formal group laws.

There is a universal formal group law

$$F_L(t_1, t_2) = \sum_{i,j\geq 0} a_{i,j} t_1^i t_2^j$$

defined over the commutative ring

$$L = \mathbb{Z}[a_{i,j} \mid i, j \geq 0]/{\sim}$$

where $\sim$ denotes the relations among the $a_{i,j}$ that are required for F_L to be a formal group law.

Lemma 7.13. *The rule that to a ring homomorphism $\phi\colon L \to R$ associates the formal group law $\phi_* F_L$ over R, with*

$$(\phi_* F_L)(t_1, t_2) = \sum_{i,j\geq 0} \phi(a_{i,j}) t_1^i t_2^j ,$$

induces a natural bijection

$$\operatorname{Hom}(L, R) \cong \operatorname{FGL}(R)$$

where $\operatorname{FGL}(R)$ *denotes the set of formal group laws over R.*

Theorem 7.14 (Lazard)**.**

$$L \cong \mathbb{Z}[x_1, x_2, \dots]$$

is a polynomial ring on countably many generators, with $x_1 = a_{1,1}$, $x_2 = a_{1,2}$, $x_3 = a_{2,2} - a_{1,3}$, etc.

Theorem 7.15 (Quillen)**.** *The formal group law F_{MU} over MU_* is isomorphic to Lazard's universal formal group law F_L over L: The ring homomorphism*

$$\phi\colon L \xrightarrow{\cong} MU_*$$

such that $\phi_* F_L = F_{MU}$ is an isomorphism.

In particular, the Lazard ring $L = \mathbb{Z}[x_1, x_2, \dots]$ is isomorphic to the complex bordism ring $MU_* = \Omega^U_*$.

Remark 7.16. It is an interesting question which (1-dimensional, commutative) formal group laws F over a graded ring R, with $F(t_1, t_2)$ in homogeneous degree (-2), can be realized as

$$m^*(t) \in E^2(BU(1) \times BU(1))$$

for a ring spectrum E with $\pi_*(E) = R$, $E^*(BU(1)) \cong R[[t]]$ and $E^*(BU(1) \times BU(1)) \cong R[[t_1, t_2]]$.

This is the case for $F = F_a$ over $R = \mathbb{Z}$ with $E = H\mathbb{Z}$ and $t = c_1(\eta) \in H^2(BU(1); \mathbb{Z})$ is the first Chern class of the Hopf $\mathbb{C}$-bundle η.

It is also the case for $F = F_u$ over $R = \mathbb{Z}[u, u^{-1}]$ with $E = KU$ and $t = u^{-1}(1 - [\eta]) \in KU^2(BU(1))$, where $[\eta] \in KU^0(BU(1))$ is the topological K-theory class of η.

(Elliptic cohomology.)

(Conner–Floyd theorem $\mu\colon MU \to KU$ inducing isomorphisms $\pi_*(KU) \otimes_{\pi_*(MU)} MU_*(X) \cong KU_*(X)$, where $\pi_{2n}(MU) = \Omega^U_{2n} \to \pi_{2n}(KU) \cong \mathbb{Z}\{u^n\}$ sends an almost complex manifold M to $(-1)^n$ times its Todd class $Td(M)$.)

(Landweber exactness.)

7.5 Topological K-theory spectra

(Real and complex topological K-theory.)

(Adams operations. The image of J.)

(Algebraic K-theory spectra?)

7.6 Topological Hochschild homology

For any orthogonal spectrum X the n-fold smash power

$$X^{\wedge n} = X \wedge \cdots \wedge X$$

is an orthogonal spectrum, where we take n copies of X. We let $X^{\wedge 0} = S$, so that $X^{\wedge k} \wedge X^{\wedge \ell} \cong X^{\wedge n}$ for $k + \ell = n$. Another common notation for the n-fold smash power is $X^{(n)}$.

(Beware that $X^{\wedge n}$ is only homotopically meaningful under suitable cofibrancy conditions on X, e.g., if X is flat or projectively cofibrant as an orthogonal spectrum.)

Let R be an orthogonal ring spectrum, with unit $\eta\colon S \to R$ and product $\phi\colon R \wedge R \to R$. We can view R as a simultaneous left and right R-module, i.e., an R-R-bimodule, via the action

$$\phi(\phi \wedge 1) = \phi(1 \wedge \phi)\colon R \wedge R \wedge R \longrightarrow R\,.$$

There is a simplicial resolution $C_\bullet$ of R in R-R-bimodules, of the form

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} R \wedge R \wedge R \wedge R \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} R \wedge R \wedge R \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} R \wedge R \xrightarrow{\ \epsilon\ } R\,.$$

Here

$$C_q = R \wedge R^{\wedge q} \wedge R$$

for each $q \ge 0$. For each $0 \le i \le q$ there is a face operator

$$d_i = R^{\wedge i} \wedge \phi \wedge R^{\wedge q-i}\colon C_q \longrightarrow C_{q-1}$$

and for each $0 \le j \le q$ there is a degeneracy operator

$$s_j = R^{\wedge 1+j} \wedge \eta \wedge R^{q+1-j}\colon C_q \longrightarrow C_{q+1}$$

that make $[q] \mapsto C_q$ into a simplicial object in $\mathrm{Sp}^{\mathbb{O}}$. The map $\epsilon = \phi\colon C_0 \to R$ defines an augmentation $C_\bullet \to R$, which can be viewed as a simplicial map from $C_\bullet$ to R viewed as a constant simplicial object. The augmentation induces a homotopy equivalence from the geometric realization

$$|C_\bullet| = \bigvee C_q \wedge \Delta^q_+/\!\sim$$

to R. (Explain using a simplicial contraction?) For each R-R-bimodule spectrum M, with commuting left action $\lambda\colon R \wedge M \to M$ and right action $\rho\colon M \wedge R \to M$, we can form the smash product

$$THH_q(R, M) = M \underset{R-R}{\wedge} C_q = M \wedge R^{\wedge q}$$

and obtain a simplicial orthogonal spectrum $THH_\bullet(R, M)$. The face operators

$$d_i\colon M \wedge R^{\wedge q} \longrightarrow M \wedge R^{\wedge q-1}$$

are given by

$$\begin{aligned} d_0 &= \rho \wedge R^{\wedge q-1} \\ d_i &= M \wedge R^{\wedge i-1} \wedge \phi \wedge R^{\wedge q-1-i} \\ d_q &= (\lambda \wedge R^{\wedge q-1})\gamma \end{aligned}$$

(where $0 < i < q$, and γ transposes $M \wedge R^{\wedge q-1}$ and R), and the degeneracy operators

$$s_j\colon M \wedge R^{\wedge q} \longrightarrow M \wedge R^{\wedge q+1}$$

are given by

$$s_j = M \wedge R^{\wedge j} \wedge \eta \wedge R^{q-j}\,.$$

We write

$$THH_\bullet(R) = THH_\bullet(R, R)$$

in the special case $M = R$ (with the bimodule structure mentioned above). Hence

$$THH_q(R) = R \wedge R^{\wedge q}$$

for each $q \ge 0$, with face operators symbolically given by

$$d_0(r_0 \wedge r_1 \wedge \cdots \wedge r_q) = r_0 r_1 \wedge r_2 \wedge \cdots \wedge r_q\,,$$

$$d_i(r_0 \wedge r_1 \wedge \cdots \wedge r_q) = r_0 \wedge \cdots \wedge r_i r_{i+1} \wedge \cdots \wedge r_q$$

for $0 < i < q$ and

$$d_q(r_0 \wedge r_1 \wedge \cdots \wedge r_q) = r_q r_0 \wedge r_1 \wedge \cdots \wedge r_{q-1}\,.$$

The degeneracy operators are symbolically given by

$$s_j(r_0 \wedge r_1 \wedge \cdots \wedge r_q) = r_0 \wedge \cdots \wedge r_j \wedge 1 \wedge r_{j+1} \wedge \cdots \wedge r_q\,.$$

We let

$$THH(R, M) = |THH_\bullet(R, M)| = \bigvee_{q \ge 0} THH_q(R, M) \wedge \Delta^q_+/{\sim}$$

and

$$THH(R) = |THH_\bullet(R)| = \bigvee_{q \ge 0} THH_q(R) \wedge \Delta^q_+/{\sim}$$

be the orthogonal spectra given by the geometric realization of these simplicial objects.

(Beware that $THH(R)$ is only homotopically meaningful under suitable cofibrancy conditions on R, e.g., if R is flat or projectively cofibrant as an orthogonal spectrum under S.)

8 Equivariant spaces and spectra

8.1 G-spaces

Let G be a compact Lie group, with unit element e. We do not assume that G is connected, so the case where G is a finite (discrete) group in included. We only consider closed subgroups of G, so when we say that H is a subgroup of G it is implicitly assumed that H is closed.

A (left) based G-space X is a based space with a continuous map

$$\lambda\colon G_+ \wedge X \longrightarrow X\,,$$

satisfying associativity and unitality. Writing gx for $\lambda(g \wedge x)$ this means that $g_1(g_2x) = (g_1g_2)x$ and $ex = x$ for all $g_1, g_2 \in G$ and $x \in X$. A based G-map $f\colon X \to Y$ is a based map such that the diagram

$$\begin{tikzcd}
G_+ \wedge X \arrow[r, "\mathbb{1}\wedge f"] \arrow[d, "\lambda"'] & G_+ \wedge Y \arrow[d, "\lambda"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes, i.e., $gf(x) = f(gx)$ for all $g \in G$ and $x \in X$. Such a map is also said to be G-equivariant. We write $G\mathscr{T}$ for the topological category of based G-spaces and based G-maps. ((It is a subcategory of the G-topological category $\mathscr{T}_G$ of based G-spaces and based maps.))

Example 8.1. A homomorphism $\rho\colon G \to O(n)$ defines an unbased action of G on $\mathbb{R}^n$, by $(g, v) \mapsto \rho(g)(v)$, which induces a based action of G on the one-point compactification $S^n = \mathbb{R}^n \cup \{\infty\}$, based at ∞. We write V and S^V for $\mathbb{R}^n$ and S^n with these implicit G-actions. Such a vector space V is called an orthogonal G-representation, and S^V is called a G-representation sphere. If $\phi\colon V \to W$ is injective, hence proper, we write $S^\phi\colon S^V \to S^W$ for its based extension.

Hereafter we omit to say 'based'.

Let $\theta\colon G_1 \to G_2$ be a group homomorphism. We get a continuous functor

$$\theta^*\colon G_2\mathscr{T} \longrightarrow G_1\mathscr{T}$$

mapping each G_2-space X to the same topological space, with the G_1-action given by composition with θ, taking $g \in G_1$ and $x \in X$ to $\theta(g)x$ in X.

Any homomorphism $\theta\colon G_1 \to G_2$ factors as the composite of a surjection of the form $\pi\colon G \to G/N$, with N normal in G, an isomorphism of groups, and an inclusion of the form $\iota\colon H \subset G$. The functor $\pi^*\colon (G/N)\mathscr{T} \to G\mathscr{T}$ views a G/N-space as a G-space with trivial N-action. The functor $\iota^*\colon G\mathscr{T} \to H\mathscr{T}$ restricts the G-action to an H-action. In particular we have the functors $\mathscr{T} \to G\mathscr{T}$ that gives a non-equivariant space the trivial G-action, and $G\mathscr{T} \to \mathscr{T}$ that takes a G-space to the underlying non-equivariant space.

These functors admit left and right adjoints.

For $\pi\colon G \to G/N$ the left adjoint of π^* is the N-orbit space functor

$$X \longmapsto X/N$$

where $X/N = X/{\sim}$ is the quotient space of X given by $x \sim nx$ for each $x \in X$ and $n \in N$. The natural homeomorphism

$$(G/N)\mathscr{T}(X/N, Y) \cong G\mathscr{T}(X, \pi^*Y)$$

has unit the surjective G-map $\eta\colon X \to \pi^*(X/N)$ and counit the G/N-homeomorphism $\epsilon\colon (\pi^*Y)/N \to Y$. (Notation for image of $x \in X$ in X/N?)

The right adjoint of π^* is the N-fixed point functor

$$Z \longmapsto Z^N$$

where $Z^N = \{z \in Z \mid nz = z \text{ for each } n \in N\}$ is a subspace of Z. The natural homeomorphism

$$G\mathscr{T}(\pi^*Y, Z) \cong (G/N)\mathscr{T}(Y, Z^N)$$

has unit the G/N-homeomorphism $\eta\colon Y \to (\pi^*Y)^N$ and counit the injective G-map $\epsilon\colon \pi^*(Z^N) \to Z$.

$$
\begin{tikzcd}
G\mathscr{T} \arrow[r, shift left=2, "(-)/N"] \arrow[r, shift right=2, "(-)^N"'] & (G/N)\mathscr{T} \arrow[l, "\pi^*" description]
\end{tikzcd}
$$

For $\iota\colon H \subset G$ the left adjoint of ι^* is the induction functor

$$X \longmapsto G_+ \wedge_H X$$

where $G_+ \wedge_H X$ is the quotient space of $G_+ \wedge X$ by $\gamma h \wedge x \sim \gamma \wedge hx$, where $\gamma \in G$, $h \in H$ and $x \in X$. (Notation for image of $\gamma \wedge x \in G_+ \wedge X$ in $G_+ \wedge_H X$?) The G-action on $G_+ \wedge_H X$ is given by $g(\gamma \wedge x) = g\gamma \wedge x$ for $g \in G$. (Note that this is not the diagonal action.) The natural homeomorphism

$$G\mathscr{T}(G_+ \wedge_H X, Y) \cong H\mathscr{T}(X, \iota^*Y)$$

has unit the injective H-map $\eta\colon X \to \iota^*(G_+ \wedge_H X)$ sending $x \in X$ to the class of $e \wedge x$, and counit the surjective G-map $G_+ \wedge_H \iota^*Y \to Y$ sending the class of $\gamma \wedge y$ to γy.

The right adjoint of ι^* is the coinduction functor

$$Z \longmapsto F_H(G_+, Z)$$

where $F_H(G_+, Z)$ is the subspace of $F(G_+, Z)$ consisting of maps $f\colon G_+ \to Z$ such that $hf(\gamma) = f(h\gamma)$, for all $h \in H$ and $\gamma \in G$. The G-action on $F_H(G_+, Z)$ is given by sending f to $gf\colon G_+ \to Z$ with $(gf)(\gamma) = f(\gamma g)$. (Note that this is not the conjugation action.) The natural homeomorphism

$$H\mathscr{T}(\iota^*Y, Z) \cong G\mathscr{T}(Y, F_H(G_+, Z))$$

has unit the injective G-map $\eta\colon Y \to F_H(G_+, \iota^*Y)$ taking $y \in Y$ to the map $f\colon G_+ \to Y$ given by $f(\gamma) = \gamma y$ for all $\gamma \in G$, and counit the surjective H-map $\epsilon\colon i^*F_H(G_+, Z) \to Z$ taking $f\colon G_+ \to Z$ to $f(e) \in Z$.

$$
\begin{tikzcd}
H\mathscr{T} \arrow[r, shift left=2, "G_+\wedge_H(-)"] \arrow[r, shift right=2, "F_H(G_+{,}-)"'] & G\mathscr{T} \arrow[l, "\iota^*" description]
\end{tikzcd}
$$

When $\phi\colon G_1 \to G_2$ is an isomorphism, with inverse ψ, the functor $\phi^*\colon G_2\mathscr{T} \to G_1\mathscr{T}$ is an isomorphism of categories, with left and right adjoint given by ψ^*. Combining these constructions, for a general homomorphism $\theta\colon G_1 \to G_2$ with kernel $N \subset G_1$, image $H \subset G_2$, isomorphism $\phi\colon G_1/N \cong H$ and inverse isomorphism $\psi\colon H \cong G_1/N$, the functor θ^* has left adjoint $X \mapsto G_{2+} \wedge_H \psi^*(X/N)$ and right adjoint $Z \mapsto F_H(G_{2+}, \psi^*(Z^N))$.

So far we have only discussed the H-orbits and H-fixed points of a G-space when H is normal in G. The definition $X/H = X/{\sim}$ with $x \sim hx$ for $x \in X$, $h \in H$ works for all $H \subset G$, and likewise for $Z^H = \{z \in Z \mid hz = z \text{ for all } h \in H\}$. The H-orbit functor and H-fixed point functors can be viewed as the composites

$$G\mathscr{T} \xrightarrow{\iota_1^*} N_GH\mathscr{T} \xrightarrow{(-)/H} W_GH\mathscr{T} \xrightarrow{\iota_2^*} \mathscr{T}$$

and

$$G\mathscr{T} \xrightarrow{\iota_1^*} N_GH\mathscr{T} \xrightarrow{(-)^H} W_GH\mathscr{T} \xrightarrow{\iota_2^*} \mathscr{T}$$

where $N_GH = \{n \in G \mid nH = Hn\}$ is the normalizer of H in G, so that H is normal in N_GH, with quotient the Weyl group $W_GH = N_GH/H$. The left hand functors are restriction along $\iota_1\colon N_GH \subset G$, the right hand functors are restriction along $\iota_2\colon \{e\} \subset W_GH$.

(Identify $G_+ \wedge_H Z$ with the H-orbit space of $G_+ \wedge Z$ with the diagonal action, where $h \in H$ acts on G_+ through right multiplication by h^{-1}. Identify $F_H(G_+, Z)$ with the space of left H-maps $G_+ \to Z$, and with the H-fixed points of $F(G_+, Z)$ with the conjugation action.)

8.2 G-homotopies

Given two G-spaces X and Y we give $X \wedge Y$ the diagonal G-action, so that $g(x \wedge y) = gx \wedge gy$. We give the space $F(X, Y)$ of maps $f : X \to Y$ the conjugate G-action, so that $gf : X \to Y$ is given by $(gf)(x) = g(f(g^{-1}x))$. This defines a closed symmetric monoidal structure on $G\mathscr{T}$, so that there are coherent G-equivariant homeomorphisms $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$, $S^0 \wedge Y \cong Y \cong Y \wedge S^0$, $X \wedge Y \cong Y \wedge X$ and

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)) .$$

For any finite-dimensional orthogonal G-representation V and G-spaces X and Z we let

$$\Sigma^V X = X \wedge S^V \qquad \text{and} \qquad \Omega^V Z = F(S^V, Z) .$$

There is a natural homeomorphism

$$F(\Sigma^V X, Z) \cong F(X, \Omega^V Z)$$

with adjunction unit and counit

$$\eta : X \longrightarrow \Omega^V \Sigma^V X \qquad \text{and} \qquad \epsilon : \Sigma^V \Omega^V Z \longrightarrow Z .$$

Furthermore, $G\mathscr{T}$ is tensored over $\mathscr{T}$, with natural homeomorphisms

$$\mathscr{T}(T, G\mathscr{T}(X, Z)) \cong G\mathscr{T}(X \wedge T, Z) \cong G\mathscr{T}(X, F(T, Z))$$

for all G-spaces X and Z and spaces T. Taking $T = I_+$ with $I = [0, 1]$ this lets us define G-homotopy of G-maps. A G-map $f : X \to Y$ is a G-homotopy equivalence if there exists a G-map $f' : Y \to X$ and G-homotopies $f'f \simeq \mathbb{1}_X$ and $ff' \simeq \mathbb{1}_Y$. This implies that $f^H : X^H \to Y^H$ is a homotopy equivalence for each $H \subset G$.

A G-map $f : X \to Y$ is a G-Hurewicz cofibration if for every commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & F(I_+, Z) \arrow[d, "p_0"] \\ Y \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

in $G\mathscr{T}$ there is a dashed arrow making both triangles commute. Equivalently,

$$i_0 \cup (f \times \mathbb{1}) : Y \cup_X X \wedge I_+ \to Y \wedge I_+$$

admits a left inverse in $G\mathscr{T}$. This implies that $f^H : X^H \to Y^H$ is a Hurewicz cofibration for each $H \subset G$. (Also G-HLP and G-Hurewicz fibration.)

Definition 8.2. Let $X \in G\mathscr{T}$ be a G-space. For each closed subgroup $H \subset G$ and each integer $k \geq 0$ we let

$$\pi_k^H(X) = \pi_k(X^H)$$

denote the set of homotopy classes of maps $S^k \to X^H$. This is a group for $k = 1$ and an abelian group for $k \geq 2$. Each G-map $f : X \to Y$ induces a function $f_* = \pi_k^H(f) : \pi_k^H(X) \to \pi_k^H(Y)$ for each $H \subset G$ and $k \geq 0$, which is a group homomorphism for $k \geq 1$. G-homotopic maps f and f' induce the same functions $f_* = f'_*$.

We say that $f : X \to Y$ is a weak G-homotopy equivalence if for each $H \subset G$ the restricted map $f^H : X^H \to Y^H$ is a weak homotopy equivalence. In this case, $\pi_k^H(f) : \pi_k^H(X) \to \pi_k^H(Y)$ is a bijection for each $H \subset G$ and $k \geq 0$, and an isomorphism for each $k \geq 1$. A G-homotopy equivalence is a weak G-homotopy equivalence.

For G-spaces X and Y note that $F(X, Y)^H$ is the space of H-maps $f : X \to Y$. Let

$$\pi^H(X \to Y) = \pi_0^H F(X, Y) = \pi_0 F(X, Y)^H$$

be the set of H-homotopy classes of H-maps $X \to Y$. Thus $\pi_k^H(X) = \pi^H(S^k \to X)$ for each $k \geq 0$, where S^k has the trivial G-action.

A G-CW complex is built by attaching G-n-cells of the form $(G/H \times D^n)_+$ along their boundaries $(G/H \times \partial D^n)_+$.

Definition 8.3. A G-CW complex (with base point) X is a G-space with a filtration

$$* = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X$$

by G-subspaces, such that there is a pushout square

$$\begin{tikzcd}
\bigvee_\alpha (G/H_\alpha \times \partial D^n)_+ \arrow[r] \arrow[d, "\phi"'] & \bigvee_\alpha (G/H_\alpha \times D^n)_+ \arrow[d, "\Phi"] \\
X^{(n-1)} \arrow[r] & X^{(n)}
\end{tikzcd}$$

for each integer $n \geq 0$, and $X = \operatorname{colim}_n X^{(n)}$. We call $X^{(n)}$ the G-n-skeleton of X. A G-map $f\colon X \to Y$ of G-CW complexes is G-cellular if $f(X^{(n)}) \subset Y^{(n)}$ for each $n \geq 0$.

Proposition 8.4. *If X is a G-CW complex and $f\colon Y \to Z$ a weak G-homotopy equivalence, then*

$$f_*\colon \pi^G(X \to Y) \longrightarrow \pi^G(X \to Z)$$

is a bijection. Hence, if Y and Z are G-CW complexes then f is a G-homotopy equivalence.

(The following argument is imprecise in low degrees.)

Proof. Each inclusion $(G/H_\alpha \times \partial D^n)_+ \to (G/H_\alpha \times D^n)_+$ is a G-Hurewicz cofibration, hence so is each pushout $X^{(n-1)} \to X^{(n)}$. The G-homotopy cofiber sequence

$$X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \bigvee_\alpha (G/H_\alpha)_+ \wedge D^n/\partial D^n$$

and the map f induce a map of long exact sequences

$$\begin{tikzcd}
\pi^G(\Sigma X^{(n-1)} \to Y) \arrow[r] \arrow[d, "f_*"'] & \prod_\alpha \pi_n^{H_\alpha}(Y) \arrow[r] \arrow[d, "f_*"'] & \pi^G(X^{(n)} \to Y) \arrow[r] \arrow[d, "f_*"'] & \pi^G(X^{(n-1)} \to Y) \arrow[r] \arrow[d, "f_*"'] & \prod_\alpha \pi_{n-1}^{H_\alpha}(Y) \arrow[d, "f_*"'] \\
\pi^G(\Sigma X^{(n-1)} \to Z) \arrow[r] & \prod_\alpha \pi_n^{H_\alpha}(Z) \arrow[r] & \pi^G(X^{(n)} \to Z) \arrow[r] & \pi^G(X^{(n-1)} \to Z) \arrow[r] & \prod_\alpha \pi_{n-1}^{H_\alpha}(Z)
\end{tikzcd}$$

so by the five-lemma and induction on n it follows that

$$f_*\colon \pi^G(X^{(n)} \to Y) \longrightarrow \pi^G(X^{(n)} \to Z)$$

is a bijection for each $n \geq 0$. The claim for X then follows by passage to limits (using Milnor's lim-lim^1-sequence). The final conclusion then follows by the Yoneda lemma. □

Definition 8.5. For G-spaces X and Y let

$$[X, Y]^G = \pi^G(\Gamma X \to Y)$$

where $\Gamma X \to X$ is a weak G-homotopy equivalence from a G-CW complex.

8.3 Orthogonal G-spectra

Let G be a compact Lie group. Following Schwede we work with a model for G-equivariant spectra where the objects are simply spectra with a G-action.

Definition 8.6. An orthogonal G-spectrum is an orthogonal spectrum X with a continuous G-action

$$\lambda\colon G_+ \wedge X \longrightarrow X$$

through orthogonal spectrum maps, i.e., for each $g \in G$ the composite

$$g\cdot\colon X \cong \{g\}_+ \wedge X \xrightarrow{\lambda} X$$

is map of orthogonal spectra. (Being an action, λ satisfies associativity and unitality.) A G-map $f\colon X \to Y$ of orthogonal G-spectra is a map $f\colon X \to Y$ of orthogonal spectra that commutes with the G-actions, so that $\lambda(\mathbb{1} \wedge f) = f\lambda\colon G_+ \wedge X \to Y$. Let $G\,\mathrm{Sp}^{\mathbb{O}}$ denote the topological category of orthogonal G-spectra and G-maps.

In more detail, X is a sequence of based $G \times O(n)$-spaces X_n for $n \geq 0$ and a sequence of structure G-maps $\sigma\colon X_n \wedge S^1 \to X_{n+1}$, where G acts trivially on S^1, such that each ℓ-fold composite

$$\sigma^\ell\colon X_k \wedge S^\ell \longrightarrow X_{k+\ell}$$

is $O(k) \times O(\ell)$-equivariant. A G-map $f\colon X \to Y$ is a sequence of $G \times O(n)$-maps $f_n\colon X_n \to Y_n$ that commute with the structure G-maps.

From another point of view, λ is adjoint to a map

$$\tilde{\lambda}\colon G \longrightarrow \mathrm{Sp}^{\mathbb{O}}(X, X) \subset \prod_n \mathscr{T}(X_n, X_n)^{O(n)}$$

of topological monoids, where the monoid structure in the target is given by composition of maps.

Definition 8.7. An orthogonal ring G-spectrum is an orthogonal ring spectrum R with a continuous G-action

$$\lambda\colon G_+ \wedge R \longrightarrow R$$

through orthogonal ring spectrum maps, i.e., for each $g \in G$ the composite

$$g\cdot\colon R \cong \{g\}_+ \wedge R \xrightarrow{\lambda} R$$

is map of orthogonal ring spectra. (By assumption, $g_1(g_2x) = (g_1g_2)x$ and $ex = x$ for $x \in R_n$ at each level $n \geq 0$.) A G-map $f\colon Q \to R$ of orthogonal ring G-spectra is a map $f\colon Q \to R$ of orthogonal ring spectra that commutes with the G-actions, so that $\lambda(\mathbb{1} \wedge f) = f\lambda\colon G_+ \wedge Q \to R$.

In G-equivariant homotopy theory, we will wish to be able to embed G-spaces like G/H equivariantly in inner product spaces like $\mathbb{R}^n$. We must then permit non-trivial G-actions on these vector spaces, and therefore consider G-embeddings in orthogonal G-representations W, extending to based maps $(G/H)_+ \to S^W$. In the stable theory we wish to make the operation of smashing with S^W into an equivalence, which we can achieve by stabilizing with respect to structure G-maps of the form $\sigma\colon X(U) \wedge S^W \to X(V)$, for G-representations $U \subset V$, where $W = V - U$ is the orthogonal complement of U in V. Here $X(V)$ will be equal to X_n in the case when $V = \mathbb{R}^n$ with the trivial G-action, but in general the G-action on V should be reflected in the G-action on $X(V)$.

Definition 8.8. An inner product space is a finite-dimensional real vector space V equipped with an Euclidean inner product $\langle -, - \rangle\colon V \times V \to \mathbb{R}$. Let $\mathbb{I}(V, W)$ be the space of linear isometries $\phi\colon V \to W$.

An orthogonal G-representation is an action through linear isometries of G on an inner product space V. Let G act by conjugation on $\mathbb{I}(V, W)$, sending ϕ to $g\phi$ given by $(g\phi)(v) = g(\phi(g^{-1}(v)))$ for $v \in V$.

Example 8.9. Let $V = \mathbb{R}^n$ with the usual dot product. The orthogonal group $O(n)$ is equal to the group of linear isometries $\mathbb{R}^n \to \mathbb{R}^n$. An orthogonal G-representation on $\mathbb{R}^n$ is equivalent to a group homomorphism $\rho\colon G \to O(n)$, with $g \cdot v = \rho(g)(v)$ for each $g \in G$ and $v \in V = \mathbb{R}^n$.

Hereafter we simply refer to an orthogonal G-representation V as a G-representation. The following important construction defines a prolongation of orthogonal G-sequences $X\colon \mathbb{O} \to G\mathscr{T}$ from the trivial G-representations $\mathbb{R}^n$ to general G-representations V.

Definition 8.10. Let X_n be a $G \times O(n)$-space, and let V be an n-dimensional G-representation. Let

$$X(V) = \mathbb{I}(\mathbb{R}^n, V)_+ \underset{O(n)}{\wedge} X_n$$

be the balanced product, where $O(n)$ acts on the right on $\mathbb{I}(\mathbb{R}^n, V)$ by $(\phi, A) \mapsto \phi \circ A \colon \mathbb{R}^n \to \mathbb{R}^n \to V$.

Let G act diagonally on the balanced product, by $g(\phi \wedge x) = g\phi \wedge gx$. Here $(g\phi)(v) = g(\phi(v))$ for $v \in \mathbb{R}^n$, since G acts trivially on $\mathbb{R}^n$.

Any choice of linear isometry $\phi \colon \mathbb{R}^n \to V$ determines a homeomorphism $X(\phi) \colon X_n \cong X(\mathbb{R}^n) \to X(V)$, sending $x \in X_n$ to the class of $\phi \wedge x$. The G-action on $X(V)$ depends on the G-actions on both X_n and V. The following construction defines the prolongation of an orthogonal G-spectrum to a "coordinate free" G-spectrum, indexed on general G-representations.

Definition 8.11. Let X be an orthogonal G-spectrum, and $U \subset V$ be k- and $(k+\ell)$-dimensional G-representations, and let $W = V - U$ be the ℓ-dimensional orthogonal complement of U in V. Choose linear isometries $\phi \colon \mathbb{R}^k \to U$ and $\psi \colon \mathbb{R}^\ell \to W$, with sum a linear isometry $\phi \oplus \psi \colon \mathbb{R}^{k+\ell} \to U + W = V$. Let the generalized structure map

$$\sigma \colon X(U) \wedge S^{V-U} \longrightarrow X(V)$$

for $U \subset V$ be characterized by the commutative diagram

$$\begin{tikzcd}
X_k \wedge S^\ell \arrow[r, "\sigma^\ell"] \arrow[d, "X(\phi)\wedge S^\psi"', "\cong"] & X_{k+\ell} \arrow[d, "\cong"', "X(\phi\oplus\psi)"] \\
X(U) \wedge S^{V-U} \arrow[r, "\sigma"] & X(V) .
\end{tikzcd}$$

In other words, σ maps $(\phi \wedge x) \wedge \psi(s)$ for $x \in X_k$ and $s \in S^\ell$ to $\phi \oplus \psi \wedge \sigma^\ell(x \wedge s)$.

Lemma 8.12. *σ is a well-defined G-map.*

Proof. Any other choices of linear isometries $\phi' \colon \mathbb{R}^k \to U$ and $\psi' \colon \mathbb{R}^\ell \to W$ have the form $\phi' = \phi A$ and $\psi' = \psi B$ for some $A \in O(k)$ and $B \in O(\ell)$. In view of the commutative diagram

$$\begin{tikzcd}
X_k \wedge S^\ell \arrow[rr, "\sigma^\ell"] \arrow[dr, "X(\phi)\wedge S^\psi"] & & X_{k+\ell} \arrow[dl, "X(\phi\oplus\psi)"'] \\
& X(U) \wedge S^{V-U} \arrow[r, "\sigma"] & X(V) \\
X_k \wedge S^\ell \arrow[uu, "A\wedge B"] \arrow[ur, "X(\phi')\wedge S^{\psi'}"'] \arrow[rr, "\sigma^\ell"] & & X_{k+\ell} \arrow[uu, "A\oplus B"'] \arrow[ul, "X(\phi'\oplus\psi')"]
\end{tikzcd}$$

where the outer rectangle commutes by the $O(k)\times O(\ell)$-equivariance of σ^ℓ, the two maps $X(U)\wedge S^{V-U} \to X(V)$ corresponding to σ^ℓ under $X(\phi) \wedge S^\psi$ and $X(\phi \oplus \psi)$, and to σ^ℓ under $X(\phi') \wedge S^{\psi'}$ and $X(\phi' \oplus \psi')$, are the same.

The group G acts diagonally on $X(U)$ and S^{V-U}, with $g \in G$ mapping $(\phi\wedge x)\wedge\psi(s)$ to $(g\phi\wedge gx)\wedge g\psi(s)$. Here $\phi' = g\phi \colon \mathbb{R}^k \to U$ and $\psi' = g\psi \colon \mathbb{R}^\ell \to W$ are linear isometries, so σ takes $(g\phi \wedge gx) \wedge g\psi(s) = (\phi' \wedge gx) \wedge \psi'(s)$ to

$$(\phi' \oplus \psi') \wedge \sigma^\ell(gx \wedge s) .$$

On the other hand, σ maps $(\phi \wedge x) \wedge \psi(s)$ to $\phi \oplus \psi \wedge \sigma^\ell(x \wedge s)$, which g takes to

$$g(\phi \oplus \psi) \wedge g\sigma^\ell(x \wedge s) .$$

These expressions are equal, since $\phi' \oplus \psi' = g(\phi \oplus \psi)$ and $\sigma^\ell \colon X_k \wedge S^\ell \to X_{k+\ell}$ is assumed to be a G-map with respect to the trivial G-action on S^ℓ, so that $\sigma^\ell(gx \wedge s) = g\sigma^\ell(x \wedge s)$. □

We think of $X(V)$ as the G-space at level V of the orthogonal G-spectrum X.

Lemma 8.13. *$\sigma\colon X(V) \wedge S^0 \to X(V)$ for $V = V$ is the canonical isomorphism, and the diagram*

$$\begin{tikzcd} X(U) \wedge S^{V-U} \wedge S^{W-V} \arrow[r, "\sigma\wedge 1"] \arrow[d, "\cong"'] & X(V) \wedge S^{W-V} \arrow[d, "\sigma"] \\ X(U) \wedge S^{W-U} \arrow[r, "\sigma"] & X(W) \end{tikzcd}$$

commutes for $U \subset V \subset W$.

The prolongation $V \mapsto X(V)$ of an orthogonal spectrum $n \mapsto X_n$ can be viewed as a left Kan extension, i.e., a left adjoint to a restriction functor.

Recall that $\mathbb{O}$ is the topological category with objects $n \geq 0$ and morphisms $\mathbb{O}(n,n) = O(n)$ and $\mathbb{O}(m,n) = \emptyset$ for $m \neq n$. Let $\mathbb{I}$ be the topological category of (real, finite-dimensional) inner product spaces $(V, \langle -,- \rangle)$, usually denoted V, and isometric isomorphisms $\phi\colon V \to W$. The functor $i\colon \mathbb{O} \to \mathbb{I}$ mapping n to $(\mathbb{R}^n, \cdot)$ and $A \in O(n)$ to the linear isometry $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is an equivalence of topological categories.

To each continuous functor $Y\colon \mathbb{I} \to \mathscr{T}$ we can associate its restriction $i^*Y\colon \mathbb{O} \to \mathscr{T}$. The resulting functor $i^*\colon \mathscr{T}^{\mathbb{I}} \to \mathscr{T}^{\mathbb{O}}$ has left and right adjoints, called the left and right Kan extensions.

$$\begin{tikzcd} \mathbb{O} \arrow[r, "X"] \arrow[d, "i"'] & \mathscr{T} \\ \mathbb{I} \arrow[ur, "i_*X"'] & \end{tikzcd}$$

For an orthogonal sequence $X\colon \mathbb{O} \to \mathscr{T}$ the left Kan extension $i_*X\colon \mathbb{I} \to \mathscr{T}$ is given by the topological colimit

$$(i_*X)(V) = \operatorname*{colim}_{n,\phi\colon \mathbb{R}^n \to V} X_n \cong \mathbb{I}(\mathbb{R}^n, V)_+ \underset{O(n)}{\wedge} X_n$$

over the left fiber category i/V, with objects pairs (n, ϕ) with n an object in $\mathbb{O}$ and $\phi\colon i(n) = \mathbb{R}^n \to V$ a morphism in $\mathbb{I}$. There is a morphism $A\colon (n, \phi A) \to (n, \phi)$ for each $A \in O(n)$.

The adjunction isomorphism

$$\mathscr{T}^{\mathbb{I}}(i_*X, Y) \cong \mathscr{T}^{\mathbb{O}}(X, i^*Y)$$

has unit $\eta\colon X \to i^*(i_*X)$, which at level n is the isomorphism $X_n \cong X(\mathbb{R}^n)$, and counit $\epsilon\colon i_*(i^*Y) \to Y$, which at level V is the isomorphism $\mathbb{I}(\mathbb{R}^n, V)_+ \underset{O(n)}{\wedge} Y(\mathbb{R}^n) \cong Y(V)$. Hence i_* and i^* are inverse equivalences of topological categories $\mathscr{T}^{\mathbb{O}} \simeq \mathscr{T}^{\mathbb{I}}$.

The right Kan extension $i_\star\colon \mathscr{T}^{\mathbb{O}} \to \mathscr{T}^{\mathbb{I}}$ is given by the topological limit

$$(i_\star X)(V) = \lim_{n,\psi\colon V \to \mathbb{R}^n} X_n \cong F(\mathbb{I}(V, \mathbb{R}^n)_+, X_n)^{O(n)}$$

over the right fiber category V/i. This is the space of $O(n)$-maps $\mathbb{I}(V, \mathbb{R}^n)_+ \to X_n$, where $O(n)$ acts from the left on $\mathbb{I}(V, \mathbb{R}^n)$ by composition. In this case the left and right Kan extensions are isomorphic, $i_* \cong i_\star$, so i^* is both a right and a left adjoint, and preserves all small limits and colimits.

To account for the diagonal G-action on $X(V)$, we use categories enriched in $G\mathscr{T}$, i.e., topological G-categories and continuous G-functors.

Let $\mathbb{I}_G$ be the topological G-category of (orthogonal) G-representations V, and isometric isomorphisms $\phi\colon V \to W$, not necessarily commuting with the G-actions. The morphism G-space $\mathbb{I}_G(V, W) = \mathbb{I}(V, W)$ has the G-action given by conjugation: $(g\phi)(v) = g(\phi(g^{-1}(v)))$. The forgetful functor $\mathbb{I}_G \to \mathbb{I}$ is an equivalence of topological categories. (The category $G\mathbb{I}$ of G-representations and G-linear isometric isomorphisms arises as the G-fixed points $(\mathbb{I}_G)^G$ of this G-category, with morphism spaces $G\mathbb{I}(V, W) = \mathbb{I}(V, W)^G$.)

Let $\mathscr{T}_G$ be the topological G-category of (based) G-spaces X and (based) maps $f\colon X \to Y$, not necessarily commuting with the G-actions. The morphism G-space $\mathscr{T}_G(X, Y) = F(X, Y)$ has the G-action given by conjugation: $(gf)(x) = g(f(g^{-1}(x)))$. The forgetful functor $\mathscr{T}_G \to \mathscr{T}$ is an equivalence of topological categories. (The category $G\mathscr{T}$ of G-spaces and G-maps arises as the G-fixed points $(\mathscr{T}_G)^G$ of this G-category, with morphism spaces $G\mathscr{T}(X, Y) = F(X, Y)^G$.)

An orthogonal sequence $X : \mathbb{O} \to \mathscr{T}$ equipped with a G-action is equivalent to an orthogonal sequence of G-spaces, i.e., a continuous functor $X : \mathbb{O} \to G\mathscr{T} \subset \mathscr{T}_G$. The functor $i : \mathbb{O} \to \mathbb{I}$ factors through $G\mathbb{I} \subset \mathbb{I}_G$.

$$\begin{tikzcd} \mathbb{O} \arrow[r, "X"] \arrow[d, "i"'] & \mathscr{T}_G \\ \mathbb{I}_G \arrow[ur, "i_* X"'] & \end{tikzcd}$$

The left Kan extension of the continuous G-functor $X : \mathbb{O} \to \mathscr{T}_G$ along the continuous G-functor $i : \mathbb{O} \to \mathbb{I}_G$ is the continuous G-functor $i_* X : \mathbb{I}_G \to \mathscr{T}_G$ given at level V by the same topological colimit

$$(i_* X)(V) = \operatorname*{colim}_{n,\phi : \mathbb{R}^n \to V} X_n \cong \mathbb{I}(\mathbb{R}^n, V)_+ \underset{O(n)}{\wedge} X_n$$

as before, but now with the diagonal G-action arising from the G-actions on i/V and on X_n. Simplifying the notation from $(i_* X)(V)$ to $X(V)$ we recover the definition given above.

((Discuss how right S-module action prolongs to the generalized structure maps.))

8.4 Examples of orthogonal G-spectra

Example 8.14. The sphere orthogonal G-spectrum S is the sphere spectrum equipped with the trivial G-actions. In other words, $S_n = S^n$ for each integer $n \geq 0$, with trivial G-action and the usual $O(n)$-action, and $\sigma : S_n \wedge S^1 \to S_{n+1}$ is the identification $S^n \wedge S^1 \cong S^{n+1}$, which is clearly G-equivariant.

For each G-representation V there is a G-homeomorphism

$$S(V) = \mathbb{I}(\mathbb{R}^n, V)_+ \underset{O(n)}{\wedge} S_n \xrightarrow{\cong} S^V$$

that sends $\phi \wedge s$ to $S^\phi(s)$, for $\phi : \mathbb{R}^n \to V$ and $s \in S^n$. In particular, G does not act trivially on S^V, unless it acts trivially on V. The generalized structure map $\sigma : S(U) \wedge S^{V-U} \to S(V)$ corresponds to the G-homeomorphism

$$S^U \wedge S^{V-U} \xrightarrow{\cong} S^V$$

obtained by one-point compactification from the G-linear isomorphism $U \oplus (V - U) \cong V$.

Example 8.15. For a finite group G, a $\mathbb{Z}[G]$-module M is an abelian group with an additive G-action. For each $n \geq 0$ the space $HM_n = B(M, S^n)$ of finite sums

$$u = \sum_i m_i x_i$$

with $m_i \in M$ and $x_i \in S^n$ admits a natural G-action, with $g \in G$ taking u to

$$gu = \sum_i (gm_i) x_i \,.$$

It commutes with the $O(n)$-action arising from the standard action on S^n, and the structure maps $B(M, S^n) \wedge S^1 \to B(M, S^{n+1})$ are G-linear. We call HM the Eilenberg–Mac Lane G-spectrum of M. The prolonged functor has

$$HM(V) = B(M, S^V)$$

with $g \in G$ taking $u = \sum_i m_i x_i$ to $gu = \sum_i (gm_i)(gx_i)$, where $m_i \in M$ and $x_i \in S^V$. The generalized structure maps are of the form

$$\sigma : B(M, S^U) \wedge S^{V-U} \longrightarrow B(M, S^V)$$

taking $(\sum_i m_i x_i) \wedge y$ to $\sum_i m_i (x_i \wedge y)$ for $m_i \in M$, $x_i \in S^U$ and $y \in S^{V-U}$.

The category $G\,\mathrm{Sp}^{\mathbb{O}}$ of orthogonal G-spectra has all small colimits and limits, and is tensored and cotensored over the category $G\mathscr{T}$ of G-spaces and G-maps. These colimits, limits, tensors and cotensors are all created levelwise:

$$\begin{aligned}
(\operatorname*{colim}_\alpha X_\alpha)_n &= \operatorname*{colim}_\alpha (X_\alpha)_n \\
(\lim_\alpha X_\alpha)_n &= \lim_\alpha (X_\alpha)_n \\
(X \wedge T)_n &= X_n \wedge T \\
F(T, X)_n &= F(T, X_n)\,.
\end{aligned}$$

The G-actions on $\operatorname{colim}_\alpha (X_\alpha)_n$ and $\lim_\alpha (X_\alpha)_n$ are determined by the termwise coactions on $\bigvee_\alpha (X_\alpha)_n$ and $\prod_\alpha (X_\alpha)_n$. The G-action on $X_n \wedge T$ is the diagonal action

$$g(x \wedge t) = gx \wedge gt$$

and the G-action on $F(T, X_n)$ is the conjugation action

$$(gf)(t) = g(f(g^{-1}(t)))\,,$$

for $g \in G$, $x \in X_n$, $t \in T$ and $f \colon T \to X_n$.

Example 8.16. For any G-space $T \in G\mathscr{T}$ the suspension spectrum $T \wedge S = \Sigma^\infty T$ is given by

$$(\Sigma^\infty T)_n = T \wedge S^n$$

with G acting only on T and $O(n)$ acting only on S^n. The structure maps are the identifications $T \wedge S^n \wedge S^1 \cong T \wedge S^{n+1}$, which are clearly G-equivariant. The prolonged functor is given by

$$(\Sigma^\infty T)(V) = T \wedge S^V$$

with the diagonal G-action, and the generalized structure map for $U \subset V$ is the G-homeomorphism

$$T \wedge S^U \wedge S^{V-U} \cong T \wedge S^V\,.$$

Given a group homomorphism $\theta \colon G_1 \to G_2$, any orthogonal G_2-spectrum Y gives rise to an orthogonal G_1-spectrum $X = \theta^* Y$, having the same underlying orthogonal spectrum, and the G_1-action given by composition with θ. We get a functor $\theta^* \colon G_2\,\mathrm{Sp}^{\mathbb{O}} \to G_1\,\mathrm{Sp}^{\mathbb{O}}$.

Example 8.17. Any orthogonal spectrum X can be viewed as an orthogonal G-spectrum by giving each space X_n the trivial G-action. When $V = \mathbb{R}^n$ with G-action given by a homomorphism $\rho \colon G \to O(n)$, the G-action on $X(V) = X_n$ is the restriction of the $O(n)$-action on X_n along ρ.

The functor θ^* admits left and right adjoints. As in the case of G-spaces, it is easiest to discuss the cases $\pi \colon G \to G/N$ and $\iota \colon H \to G$ separately.

Definition 8.18. For N a normal subgroup of G, the functor

$$\pi^* \colon (G/N)\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow G\,\mathrm{Sp}^{\mathbb{O}}$$

has the left adjoint

$$(-)/N \colon G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow (G/N)\,\mathrm{Sp}^{\mathbb{O}}$$

mapping an orthogonal G-spectrum X to the N-orbit orthogonal G/N-spectrum X/N with n-th G/N-space $(X/N)_n = X_n/N$, and n-th structure G/N-map

$$X_n/N \wedge S^1 \cong (X_n \wedge S^1)/N \xrightarrow{\sigma/N} X_{n+1}/N\,.$$

The unit $\eta \colon X \to \pi^*(X/N)$ is the canonical surjection $X_n \to X_n/N$ at each level, and the counit $\epsilon \colon (\pi^* Y)/N \to Y$ is an isomorphism.

Definition 8.19. The functor π^* also has the right adjoint

$$(-)^N : G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow (G/N)\,\mathrm{Sp}^{\mathbb{O}}$$

mapping an orthogonal G-spectrum Z to the N-fixed orthogonal G/N-spectrum Z^N with n-th G/N-space $(Z^N)_n = (Z_n)^N$, and n-th structure G/N-map

$$Z_n^N \wedge S^1 \cong (Z_n \wedge S^1)^N \xrightarrow{\sigma^N} Z_{n+1}^N\,.$$

The unit $\eta\colon Y \to (\pi^*Y)^N$ is an isomorphism, and the counit $\epsilon\colon \pi^*(Z^N) \to Z$ is the canonical inclusion $Z_n^N \to Z_n$ at each level.

Definition 8.20. For H and subgroup of G, the functor

$$\iota^* : G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow H\,\mathrm{Sp}^{\mathbb{O}}$$

has the left adjoint

$$G \ltimes_H (-) \colon H\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow G\,\mathrm{Sp}^{\mathbb{O}}$$

mapping an orthogonal H-spectrum X to the induced orthogonal G-spectrum $G \ltimes_H X$ with n-th G-space $(G \ltimes_H X)_n = G_+ \wedge_H X_n$, and n-th structure G-map

$$(G_+ \wedge_H X_n) \wedge S^1 \cong G_+ \wedge_H (X_n \wedge S^1) \xrightarrow{1\wedge\sigma} G_+ \wedge_H X_{n+1}\,.$$

The unit $\eta\colon X \to \iota^*(G \ltimes_H X)$ is the inclusion $X_n \cong H_+ \wedge_H X_n \subset G_+ \wedge_H X_n$ at each level, and the counit $G \ltimes_H (\iota^*Y) \to Y$ is given by the G-action $G_+ \wedge_H Y_n \to Y_n$ at each level.

Definition 8.21. The functor ι^* also has the right adjoint

$$F_H[G,-) \colon H\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow G\,\mathrm{Sp}^{\mathbb{O}}$$

mapping an orthogonal H-spectrum Z to the coinduced orthogonal G-spectrum $F_H[G,Z)$ with n-th G-space $F_H(G,Z)_n = F_H(G_+, Z_n)$, and n-th structure G-map

$$F_H(G_+, Z_n) \wedge S^1 \longrightarrow F_H(G_+, Z_n \wedge S^1) \xrightarrow{F(1,\sigma)} F_H(G_+, Z_{n+1})\,.$$

The unit $\eta\colon Y \to F_H[G, \iota^*Y)$ is given by the adjoint G-action $Y_n \to F_H(G_+, Y_n)$ at each level, and the counit $\epsilon\colon \iota^*F_H[G,Z) \to Z$ is the projection $F_H(G_+, Z_n) \to F_H(H_+, Z_n) \cong Z_n$ at each level.

Remark 8.22. Following Lewis-May-Steinberger, we write $G \ltimes_H X$ in place of $G_+ \wedge_H X$, in part to remember that the generalized structure maps $\sigma\colon (G \ltimes_H X)(U) \wedge S^{V-U} \to (G \ltimes_H X)(V)$ involves an untwisting isomorphism and is more complicated than the notation $G_+ \wedge_H X$ might suggest. Similarly we write $F_H[G,Z)$ in place of $F_H(G_+, Z)$ to emphasize that the adjoint generalized structure map $\tilde\sigma\colon F_H[G,Z)(U) \to \Omega^{V-U} F_H[G,Z)(V)$ also involves an untwisting isomorphism that might not be expected from the notation $F_H(G_+, Z)$.

Remark 8.23. For each orthogonal G-spectrum X and subgroup $H \subset G$ the H-fixed orthogonal spectrum X^H is defined by the same formulas as in the case when $H = N$ is normal in G, with $(X^H)_n = (X_n)^H$. The functor $(-)^H$ can be viewed as the composite

$$G\,\mathrm{Sp}^{\mathbb{O}} \xrightarrow{\iota^*} (N_G H)\,\mathrm{Sp}^{\mathbb{O}} \xrightarrow{(-)^H} (W_G H)\,\mathrm{Sp}^{\mathbb{O}} \xrightarrow{\iota^*} \mathrm{Sp}^{\mathbb{O}}$$

as in the case of G-spaces. Similar remarks apply to X/H.

Remark 8.24. The H-orbit and H-fixed functors will not preserve $\underline{\pi}_*$-isomorphisms when applied to general orthogonal G-spectra, hence are only homotopically meaningful in restricted settings. On the other hand, induction and coinduction along $H \subset G$ will have good homotopical properties, since G admits the structure of a finite H-CW complex.

8.5 Closed symmetric monoidal structure

The smash product $X \wedge Y$ of an orthogonal G_1-spectrum X and an orthogonal G_2-spectrum Y is naturally an orthogonal $(G_1 \times G_2)$-spectrum, with (g_1, g_2) acting as the smash product

$$g_1 \wedge g_2 \colon X \wedge Y \longrightarrow X \wedge Y$$

of $g_1 \colon X \to X$ and $g_2 \colon Y \to Y$. When $G_1 = G_2 = G$, we can restrict the $(G \times G)$-action over the diagonal $\Delta \colon G \to G \times G$, to view the smash product $X \wedge Y$ of two G-spectra X and Y as a G-spectrum. Here $g \in G$ acts as the smash product

$$g \wedge g \colon X \wedge Y \longrightarrow X \wedge Y\,.$$

The function spectrum $F(X, Y)$ from an orthogonal G_1-spectrum X to an orthogonal G_2-spectrum Y is naturally an orthogonal $(G_1^{op} \times G_2)$-spectrum, where G_1^{op} denotes the opposite group of G_1, i.e., the group with the same underlying set as G_1, in which the product gg' of two elements $g, g' \in G_1$ is defined to be the product $g'g$ as formed in G_1. An element $(g_1, g_2) \in G_1^{op} \times G_2$ acts on $F(X, Y)$ by

$$F(g_1, g_2) \colon F(X, Y) \longrightarrow F(X, Y)\,.$$

When $G_1 = G_2 = G$ we can restrict the $(G^{op} \times G)$-action over the anti-diagonal $\tilde{\Delta} = (\chi \times 1)\Delta \colon G \to G^{op} \times G$, given by $\tilde{\Delta}(g) = (g^{-1}, g)$, to view the function spectrum $F(X, Y)$ of two orthogonal G-spectra X and Y as a G-spectrum. Here $g \in G$ acts on $F(X, Y)$ by

$$F(g^{-1}, g) \colon F(X, Y) \longrightarrow F(X, Y)\,.$$

We get a closed symmetric monoidal structure on each category $G\,\mathrm{Sp}^{\mathbb{O}}$.

Example 8.25. An orthogonal ring G-spectrum R is a monoid in the category $G\,\mathrm{Sp}^{\mathbb{O}}$, with unit G-map $\eta \colon S \to R$ and multiplication G-map $\mu \colon R \wedge R \to R$, satisfying unitality and associativity. An orthogonal commutative ring G-spectrum is a commutative monoid in the same category.

Example 8.26. For $\mathbb{Z}[G]$-modules M and N, the tensor product $M \otimes N$ has a diagonal $\mathbb{Z}[G]$-module structure with $g(m \otimes n) = gm \otimes gn$, and the natural map

$$HM \wedge HN \longrightarrow H(M \otimes N)$$

is a G-map of orthogonal G-spectra. Similarly, $\mathrm{Hom}(M, N)$ has a conjugation $\mathbb{Z}[G]$-module structure, and the natural map

$$H(\mathrm{Hom}(M, N)) \longrightarrow F(HM, HN)$$

is a G-map of orthogonal G-spectra.

Example 8.27. For any orthogonal G-spectrum X, the n-fold smash power

$$X^{\wedge n} = X \wedge \cdots \wedge X$$

is an orthogonal $\Sigma_n \wr G$-spectrum, where the wreath product

$$\Sigma_n \wr G = \Sigma_n \ltimes G^n$$

is the semidirect product for the permutation action of the symmetric group Σ_n on the n-th power $G^n = G \times \cdots \times G$. Here $(g_1, \dots, g_n) \in G^n$ acts on $X^{\wedge n}$ with g_i acting on the i-th smash factor, and the transposition $(i, i+1) \in \Sigma_n$ acts on $X^{\wedge n}$ by the twist map $X^{\wedge(i-1)} \wedge \gamma \wedge X^{\wedge(n-i-1)}$, for $1 \le i < n$. These transpositions generate Σ_n, subject to the Coxeter group relations $(i, i+1)^2 = e$ and

$$(i-1, i)(i, i+1)(i-1, i) = (i, i+1)(i-1, i)(i, i+1)\,,$$

together with the condition that $(i, i+1)$ and $(j, j+1)$ commute for $|i - j| \ge 2$. The analogous relations $\gamma^2 = 1 \colon X \wedge X \to X \wedge X$ and

$$(\gamma \wedge 1)(1 \wedge \gamma)(\gamma \wedge 1) = (1 \wedge \gamma)(\gamma \wedge 1)(1 \wedge \gamma) \colon X^{\wedge 3} \to X^{\wedge 3}$$

are satisfied in any symmetric monoidal category (when we suppress the associativity isomorphisms α).

In particular, for any orthogonal spectrum X, the smash power $X^{\wedge n}$ is naturally an orthogonal Σ_n-spectrum. By restriction to cyclic permutations, $X^{\wedge n}$ is also naturally an orthogonal C_n-spectrum. When R is an orthogonal commutative ring G-spectrum the n-fold multiplication $\mu^{(n)} \colon R^{\wedge n} \to R$ factors through the Σ_n-orbit spectrum to give a G-map $R^{\wedge n}/\Sigma_n \to R$ for each $n \ge 0$. This orbit construction is not generally homotopically meaningful, but the corresponding "extended power" construction

$$D_n(R) = E\Sigma_{n+} \wedge_{\Sigma_n} R^{\wedge n}$$

and the resulting maps

$$\begin{tikzcd}
R^{\wedge n} \arrow[r] \arrow[dr, "\mu^{(n)}"'] & E\Sigma_{n+} \wedge_{\Sigma_n} R^{\wedge n} \arrow[r] \arrow[d] & R^{\wedge n}/\Sigma_n \arrow[dl] \\
& R &
\end{tikzcd}$$

play an important role in multiplicative stable homotopy theory.

(Multiplicative norms.)

(Topological Hochschild homology.)

9 Stable equivariant homotopy theory

9.1 Homotopy groups

The homotopy groups of an orthogonal G-spectrum are defined as filtered colimits over suitable G-representations. To specify the (small) category over which this colimit is formed, we can use the notion of a G-universe (Lewis-May-Steinberger, p. 11).

Definition 9.1. Given a collection of irreducible (real, finite-dimensional, orthogonal) G-representations, including the trivial representation $\mathbb{R}$, the associated G-universe $\mathscr{U}$ is the direct sum of a countably infinite number of copies of each of these irreducible G-representations. It is itself a real inner product space of countably infinite dimension, G acts on $\mathscr{U}$ by isometries, and the fixed subspace $\mathscr{U}^G$ is identified with a countably infinite direct sum of copies of $\mathbb{R}$.

If $\mathscr{U} \cong \mathbb{R}^\infty$ only contains the trivial G-representation, we call it the trivial G-universe. If $\mathscr{U}$ contains every irreducible G-representation, we call it a complete G-universe.

Example 9.2. If G is a finite group, the complex regular representation $\mathbb{C}[G]$ contains one or more copies of each irreducible G-representation, and $\mathbb{C}[G] \cong \mathbb{R}[G] \oplus \mathbb{R}[G]$. Hence the direct sum

$$\mathscr{U} = \bigoplus_{i=1}^{\infty} \mathbb{R}[G]$$

is a complete G-universe, with $\mathscr{U}^G \cong \bigoplus_{i=1}^{\infty} \mathbb{R} = \mathbb{R}^\infty$. In this case the n-fold direct sums

$$n\rho_G = \bigoplus_{i=1}^{n} \mathbb{R}[G]$$

of the real regular representation $\rho_G = \mathbb{R}[G]$ form a cofinal sequence

$$0 \subset \rho_G \subset 2\rho_G \subset \cdots \subset n\rho_G \subset \dots$$

of finite-dimensional subrepresentations of $\mathscr{U} = \infty\rho_G$. In other words, for any finite-dimensional representation $V \subset \mathscr{U}$ there exists an n such that $V \subset n\rho_G$.

Example 9.3. If G is a positive-dimensional compact Lie group, it has a countable number of isomorphism classes of irreducible representations. For instance, the irreducible complex representations of the circle group $\mathbb{T} = U(1)$ are the integral tensor powers $\mathbb{C}(n) = \mathbb{C}(1)^{\otimes n}$ of the standard representation. Here $z \in \mathbb{T}$ acts on $\mathbb{C}(n)$ by multiplication by z^n. Viewed as real representations these are irreducible for $n \neq 0$, while $\mathbb{C}(0) = \mathbb{R} \oplus \mathbb{R}$ is the sum of two copies of the trivial real representation.

Let $X \in G\,\mathrm{Sp}^{\mathbb{O}}$ be an orthogonal G-spectrum, with generalized structure maps $\sigma\colon X(U) \wedge S^{V-U} \to X(V)$. The following definition depends on an chosen G-universe $\mathscr{U}$, or more specifically on the collection of irreducible G-representations contained in $\mathscr{U}$.

Definition 9.4. For each closed subgroup $H \subset G$ and each non-negative integer $k \geq 0$ we let

$$\pi_k^H(X) = \operatorname*{colim}_{V \subset \mathscr{U}} \pi_k^H(\Omega^V X(V)) .$$

Here V ranges over the partially ordered set of G-representations $V \subset \mathscr{U}$,

$$\pi_k^H(\Omega^V X(V)) = \pi^H(S^k \wedge S^V \to X(V))$$

is the set of homotopy classes of H-maps $S^k \wedge S^V \to X(V)$, and the colimit is formed over the functions $\pi_k^H(\Omega^U X(U)) \longrightarrow \pi_k^H(\Omega^V X(V))$ for $U \subset V \subset \mathscr{U}$, induced on π_k^H by the G-map $\Omega^U X(U) \to \Omega^V X(V)$ that takes $f\colon S^U \to X(U)$ to the composite

$$S^V \cong S^U \wedge S^{V-U} \xrightarrow{f\wedge 1} X(U) \wedge S^{V-U} \xrightarrow{\sigma} X(V) .$$

For negative $k = -\ell < 0$, we let

$$\pi_k^H(X) = \operatorname*{colim}_{\mathbb{R}^\ell \subset V \subset \mathscr{U}} \pi_0^H(\Omega^{V-\mathbb{R}^\ell} X(V)) .$$

Here V ranges over the partially ordered set of G-representations $V \subset \mathscr{U}$ that contain the standard copy of $\mathbb{R}^\ell \subset \mathscr{U}$,

$$\pi_0^H(\Omega^{V-\mathbb{R}^\ell} X(V)) = \pi^H(S^{V-\mathbb{R}^\ell} \to X(V))$$

is the set of homotopy classes of H-maps $S^{V-\mathbb{R}^\ell} \to X(V)$, where $V - \mathbb{R}^\ell$ denotes the orthogonal complement of $\mathbb{R}^\ell$ in V, and the colimit is formed over the functions $\pi_0^H(\Omega^{U-\mathbb{R}^\ell} X(U)) \longrightarrow \pi_0^H(\Omega^{V-\mathbb{R}^\ell} X(V))$ for $\mathbb{R}^\ell \subset U \subset V \subset \mathscr{U}$, induced on π_0^H by the G-map $\Omega^{U-\mathbb{R}^\ell} X(U) \to \Omega^{V-\mathbb{R}^\ell} X(V)$ that takes $f\colon S^{U-\mathbb{R}^\ell} \to X(U)$ to the composite

$$S^{V-\mathbb{R}^\ell} \cong S^{U-\mathbb{R}^\ell} \wedge S^{V-U} \xrightarrow{f\wedge 1} X(U) \wedge S^{V-U} \xrightarrow{\sigma} X(V) .$$

Remark 9.5. For $k \geq 0$ there are evident isomorphisms $\pi_k^H(\Omega X) \cong \pi_{k+1}^H(X)$. For $k = -\ell < 0$ and $\mathbb{R}^{\ell-1} \subset \mathbb{R}^\ell \subset V$ we can identify $(V - \mathbb{R}^\ell) \oplus \mathbb{R}$ with $V - \mathbb{R}^{\ell-1}$, and make compatible identifications

$$\Omega^{V-\mathbb{R}^\ell} \Omega X(V) \cong \Omega^{V-\mathbb{R}^{\ell-1}} X(V)$$

that induce isomorphisms $\pi_k^H(\Omega X) \cong \pi_{k+1}^H(X)$, also in these cases. (Can we make preferred identifications? Do they matter for product pairings?)

Example 9.6. When G is finite and $\mathscr{U} = \infty\rho_G$, these colimits can be calculated using the cofinal sequence of G-representations $n\rho_G$:

$$\pi_k^H(X) = \operatorname*{colim}_n \pi_k^H(\Omega^{n\rho_G} X(n\rho_G))$$

for $k \geq 0$ and

$$\pi_k^H(X) = \operatorname*{colim}_n \pi_0^H(\Omega^{n\rho_G - \mathbb{R}^\ell} X(n\rho_G))$$

for $k = -\ell < 0$. Here $\mathbb{R}^\ell \cong (\ell\rho_G)^G \subset n\rho_G$ for each $n \geq \ell$.

Whenever V is sufficiently large the sets $\pi_k^H \Omega^V X(V)$ and $\pi_0^H(\Omega^{V-\mathbb{R}^\ell} X(V))$ are naturally abelian groups, so in each case the colimit $\pi_k^H(X)$ is also naturally an abelian group. We write $\pi_*^H(X)$ for the resulting graded abelian group, and obtain a functor

$$\pi_*^H \colon G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow \mathrm{grAb}$$

for each $H \subset G$, taking a G-map $f\colon X \to Y$ to the homomorphisms $f_* = \pi_*^H(f)\colon \pi_*^H(X) \to \pi_*^H(Y)$.

Definition 9.7. The collection of abelian groups $\underline{\pi}_k(X) = \{H \mapsto \pi_k^H(X)\}$ is part of a structure called a Mackey functor. (The additional structure is given by restriction maps $\pi_k^L(X) \to \pi_k^H(X)$ and transfer maps $\pi_k^H(X) \to \pi_k^L(X)$ for G-maps $G/H \to G/L$, which we do not specify here.) We write $\underline{\pi}_*(X) = \{H \mapsto \pi_*^H(X)\}$ for the resulting graded Mackey functor.

9.2 Level equivalences and π_*-isomorphisms

The following definitions depend on an chosen G-universe $\mathscr{U}$. Compare Mandell-May, §III.3.

Definition 9.8. A G-map $f\colon X \to Y$ of orthogonal G-spectra is a level G-equivalence (= level equivalence in Mandell-May) if $f(V)\colon X(V) \to Y(V)$ is a weak G-homotopy equivalence, for each G-representation $V \subset \mathscr{U}$.

A G-map $f\colon X \to Y$ is a $\underline{\pi}_*$-isomorphism (= π_*-isomorphism in Mandell-May) if $\pi_k^H(X) \to \pi_k^H(Y)$ is an isomorphism for each subgroup $H \subset G$ and each integer k.

Lemma 9.9. *Each level G-equivalence $f\colon X \to Y$ is a $\underline{\pi}_*$-isomorphism.*

Proof. By assumption, each G-map $X(V) \to Y(V)$ is a weak G-homotopy equivalence. Since S^V and $S^{V-\mathbb{R}^\ell}$ are finite G-CW spaces it follows that $\Omega^V X(V) \to \Omega^V Y(V)$ and $\Omega^{V-\mathbb{R}^\ell} X(V) \to \Omega^{V-\mathbb{R}^\ell} Y(V)$ are weak G-homotopy equivalences. Hence $\pi_k^H(\Omega^V X(V)) \to \pi_k^H(\Omega^V Y(V))$ and $\pi_0^H(\Omega^{V-\mathbb{R}^\ell} X(V)) \to \pi_0^H(\Omega^{V-\mathbb{R}^\ell} Y(V))$ are isomorphisms for all $H \subset G$, $k \geq 0$ and $\ell > 0$. Passing to colimits, $\pi_k^H(X) \to \pi_k^H(Y)$ is an isomorphism for each $H \subset G$ and each integer k. □

The following definition also depends on the choice of G-universe $\mathscr{U}$.

Definition 9.10. An orthogonal G-spectrum X is a G-Ω-spectrum if for each pair of G-representations $U \subset V \subset \mathscr{U}$ the adjoint generalized structure G-map

$$\tilde{\sigma}\colon X(U) \longrightarrow \Omega^{V-U} X(V)$$

is a weak G-homotopy equivalence.

Definition 9.11. The G-equivariant stable homotopy category (associated to the G-universe $\mathscr{U}$) is the localization of the category $G\,\mathrm{Sp}^{\mathbb{O}}$ of orthogonal G-spectra obtained by inverting the subcategory $\mathscr{W}$ of $\underline{\pi}_*$-isomorphisms:

$$\mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}}) = G\,\mathrm{Sp}^{\mathbb{O}}[\mathscr{W}^{-1}]$$

It is equivalent to the localization of the full subcategory of Ω-G-spectra where the level G-equivalences have been inverted, because every orthogonal G-spectrum is $\underline{\pi}_*$-isomorphic to an Ω-G-spectrum.

Remark 9.12. The converse to Lemma 9.9 for maps between G-Ω-spectra will be proved as Theorem 9.29 below. The non-equivariant case was trivial, but the equivariant case requires some work.

Proposition 9.13 (Mandell-May, III.3.8)**.** *Let X be any orthogonal G-spectrum, and W any G-representation. The adjunction unit $\eta\colon X \to \Omega^W \Sigma^W X$ is a $\underline{\pi}_*$-isomorphism.*

Proof. Write $\mathscr{U}$ as an orthogonal sum $\mathscr{U}' \oplus \infty W$, such that each irreducible G-representation in $\mathscr{U}$ can either be embedded in $\mathscr{U}'$ or in W, but not both.

For $k \geq 0$

$$\pi_k^H(X) = \operatorname*{colim}_{V' \subset \mathscr{U}', n} \pi_k^H(\Omega^{V' \oplus nW} X(V' \oplus nW))$$

maps by η_* to

$$\pi_k^H(\Omega^W \Sigma^W X) = \operatorname*{colim}_{V' \subset \mathscr{U}', n} \pi_k^H(\Omega^{V' \oplus nW} \Omega^W \Sigma^W X(V' \oplus nW)) \,.$$

The generalized structure G-maps

$$\sigma\colon \Sigma^W X(V' \oplus nW) \longrightarrow X(V' \oplus (n+1)W)$$

induce a map from the second colimit to the first. These are inverse isomorphisms (check!).

For $k = -\ell < 0$

$$\pi_k^H(X) = \operatorname*{colim}_{V' \subset \mathscr{U}', n} \pi_0^H(\Omega^{V' \oplus nW - \mathbb{R}^\ell} X(V' \oplus nW))$$

where V' and n are such that $\mathbb{R}^\ell \subset V'$ or $\mathbb{R}^\ell \subset nW$, and maps by η_* to

$$\pi_k^H(\Omega^W \Sigma^W X) = \operatorname*{colim}_{V' \subset \mathscr{U}', n} \pi_0^H(\Omega^{V' \oplus nW - \mathbb{R}^\ell} \Omega^W \Sigma^W X(V' \oplus nW)) \,.$$

The generalized structure G-maps σ again induce a map from the second colimit to the first, and these are inverse isomorphisms (check!). □

Corollary 9.14. *Let X be any orthogonal G-spectrum.*

$$S^1 \wedge - \colon \pi_k^H(X) \longrightarrow \pi_{1+k}^H(S^1 \wedge X)$$

is an isomorphism for each $H \subset G$ and each $k \in \mathbb{Z}$.

((The case of $\epsilon\colon \Sigma^W \Omega^W X \to X$ seems to require more effort, either by model category theory, or by using $RO(G)$-graded homotopy groups. Or does the following direct argument work?))

Proposition 9.15. *Let X be any G-spectrum and W any G-representation. The adjunction counit $\epsilon\colon \Sigma^W \Omega^W X \to X$ is a $\underline{\pi}_*$-isomorphism.*

Proof. For $k \geq 0$,

$$\pi_k^H(\Sigma^W \Omega^W X) = \operatorname*{colim}_V \pi_k^H(\Omega^V \Sigma^W \Omega^W X(V))$$

maps by ϵ_* to

$$\pi_k^H(X) = \operatorname*{colim}_V \pi_k^H(\Omega^V X(V)) \cong \operatorname*{colim}_{W \subset V} \pi_k^H(\Omega^{V-W} \Omega^W X(V)) .$$

The adjunction unit $\eta\colon \mathbb{1} \to \Omega^W \Sigma^W$ induces a homomorphism $\eta_\#$ from the second colimit to

$$\operatorname*{colim}_{W \subset V} \pi_k^H(\Omega^{V-W} \Omega^W \Sigma^W \Omega^W X(V)) \cong \pi_k^H(\Sigma^W \Omega^W X) .$$

This makes sense because of the following commutative diagram, for $W \subset U \subset V$.

$$\begin{tikzcd}
\Omega^U X(U) \arrow[r, "\Omega^{U-W}\eta"] \arrow[d, "\Omega^U \tilde\sigma"'] & \Omega^U \Sigma^W \Omega^W X(U) \arrow[r, "\Omega^U \epsilon"] \arrow[d, "\Omega^U \tilde\sigma"] & \Omega^U X(U) \arrow[d, "\Omega^U \tilde\sigma"] \\
\Omega^V X(V) \arrow[r, "\Omega^{V-W}\eta"] & \Omega^V \Sigma^W \Omega^W X(V) \arrow[r, "\Omega^V \epsilon"] & \Omega^V X(V)
\end{tikzcd}$$

Here $\tilde\sigma\colon X(U) \to \Omega^{V-U} X(V)$ and $\tilde\sigma\colon \Sigma^W \Omega^W X(U) \to \Omega^{V-U} \Sigma^W \Omega^W X(V)$ are the generalized adjoint structure maps of X and $\Sigma^W \Omega^W X$, respectively.

The composite $\Omega^U \epsilon \circ \Omega^{U-W} \eta$ is the identity of $\Omega^U X(U)$, for each $W \subset U$, so $\epsilon_* \circ \eta_\#$ is the identity.

Let $\mathrm{sh}^W X$ denote the W-shift of X, i.e., the orthogonal G-spectrum with $(\mathrm{sh}^W X)(V) = X(W \oplus V)$. Hence $(\mathrm{sh}^W X)(U - W) = X(U)$ for $W \subset U$. The horizontal maps above correspond to G-maps

$$\Omega^W \mathrm{sh}^W X \xrightarrow{\eta} \Omega^W \Sigma^W \Omega^W \mathrm{sh}^W X \xrightarrow{\Omega^W \epsilon} \Omega^W \mathrm{sh}^W X .$$

The homomorphism $\eta_\#$ for X is the homomorphism η_* for $\Omega^W \mathrm{sh}^W X$, hence is an isomorphism by Proposition 9.13. Thus ϵ_* for X is also an isomorphism.

((Handle $k = -\ell < 0$.)) □

9.3 Puppe sequences and gluing lemmas

The homotopy cofiber Cf and homotopy fiber Ff of a G-map $f\colon X \to Y$ of orthogonal G-spectra are again orthogonal G-spectra. Compare Mandell-May, Theorem III.3.5.

Proposition 9.16. *For any G-map $f\colon X \to Y$ there are natural long exact sequences*

$$\dots \to \pi_{1+k}^H(Y) \xrightarrow{\iota_*} \pi_k^H(Ff) \xrightarrow{p_*} \pi_k^H(X) \xrightarrow{f_*} \pi_k(Y) \to \pi_{-1+k}^H(Ff) \to \dots$$

and

$$\dots \to \pi_{1+k}^H(Cf) \longrightarrow \pi_k^H(X) \xrightarrow{f_*} \pi_k^H(Y) \xrightarrow{i_*} \pi_k^H(Cf) \xrightarrow{\pi_*} \pi_{-1+k}(X) \to \dots$$

for all subgroups $H \subset G$. The natural map $\eta\colon Ff \to \Omega Cf$ is a $\underline{\pi}_*$-isomorphism.

Proof. (Same as in the non-equivariant case.) □

Lemma 9.17. *For any finite collection $(X_\alpha)_\alpha$ of orthogonal G-spectra the canonical homomorphisms*

$$\bigoplus_\alpha \pi_k^H(X_\alpha) \xrightarrow{\cong} \pi_k^H(\bigvee_\alpha X_\alpha)$$

and

$$\pi_k^H(\prod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha \pi_k^H(X_\alpha)$$

are isomorphisms, for all $H \subset G$ and $k \in \mathbb{Z}$. Hence $\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$ is a $\underline{\pi}_*$-isomorphism.

Proof. (Same as in the non-equivariant case.) □

Lemma 9.18. *For any collection $(X_\alpha)_\alpha$ of orthogonal G-spectra the canonical homomorphism*

$$\bigoplus_\alpha \pi_k^H(X_\alpha) \xrightarrow{\cong} \pi_k^H(\bigvee_\alpha X_\alpha)$$

are isomorphism, for all $H \subset G$ and $k \in \mathbb{Z}$.

Proof. (Same as in the non-equivariant case.) □

Proposition 9.19 (Cobase change)**.** *Consider a pushout square*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "\bar{f}"] \\
Y \arrow[r, "\bar{g}"] & Y \cup_X Z
\end{tikzcd}$$

of orthogonal G-spectra, where $f\colon X \to Y$ is a G-Hurewicz cofibration. If f (resp. g) is a $\underline{\pi}_*$-isomorphism then $\bar{f}$ (resp. $\bar{g}$) is a $\underline{\pi}_*$-isomorphism.

Proof. (Same as in the non-equivariant case.) □

Proposition 9.20 (Gluing lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
Y \arrow[d] & X \arrow[l, "f"'] \arrow[r] \arrow[d] & Z \arrow[d] \\
Y' & X' \arrow[l, "f'"'] \arrow[r] & Z'
\end{tikzcd}$$

of orthogonal G-spectra, where f and f' are G-Hurewicz cofibrations. If $X \to X'$, $Y \to Y'$ and $Z \to Z'$ are $\underline{\pi}_*$-isomorphisms, then so is the induced map $Y \cup_X Z \to Y' \cup_{X'} Z'$.

Proof. (Same as in the non-equivariant case.) □

Proposition 9.21. *If Y is the colimit of a sequence*

$$X_0 \to X_1 \to \dots \to X_\alpha \to X_{\alpha+1} \to \dots$$

of G-Hurewicz cofibrations, then

$$\operatorname*{colim}_\alpha \pi_k^H(X_\alpha) \xrightarrow{\cong} \pi_k^H(Y)$$

is an isomorphism, for each $H \subset G$ and $k \in \mathbb{Z}$.

Proof. (Same as in the non-equivariant case.) □

((Discuss pairings $\pi_k^H(X) \otimes \pi_\ell^H(Y) \to \pi_{k+\ell}^H(X \wedge Y)$, especially when k and ℓ have opposite signs.))

9.4 Untwisting isomorphisms

Lewis-May-Steinberger, §II.4, show how the induction and coinduction functors interact with the closed symmetric monoidal structures, first at the space level and then at the spectrum level.

Let $\iota\colon H \subset G$, let X and Z be H-spaces and let Y be a G-space.

Lemma 9.22. *(i) The left adjoint*

$$\begin{aligned} \zeta\colon G_+ \wedge_H (X \wedge \iota^* Y) &\xrightarrow{\cong} (G_+ \wedge_H X) \wedge Y \\ \gamma \wedge (x \wedge y) &\longmapsto (\gamma \wedge x) \wedge \gamma y \end{aligned}$$

to the H-map $\eta \wedge Y\colon X \wedge Y \to (G_+ \wedge_H X) \wedge Y$ is a G-homeomorphism, with inverse

$$\zeta^{-1}\colon (\gamma \wedge x) \wedge y \longmapsto \gamma \wedge (x \wedge \gamma^{-1} y) .$$

(ii) The right adjoint

$$\begin{aligned} \phi\colon F(Y, F_H(G_+, Z)) &\xrightarrow{\cong} F_H(G_+, F(\iota^* Y, Z)) \\ f &\longmapsto (\gamma \mapsto (y \mapsto f(\gamma^{-1} y)(\gamma))) \end{aligned}$$

to the H-map $F(Y, \epsilon)\colon F(Y, F_H(G_+, Z)) \to F(Y, Z)$ is a G-homeomorphism, with inverse

$$\phi^{-1}\colon f' \longmapsto (y \mapsto (\gamma \mapsto f'(\gamma)(\gamma y))) .$$

Equivalently,

$$\phi(f)(\gamma)(y) = f(\gamma^{-1} y)(\gamma) \qquad \text{and} \qquad \phi^{-1}(f')(y)(\gamma) = f'(\gamma)(\gamma y) .$$

(iii) The right adjoint

$$\begin{aligned} \kappa\colon F(G_+ \wedge_H X, Y) &\xrightarrow{\cong} F_H(G_+, F(X, \iota^* Y)) \\ f &\longmapsto (\gamma \mapsto (x \mapsto \gamma \cdot f(\gamma^{-1} \wedge x))) \end{aligned}$$

to the H-map $F(\eta, Y)\colon F(G_+ \wedge_H X, Y) \to F(X, Y)$ is a G-homeomorphism, with inverse

$$\kappa^{-1}\colon f' \longmapsto (\gamma \wedge x \mapsto \gamma \cdot f'(\gamma^{-1})(x)) .$$

Equivalently,

$$\kappa(f)(\gamma)(x) = \gamma \cdot f(\gamma^{-1} \wedge x) \qquad \text{and} \qquad \kappa^{-1}(f')(\gamma \wedge x) = \gamma \cdot f'(\gamma^{-1})(x) .$$

Example 9.23. For G-spaces Y, there are natural G-homeomorphisms

$$\zeta\colon G_+ \wedge_H (\iota^* Y) \cong (G/H)_+ \wedge Y$$

and

$$\kappa\colon F(G/H_+, Y) \cong F_H(G_+, \iota^* Y) .$$

Example 9.24. For orthogonal H-spectra X and Z, the induced orthogonal G-spectrum $G \ltimes_H X$ has generalized structure G-maps

$$\sigma\colon (G_+ \wedge_H X(U)) \wedge S^W \xrightarrow{\zeta^{-1}} G_+ \wedge_H (X(U) \wedge \iota^* S^W) \xrightarrow{1 \wedge \sigma} G_+ \wedge_H X(V) ,$$

and the coinduced orthogonal G-spectrum $F_H[G, Z)$ has adjoint generalized structure G-maps

$$\tilde\sigma\colon F_H(G_+, Z(U)) \xrightarrow{F(1, \tilde\sigma)} F_H(G_+, F(\iota^* S^W, Z(V)) \xrightarrow{\phi^{-1}} F(S^W, F_H(G_+, Z(V))) ,$$

for G-representations $U \subset V$ with $W = V - U$.

Proposition 9.25. *Let $\iota\colon H \subset G$, let X and Z be orthogonal H-spectra and let Y be an orthogonal G-spectrum. (i) The left adjoint*

$$\zeta\colon G \ltimes_H (X \wedge \iota^* Y) \xrightarrow{\cong} (G \ltimes_H X) \wedge Y$$

to the H-map $\eta \wedge 1\colon X \wedge Y \to (G \ltimes_H X) \wedge Y$ is a G-isomorphism. In particular,

$$\zeta\colon G \ltimes_H \iota^* Y \xrightarrow{\cong} (G/H)_+ \wedge Y\,.$$

(ii) The right adjoint

$$\phi\colon F(Y, F_H[G, Z]) \xrightarrow{\cong} F_H[G, F(\iota^* Y, Z))$$

to the H-map $F(1, \epsilon)\colon F(Y, F_H[G, Z)) \to F(Y, Z)$ is a G-isomorphism.

(iii) The right adjoint

$$\kappa\colon F(G \ltimes_H X, Y) \xrightarrow{\cong} F_H[G, F(X, \iota^* Y))$$

to the H-map $F(\eta, 1)\colon F(G \ltimes_H X, Y) \to F(X, Y)$ is a G-isomorphism. In particular,

$$\kappa\colon F(G/H_+, Y) \cong F_H[G, \iota^* Y)\,.$$

Proposition 9.26. *If $f\colon X \to Y$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, and B is a finite G-CW complex, then*

$$F(1, f)\colon F(B, X) \longrightarrow F(B, Y)$$

is a $\underline{\pi}_*$-isomorphism.

Proof. There is a natural isomorphism

$$\pi_k^H(F(B, X)) \cong \pi_k^G(F(G_+ \wedge_H B, X))$$

and $G_+ \wedge_H B \cong (G/H)_+ \wedge B$ is a finite G-CW complex, so it suffices to prove that $F(1, f)$ induces an isomorphism on π_k^G for each $k \in \mathbb{Z}$. By induction over the G-cells of B, it suffices to prove this for $B = (G/K)_+ \wedge S^n$, for each $K \subset G$ and $n \ge 0$. The natural isomorphisms

$$\pi_k^G(F((G/K)_+ \wedge S^n, X)) \cong \pi_k^K(\Omega^n X) \cong \pi_{k+n}^K(X)$$

then reduce this to the assumption that f induces an isomorphism on π_{k+n}^K. □

Corollary 9.27. *If $\Sigma^W f\colon \Sigma^W X \to \Sigma^W Y$ is a $\underline{\pi}_*$-isomorphism, then $f\colon X \to Y$ is a $\underline{\pi}_*$-isomorphism.*

Proof. By the case $B = S^W$ of the proposition, $\Omega^W \Sigma^W f$ is a $\underline{\pi}_*$-isomorphism. This uses that S^W admits the structure of a finite G-CW complex. By naturality of the $\underline{\pi}_*$-isomorphism η, it follows that f is a $\underline{\pi}_*$-isomorphism.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "f"'] & \Omega^W \Sigma^W X \arrow[d, "\Omega^W \Sigma^W f"] \\
Y \arrow[r, "\eta"] & \Omega^W \Sigma^W Y
\end{tikzcd}$$

□

Remark 9.28. The generalized converse assertion, that $\underline{\pi}_*$-isomorphisms are preserved by $A \wedge -$ for G-CW complexes A, is harder to prove. This can be obtained model-categorically, as in Mandell-May. For finite or abelian groups G it can be deduced from the Wirthmüller equivalence, as in Schwede's notes. See Proposition 10.27.

Here is the promised partial converse to Lemma 9.9. See Lewis-May-Steinberger (I.7.12) or Mandell-May (III.3.4 and §III.9), who refer to Henning Hauschild (a student of Tammo tom Dieck) for the idea of the argument.

Theorem 9.29. *Each $\underline{\pi}_*$-isomorphism $f\colon X \to Y$ between G-Ω-spectra X and Y is a level G-equivalence.*

Proof. Let Ff be the homotopy fiber of $f\colon X \to Y$, which is an Ω-G-spectrum if X and Y have this property. In view of the homotopy fiber sequence

$$Ff(V) \to X(V) \xrightarrow{f(V)} Y(V) \to Ff(V \oplus \mathbb{R})$$

and the long exact sequences

$$\dots \to \pi_{1+k}^H(X) \xrightarrow{f_*} \pi_{1+k}^H(Y) \to \pi_k^H(Ff) \to \pi_k^H(X) \xrightarrow{f_*} \pi_k^H(Y) \to \dots$$

it suffices to prove the following lemma (with Ff renamed as X). □

Lemma 9.30. *If X is an Ω-G-spectrum such that $\pi_k^H(X) = 0$ for all $H \subset G$ and $k \in \mathbb{Z}$, then $\pi_i^H(X(V)) = 0$ for all $H \subset G$, $V \subset \mathscr{U}$ and $i \geq 0$.*

Proof. For each $H \subset G$ the case of π_*^H with $V = V^H$ follows from the case $V = \mathbb{R}^n$, with $n \geq 0$. Here

$$\pi_i^H(X_n) = \pi_{i-n}^H(\Omega^n X_n) \cong \pi_{i-n}^H(X) = 0$$

for each $i \geq n$. Furthermore,

$$\pi_i^H(X_n) = \pi_0^H(\Omega^i X_n) = \pi_0^H(\Omega^{\mathbb{R}^n - \mathbb{R}^{n-i}} X_n) \cong \pi_{-(n-i)}^H(X) = 0$$

for each $0 \leq i < n$.

The compact Lie group G does not contain any infinite descending chain of (closed) subgroups, so by induction we may assume for a given $H \subset G$ that $\pi_*^K(X(V)) = 0$ for all proper subgroups K of H and all finite G-representations $V \subset \mathscr{U}$. We must prove that $\pi_*^H(X(V))$ for all these V. The inductive beginning, for $H = \{e\}$, was established in the previous paragraph.

Let $W = V - V^G$ and $Z = W - W^H$, so that we have direct sum decompositions

$$V = V^G \oplus W \qquad \text{and} \qquad W = Z \oplus W^H$$

of G and $H \subset N_G H$-representations, respectively. Let $d = \dim(W^H)$.

Claim: $\pi_i^H(X(V)) = 0$ for $i > d$.

Let $S(Z) \subset D(Z)$ be the unit sphere and unit disc of the H-representation Z. We have an H-homotopy cofiber sequence

$$S(Z)_+ \longrightarrow D(Z)_+ \longrightarrow D(Z)/S(Z) \cong S^Z$$

that induces a homotopy fiber sequence

$$F(S^Z, \Omega^{W^H} X(V))^H \longrightarrow F(D(Z)_+, \Omega^{W^H} X(V))^H \longrightarrow F(S(Z)_+, \Omega^{W^H} X(V))^H .$$

Here

$$F(S^Z, \Omega^{W^H} X(V))^H \cong (\Omega^W X(V))^H$$

since $Z \oplus W^H = W$, and

$$F(D(Z)_+, \Omega^{W^H} X(V))^H \simeq \Omega^{W^H} X(V)^H$$

since $D(Z)$ is H-equivariantly contractible. By the assumption that X is an Ω-G-spectrum, for the case of $V^G \subset V$ with orthogonal complement W, the map

$$(\tilde{\sigma})^H \colon X(V^G)^H \xrightarrow{\simeq} (\Omega^W X(V))^H$$

is a weak equivalence. Hence we have a homotopy fiber sequence

$$X(V^G)^H \longrightarrow \Omega^{W^H} X(V)^H \longrightarrow F(S(Z)_+, \Omega^{W^H} X(V))^H .$$

We have shown that $\pi_*(X(V^G)^H) = \pi_*^H(X(V^G)) = 0$, since V^G is a trivial H-representation. Likewise, $W^H \cong \mathbb{R}^d$ is a trivial H-representation, so to show the claim that

$$\pi_i^H(X(V)) = \pi_{i-d}(\Omega^{W^H} X(V)^H) = 0$$

for $i > d$ it suffices to show that

$$\pi_j^H(F(S(Z)_+, \Omega^{W^H} X(V))) = 0$$

for $j > 0$. We may triangulate $S(Z)$ as a finite H-CW complex, with cells of the type $H/K \times D^n$ for $K \subset H$. By definition $Z^H = \{0\}$, so $S(Z)^H = \emptyset$, hence only cells with K a proper subgroup of H will occur in this cell structure. By the inductive hypothesis on H,

$$\pi_j^H(F((H/K)_+ \wedge S^n, \Omega^{W^H} X(V))) \cong \pi_j^K(\Omega^n \Omega^{W^H} X(V)) \cong \pi_{j+n+d}^K(X(V)) = 0$$

for all $j \geq 0$ and $n \geq 0$, so by induction over the H-cells of $S(V)$ we deduce that

$$\pi_j^H(F(S(Z)_+, \Omega^{W^H} X(V))) = 0$$

for $j > 0$, as required to finish the proof of the claim.

It remains to prove that $\pi_i^H(X(V)) = 0$ for $0 \leq i \leq d$. Choose a trivial G-representation $U \cong \mathbb{R}^{d+1}$ that is orthogonal to V, and apply the argument above to $U \oplus V$ in place of V. Then $(U \oplus V)^G = U \oplus V^G$, so $W = (U \oplus V) - (U \oplus V)^G$ and $Z = W - W^H$ are unchanged. In particular, $d = \dim(W^H)$ is unchanged, so $\pi_j^H(X(U \oplus V)) = 0$ for $j > d$. In view of the isomorphism

$$\tilde{\sigma}_* \colon \pi_i^H(X(V)) \xrightarrow{\cong} \pi_i^H(\Omega^U X(U \oplus V)) = \pi_{i+d+1}^H(X(U \oplus V))$$

we conclude that $\pi_i^H(X(V)) = 0$ for all $i \geq 0$. $\square$

10 The Wirthmüller equivalence

Let G be a compact Lie group, $H \subset G$ a closed subgroup, and $L = T_{eH}(G/H)$ the tangent space H-representation. Suppose that $\mathscr{U}$ is a G-universe such that G/H embeds in $\mathscr{U}$. The following theorem was proved by Wirthmüller for suspension spectra of H-spaces, and extended to general H-spaces by Lewis-May-Steinberger.

Theorem 10.1. *Let X be an orthogonal H-spectrum. There is a natural $\underline{\pi}_*$-isomorphism*

$$G \ltimes_H X \xrightarrow{\simeq} F_H[G, \Sigma^L X)$$

of orthogonal G-spectra.

Example 10.2. For each orthogonal spectrum X there is a natural $\underline{\pi}_*$-isomorphism

$$\mathbb{T} \ltimes X \simeq F[\mathbb{T}, \Sigma X)$$

of orthogonal $\mathbb{T}$-spectra, so that $\pi_k^H(\mathbb{T} \ltimes X) \cong \pi_k^H(F[\mathbb{T}, \Sigma X))$ for all $H \subset \mathbb{T}$ and $k \in \mathbb{Z}$. Here $\Sigma \cong \Sigma^L$, where $L = T_1\mathbb{T} = i\mathbb{R}$.

10.1 Algebraic prototype

Let G be a finite group, $H \subset G$ a subgroup, and M a $\mathbb{Z}[H]$-module. The induced and coinduced $\mathbb{Z}[G]$-modules

$$\psi \colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \xleftrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \colon \omega$$

are naturally isomorphic, by a pair of mutually inverse homomorphisms ψ and ω that we now make explicit.

Definition 10.3. Let

$$N = \sum_{kH \in G/H} kH \in \mathbb{Z}[G/H]$$

be the norm element, and let $t\colon \mathbb{Z} \to \mathbb{Z}[G/H]$ be the $\mathbb{Z}[G]$-module homomorphism given by $t(1) = N$. (Note that $gN = N$ for each $g \in G$, since $kH \mapsto gkH$ permutes the left cosets of H in G.) Consider the $\mathbb{Z}[G]$-module homomorphism

$$t \otimes 1\colon \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \cong \mathbb{Z} \otimes \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \xrightarrow{t\otimes 1} \mathbb{Z}[G/H] \otimes \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$$

mapping $f\colon \mathbb{Z}[G] \to M$ to $N \otimes f$. Consider also the $\mathbb{Z}[G]$-linear untwisting isomorphism

$$\begin{aligned} \zeta\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (\iota^* \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)) &\xrightarrow{\cong} \mathbb{Z}[G/H] \otimes \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \\ \gamma \otimes f &\longmapsto \gamma \otimes \gamma f \end{aligned}$$

and the $\mathbb{Z}[G]$-linear homomorphism

$$1 \otimes \epsilon\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (\iota^* \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)) \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \,,$$

where $\epsilon\colon \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \to M$ is the counit mapping $f\colon \mathbb{Z}[G] \to M$ to $f(e) \in M$. Let

$$\omega = (1 \otimes \epsilon)\zeta^{-1}(t \otimes 1)\colon \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$$

be the composite $\mathbb{Z}[G]$-linear homomorphism. More explicitly, a $\mathbb{Z}[H]$-linear homomorphism $f\colon \mathbb{Z}[G] \to M$ maps under $(t \otimes 1)$ to $\sum_{kH\in G/H} kH \otimes f$, which is the image under ζ of $\sum_{kH\in G/H} k \otimes k^{-1}f$, which maps under $1 \otimes \epsilon$ to

$$\sum_{kH\in G/H} k \otimes f(k^{-1})\,.$$

(Note that $\epsilon(k^{-1}f) = (k^{-1}f)(e) = f(ek^{-1}) = f(k^{-1})$.)

Definition 10.4. Let $u\colon \mathbb{Z}[G] \to \mathbb{Z}[H]$ be the $\mathbb{Z}[H]$-$\mathbb{Z}[H]$-bimodule homomorphism given by

$$u(g) = \begin{cases} g & \text{for } g \in H, \\ 0 & \text{otherwise,} \end{cases}$$

let $u \otimes 1\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \longrightarrow \mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} M \cong M$ be the $\mathbb{Z}[H]$-module homomorphism given by

$$(u \otimes 1)(g \otimes m) = \begin{cases} gm & \text{for } g \in H, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\psi\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \longrightarrow \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$$

be the $\mathbb{Z}[G]$-module homomorphism that is right adjoint to $u \otimes 1$. Hence $\psi(g \otimes m) \in \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ is given by

$$\psi(g \otimes m)\colon \gamma \mapsto \begin{cases} \gamma g m & \text{if } \gamma g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 10.5. *The natural $\mathbb{Z}[G]$-module homomorphisms ψ and ω are mutually inverse isomorphisms.*

Proof. $\omega\psi$ maps $g \otimes m$ to

$$\omega(f) = \sum_{kH\in G/H} k \otimes f(k^{-1})\,,$$

where $f(k^{-1}) = k^{-1}gm$ if $k^{-1}g \in H$ and $f(k^{-1}) = 0$ otherwise. Hence

$$k \otimes f(k^{-1}) = k \otimes k^{-1}gm = kk^{-1}g \otimes m = g \otimes m$$

if $kH = gH$, and $k \otimes f(k^{-1}) = 0$ otherwise. Thus $\omega\psi(g \otimes m) = g \otimes m$.

Conversely, $\psi\omega$ maps f to

$$\sum_{kH\in G/H} \psi(k \otimes f(k^{-1}))\,,$$

which takes γ to $\gamma k f(k^{-1})$ where $kH \in G/H$ is such that $\gamma k \in H$, i.e. $kH = \gamma^{-1}H$. By assumption f is $\mathbb{Z}[H]$-linear, so $\psi\omega(f)$ takes γ to

$$\gamma k f(k^{-1}) = f(\gamma k k^{-1}) = f(\gamma)\,,$$

i.e., is equal to f again. $\square$

10.2 Space level constructions

(See Lewis-May-Steinberger, §II.5.)

Let G be a compact Lie group and let $H \subset G$ be a closed subgroup.

Definition 10.6. Let $L = T_{eH}(G/H)$ be the tangent space of G/H at the point eH. The left H-action on G/H induces a left H-action on L, making L an H-representation. We may equip L with an inner product so that H acts through isometries; hence L is a finite-dimensional orthogonal L-representation.

The following G-map t will be used in the construction of the Wirthmüller equivalence ψ.

Definition 10.7. Let $j\colon G/H \to W$ be a G-equivariant smooth embedding of G/H in a G-representation W contained in the G-universe $\mathscr{U}$. The induced map $T_{eH}(G/H) \to T_{j(eH)}(W)$ is an H-linear embedding $L \to W$, with orthogonal complement $W - L$. The normal bundle of j is $G \times_H (W - L) \to G/H$, so we can extend j to an embedding

$$\tilde{\jmath}\colon G \times_H (W - L) \longrightarrow W$$

of a tubular neighborhood. By the Pontryagin-Thom construction, collapsing the complement of that neighborhood to a point, we get a G-map

$$t\colon S^W \longrightarrow G_+ \wedge_H S^{W-L} .$$

(The case G finite?)

Example 10.8. For $G = \mathbb{T}$ and $H = 1$, we can take $j\colon G \to W$ to be the standard embedding $\mathbb{T} \subset \mathbb{C}$, so that $L = T_e(\mathbb{T}) = i\mathbb{R}$ is the imaginary axis, with orthogonal complement $W - T = \mathbb{R}$, the real axis. A tubular neighborhood of $\mathbb{T}$ is $\tilde{\jmath}\colon \mathbb{T} \times \mathbb{R} \cong \mathbb{C} \setminus \{0\}$, and the Pontryagin-Thom construction gives the $\mathbb{T}$-equivariant map

$$t\colon S^{\mathbb{C}} \longrightarrow S^{\mathbb{C}}/S^0 \cong \mathbb{T}_+ \wedge S^{\mathbb{R}}$$

that collapses $S^0 = \{0, \infty\} \subset S^{\mathbb{C}}$ to the base point. With the choice $\tilde{\jmath}(e^{i\theta}, m) = e^{m+i\theta}$, we get $t(re^{i\theta}) = e^{i\theta} \wedge \log(r)$.

The following H-H-map u will be used in the construction of the inverse Wirthmüller equivalence ω.

Definition 10.9. The normal bundle of $eH \in G/H$ is L. Let the H-map

$$k\colon L \longrightarrow G/H$$

be the embedding of a tubular neighborhood. We can lift k to an H-H-map

$$\tilde{k}\colon L \times H \longrightarrow G .$$

The Pontryagin-Thom construction, collapsing the complement of the image of $\tilde{k}$ to the base point, gives an H-H-map

$$u\colon G_+ \longrightarrow S^L \wedge H_+ .$$

Here $h_1 \in H$ and $h_2 \in H$ act from the left and from the right, respectively, on G by $h_1 \cdot g \cdot h_2 = h_1 g h_2$, and on $S^L \wedge H_+$ by $h_1 \cdot (\ell, h) \cdot h_2 = (h_1\ell, h_1 h h_2)$.

(The case G finite?)

Example 10.10. For $G = \mathbb{T}$ and $H = 1$, we can take $k\colon L \to G$ to be the inverse $i\mathbb{R} \to \mathbb{T}$ of the stereographic projection from -1 to the tangent line $1 + i\mathbb{R}$. The image of $\tilde{k}$ is $\mathbb{T} \setminus \{-1\}$, and the Pontryagin-Thom construction gives the map

$$u\colon \mathbb{T}_+ \longrightarrow \mathbb{T}_+/\{-1\}_+ \cong S^{i\mathbb{R}}$$

given by

$$u(e^{i\theta}) = i \cdot 2 \tan \frac{\theta}{2} .$$

Lemma 10.11. *The composite*

$$S^W \xrightarrow{t} G_+ \wedge_H S^{W-L} \xrightarrow{u\wedge 1} (S^L \wedge H_+) \wedge_H S^{W-L} \cong S^L \wedge S^{W-L} \cong S^W$$

is H-equivariantly homotopic to the identity.

Example 10.12. For $G = \mathbb{T}$ and $H = 1$, the composite map $S^{\mathbb{C}} \to S^{\mathbb{C}}$ takes $re^{i\theta}$ to $\log(r) + i \cdot 2\tan(\theta/2)$. This map sends the ray $(-\infty, 0] \subset \mathbb{C}$ to the base point, and is homotopic to the identity.

Definition 10.13. For each H-space Z, let

$$\omega_Z = (1 \wedge \epsilon)\zeta^{-1}(t \wedge 1) \colon S^W \wedge F_H(G_+, Z) \longrightarrow G_+ \wedge_H (S^{W-L} \wedge Z)$$

be the composite

$$\begin{aligned} S^W \wedge F_H(G_+, Z) &\xrightarrow{t\wedge 1} (G_+ \wedge_H S^{W-L}) \wedge F_H(G_+, Z) \\ &\xrightarrow{\zeta^{-1}} G_+ \wedge_H (S^{W-L} \wedge \iota^* F_H(G_+, Z)) \\ &\xrightarrow{1\wedge\epsilon} G_+ \wedge_H (S^{W-L} \wedge Z)\,. \end{aligned}$$

Here $t \colon S^W \to G_+ \wedge_H S^{W-L}$ is the G-map defined earlier, ζ is the untwisting isomorphism, and the H-map $\epsilon \colon \iota^* F_H(G_+, Z) \to Z$ is the adjunction counit.

Example 10.14. For $G = \mathbb{T}$ and $H = 1$, the map

$$\omega_Z \colon S^{\mathbb{C}} \wedge F_1(\mathbb{T}_+, Z) \longrightarrow \mathbb{T}_+ \wedge_1 (S^{\mathbb{R}} \wedge Z)$$

is given by

$$\omega_Z(re^{i\theta} \wedge f) = e^{i\theta} \wedge \log(r) \wedge f(e^{-i\theta})\,.$$

Definition 10.15. For each H-space X, form the (left) H-map

$$u \wedge 1 \colon G_+ \wedge_H X \longrightarrow (S^L \wedge H_+) \wedge_H X \cong S^L \wedge X$$

and let

$$\psi_X \colon G_+ \wedge_H X \longrightarrow F_H(G_+, S^L \wedge X)$$

given by

$$\psi_X(g \wedge x) \colon \gamma \longmapsto (u \wedge 1)(\gamma g \wedge x)$$

be the G-map that is right adjoint to $u \wedge 1$. Hence $\epsilon \circ \psi_X = u \wedge 1$.

Example 10.16. For $G = \mathbb{T}$ and $H = 1$, the map

$$\psi_X \colon \mathbb{T}_+ \wedge_1 X \longrightarrow F_1(\mathbb{T}_+, S^{i\mathbb{R}} \wedge X)$$

is given by

$$\psi_X(e^{i\alpha} \wedge x) \colon e^{i\beta} \longmapsto i \cdot 2\tan\frac{\alpha+\beta}{2} \wedge x\,.$$

Definition 10.17. Let

$$\begin{aligned} \nu \colon Y \wedge F_H(G_+, Z) &\longrightarrow F_H(G_+, \iota^* Y \wedge Z) \\ y \wedge f &\longmapsto (\gamma \mapsto \gamma y \wedge f(\gamma)) \end{aligned}$$

be the G-map right adjoint to the H-map $Y \wedge \epsilon \colon Y \wedge F_H(G_+, Z) \to Y \wedge Z$.

Proposition 10.18. *The following triangle commutes up to G-homotopy.*

$$\begin{tikzcd} S^W \wedge F_H(G_+, Z) \arrow[d, "\omega_Z"'] \arrow[dr, "\nu"] & \\ G_+ \wedge_H (S^{W-L} \wedge Z) \arrow[r, "\psi_{S^{W-L}\wedge Z}"'] & F_H(G_+, \iota^* S^W \wedge Z) \end{tikzcd}$$

Here $\nu(w \wedge f) \colon \gamma \mapsto \gamma w \wedge f(\gamma)$, and we used the identification $S^L \wedge S^{W-L} \cong \iota^* S^W$.

Proof. To prove that ν and $\psi_{S^{W-L}\wedge Z} \circ \omega_Z$ are G-homotopic is equivalent to proving that $\epsilon \circ \nu = 1 \wedge \epsilon$ and $\epsilon \circ \psi_{S^{W-L}\wedge Z} \circ \omega_Z = (u \wedge 1) \circ \omega_Z$ are H-homotopic.

$$\begin{tikzcd}
S^W \wedge F_H(G_+, Z) \arrow[r, "t\wedge 1"] \arrow[d, "\omega_Z"] \arrow[dd, bend right=60, "\nu"'] \arrow[ddd, bend right=80, "1\wedge\epsilon"'] & (G_+ \wedge_H S^{W-L}) \wedge F_H(G_+, Z) \arrow[d, "\cong"', "\zeta^{-1}"] \\
G_+ \wedge_H (S^{W-L} \wedge Z) \arrow[d, "\psi_{S^{W-L}\wedge Z}"] \arrow[dd, bend left=60, "u\wedge 1"] & G_+ \wedge_H (S^{W-L} \wedge F_H(G_+, Z)) \arrow[l, "1\wedge\epsilon"'] \arrow[dd, "u\wedge 1"] \\
F_H(G_+, S^W \wedge Z) \arrow[d, "\epsilon"] & \\
S^W \wedge Z & S^W \wedge F_H(G_+, Z) \arrow[l, "1\wedge\epsilon"']
\end{tikzcd}$$

Here

$$(u \wedge 1) \circ \omega_Z = (1 \wedge \epsilon) \circ (u \wedge 1)\zeta^{-1}(t \wedge 1)\,,$$

so it suffices to prove that $(u \wedge 1)\zeta^{-1}(t \wedge 1)$ is H-homotopic to the identity of $S^W \wedge F_H(G_+, Z)$. This is the content of Lemma 10.19 below. $\square$

Lemma 10.19. *For each G-space Y, the composite*

$$S^W \wedge Y \xrightarrow{t\wedge 1} (G_+ \wedge_H S^{W-L}) \wedge Y \xrightarrow{\zeta^{-1}} G_+ \wedge_H (S^{W-L} \wedge Y) \xrightarrow{u\wedge 1} S^L \wedge S^{W-L} \wedge Y \cong S^W \wedge Y$$

is H-homotopic to the identity.

Example 10.20. For $G = \mathbb{T}$ and $H = 1$, the maps

$$S^{\mathbb{C}} \wedge Y \xrightarrow{t\wedge 1} (\mathbb{T}_+ \wedge_1 S^{\mathbb{R}}) \wedge Y \xrightarrow{\zeta^{-1}} \mathbb{T}_+ \wedge_1 (S^{\mathbb{R}} \wedge Y) \xrightarrow{u\wedge 1} S^{i\mathbb{R}} \wedge S^{\mathbb{R}} \wedge Y \cong S^{\mathbb{C}} \wedge Y$$

take $re^{i\theta} \wedge y$ by $t \wedge 1$ to $e^{i\theta} \wedge \log(r) \wedge y$, by ζ^{-1} to $e^{i\theta} \wedge \log(r) \wedge e^{-i\theta}y$, and by $u \wedge 1$ to

$$(\log(r) + i \cdot 2\tan\frac{\theta}{2}) \wedge e^{-i\theta}y\,.$$

The composite map is homotopic to the identity.

Proposition 10.21. *The following diagram commutes up to G-homotopy.*

$$\begin{tikzcd}
S^W \wedge (G_+ \wedge_H X) \arrow[r, "1\wedge\psi_X"] \arrow[d, "\zeta^{-1}"', "\cong"] & S^W \wedge F_H(G_+, S^L \wedge X) \arrow[d, "\omega_{S^L\wedge X}"] \\
G_+ \wedge_H (\iota^* S^W \wedge X) \arrow[r, "1\wedge\sigma\wedge 1", "\cong"'] & G_+ \wedge_H (\iota^* S^W \wedge X)
\end{tikzcd}$$

Here $\zeta^{-1} \colon w \wedge (\gamma \wedge x) \mapsto \gamma \wedge (\gamma^{-1} w \wedge x)$, we use the identification $S^{W-L} \wedge S^L \cong \iota^* S^W$, and $\sigma \colon S^W \to S^W$ is the identity on S^{W-L} and reverses sign on S^L.

Proof. It suffices to prove that $\omega_{S^L\wedge X} \circ (1 \wedge \psi_X) \circ \zeta$ is G-homotopic to $1 \wedge \sigma \wedge 1$.

$$\begin{tikzcd}
G_+ \wedge_H (S^W \wedge X) \arrow[d, "\cong"', "\zeta"] \arrow[ddd, bend right=70, "1\wedge\sigma\wedge 1"'] & G_+ \wedge_H (S^{W-L} \wedge (G_+ \wedge_H X)) \arrow[ddd, bend left=70, "1\wedge\psi_X"] \\
S^W \wedge (G_+ \wedge_H X) \arrow[r, "t\wedge 1"] \arrow[d, "1\wedge\psi_X"] & (G_+ \wedge_H S^{W-L}) \wedge (G_+ \wedge_H X) \arrow[u, "\cong", "\zeta^{-1}"'] \arrow[d, "1\wedge\psi_X"] \\
S^W \wedge F_H(G_+, S^L \wedge X) \arrow[r, "t\wedge 1"] \arrow[d, "\omega_{S^L\wedge X}"] & (G_+ \wedge_H S^{W-L}) \wedge F_H(G_+, S^L \wedge X) \arrow[d, "\cong"', "\zeta^{-1}"] \\
G_+ \wedge_H (S^W \wedge X) & G_+ \wedge_H (S^{W-L} \wedge F_H(G_+, S^L \wedge X)) \arrow[l, "1\wedge\epsilon"']
\end{tikzcd}$$

Chasing the diagram above, using that $(1 \wedge \epsilon) \circ (1 \wedge \psi_X) = 1 \wedge u \wedge 1$, we see that this reduces to Lemma 10.22 below. $\square$

Lemma 10.22. *For each H-space X, the diagram*

$$\begin{tikzcd}
G_+ \wedge_H (S^W \wedge X) \arrow[r, "\zeta", "\cong"'] \arrow[d, "1\wedge\sigma\wedge 1"] & S^W \wedge (G_+ \wedge_H X) \arrow[r, "t\wedge 1"] & (G_+ \wedge_H S^{W-L}) \wedge (G_+ \wedge_H X) \arrow[d, "\cong"', "\zeta^{-1}"] \\
G_+ \wedge_H S^W & G_+ \wedge_H (S^{W-L} \wedge S^L \wedge X) \arrow[l, "\cong"'] & G_+ \wedge_H (S^{W-L} \wedge (G_+ \wedge_H X)) \arrow[l, "1\wedge u\wedge 1"']
\end{tikzcd}$$

is G-homotopy commutative, where $\sigma \colon S^W \to S^W$ is the identity on S^{W-L} and reverses sign on S^L.

Example 10.23. For $G = \mathbb{T}$ and $H = 1$, the diagram appears as follows.

$$\begin{tikzcd}
\mathbb{T}_+ \wedge_1 (S^{\mathbb{C}} \wedge X) \arrow[r, "\zeta", "\cong"'] \arrow[d, "1\wedge\sigma\wedge 1"] & S^{\mathbb{C}} \wedge (\mathbb{T}_+ \wedge_1 X) \arrow[r, "t\wedge 1"] & (\mathbb{T}_+ \wedge_1 S^{\mathbb{R}}) \wedge (\mathbb{T}_+ \wedge_1 X) \arrow[d, "\cong"', "\zeta^{-1}"] \\
\mathbb{T}_+ \wedge_1 (S^{\mathbb{C}} \wedge X) & \mathbb{T}_+ \wedge_1 (S^{\mathbb{R}} \wedge S^L \wedge X) \arrow[l, "\cong"'] & \mathbb{T}_+ \wedge_1 (S^{\mathbb{R}} \wedge (\mathbb{T}_+ \wedge_1 X)) \arrow[l, "1\wedge u\wedge 1"']
\end{tikzcd}$$

The clockwise route maps $e^{i\alpha} \wedge re^{i\theta} \wedge x$ by ζ to $re^{i(\alpha+\theta)} \wedge e^{i\alpha} \wedge x$, by $t \wedge 1$ to $e^{i(\alpha+\theta)} \wedge \log(r) \wedge e^{i\alpha} \wedge x$, by ζ^{-1} to $e^{i(\alpha+\theta)} \wedge \log(r) \wedge e^{-i\theta} \wedge x$, and by $1 \wedge u \wedge 1$ to

$$e^{i(\alpha+\theta)} \wedge (\log(r) - i \cdot 2\tan\frac{\theta}{2}) \wedge x\,.$$

At the left hand side, $1 \wedge \sigma \wedge 1$ takes $e^{i\alpha} \wedge re^{i\theta} \wedge x$ to

$$e^{i\alpha} \wedge re^{-i\theta} \wedge x\,.$$

These maps are $\mathbb{T}$-homotopic, since their restrictions to $\alpha = 0$ are (non-equivariantly) homotopic.

10.3 Spectrum level maps

Definition 10.24. For each orthogonal H-spectrum Z, let the G-map of orthogonal G-spectra

$$\omega \colon S^W \wedge F_H[G, Z] \longrightarrow G \ltimes_H (S^{W-L} \wedge Z)$$

be given at level $n \ge 0$ by

$$\omega_{Z_n} \colon S^W \wedge F_H(G_+, Z_n) \longrightarrow G_+ \wedge_H (S^{W-L} \wedge Z_n)\,.$$

(Discuss compatibility with structure maps.)

Definition 10.25. For each orthogonal H-spectrum X, let the G-map of orthogonal G-spectra

$$\psi \colon G \ltimes_H X \longrightarrow F_H[G, S^L \wedge X)$$

be given at level $n \ge 0$ by

$$\psi_{X_n} \colon G_+ \wedge_H X_n \longrightarrow F_H(G_+, S^L \wedge X_n)\,.$$

(Discuss compatibility with structure maps.)

Theorem 10.26 (Wirthmüller, Lewis-May-Steinberger)**.** *Let H be a closed subgroup of a compact Lie group G, let $L = T_{eH}(G/H)$ be the H-representation given by the tangent space of G/H at eH, and let X be any orthogonal H-spectrum. The natural G-map*

$$\psi \colon G \ltimes_H X \xrightarrow{\sim} F_H[G, S^L \wedge X)$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.

Proof. (Compare Schwede's Theorem 4.9) We prove that

$$\psi_*\colon \pi_n^K(G \ltimes_H X) \longrightarrow \pi_n^K(F_H[G, S^L \wedge X))$$

is an isomorphism for each $K \subset G$ and $n \in \mathbb{Z}$. First consider the case $n \geq 0$, when

$$\pi_n^K(G \ltimes_H X) = \operatorname*{colim}_U \pi_n^K(\Omega^U(G_+ \wedge_H X(U)))$$

and

$$\pi_n^K(F_H[G, S^L \wedge X)) = \operatorname*{colim}_V \pi_n^K(\Omega^V F_H(G_+, S^L \wedge X(V)))\,.$$

To prove that ψ_* is injective, we consider a class $[f]$ in the source of ψ_*. It is represented at level U, for some G-representation U, by a K-map

$$f\colon S^n \wedge S^U \longrightarrow G_+ \wedge_H X(U)\,.$$

The image $\psi_*[f]$ is represented by the composite K-map

$$S^n \wedge S^U \xrightarrow{f} G_+ \wedge_H X(U) \xrightarrow{\psi_{X(U)}} F_H(G_+, S^L \wedge X(U))\,.$$

This image is zero, so that $[f]$ is in the kernel of ψ_*, precisely if

$$\sigma \circ (\psi_{X(U)} f \wedge 1)\colon S^n \wedge S^V \longrightarrow F_H(G_+, S^L \wedge X(V))$$

is K-equivariantly null-homotopic, for some G-representation V that contains U. Replacing U and f by V and the stabilized K-map

$$\sigma \circ (f \wedge 1)\colon S^n \wedge S^V \longrightarrow G_+ \wedge_H X(V)\,,$$

respectively, we may assume that $U = V$, and that $\psi_{X(U)} f$ is K-equivariantly null-homotopic. By Proposition 10.21 the following diagram is G-homotopy commutative.

$$\begin{tikzcd}
S^W \wedge S^n \wedge S^U \arrow[d, "1\wedge f"'] \arrow[dr, bend left, "1\wedge\psi_{X(U)} f"] & \\
S^W \wedge (G_+ \wedge_H X(U)) \arrow[r, "1\wedge\psi_{X(U)}"] \arrow[d, "\zeta^{-1}"', "\cong"] & S^W \wedge F_H(G_+, S^L \wedge X(U)) \arrow[d, "\omega_{S^L \wedge X(U)}"] \\
G_+ \wedge_H (\iota^* S^W \wedge X(U)) \arrow[r, "1\wedge\sigma\wedge 1", "\cong"'] & G_+ \wedge_H (\iota^* S^W \wedge X(U))
\end{tikzcd}$$

Hence $1 \wedge f\colon S^W \wedge S^n \wedge S^U \to S^W \wedge (G_+ \wedge_H X(U))$ is K-equivariantly null-homotopic. It follows that the stabilization

$$\sigma(f \wedge 1)\colon S^n \wedge S^{U\oplus W} \longrightarrow G_+ \wedge_H X(U \oplus W)\,,$$

of f at level $U \oplus W$ of the colimit system defining $\pi_n^K(G \ltimes_H X)$, is K-equivariantly null-homotopic. Hence $[f] = 0$ and ψ_* is injective.

To prove that ψ_* is surjective, we consider a class $[g]$ in its target. It is represented at level U, for some G-representation U, by a K-map

$$g\colon S^n \wedge S^U \longrightarrow F_H(G_+, S^L \wedge X(U))\,.$$

The composite K-map $f = (1 \wedge \sigma)\omega_{S^L \wedge X(U)}(1 \wedge g)$:

$$\begin{aligned} S^W \wedge S^n \wedge S^U &\xrightarrow{1\wedge g} S^W \wedge F_H(G_+, S^L \wedge X(U)) \\ &\xrightarrow{\omega_{S^L \wedge X(U)}} G_+ \wedge_H (S^W \wedge X(U)) \xrightarrow{1\wedge\sigma} G_+ \wedge_H X(W \oplus U) \end{aligned}$$

corresponds to an element $[f] \in \pi_n^K(G \ltimes_H X)$. The image $\psi_*[f]$ is represented by the composite K-map

$$S^W \wedge S^n \wedge S^U \xrightarrow{f} G_+ \wedge_H X(W \oplus U) \xrightarrow{\psi_{X(W \oplus U)}} F_H(G_+, S^L \wedge X(W \oplus U)) .$$

By the naturality of ψ_X in X, and Proposition 10.18, the following diagram is G-homotopy commutative.

$$\begin{tikzcd}
S^W \wedge S^n \wedge S^U \arrow[d, "1\wedge g"'] \arrow[rr, bend left=10, "f"] & & G_+ \wedge_H X(W \oplus U) \\
S^W \wedge F_H(G_+, S^L \wedge X(U)) \arrow[r, "\omega_{S^L \wedge X(U)}"] \arrow[dr, "\nu"'] & G_+ \wedge_H (S^W \wedge X(U)) \arrow[r, "1\wedge\sigma"] \arrow[d, "\psi_{S^W \wedge X(U)}"] & G_+ \wedge_H X(W \oplus U) \arrow[d, "\psi_{X(W \oplus U)}"] \\
& F_H(G_+, S^L \wedge S^W \wedge X(U)) \arrow[r, "F(1,\sigma)"] & F_H(G_+, S^L \wedge X(W \oplus U))
\end{tikzcd}$$

Up to transpositions, the composite $F(1, \sigma)\nu(1 \wedge g)$ corresponds to the stabilization

$$\sigma(g \wedge 1) \colon S^n \wedge S^U \wedge S^W \longrightarrow F_H(G_+, S^L \wedge X(U)) \wedge S^W \longrightarrow F_H(G_+, S^L \wedge X(U \oplus W))$$

of g at level $U \oplus W$ in the colimit system defining $\pi_n^K(F_H[G, S^L \wedge X))$. Hence $\psi_*[f]$ equals $[g]$ (up to a sign?) and ψ_* is surjective. $\square$

If G is finite, or abelian, then $L = T_{eH}(G/H)$ is a trivial H-representation for each subgroup $H \subset G$. In particular, L extends to a (trivial) G-representation. In these cases we can give the promised generalized converse to Proposition 9.26.

Proposition 10.27. *(Suppose that $L = T_{eH}(G/H)$ is trivial for each $H \subset G$.) If $f \colon X \to Y$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, and A is a G-CW complex, then*

$$1 \wedge f \colon A \wedge X \longrightarrow A \wedge Y$$

is a $\underline{\pi}_*$-isomorphism.

Proof. By induction over the skeleta of A, and a passage to colimits, it suffices to prove this for $A = (G/H)_+ \wedge S^n$. The $\underline{\pi}_*$-isomorphisms

$$(G/H)_+ \wedge S^n \wedge X \cong S^n \wedge (G \ltimes_H X) \xrightarrow{1\wedge\psi} S^n \wedge F_H[G, S^L \wedge X) \cong F(G/H_+, S^n \wedge S^L \wedge X)$$

are natural in X. The $\underline{\pi}_*$-isomorphism $f \colon X \to Y$ induces a $\underline{\pi}_*$-isomorphism $1 \wedge f \colon S^n \wedge S^L \wedge X \to S^n \wedge S^L \wedge Y$. By Proposition 9.26, the map $F(1, 1 \wedge f) \colon F(G/H_+, S^n \wedge S^L \wedge X) \to F(G/H_+, S^n \wedge S^L \wedge Y)$ is a $\underline{\pi}_*$-isomorphism, and by the Wirthmüller equivalences and untwisting isomorphisms displayed above, it follows that $1 \wedge f \colon (G/H)_+ \wedge S^n \wedge X \to (G/H)_+ \wedge S^n \wedge Y$ is a $\underline{\pi}_*$-isomorphism. $\square$

Corollary 10.28. *(Suppose that $L = T_{eH}(G/H)$ is trivial for each $H \subset G$.) A G-map $f \colon X \to Y$ is a $\underline{\pi}_*$-isomorphism if and only if $\Sigma^W f \colon \Sigma^W X \to \Sigma^W Y$ is a $\underline{\pi}_*$-isomorphism.*

Theorem 10.29. *(Suppose that $L = T_{eH}(G/H)$ is trivial for each $H \subset G$.) For each G-representation W, the functors*

$$\Sigma^W : G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow G\,\mathrm{Sp}^{\mathbb{O}} \qquad \textit{and} \qquad \Omega^W : G\,\mathrm{Sp}^{\mathbb{O}} \longrightarrow G\,\mathrm{Sp}^{\mathbb{O}}$$

preserve $\underline{\pi}_*$-isomorphisms. The induced functors

$$\Sigma^W \colon \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}}) \longrightarrow \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}}) \qquad \textit{and} \qquad \Omega^W \colon \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}}) \longrightarrow \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}})$$

are mutually inverse equivalences of categories.

Proof. The first assertions follow from the cases $A = S^W$ and $B = S^W$ of Propositions 10.27 and 9.26, respectively. The unit

$$\eta \colon X \longrightarrow \Omega^W \Sigma^W X$$

and counit

$$\epsilon \colon \Sigma^W \Omega^W Y \longrightarrow Y$$

are $\underline{\pi}_*$-isomorphisms by Propositions 9.13 and 9.15. $\square$

11 Spanier–Whitehead duality

Definition 11.1. For each H-representation V, define

$$S^{-V} = \Omega^V S = F(S^V, S)$$

as an orthogonal H-spectrum.

Proposition 11.2. *There are stable G-equivalences*

$$F(G/H_+, S) \simeq G \ltimes_H S^{-L}$$

and

$$(G/H)_+ \simeq F_H[G, S^L) .$$

Proof. The counit $\epsilon\colon \Sigma^L \Omega^L S \to S$ is a stable H-equivalence, and induces a stable G-equivalence

$$F_H[G, \Sigma^L \Omega^L S) \longrightarrow F_H[G, S) .$$

Using the Wirthmüller equivalence $F_H[G, \Sigma^L \Omega^L S) \simeq G \ltimes_H \Omega^L S = G \ltimes_H S^{-L}$ and the untwisting isomorphism $F_H[G, S) \cong F(G/H_+, S)$ gives the first equivalence.

The Wirthmüller equivalence $G \ltimes_H S \simeq F_H[G, S^L)$ and the untwisting isomorphism $G \ltimes_H S \cong (G/H)_+ \wedge S$ gives the second equivalence. □

Example 11.3. For $G = \mathbb{T}$, $H = 1$ we have a stable $\mathbb{T}$-equivalence $F(\mathbb{T}_+, S) \simeq \Sigma^{-1}\mathbb{T}_+$.

For an orthogonal G-spectrum X we call $DX = F(X, S)$ the functional dual of X. The result above calculates DX for $X = (G/H)_+$. More generally, the functional dual of X has duality properties similar to those of finite-dimensional vector spaces or finitely generated projective modules, when X is a finite G-CW spectrum. These duality properties are called Spanier–Whitehead duality. Following Dold–Puppe and §III.1 and §III.2 of Lewis–May-Steinberger we will first discuss Spanier–Whitehead duality in general closed symmetric monoidal categories, and then specialize to the stable G-equivariant homotopy category.

11.1 Categorical duality theory

(See Lewis–May-Steinberger, §III.1.)

Let $\mathscr{C}$ be a closed symmetric monoidal category, with unit object S, product $\wedge\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and internal hom functor $F\colon \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{C}$, with coherently compatible unit isomorphisms $S \wedge Y \cong Y \cong Y \wedge S$, associativity isomorphisms $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$, commutativity isomorphism

$$\gamma\colon X \wedge Y \xrightarrow{\cong} Y \wedge X$$

and adjunction isomorphism

$$\mathscr{C}(X \wedge Y, Z) \cong \mathscr{C}(X, F(Y, Z)) .$$

As reflected by the notation, the main example we have in mind is the stable G-equivariant homotopy category $\mathscr{C} = \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}})$. Other examples would be the categories of graded modules or chain complexes over a commutative ring, or over a Hopf algebra.

Let $DX = F(X, S)$ be the functional dual of X, and let

$$\eta\colon X \longrightarrow F(Y, X \wedge Y) \qquad \text{and} \qquad \epsilon\colon F(Y, Z) \wedge Y \longrightarrow Z$$

be the adjunction unit and counit, respectively.

The maps

$$\eta\colon X \xrightarrow{\cong} F(S, X \wedge S) \cong F(S, X) \qquad \text{and} \qquad \epsilon\colon F(S, Z) \cong F(S, Z) \wedge S \xrightarrow{\cong} Z$$

are inverse isomorphisms (for $X = Z$). Applying adjunction twice to the "evaluation" map $\epsilon\colon F(X \wedge Y, Z) \wedge X \wedge Y \to Z$ defines an isomorphism

$$F(X \wedge Y, Z) \xrightarrow{\cong} F(X, F(Y, Z)) .$$

A pairing

$$\wedge\colon F(X,Y) \wedge F(X',Y') \longrightarrow F(X \wedge X', Y \wedge Y')$$

is defined as the right adjoint of the composite

$$F(X,Y) \wedge F(X',Y') \wedge X \wedge X' \xrightarrow{1\wedge\gamma\wedge 1} F(X,Y) \wedge X \wedge F(X',Y') \wedge X' \xrightarrow{\epsilon\wedge\epsilon} Y \wedge Y' .$$

As a special case, we have the natural map

$$\nu\colon F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z)$$

which is right adjoint to the composite

$$F(X,Y) \wedge Z \wedge X \xrightarrow{1\wedge\gamma} F(X,Y) \wedge X \wedge Z \xrightarrow{\epsilon\wedge 1} Y \wedge Z .$$

There is also a natural map

$$\rho\colon X \longrightarrow DDX$$

that is right adjoint to the composite

$$X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\epsilon} S .$$

Under suitable hypotheses, the pairing $\wedge$ and the maps ν and ρ become isomorphisms. Dold–Puppe use the term “strongly dualizable”, and Lewis–May–Steinberger write “finite”, for what we will simply call “dualizable” objects. (Hovey–Strickland use the same terminology.)

Definition 11.4. An object Y of $\mathscr{C}$ is dualizable if there is a “coevaluation” map

$$\eta\colon S \longrightarrow Y \wedge DY$$

such that the diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & Y \wedge DY \arrow[d, "\gamma", "\cong"'] \\ F(Y,Y) & DY \wedge Y \arrow[l, "\nu"'] \end{tikzcd}$$

commutes.

Proposition 11.5. *Let Y be a dualizable object, with coevaluation map $\eta\colon S \to Y \wedge DY$.*

(i) The functional dual DY is a dualizable object, with coevaluation map $\eta\colon S \to DY \wedge DDY$ the composite

$$S \xrightarrow{\eta} Y \wedge DY \xrightarrow{\gamma} DY \wedge Y \xrightarrow{1\wedge\rho} DY \wedge DDY .$$

(ii) The composites

$$Y \cong S \wedge Y \xrightarrow{\eta\wedge 1} Y \wedge DY \wedge Y \xrightarrow{1\wedge\epsilon} Y \wedge S \cong Y$$

and

$$DY \cong DY \wedge S \xrightarrow{1\wedge\eta} DY \wedge Y \wedge DY \xrightarrow{\epsilon\wedge 1} S \wedge DY \cong DY$$

are the identity maps.

(iii) There is a natural bijection

$$\eta_\#\colon \mathscr{C}(X \wedge Y, Z) \xrightarrow{\cong} \mathscr{C}(X, Z \wedge DY)$$

with inverse

$$\epsilon_\#\colon \mathscr{C}(X, Z \wedge DY) \xrightarrow{\cong} \mathscr{C}(X \wedge Y, Z) ,$$

where $\eta_\#(f)$ is the composite

$$X \cong X \wedge S \xrightarrow{1\wedge\eta} X \wedge Y \wedge DY \xrightarrow{f\wedge 1} Z \wedge DY$$

and $\epsilon_\#(g)$ is the composite

$$X \wedge Y \xrightarrow{g\wedge 1} Z \wedge DY \wedge Y \xrightarrow{1\wedge\epsilon} Z \wedge S \cong Z .$$

(iv) The functor $-\wedge DY$ is right adjoint to the functor $-\wedge Y$, and is therefore isomorphic to $F(Y,-)$.

Proposition 11.6. *(i) If Y is dualizable then*

$$\rho \colon Y \longrightarrow DDY$$

is an isomorphism.

(ii) If X or Z is dualizable then

$$\nu \colon F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z)$$

is an isomorphism.

(iii) If X and X' are dualizable, or if X is dualizable and $Y = S$, then

$$\wedge \colon F(X,Y) \wedge F(X',Y') \longrightarrow F(X \wedge X', Y \wedge Y')$$

is an isomorphism.

Proof. (i) The composite

$$S \wedge DDY \xrightarrow{\eta\wedge 1} Y \wedge DY \wedge DDY \xrightarrow{1\wedge\gamma} Y \wedge DDY \wedge DY \xrightarrow{1\wedge\epsilon} Y \wedge S$$

gives the inverse of ρ.

(ii) When X is dualizable, the composite

$$F(X, Y\wedge Z) \wedge S \xrightarrow{1\wedge\eta} F(X, Y \wedge Z) \wedge X \wedge DX \xrightarrow{\epsilon\wedge 1} Y \wedge Z \wedge DX \xrightarrow{\gamma} DX \wedge Y \wedge Z \xrightarrow{\nu\wedge 1} F(X,Y) \wedge Z$$

gives the inverse of ν.

(iii) (ETC).

(ii) When Z is dualizable (ETC). □

Corollary 11.7. *An object Y is dualizable if and only if*

$$\nu \colon DY \wedge Y \longrightarrow F(Y,Y)$$

is an isomorphism. In this case the coevaluation map $\eta \colon S \to Y \wedge DY$ is the composite

$$S \xrightarrow{\eta} F(Y,Y) \xrightarrow{\nu^{-1}} DY \wedge Y \xrightarrow{\gamma} Y \wedge DY\,.$$

Definition 11.8. The composite

$$\chi(Y) \colon S \xrightarrow{\eta} Y \wedge DY \xrightarrow{\gamma} DY \wedge Y \xrightarrow{\epsilon} S$$

is the Euler characteristic of Y, viewed as an element of $\mathscr{C}(S,S)$.

11.2 Duality for G-spectra

(See Lewis–May-Steinberger, §III.2.)

We now consider the case $\mathscr{C} = \mathrm{Ho}(G\,\mathrm{Sp}^{\mathbb{O}})$, the stable G-equivariant homotopy category, where the stable G-equivalences ($= \underline{\pi}_*$-isomorphisms) have been inverted.

Lemma 11.9. *For a map $f \colon X \to Y$ of orthogonal G-spectra, and any orthogonal G-spectrum Z, the sequence*

$$F(\Sigma X, Z) \xrightarrow{\pi^*} F(Cf, Z) \xrightarrow{i^*} F(Y,Z) \xrightarrow{f^*} F(X,Z)$$

is stably G-equivalent to the sequence

$$\Omega F(X,Z) \xrightarrow{\iota} F(f^*) \xrightarrow{p} F(Y,Z) \xrightarrow{f^*} F(X,Z)\,.$$

Proposition 11.10. *The functional dual of a finite G-CW spectrum is stably G-equivalent to a finite G-CW spectrum.*

Proof. By Proposition 11.2, the dual of G/H_+ is stably G-equivalent to $G \ltimes_H S^{-L}$. ((Why is this a finite G-CW spectrum?)) General finite G-CW spectra are built from suspensions of G/H_+ using cofiber sequences, so their functional duals are built from desuspensions of $D(G/H_+)$ using fiber sequences. By the lemma above, these are again finite G-CW complexes. ((Elaborate?)) $\square$

Theorem 11.11. *Any finite G-CW spectrum is dualizable.*

Proof. We first consider the case of a single G-cell G/H_+. For any orthogonal G-spectrum Z the following diagram commutes.

$$\begin{tikzcd}
D(G/H_+) \wedge Z \arrow[r, "\cong"] \arrow[d, "\nu"'] & F_H[G,S] \wedge Z \arrow[r, "\omega\wedge 1"] & (G \ltimes_H S^{-L}) \wedge Z \arrow[d, "\cong"', "\zeta^{-1}"] \\
F(G/H_+, Z) \arrow[r, "\cong"] & F_H[G, S \wedge Z] \arrow[r, "\omega"] & G \ltimes_H (S^{-L} \wedge Z)
\end{tikzcd}$$

(Check?) The Wirthmüller equivalences ω are stable G-equivalences, hence ν is a stable G-equivalence.

For any map $f\colon X \to Y$ the following diagram commutes

$$\begin{tikzcd}
D(\Sigma X) \wedge Z \arrow[r] \arrow[d, "\nu"'] & D(Cf) \wedge Z \arrow[r] \arrow[d, "\nu"'] & DY \wedge Z \arrow[r] \arrow[d, "\nu"'] & DX \wedge Z \arrow[d, "\nu"'] \\
F(\Sigma X, Z) \arrow[r] & F(Cf, Z) \arrow[r] & F(Y, Z) \arrow[r] & F(X, Z)
\end{tikzcd}$$

If X is dualizable, and Y is built from X by attaching a single G-cell, so that $Cf \simeq G/H_+ \wedge S^n$ for some integer n, then ν is a stable G-equivalence for X and for Cf, so by the five-lemma it is a stable G-equivalence for Y.

By induction over the number of G-cells, it follows that $\nu\colon DY \wedge Z \to F(Y, Z)$ is a stable G-equivalence for each finite G-CW spectrum Y. The special case $Y = Z$ then tells us that Y is dualizable, by Corollary 11.7. $\square$

Proposition 11.12. *(i) If Y is a finite G-CW spectrum then*

$$\rho\colon Y \longrightarrow DDY$$

is a stable G-equivalence.

(ii) If X or Z is a finite G-CW spectrum then

$$\nu\colon F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z)$$

is a stable G-equivalence.

(iii) If X and X' are finite G-CW spectra, or if X is a finite G-CW spectrum and $Y = S$, then

$$\wedge\colon F(X,Y) \wedge F(X',Y') \longrightarrow F(X \wedge X', Y \wedge Y')$$

is a stable G-equivalence.

Example 11.13. If A is a finite G-CW complex, and B is any G-CW complex, then

$$\nu\colon A \wedge F(B,X) \longrightarrow F(B, A \wedge X)$$

is a stable G-equivalence ($= \underline{\pi}_*$-isomorphism).

Corollary 11.14 (Spanier–Whitehead duality)**.** *For any finite G-CW spectrum X and G-spectrum E, the canonical map*

$$E^H_k(DX) = \pi^H_k(DX \wedge E) \xrightarrow{\nu_*} \pi^H_k F(X, E) = E^{-k}_H(X)$$

is an isomorphism, for each $H \subset G$ and $k \in \mathbb{Z}$.

Remark 11.15. Conversely, each dualizable G-spectrum is a retract (in the stable G-equivariant homotopy category) of a finite G-CW spectrum. For a proof, see May et al (Alaska notes), §XVI, Theorem 7.4.

Example 11.16. The G-equivariant Euler characteristic of a finite G-CW spectrum Y is $\chi(Y) \in \mathscr{C}(S,S) = \pi^G_0(S)$, where $\pi^G_0(S) \cong A(G)$ is the Burnside ring of G. ((Give calculation of $A(G)$ and explain how G-cells of the type $G/H_+ \wedge (D^n, \partial D^n)$ are counted?))