Spectral Sequences

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Contents

Introduction

These are notes for the course MAT9580, Algebraic Topology III, in the spring term of 2021. The emphasis will be on the theory and applications of spectral sequences. Some of the key articles are:

- [**Ser51**] Serre: “Homologie singulière des espaces fibrés. Applications”
- [**Ada58**] Adams: “On the structure and applications of the Steenrod algebra”
- [**Boa99**] Boardman: “Conditionally convergent spectral sequences”

Book-length sources for this material include the following:

- [**CE56**] Cartan and Eilenberg: “Homological algebra”
- [**HW60**] Hilton and Wylie: “Homology theory: An introduction to algebraic topology”
- [**Bou63**] Bourgin: “Modern algebraic topology”
- [**ML63**] Mac Lane: “Homology”
- [**Spa66**] Spanier: “Algebraic topology”
- [**GZ67**] Gabriel and Zisman: “Calculus of fractions and homotopy theory”
- [**May67**] May: “Simplicial objects in algebraic topology”
- [**MT68**] Mosher and Tangora: “Cohomology operations and applications in homotopy theory”
- [**Ada72**] Adams: “Algebraic topology—a student’s guide”
- [**MS74**] Milnor and Stasheff: “Characteristic classes”
- [**Swi75**] Switzer: “Algebraic topology—homotopy and homology”
- [**Whi78**] Whitehead: “Elements of homotopy theory”
- [**BT82**] Bott and Tu: “Differential forms in algebraic topology”
- [**McC85**] McCleary: “User’s guide to spectral sequences”
- [**Rav86**] Ravenel: “Complex cobordism and stable homotopy groups of spheres”
- [**DP97**] Dodson and Parker: “A user’s guide to algebraic topology”
- [**Boa99**] Sato: “Algebraic topology: an intuitive approach”
- [**DK01**] Davis and Kirk: “Lecture notes in algebraic topology”
- [**Hat**] Hatcher: “Spectral sequences”
- [**Bru**] Bruner: “An Adams spectral sequence primer”
- [**MP12**] May and Ponto: “More concise algebraic topology”
- [**Sha14**] Shastri: “Basic algebraic topology”

The author is most familiar with the books [**CE56**], [**ML63**], [**Spa66**], [**MT68**], [**MS74**], [**Swi75**], [**Whi78**], [**McC85**] and [**Rav86**]. Most of these sources cover the Serre spectral sequence, while the Adams spectral sequence is discussed in [**Swi75**], [**McC85**], [**Rav86**] and [**Bru**]. None of these make use of the modern categories of spectra, so one aim of these notes is use orthogonal spectra as models for stable homotopy theory, and to benefit from these when treating the behavior of products and other operations in the Adams spectral sequence.

CHAPTER 1

Spectral Sequences

We start with the abstract definition of a spectral sequence. It involves the same concepts as the definition of a chain complex and its homology, but involves multiple indices. In the next section we discuss in what sense a spectral sequence can calculate, or converge to, a given abutment. Thereafter we consider some relatively simple examples, which may help the reader get accustomed to the different roles of the various indices, and the meaning of convergence.

1.1. Homological spectral sequences

DEFINITION 1.1.1. A *bigraded abelian group* $A = A_{*,*}$ is a doubly-indexed sequence

$$A_{*,*} = (A_{s,t})_{s,t}$$

of abelian groups, where s and t range over the integers. We say that $A_{s,t}$ is the group in bidegree (s,t). A *morphism* $f\colon A \to B$ of bigraded abelian groups is a sequence of group homomorphisms

$$f_{s,t}\colon A_{s,t} \longrightarrow B_{s,t}$$

for all $s,t \in \mathbb{Z}$. More generally, a morphism $f\colon A \to B$ of *bidegree* (u,v) is a sequence of group homomorphisms

$$f_{s,t}\colon A_{s,t} \longrightarrow B_{s+u,t+v}$$

for all $s,t \in \mathbb{Z}$. The composite of f followed by a morphism $g\colon B \to C$ of bidegree (u',v') is a morphism $gf\colon A \to C$ of bidegree $(u+u', v+v')$. To emphasize that a morphism has bidegree $(0,0)$ we may say that it is *degree-preserving*.

DEFINITION 1.1.2. Let $E = E_{*,*}$ be a bigraded abelian group, and let r be an integer. A *differential* $d\colon E \to E$ of bidegree (u,v) is a morphism of bidegree (u,v) such that $dd = 0$. More explicitly, for each pair $s,t \in \mathbb{Z}$ we have a homomorphism

$$d_{s,t}\colon E_{s,t} \longrightarrow E_{s+u,t+v}$$

and the composite

$$E_{s-u,t-v} \xrightarrow{d_{s-u,t-v}} E_{s,t} \xrightarrow{d_{s,t}} E_{s+u,t+v}$$

is the zero homomorphism. Let the kernel $\ker(d) = \ker(d)_{*,*}$ be the bigraded abelian group

$$\ker(d)_{s,t} = \ker(d_{s,t})$$

and let the image $\mathrm{im}(d) = \mathrm{im}(d)_{s,t}$ be the bigraded abelian group

$$\mathrm{im}(d)_{s,t} = \mathrm{im}(d_{s-u,t-v})\,.$$

Then

$$\mathrm{im}(d)_{s,t} \subset \ker(d)_{s,t} \subset E_{s,t}$$

for all $s, t \in \mathbb{Z}$. We call $\ker(d)$ and $\operatorname{im}(d)$ the *d-cycles* and *d-boundaries* in E, respectively. The *homology* of (E, d) is the bigraded abelian group

$$H(E, d) = \frac{\ker(d)}{\operatorname{im}(d)}$$

given in bidegree (s, t) by the subquotient

$$H_{s,t}(E, d) = \frac{\ker(d)_{s,t}}{\operatorname{im}(d)_{s,t}} = \frac{\ker(d_{s,t})}{\operatorname{im}(d_{s-u,t-v})}$$

of $E_{s,t}$. We write $[x] \in H(E, d)$ for the homology class of a d-cycle $x \in \ker(d)$.

Definition 1.1.3. A *homological spectral sequence* $(E^r, d^r)_{r\geq 1}$ is a sequence of bigraded abelian groups $E^r = E^r_{*,*}$ and differentials

$$d^r : E^r \longrightarrow E^r$$

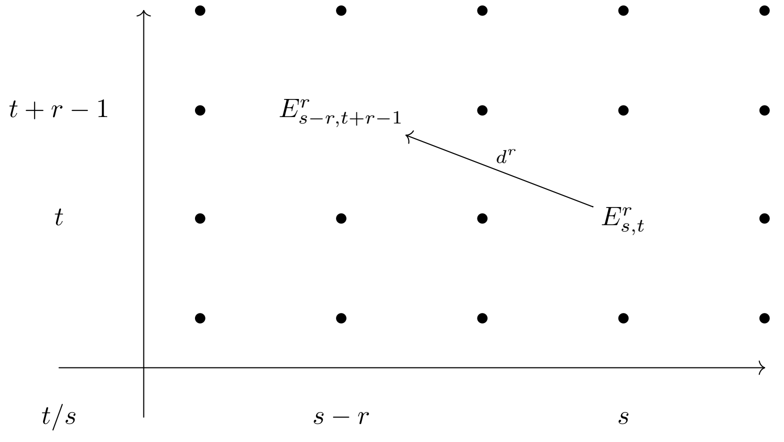
of bidegree $(-r, r-1)$, together with isomorphisms

$$H(E^r, d^r) \cong E^{r+1}$$

for all integers $r \geq 1$.

Remark 1.1.4. We call E^r and d^r the *E^r-term* and *d^r-differential* of the spectral sequence, respectively. In each bidegree (s, t) we refer to s as the *filtration degree*, t as the *complementary degree*, and $s + t$ as the *total degree*. Each d^r-differential sends classes in total degree $s + t$ to classes in total degree $(s - r) + (t + r - 1) = s + t - 1$, hence reduces the total degree by 1. We do not introduce notation for the isomorphisms $H(E^r, d^r) \cong E^{r+1}$, but they are part of the structure of the spectral sequence. More generally, an E^p-spectral sequence $(E^r, d^r)_{r\geq p}$ is a sequence of bigraded abelian groups and differentials, as above, but indexed on the integers $r \geq p$. Usually $p = 1$ or $p = 2$.

Remark 1.1.5. We usually visualize a bigraded group $A_{*,*}$ as being spread out over the (s, t)-plane, with $A_{s,t}$ located in the position with horizontal coordinate s and vertical coordinate t. We visualize the component $d_{s,t} \colon E_{s,t} \to E_{s+u,t+v}$ of a differential d of bidegree (u, v) as an arrow from position (s, t) to position $(s+u, t+v)$. When d is a homological d^r-differential, this arrow points to the left and up, from position (s, t) to position $(s - r, t + r - 1)$.



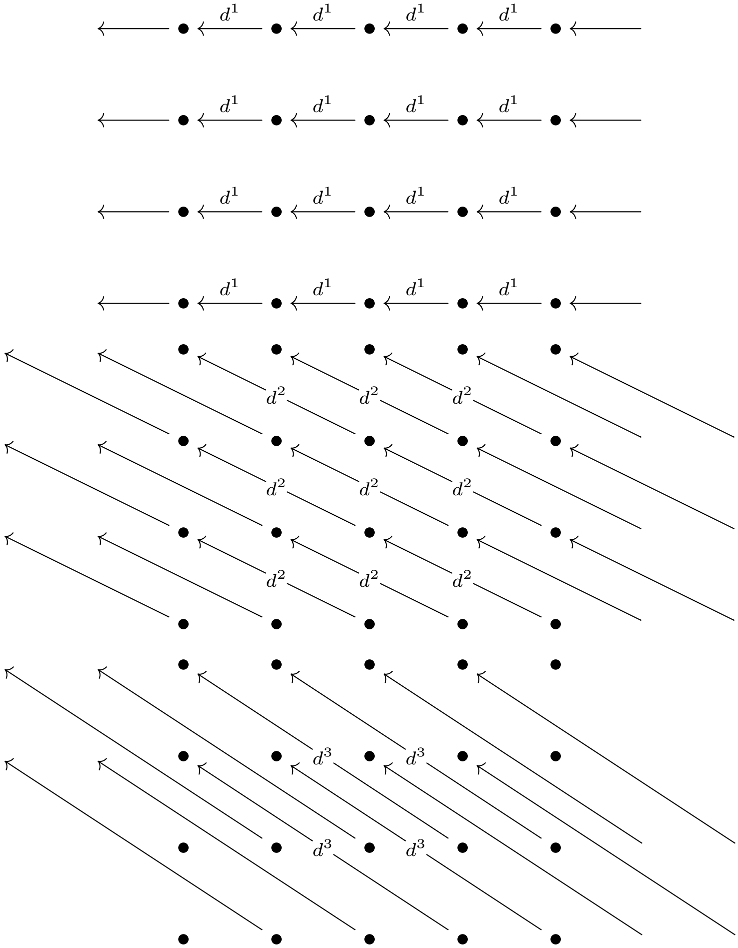
If $d^r_{s,t}(x) = y$ we say that x *supports* a d^r-differential, and that y is *hit* (or "killed") by a d^r-differential. The classes that support a nonzero d^r-differential are not present at the E^{r+1}-term, and the classes that are hit by a d^r-differential are set equal to zero at the E^{r+1}-term. Informally, the classes that support differentials, or are hit by differentials, do not "survive" to the next term.

REMARK 1.1.6. Some authors refer to the E^r-term as the E^r-*page*. If we think of the index r as measuring procession through a number of stages, then the transition from E^r to its subquotient E^{r+1}, by passage to homology with respect to d^r, can be viewed as turning one page over to reveal the next.

(E^1, d^1):

(E^2, d^2):

(E^3, d^3):



REMARK 1.1.7. The most common spectral sequences are bigraded, as in the definition above. Often one grading comes from a filtration and the other comes from a degree shift present in a long exact sequence. However, there are also cases where the complementary degree t is not present, or appears with the opposite sign, or is itself a multigrading. The key feature of a homological spectral sequence is that the d^r-differential reduces the filtration degree from s to $s - r$.

DEFINITION 1.1.8. Let (E, d) and $('E, 'd)$ be bigraded abelian groups with differentials of bidegree (u, v). A morphism $\phi\colon (E, d) \to ('E, 'd)$ is a morphism

$\phi\colon E \to {}'E$ that commutes with the differentials, in the sense that the diagram

$$\begin{tikzcd} E_{s,t} \arrow[r, "\phi_{s,t}"] \arrow[d, "d_{s,t}"'] & {}'E_{s,t} \arrow[d, "{}'d_{s,t}"] \\ E_{s+u,t+v} \arrow[r, "\phi_{s+u,t+v}"] & {}'E_{s+u,t+v} \end{tikzcd}$$

commutes for all bidegrees (s,t). There is then an induced morphism

$$\phi_*\colon H(E,d) \longrightarrow H({}'E,{}'d)$$

given by $\phi_*[x] = [\phi(x)]$ for each d-cycle x in E.

DEFINITION 1.1.9. Let $E = (E^r, d^r)_{r\geq 1}$ and ${}'E = ({}'E^r, {}'d^r)_{r\geq 1}$ be spectral sequences. A *morphism* $\phi\colon E \to {}'E$ of spectral sequences is a sequence of morphisms

$$\phi^r\colon (E^r, d^r) \longrightarrow ({}'E^r, {}'d^r)$$

of differential bigraded abelian groups, such that the diagram

$$\begin{tikzcd} H(E^r, d^r) \arrow[r, "\phi^r_*"] \arrow[d, "\cong"'] & H({}'E^r, {}'d^r) \arrow[d, "\cong"] \\ E^{r+1} \arrow[r, "\phi^{r+1}"] & {}'E^{r+1} \end{tikzcd}$$

commutes for each $r \geq 1$.

REMARK 1.1.10. Sheaves, sheaf cohomology and spectral sequences were invented by Jean Leray while a prisoner of war around 1943, with the first published references being the notes [**Ler46a**] and [**Ler46b**]. For a map $f\colon X \to Y$ of spaces, Leray constructed (what we now call) a sheaf of graded abelian groups over Y, and obtained (what we now call) a spectral sequence with initial term given by the cohomology of Y with coefficients in this sheaf, converging to the cohomology of X. The current algebraic formalism, where the E^{r+1}-term is expressed as the homology of a d^r-differential acting on the E^r-term, is due to Jean-Louis Koszul [**Kos47**]. Similar structures were implicitly present in the 1946 PhD thesis of Roger C. Lyndon [**Lyn48**]. The name "suite spectrale" is due to Jean–Pierre Serre [**Ser51**], merging the names "anneau spectral" of [**Ler50**] and "suite de Leray–Koszul". See the articles by John McCleary [**McC99**] and Haynes Miller [**Mil00**] for more on the history of spectral sequences.

1.2. Bounded convergence

To each spectral sequence (E^r, d^r) we will associate a limiting bigraded abelian group $E^\infty = E^\infty_{*,*}$, called the E^∞-term. The general definition requires some details that we will discuss later in Definition 2.3.3, so for now we will instead describe some special cases for which the E^∞-term can be read off from the E^r-terms for finite r.

DEFINITION 1.2.1. A spectral sequence (E^r, d^r) *collapses at the E^q-term* if $d^r = 0$ for all $r \geq q$. It *stabilizes in each bidegree* if for each bidegree (s,t) there is a $q(s,t)$ such that both $d^r_{s,t}\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ and $d^r_{s+r,t-r+1}\colon E^r_{s+r,t-r+1} \to E^r_{s,t}$ are zero for all $r \geq q(s,t)$.

Setting $q(s,t) = q$, we see that a spectral sequence that collapses at the E^q-term also stabilizes in each bidegree. The latter condition is strictly weaker.

LEMMA 1.2.2. *If (E^r, d^r) collapses at the E^q-term, then $E^r \cong H(E^r, d^r) \cong E^{r+1}$ for all $r \ge q$, so that there are isomorphisms*

$$E^q \cong E^{q+1} \cong \cdots \cdot E^r \cong \dots$$

for all $r \ge q$. More generally, if (E^r, d^r) stabilizes in each bidegree, then for each bidegree (s,t) there are isomorphisms

$$E^q_{s,t} \cong E^{q+1}_{s,t} \cong \cdots \cdot E^r_{s,t} \cong \dots$$

for all $r \ge q = q(s,t)$.

PROOF. If $d^r = 0$ then $\ker(d^r) = E^r$ and $\operatorname{im}(d^r) = 0$, so $H(E^r, d^r) = E^r/0 \cong E^r$. By the assumption that (E^r, d^r) is a spectral sequence, this is isomorphic to E^{r+1}.

In the general case, for each (s,t) and $r \ge q(s,t)$ we have $\ker(d^r)_{s,t} = E^r_{s,t}$ and $\operatorname{im}(d^r)_{s,t} = 0$, so $H(E^r, d^r)_{s,t} = E^r_{s,t}/0 \cong E^r_{s,t}$, and this is isomorphic to $E^{r+1}_{s,t}$. □

In other words, if all the d^r-differentials for $r \ge q$ mapping into or out of a given bidegree (s,t) are trivial, then the groups $E^r_{s,t}$ remain the same for all $r \ge q$. Here $q = q(s,t)$ may vary with (s,t).

LEMMA 1.2.3. *If (E^r, d^r) collapses at the E^q-term, then $E^\infty \cong E^q$ is isomorphic to the common value of E^r for $r \ge q$. More generally, if (E^r, d^r) stabilizes in each bidegree, then for each bidegree (s,t) there are isomorphisms $E^\infty_{s,t} \cong E^r_{s,t}$ for all sufficiently large r.*

We will also see that a morphism $\phi\colon E \to {}'E$ of spectral sequences induces a limiting homomorphism $\phi^\infty\colon E^\infty \to {}'E^\infty$, with components $\phi^\infty_{s,t}\colon E^\infty_{s,t} \to {}'E^\infty_{s,t}$.

LEMMA 1.2.4. *If (E^r, d^r) and $({}'E^r, {}'d^r)$ both collapse at the E^q-term, then $\phi^\infty\colon E^\infty \to {}'E^\infty$ corresponds to $\phi^r\colon E^r \to {}'E^r$ for each $r \ge q$. More generally, if (E^r, d^r) and $({}'E^r, {}'d^r)$ stabilize in each bidegree, then $\phi^\infty_{s,t}\colon E^\infty_{s,t} \to {}'E^\infty_{s,t}$ corresponds, for each bidegree (s,t), to $\phi^r_{s,t}\colon E^r_{s,t} \to {}'E^r_{s,t}$ for all sufficiently large r.*

We postpone the proofs of these two lemmas until we give the general definition of the E^∞-term. In the meantime, their conclusions can be taken to characterize E^∞ and ϕ^∞, in the cases at hand.

Henri Cartan [**Car48**] clarified the distinction between a filtered group and its associated graded group, as defined below.

DEFINITION 1.2.5. An *increasing filtration* $(F_sG)_s$ of an abelian group G is a sequence of subgroups

$$\cdots \subset F_{s-1}G \subset F_sG \subset \cdots \subset G$$

where $s \in \mathbb{Z}$. For each *filtration degree* s there is a short exact sequence

$$0 \to F_{s-1}G \longrightarrow F_sG \longrightarrow \frac{F_sG}{F_{s-1}G} \to 0 \tag{1.1}$$

that expresses the abelian group F_sG as an extension of the *filtration quotient* $F_sG/F_{s-1}G$ by the preceding group $F_{s-1}G$. ((ETC: Do we also refer to $F_sG/F_{s-r}G$ as a filtration (sub-)quotient?)) The graded abelian group

$$(F_sG/F_{s-1}G)_s$$

is often called the *associated graded* of the filtration $(F_sG)_s$. ((ETC: Maybe introduce notation for the associated graded.))

We say that the filtration is *bounded* if there are integers a and b such that $F_{a-1}G = 0$ and $F_bG = G$, in which case the sequence is determined by the finitely many terms

$$0 = F_{a-1}G \subset F_aG \subset \cdots \subset F_{b-1}G \subset F_bG = G\,,$$

extended by identities on both sides.

REMARK 1.2.6. If we have inductively determined $F_{s-1}G$, and know the filtration quotient $F_sG/F_{s-1}G$, then the next term F_sG is partially determined by the short exact sequence (1.1). There can be several non-isomorphic abelian group extensions with the same subgroup and quotient group, and the task of determining which of these is realized by F_sG is known as the *extension problem* in filtration s. If the filtration is bounded, then this inductive argument involves finitely many extension problems, starting with $s = a$ and ending with $s = b$. We will return in Theorem 2.4.5 and later ((ETC: where?)) to extension problems for discrete (= bounded below) and unbounded filtrations, respectively.

When studying bigraded spectral sequences we must consider filtrations of graded abelian groups.

DEFINITION 1.2.7. An increasing filtration of a graded abelian group $G_* = (G_n)_n$, where $n \in \mathbb{Z}$, is a sequence of graded subgroups

$$\cdots \subset F_{s-1}G_* \subset F_sG_* \subset \cdots \subset G_*$$

where $s \in \mathbb{Z}$. We call s the filtration degree and n the *total degree*. For each s there is a short exact sequence

$$0 \to F_{s-1}G_* \longrightarrow F_sG_* \longrightarrow \frac{F_sG_*}{F_{s-1}G_*} \to 0$$

that expresses the graded abelian group F_sG as an extension of the filtration quotient $F_sG_*/F_{s-1}G_*$ by the preceding graded group $F_{s-1}G_*$. This consists of an extension

$$0 \to F_{s-1}G_n \longrightarrow F_sG_n \longrightarrow \frac{F_sG_n}{F_{s-1}G_n} \to 0$$

in each total degree n. In this case, the associated graded of the filtration is bigraded, either by (s, n) or by $(s, t) = (s, n - s)$.

The filtration of G_* is bounded if there are integers a and b such that $F_{a-1}G_* = 0$ and $F_bG_* = G_*$. More generally, it is *degreewise bounded* if for each total degree n there are integers $a = a(n)$ and $b = b(n)$ such that $F_{a-1}G_n = 0$ and $F_bG_n = G_n$. In these cases the filtration in total degree n is determined by finitely many terms, extended by identities in both directions.

We can now express in what sense a spectral sequence may calculate a given graded abelian group.

DEFINITION 1.2.8. Let $(E^r_{*,*}, d^r)$ be a spectral sequence and let $(F_sG_*)_s$ be a filtration of a graded abelian group G_*. Suppose that the spectral sequence stabilizes in each bidegree, and that the filtration is degreewise bounded. Then we say that the spectral sequence *converges* to G_*, written

$$E^r_{*,*} \Longrightarrow G_*\,,$$

if there are isomorphisms

$$E^\infty_{s,t} \cong \frac{F_sG_{s+t}}{F_{s-1}G_{s+t}}$$

in all bidegrees (s,t). The choice of filtration of G_*, and of the isomorphisms displayed above, are implicitly part of the convergence assertion. We call G_* the *abutment* of the spectral sequence. To emphasize the filtration degree s, and the relation between the complementary degree and the total degree, we may write

$$E^r_{s,t} \Longrightarrow_s G_{s+t}\,.$$

((ETC: Does the term "abutment" assume convergence?))

When the filtration is degreewise bounded, we may also say that the spectral sequence converges strongly to G_*. ((ETC: Later we will consider filtrations that are not degreewise bounded, and define weak convergence, convergence and strong convergence, respectively. Terminology from Cartan–Eilenberg.))

REMARK 1.2.9. When $E^r_{*,*} \Longrightarrow G_*$, the strategy for using the spectral sequence $(E^r_{*,*}, d^r)_{r\ge p}$ to calculate G_* is the following: We assume that the initial term $E^p_{*,*}$ can somehow be calculated. Furthermore, for each $r \ge p$ we assume that the differentials d^r can be calculated, so that we can inductively obtain $E^{r+1}_{*,*}$ as $H(E^r, d^r)_{*,*}$, for each $r \ge p$. Under the hypothesis that the spectral sequence stabilizes in each bidegree, we can let $E^\infty_{s,t} = E^r_{s,t}$ for $r \ge q(s,t)$ sufficiently large. By convergence, these are also the groups $F_sG_n/F_{s-1}G_n$ for $n = s+t$. Consider one total degree n. Assuming that the filtration is degreewise bounded, we know that $F_sG_n = 0$ for $s < a(n)$ sufficiently small. For each $s \ge a(n)$ we must inductively solve an extension problem to determine F_sG_n from $F_{s-1}G_n$ and $E^\infty_{s,n-s}$. Once $s = b(n)$ is sufficiently large, this recovers $F_sG_n = G_n$, which is the total degree n component of the abutment of the spectral sequence.

DEFINITION 1.2.10. Let G and $'G$ be abelian groups, filtered by $(F_sG)_s$ and $(F_s{}'G)_s$, respectively. A homomorphism $\psi\colon G \to {}'G$ is *filtration-preserving* if $\psi(F_sG) \subset F_s{}'G$ for each s. Let $\psi_s\colon F_sG \to F_s{}'G$ be the restriction of ψ, and let $\bar\psi_s\colon F_sG/F_{s-1}G \to F_s{}'G/F_{s-1}{}'G$ be the induced homomorphism between the filtration quotients. ((ETC: We may also write $\psi_{s,1}$ for $\bar\psi_s$.)) We obtain a vertical map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_{s-1}G \arrow[r, tail] \arrow[d, "\psi_{s-1}"'] & F_sG \arrow[r, two heads] \arrow[d, "\psi_s"'] & \dfrac{F_sG}{F_{s-1}G} \arrow[r] \arrow[d, "\bar\psi_s"'] & 0 \\
0 \arrow[r] & F_{s-1}{}'G \arrow[r, tail] & F_s{}'G \arrow[r, two heads] & \dfrac{F_s{}'G}{F_{s-1}G} \arrow[r] & 0
\end{tikzcd} \tag{1.2}$$

for each s. If G_* and $'G_*$ are filtered graded abelian groups, and $\psi\colon G_* \to {}'G_*$ is a degree-preserving morphism, then the same definitions apply.

DEFINITION 1.2.11. Let $(E^r_{*,*}, d^r)$ and $('E^r_{*,*}, {}'d^r)$ be spectral sequences converging to G_* and $'G_*$, respectively. Let $\phi\colon E \to {}'E$ be a morphism of bigraded spectral sequences, and let $\psi\colon G_* \to {}'G_*$ be a morphism of filtered graded abelian groups. Then we say that the spectral sequence morphism ϕ *converges* to the filtration-preserving morphism ψ if the diagram

$$\begin{tikzcd}
E^\infty_{s,*} \arrow[r, "\cong"] \arrow[d, "\phi^\infty_{s,*}"'] & \dfrac{F_sG_*}{F_{s-1}G_*} \arrow[d, "\bar\psi_{s,*}"'] \\
{'E^\infty_{s,*}} \arrow[r, "\cong"] & \dfrac{F_s{'G_*}}{F_{s-1}{'G_*}}
\end{tikzcd} \tag{1.3}$$

commutes for each s.

REMARK 1.2.12. If we have resolved the extension problems for spectral sequences (E^r, d^r) and $('E^r, 'd^r)$ converging to $G = G_*$ and $'G = 'G_*$, respectively, and there is a morphism $\phi\colon E \to 'E$ converging to $\psi\colon G \to 'G$, then we can inductively attempt to determine ψ from ϕ^∞. Assuming that we have determined ψ_{s-1}, we obtain $\bar\psi_s$ from ϕ^∞_s via the commutative diagram (1.3). It then remains to identify ψ_s from diagram (1.2). In general there can be several different homomorphisms $F_sG \to F_s{'G}$ that make the diagram commute. Any two possible choices of ψ_s differ by a composite of the form

$$F_sG \longrightarrow \frac{F_sG}{F_{s-1}G} \xrightarrow{f} F_{s-1}{'G} \longrightarrow F_s{'G}\,,$$

where f is any homomorphism. Having determined ψ_s for all finite s, we can then pass to a colimit to obtain ψ.

((ETC: Discuss examples of *filtration shifts*, later.))

REMARK 1.2.13. In diagram (1.2), if ψ_{s-1} and $\bar\psi_s$ are isomorphisms, then so is ψ_s. In Theorem 2.4.5 ((ETC: and later)) we use this to give conditions which ensure that $\psi\colon G \to 'G$ is an isomorphism.

1.3. Long exact sequences as spectral sequences

The associated long exact sequence in homology

$$\dots \to H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{j} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \dots$$

of a pair (X, A) lets us analyze the homology of X in terms of the homology of A and the relative homology of (X, A). This requires determining the connecting homomorphisms ∂_n, calculating their kernels and cokernels, and synthesizing the result from the extension

$$0 \to \mathrm{cok}(\partial_{n+1}) \longrightarrow H_n(X) \longrightarrow \ker(\partial_n) \to 0\,.$$

Spectral sequences provide a framework for a similar analysis and synthesis, when the pair $A \subset X$ is generalized to a longer sequence

$$\dots \subset X_{s-1} \subset X_s \subset \dots \subset X$$

of subspaces of X. In this section we spell out how the study of $H_*(X)$ in terms of the long exact sequence above can be expressed in terms of the spectral sequence formalism.

Let (X, A) be a pair of spaces. We will specify an associated spectral sequence $(E^r, d^r)_{r\geq 1}$. First, let $E^1 = E^1_{*,*}$ be given by

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{if } s = 0, \\ H_{1+t}(X, A) & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, let $d^1 \colon E^1_{s,t} \to E^1_{s-1,t}$ be given by

$$d^1_{1,t} = \partial_{1+t} \colon H_{1+t}(X, A) \longrightarrow H_t(A)$$

for $s = 1$, and $d^1_{s,t} = 0$ otherwise. We can depict the (E^1, d^1)-term in the (s, t)-plane, with horizontal coordinate s and vertical coordinate t, as on the left hand side below.

$$\begin{array}{c|cc} & \vdots & \vdots \\ t & H_t(A) \xleftarrow{\partial_{1+t}} & H_{1+t}(X, A) \\ & \vdots & \vdots \\ 1 & H_1(A) \xleftarrow{\partial_2} & H_2(X, A) \\ 0 & H_0(A) \xleftarrow{\partial_1} & H_1(X, A) \\ \hline t/s & 0 & 1 \end{array} \qquad \begin{array}{c|cc} & \vdots & \vdots \\ t & E^1_{0,t} \xleftarrow{d^1_{1,t}} & E^1_{1,t} \\ \vdots & \vdots & \vdots \\ 1 & E^1_{0,1} \xleftarrow{d^1_{1,1}} & E^1_{1,1} \\ 0 & E^1_{0,0} \xleftarrow{d^1_{1,0}} & E^1_{1,0} \\ \hline t/s & 0 & 1 \end{array}$$

In the abstract notation, these correspond to the groups and homomorphisms on the right hand side above. The columns with $s < 0$ or $s > 1$ consist of trivial groups, so we have a *two-column* spectral sequence. To simplify drawing these diagrams let us assume that $H_0(X, A) = 0$, so that the rows with $t < 0$ also consist of trivial groups, even if this is not strictly necessary for the argument. In this case the E^1-term is concentrated in the first quadrant in the (s, t)-plane, and we speak of a *first quadrant* homological spectral sequence. Clearly $d^1 d^1 = 0$, since $d^1_{s,t} d^1_{s+1,t} \colon E^1_{s+1,t} \to E^1_{s-1,t}$ maps from a trivial group, or to a trivial group, or both, for each pair (s, t), so (E^1, d^1) is a bigraded abelian group with differential of bidegree $(-1, 0)$.

The E^2-term of this spectral sequence must be given by the homology groups $E^2_{s,t} = H(E^1, d^1)_{s,t}$. The d^1-cycles are

$$\ker(d^1)_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the d^1-boundaries are

$$\operatorname{im}(d^1)_{s,t} = \begin{cases} \operatorname{im}(\partial_{1+t}) & \text{for } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$E^2_{s,t} \cong \begin{cases} \operatorname{cok}(\partial_{1+t}) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We depict this in the (s,t)-plane, as on the left hand side below, with $E^2_{s,t}$ in the position where we had $E^1_{s,t}$ earlier.

$$\begin{array}{c|cc}
 & \vdots & \vdots \\
t & \operatorname{cok}(\partial_{1+t}) & \ker(\partial_{1+t}) \\
 & \vdots & \vdots \\
1 & \operatorname{cok}(\partial_2) & \ker(\partial_2) \\
0 & \operatorname{cok}(\partial_1) & \ker(\partial_1) \\
\hline
t/s & 0 & 1
\end{array}
\qquad\qquad
\begin{array}{c|cc}
 & \vdots & \vdots \\
t & E^2_{0,t} & E^2_{1,t} \\
\vdots & \vdots & \vdots \\
1 & E^2_{0,1} & E^2_{1,1} \\
0 & E^2_{0,0} & E^2_{1,0} \\
\hline
t/s & 0 & 1
\end{array}$$

These groups have the names given on the right hand side above, in the spectral sequence notation. Since the E^2-term consists of subquotients of the E^1-term, it remains concentrated in the first quadrant, under our assumption that $H_0(X, A)$ vanishes.

All components $d^2_{s,t} \colon E^2_{s,t} \to E^2_{s-2,t+1}$ of the d^2-differential must be zero, because the source $E^2_{s,t}$ can only be nonzero for $0 \le s \le 1$, in which case $s - 2 < 0$ and the target $E^2_{s-2,t+1}$ is trivial. Hence we must have $d^2 = 0$, and then $d^2 d^2 = 0$ is obvious. It follows that $H(E^2, d^2) \cong E^2$, since $\ker(d^2) = E^2$ and $\operatorname{im}(d^2) = 0$, so that $E^3 \cong E^2$. In the same way it follows that $d^r = 0$ for all $r \ge 2$, and that $E^r \cong E^2$ for all $r \ge 2$. In other words, the spectral sequence collapses at the E^2-term. The limiting term is thus $E^\infty \cong E^2$, with components

$$E^\infty_{s,t} \cong \begin{cases} \operatorname{cok}(\partial_{1+t}) & \text{for } s = 0, \\ \ker(\partial_{1+t}) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The picture of the E^∞-term in the (s,t)-plane is identical to that of the E^2-term, except that the group labeled $E^2_{s,t}$ is now labeled $E^\infty_{s,t}$.

We have now specified a spectral sequence (E^r, d^r) with E^1-term given in terms of $H_*(A)$ and $H_*(X, A)$. To make sense of the assertion that this spectral sequence converges to $G_* = H_*(X)$, we must also specify a filtration of $H_*(X)$. This is done by setting

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < 0, \\ \operatorname{im}(i \colon H_n(A) \to H_n(X)) & \text{for } s = 0, \\ H_n(X) & \text{for } s \ge 1. \end{cases}$$

Then

$$0 = F_{-1}H_*(X) \subset F_0H_*(X) \subset F_1H_*(X) = H_*(X)$$

is a bounded filtration of the graded abelian group $H_*(X)$. The claim that the spectral sequence $(E^r_{*,*}, d^r)$ converges to $H_*(X)$, denoted

$$E^r_{s,t} \Longrightarrow H_{s+t}(X)\,,$$

is the assertion that there are isomorphisms

$$E^\infty_{s,t} \cong \frac{F_sH_{s+t}(X)}{F_{s-1}H_{s+t}(X)}$$

for all (s,t). Both sides are trivial if $s < 0$ or $s > 1$, hence are obviously isomorphic. When $s = 0$, the assertion is that

$$\mathrm{cok}(\partial_{1+t}) \cong \frac{\mathrm{im}(i\colon H_t(A) \to H_t(X))}{0}$$

for each t. When $s = 1$, the assertion is that

$$\ker(\partial_{1+t}) \cong \frac{H_{1+t}(X)}{\mathrm{im}(i\colon H_{1+t}(A) \to H_{1+t}(X))}$$

for each t. Both of these follow from the part

$$H_{1+t}(A) \xrightarrow{i_{1+t}} H_{1+t}(X) \xrightarrow{j_{1+t}} H_{1+t}(X,A) \xrightarrow{\partial_{1+t}} H_t(A) \xrightarrow{i_t} H_t(X)$$

of the long exact sequence in homology for the pair (X, A), in view of the isomorphisms

$$\mathrm{cok}(\partial_{1+t}) = \frac{H_t(A)}{\mathrm{im}(\partial_{1+t})} = \frac{H_t(A)}{\ker(i_t)} \cong \mathrm{im}(i_t)$$

and

$$\ker(\partial_{1+t}) = \mathrm{im}(j_{1+t}) \cong \frac{H_{1+t}(X)}{\ker(j_{1+t})} = \frac{H_{1+t}(X)}{\mathrm{im}(i_{1+t})}\,.$$

What remains to be done, in order to determine $H_*(X)$, is to resolve the extension problems, i.e., to find $F_0H_*(X)$ and $F_1H_*(X) = H_*(X)$. The initial step is easy, since convergence in bidegree $(s,t) = (0,n)$ tells us that

$$F_0H_n(X) = \mathrm{im}(i\colon H_n(A) \to H_n(X)) \cong E^\infty_{0,n} = \mathrm{cok}(\partial_{1+n})\,.$$

The next, and final, step is to determine $F_1H_n(X) = H_n(X)$ from the extension

$$0 \to F_0H_n(X) \longrightarrow H_n(X) \longrightarrow \frac{H_n(X)}{F_0H_n(X)} \to 0\,.$$

By convergence in bidegree $(s,t) = (1, n-1)$

$$\frac{H_n(X)}{F_0H_n(X)} \cong \ker(\partial_n)\,.$$

Hence, this extension is nothing but the short exact sequence

$$0 \to \mathrm{cok}(\partial_{1+n}) \longrightarrow H_n(X) \longrightarrow \ker(\partial_n) \to 0\,,$$

which we recalled at the beginning of this section. The role of the spectral is thus to calculate $\mathrm{cok}(\partial_{1+n})$ and $\ker(\partial_n)$, and convergence tells us that we have this short exact sequence, while the actual resolution of this extension problem is in a sense external to the spectral sequence.

Nonetheless, we can often visualize the extension problem in the same coordinate system as the spectral sequence, by placing these short exact sequences along lines of constant total degree. In the (s,t)-plane, this amounts to lines of slope -1.

$$
\begin{tikzcd}[column sep=small]
 & \vdots & \arrow[dr, two heads] & \vdots \\
t & \mathrm{cok}(\partial_{1+t}) \arrow[dr, tail] & & \ker(\partial_{1+t}) \\
 & & H_t(X) \arrow[dr, two heads] & \\
t-1 & \mathrm{cok}(\partial_t) \arrow[dr, tail] & & \ker(\partial_t) \\
 & \vdots & \cdots \arrow[dr, two heads] & \vdots \\
1 & \mathrm{cok}(\partial_2) \arrow[dr, tail] & & \ker(\partial_2) \\
 & & H_1(X) \arrow[dr, two heads] & \\
0 & H_0(X) & & \ker(\partial_1) \\
t/s & 0 & & 1
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \vdots & \arrow[dr, two heads] & \vdots \\
t & E^\infty_{0,t} \arrow[dr, tail] & & E^\infty_{1,t} \\
 & & H_t(X) \arrow[dr, two heads] & \\
t-1 & E^\infty_{0,t-1} \arrow[dr, tail] & & E^\infty_{1,t-1} \\
 & \vdots & \cdots \arrow[dr, two heads] & \vdots \\
1 & E^\infty_{0,1} \arrow[dr, tail] & & E^\infty_{1,1} \\
 & & H_1(X) \arrow[dr, two heads] & \\
0 & E^\infty_{0,0} & & E^\infty_{1,0} \\
t/s & 0 & & 1
\end{tikzcd}
$$

If we draw the filtration and the filtration quotients as follows

$$
\begin{tikzcd}
0 \arrow[r, tail] & F_0H_n(X) \arrow[r, tail] \arrow[d, equal] & H_n(X) \arrow[d, two heads] \\
 & F_0H_n(X) & H_n(X)/F_0H_n(X)\,,
\end{tikzcd}
$$

then we can imagine the upper row as being placed along the line $s+t=n$, with $F_sH_n(X)$ in bidegree $(s,t)=(s,n-s)$, and with the quotients in the lower row appearing as the E^∞-term in the same bidegree. Note that in a homological spectral sequence, the differentials map to the left, while the inclusions in the filtration map to the right.

To summarize, this section has spelled out what we have in mind when we say that there is a convergent spectral sequence

$$E^r_{s,t} \Longrightarrow H_{s+t}(X)$$

with

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{for } s=0, \\ H_{1+t}(X,A) & \text{for } s=1, \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we might add detail, such as saying that the d^1-differential is given by $d^1_{1,t} = \partial_{1+t}\colon H_{1+t}(X,A) \to H_t(A)$, or that the convergence is with respect to the filtration with $F_0H_n(X) = \mathrm{im}(i\colon H_n(A) \to H_n(X))$ and $F_1H_n(X) = H_n(X)$.

REMARK 1.3.1. If (X, A, x_0) is a pair of based spaces, the long exact sequence in homotopy

$$\cdots \to \pi_n(A,x_0) \xrightarrow{i} \pi_n(X,x_0) \xrightarrow{j} \pi_n(X,A,x_0) \xrightarrow{\partial_n} \pi_{n-1}(A,x_0) \to \dots$$

can also, usually, be viewed as a spectral sequence converging to $\pi_*(X, x_0)$, with

$$E^1_{s,t} = \begin{cases} \pi_t(A, x_0) & \text{for } s = 0, \\ \pi_{1+t}(X, A, x_0) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

To ensure that this is a bigraded abelian group, and that $\pi_*(X, x_0)$ is a filtered graded abelian group, we may assume that X and A are 0-connected with abelian fundamental groups. This then implies that $\pi_1(X, A, x_0)$ inherits an abelian group structure from $\pi_1(X, x_0)$. ((ETC: Without these hypotheses one has a *fringed* spectral sequence, considered principally by Bousfield and Kan.))

1.4. Two linked long exact sequences

We now consider the case of a triple (X, K, A) of spaces, with $A \subset K \subset X$. This leads to the following diagram of (spaces and) pairs of spaces

$$\begin{tikzcd} A \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[d] \\ & (K, A) \arrow[r] & (X, A) \arrow[d] \\ & & (X, K) \end{tikzcd}$$

to which we can associate the long exact sequences

$$\cdots \to H_n(A) \xrightarrow{i_{K,A}} H_n(K) \xrightarrow{j_{K,A}} H_n(K, A) \xrightarrow{\partial_{K,A}} H_{n-1}(A) \to \dots$$

$$\cdots \to H_n(A) \xrightarrow{i_{X,A}} H_n(X) \xrightarrow{j_{X,A}} H_n(X, A) \xrightarrow{\partial_{X,A}} H_{n-1}(A) \to \dots$$

$$\cdots \to H_n(K) \xrightarrow{i_{X,K}} H_n(X) \xrightarrow{j_{X,K}} H_n(X, K) \xrightarrow{\partial_{X,K}} H_{n-1}(K) \to \dots$$

and

$$\cdots \to H_n(K, A) \xrightarrow{i_{X,K,A}} H_n(X, A) \xrightarrow{j_{X,K,A}} H_n(X, K) \xrightarrow{\partial_{X,K,A}} H_{n-1}(K, A) \to \dots .$$

The last connecting homomorphism can be factored as the composite

$$\partial_{X,K,A} = j_{K,A}\partial_{X,K} \colon H_n(X, K) \xrightarrow{\partial_{X,K}} H_{n-1}(K) \xrightarrow{j_{K,A}} H_{n-1}(K, A) .$$

We would like to calculate $H_*(X)$, supposing that we know the homologies $H_*(A)$, $H_*(K, A)$ and $H_*(X, K)$ of the "minimal" pairs along the diagonal in the diagram above. These involve pairs that are closer together than $H_*(K)$, $H_*(X, A)$ and $H_*(X)$, and may therefore be easier to determine.

Using only exact sequences, the calculation might be done in two steps, in two different ways. On one hand, we might first calculate $H_*(K)$ from $H_*(A)$ and $H_*(K, A)$, and then calculate $H_*(X)$ from $H_*(K)$ and $H_*(X, K)$. On the other hand, we might first calculate $H_*(X, A)$ from $H_*(K, A)$ and $H_*(X, K)$, and then calculate $H_*(X)$ from $H_*(A)$ and $H_*(X, A)$. Either approach involves passing to subquotients, resolving extensions, passing to subquotients again, and resolving extensions again. Instead, we will express the calculation in terms of a single spectral sequence, where all of the passages to subquotients is performed first, in a symmetric manner, and only thereafter are the extension problems resolved.

Here is the basic result that we will prove.

PROPOSITION 1.4.1. *Let (X, K, A) be a triple of spaces. There is a convergent spectral sequence*

$$E^r_{s,t} \Longrightarrow_s H_{s+t}(X)$$

with

$$E^1_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ H_{1+t}(K, A) & \text{for } s = 1, \\ H_{2+t}(X, K) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The d^1-differentials are given by the connecting homomorphisms

$$d^1_{s,t} = \begin{cases} \partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A) & \text{for } s = 1, \\ \partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The abutment is filtered by

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < 0, \\ \operatorname{im}(i_{X,A} \colon H_n(A) \to H_n(X)) & \text{for } s = 0, \\ \operatorname{im}(i_{X,K} \colon H_n(K) \to H_n(X)) & \text{for } s = 1, \\ H_n(X) & \text{for } s \geq 2. \end{cases}$$

PROOF. Note that the description of the E^1-term and the d^1-differential only depend on two of the long exact sequences listed above, namely the ones associated to the pairs (K, A) and (X, K). We can wrap each of these up into an exact triangle, and the two exact triangles are then linked together at a common vertex, given by $H_*(K)$.

(1.4)
$$\begin{tikzcd}
H_*(A) \arrow[r, "i_{K,A}"] & H_*(K) \arrow[r, "i_{X,K}"] \arrow[d, "j_{K,A}"] & H_*(X) \arrow[d, "j_{X,K}"] \\
 & H_*(K, A) \arrow[ul, dashed, "\partial_{K,A}"] & H_*(X, K) \arrow[ul, dashed, "\partial_{X,K}"]
\end{tikzcd}$$

Here the dashed arrows denote homomorphisms of degree -1, sending $H_n(K, A)$ to $H_{n-1}(A)$ and $H_n(X, K)$ to $H_{n-1}(K)$, respectively. The E^1-term is then given by $H_*(A)$ and the groups in the lower row, while the d^1-differentials are given by $\partial_{K,A}$ and the composite $j_{K,A}\partial_{X,K}$, all of which are visible in this diagram.

The filtration on the abutment is also visible in this diagram, being given by the image of the composite $i_{X,K} i_{K,A}$ for $s = 0$, the image of $i_{X,K}$ for $s = 1$, and by $H_*(X)$ itself for $s = 2$.

As before, we can depict the (E^1, d^1)-term in the (s,t)-plane, as in the following diagram.

$$
\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \vdots \\
t+1 & H_{t+1}(A) & \xleftarrow{\partial_{K,A}} & H_{t+2}(K,A) & \xleftarrow{\partial_{X,K,A}} & H_{t+3}(X,K) \\
t & H_t(A) & \xleftarrow{\partial_{K,A}} & H_{t+1}(K,A) & \xleftarrow{\partial_{X,K,A}} & H_{t+2}(X,K) \\
 & \vdots & & \vdots & & \vdots \\
1 & H_1(A) & \xleftarrow{\partial_{K,A}} & H_2(K,A) & \xleftarrow{\partial_{X,K,A}} & H_3(X,K) \\
0 & H_0(A) & \xleftarrow{\partial_{K,A}} & H_1(K,A) & \xleftarrow{\partial_{X,K,A}} & H_2(X,K) \\
\hline
t/s & 0 & & 1 & & 2
\end{array}
$$

The columns with $s < 0$ or $s > 2$ consist of trivial groups. In abstract notation, this appears as below.

$$
\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \vdots \\
t+1 & E^1_{0,t+1} & \xleftarrow{d^1_{1,t+1}} & E^1_{1,t+1} & \xleftarrow{d^1_{2,t+1}} & E^1_{2,t+1} \\
t & E^1_{0,t} & \xleftarrow{d^1_{1,t}} & E^1_{1,t} & \xleftarrow{d^1_{2,t}} & E^1_{2,t} \\
 & \vdots & & \vdots & & \vdots \\
1 & E^1_{0,1} & \xleftarrow{d^1_{1,1}} & E^1_{1,1} & \xleftarrow{d^1_{2,1}} & E^1_{2,1} \\
0 & E^1_{0,0} & \xleftarrow{d^1_{1,0}} & E^1_{1,0} & \xleftarrow{d^1_{2,0}} & E^1_{2,0} \\
\hline
t/s & 0 & & 1 & & 2
\end{array}
$$

Again, let us assume that $H_0(K,A)$, $H_0(X,K)$ and $H_1(X,K)$ vanish, so that the rows with $t < 0$ are trivial and the spectral sequence is concentrated in the first quadrant. The proposition still holds without this assumption, but there may then be nonzero groups in bidegrees $(1,-1)$, $(2,-2)$ and $(2,-1)$, respectively.

The condition that $d^1_{s,t} d^1_{s+1,t} = 0$ needs only be verified for $s = 1$, when it asserts that the composite

$$\partial_{K,A}\partial_{X,K,A} \colon H_{n+1}(X,K) \xrightarrow{\partial_{X,K,A}} H_n(K,A) \xrightarrow{\partial_{K,A}} H_{n-1}(A)$$

is zero. This follows from the factorization $\partial_{X,K,A} = j_{K,A}\partial_{X,K}$ and the fact that $\partial_{K,A} j_{K,A} = 0$, both of which are visible in the diagram (1.4) with two linked exact triangles.

By the defining property of a spectral sequence, the E^2-term must now be given by the homology of the E^1-term with respect to these d^1-differentials. The d^1-cycles are

$$\ker(d^1)_{s,t} = \begin{cases} H_t(A) & \text{for } s = 0, \\ \ker(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A)) & \text{for } s = 1, \\ \ker(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A)) & \text{for } s = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The d^1-boundaries are

$$\operatorname{im}(d^1)_{s,t} = \begin{cases} \operatorname{im}(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A)) & \text{for } s = 0, \\ \operatorname{im}(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A)) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the E^2-term satisfies

$$E^2_{s,t} \cong \begin{cases} \operatorname{cok}(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A)) & \text{for } s = 0, \\ \dfrac{\ker(\partial_{K,A} \colon H_{1+t}(K, A) \to H_t(A))}{\operatorname{im}(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A))} & \text{for } s = 1, \\ \ker(\partial_{X,K,A} \colon H_{2+t}(X, K) \to H_{1+t}(K, A)) & \text{for } s = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and we may assume that this isomorphism is the identity.

We must now specify the d^2-differentials in the spectral sequence. They can only be nonzero when mapping from bidegree (s, t) with $s = 2$, since for other values of s the source or target (or both) is a trivial group. The interesting case is therefore

$$d^2_{2,t} \colon E^2_{2,t} = \ker(\partial_{X,K,A})_{t+2} \longrightarrow \operatorname{cok}(\partial_{K,A})_{t+1} = E^2_{0,t+1}$$

of bidegree $(-2, 1)$. Here $\ker(\partial_{X,K,A})_{t+2} \subset H_{t+2}(X, K)$ while $\operatorname{cok}(\partial_{K,A})_{t+1}$ is a quotient of $H_{t+1}(A)$.

Since $\partial_{X,K,A} = j_{K,A}\partial_{X,K}$, the restriction of $\partial_{X,K}$ defines a homomorphism

$$\tilde\partial_{X,K} \colon \ker(\partial_{X,K,A})_{t+2} \longrightarrow \ker(j_{K,A})_{t+1} = \operatorname{im}(i_{K,A})_{t+1}$$

where $\operatorname{im}(i_{K,A})_{t+1} \subset H_{t+1}(K)$. Furthermore, $i_{K,A}$ induces an isomorphism

$$\bar\imath_{K,A} \colon \operatorname{cok}(\partial_{K,A})_{t+1} = \frac{H_{t+1}(A)}{\ker(i_{K,A})_{t+1}} \xrightarrow{\cong} \operatorname{im}(i_{K,A})_{t+1} .$$

We then define

$$d^2_{2,t} = \bar\imath^{-1}_{K,A}\tilde\partial_{X,K}$$

to be $\tilde{\partial}_{X,K}$ followed by the inverse of $\bar{i}_{K,A}$:

$$
\begin{tikzcd}
\operatorname{cok}(\partial_{K,A})_{t+1} \arrow[r, "\bar{i}_{K,A}", "\cong"'] & \operatorname{im}(i_{K,A})_{t+1} & \\
& & \ker(\partial_{X,K,A})_{t+2} \arrow[ul, "\tilde{\partial}_{X,K}"'] \arrow[ull, bend left, "d^2_{2,t}"]
\end{tikzcd}
\tag{1.5}
$$

In terms of diagram (1.4), we calculate $d^2_{2,t}(x)$ for a class

$$x \in E^2_{2,t} = \ker(\partial_{X,K,A})_{t+2} \subset H_{t+2}(X, K)$$

by applying $\partial_{X,K}$ to get an element $\partial_{X,K}(x) \in \ker(j_{K,A})_{t+1} = \operatorname{im}(i_{K,A})_{t+1} \subset H_{t+1}(K)$, writing this in the form

$$\partial_{X,K}(x) = i_{K,A}(y)$$

for an element $y \in H_{t+1}(A)$, and setting $d^2_{2,t}(x) = [y]$ to be the homology class of y in the quotient $E^2_{0,t+1} = \operatorname{cok}(\partial_{K,A})_{t+1}$ of $H_{t+1}(A)$. Any two choices y and y' with the same image under $i_{K,A}$ differ by an element in $\ker(i_{K,A}) = \operatorname{im}(\partial_{K,A})$, hence define the same class $[y] = [y']$ in $\operatorname{cok}(\partial_{K,A})$.

The (E^2, d^2)-term has the following shape.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
t+1 & \operatorname{cok}(\partial_{K,A})_{t+1} & E^2_{1,t+1} & \ker(\partial_{X,K,A})_{t+3} \arrow[ull, "d^2_{2,t+1}"'] \\
t & \operatorname{cok}(\partial_{K,A})_{t} & E^2_{1,t} & \ker(\partial_{X,K,A})_{t+2} \arrow[ull, "d^2_{2,t}"'] \\
 & \vdots & \vdots & \vdots \arrow[ull, "d^2_{2,t-1}"'] \\
1 & \operatorname{cok}(\partial_{K,A})_{1} & E^2_{1,1} & \ker(\partial_{X,K,A})_{1} \arrow[ull, "d^2_{2,1}"'] \\
0 & \operatorname{cok}(\partial_{K,A})_{0} & E^2_{1,0} & \ker(\partial_{X,K,A})_{2} \arrow[ull, "d^2_{2,0}"'] \\
t/s & 0 & 1 & 2
\end{tikzcd}
$$

where each subquotient $E^2_{1,t} = \ker(\partial_{K,A})_{1+t} / \operatorname{im}(\partial_{X,K,A})_{1+t}$ of $E^1_{1,t} = H_{1+t}(K, A)$ is unaffected by the d^2-differentials. In generic notation, this appears as below.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
t+1 & E^2_{0,t+1} & E^2_{1,t+1} & E^2_{2,t+1} \arrow[ull, "d^2_{2,t+1}"'] \\
t & E^2_{0,t} & E^2_{1,t} & E^2_{2,t} \arrow[ull, "d^2_{2,t}"'] \\
 & \vdots & \vdots & \vdots \arrow[ull, "d^2_{2,t-1}"'] \\
1 & E^2_{0,1} & E^2_{1,1} & E^2_{2,1} \arrow[ull, "d^2_{2,1}"'] \\
0 & E^2_{0,0} & E^2_{1,0} & E^2_{2,0} \arrow[ull, "d^2_{2,0}"'] \\
t/s & 0 & 1 & 2
\end{tikzcd}
$$

It is clear that $d^2 d^2 = 0$, and that $d^r = 0$ for $r \geq 3$, since for each of these homomorphisms the source or target, or both, must be a trivial group. Hence the spectral sequence collapses at the E^3-term, which equals the E^r-term for each $3 \leq r \leq \infty$, and has the following form.

$$
\begin{array}{c|ccc}
 & \vdots & \vdots & \vdots \\
t+1 & \operatorname{cok}(d^2_{2,t}) & \ker(\partial_{K,A})_{t+2} / \operatorname{im}(\partial_{X,K,A})_{t+2} & \ker(d^2_{2,t+1}) \\
t & \operatorname{cok}(d^2_{2,t-1}) & \ker(\partial_{K,A})_{t+1} / \operatorname{im}(\partial_{X,K,A})_{t+1} & \ker(d^2_{2,t}) \\
 & \vdots & \vdots & \vdots \\
1 & \operatorname{cok}(d^2_{2,0}) & \ker(\partial_{K,A})_{2} / \operatorname{im}(\partial_{X,K,A})_{2} & \ker(d^2_{2,1}) \\
0 & \operatorname{cok}(\partial_{K,A})_0 & \ker(\partial_{K,A})_{1} / \operatorname{im}(\partial_{X,K,A})_{1} & \ker(d^2_{2,0}) \\
\hline
t/s & 0 & 1 & 2
\end{array}
$$

We shall make these entries more explicit in a moment, and connect them to the subquotients of the filtration of $H_*(X)$ given in the statement of the proposition.

First, let us exhibit the generic notation for the components of this E^∞-term.

$$
\begin{array}{c|ccc}
 & \vdots & \vdots & \vdots \\
t+1 & E^\infty_{0,t+1} & E^\infty_{1,t+1} & E^\infty_{2,t+1} \\
t & E^\infty_{0,t} & E^\infty_{1,t} & E^\infty_{2,t} \\
 & \vdots & \vdots & \vdots \\
1 & E^\infty_{0,1} & E^\infty_{1,1} & E^\infty_{2,1} \\
0 & E^\infty_{0,0} & E^\infty_{1,0} & E^\infty_{2,0} \\
\hline
t/s & 0 & 1 & 2
\end{array}
$$

Recall that

$$
\begin{aligned}
F_0 H_n(X) &= \operatorname{im}(i_{X,A})_n \\
F_1 H_n(X) &= \operatorname{im}(i_{X,K})_n \\
F_2 H_n(X) &= H_n(X)\,,
\end{aligned}
$$

so that

$$0 \subset F_0 H_*(X) \subset F_1 H_*(X) \subset F_2 H_*(X) = H_*(X)$$

is a bounded filtration of $H_*(X)$. The following three lemmas will therefore complete the proof of the proposition. □

LEMMA 1.4.2. *There is a preferred isomorphism*

$$E^\infty_{0,n} \cong F_0 H_n(X)\,.$$

PROOF. Recall diagram (1.5). The cokernel

$$E^\infty_{0,n} = E^3_{0,n} = \operatorname{cok}(d^2_{2,n-1})$$

maps isomorphically by $\bar{i}_{K,A}$ to the cokernel

$$\frac{\operatorname{im}(i_{K,A})_n}{\operatorname{im}(\tilde{\partial}_{X,K})_n} = \frac{\operatorname{im}(i_{K,A})_n}{\operatorname{im}(i_{K,A})_n \cap \operatorname{im}(\partial_{X,K})_n} = \frac{\operatorname{im}(i_{K,A})_n}{\operatorname{im}(i_{K,A})_n \cap \ker(i_{X,K})_n}\,,$$

which maps isomorphically by $i_{X,K}$ to

$$i_{X,K}(\operatorname{im}(i_{K,A}))_n = \operatorname{im}(i_{X,A})_n = F_0 H_n(X)\,.$$

□

LEMMA 1.4.3. *There is a preferred isomorphism*

$$E^\infty_{1,n-1} \cong \frac{F_1 H_n(X)}{F_0 H_n(X)}\,.$$

PROOF. The quotient group

$$E^\infty_{1,n-1} = E^2_{1,n-1} = \frac{\ker(\partial_{K,A})_n}{\mathrm{im}(\partial_{X,K,A})_n} = \frac{\mathrm{im}(j_{K,A})_n}{\mathrm{im}(j_{K,A}\partial_{X,K})_n} = \frac{\mathrm{im}(j_{K,A})_n}{j_{K,A}(\ker(i_{X,K}))_n}\,.$$

receives an isomorphism induced by $j_{K,A}$ from

$$\frac{H_n(K)}{\ker(j_{K,A})_n + \ker(i_{X,K})_n} = \frac{H_n(K)}{\mathrm{im}(i_{K,A})_n + \ker(i_{X,K})_n}\,,$$

and this group maps isomorphically under $i_{X,K}$ to

$$\frac{i_{X,K}(H_n(K))}{i_{X,K}(\mathrm{im}(i_{K,A}))_n} = \frac{F_1H_n(X)}{F_0H_n(X)}\,.$$

□

LEMMA 1.4.4. *There is a preferred isomorphism*

$$E^\infty_{2,n-2} \cong \frac{H_n(X)}{F_1H_n(X)}\,.$$

PROOF. The subgroup

$$E^\infty_{2,t} = E^3_{2,t} = \ker(d^2_{2,t}) = \ker(\tilde\partial_{X,K})_{t+2} = \ker(\partial_{X,K})_{t+2} = \mathrm{im}(j_{X,K})_{t+2}$$

of $H_n(X,K)$ receives an isomorphism induced by $j_{X,K}$ from

$$\frac{H_n(X)}{\ker(j_{X,K})_n} = \frac{H_n(X)}{\mathrm{im}(i_{X,K})_n} = \frac{H_n(X)}{F_1H_n(X)}\,.$$

□

REMARK 1.4.5. The d^2-differentials in this *three-column* spectral sequence were not fully determined by the statement of the proposition. For instance, we could have reversed the sign of some of the d^2-differentials and obtained a slightly different spectral sequence, with the same (E^1,d^1)-term and filtered abutment. In order to be clear about which spectral sequence one has in mind one must therefore be more specific about how the spectral sequence arises, beyond just giving the initial term. In many cases this complete precision is not necessary, but one should be aware of the issue.

REMARK 1.4.6. Another way to depict the two exact triangles in (1.4) is the following pair of long exact sequences, each shown as a "staircase" shape.

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(K,A) \arrow[r] & H_n(A) \arrow[d] & & & & \\
\dots \arrow[r] & H_{n+1}(X,K) \arrow[r] & H_n(K) \arrow[r] \arrow[d] & H_n(K,A) \arrow[r] & H_{n-1}(A) \arrow[d] & & \\
& & H_n(X) \arrow[r] & H_n(X,K) \arrow[r] & H_{n-1}(K) \arrow[r] \arrow[d] & H_{n-1}(K,A) \arrow[r] & \dots \\
& & & & H_{n-1}(X) \arrow[r] & H_{n-1}(X,K) \arrow[r] & \dots
\end{tikzcd}$$

CHAPTER 2

Exact Couples

Almost every spectral sequences arises from a generalization of diagram (1.4) to the case where there are infinitely many long exact sequences that are chained together at common vertices. This algebraic structure is called an *exact couple*, and was introduced by William Massey [**Mas52**], [**Mas53**]. We prefer to display exact couples in an unrolled form, as in Michael Boardman's paper [**Boa99**, (0.1)].

2.1. Unrolled exact couples

Definition 2.1.1. An *unrolled exact couple* $(A, E) = (A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ is a diagram of the form

$$\begin{tikzcd}
\dots \arrow[r] & A_{s-2} \arrow[r, "\alpha_{s-1}"] \arrow[d] & A_{s-1} \arrow[r, "\alpha_s"] \arrow[d, "\beta_{s-1}"] & A_s \arrow[r, "\alpha_{s+1}"] \arrow[d, "\beta_s"] & A_{s+1} \arrow[r] \arrow[d, "\beta_{s+1}"] & \dots \\
& \dots & E_{s-1} \arrow[ul, "\gamma_{s-1}"] & E_s \arrow[ul, "\gamma_s"] & E_{s+1} \arrow[ul, "\gamma_{s+1}"] & \dots , \arrow[ul]
\end{tikzcd}$$

in which each triangle forms a long exact sequence

$$\dots \to A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\beta_s} E_s \xrightarrow{\gamma_s} A_{s-1} \to \dots .$$

Here each A_s and E_s is a graded abelian group, and α_s, β_s and γ_s are graded morphisms of graded abelian groups.

Remark 2.1.2. In the long circulated preprint form of Boardman's paper, this structure was called an *unraveled exact couple*. Frequently, α_s and β_s preserve the total degree, and γ_s reduces the total degree by 1, so that we have a long exact sequence of abelian groups

$$\dots \to (A_{s-1})_n \xrightarrow{\alpha_s} (A_s)_n \xrightarrow{\beta_s} (E_s)_n \xrightarrow{\gamma_s} (A_{s-1})_{n-1} \to \dots$$

for each s. If we set $A_{s,t} = (A_s)_{s+t}$ and $E_{s,t} = (E_s)_{s+t}$, with t a complementary degree, this appears as follows

$$\dots \to A_{s-1,t+1} \xrightarrow{\alpha_s} A_{s,t} \xrightarrow{\beta_s} E_{s,t} \xrightarrow{\gamma_s} A_{s-1,t} \to \dots ,$$

so that each α_s has (s, t)-bidegree $(1, -1)$, each β_s has bidegree $(0, 0)$, and each γ_s has bidegree $(-1, 0)$.

Definition 2.1.3. A *morphism of exact couples* $\phi\colon (A, E) \to ('A, 'E)$ consists of degree-preserving homomorphisms $\phi_s\colon A_s \to 'A_s$ and $\phi_s\colon E_s \to 'E_s$, for $s \in \mathbb{Z}$, making each diagram

$$\begin{tikzcd}
A_{s-1} \arrow[r, "\alpha_s"] \arrow[d, "\phi_{s-1}"'] & A_s \arrow[r, "\beta_s"] \arrow[d, "\phi_s"'] & E_s \arrow[r, "\gamma_s"] \arrow[d, "\phi_s"'] & A_{s-1} \arrow[d, "\phi_{s-1}"'] \\
'A_{s-1} \arrow[r, "'\alpha_s"] & 'A_s \arrow[r, "'\beta_s"] & 'E_s \arrow[r, "'\gamma_s"] & 'A_{s-1}
\end{tikzcd}$$

commute.

Example 2.1.4. A filtration of a space X is a sequence of subspaces

$$\cdots \subset X_{s-1} \subset X_s \subset \dots$$

where $s \in \mathbb{Z}$. The (unrolled) exact couple in homology associated to such a filtration $(X_s)_s$ is the chain of exact triangles

$$\begin{tikzcd}
\dots \arrow[r] & H_*(X_{s-2}) \arrow[r, "\alpha_{s-1}"] & H_*(X_{s-1}) \arrow[r, "\alpha_s"] \arrow[d, "\beta_{s-1}"'] & H_*(X_s) \arrow[r, "\alpha_{s+1}"] \arrow[d, "\beta_s"'] & H_*(X_{s+1}) \arrow[r] \arrow[d, "\beta_{s+1}"'] & \dots \\
& & H_*(X_{s-1}, X_{s-2}) \arrow[ul, dashed, "\gamma_{s-1}"] & H_*(X_s, X_{s-1}) \arrow[ul, dashed, "\gamma_s"] & H_*(X_{s+1}, X_s) \arrow[ul, dashed, "\gamma_{s+1}"] &
\end{tikzcd}$$

with

$$\begin{aligned}
A_s &= H_*(X_s) \\
E_s &= H_*(X_s, X_{s-1})
\end{aligned}$$

and $\alpha_s = i_{X_s, X_{s-1}}$, $\beta_s = j_{X_s, X_{s-1}}$, $\gamma_s = \partial_{X_s, X_{s-1}}$, so that

$$\cdots \to H_*(X_{s-1}) \xrightarrow{\alpha_s} H_*(X_s) \xrightarrow{\beta_s} H_*(X_s, X_{s-1}) \xrightarrow{\gamma_s} H_{*-1}(X_{s-1}) \to \dots$$

is the long exact sequence in homology of the pair (X_s, X_{s-1}). The solid arrows α_s and β_s preserve the total grading, while the dashed arrows γ_s have total degree -1.

Example 2.1.5. Let p be a prime. There is an (unrolled) exact couple (A, E) with $A_s = \mathbb{Z}$, $E_s = \mathbb{Z}/p$ and $\alpha_s \colon \mathbb{Z} \to \mathbb{Z}$ given by multiplication by p, for each $s \in \mathbb{Z}$. The morphisms $\beta_s \colon \mathbb{Z} \to \mathbb{Z}/p$ are the canonical surjections, while each γ_s is zero. The whole diagram

$$\begin{tikzcd}
\dots \arrow[r, tail, "p"] & \mathbb{Z} \arrow[r, tail, "p"] & \mathbb{Z} \arrow[r, tail, "p"] \arrow[d, two heads] & \mathbb{Z} \arrow[r, tail, "p"] \arrow[d, two heads] & \mathbb{Z} \arrow[r, tail, "p"] \arrow[d, two heads] & \dots \\
& & \mathbb{Z}/p \arrow[ul, dashed, "0"] & \mathbb{Z}/p \arrow[ul, dashed, "0"] & \mathbb{Z}/p \arrow[ul, dashed, "0"] &
\end{tikzcd}$$

is concentrated in total degree 0. Let $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ denote the p-adic integers. There is a second exact couple with the same groups E_s, given by the diagram

$$\begin{tikzcd}
\dots \arrow[r, tail, "p"] & \mathbb{Z}_p \arrow[r, tail, "p"] & \mathbb{Z}_p \arrow[r, tail, "p"] \arrow[d, two heads] & \mathbb{Z}_p \arrow[r, tail, "p"] \arrow[d, two heads] & \mathbb{Z}_p \arrow[r, tail, "p"] \arrow[d, two heads] & \dots \\
& & \mathbb{Z}/p \arrow[ul, dashed, "0"] & \mathbb{Z}/p \arrow[ul, dashed, "0"] & \mathbb{Z}/p \arrow[ul, dashed, "0"] &
\end{tikzcd} .$$

EXAMPLE 2.1.6. Let $(X_s)_s$ and $(Y_s)_s$ be filtrations of the spaces X and Y, respectively. A map $\phi\colon X \to Y$ is *filtration-preserving* if $\phi(X_s) \subset Y_s$ for each s. Such a map induces a morphism ϕ of exact couples, given by the homomorphisms

$$\phi_s\colon H_*(X_s) \longrightarrow H_*(Y_s)$$
$$\phi_s\colon H_*(X_s, X_{s-1}) \longrightarrow H_*(Y_s, Y_{s-1})$$

induced by the evident restrictions of ϕ.

REMARK 2.1.7. In Massey's paper, the exact triangles are rolled up further, by setting

$$A = \bigoplus_s A_s \qquad \text{and} \qquad E = \bigoplus_s E_s\,.$$

An *exact couple* is then a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A \arrow[d, "\beta"] \\ & E \arrow[ul, "\gamma"] \end{tikzcd}$$

that is exact at each point, meaning that $\mathrm{im}(\alpha) = \ker(\beta)$, $\mathrm{im}(\beta) = \ker(\gamma)$ and $\mathrm{im}(\gamma) = \ker(\alpha)$. Boardman's unrolled presentation has the advantage that it visually emphasizes the filtration degree s. As a matter of notation, Massey writes $(A, C; f, g, h)$, Saunders Mac Lane [**ML63**, Ch. XI, (5.1)] and George Whitehead [**Whi78**, §XIII.2] write $(D, E; i, j, \partial)$, and Boardman writes $(A, E; i, j, k)$, where we write $(A, E; \alpha, \beta, \gamma)$.

REMARK 2.1.8. Exact couples can also be fully unrolled in the plane, so that each long exact sequence appears as a staircase in the following whole-plane diagram.

$$\begin{tikzcd}
 & & \cdots \arrow[d] & & \cdots \arrow[d] & & \\
\cdots \arrow[r] & E_{s-1,t+2} \arrow[r, "\gamma_{s-1}"] & A_{s-2,t+2} \arrow[r, "\beta_{s-2}"] \arrow[d, "\alpha_{s-1}"] & E_{s-2,t+2} \arrow[r, "\gamma_{s-2}"] & A_{s-3,t+2} \arrow[r, "\beta_{s-3}"] \arrow[d, "\alpha_{s-2}"] & E_{s-3,t+2} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_{s,t+1} \arrow[r, "\gamma_{s}"] & A_{s-1,t+1} \arrow[r, "\beta_{s-1}"] \arrow[d, "\alpha_{s}"] & E_{s-1,t+1} \arrow[r, "\gamma_{s-1}"] & A_{s-2,t+1} \arrow[r, "\beta_{s-2}"] \arrow[d, "\alpha_{s-1}"] & E_{s-2,t+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_{s+1,t} \arrow[r, "\gamma_{s+1}"] & A_{s,t} \arrow[r, "\beta_{s}"] \arrow[d, "\alpha_{s+1}"] & E_{s,t} \arrow[r, "\gamma_{s}"] & A_{s-1,t} \arrow[r, "\beta_{s-1}"] \arrow[d, "\alpha_{s}"] & E_{s-1,t} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_{s+2,t-1} \arrow[r, "\gamma_{s+2}"] & A_{s+1,t-1} \arrow[r, "\beta_{s+1}"] \arrow[d, "\alpha^{s+2}"] & E_{s+1,t-1} \arrow[r, "\gamma_{s+1}"] & A_{s,t-1} \arrow[r, "\beta_{s}"] \arrow[d, "\alpha^{s+1}"] & E_{s,t-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_{s+3,t-2} \arrow[r, "\gamma_{s+3}"] & A_{s+2,t-2} \arrow[r, "\beta_{s+2}"] \arrow[d] & E_{s+2,t-2} \arrow[r, "\gamma_{s+2}"] & A_{s+1,t-2} \arrow[r, "\beta_{s+1}"] \arrow[d] & E_{s+1,t-2} \arrow[r] & \cdots \\
 & & \cdots & & \cdots & &
\end{tikzcd}$$

See [**Spa66**, §9.1.6], [**Whi78**, Fig. 13.1]. In most cases the groups $A_{s,t}$ for varying t play a similar role, but appear in many places in this diagram. The same issue applies for the groups $E_{s,t}$ for varying t.

2.2. The spectral sequence associated to an exact couple

Given an exact couple, we can construct a spectral sequence.

THEOREM 2.2.1. *Let $(A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ be an exact couple. Then there is a spectral sequence $(E^r, d^r)_{r\geq 1}$ with*

$$E^1_s = E_s$$

and

$$d^1_s = \beta_{s-1}\gamma_s \colon E^1_s \longrightarrow E^1_{s-1}$$

for all $s \in \mathbb{Z}$. If α_s and β_s have total degree 0 and γ_s has total degree -1, then

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

has bidegree $(-r, r+1)$, where $E^r_{s,t} = (E^r_s)_{s+t}$ is a subquotient of $E^1_{s,t} = (E_s)_{s+t}$.

The E^1-term of the spectral sequence is thus visible in the lower row of the unrolled exact couple, with each d^1-differential being given by the composite of two homomorphisms.

$$\begin{tikzcd}
\dots & E^1_{s-1} \arrow[l] \arrow[d, equal] & E^1_s \arrow[l, "d^1_s"'] \arrow[d, equal] & E^1_{s+1} \arrow[l, "d^1_{s+1}"'] \arrow[d, equal] & \dots \arrow[l] \\
\dots & E_{s-1} \arrow[l] & E_s \arrow[l, "\beta_{s-1}\gamma_s"'] & E_{s+1} \arrow[l, "\beta_s\gamma_{s+1}"'] & \dots \arrow[l]
\end{tikzcd}$$

To construct the E^r-term of the spectral sequence, we consider the following part of the unrolled exact couple.

$$\begin{tikzcd}
A_{s-r} \arrow[r, "\alpha^{r-1}"] & A_{s-1} \arrow[r, "\alpha_s"] & A_s \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta_s"] & A_{s+r-1} \\
& & E_s \arrow[ul, dashed, "\gamma_s"] &
\end{tikzcd}$$

Here α^{r-1} denotes the composite of $(r-1)$ instances of the maps α_s, for s in a suitable range.

DEFINITION 2.2.2. For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$Z^r_s = \gamma_s^{-1} \operatorname{im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1})$$

be the r-th *cycle group*, and let

$$B^r_s = \beta_s \ker(\alpha^{r-1} \colon A_s \to A_{s+r-1})$$

be the r-th *boundary group*, both in filtration s. Here Z^r_s is the preimage under $\gamma_s \colon E_s \to A_{s-1}$ of the image of $\alpha^{r-1} \colon A_{s-r} \to A_{s-1}$, while B^r_s is the image under $\beta_s \colon A_s \to E_s$ of the kernel of $\alpha^{r-1} \colon A_s \to A_{s+r-1}$. These are both graded subgroups of E_s, with components $Z^r_{s,t}$ and $B^r_{s,t}$ contained in $E_{s,t} = (E_s)_{s+t}$.

LEMMA 2.2.3. *There are inclusions*

$$0 = B^1_s \subset \cdots \subset B^r_s \subset B^{r+1}_s \subset \cdots \subset \operatorname{im}(\beta_s)$$
$$= \ker(\gamma_s) \subset Z^{r+1}_s \subset Z^r_s \subset \cdots \subset Z^1_s = E_s\,.$$

PROOF. The inclusions of $\ker(\gamma_s)$ and the cycle groups follow from the inclusions

$$0 \subset \operatorname{im}(\alpha^r \colon A_{s-r-1} \to A_{s-1}) \subset \operatorname{im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1})\,.$$

The preimage Z^1_s of $\operatorname{im}(\alpha^0) = A_{s-1}$ is the whole of E_s. The inclusions of boundary groups and $\operatorname{im}(\beta_s)$ follow from the inclusions

$$\ker(\alpha^{r-1} \colon A_s \to A_{s+r-1}) \subset \ker(\alpha^r \colon A_s \to A_{s+r}) \subset A_s\,.$$

The image B^1_s of $\ker(\alpha^0) = 0$ is trivial. For each finite $r \ge 1$ we have

$$B^r_s \subset \operatorname{im}(\beta_s) = \ker(\gamma_s) \subset Z^r_s$$

by exactness at E_s. $\square$

DEFINITION 2.2.4. For $r \ge 1$ and $s \in \mathbb{Z}$ let

$$E^r_s = Z^r_s / B^r_s$$

and $E^r_{s,t} = Z^r_{s,t}/B^r_{s,t}$ so that $E^r = E^r_{*,*}$ is the E^r*-term* of the spectral sequence. In particular, $E^1_s = E_s/0 \cong E_s$.

REMARK 2.2.5. As r increases, each E^r-term is a successively smaller subquotient of the E^1-term, since the cycle groups Z^r_s decrease and the boundary groups B^r_s increase in size. Each term E^q thus gives an upper bound for the subsequent terms E^r with $r \ge q$. If $E^q_{s,t} = 0$ for (s,t) in some region of the (s,t)-plane, then $E^r_{s,t} = 0$ for all $r \ge q$ and (s,t) in this region. In other words, if a term of a spectral sequence is concentrated in some region, such as the first quadrant, then so is the remainder of the spectral sequence.

In order to have a spectral sequence, we must identify E^{r+1} as the homology of E^r with respect to a d^r-differential. To define the d^r-differential, we use the following part of the unrolled exact couple.

$$\begin{tikzcd}
A_{s-r-1} \arrow[r, "\alpha_{s-r}"] & A_{s-r} \arrow[rr, "\alpha^{r-1}"] \arrow[d, "\beta_{s-r}"] & & A_{s-1} \arrow[r, "\alpha_s"] & A_s \arrow[d, "\beta_s"] \\
& E_{s-r} \arrow[ul, "\gamma_{s-r}"] & & & E_s \arrow[ul, "\gamma_s"]
\end{tikzcd}$$

DEFINITION 2.2.6. For each $x \in Z^r_s \subset E_s$ in the r-th cycle group we write $[x] \in E^r_s$ for its equivalence class modulo the r-th boundary group. Let the d^r-*differential*

$$d^r_s \colon E^r_s \longrightarrow E^r_{s-r}$$

be defined by

$$d^r_s \colon [x] \longmapsto [\beta_{s-r}(y)]$$

where $y \in A_{s-r}$ is chosen to satisfy $\gamma_s(x) = \alpha^{r-1}(y)$. In particular, $d^1_s = \beta_{s-1}\gamma_s$.

LEMMA 2.2.7. *d^r_s is well defined.*

Proof. Since $x \in Z^r_s$ we have $\gamma_s(x) \in \operatorname{im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1})$, so there exists a $y \in A_{s-r}$ with $\alpha^{r-1}(y) = \gamma_s(x)$. The image $\beta_{s-r}(y)$ then lies in $\operatorname{im}(\beta_{s-r}) \subset Z^r_{s-r}$, hence defines a class $[\beta_{s-r}(y)]$ in E^r_{s-r}.

Another choice of $y' \in A_{s-r}$ with $\alpha^{r-1}(y') = \gamma_s(x)$ differs from y by a class $y' - y \in \ker(\alpha^{r-1} \colon A_{s-r} \to A_{s-1})$, hence $\beta_{s-r}(y')$ differs from $\beta_{s-r}(y)$ by a class

$$\beta_{s-r}(y' - y) \in \beta_{s-r} \ker(\alpha^{r-1} \colon A_{s-r} \to A_{s-1}) = B^r_{s-r} .$$

This means that $[\beta_{s-r}(y)] = [\beta_{s-r}(y')]$ as elements of E^r_{s-r}.

Any other choice of $x' \in Z^r_s$ representing the same class $[x'] = [x]$ in E^r_s differs from x by an element $x' - x \in B^r_s$. Since $B^r_s \subset \ker(\gamma_s)$, it follows that $\gamma_s(x') = \gamma_s(x)$, so x and x' lead to the same choices for y and the same value of $[\beta_{s-r}(y)]$. □

To discuss d^r-cycles in filtration s we use the diagram above, while for d^r-boundaries in filtration s we can use the following variant.

$$\begin{tikzcd}
A_{s-1} \arrow[r, "\alpha_s"] & A_s \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta_s"] & A_{s+r-1} \arrow[r, "\alpha_{s+r}"] & A_{s+r} \arrow[d, "\beta_{s+r}"] \\
& E_s \arrow[ul, "\gamma_s"] & & E_{s+r} \arrow[ul, "\gamma_{s+r}"]
\end{tikzcd}$$

Lemma 2.2.8.

$$\ker(d^r)_s = \ker(d^r_s) = Z^{r+1}_s / B^r_s$$
$$\operatorname{im}(d^r)_s = \operatorname{im}(d^r_{s+r}) = B^{r+1}_s / B^r_s .$$

Hence $d^r d^r = 0$ and the projection $Z^{r+1}_s \to \ker(d^r_s)$ induces an isomorphism $E^{r+1}_s \cong H(E^r, d^r)_s$.

Proof. The displayed identities compare subgroups of $E^r_s = Z^r_s / B^r_s$.

First, let $x \in Z^r_s$, choose $y \in A_{s-r}$ with $\alpha^{r-1}(y) = \gamma_s(x)$, and suppose that $[x] \in \ker(d^r_s)$. This means that $\beta_{s-r}(y) \in B^r_{s-r}$, so there exists a $y' \in \ker(\alpha^{r-1}) \subset A_{s-r}$ with $\beta_{s-r}(y) = \beta_{s-r}(y')$. Then $y - y' \in \ker(\beta_{s-r}) = \operatorname{im}(\alpha_{s-r})$ equals $\alpha_{s-r}(z)$ for some $z \in A_{s-r-1}$, and $\alpha^r(z) = \alpha^{r-1}(y - y') = \alpha^{r-1}(y) - \alpha^{r-1}(y') = \gamma_s(x) - 0 = \gamma_s(x)$, which proves that $x \in Z^{r+1}_s$. Hence $\ker(d^r_s) \subset Z^{r+1}_s / B^r_s$.

Conversely, if $x \in Z^{r+1}_s$ then we can write $\gamma_s(x) = \alpha^r(z) = \alpha^{r-1}(y)$ for some $z \in A_{s-r-1}$ and $y = \alpha_{s-r}(z) \in \operatorname{im}(\alpha_{s-r}) = \ker(\beta_{s-r})$. Then $\beta_{s-r}(y) = 0$, so d^r_s maps $[x]$ to $[0]$, and $[x] \in \ker(d^r_s)$. Hence $Z^{r+1}_s / B^r_s \subset \ker(d^r_s)$.

Next, let $x \in Z^r_{s+r}$, choose $y \in A_s$ with $\alpha^{r-1}(y) = \gamma_{s+r}(x)$, and consider $[\beta_s(y)] \in \operatorname{im}(d^r_{s+r})$. Then $\alpha^r(y) = \alpha_{s+r}\alpha^{r-1}(y) = \alpha_{s+r}\gamma_{s+r}(x) = 0$, so $y \in \ker(\alpha^r \colon A_s \to A_{s+r})$ and $\beta_s(y) \in B^{r+1}_s$. Hence $\operatorname{im}(d^r_{s+r}) \subset B^{r+1}_s / B^r_s$.

Conversely, if $\beta_s(y) \in B^{r+1}_s$ with $y \in \ker(\alpha^r)$, then $\alpha^{r-1}(y) \in \ker(\alpha_{s+r}) = \operatorname{im}(\gamma_{s+r})$, so we can write $\alpha^{r-1}(y) = \gamma_{s+r}(x)$. Then $x \in Z^r_{s+r}$ and d^r_{s+r} maps $[x]$ to $[\beta_s(y)]$. Hence $B^{r+1}_s / B^r_s \subset \operatorname{im}(d^r_{s+r})$.

It follows from $B^{r+1}_s \subset Z^{r+1}_s$ that $\operatorname{im}(d^r)_s \subset \ker(d^r)_s$, so $d^r_s d^r_{s+r} = 0$ and $d^r \colon E^r \to E^r$ is a differential of filtration degree $-r$. The isomorphism

$$Z^{r+1}_s / B^{r+1}_s \xrightarrow{\cong} \frac{Z^{r+1}_s / B^r_s}{B^{r+1}_s / B^r_s}$$

shows that $E^{r+1}_s \cong H(E^r, d^r)_s$, as claimed. □

PROOF OF THEOREM 2.2.1. We specified the E^r-terms and d^r-differentials in Definitions 2.2.4 and 2.2.6, and checked the spectral sequence condition in Lemma 2.2.8.

The explicit form of the E^1-differential and d^1-differential follows easily by inspection of the definitions.

If α_s and β_s have total degree 0 while γ_s has total degree -1, then $d^r_s \colon E^r_s \to E^r_{s-r}$ has total degree -1 and reduces the filtration degree s by r. Hence it must increase the complementary degree t by $(r-1)$. $\square$

LEMMA 2.2.9. *Each morphism $\phi\colon (A, E) \to ('A, 'E)$ of exact couples induces a morphism $\phi\colon (E^r, d^r) \to ('E^r, 'd^r)$ of spectral sequences. Hence the associated spectral sequence defines a functor*

$$\textit{Exact Couples} \longrightarrow \textit{Spectral Sequences}\,.$$

PROOF. It is straightforward to check that $\phi_s \colon E_s \to 'E_s$ restricts to homomorphisms $\phi_s \colon Z^r_s \to 'Z^r_s$, $\phi_s \colon B^r_s \to 'B^r_s$ and $\phi_s \colon E^r_s \to 'E^r_s$ for all $r \ge 1$ and s, and that these commute with the differentials d^r and $'d^r$, as well as the isomorphisms $H(E^r, d^r) \cong E^{r+1}$ and $H('E^r, d^r) \cong 'E^{r+1}$. $\square$

REMARK 2.2.10. We are following the notation of [**Boa99**, §0], but translated into homological indexing. (We will explain cohomological indexing later ((ETC: where?)).) Beware that the d^r-cycles $\ker(d^r)$ are the quotient Z^{r+1}/B^r of the $(r+1)$-th cycle group, and the d^r-boundaries $\mathrm{im}(d^r)$ are the quotient B^{r+1}/B^r of the $(r+1)$-th boundary group, so that there is an offset by one from r to $(r+1)$ in the indexing of these bigraded groups.

In [**ML63**, Ch. XI, (1.4)], our Z^r and B^r are denoted C^{r-1} and B^{r-1}, respectively. In that notation, the d^{r-1}-cycles are a quotient of C^{r-1} and the d^{r-1}-boundaries are a quotient of B^{r-1}. Note, however, that Mac Lane uses different indexing in [**ML63**, §XI.3].

2.3. The E^∞-term of a spectral sequence

In every spectral sequence starting at the E^1-term, the E^r-terms can be expressed as in Definition 2.2.4 in terms of chains of cycle groups and boundary groups, as in Lemma 2.2.3.

LEMMA 2.3.1. *Let $(E^r, d^r)_{r\ge p}$ be an E^p-spectral sequence. There are inclusions*

$$0 = B^p_s \subset \dots \subset B^r_s \subset B^{r+1}_s \subset \dots \subset Z^{r+1}_s \subset Z^r_s \subset \dots \subset Z^p_s = E^p_s$$

and isomorphisms $Z^r_s/Z^{r+1}_s \cong B^{r+1}_{s-r}/B^r_{s-r}$ such that

$$E^r_s \cong Z^r_s/B^r_s$$

and $d^r_s \colon E^r_s \to E^r_{s-r}$ corresponds to the composite

$$Z^r_s/B^r_s \xrightarrow{\pi} Z^r_s/Z^{r+1}_s \cong B^{r+1}_{s-r}/B^r_{s-r} \xrightarrow{\iota} Z^r_{s-r}/B^r_{s-r}$$

for all $r \ge p$ and $s \in \mathbb{Z}$.

Here π and ι denote the canonical projection and inclusion, respectively.

PROOF. We show this by induction on $r \ge p$. Suppose that $E^r_s \cong Z^r_s/B^r_s$ for some r and all s. Then the subgroup $\ker(d^r)_s \subset E^r_s$ corresponds to a subgroup of Z^r_s/B^r_s, which must have the form Z^{r+1}_s/B^r_s for some $Z^{r+1}_s \subset Z^r_s$. Similarly, the subgroup $\mathrm{im}(d^r)_s \subset \ker(d^r)_s$ corresponds to a subgroup of Z_s^{r+1}/B_s^r, which must be of the form B_s^{r+1}/B_s^r for some $B_s^{r+1} \subset Z_s^{r+1}$. We have the following inclusions

$$B_s^r \subset B_s^{r+1} \subset Z_s^{r+1} \subset Z_s^r$$

and isomorphisms

$$E_s^{r+1} \cong H(E^r, d^r)_s = \frac{\ker(d^r)_s}{\mathrm{im}(d^r)_s} \cong \frac{Z_s^{r+1}/B_s^r}{B_s^{r+1}/B_s^r} \cong Z_s^{r+1}/B_s^{r+1} .$$

This completes the inductive step. The d^r-differential factors as

$$E_s^r \xrightarrow{\pi} \frac{E_s^r}{\ker(d^r)_s} \xrightarrow{\cong} \mathrm{im}(d^r)_{s-r} \xrightarrow{\iota} E_{s-r}^r$$

and corresponds to the composition

$$Z_s^r/B_s^r \xrightarrow{\pi} Z_s^r/Z_s^{r+1} \xrightarrow{\cong} \frac{Z_s^r/B_s^r}{Z_s^{r+1}/B_s^r} \xrightarrow{\cong} B_{s-r}^{r+1}/B_{s-r}^r \xrightarrow{\iota} Z_{s-r}^r/B_{s-r}^r .$$

The composite of the two inner isomorphisms is the required isomorphism from Z_s^r/Z_s^{r+1} to B_{s-r}^{r+1}/B_{s-r}^r, which leads to the asserted expression for d_s^r. □

LEMMA 2.3.2. *When (E^r, d^r) is the E^1-spectral sequence associated to an exact couple (A, E), then the subgroups Z^r and B^r of E^1 in Lemma 2.3.1 agree with the subgroups Z^r and B^r of E in Lemma 2.2.3.*

PROOF. This follows directly from Lemma 2.2.8. □

DEFINITION 2.3.3. Let (E^r, d^r) be an E^p-spectral sequence. For each $s \in \mathbb{Z}$ we let the *infinite cycles*

$$Z_s^\infty = \bigcap_{r \ge p} Z_s^r$$

be the intersection (or limit) of the r-th cycle groups, and we let the *infinite boundaries*

$$B_s^\infty = \bigcup_{r \ge p} B_s^r$$

be the union (or colimit) of the r-th boundary groups. In each case r ranges over the integers $\ge p$. Hence there are inclusions

$$0 \subset \dots \subset B_s^r \subset \dots \subset B_s^\infty \subset Z_s^\infty \subset \dots \subset Z_s^r \subset \dots \subset E_s^p$$

for all $r \ge p$ and $s \in \mathbb{Z}$. We define the *E^∞-term* of the spectral sequence to be the (bi-)graded group

$$E^\infty = (E_s^\infty)_s = E_{*,*}^\infty$$

with

$$E_s^\infty = Z_s^\infty / B_s^\infty$$

for each $s \in \mathbb{Z}$.

REMARK 2.3.4. Since E_s^∞ is a subquotient of E_s^r for each r, we can think of the earlier terms in the spectral sequence as a shrinking sequence of majorizing bounds for the E^∞-term. ((ETC: In what sense do the E^r-terms converge to the E^∞-term? The formation of Z^∞ as the limit $\lim_r Z^r$ is not exact, and creates difficulties that are best managed by also introducing the right derived limit group $\mathrm{Rlim}_r\, Z^r$.))

PROOF OF LEMMA 1.2.3. Fix a bidegree (s,t). If $d^r_{s,t}$ and $d^r_{s+r,t-r+1}$ are both zero for all $r \ge q(s,t)$ then $Z^r_{s,t}/Z^{r+1}_{s,t} = 0$ and $B^{r+1}_{s,t}/B^r_{s,t} = 0$ by Lemma 2.3.1, so $Z^r_{s,t} = Z^{r+1}_{s,t} = Z^\infty_{s,t}$ and $B^r_{s,t} = B^{r+1}_{s,t} = B^\infty_{s,t}$ for all $r \ge q(s,t)$, and $E^r_{s,t} \cong E^{r+1}_{s,t} \cong E^\infty_{s,t}$ for all $r \ge q(s,t)$. □

LEMMA 2.3.5. *A morphism $\phi\colon E \to {}'E$ of spectral sequences induces compatible morphisms $\phi^r\colon Z^r \to {}'Z^r$ and $\phi^r\colon B^r \to {}'B^r$ for all $r \ge p$, including $r = \infty$. This also defines a morphism $\phi^\infty\colon E^\infty \to {}'E^\infty$.*

PROOF. By induction on $r \ge p$ we have vertical maps ϕ^r_s, as shown in the following commutative diagram.

$$\begin{tikzcd}
B^r_s \arrow[r, tail] \arrow[dd, "\phi^r_s"'] & B^{r+1}_s \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd, dotted, "\phi^{r+1}_s"'] & & Z^{r+1}_s \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd, dotted, "\phi^{r+1}_s"'] & & Z^r_s \arrow[dr, two heads] \arrow[dd, "\phi^r_s"'] & \\
& & \mathrm{im}(d^r)_s \arrow[rr, tail] \arrow[dd, "\phi^r_s"'] & & \ker(d^r)_s \arrow[rr, tail] \arrow[dd, "\phi^r_s"'] & & E^r_s \arrow[dd, "\phi^r_s"'] \\
{}'B^r_s \arrow[r, tail] & {}'B^{r+1}_s \arrow[rr, tail] \arrow[dr, two heads] & & {}'Z^{r+1}_s \arrow[rr, tail] \arrow[dr, two heads] & & {}'Z^r_s \arrow[dr, two heads] & \\
& & \mathrm{im}({}'d^r)_s \arrow[rr, tail] & & \ker({}'d^r)_s \arrow[rr, tail] & & {}'E^r_s
\end{tikzcd}$$

There are unique dotted maps ϕ^{r+1}_s making the whole diagram commute, because the lower parallelograms are pullbacks. Hence the maps $B^{r+1}_s \to {}'Z^r_s$ and $B^{r+1}_s \to \mathrm{im}({}'d^r)_s$ with equal composites to ${}'E^r_s$ admit a unique common lift to ${}'B^{r+1}_s$. Likewise, the maps $Z^{r+1}_s \to {}'Z^r_s$ and $Z^{r+1}_s \to \ker({}'d^r)_s$ with equal composites to ${}'E^r_s$ admit a unique common lift to ${}'Z^{r+1}_s$.

The map $\phi^\infty_s\colon Z^\infty_s \to {}'Z^\infty_s$ is then given by the intersection (= limit) of the maps $\psi^r_s\colon Z^r_s \to {}'Z^r_s$, and $\phi^\infty_s\colon B^\infty_s \to {}'B^\infty_s$ is given by the union (= colimit) of the maps $\psi^r_s\colon B^r_s \to {}'B^r_s$. The induced map of quotient groups is $\phi^\infty_s\colon E^\infty_s \to {}'E^\infty_s$. □

PROOF OF LEMMA 1.2.4. Fix a bidegree (s,t). If $(E^r_{s,t})_r$ and $({}'E^r_{s,t})_r$ both stabilize for $r \ge q = q(s,t)$, then $Z^\infty_s = Z^r_s$, $B^r_s = B^\infty_s$, ${}'Z^\infty_s = {}'Z^r_s$ and ${}'B^r_s = {}'B^\infty_s$ for $r \ge q$, hence $\phi^r_s = \phi^\infty_s$ as maps of infinite cycles, infinite boundaries and E^∞-terms. □

The E^∞-term does not depend on where we start indexing the spectral sequence.

LEMMA 2.3.6. *Let $(E^r, d^r)_{r \ge p}$ be an E^p-spectral sequence, let $q \ge p$, and let $({}'E^r, {}'d^r)_{r \ge q}$ be the E^q-spectral sequence with $E^r = {}'E^r$ and $d^r = {}'d^r$ for $r \ge q$. Then there is a canonical isomorphism*

$$E^\infty \cong {}'E^\infty\,.$$

PROOF. The sequence

$$0 = {}'B^q_s \subset \dots \subset {}'B^r_s \subset {}'B^{r+1}_s \subset \dots \subset {}'Z^{r+1}_s \subset {}'Z^r_s \subset \dots \subset {}'Z^q_s = {}'E^q_s$$

equals

$$0 = B^q_s/B^q_s \subset \cdots \subset B^r_s/B^q_s \subset B^{r+1}_s/B^q_s \subset \dots$$
$$\cdots \subset Z^{r+1}_s/B^q_s \subset Z^r_s/B^q_s \subset \cdots \subset Z^q_s/B^q_s = E^q_s$$

so

$$'Z^\infty_s = \bigcap_r Z^r_s/B^q_s \cong Z^\infty_s/B^q_s$$
$$'B^\infty_s = \bigcup_r B^r_s/B^q_s \cong B^\infty_s/B^q_s$$

and

$$'E^\infty_s \cong \frac{Z^\infty_s/B^q_s}{B^\infty_s/B^q_s} \cong E^\infty_s\,.$$

□

REMARK 2.3.7. The only slightly tricky step here is the commutation of quotients (which are colimits) and intersections (which are limits), giving the isomorphism

$$\kappa\colon (\bigcap_r Z^r_s)/B^q_s \xrightarrow{\cong} \bigcap_r (Z^r_s/B^q_s)\,.$$

((ETC: Under what hypotheses does this hold in an abelian category?))

The following result allows us to make deductions about a morphism between two spectral sequences, even if we are not able to calculate all of their differentials.

PROPOSITION 2.3.8. *Let $\phi\colon (E^r, d^r)_{r\ge p} \to ('E^r, 'd^r)_{r\ge p}$ be a morphism of E^p-spectral sequences. Suppose that there is a $q < \infty$ such that*

$$\phi^q\colon E^q_{*,*} \xrightarrow{\cong} 'E^q_{*,*}$$

is an isomorphism. Then

$$\phi^r\colon E^r_{*,*} \xrightarrow{\cong} 'E^r_{*,*}$$

is an isomorphism for all $r \ge q$, including $r = \infty$.

PROOF. Ignoring the E^r-terms for $r < q$, we may assume that $p = q$ and that $\phi^p\colon E^p \to 'E^p$ is an isomorphism. It then follows for each $r \ge p$, by induction, that $\phi^r\colon E^r \to 'E^r$, $\phi^r\colon \ker(d^r) \to \ker('d^r)$ and $\phi^r\colon \operatorname{im}(d^r) \to \operatorname{im}('d^r)$ are isomorphisms, in view of the commutative diagrams

$$\begin{tikzcd} E^r \arrow[r, "\phi^r", "\cong"'] \arrow[d, "d^r"'] & 'E^r \arrow[d, "'d^r"] \\ E^r \arrow[r, "\phi^r", "\cong"'] & 'E^r \end{tikzcd}$$

and

$$\begin{tikzcd} H(E^r, d^r) \arrow[r, "\phi^r_*", "\cong"'] \arrow[d, "\cong"'] & H('E^r, 'd^r) \arrow[d, "\cong"] \\ E^{r+1} \arrow[r, "\phi^{r+1}"] & 'E^{r+1}\,. \end{tikzcd}$$

Since $\ker(d^r) = Z^{r+1}/B^r$ and $\mathrm{im}(d^r) = B^{r+1}/B^r$ with $0 = B^p \subset Z^p = E^p$, and likewise for $'d^r$, it follows that

$$\phi^r \colon Z^r \xrightarrow{\cong} {}'Z^r$$

$$\phi^r \colon B^r \xrightarrow{\cong} {}'B^r$$

are isomorphisms for all $r \ge p$. Passing to intersections and unions, we deduce that

$$\phi^\infty \colon Z^\infty \xrightarrow{\cong} {}'Z^\infty$$

$$\phi^\infty \colon B^\infty \xrightarrow{\cong} {}'B^\infty$$

are isomorphisms, which implies that $\phi^\infty \colon E^\infty \to {}'E^\infty$ is an isomorphism, as claimed. □

Remark 2.3.9. This proposition shows that if $\phi \colon (A, E) \to ('A, 'E)$ is a morphism of exact couples such that $\phi \colon E \to {}'E$ is an isomorphism, then the induced morphism of E^1-spectral sequences $\phi \colon (E^r, d^r) \to ('E^r, 'd^r)$ is an isomorphism. This may well happen even if $\phi \colon A \to {}'A$ is not an isomorphism, so different exact couples may give rise to the same spectral sequence. ((ETC: Forward reference to examples.))

((ETC: One can weaken the hypothesis that $\phi^q_{s,t}$ is an isomorphism to allow q to vary with s and t, but subject to further compatibility conditions. How can these conditions be formulated?))

2.4. Discrete and exhaustive convergence

We now generalize Definition 1.2.8, by weakening the bounded above condition.

Definition 2.4.1. A filtration $(F_s G_*)_s$ of a graded abelian group G_* is *exhaustive* if

$$\bigcup_s F_s G_* = G_* \,.$$

It is *discrete* if there is an integer a such that $F_{a-1} G_* = 0$, and it is *degreewise discrete* if for each total degree n there is an integer $a = a(n)$ such that $F_{a-1} G_n = 0$.

Remark 2.4.2. We might say "bounded below" in place of "discrete", but this may become confusing when we also discuss decreasing filtrations. The terminology "degreewise discrete" is suggested by thinking of the subgroups $F_s G_n$ for $s \in \mathbb{Z}$ as forming a neighborhood basis of the origin for a linear topology on G_n. The cosets $x + F_s G_n$ for $s \in \mathbb{Z}$ then form a neighborhood basis around x. The resulting topology is discrete if and only if $F_s G_n = 0$ for some s. ((ETC: This is related to, but not equal to, the condition on a filtered chain complex called "regularity" in [**CE56**, §XV.4].)) Later, we shall define "Hausdorff" and "complete" filtrations, in such a way that the algebraic terminology matches that from point set topology.

Definition 2.4.3. Let $(E^r_{*,*}, d^r)_r$ be a spectral sequence and let $(F_s G_*)_s$ be a filtration of a graded abelian group G_*. Suppose that the filtration is exhaustive and degreewise discrete. Then we say that the spectral sequence *converges* to G_*, written

$$E^r_{*,*} \Longrightarrow G_* \,,$$

if there are isomorphisms

$$E^{\infty}_{s,t} \cong \frac{F_s G_{s+t}}{F_{s-1} G_{s+t}}$$

for all (s, t).

REMARK 2.4.4. The following isomorphism result is often used in conjunction with Proposition 2.3.8 to show that a map of spectral sequences can be used to establish an isomorphism $G_* \cong {}'G_*$, even if we do not know enough about the differentials d^r and ${}'d^r$ in these spectral sequences to actually calculate their abutments.

THEOREM 2.4.5. *Let $\phi\colon (E^r, d^r)_{r\geq p} \to ({}'E^r, {}'d^r)_{r\geq p}$ be a morphism of E^p-spectral sequences, converging to a morphism $\psi\colon G_* \to {}'G_*$ of filtered graded abelian groups. Suppose that each filtration is degreewise discrete and exhaustive, and suppose that*

$$\phi^{\infty}\colon E^{\infty}_{*,*} \xrightarrow{\cong} {}'E^{\infty}_{*,*}$$

is an isomorphism. Then

$$\psi\colon G_* \xrightarrow{\cong} {}'G_*$$

is an isomorphism.

PROOF. Fix a total degree n. We prove for each s, by induction, that

$$\psi_s\colon F_s G_n \longrightarrow F_s{}'G_n$$

is an isomorphism. The assumption that the filtrations $(F_s G_*)_s$ and $(F_s{}'G_*)_s$ are degreewise discrete ensures that the (ungraded) filtrations $(F_s G_n)_s$ and $(F_s{}'G_n)_s$ are discrete, so that there is an integer a with $F_{a-1} G_n = 0$ and $F_{a-1}{}'G_n = 0$. Hence ψ_{a-1} is trivially an isomorphism. Consider the vertical map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_{s-1} G_n \arrow[r] \arrow[d, "\psi_{s-1}"'] & F_s G_n \arrow[r] \arrow[d, "\psi_s"'] & \dfrac{F_s G_n}{F_{s-1} G_n} \arrow[r] \arrow[d, "\bar\psi_s"'] & 0 \\
0 \arrow[r] & F_{s-1}{}'G_n \arrow[r] & F_s{}'G_n \arrow[r] & \dfrac{F_s{}'G_n}{F_{s-1}{}'G_n} \arrow[r] & 0\,.
\end{tikzcd}$$

We may assume, by induction starting with $s = a$, that ψ_{s-1} is an isomorphism. Furthermore, by convergence, the commutative diagram

$$\begin{tikzcd}
E^{\infty}_{s,n-s} \arrow[r, "\phi^{\infty}", "\cong"'] \arrow[d, "\cong"'] & {}'E^{\infty}_{s,n-s} \arrow[d, "\cong"] \\
\dfrac{F_s G_n}{F_{s-1} G_n} \arrow[r, "\bar\psi_s"] & \dfrac{F_s{}'G_n}{F_{s-1}{}'G_n}
\end{tikzcd}$$

and the assumption that ϕ^{∞} is an isomorphism, we know that $\bar\psi_s$ is an isomorphism. It then follows (by a very special case of the snake lemma) that ψ_s is an isomorphism.

To complete the proof we use that both filtrations are exhaustive to pass to unions over s and conclude that

$$\psi\colon G_n = \bigcup_s F_s G_n \xrightarrow{\cong} \bigcup_s F_s{}' G_n = {}' G_n$$

is an isomorphism. $\square$

2.5. Discrete convergence for exact couples

We return to the setting of the spectral sequence (E^r, d^r) associated to an exact couple (A, E), where we assume that each α_s preserves the total degree. We will show that if the sequence of graded abelian groups

$$\dots \to A_{s-2} \xrightarrow{\alpha_{s-1}} A_{s-1} \xrightarrow{\alpha_s} A_s \xrightarrow{\alpha_{s+1}} A_{s+1} \to \dots \tag{2.1}$$

is (degreewise) discrete, then the spectral sequence converges (strongly) to the colimit

$$A_\infty = \operatorname*{colim}_s A_s$$

of this sequence. In a later section ((ETC: where?)) we will discuss what happens when the sequence is not discrete.

Definition 2.5.1. The sequence (2.1) is *discrete* if there is an integer a such that $A_s = 0$ for all $s < a$. More generally, it is *degreewise discrete* if for each total degree n there is an integer $a(n)$ such that $(A_s)_n = 0$ for all $s < a(n)$.

Definition 2.5.2. The *colimit* $A_\infty = \operatorname{colim}_s A_s$ of the sequence (2.1) is the initial graded abelian group that receives compatible structure morphisms

$$i_s\colon A_s \longrightarrow A_\infty$$

for each $s \in \mathbb{Z}$. Explicitly,

$$A_\infty = \bigoplus_s A_s/(\sim)$$

where $\sim$ identifies $x \in A_{s-1}$ with $\alpha_s(x) \in A_s$, for all s.

Remark 2.5.3. Hatcher [**Hat02**, p. 243] writes "direct limit" for what we call directed colimits, of which the sequential colimit formed here is a special case. See Mac Lane's book [**ML71**, Ch. III] for the categorical context behind algebraic colimits and limits. By "compatible" we mean that $i_s\alpha_s = i_{s-1}$ for each s. By "initial" we mean that for any other graded abelian group B with compatible homomorphisms $j_s\colon A_s \to B$ there exists a unique homomorphism $j\colon A_\infty \to B$ such that $j_s = j i_s$ for each s. This characterizes A_∞, with the structure morphisms i_s, up to unique isomorphism.

$$\begin{tikzcd}
 & & & A_\infty \arrow[dd, dotted, "j"] \\
\dots \arrow[r] & A_{s-1} \arrow[r, "\alpha_s"] \arrow[urr, bend left, "i_{s-1}"] \arrow[drr, bend right, "j_{s-1}"'] & A_s \arrow[r] \arrow[ur, "i_s"] \arrow[dr, "j_s"'] & \dots \\
 & & & B
\end{tikzcd}$$

LEMMA 2.5.4. *Each element $y \in A_\infty$ is of the form $y = i_s(x)$ for some $s \in \mathbb{Z}$ and $x \in A_s$. An element $x \in A_s$ maps to zero in A_∞, meaning that $i_s(x) = 0$, only if there is some $u \ge 0$ with $\alpha_{s+u} \cdots \alpha_{s+1}(x) = 0$ in A_{s+u}.*

PROOF. ((ETC)) □

LEMMA 2.5.5. *There is a short exact sequence*

$$0 \to \bigoplus_s A_s \xrightarrow{1-\alpha} \bigoplus_s A_s \xrightarrow{\pi} A_\infty \to 0 \,,$$

where 1 denotes the identity map and

$$\alpha \colon (x_s)_s \longmapsto (\alpha_s(x_{s-1}))_s$$

for each sequence $(x_s)_s$ with only finitely many nonzero terms.

PROOF. In view of the explicit formula for $A_\infty = \operatorname{colim}_s A_s$, we only need to argue that $1-\alpha$ is injective. Let $x = (x_s)_s \in \bigoplus_s A_s$, and choose a such that $x_s = 0$ for all $s < a$. If $(1-\alpha)(x) = 0$ then $x_s = \alpha_s(x_{s-1})$ for all s. It follows by induction on s, starting with $s = a$, that $x_s = 0$ for all s. Hence $x = 0$. □

DEFINITION 2.5.6. For $s \in \mathbb{Z}$ let

$$F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty) \,.$$

This defines an increasing filtration

$$\cdots \subset F_{s-1}A_\infty \subset F_s A_\infty \subset \cdots \subset A_\infty$$

of graded abelian groups.

LEMMA 2.5.7. *The filtration of $A_\infty = \operatorname{colim}_s A_s$ by $F_s A_\infty = \operatorname{im}(i_s \colon A_s \to A_\infty)$ is exhaustive.*

PROOF. Each $y \in A_\infty$ has the form $y = i_s(x)$ for some $x \in A_s$, and then $y \in F_s A_\infty$. Hence $\bigcup_s F_s A_\infty = A_\infty$. □

LEMMA 2.5.8. *Consider an exact couple (A, E) such that the sequence (2.1) is degreewise discrete. Then*

$$Z_s^\infty = \ker(\gamma_s)$$

for each s, and the filtration $(F_s A_\infty)_s$ is degreewise discrete.

PROOF. We always have $\ker(\gamma_s) \subset Z_s^\infty$. If $x \in Z_s^\infty$ then $x \in Z_s^r$ for each r, so $\gamma_s(x) \in A_{s-1}$ lies in the image of $\alpha^{r-1} \colon A_{s-r} \to A_{s-1}$ for each r. Let n be the total degree of $\gamma_s(x)$. By assumption there is an $a(n)$ such that $(A_{s-r})_n = 0$ whenever $s - r < a(n)$. It follows that the image of $(A_{s-r})_n$ in $(A_{s-1})_n$ is trivial for all sufficiently large r, which means that $\gamma_s(x) = 0$. Hence $x \in \ker(\gamma_s)$.

If $(A_s)_n = 0$ for all $s < a(n)$ then $(F_s A_\infty)_n = 0$ for $s < a(n)$, so the filtration is degreewise discrete whenever the sequence is. □

LEMMA 2.5.9. *Let (A, E) be any exact couple, and set $A_\infty = \operatorname{colim}_s A_s$. Then*

$$B_s^\infty = \beta_s \ker(i_s \colon A_s \to A_\infty)$$

for each s.

PROOF. It is elementary that B_s^∞ equals

$$\bigcup_r B_s^r = \bigcup_r \beta_s \ker(\alpha^{r-1}\colon A_s \to A_{s+r-1}) = \beta_s \bigcup_r \ker(\alpha^{r-1}\colon A_s \to A_{s+r-1})\,,$$

which equals

$$\beta_s \ker(i_s\colon A_s \to A_\infty)$$

by Lemma 2.5.4, since $x \in A_s$ maps to zero under some α^{r-1} if and only if it maps to zero under i_s. □

LEMMA 2.5.10. *Let (A, E) be any exact couple, and filter $A_\infty = \operatorname{colim}_s A_s$ by $F_s A_\infty = \operatorname{im}(i_s\colon A_s \to A_\infty)$. There is a preferred isomorphism*

$$\frac{\ker(\gamma_s)}{\beta_s \ker(i_s\colon A_s \to A_\infty)} \cong \frac{F_s A_\infty}{F_{s-1} A_\infty}$$

for each $s \in \mathbb{Z}$.

PROOF.

$$\frac{\ker(\gamma_s)}{\beta_s \ker(i_s\colon A_s \to A_\infty)} = \frac{\operatorname{im}(\beta_s)}{\beta_s \ker(i_s\colon A_s \to A_\infty)}$$

receives an isomorphism induced by β_s from

$$\frac{A_s}{\ker(\beta_s) + \ker(i_s\colon A_s \to A_\infty)} = \frac{A_s}{\operatorname{im}(\alpha_s) + \ker(i_s\colon A_s \to A_\infty)}$$

which maps isomorphically by i_s to

$$\frac{\operatorname{im}(i_s\colon A_s \to A_\infty)}{i_s \operatorname{im}(\alpha_s)} = \frac{\operatorname{im}(i_s\colon A_s \to A_\infty)}{\operatorname{im}(i_{s-1}\colon A_{s-1} \to A_\infty)} = \frac{F_s A_\infty}{F_{s-1} A_\infty}\,.$$

□

PROPOSITION 2.5.11. *Let (A, E) be an exact couple with associated spectral sequence (E^r, d^r) and E^∞-term $(E_s^\infty)_s$. Let $A_\infty = \operatorname{colim}_s A_s$ be filtered by $F_s A_\infty = \operatorname{im}(i_s\colon A_s \to A_\infty)$.*

(1) There is always a preferred injective homomorphism

$$\frac{F_s A_\infty}{F_{s-1} A_\infty} \overset{\zeta}{\rightarrowtail} E_s^\infty\,,$$

which is an isomorphism if $Z_s^\infty = \ker(\gamma_s)$.

(2) In particular, if each α_s preserves total degree and the sequence

$$\dots \to A_{s-1} \xrightarrow{\alpha_s} A_s \to \dots$$

is degreewise discrete, then ζ is an isomorphism and the spectral sequence

$$E_s^r \Longrightarrow_s A_\infty$$

converges.

PROOF. This summarizes the previous four lemmas, keeping in mind that we always have the inclusion $\ker(\gamma_s) \subset Z_s^\infty$. □

For filtrations that are discrete, the notions of weak convergence, convergence and strong convergence coincide. We may therefore replace "convergence" with "strong convergence" in the definition and proposition above.

Example 2.5.12. For the first exact couple in Example 2.1.5, the colimit $A_\infty = \mathbb{Z}[1/p]$ is exhaustively filtered by $F_s A_\infty = p^{-s}\mathbb{Z} \subset \mathbb{Z}[1/p]$. For the second exact couple, $A_\infty = \mathbb{Z}_p[1/p] = \mathbb{Q}_p$ equals the group of p-adic (rational) numbers. It is exhaustively filtered by $F_s A_\infty = p^{-s}\mathbb{Z}_p \subset \mathbb{Z}_p[1/p]$. Neither of these filtrations are discrete. However, the second is "Hausdorff" and "complete", in a sense we will discuss in Chapter 8. In both cases the associated spectral sequence collapses at the E^1-term, since the connecting homomorphisms γ_s are all zero.

2.6. Derived exact couples

((ETC: Massey's alternative approach to the spectral sequence associated to an exact couple, given iteratively by deriving the exact couple.))

CHAPTER 3

Filtrations

We now consider how filtered chain complexes and filtered spaces give rise to exact couples, with associated spectral sequences.

3.1. Filtered chain complexes

DEFINITION 3.1.1. An *increasing filtration* $(F_sC_*)_s = (F_sC_*, \partial)_s$ of a chain complex $C_* = (C_*, \partial)$ is a sequence of subcomplexes

$$\cdots \subset (F_{s-1}C_*, \partial) \subset (F_sC_*, \partial) \subset \cdots \subset (C_*, \partial)\,.$$

For each $s \in \mathbb{Z}$ there is a short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \xrightarrow{i} F_sC_* \xrightarrow{j} \frac{F_sC_*}{F_{s-1}C_*} \to 0\,. \tag{3.1}$$

We refer to the grading of $C_* = (C_n)_n$, and of each subcomplex $F_sC_* = (F_sC_n)_n$, as the *total degree*, while s is the *filtration degree*. We say that the filtration is *exhaustive* if

$$\bigcup_s F_sC_* = C_*\,.$$

It is *degreewise discrete* if for each degree n there is an integer $a = a(n)$ such that $F_{a-1}C_n = 0$.

DEFINITION 3.1.2. The exact couple $(A_s, E_s; \alpha_s, \beta_s, \gamma_s)_s$ associated to a filtered chain complex $(F_sC_*)_s$ is the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha_{s-1}"] & H_*(F_{s-1}C_*) \arrow[r, "\alpha_s"] & H_*(F_sC_*) \arrow[r, "\alpha_{s+1}"] \arrow[d, "\beta_s"] & \cdots \\
\cdots & & H_*(F_sC_*/F_{s-1}C_*) \arrow[ul, dashed, "\gamma_s"] & \cdots,
\end{tikzcd}$$

where

$$(A_s)_* = H_*(F_sC_*)$$
$$(E_s)_* = H_*(F_sC_*/F_{s-1}C_*)$$

with α_s and β_s induced by i and j, and γ_s equal to the connecting homomorphism associated to the short exact sequence (3.1).

More explicitly, the bigrading is given by

$$A_{s,t} = H_{s+t}(F_sC_*)$$
$$E_{s,t} = H_{s+t}(F_sC_*/F_{s-1}C_*)\,,$$

so that α_s has bidegree $(1, -1)$, β_s has bidegree $(0, 0)$ and γ_s has bidegree $(-1, 0)$. Thus α_s and β_s preserve the total degree $n = s + t$, while γ_s reduces it by 1.

DEFINITION 3.1.3. Given a filtration $(F_sC_*)_s$ of $C_* = (C_*, \partial)$, let

$$F_sH_*(C_*) = \mathrm{im}(H_*(F_sC_*) \to H_*(C_*))$$

for each s.

Note the two different roles played by the notation "F_s" in this definition. On the left hand side it refers to the filtration of the abutment $H_*(C_*)$, while on the right hand side it refers to the filtration of the chain complex (C_*, ∂).

LEMMA 3.1.4. *If $(F_sC_*)_s$ exhausts C_*, then the canonical morphism*

$$A_\infty = \operatorname*{colim}_s H_*(F_sC_*) \xrightarrow{\cong} H_*(C_*)$$

is an isomorphism, which restricts to isomorphisms

$$F_sA_\infty \cong F_sH_*(C_*)$$

for all s. If $(F_sC_*)_s$ is degreewise discrete, then the sequence

$$\dots \to A_{s-1} \xrightarrow{\alpha_s} A_s \to \dots$$

is degreewise discrete.

PROOF. The first claims follow from the well-known isomorphism

$$\operatorname*{colim}_s H_*(F_sC_*) \xrightarrow{\cong} H_*(\operatorname*{colim}_s F_sC_*)\,.$$

If $F_{a-1}C_n = 0$ for some n and $a = a(n)$, then $F_sC_n = 0$ and $H_n(F_sC_*) = (A_s)_n = 0$ for all $s < a$, which implies the last claim. □

LEMMA 3.1.5. *Each morphism $\psi\colon (F_sC_*)_s \to (F_s{}'C_*)_s$ of filtered chain complexes induces a morphism $\phi\colon (A, E) \to ('A, 'E)$ of exact couples. Hence the associated exact couple defines a functor*

$$\textit{Filtered Chain Complexes} \longrightarrow \textit{Exact Couples}\,.$$

PROOF. $\phi\colon A_s \to {}'A_s$ and $\phi\colon E_s \to {}'E_s$ are induced by the chain maps

$$\psi_s\colon F_sC_* \longrightarrow F_s{}'C_*$$

$$\bar\psi_s\colon \frac{F_sC_*}{F_{s-1}C_*} \longrightarrow \frac{F_s{}'C_*}{F_{s-1}{}'C_*}$$

by passage to homology. □

PROPOSITION 3.1.6. *Let $C_* = (C_*, \partial)$ be a chain complex with a filtration $(F_sC_*)_s = (F_sC_*, \partial)_s$ that is exhaustive and degreewise discrete. The associated spectral sequence has E^1-term*

$$E^1_{s,*} = H_*(F_sC_*/F_{s-1}C_*)$$

and d^1-differential $d^1_s\colon E^1_{s,*} \to E^1_{s-1,*}$ the composite

$$d^1_s = \beta_{s-1}\gamma_s\colon H_*(F_sC_*/F_{s-1}C_*) \longrightarrow H_{*-1}(F_{s-1}C_*) \longrightarrow H_{*-1}(F_{s-1}C_*/F_{s-2}C_*)\,,$$

which equals the connecting homomorphism associated to the short exact sequence

$$0 \to F_{s-1}C_*/F_{s-2}C_* \xrightarrow{i} F_sC_*/F_{s-2}C_* \xrightarrow{j} F_sC_*/F_{s-1}C_* \to 0$$

of chain complexes. The spectral sequence converges to $H_*(C_*)$, with the filtration given by

$$F_sH_*(C_*) = \mathrm{im}(H_*(F_sC_*) \to H_*(C_*))\,.$$

PROOF. The spectral sequence is the one associated in Theorem 2.2.1 to the exact couple of Definition 3.1.2. The (strong) convergence follows from Lemma 3.1.4 and Proposition 2.5.11. □

COROLLARY 3.1.7. *Let* C_* *and* $'C_*$ *be chain complexes, with filtrations* $(F_sC_*)_s$ *and* $(F_s{}'C_*)_s$ *that are exhaustive and degreewise discrete. Let* $\psi\colon C_* \to {}'C_*$ *be a filtration-preserving map of filtered chain complexes, and suppose that the induced map*

$$\phi^r : E^r \longrightarrow {}'E^r ,$$

of E^r*-terms of the associated homology spectral sequences, is an isomorphism for some* $r = q$*. Then*

$$\psi_* : H_*(C_*) \longrightarrow H_*('C_*)$$

is an isomorphism.

3.2. Filtered spaces

Recall Example 2.1.4. The following terminology from Neil Strickland's note [**Str**, Def. 3.4] may not be standard, but is useful.

DEFINITION 3.2.1. A space X is *strongly filtered* by a sequence of subspaces

$$\dots \subset X_{s-1} \subset X_s \subset \dots \subset X$$

if for each compact subset $K \subset X$ there is an s with $K \subset X_s$.

LEMMA 3.2.2. *If* X *is strongly filtered by* $(X_s)_s$*, then the singular chain complex* $(C_*(X), \partial)$ *is exhaustively filtered by the subcomplexes*

$$\dots \subset C_*(X_{s-1}) \subset C_*(X_s) \subset \dots \subset C_*(X) .$$

If $X_{a-1} = \emptyset$ *for some* a*, then the filtration* $(C_*(X_s))_s$ *is discrete.*

PROOF. The only thing to prove is that each singular simplex $\sigma\colon \Delta^n \to X$, viewed as an element of $C_n(X)$, lies in the image from some $C_n(X_s)$. Since the image $\sigma(\Delta^n) \subset X$ is compact, this follows from the assumption that the filtration is strong. □

In this situation, concerning singular complexes of topological spaces, there are no examples where $(C_*(X_s))_s$ is degreewise discrete but not (uniformly) discrete. The two- and three-column spectral sequences from Sections 1.3 and 1.4 are now special cases of the following result.

PROPOSITION 3.2.3. *If* X *is strongly filtered by* $(X_s)_s$*, then the associated homology spectral sequence*

$$E^r_{s,*} \Longrightarrow_s H_*(X)$$

has E^1*-term*

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$$

and the differential d^1_s *equals the connecting homomorphism in the long exact sequence of the triple* (X_s, X_{s-1}, X_{s-2})*. If* $X_{a-1} = \emptyset$*, then* $E^1_s = 0$ *for all* $s < a$*, and the spectral sequence converges to* $H_*(X)$ *with the filtration*

$$F_sH_*(X) = \operatorname{im}(H_*(X_s) \to H_*(X)) .$$

PROOF. Combine Proposition 3.1.6 with Lemma 3.2.2. □

REMARK 3.2.4. The convergence statement tells us that there is an exhaustive filtration

$$0 = F_{a-1}H_n(X) \subset \cdots \subset F_{s-1}H_n(X) \subset F_sH_n(X) \subset \cdots \subset H_n(X)$$

in each total degree n, with filtration quotients determined by the E^∞-term, through isomorphisms

$$E^\infty_{s,n-s} \cong \frac{F_sH_n(X)}{F_{s-1}H_n(X)}$$

for all s. Hence the components of $E^\infty_{*,*}$ in bidegrees $(s, n-s)$, on a line of slope -1, give the associated graded of this exhaustive filtration. By induction on s, starting at $s = a$, we can thus attempt to determine $F_sH_n(X)$ as an extension of $E^\infty_{s,n-s}$ by $F_{s-1}H_n(X)$. The union of these groups, over all s, then gives us $H_n(X)$.

((ETC: Draw a generic chart for this?))

Many strongly filtered spaces are of the following form.

LEMMA 3.2.5 ([**Ste67**, Lem. 9.3]). *Let X be filtered by an exhaustive sequence of T_1 subspaces*

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

such that X_{s-1} is closed in X_s for each s, and suppose that X has the weak (= colimit) topology. Then X is strongly filtered by these $(X_s)_s$.

PROOF. We have $X = \bigcup_s X_s$ since the filtration is exhaustive. To be a T_1 space is equivalent to asking that each singleton subset is closed. This is satisfied by all (weak) Hausdorff spaces. The weak topology on X is defined so that a subset $A \subset X$ is closed in X if and only if $A \cap X_s$ is closed in X_s for each s.

Following Steenrod, we argue that if $K \subset X$ is compact, then $K \subset X_s$ for some s. If not, we can choose a point $x_s \in K \cap (X - X_s)$ for each s. Let

$$A_m = \{x_s \mid s \geq m\} \subset K \cap (X - X_m),$$

so that

$$\cdots \supset A_{m-1} \supset A_m \supset \dots$$

is a collection of subsets of K, such that each finite subcollection has nonempty intersection $A_{m_1} \cap \cdots \cap A_{m_n} = A_m$ (with $m = \max\{m_1, \dots, m_n\}$), but the whole collection satisfies $\bigcap_m A_m = \emptyset$. If we show that each A_m is closed in K, then this contradicts the finite intersection property of compact spaces, and proves that $K \subset X_s$ for some s. To see that each A_m is closed, note that each intersection $A_m \cap X_s \subset \{x_m, \dots, x_{s-1}\}$ is finite, hence is closed in X_s since this is a T_1 space. By the definition of the weak topology this proves that A_m is closed in X, hence also in the subspace K. □

The cellular complex $(C^{CW}_*(X), \partial)$ calculating the homology of a CW complex X is a very special case of this spectral sequence. Other notations for the cellular complex are $\Gamma_*(X)$, as in [**Whi78**, §II.2], or $W_*(X)$. Let us write $H^{CW}_n(X) = H_n(C^{CW}_*(X), \partial)$ for the cellular homology groups. The usual argument for why cellular homology is isomorphic to singular homology [**Whi78**, Thm. II.2.19], [**Hat02**, Thm. 2.35], is contained within our more elaborate algebraic work, as we can now spell out.

PROPOSITION 3.2.6. *Let X be a CW complex, with skeleton filtration*

$$\emptyset = X^{(-1)} \subset \cdots \subset X^{(s-1)} \subset X^{(s)} \subset \cdots \subset X\,.$$

The associated homology spectral sequence has $(E^1, d^1) = (C_*^{CW}(X), \partial)$, concentrated on the line $t = 0$. Hence

$$E^2_{s,t} = \begin{cases} H_s^{CW}(X) & \text{for } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the spectral sequence collapses at $E^2 = E^\infty$. The filtration of $H_n(X)$ satisfies

$$F_s H_n(X) = \begin{cases} 0 & \text{for } s < n, \\ H_n^{CW}(X) & \text{for } s \geq n. \end{cases}$$

Hence $H_*^{CW}(X) \cong H_*(X)$.

PROOF. The CW complex X is strongly filtered by its skeleta. By definition, $E^1_{s,t} = H_{s+t}(X^{(s)}, X^{(s-1)})$ equals

$$C_s^{CW}(X) \cong \mathbb{Z}\{n\text{-cells of } X\}$$

when $t = 0$, and is trivial when $t \neq 0$. Likewise, $d^1_{s,t} = \partial_s$ when $t = 0$ and is zero otherwise.

$$\begin{array}{c|ccccccc}
2 & 0 & & \cdots & & 0 & & 0 & \cdots \\
1 & 0 & & \cdots & & 0 & & 0 & \cdots \\
0 & C_0^{CW}(X) & \xleftarrow{\partial_1} & \cdots & \xleftarrow{\partial_{s-1}} & C_{s-1}^{CW}(X) & \xleftarrow{\partial_s} & C_s^{CW}(X) & \xleftarrow{\partial_{s+1}} \cdots \\
\hline
t/s & 0 & & \cdots & & s-1 & & s & \cdots
\end{array}$$

Hence $E^2_{s,t} = H_s(C_*^{CW}(X), \partial) = H_s^{CW}(X)$ equals the cellular homology of X when $t = 0$, and is trivial otherwise. Each d^r-differential for $r \geq 2$ increases t, hence must be zero, so $E^2 = E^\infty$. In each total degree n there is only one nonzero group of the form $E^\infty_{s,n-s}$, namely $E^\infty_{n,0} = E^2_{n,0} = H_n^{CW}(X)$. The short exact sequences

$$0 \to F_{s-1}H_n(X) \longrightarrow F_s H_n(X) \longrightarrow E^\infty_{s,n-s} \to 0$$

for $s < n$ simplify to

$$0 \to 0 \longrightarrow F_s H_n(X) \longrightarrow 0 \to 0$$

and imply that $F_s H_n(X) = 0$ for $s < n$ by induction on s. The short exact sequence sequence for $s = n$ simplifies to an isomorphism

$$0 \to 0 \longrightarrow F_n H_n(X) \xrightarrow{\cong} H_n^{CW}(X) \to 0\,.$$

Thereafter, for $s > n$ they simplify to isomorphisms

$$0 \to H_n^{CW}(X) \xrightarrow{\cong} F_s H_n(X) \longrightarrow 0 \to 0\,.$$

Hence $F_s H_n(X) \cong H_n^{CW}(X)$ for $s > n$. □

3.3. The Atiyah–Hirzebruch spectral sequence

Let G be an abelian group. Singular homology with coefficients in G is an example of a homology theory, sometimes referred to as "ordinary" homology. Since ca. 1960 many other "generalized" or "extraordinary" homology theories have come to play important roles in algebraic topology. The following definition is close to the axiomatization by Samuel Eilenberg and Norman Steenrod from [**ES52**, §I.3], but omits their dimension axiom and adds John Milnor's additivity axiom [**Mil62**].

Definition 3.3.1. A *(generalized) homology theory M* on the category of CW pairs is a functor assigning to each CW pair (X, A) a graded abelian group

$$M_*(X, A) = (M_n(X, A))_n \,,$$

and a natural transformation

$$\partial \colon M_*(X, A) \longrightarrow M_{*-1}(A)$$

of degree -1, such that

(1) Exactness: the sequence

$$\dots \to M_*(A) \xrightarrow{i_*} M_*(X) \xrightarrow{j_*} M_*(X, A) \xrightarrow{\partial} M_{*-1}(A) \to \dots$$

is long exact.

(2) Homotopy invariance: if $f \simeq g \colon (X, A) \to (Y, B)$ are homotopic, then $f_* = g_*$.

(3) Excision: if $X = A \cup B$ is a union of subcomplexes, then the inclusion induces an isomorphism

$$M_*(B, A \cap B) \xrightarrow{\cong} M_*(X, A) \,.$$

(4) Additivity: the canonical map

$$\bigoplus_\alpha M_*(X_\alpha) \xrightarrow{\cong} M_*(\coprod_\alpha X_\alpha)$$

is an isomorphism.

Definition 3.3.2. The *coefficient groups* of a homology theory M is the graded abelian group

$$M_* = (M_n(\text{point}))_n \,.$$

We say that M_* is *bounded below* if there is an a such that $M_n = 0$ for all $n < a$. We say that M_* is *bounded above* if there is a b such that $M_n = 0$ for all $n > b$.

Example 3.3.3. Let G be an abelian group. The coefficient groups of ordinary homology with coefficients in G, i.e., the homology theory HG given by

$$HG_n(X) = H_n(X; G)$$

for all n, equals G in degree 0 and 0 in all other degrees. This is the content of the Eilenberg–Steenrod dimension axiom.

Lemma 3.3.4. *For any homology theory M there are isomorphisms*

$$M_{s+t}(D^s, \partial D^s) \cong \tilde{M}_{s+t}(S^s) \cong M_t$$

for all $s \ge 0$, $t \in \mathbb{Z}$.

Proof. This is clear for $s = 0$, and follows by induction for $s \ge 1$. ☐

REMARK 3.3.5. For any graded abelian group G_* there is a generalized homology theory with $M_n(X) = \bigoplus_{i+j=n} H_i(X; G_j)$, but it carries more-or-less the same information as ordinary homology. Other important examples of (co-)homology theories include the topological K-theories $KO^*(X)$ and $K^*(X) = KU^*(X)$ defined by Michael Atiyah and Friedrich Hirzebruch [**AH59**], following Alexander Grothendick [**BS58**], and the bordism theories $N_*(X) = MO_*(X)$ and $\Omega_*(X) = MSO_*(X)$ defined by Atiyah [**Ati61a**], building on the work of René Thom [**Tho54**]. By construction, these involve vector bundles over X and closed manifolds mapping to X, respectively, rather than simplices in X, and often turn out to emphasize different information than the ordinary homology of X. We will later ((ETC: where?)) present generalized (co-)homology theories by the objects, called spectra, of a stable (homotopy) category, and analyze the coefficient groups (and rings) of some of these homology theories.

DEFINITION 3.3.6. Let X be a CW complex exhaustively filtered by subcomplexes

$$\dots \subset X_{s-1} \subset X_s \subset \dots \subset X \,,$$

and let M be a homology theory. The associated exact couple is the diagram

$$\begin{tikzcd} \dots \arrow[r] & M_*(X_{s-1}) \arrow[r, "i_*"] & M_*(X_s) \arrow[r] \arrow[d, "j_*"] & \dots \\ & & M_*(X_s, X_{s-1}) \arrow[ul, dashed, "\partial"] & \end{tikzcd}$$

with

$$(A_s)_* = M_*(X_s)$$
$$(E_s)_* = M_*(X_s, X_{s-1})\,.$$

LEMMA 3.3.7 ([**Mil62**, Lem. 1]). *The canonical homomorphism*

$$\operatorname*{colim}_s M_*(X_s) \xrightarrow{\cong} M_*(X)$$

is an isomorphism.

PROOF. There is a homotopy cofiber sequence

$$\bigvee_s \Sigma_+ X_s \xrightarrow{1-\alpha} \bigvee_s \Sigma_+ X_s \longrightarrow \Sigma_+ T$$

where $\Sigma_+ Y = \Sigma(Y_+)$, and $T \simeq X$ is the mapping telescope of $(X_s)_s$. In view of Lemma 2.5.5 the associated long exact sequence in reduced M-homology breaks up into short exact sequences

$$0 \to \bigoplus_s M_*(X_s) \xrightarrow{1-\alpha} \bigoplus_s M_*(X_s) \longrightarrow M_*(T) \to 0$$

that exhibit $M_*(T)$ as $\operatorname{colim}_s M_*(X_s)$. □

PROPOSITION 3.3.8. *Let $(X_s)_s$ and M_* be as in Definition 3.3.6. The associated spectral sequence has*

$$E^1_{s,t} = M_{s+t}(X_s, X_{s-1})$$

and $d^1_{s,t}$ is equal to the composite

$$M_{s+t}(X_s, X_{s-1}) \xrightarrow{\partial} M_{s+t-1}(X_{s-1}) \xrightarrow{j_*} M_{s+t-1}(X_{s-1}, X_{s-2})\,.$$

If $X_{a-1} = \emptyset$ for some a, then the spectral sequence converges to $M_*(X)$ with the filtration

$$F_s M_*(X) = \operatorname{im}(M_*(X_s) \to M_*(X)) .$$

PROOF. This follows from Proposition 2.5.11. $\square$

When X is equipped with its skeleton filtration, we can make the E^1- and E^2-term explicit.

PROPOSITION 3.3.9. *Let X be a CW complex filtered by its skeleta*

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(s-1)} \subset X^{(s)} \subset \cdots \subset X ,$$

and let M be a homology theory. The associated spectral sequence

$$E^r_{s,*} \Longrightarrow_s M_*(X)$$

has (E^1, d^1)-term given by the cellular complex $(C^{CW}_*(X; M_*), \partial)$, with

$$E^1_{s,t} \cong C^{CW}_s(X; M_t) = H_s(X^{(s)}, X^{(s-1)}; M_t)$$

and $d^1_{s,t}$ equal to the connecting homomorphism

$$\partial_s \colon H_s(X^{(s)}, X^{(s-1)}; M_t) \longrightarrow H_{s-1}(X^{(s-1)}, X^{(s-2)}; M_t)$$

for homology with coefficients in the group M_t. Hence

$$E^2_{s,t} \cong H^{CW}_s(X; M_t) \cong H_s(X; M_t)$$

is given by the cellular (or singular) homology of X in degree s, with coefficients in M_t.

$$\begin{array}{c|ccccccccc}
 & \dots & & \dots & & \dots & & \dots & & \dots \\
t & C^{CW}_0(X; M_t) & \xleftarrow{\partial_1} & \dots & \xleftarrow{\partial_{s-1}} & C^{CW}_{s-1}(X; M_t) & \xleftarrow{\partial_s} & C^{CW}_s(X; M_t) & \xleftarrow{\partial_{s+1}} & \dots \\
 & \dots & & \dots & & \dots & & \dots & & \dots \\
0 & C^{CW}_0(X; M_0) & \xleftarrow{\partial_1} & \dots & \xleftarrow{\partial_{s-1}} & C^{CW}_{s-1}(X; M_0) & \xleftarrow{\partial_s} & C^{CW}_s(X; M_0) & \xleftarrow{\partial_{s+1}} & \dots \\
\hline
t/s & 0 & & & & s-1 & & s & &
\end{array}$$

PROOF. To identify the E^1-term we use the excision and additivity isomorphisms

$$E^1_{s,t} = M_{s+t}(X^{(s)}, X^{(s-1)}) \cong M_{s+t}(\coprod_\alpha (D^s, \partial D^s)) \cong \bigoplus_\alpha M_{s+t}(D^s, \partial D^s) ,$$

where α indexes the s-cells of X. By Lemma 3.3.4 the right hand side is isomorphic to

$$\bigoplus_\alpha M_t \cong C^{CW}_s(X; M_t) .$$

The degree formula for the connecting homomorphism ∂_s implies that $d^1_{s,t}$ corresponds to the cellular boundary homomorphism

$$\partial_s \colon C^{CW}_s(X; M_t) \longrightarrow C^{CW}_{s-1}(X; M_t) .$$

Granting this, we can pass to homology to deduce that

$$E^2_{s,t} \cong H^{CW}_s(X; M_t)\,.$$

By Proposition 3.2.6, and its evident analogue for homology with coefficients, we know that this cellular homology is isomorphic to singular homology with coefficients in M_t. □

DEFINITION 3.3.10. The spectral sequence

$$E^2_{s,t} = H_s(X; M_t) \Longrightarrow_s M_{s+t}(X)$$

is called the *Atiyah–Hirzebruch spectral sequence* of X for the homology theory M.

This spectral sequence can be defined for general spaces X by CW approximation. It is then natural in the homology theory M and in the space X.

COROLLARY 3.3.11. *If $\theta\colon M \to N$ is a morphism of homology theories that induces an isomorphism of coefficient groups, then $\theta_*\colon M_*(X) \cong N_*(X)$ for any CW complex X.*

PROOF. The natural transformation θ induces an isomorphism $C^{CW}_*(X; M_*) \cong C^{CW}_*(X; N_*)$ of Atiyah–Hirzebruch E^1-terms, which implies the result by Proposition 2.3.8 and Theorem 2.4.5. □

COROLLARY 3.3.12. *If $f\colon X \to Y$ induces an isomorphism $f_*\colon H_*(X) \cong H_*(Y)$ in integral homology, then it induces an isomorphism $f_*\colon M_*(X) \cong M_*(Y)$ for any generalized homology theory M.*

PROOF. The map f induces an isomorphism

$$H_*(X; M_*) \xrightarrow{\cong} H_*(Y; M_*)$$

of Atiyah–Hirzebruch E^2-terms, which implies the result by Proposition 2.3.8 and Theorem 2.4.5. □

The *Eilenberg–Steenrod uniqueness theorem* [**ES52**, Thm. III.10.1] also follows easily from this formalism. ((ETC: Also discuss compatibility of connecting homomorphisms?))

THEOREM 3.3.13. *Let G be an abelian group and let M be a homology theory with coefficient groups $M_0 = G$ and $M_t = 0$ for $t \neq 0$. Then M is naturally isomorphic to HG, so that*

$$M_n(X) \cong H_n(X; G)$$

for all n.

PROOF. The Atiyah–Hirzebruch spectral sequence of X for M has E^2-term

$$E^2_{s,t} = \begin{cases} H_s(X; G) & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since this is concentrated on the line $t = 0$, the d^r-differentials for $r \geq 2$ must vanish, so that $E^2 = E^\infty$ is concentrated on the line $t = 0$. Since $E^\infty_{n,0}$ is the only group in total degree n, the extension problems are very easy, and we conclude that $M_n(X) \cong E^\infty_{n,0} \cong H_n(X; G)$ for each n. □

According to Whitehead [**Whi78**, p. 604] the existence of the spectral sequence in Definition 3.3.10 was folklore by 1955, but Atiyah and Hirzebruch [**AH61**] were the first to make significant use of it, in the case of topological K-theory.

EXAMPLE 3.3.14. Complex K-theory is a (co-)homology theory $K = KU$ with coefficient groups

$$KU_n \cong \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

If $H_*(X)$ is concentrated in even degrees, it follows that the E^2-term of the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; KU_t) \Longrightarrow_s KU_{s+t}(X)$$

is concentrated in even total degrees $s+t$. Since each d^r-differential reduces the total degree by one, they must all vanish, so the Atiyah–Hirzebruch spectral sequence collapses at the E^2-term. If, furthermore, $H_*(X)$ is free in each degree, then there exists a (non-canonical) sum formula

$$KU_n(X) \cong \bigoplus_{s \equiv n \mod 2} H_s(X) \,,$$

since each extension

$$0 \to F_{s-1}KU_n(X) \longrightarrow F_sKU_n(X) \longrightarrow H_s(X; KU_{n-s}) \to 0$$

satisfies $H_s(X; KU_{n-s}) \cong H_s(X)$ for $n - s$ even and $H_s(X; KU_{n-s}) = 0$ for $n - s$ odd. This applies, for instance, when $X = \mathbb{C}P^\infty$.

3.4. Mapping cones and telescopes

((ETC: Also consider sequences of chain complexes or spaces, using mapping cones to form relative homology. Compare with mapping telescopes to discuss convergence.))

3.5. Cartan–Eilenberg systems

The exact couple associated to a filtration of chain complexes, or of spaces, is part of a larger web of exact sequences, which we call a *Cartan–Eilenberg system*. This structure was introduced in [**CE56**, §XV.7], and will be our formalism of choice when we construct products in spectral sequences in Chapter 5.

DEFINITION 3.5.1. A (homological) *finite Cartan–Eilenberg system* (H_*, η, ∂) consists of graded abelian groups

$$H_*(i, j)$$

for all integers $i \le j$, structure homomorphisms preserving degree

$$\eta \colon H_*(i, j) \longrightarrow H_*(i', j')$$

for all integers $i \le j$ and $i' \le j'$ with $i \le i'$ and $j \le j'$, and connecting homomorphisms reducing degree by 1

$$\partial \colon H_*(j, k) \longrightarrow H_{*-1}(i, j)$$

for all integers $i \le j \le k$. These must satisfy:

(1) Functoriality: $\eta\colon H_*(i,j) \to H_*(i,j)$ equals the identity, and

$$\eta \circ \eta\colon H_*(i,j) \to H_*(i',j') \to H_*(i'',j'')$$

equals $\eta\colon H_*(i,j) \to H_*(i'',j'')$ for all integers $i \le j$, $i' \le j'$ and $i'' \le j''$ with $i \le i' \le i''$ and $j \le j' \le j''$.

(2) Naturality: The diagrams

$$\begin{tikzcd} H_*(j,k) \arrow[r, "\partial"] \arrow[d, "\eta"'] & H_{*-1}(i,j) \arrow[d, "\eta"] \\ H_*(j',k') \arrow[r, "\partial"] & H_{*-1}(i',j') \end{tikzcd}$$

commute, for all integers $i \le j \le k$ and $i' \le j' \le k'$ with $i \le i'$, $j \le j'$ and $k \le k'$.

(3) Exactness: The sequence

$$\dots \xrightarrow{\partial} H_*(i,j) \xrightarrow{\eta} H_*(i,k) \xrightarrow{\eta} H_*(j,k) \xrightarrow{\partial} H_{*-1}(i,j) \xrightarrow{\eta} \dots$$

is exact, for all integers $i \le j \le k$.

Definition 3.5.2. By an *extended integer* we mean an element of

$$\{-\infty\} \cup \mathbb{Z} \cup \{\infty\},$$

linearly ordered with $-\infty$ minimal and ∞ maximal.

An *extended Cartan–Eilenberg system* (H_*, η, ∂) is defined as a finite Cartan–Eilenberg system, except that all references to "integers" are replaced with "extended integers", and subject to the following additional condition.

(4) Colimit: For each extended integer i the canonical homomorphism

$$\operatorname*{colim}_j H_*(i,j) \xrightarrow{\cong} H_*(i,\infty)$$

is an isomorphism.

Example 3.5.3. Let $(F_sC_*)_s$ be an increasing filtration of a chain complex C_*. We obtain a finite Cartan–Eilenberg system by setting

$$H_*(i,j) = H_*(F_jC_*/F_iC_*)$$

for integers $i \le j$, letting $\eta\colon H_*(i,j) \to H_*(i',j')$ be induced by the chain map $F_jC_*/F_iC_* \to F_{j'}C_*/F_{i'}C_*$, and setting $\partial\colon H_*(j,k) \to H_{*-1}(i,j)$ equal to the connecting homomorphism associated to the short exact sequence

$$0 \to F_jC_*/F_iC_* \longrightarrow F_kC_*/F_iC_* \longrightarrow F_kC_*/F_jC_* \to 0$$

of chain complexes.

Suppose also that the filtration exhausts C_*. Letting $F_{-\infty}C_* = 0$ and $F_\infty C_* = C_*$, the same expressions then define an extended Cartan–Eilenberg system. In particular $H_*(-\infty,s) = H_*(F_sC_*)$ and $H_*(-\infty,\infty) = H_*(C_*)$.

Example 3.5.4. Let $(X_s)_s$ be an increasing filtration of a space X. We obtain a finite Cartan–Eilenberg system by setting

$$H_*(i,j) = H_*(X_j, X_i)$$

for integers $i \le j$, letting $\eta\colon H_*(i,j) \to H_*(i',j')$ be induced by the inclusion $(X_j, X_i) \to (X_{j'}, X_{i'})$, and setting $\partial\colon H_*(j,k) \to H_{*-1}(i,j)$ equal to the connecting homomorphism in the long exact homology sequence for the triple (X_k, X_j, X_i).

Suppose also that X is strongly filtered. Letting $X_{-\infty} = \emptyset$ and $X_\infty = X$, the same expressions then define an extended Cartan–Eilenberg system. In particular $H_*(-\infty, s) = H_*(X_s)$ and $H_*(-\infty, \infty) = H_*(X)$.

Remark 3.5.5. It follows from exactness that $H_*(j, j) = 0$ for each j. We can visualize a finite Cartan–Eilenberg system as a triangular diagram with $H_*(i, j)$ in "matrix" position (i, j) (going i steps down and j steps to the right), and with a connecting homomorphism $\partial \colon H_*(j, k) \to H_*(i, j)$ for each rectangle with corners at (i, j), (i, k), (j, j) and (j, k).

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \\
i & 0 \arrow[r] & H_*(i,j) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & H_*(i,k) \arrow[r] \arrow[d] & \cdots \\
j & & 0 \arrow[r] & \cdots \arrow[r] \arrow[d] & H_*(j,k) \arrow[r] \arrow[d] \arrow[ull, dashed, "\partial"'] & \cdots \\
 & & & 0 \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \\
 & & & & 0 \arrow[r] & \cdots \\
 & & j & & k &
\end{tikzcd}$$

An extended Cartan–Eilenberg system is then augmented with a top row (for $i = -\infty$) and a right hand column (for $j = \infty$). The colimit condition specifies the right hand column in terms of the remainder of the diagram.

$$\begin{tikzcd}
-\infty & \dots \arrow[r]\arrow[d] & H_*(-\infty,j) \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & H_*(-\infty,k) \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & H_*(-\infty,\infty) \arrow[d] \\
 & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[d] \\
i & 0 \arrow[r] & H_*(i,j) \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & H_*(i,k) \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & H_*(i,\infty) \arrow[d] \\
j & & 0 \arrow[r] & \dots \arrow[r]\arrow[d] & H_*(j,k) \arrow[r]\arrow[d] \arrow[uuull, dashed, "\partial"'] \arrow[ull, dashed, "\partial"] & \dots \arrow[r]\arrow[d] & H_*(j,\infty) \arrow[d] \arrow[uuullll, dashed, "\partial"] \arrow[ullll, dashed, "\partial"] \\
 & & & 0 \arrow[r] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[d] \\
 & & & & 0 \arrow[r] & \dots \arrow[r] & \dots \\
 & & j & & k & & \infty
\end{tikzcd}$$

((ETC: Define a morphism $\phi\colon H_* \to {}'H_*$ of (finite or extended) Cartan–Eilenberg systems.))

Example 3.5.6. The exact couples in Example 2.1.5 arise from Cartan–Eilenberg systems. Let $H_*(-\infty,j) = p^{-j}\mathbb{Z} \subset \mathbb{Z}[1/p] = H_*(-\infty,\infty)$, and set $H_*(i,j) = p^{-j}\mathbb{Z}/p^{-i}\mathbb{Z}$ for $i \le j$. Then $H_*(i,j) \cong \mathbb{Z}/p^{j-i}$.

$$\begin{tikzcd}
-\infty & \dots \arrow[r] & p\mathbb{Z} \arrow[r]\arrow[d] & \mathbb{Z} \arrow[r]\arrow[d] & p^{-1}\mathbb{Z} \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \mathbb{Z}[1/p] \arrow[d] \\
 & \dots \arrow[r] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \dots \arrow[d] \\
-1 & & 0 \arrow[r] & \mathbb{Z}/p \arrow[r]\arrow[d] & \mathbb{Z}/p^2 \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \mathbb{Z}[1/p]/p\mathbb{Z} \arrow[d] \\
0 & & & 0 \arrow[r] & \mathbb{Z}/p \arrow[r]\arrow[d] & \dots \arrow[r]\arrow[d] & \mathbb{Z}[1/p]/\mathbb{Z} \arrow[d] \\
1 & & & & 0 \arrow[r] & \dots \arrow[r] & \mathbb{Z}[1/p]/p^{-1}\mathbb{Z} \\
 & & -1 & 0 & 1 & & \infty
\end{tikzcd}$$

For the second example, replace $\mathbb{Z}$ with $\mathbb{Z}_p$ everywhere. The finite parts of these Cartan–Eilenberg systems are isomorphic, since $p^{-j}\mathbb{Z}/p^{-i}\mathbb{Z} \cong p^{-j}\mathbb{Z}_p/p^{-i}\mathbb{Z}_p$.

There are two exact couples associated to any extended Cartan–Eilenberg system, generating the same spectral sequence. We concentrate on the one given by the top row and the superdiagonal.

Definition 3.5.7. The (top) exact couple $(A_s, E_s)_s$ associated to an extended Cartan–Eilenberg system (H_*, η, ∂) is given by the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\eta"] & H_*(-\infty, s-1) \arrow[r, "\eta"] & H_*(-\infty, s) \arrow[r, "\eta"] \arrow[d, "\eta"] & \dots \\
& \dots & H_*(s-1, s) \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where

$$\begin{aligned}
(A_s)_* &= H_*(-\infty, s) \\
(E_s)_* &= H_*(s-1, s)
\end{aligned}$$

with α_s and β_s given by the structure homomorphisms η, while γ_s is given by the connecting homomorphism ∂.

The spectral sequence $(E^r, d^r)_{r\geq 1}$ associated to (H_*, η, ∂) is the spectral sequence associated to the exact couple $(A_s, E_s)_s$.

Lemma 3.5.8. *Each morphism $\psi\colon C_* \to {}'C_*$ of filtered chain complexes induces a morphism $\phi\colon H_* \to {}'H_*$ of extended Cartan–Eilenberg systems, and each such morphism induces a morphism $\phi\colon (A, E) \to ({}'A, {}'E)$ of exact couples. Hence the functor of Lemma 3.1.5 factors as a composite*

$$\textit{Filtered Chain Complexes} \longrightarrow \textit{Cartan–Eilenberg Systems} \longrightarrow \textit{Exact Couples}\,.$$

Proposition 3.5.9. *In the spectral sequence $(E^r, d^r)_{r\geq 1}$ associated to an extended Cartan–Eilenberg system (H_*, η, ∂) we have*

$$\begin{aligned}
Z_s^r &= \partial^{-1}\operatorname{im}(\eta\colon H_{*-1}(-\infty, s-r) \to H_{*-1}(-\infty, s-1)) \\
&= \ker(\partial\colon H_*(s-1, s) \to H_{*-1}(s-r, s-1)) \\
&= \operatorname{im}(\eta\colon H_*(s-r, s) \to H_*(s-1, s))
\end{aligned}$$

and

$$\begin{aligned}
B_s^r &= \eta\ker(\eta\colon H_*(-\infty, s) \to H_*(-\infty, s+r-1)) \\
&= \operatorname{im}(\partial\colon H_{*+1}(s, s+r-1) \to H_*(s-1, s)) \\
&= \ker(\eta\colon H_*(s-1, s) \to H_*(s-1, s+r-1))\,,
\end{aligned}$$

so that η induces an isomorphism

$$E_s^r \xrightarrow{\cong} \operatorname{im}(\eta\colon H_*(s-r, s) \to H_*(s-1, s+r-1))\,.$$

The d^r-differential is given by

$$\begin{aligned}
d_s^r\colon E_s^r &\longrightarrow E_{s-r}^r \\
[x] &\longmapsto [\partial(z)]
\end{aligned}$$

where $z \in H_*(s-r, s)$, $x = \eta(z) \in H_*(s-1, s)$ and $\partial(z) \in H_{*-1}(s-r-1, s-r)$.

Proof. For the r-th cycles,

$$
\begin{aligned}
&\partial^{-1}\operatorname{im}(\eta\colon H_{*-1}(-\infty, s-r) \to H_{*-1}(-\infty, s-1)) \\
&\qquad = \partial^{-1}\ker(\eta\colon H_{*-1}(-\infty, s-1) \to H_{*-1}(s-r, s-1)) \\
&\qquad\qquad = \ker(\partial\colon H_*(s-1, s) \to H_{*-1}(s-r, s-1))
\end{aligned}
$$

by exactness and naturality.

$$
\begin{tikzcd}
H_{*-1}(-\infty, s-r) \arrow[r, "\eta"] & H_{*-1}(-\infty, s-1) \arrow[d, "\eta"] & \\
0 & H_{*-1}(s-r, s-1) & H_*(s-r, s) \arrow[d, "\eta"] \\
 & 0 & H_*(s-1, s) \arrow[ul, "\partial"] \arrow[uul, "\partial"']
\end{tikzcd}
$$

For the r-th boundaries,

$$
\begin{aligned}
&\eta\ker(\eta\colon H_*(-\infty, s) \to H_*(-\infty, s+r-1)) \\
&\qquad = \eta\operatorname{im}(\partial\colon H_{*+1}(s, s+r-1) \to H_*(-\infty, s)) \\
&\qquad\qquad = \operatorname{im}(\partial\colon H_{*+1}(s, s+r-1) \to H_*(s-1, s))
\end{aligned}
$$

for the same reasons.

$$
\begin{tikzcd}
H_*(-\infty, s) \arrow[r, "\eta"] \arrow[d, "\eta"] & H_*(-\infty, s+r-1) \\
H_*(s-1, s) \arrow[r, "\eta"'] & H_*(s-1, s+r-1) \\
0 & H_{*+1}(s, s+r-1) \arrow[ul, "\partial"] \arrow[uul, "\partial"']
\end{tikzcd}
$$

Considering the composition $\eta'' \circ \eta'$ (where the primes only serve to keep the two homomorphisms apart),

$$
\begin{tikzcd}
 & H_*(-\infty, s) \arrow[d, "\eta"] & & \\
H_{*-1}(s-r, s-1) & H_*(s-r, s) \arrow[d, "\eta'"] & & \\
0 & H_*(s-1, s) \arrow[r, "\eta''"] \arrow[ul, "\partial"] & H_*(s-1, s+r-1) \arrow[r, "\eta"] & H_*(s-1, \infty) \\
 & 0 & H_{*+1}(s, s+r-1) \arrow[ul, "\partial"] &
\end{tikzcd}
$$

the isomorphism

$$
\eta''\colon H_*(s-1, s)/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'')
$$

restricts to the asserted isomorphism

$$E_s^r = Z_s^r / B_s^r = \operatorname{im}(\eta') / \ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'' \circ \eta') \,.$$

Note that we already know that $B_s^r = \ker(\eta'') \subset \operatorname{im}(\eta') = Z_s^r$, so that $\operatorname{im}(\eta') \cap \ker(\eta'') = \ker(\eta'')$.

If $x = \eta(z) \in Z_s^r \subset H_*(s-1, s)$ with $z \in H_*(s-r, s)$, then $\partial(x) = \eta(y) \in H_{*-1}(-\infty, s-1)$ with $y = \partial(z) \in H_{*-1}(-\infty, s-r)$, by naturality. Hence $\eta(y) = \partial(z) \in H_{*-1}(s-r-1, s-r)$, also by naturality. Thus $d_s^r([x]) = [\eta(y)] = [\partial(z)]$.

$$\begin{tikzcd}
H_{*-1}(-\infty, s-r) \arrow[r, "\eta"] \arrow[d, "\eta"'] & H_{*-1}(-\infty, s-1) & \\
H_{*-1}(s-r-1, s-r) & & \\
0 & & H_*(s-r, s) \arrow[uull, "\partial"'] \arrow[ull, "\partial"] \arrow[d, "\eta"] \\
& 0 & H_*(s-1, s) \arrow[uuul, "\partial"']
\end{tikzcd}$$

□

REMARK 3.5.10. The formulas for Z_s^r, B_s^r and d_s^r in the proposition above show that only the finite part of a Cartan–Eilenberg system is needed in order to define the associated spectral sequence, with $E_s^1 = H_*(s-1, s)$. The groups $H_*(-\infty, s)$ are not needed in order to construct the spectral sequence. However, they do play a role in the description of the target group, hence also for the questions of convergence. ((ETC: Technically speaking, we have not shown that the formulas give a spectral sequence, since the argument for this used the terms $(A_s)_s$ of the exact couple.))

LEMMA 3.5.11. *The colimit*

$$G_* = H_*(-\infty, \infty) \cong \operatorname*{colim}_s H_*(-\infty, s)$$

is exhaustively filtered by

$$F_s G_* = \operatorname{im}(\eta \colon H_*(-\infty, s) \to H_*(-\infty, \infty)) \,.$$

LEMMA 3.5.12. *Consider an extended Cartan–Eilenberg system* (H_*, η, ∂) *such that the sequence*

$$\dots \xrightarrow{\eta} H_*(-\infty, s-1) \xrightarrow{\eta} H_*(-\infty, s) \xrightarrow{\eta} \dots$$

is degreewise discrete. Then

$$\begin{aligned} Z_s^\infty &= \ker(\partial \colon H_*(s-1, s) \to H_{*-1}(-\infty, s-1)) \\ &= \operatorname{im}(\eta \colon H_*(-\infty, s) \to H_*(s-1, s)) \end{aligned}$$

and the filtration $(F_s G_*)_s$ *is degreewise discrete.*

PROOF. If $H_{n-1}(-\infty, j) = 0$ for $j < a = a(n-1)$ then

$$\ker(\partial \colon H_n(s-1, s) \to H_{n-1}(-\infty, s-1)) = \ker(\partial \colon H_n(s-1, s) \to H_{n-1}(s-r, s-1))$$

for all $s - r < a$, i.e., for all $r > s - a$, so $(Z_s^\infty)_n$ equals this common value of $(Z_s^r)_n$. □

LEMMA 3.5.13. *Let* (H_*, η, ∂) *be any extended Cartan–Eilenberg system. Then*

$$\begin{aligned} B_s^\infty &= \operatorname{im}(\partial\colon H_{*+1}(s,\infty) \to H_*(s-1,s)) \\ &= \ker(\eta\colon H_*(s-1,s) \to H_*(s-1,\infty)) \,. \end{aligned}$$

PROOF. The union $B_s^\infty \cong \operatorname{colim}_r B_s^r$ equals

$$\operatorname*{colim}_r \ker(\eta\colon H_*(s-1,s) \to H_*(s-1,s+r-1)) \cong \ker(\eta\colon H_*(s-1,s) \to H_*(s-1,\infty))$$

since $H_*(s-1,\infty) \cong \operatorname{colim}_r H_*(s-1,s+r-1)$. □

LEMMA 3.5.14. *Let* (H_*, η, ∂) *be any extended Cartan–Eilenberg system. There is a preferred isomorphism*

$$\frac{\operatorname{im}(\eta\colon H_*(-\infty,s) \to H_*(s-1,s))}{\ker(\eta\colon H_*(s-1,s) \to H_*(s-1,\infty))} \cong \frac{F_sG_*}{F_{s-1}G_*}$$

for each $s \in \mathbb{Z}$.

PROPOSITION 3.5.15. *Let* (H_*, η, ∂) *be an extended Cartan–Eilenberg system, with associated spectral sequence* (E^r, d^r) *and filtered target* $G_* = H_*(-\infty,\infty)$.

(1) There is always a preferred injective homomorphism

$$\frac{F_sG_*}{F_{s-1}G_*} \overset{\zeta}{\rightarrowtail} E^\infty_{s,*} \,,$$

which is an isomorphism if $Z_s^\infty = \operatorname{im}(\eta\colon H_*(-\infty,s) \to H_*(s-1,s))$.

(2) In particular, if the sequence

$$\dots \xrightarrow{\eta} H_*(-\infty,s-1) \xrightarrow{\eta} H_*(-\infty,s) \xrightarrow{\eta} \dots$$

is degreewise discrete, then ζ *is an isomorphism and the spectral sequence*

$$E^r_{s,*} \Longrightarrow_s G_*$$

converges.

CHAPTER 4

The Serre Spectral Sequence

4.1. Maps, fiber bundles and fibrations

Leray [**Ler46a**], [**Ler46b**] was led to spectral sequences by studying the relation between $H^*(B)$ and $H^*(E)$, where $p\colon E \to B$ is a given map. To outline the main features we use the modern language of sheaf theory, as it was reworked by Cartan in his 1951 seminar. For each open $U \subset B$ let $E_U = p^{-1}(U)$ be the part of E above B. In each degree t the association

$$U \longmapsto \mathscr{F}^t(U) = H^t(E_U)$$

is a contravariant functor from the category of open subsets of B, partially ordered by inclusions, to the category of abelian groups, i.e., an abelian presheaf on B. It is not a sheaf, because

$$H^t(E_{U\cup V}) \longrightarrow H^t(E_U) \oplus H^t(E_V)$$

is not generally injective, but it can be sheafified. For each point $b \in B$ let $F_b = p^{-1}(b)$ be the fiber at b. The stalk of this presheaf (and the associated sheaf $\widetilde{\mathscr{F}^t}$) at this point is the colimit

$$\operatorname*{colim}_{b\in U} H^t(E_U)\,,$$

which canonically maps to $H^t(F_b)$, and for "nice" $p\colon E \to B$ this map is an isomorphism. There results a cohomologically indexed *Leray spectral sequence*

$$E_2^{s,t} = H^s(B; \widetilde{\mathscr{F}^t}) \Longrightarrow_s H^{s+t}(E)\,,$$

where the E_2-term is given in terms of sheaf cohomology.

To stay within the realm of topological spaces and their (co-)homology, one would like to replace sheaf cohomology with ordinary cohomology of the base space B, and to replace the coefficient sheaf with the ordinary cohomology of the fiber F_b. Some hypothesis on the map $p\colon E \to B$ will be needed in order to control how the fiber varies with b.

When $p\colon E \to B$ is a fiber bundle with fiber F, so that B is covered by open subsets U for which there are homeomorphisms h_U making the diagram

$$\begin{tikzcd} U \times F \arrow[rr, "h_U", "\cong"'] \arrow[dr, "p_1"'] & & E_U \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

commute, this problem was considered by Guy Hirsch [**Hir47**], [**Hir48**] and by Tatsuji Kudo [**Kud50**], [**Kud52**]. Here Kudo adapted Leray's algebraic framework to the case where the base space B is a simplicial complex with skeleton filtration

$$\emptyset = B^{(-1)} \subset B^{(0)} \subset \cdots \subset B^{(s-1)} \subset B^{(s)} \subset \cdots \subset B\,.$$

He filtered the total space E by the preimages $E_s = p^{-1}(B^{(s)})$, so that

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E\,.$$

Kudo thus obtained a convergent homological spectral sequence

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) \Longrightarrow H_{s+t}(E)\,.$$

For each s-simplex $\sigma \subset B$ there is a homeomorphism of pairs

$$h_\sigma \colon (\sigma \times F_b, \partial\sigma \times F_b) \xrightarrow{\cong} (p^{-1}(\sigma), p^{-1}(\partial\sigma))$$

where $F_b = p^{-1}(b)$ for a chosen point $b \in \partial\sigma$. By excision and the Künneth theorem, this induces isomorphisms

$$\begin{aligned} E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) &\cong \bigoplus_\sigma H_{s+t}(p^{-1}(\sigma), p^{-1}(\partial\sigma)) \\ &\cong \bigoplus_\sigma H_{s+t}(\sigma \times F_b, \partial\sigma \times F_b) \cong \bigoplus_\sigma H_s(\sigma, \partial\sigma) \otimes H_t(F_b)\,, \end{aligned}$$

where σ ranges over the s-simplices in B. When the fiber bundle admits a connected structure group G, e.g. if B is 1-connected, then there are preferred isomorphisms $H_t(F_b) \cong H_t(F)$, so that the E^1-term can be identified with

$$\bigoplus_\sigma H_s(\sigma, \partial\sigma) \otimes H_t(F) \cong \bigoplus_\sigma H_t(F) = \Delta_s(B; H_t(F))\,,$$

i.e., the simplicial s-chains of B with coefficients in $H_t(F)$. Moreover, Kudo verified that the d^1-differential

$$d^1_{s,t} \colon H_{s+t}(E_s, E_{s-1}) \longrightarrow H_{s+t-1}(E_{s-1}, E_{s-2})$$

corresponds to the simplicial boundary homomorphism

$$\partial_s \colon \Delta_s(B; H_t(F)) \longrightarrow \Delta_{s-1}(B; H_t(F))$$

under these identifications. Hence the spectral sequence E^2-term satisfies

$$E^2_{s,t} \cong H^\Delta_s(B; H_t(F))\,.$$

Since simplicial and singular homology agree for simplicial complexes, this establishes a spectral sequence of the form

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)\,,$$

converging to the homology of the total space. Kudo also discusses the case of non-connected structure group G, which is relevant for non-simply connected bases B, and which leads to an E^2-term expressed in terms of Steenrod's (co-)homology with local coefficients [**Ste43**], [**Hat02**, §3.H].

There are many geometrically interesting examples of such fiber bundles, arising from the theory of Lie groups and their homogeneous spaces. However, to analyze the (co-)homology of Eilenberg–MacLane spaces, Jean–Pierre Serre [**Ser51**] was led to consider the more general situation of the path–loop fibration $p \colon PX \to X$, with fiber ΩX, which is not a fiber bundle. However, this map has the homotopy lifting property with respect to arbitrary source spaces, hence is a fibration $p \colon E \to B$ in the sense of Witold Hurewicz [**Hur55**]. Serre recognized that this lifting property allowed him to construct a spectral sequence of the same form

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)$$

as before, by filtering a version of the singular chain complex $(C_*(E), \partial)$. This filtration is different from the one used by Kudo, and does not assume that B has a skeletal filtration. To make this work, Serre uses singular cubes in place of singular simplices. Moreover, for this argument the map $p\colon E \to B$ would only need to satisfy the homotopy lifting condition with respect to maps from compact polyhedra (or finite CW complexes), and this larger class of maps is now known as Serre fibrations. Furthermore, Serre showed that the cup product in cohomology is compatible with the differentials in the cohomological version of his spectral sequence, leading to a ring spectral sequence in the sense of the "anneau spectral" of Leray. This is a key feature needed to make precise calculations with spectral sequences, which we will return to later ((ETC: where?)).

Definition 4.1.1. A map $p\colon E \to B$ is a *Hurewicz fibration* if it has the homotopy lifting property with respect to each space T. In other words, for each commutative square of solid arrows

$$\begin{tikzcd} T \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ I \times T \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

there exists a dotted arrow making both trianges commute. The map $p\colon E \to B$ is a *Serre fibration* if it has the homotopy lifting property with respect to the n-disk D^n for each $n \ge 0$.

Each fiber bundle over a paracompact (e.g., metric) base space is a Hurewicz fibration [**Hur55**, §4], and each Hurewicz fibration is a Serre fibration. Pullback preserves fiber bundles, Hurewicz fibrations and Serre fibrations.

For a Hurewicz fibration $p\colon E \to B$ with contractible base space, the inclusion $F_b \subset E$ of any fiber $F_b = p^{-1}(b)$ is a homotopy equivalence. Let $p\colon E \to B$ be a Hurewicz fibration over a general base. For any path $\beta\colon I \to B$ from $\beta(0) = b_0$ to $\beta(1) = b_1$, the pullback $\beta^*p\colon \beta^*E \to I$ is a Hurewicz fibration over a contractible base, so the inclusions

$$F_{b_0} \xrightarrow{\simeq} \beta^*E \xleftarrow{\simeq} F_{b_1}$$

are homotopy equivalences. This defines a homotopy equivalence $e\colon F_{b_0} \simeq F_{b_1}$, up to homotopy. Let $'\beta$ be another path from b_0 to b_1. A path homotopy $h\colon I \times I \to B$ from β to $'\beta$ leads to another Hurewicz fibration over a contractible base, so that all of the inclusions

$$\begin{tikzcd} & \beta^*E \arrow[d, "\simeq"] & \\ F_{b_0} \arrow[ur, "\simeq"] \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & h^*E & F_{b_1} \arrow[ul, "\simeq"'] \arrow[l, "\simeq"'] \arrow[dl, "\simeq"] \\ & {'\beta^*E} \arrow[u, "\simeq"'] & \end{tikzcd}$$

are homotopy equivalences. It follows that the composite equivalence $e\colon F_{b_0} \simeq \beta^*E \simeq F_{b_1}$ is homotopic to the composite equivalence $'e\colon F_{b_0} \simeq {'\beta^*E} \simeq F_{b_1}$. Passing to homology, a choice of β gives well-defined isomorphisms $e_*\colon H_t(F_{b_0}) \cong H_t(\beta^*E) \cong H_t(F_{b_1})$ for all t, and homotopic paths β and $'\beta$ give the same composite isomorphism $e_*\colon H_t(F_{b_0}) \cong H_t(F_{b_1})$. If B is 1-connected, with base point b_0, this gives canonical isomorphisms $H_t(F) \cong H_t(F_b)$ for each $b \in B$. In general, it gives a *local coefficient system* $\mathscr{H}_t(F)$ on B, i.e., a functor from the fundamental groupoid of B to abelian groups. If B is 0-connected, with fundamental group $\pi = \pi_1(B, b_0)$, then this structure can equivalently be encoded as an action of π on $H_t(F)$ for $F = F_{b_0}$.

The equivalent inclusions of fibers can be made compatible, for varying $b \in B$, as follows. For brevity, we write $(B, A) \times F$ for $(B \times F, A \times F)$. ((ETC: This will be generalized by the convolution product of filtrations, later.))

PROPOSITION 4.1.2 ([**Spa66**, ?], [**Whi78**, Cor. I.7.27]). *Any Hurewicz fibration $p \colon E \to B$ over a contractible base space B is fiber homotopy trivial, meaning that there are maps $f \colon B \times F \to E$ and $g \colon E \to B \times F$ and homotopies $gf \simeq 1$ and $fg \simeq 1$, all four of which commute with the projections $p_1 \colon B \times F \to B$ and $p \colon E \to B$. In particular, for any $A \subset B$ with $E_A = p^{-1}(A)$ there is a homotopy equivalence of pairs*

$$f \colon (B, A) \times F \xrightarrow{\simeq} (E, E_A) .$$

We refer to the cited sources for the proof.

4.2. Homology of fiber sequences

Consider a Hurewicz fibration $p \colon E \to B$, with B a CW complex. Let

$$E_s = p^{-1}(B^{(s)})$$

be the preimage of the s-skeleton of the base B. The total space E is then strongly filtered by the sequence

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E ,$$

since for any compact $K \subset E$ there is an s with $p(K) \subset B^{(s)}$, and then $K \subset E_s$. By [**Whi78**, Thm. I.7.14] each inclusion $E_{s-1} \subset E_s$ is a (closed) cofibration.

DEFINITION 4.2.1. The *(homological) Serre spectral sequence* of $p \colon E \to B$ is the spectral sequence

$$E^1_{s,t}(p) = H_{s+t}(E_s, E_{s-1}) \Longrightarrow_s H_{s+t}(E)$$

associated to the filtration $(E_s)_s$.

By Proposition 3.2.3 the d^1-differential equals the connecting homomorphism

$$d^1_{s,t} = (\partial_s)_{s+t} \colon H_{s+t}(E_s, E_{s-1}) \longrightarrow H_{s+t-1}(E_{s-1}, E_{s-2})$$

of the triple (E_s, E_{s-1}, E_{s-2}). The E^1-term is concentrated in the right half-plane $(s \geq 0)$, and the spectral sequence converges to $H_*(E)$ with the filtration

$$F_s H_*(E) = \operatorname{im}(H_*(E_s) \to H_*(E)) .$$

We shall see in Proposition 4.2.3 that this is a first quadrant spectral sequence, so that $(F_s H_*(X))_s$ is degreewise bounded.

REMARK 4.2.2. This construction is closer to that of Kudo [**Kud50**] than that of Serre [**Ser51**], but the E^2-terms will be isomorphic. Serre also established multiplicative properties for the cohomology version of his spectral sequence, which led him to stronger conclusions than those that follow from the additive structure. The name "Serre spectral sequence" thus reflects the extra versatility and power achieved by Serre's approach.

Since B is the disjoint union of its path components, there is a corresponding sum decomposition of E, and we can assume that B is 0-connected without significant loss of generality. We can then also assume that B is 0-*reduced*, in the sense that it only has a single 0-cell, given by the base point $b_0 \in B$. The 0-th filtration $E_0 = p^{-1}(B^{(0)})$ is then equal to the fiber

$$F = p^{-1}(b_0)$$

of $p\colon E \to B$ at the base point. We write

$$F \longrightarrow E \xrightarrow{p} B$$

to refer to this context. ((ETC: Beware the double usage of E for total space and spectral sequence terms, and the double usage of F for fiber and various filtrations.))

Proposition 4.2.3 ([**Whi78**, Thm. XIII.4.6]). *There are natural isomorphisms*

$$H_{s+t}(E_s, E_{s-1}) \cong C_s^{CW}(B; \mathscr{H}_t(F)),$$

where $\mathscr{H}_t(F)$ denotes the local coefficient system on B given by $H_t(F_b)$ at $b \in B$. If B is 1-connected, then this equals the usual cellular s-chains $C_s^{CW}(B; H_t(F))$ with coefficients in the abelian group $H_t(F)$.

Proof. Let α run over the s-cells of B, so that we have a pushout square

$$\begin{tikzcd}
\coprod_\alpha \partial D^s_\alpha \arrow[r] \arrow[d, "\phi"'] & \coprod_\alpha D^s_\alpha \arrow[d, "\Phi"] \\
B^{(s-1)} \arrow[r] & B^{(s)} ,
\end{tikzcd}$$

with attaching maps $\phi = \coprod_\alpha \phi_\alpha$ and characteristic maps $\Phi = \coprod_\alpha \Phi_\alpha$. Form the pullbacks of $p\colon E \to B$ along the evident maps to B, to obtain another pushout ((ETC: check)) square

$$\begin{tikzcd}
\coprod_\alpha \phi^*_\alpha E \arrow[r] \arrow[d] & \coprod_\alpha \Phi^*_\alpha E \arrow[d] \\
E_{s-1} \arrow[r] & E_s .
\end{tikzcd}$$

By additivity and excision we obtain isomorphisms

$$\bigoplus_\alpha H_{s+t}(\Phi^*_\alpha E, \phi^*_\alpha E) \cong H_{s+t}(\coprod_\alpha \Phi^*_\alpha E, \coprod_\alpha \phi^*_\alpha E) \cong H_{s+t}(E_s, E_{s-1}).$$

Proposition 4.1.2 applies to the pullback

$$p\colon \Phi^*_\alpha E \longrightarrow D^s_\alpha$$

Let $d_0 \in \partial D^s_\alpha \subset D^s_\alpha$ be a base point, let $b_\alpha = \phi_\alpha(d_0)$, and let $F_{b_\alpha} = p^{-1}(b_\alpha)$ be the fiber above this point. There is then a fiber homotopy equivalence

$$D^s_\alpha \times F_{b_\alpha} \xrightarrow{\simeq} \Phi^*_\alpha E$$

over D^s_α, which restricts to the identity over d_0. In particular, there is a homotopy equivalence of pairs

$$(D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha} \xrightarrow{\simeq} (\Phi^*_\alpha E, \phi^*_\alpha E)$$

and isomorphisms

$$H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}(\Phi^*_\alpha E, \phi^*_\alpha E).$$

By the Künneth theorem, the homology cross product induces an isomorphism

$$H_s(D^s_\alpha, \partial D^s_\alpha) \otimes H_t(F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) .$$

Taken together, we have isomorphisms

$$\bigoplus_\alpha H_s(D^s_\alpha, \partial D^s_\alpha) \otimes H_t(F_{b_\alpha}) \cong H_{s+t}(E_s, E_{s-1})$$

for all s and t. By definition, the left hand side is $C^{CW}_s(B; \mathscr{H}_t(F))$. If B is 1-connected, then the canonical isomorphisms $H_t(F_{b_\alpha}) \cong H_t(F)$ identify the direct sum above with

$$C^{CW}_s(B; H_t(F)) \cong \bigoplus_\alpha H_s(D^s_\alpha, \partial D^s_\alpha) \otimes H_t(F) .$$

□

Proposition 4.2.4 ([**Whi78**, Thm. XIII.4.8]). *The square*

$$\begin{tikzcd} H_{s+t}(E_s, E_{s-1}) \arrow[r, "d^1_{s,t}"] \arrow[d, "\cong"'] & H_{s+t-1}(E_{s-1}, E_{s-2}) \arrow[d, "\cong"] \\ C^{CW}_s(B; \mathscr{H}_t(F)) \arrow[r, "\partial_s"] & C^{CW}_{s-1}(B; \mathscr{H}_t(F)) \end{tikzcd}$$

commutes.

Proof sketch. Let α and β index the s- and $(s-1)$-cells of B, with characteristic maps Φ_α and Ψ_β, respectively. We have fiber homotopy equivalences $\Phi_\alpha^* E \simeq D^s_\alpha \times F_{b_\alpha}$ over D^s_α and $\Psi_\beta^* E \simeq D^{s-1}_\beta \times F_{b_\beta}$ over D^{s-1}_β.

In the cellular complex for B, the boundary ∂_s has components

$$\begin{aligned} H_s(D^s_\alpha, \partial D^s_\alpha) &\xrightarrow{\Phi i_\alpha} H_s(B^{(s)}, B^{(s-1)}) \\ &\xrightarrow{\partial} \tilde{H}_{s-1}(B^{(s-1)}) \\ &\xrightarrow{q} \tilde{H}_{s-1}(B^{(s-1)}/B^{(s-2)}) \\ &\xrightarrow{p_\beta \Psi^{-1}} \tilde{H}_{s-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta) , \end{aligned}$$

where we use the isomorphisms

$$\Phi \colon \bigoplus_\alpha H_s(D^s_\alpha, \partial D^s_\alpha) \xrightarrow{\cong} H_s(B^{(s)}, B^{(s-1)})$$

$$\Psi \colon \bigoplus_\beta \tilde{H}_{s-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta) \xrightarrow{\cong} \tilde{H}_{s-1}(B^{(s-1)}/B^{(s-2)}) .$$

This component can also be factored as

$$\begin{aligned} H_s(D^s_\alpha, \partial D^s_\alpha) &\overset{\partial}{\cong} \tilde{H}_{s-1}(\partial D^s_\alpha) \\ &\xrightarrow{\phi_\alpha} \tilde{H}_{s-1}(B^{(s-1)}) \\ &\xrightarrow{q} \tilde{H}_{s-1}(B^{(s-1)}/B^\wedge_\beta) \\ &\overset{\Psi_\beta^{-1}}{\cong} \tilde{H}_{s-1}(D^{s-1}_\beta / \partial D^{s-1}_\beta) , \end{aligned}$$

where $B^\wedge_\beta \subset B^{(s-1)}$ is the complement of $\Psi_\beta(\operatorname{int} D^{s-1}_\beta)$.

We must identify the corresponding composite

$$\begin{aligned} H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) &\xrightarrow{\Phi i_\alpha} H_{s+t}(E_s, E_{s-1}) \\ &\xrightarrow{\partial} \tilde{H}_{s+t-1}(E_{s-1}) \\ &\xrightarrow{q} \tilde{H}_{s+t-1}(E_{s-1}/E_{s-2}) \\ &\xrightarrow{p_\beta \Psi^{-1}} \tilde{H}_{s-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta \wedge F_{b_\beta +}), \end{aligned}$$

where now

$$\Phi\colon \bigoplus_\alpha H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) \xrightarrow{\cong} H_{s+t}(E_s, E_{s-1})$$

$$\Psi\colon \bigoplus_\beta \tilde{H}_{s+t-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta \wedge F_{b_\beta +}) \xrightarrow{\cong} \tilde{H}_{s+t-1}(E_{s-1}/E_{s-2}).$$

This can also be factored as

$$\begin{aligned} H_{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) &\xrightarrow{\partial} \tilde{H}_{s+t-1}(\partial D^s_\alpha \times F_{b_\alpha}) \\ &\xrightarrow{\phi_\alpha} \tilde{H}_{s+t-1}(E_{s-1}) \\ &\xrightarrow{q} \tilde{H}_{s+t-1}(E_{s-1}/E^\wedge_\beta) \\ &\overset{\Psi_\beta^{-1}}{\cong} \tilde{H}_{s+t-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta \wedge F_{b_\beta +}), \end{aligned}$$

where $E^\wedge_\beta = p^{-1}(B^\wedge_\beta) \subset E_{s-1}$. It thus suffices to verify that the following diagram commutes

$$\begin{tikzcd} \tilde{H}_{s-1}(\partial D^s_\alpha) \otimes H_t(F_{b_\alpha}) \arrow[r, "\Psi_\beta^{-1} q\phi_\alpha \otimes e"] \arrow[d, "\times"'] & \tilde{H}_{s-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta) \otimes H_t(F_{b_\beta}) \arrow[d, "\times"', "\cong"] \\ \tilde{H}_{s+t-1}(\partial D^s_\alpha \times F_{b_\alpha}) \arrow[r, "\Psi_\beta^{-1} q\phi_\alpha"] & \tilde{H}_{s+t-1}(D^{s-1}_\beta/\partial D^{s-1}_\beta \wedge F_{b_\beta +}). \end{tikzcd}$$

For this, which takes some effort, we refer to [**Whi78**, §XIII.5]. □

THEOREM 4.2.5. *The Serre spectral sequence*

$$E^r_{s,t}(p) \Longrightarrow_s H_{s+t}(E)$$

for $F \to E \xrightarrow{p} B$ has E^2-term

$$E^2_{s,t}(p) \cong H_s(B; \mathscr{H}_t(F)).$$

If B is 1-connected, this simplifies to

$$E^2_{s,t}(p) \cong H_s(B; H_t(F)).$$

PROOF. This follows from $(E^1_{*,*}, d^1) \cong (C^{CW}_*(B; \mathscr{H}_*(F)), \partial)$ by passage to homology. □

REMARK 4.2.6. In the context of the Serre spectral sequence, the filtration degree s and complementary degree t are also referred to as the *base degree* and *fiber degree*, respectively. This makes sense, since the E^2-term in bidegree (s,t) is given in terms of $H_s(B)$, suitably interpreted, and $H_t(F)$. The total degree $s+t$ then refers both to the total algebraic degree, and to the grading of the homology $H_*(E)$ of the total space.

In view of the universal coefficient exact sequence

$$0 \to H_s(B) \otimes H_t(F) \longrightarrow H_s(B; H_t(F)) \longrightarrow \operatorname{Tor}(H_{s-1}(B), H_t(F)) \to 0$$

this achieves the aim of obtaining a spectral sequence that connects the ordinary homology groups $H_*(F)$, $H_*(E)$ and $H_*(B)$ in one whole.

REMARK 4.2.7. There are also relative versions of the Serre spectral sequence. If $A \subset B$ is a subcomplex, then we can filter E by $p^{-1}(A \cup B^{(s)})$ and obtain a spectral sequence

$$E^2_{s,t} = H_s(B, A; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E, E_A)$$

where $E_A = p^{-1}(A)$. If $p' \colon E' \subset E \to B$ is a subfibration with fibers F', then we can filter (E, E') by $(p^{-1}(B^{(s)}), (p')^{-1}(B^{(s)}))$ and obtain a spectral sequence

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F, F')) \Longrightarrow_s H_{s+t}(E, E')\,.$$

When (E_A, E') is an excisive pair we can combine the previous two cases, as in

$$E^2_{s,t} = H_s(B, A; \mathscr{H}_t(F, F')) \Longrightarrow_s H_{s+t}(E, E_A \cup E')\,,$$

where $E_A \cap E' = E'_A = (p')^{-1}(A)$.

((ETC: Relax the condition that B be a CW complex, by comparison with a CW approximation.))

4.3. The Wang and Gysin sequences

Hsien-Chung Wang [**Wan49**] studied fiber bundles with base space a sphere, obtaining a long exact sequence as in the following theorem, which follows in greater generality from the Serre spectral sequence.

THEOREM 4.3.1. *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a (Hurewicz) fiber sequence, with $B \simeq S^u$ a 1-connected CW complex. There is a long exact sequence

$$\dots \to H_{n-u+1}(F) \xrightarrow{\partial} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{i_!} H_{n-u}(F) \xrightarrow{\partial} H_{n-1}(F) \to \dots$$

PROOF. Clearly $u \geq 2$. By the universal coefficient theorem,

$$H_s(B; H_t(F)) \cong \begin{cases} H_t(F) & \text{for } s \in \{0, u\}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the E^2-term of the Serre spectral sequence for $F \to E \to B$ is concentrated in the two columns $s = 0$ and $s = u$. For degree reasons $d^r = 0$ except for $r = u$, so $E^2 = E^u$.

$$
\begin{tikzcd}[row sep=small]
 & \vdots & \vdots & \vdots \\
t+u-1 & H_{t+u-1}(F) & 0 & H_{t+u-1}(F) \arrow[ull, "d^u"'] \\
t & H_t(F) & 0 & H_t(F) \arrow[ull, "d^u"'] \\
 & \vdots & \vdots & \vdots \arrow[ull, "d^u"'] \\
u-1 & H_{u-1}(F) & 0 & H_{u-1}(F) \arrow[ull, "d^u"'] \\
0 & H_0(F) & 0 & H_0(F) \arrow[ull, "d^u"'] \\
t/s & 0 & \dots & u
\end{tikzcd}
$$

At this stage we have differentials

$$d^u_{u,t} \colon H_t(F) \cong E^u_{u,t} \longrightarrow E^u_{0,t+u-1} \cong H_{t+u-1}(F)$$

leading to an E^{u+1}-term with

$$E^{u+1}_{s,t} = \begin{cases} \operatorname{cok}(d^u_{u,t-u+1}) & \text{for } s = 0, \\ \ker(d^u_{u,t}) & \text{for } s = u, \\ 0 & \text{otherwise.} \end{cases}$$

Since $d^r = 0$ for all $r > u$, the spectral sequence collapses at this term, so that $E^{u+1}_{s,t} = E^\infty_{s,t}$ in all bidegrees. By the convergence of the spectral sequence, we have isomorphisms

$$F_s H_n(E) \cong E^\infty_{0,n}$$

for $0 \le s < u$, a short exact sequence

$$0 \to F_{u-1}H_n(E) \longrightarrow F_u H_n(E) \longrightarrow E^\infty_{u,n-u} \to 0\,,$$

and identities

$$F_s H_n(E) = H_n(E)$$

for $s \ge u$. In other words, we have a short exact sequence

$$0 \to \operatorname{cok}(d^u_{u,n-u+1}) \xrightarrow{i_*} H_n(E) \xrightarrow{i_!} \ker(d^u_{u,n-u}) \to 0\,.$$

Writing out the definition of the cokernel and kernel gives the exact sequence

$$H_{n-u+1}(F) \xrightarrow{d^u} H_n(F) \longrightarrow H_n(E) \longrightarrow H_{n-u}(F) \xrightarrow{d^u} H_{n-1}(F)\,,$$

as claimed. □

REMARK 4.3.2. For $B = S^u$, and more generally for fibrations over a suspension $B = \Sigma W$, the Wang sequence can be established without spectral sequences, using the Mayer–Vietoris sequence for the covering of E by $p^{-1}(C_+W)$ and $p^{-1}(C_-W)$, where $\Sigma W = C_+W \cup_W C_-W$ is a union of two cones. See [**Whi78**, §VII.1].

Several years earlier, Werner Gysin [**Gys42**] studied fiber bundles with fiber a sphere, obtaining the equivalent of a long exact sequence as in the following theorem. This also follows in greater generality from the Serre spectral sequence. To avoid a discussion of local coefficients or orientability, we restrict to the case where B is 1-connected.

THEOREM 4.3.3. *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a (Hurewicz) fiber sequence, with $H_*(F) \cong H_*(S^v)$ and B a 1-connected CW complex. There is a long exact sequence

$$\dots \to H_{n+1}(B) \xrightarrow{\partial} H_{n-v}(B) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} H_{n-v-1}(B) \to \dots$$

((ETC: We call $p_!$ the *Gysin homomorphism.*))

PROOF. We assume $v \geq 1$. By the universal coefficient theorem

$$H_s(B; H_t(F)) \cong \begin{cases} H_s(B) & \text{for } t \in \{0, v\}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the E^2-term of the Serre spectral sequence for $F \to E \to B$ is concentrated in the two rows $t = 0$ and $t = v$. For degree reasons $d^r = 0$ except for $r = v + 1$, so $E^2 = E^{v+1}$.

$$\begin{tikzcd}
v & H_0(B) & H_{s-v-1}(B) & H_{s-v}(B) & \dots & \dots & \dots \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H_0(B) & H_{s-v-1}(B) & \dots & H_{v+1}(B) \arrow[uulll, "d^{v+1}"'] & H_s(B) \arrow[uulll, "d^{v+1}"'] & H_{s+1}(B) \arrow[uulll, "d^{v+1}"'] \\
t/s & 0 & s-v-1 & \dots & v+1 & s & s+1
\end{tikzcd}$$

At this stage we have differentials

$$d^{v+1}_{s,0} \colon H_s(B) \cong E^{v+1}_{s,0} \longrightarrow E^{v+1}_{s-v-1,v} \cong H_{s-v-1}(B)$$

leading to an E^{v+2}-term with

$$E^{v+2}_{s,t} = \begin{cases} \ker(d^{v+1}_{s,0}) & \text{for } t = 0, \\ \operatorname{cok}(d^{v+1}_{s+v+1,0}) & \text{for } t = v, \\ 0 & \text{otherwise.} \end{cases}$$

Since $d^r = 0$ for all $r > v + 1$, the spectral sequence collapses at this term, so that $E^{v+2}_{s,t} = E^\infty_{s,t}$ in all bidegrees. By the convergence of the spectral sequence, we have $F_s H_n(E) = 0$ for $s < n - v$, isomorphisms

$$F_s H_n(E) \cong E^\infty_{n-v,v}$$

for $n - v \leq s < n$, and a short exact sequence

$$0 \to F_{n-1} H_n(E) \longrightarrow H_n(E) \longrightarrow E^\infty_{n,0} \to 0\,.$$

In other words, we have a short exact sequence

$$0 \to \operatorname{cok}(d^{v+1}_{n+1,0}) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} \ker(d^{v+1}_{n,0}) \to 0\,.$$

Writing out the definition of the cokernel and kernel gives the exact sequence

$$H_{n+1}(B) \xrightarrow{\partial} H_{n-v}(B) \xrightarrow{p_!} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} H_{n-v-1}(B)\,,$$

as claimed. □

REMARK 4.3.4. The Gysin sequence can be established without spectral sequences, using the Thom isomorphism. Let

$$Mp = (I \times E) \cup_E B$$

be the *mapping cylinder* of $p\colon E \to B$, so that p factors as the inclusion $E \cong \{1\} \times E \subset Mp$ followed by the homotopy equivalence

$$q\colon Mp \xrightarrow{\simeq} B\,.$$

When $E = S(\xi)$ is the unit sphere bundle in a Euclidean $\mathbb{R}^{v+1}$-bundle $E(\xi) \to B$, the mapping cylinder can be identified with the unit disc bundle $Mp \cong D(\xi)$, so that $(Mp, E) \cong (D(\xi), S(\xi))$. When B is 1-connected (or the spherical fibration is orientable) there is a Thom isomorphism

$$\Phi\colon H_n(Mp, E) \xrightarrow{\cong} H_{n-v-1}(B)$$

given by the cap product $U \cap (-)$ with a Thom class

$$U \in H^{v+1}(Mp, E)\,.$$

In the fiber bundle case, U is characterized by the property that for each $b \in B$ the restriction

$$i_b^*\colon H^{v+1}(Mp, E) \longrightarrow H^{v+1}(CF_b, F_b) \cong \tilde{H}^v(F_b) \cong \mathbb{Z}$$

maps U to a generator. Here $F_b = p^{-1}(b)$ is the fiber in E over b, and $CF_b \cong q^{-1}(b)$ is the fiber in Mp, which is identified with the cone on F_p. In the Euclidean bundle case, $F_b \cong S^v$ is the unit sphere and $CF_b \cong D^{v+1}$ is the unit disc in the fiber of $E(\xi) \to B$ over b. Substituting $H_*(Mp) \cong H_*(B)$ and $H_*(Mp, E) \cong H_{*-v-1}(B)$ in the long exact homology sequence

$$\dots \to H_{n+1}(Mp) \longrightarrow H_{n+1}(Mp, E) \longrightarrow H_n(E) \longrightarrow H_n(Mp) \longrightarrow H_n(Mp, E) \to \dots$$

of the pair (Mp, E) then gives the Gysin sequence. See [**MS74**] or [**Whi78**, §VII.5].

The following examples show that Serre spectral sequences can sometimes be used "in reverse" to calculate $H_*(F)$ when $H_*(E)$ and $H_*(B)$ are known, or to calculate $H_*(B)$ when $H_*(F)$ and $H_*(E)$ are known. This is most feasible when $H_*(E)$ is as simple as possible, such as when E is contractible.

DEFINITION 4.3.5. Let $I = [0, 1]$. The *path–loop fibration* of a based space (X, x_0) is the fiber sequence

$$\Omega X \xrightarrow{i} PX \xrightarrow{p} X\,,$$

where PX is the *path space* of based maps $\xi\colon (I, 0) \to (X, x_0)$ and p is the Hurewicz fibration with $p(\xi) = \xi(1)$. The fiber ΩX is the *loop space* of X.

LEMMA 4.3.6. *The path space PX is contractible.*

PROOF. We deform each path $\xi\colon s \mapsto \xi(s)$ to the constant path $s \mapsto x_0$ via the paths $s \mapsto \xi(st)$ for $0 \le t \le 1$. □

Proposition 4.3.7. *For $u \geq 2$ and $n \geq 0$ there are isomorphisms*

$$H_n(\Omega S^u) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \mod u-1, \\ 0 & \text{otherwise.} \end{cases}$$

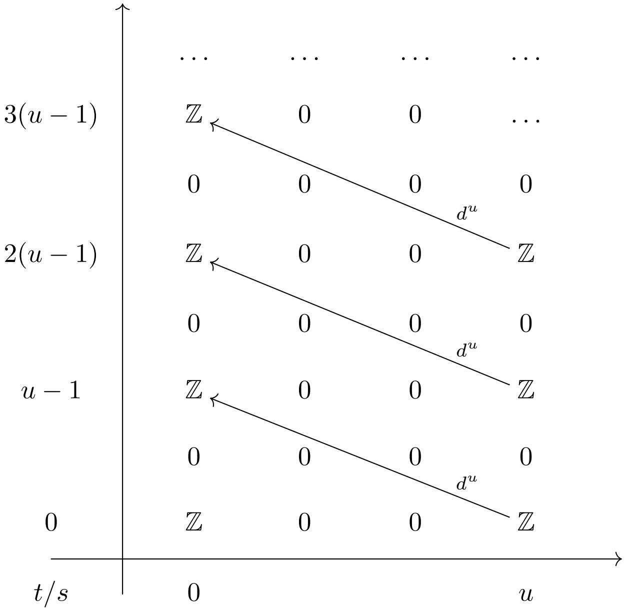
Proof. Since $H_*(PS^u) = \mathbb{Z}$ is concentrated in degree 0, the Wang sequence

$$\dots \to H_{n-u+1}(\Omega S^u) \xrightarrow{\partial} H_n(\Omega S^u) \xrightarrow{i_*} H_n(PS^u) \longrightarrow H_{n-u}(\Omega S^u) \to \dots$$

breaks up into isomorphisms

$$H_{n-u+1}(\Omega S^u) \xrightarrow{\cong} \tilde{H}_n(\Omega S^u)\,.$$

Since $H_n(\Omega S^u)$ is 0 for $n < 0$ and $\mathbb{Z}$ for $n = 0$, the proposition follows by induction on n. The differential pattern in the two-column Serre spectral sequence is shown below, with $H_*(S^u)$ on the s-axis and $H_*(\Omega S^u)$ on the t-axis.



□

Remark 4.3.8. More precise work shows that ΩS^u is equivalent to the James construction $J(S^{u-1})$, see [**Hat02**, §3.2, §4.J]. The loop composition induces a Pontryagin product in $H_*(\Omega S^u)$, and

$$H_*(\Omega S^u) \cong \mathbb{Z}[\xi]$$

is the polynomial algebra on ξ, with $|\xi| = u - 1$. In other words, ξ^k generates $H_{k(u-1)}(\Omega S^u)$ for each $k \geq 0$, and the remaining homology groups are trivial. Suppose that u is odd. Then it follows by dualization that the cup product in cohomology satisfies

$$H^*(\Omega S^u) \cong \Gamma(x)$$

with $|x| = u - 1$ even. Here

$$\Gamma(x) = \mathbb{Z}\{\gamma_k(x) \mid k \geq 0\}$$

is the *divided power algebra* on x, with the multiplication

$$\gamma_i(x) \cdot \gamma_j(x) = (i, j)\gamma_{i+j}(x)$$

where $(i, j) = (i + j)!/i!j!$ denotes the binomial coefficient. Moreover, $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $|\gamma_k(x)| = k|x|$. The terminology comes from the algebra embedding

$$\Gamma(x) \subset \mathbb{Q}[x]$$

sending $\gamma_k(x)$ to the divided k-th power $x^k/k!$.

REMARK 4.3.9. The quasi-inverse process to looping a space, $X \mapsto \Omega X$, is called *delooping*. Not every space admits a delooping, and some spaces admit multiple inequivalent deloopings, but for (almost all) topological groups G there is a well-defined space BG with a homotopy equivalence $G \simeq \Omega BG$. This delooping BG of G is called its classifying space.

DEFINITION 4.3.10. Let G be a topological group. A map $p\colon P \to B$ is a *principal G-bundle* if G acts from the right on P and B admits a cover by open subsets U such that there are G-equivariant homeomorphisms $U \times G \cong p^{-1}(U)$ that commute with the projections to U. In particular G acts freely on P, and $P/G \cong B$. A principal G-bundle $p\colon EG \to BG$ is *universal* if EG is contractible. In this case the base space BG is called a *classifying space* for G.

A universal G-bundle classifies principal G-bundles in the following sense. ((ETC: Reference to Steenrod's book [**Ste51**]? Is Dold's numerability needed for a good universal property?))

PROPOSITION 4.3.11. *Let $p\colon EG \to BG$ be a universal G-bundle, and let $p\colon P \to B$ be a principal G-bundle with B a CW complex. Then there exists a map $f\colon B \to BG$ and a G-map $\hat{f}\colon P \to EG$ such that the square*

$$\begin{tikzcd}
P \arrow[r, "\hat{f}"] \arrow[d, "p"'] & EG \arrow[d, "p"] \\
B \arrow[r, "f"] & BG
\end{tikzcd}$$

commutes, and any two such pairs $(f, \hat{f})$ are homotopic. Pullback along f defines a bijection

$$f^*\colon [B, BG] \xrightarrow{\cong} \mathrm{Bun}_G(B)$$

between the homotopy classes of maps $f\colon B \to BG$ and the isomorphism classes of principal G-bundles $p\colon P \to B$.

EXAMPLE 4.3.12. Let $G = U(1) = S(\mathbb{C})$ be the circle group, viewed as the complex numbers of unit length. It acts freely on $EG = S^\infty = S(\mathbb{C}^\infty)$, viewed as the unit sphere in $\mathbb{C}^\infty$, and the orbit space $BG = EG/G \cong \mathbb{C}P^\infty$ is infinite complex projective space. The fiber bundle

$$S^1 \longrightarrow S^\infty \xrightarrow{p} \mathbb{C}P^\infty$$

is, in particular, a (Hurewicz) fibration. As is well known, the homology of $H_*(\mathbb{C}P^\infty)$ is easily read off from a minimal cell structure on $\mathbb{C}P^\infty$, but the following proposition shows how this can be deduced from the Gysin sequence.

PROPOSITION 4.3.13. *For $n \geq 0$ there are isomorphisms*

$$H_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z} & \textit{for } n \equiv 0 \mod 2, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Since $H_*(S^\infty) = \mathbb{Z}$ is concentrated in degree 0, the Gysin sequence

$$\dots \to H_{n-1}(\mathbb{C}P^\infty) \longrightarrow H_n(S^\infty) \xrightarrow{p_*} H_n(\mathbb{C}P^\infty) \xrightarrow{\partial} H_{n-2}(\mathbb{C}P^\infty) \to \dots$$

breaks up into isomorphisms

$$\tilde{H}_n(\mathbb{C}P^\infty) \xrightarrow{\cong} H_{n-2}(\mathbb{C}P^\infty)\,.$$

Since $H_n(\mathbb{C}P^\infty)$ is 0 for $n < 0$ and $\mathbb{Z}$ for $n = 0$, the proposition follows by induction on n. The differential pattern in the two-row Serre spectral sequence is shown below, with $H_*(S^1)$ on the t-axis and $H_*(\mathbb{C}P^\infty)$ on the s-axis.

$$\begin{tikzcd}
1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "d^2"'] & 0 & \mathbb{Z} \arrow[ull, "d^2"'] & 0 & \cdots \arrow[ull, "d^2"'] \\
t/s & 0 & 1 & 2 & 3 & 4 & 5 & \cdots
\end{tikzcd}$$

□

REMARK 4.3.14. For compact Lie groups G, a universal G-bundle $p\colon EG \to BG$ can be constructed geometrically using Stiefel manifolds. Milnor [**Mil56**] gave a construction of a universal G-bundle for general topological groups G, subject to some point-set topological restrictions. Building on work of James Milgram [**Mil67**] and Norman Steenrod [**Ste67**], [**Ste68**], Michael McCord gave the following functorial construction, which we will also make use of in our discussion of orthogonal Eilenberg–MacLane spectra.

DEFINITION 4.3.15 ([**McC69**, §5]). For a monoid G and a pointed set (X, x_0) let $B(G, X)$ be the set of functions $u\colon X \to G$ with $u(x_0) = e$ and $u(x) \neq e$ only for finitely many points $x \in X$. We view u as a finite set of *points in X with labels in G*. For each $n \geq 0$ let $B_n(G, X) \subset B(G, X)$ be the subset consisting of the u such that $u(x) \neq e$ for at most n points $x \in X$.

DEFINITION 4.3.16 ([**McC69**, Def. 9.5]). Let G be a topological monoid, and consider $I = [0, 1]$ based at 0. Let $\Delta_n \subset I^n$ be the set of n-tuples $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1$, and let the surjection

$$\nu_n\colon \Delta_n \times G^n \longrightarrow B_n(G, I)$$

send $(t_1, \dots, t_n, g_1, \dots, g_n)$ to the function $u\colon I \to G$ given by $u(t) = g_a \cdots g_b$ if $t_{a-1} < t_a = t = t_b < t_{b+1}$. Give $B_n(G, I)$ the quotient topology from $\Delta_n \times G^n$, and give $B(G, I)$ the weak ($=$ colimit) topology from the $B_n(G, I)$.

Let $q\colon I \to I/\{0, 1\} \cong S^1$ be the quotient map, and give $B(G, S^1)$ the quotient topology induced by the map

$$p = B(G, q)\colon B(G, I) \longrightarrow B(G, S^1)\,.$$

((ETC: Compare this with the geometric realization of the simplicial bar construction.))

THEOREM 4.3.17 ([**McC69**, Thm. 9.17]). *Let G be a topological group, such that the inclusion $\{e\} \subset G$ is a (Hurewicz) cofibration. Then*

$$p\colon EG = B(G, I) \longrightarrow B(G, S^1) = BG$$

is a universal G-bundle.

DEFINITION 4.3.18. Let G be a topological group with classifying space BG, and let A be an abelian group. A class $c \in H^i(BG; A)$ is called a *characteristic class* for principal G-bundles. To each principal G-bundle $p\colon P \to B$ classified by a map $f\colon B \to BG$ we associate the cohomology class

$$c(p\colon P \to B) = c(f) = f^*(c) \in H^i(B; A)\,,$$

which varies naturally with $p\colon P \to B$ under pullback.

REMARK 4.3.19. The standard reference for characteristic classes is the book by Milnor and Stasheff [**MS74**]. The key examples are the Euler class $e \in H^k(BSO(k))$, the Stiefel–Whitney classes $w_i \in H^i(BO; \mathbb{F}_2)$, the Chern classes $c_i \in H^{2i}(BU)$, and the Pontryagin classes $p_i \in H^{4i}(BO)$. Here $O = \bigcup_k O(k)$ and $U = \bigcup_k U(k)$ are the infinite orthogonal and unitary groups, respectively. These define natural cohomology classes $e \in H^k(B)$ for each oriented $\mathbb{R}^k$-bundle $E \to B$, $w_i \in H^i(B; \mathbb{F}_2)$ and $p_i \in H^{4i}(B)$ for each $\mathbb{R}^k$-bundle $E \to B$, and $c_i \in H^{2i}(B)$ for each $\mathbb{C}^k$-bundle $E \to B$. The latter three are stable, i.e., do not change if we add a trivial bundle to E.

REMARK 4.3.20. The abelian groups

$$\begin{aligned} H_s^{gp}(G) &= H_s(BG) \\ H^s_{gp}(G) &= H^s(BG) \end{aligned}$$

only depend on the topological group G, and are known as the s-th *group homology* and *group cohomology* of G, respectively. When G is discrete, these admit the algebraic descriptions

$$\begin{aligned} H_s^{gp}(G) &\cong \operatorname{Tor}_s^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) \\ H^s_{gp}(G) &\cong \operatorname{Ext}^s_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})\,, \end{aligned}$$

where $\mathbb{Z}[G]$ denotes the integral group ring of G.

4.4. Edge homomorphisms and the transgression

We continue in the situation

$$F \xrightarrow{i} E \xrightarrow{p} B$$

with p a Hurewicz fibration, B a 0-reduced CW complex based at the 0-cell b_0, and $F = p^{-1}(b_0)$ the fiber above that point. The inclusion i of the fiber and the projection p to the base induce homomorphisms $i_*\colon H_*(F) \to H_*(E)$ and $p_*\colon H_*(E) \to H_*(B)$, called the *edge homomorphisms* of the Serre spectral sequence. They can be factored through the components of the E^∞-term that lie on the vertical and horizontal edges, respectively, of the first quadrant.

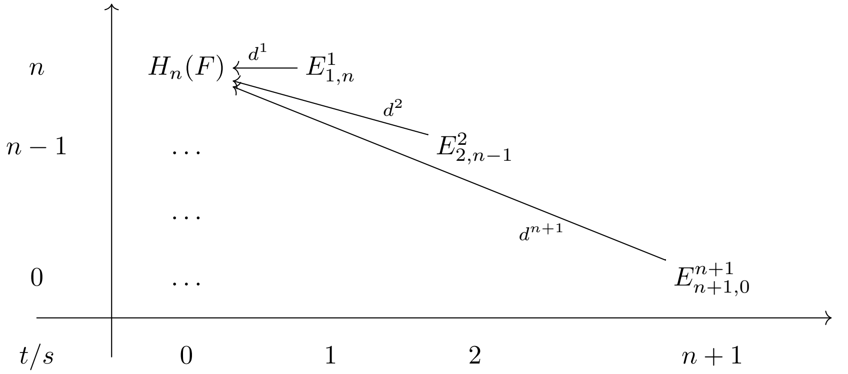
PROPOSITION 4.4.1. *The edge homomorphism $i_*\colon H_n(F) \to H_n(E)$ factors as the surjection*

$$H_n(F) \cong E^1_{0,n} \longrightarrow\!\!\!\!\!\rightarrow E^\infty_{0,n}$$

followed by the inclusion

$$E^\infty_{0,n} \cong F_0 H_n(E) \rightarrowtail H_n(E)\,.$$

PROOF. We have $F = E_0$, since B is 0-reduced, so i_* factors as the canonical surjection $H_n(F) \to F_0 H_n(E)$ followed by the inclusion $F_0 H_n(E) \subset H_n(E)$. The first isomorphism follows from $H_n(F) = H_n(E_0) \cong C_0^{CW}(B; \mathscr{H}_n(F)) = E^1_{0,n}$. By convergence, the second isomorphism follows from $E^\infty_{0,n} \cong F_0 H_n(E)/F_{-1} H_n(E)$, since $F_{-1} H_n(E)$ is trivial. $\square$



((ETC: Strictly speaking, d^2 and the later differentials land in quotient groups of $E^1_{0,n} = H_n(F)$.))

REMARK 4.4.2. Every differential $d^r_{0,n}$ lands in a trival group, so $E^1_{0,n}$ consists of infinite cycles. However, there may be differentials $d^r_{r,n-r+1}$ landing in bidegree $(0, n)$, for $1 \le r \le n+1$, and their cokernels give a sequence of surjections

$$E^1_{0,n} \twoheadrightarrow E^2_{0,n} \twoheadrightarrow \dots \twoheadrightarrow E^{n+1}_{0,n} \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n}\,.$$

If B is 1-connected, then $E^1_{1,n} = 0$, the first surjection above is the identity, and $H_n(F) \cong E^2_{0,n}$.

Suppose hereafter that F is 0-connected.

PROPOSITION 4.4.3. *The edge homomorphism* $p_* \colon H_n(E) \to H_n(B)$ *factors as the surjection*

$$H_n(E) \twoheadrightarrow E^\infty_{n,0}$$

followed by the inclusion

$$E^\infty_{n,0} \rightarrowtail E^2_{n,0} \cong H_n(B)\,.$$

PROOF. The surjection

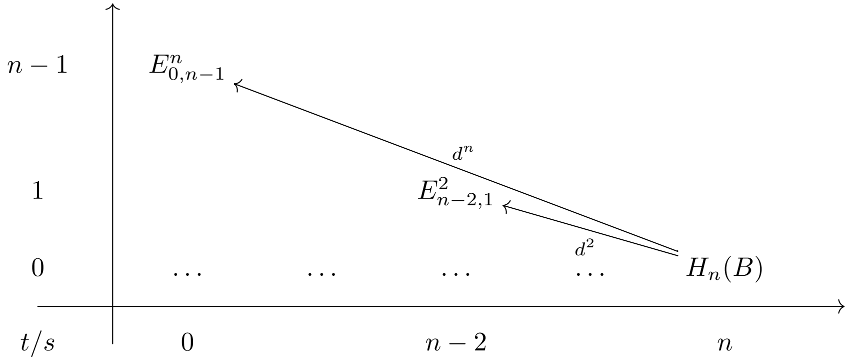
$$H_n(E) = F_n H_n(E) \twoheadrightarrow F_n H_n(E)/F_{n-1} H_n(E) \cong E^\infty_{n,0}$$

is given by convergence. For $r \ge 2$, every d^r-differential landing in bidegree $(n, 0)$ comes from a trivial group, hence is zero. However, there may be nonzero differentials $d^r_{n,0}$ for $2 \le r \le n$, and their kernels give a sequence of inclusions

$$E^\infty_{n,0} = E^{n+1}_{n,0} \subset E^n_{n,0} \subset \dots \subset E^2_{n,0}\,.$$

Since F is 0-connected, the coefficient system $\mathscr{H}_0(F) \cong \mathbb{Z}$ is constant, so $E^2_{n,0} = H_n(B; \mathscr{H}_0(F)) \cong H_n(B)$.

To see that the composite $H_n(E) \to E^\infty_{n,0} \to H_n(B)$ equals p_*, use naturality of Serre spectral sequences with respect to the map from $p\colon E \to B$ to $1\colon B \to B$. ((ETC: More detail?)) $\square$



((ETC: Strictly speaking, d^3 and later differentials are defined on subgroups of $E^2_{n,0} \cong H_n(B)$.))

These results, and the following definition, were discussed in [**Ser51**, §II.7].

Definition 4.4.4. Let $q\colon (E, F) \to (B, b_0)$ denote the map of pairs induced by p, and suppose $n \geq 1$. The additive relation

$$\partial q_*^{-1}\colon H_n(B, b_0) \xleftarrow{q_*} H_n(E, F) \xrightarrow{\partial} H_{n-1}(F)\,,$$

sending $x = q_*(y)$ to the class of $\partial(y)$, defines a homomorphism

$$\tau_n\colon \operatorname{im}(q_*) \longrightarrow H_{n-1}(F)/\partial \ker(q_*)$$

called the *homology transgression*. The elements of $\operatorname{im}(q_*)$, on which τ_n are defined, are said to be *transgressive*.

Proposition 4.4.5. *The transgression τ_n corresponds to the differential*

$$d^n_{n,0}\colon E^n_{n,0} \longrightarrow E^n_{0,n-1}$$

under isomorphisms $E^n_{n,0} \cong \operatorname{im}(q_*)$ and $E^n_{0,n-1} \cong H_{n-1}(F)/\partial \ker(q_*)$.

Proof. A relative version of Proposion 4.4.3 factors $q_*\colon H_n(E, F) \to H_n(B, b_0)$ as a surjection $H_n(E, F) \to E^n_{n,0}$ followed by an inclusion $E^n_{n,0} \subset H_n(B, b_0)$. This gives the isomorphism $\operatorname{im}(q_*) \cong E^n_{n,0}$, and shows that $\ker(q_*) = \ker(H_n(E, F) \to H_n(E, E_{n-1})) = \operatorname{im}(H_n(E_{n-1}, F) \to H_n(E, F))$. Hence $\partial \ker(q_*)$ is the image of $\partial\colon H_n(E_{n-1}, F) \to H_{n-1}(F)$, and $H_{n-1}(F)/\partial \ker(q_*)$ is the coimage of $H_{n-1}(F) \to H_{n-1}(E_{n-1})$.

Consider the following commutative diagram.

$$\begin{tikzcd}
H_n(E_n, F) \arrow[r, "i_*"] \arrow[d, two heads] & H_n(E, F) \arrow[r, "\partial"] \arrow[d, two heads] & H_{n-1}(F) \arrow[d, two heads] \\
Z^n_{n,0} \arrow[r, two heads] \arrow[d, tail] & E^n_{n,0} \arrow[r, "d^n_{n,0}"] \arrow[d, tail] & E^n_{0,n-1} \arrow[d, tail] \\
H_n(E_n, E_{n-1}) \arrow[r, "i_*"] & H_n(E, E_{n-1}) \arrow[r, "\partial"] & H_{n-1}(E_{n-1})
\end{tikzcd}$$

The inclusion $F = E_0 \subset E_{n-1}$ induces the map from the first to the third row, and the second row consists of the images of this vertical map. On the left hand side this follows by rewriting the definition of $Z^n_{n,0} = \partial^{-1} \operatorname{im}(H_{n-1}(F) \to H_{n-1}(E_{n-1}))$ as $\ker(H_n(E_n, E_{n-1}) \to H_{n-1}(E_{n-1}, F)) = \operatorname{im}(H_n(E_n, F) \to H_n(E_n, E_{n-1}))$. For the middle and right hand sides it follows by consideration of the relative fibration $q\colon (E, F) \to (B, b_0)$ and the restricted fibration $E_{n-1} \to B^{(n-1)}$, respectively. Diagram chases then confirm that $Z^n_{n,0} \to E^n_{n,0}$ is the canonical surjection, and that the induced homomorphism $E^n_{n,0} \to E^n_{0,n-1}$ equals the d^n-differential. $\square$

We can now deduce Serre's (finite length) exact homology sequence for a fibration. To avoid a discussion of local coefficients or orientability, we restrict to the case where B is 1-connected.

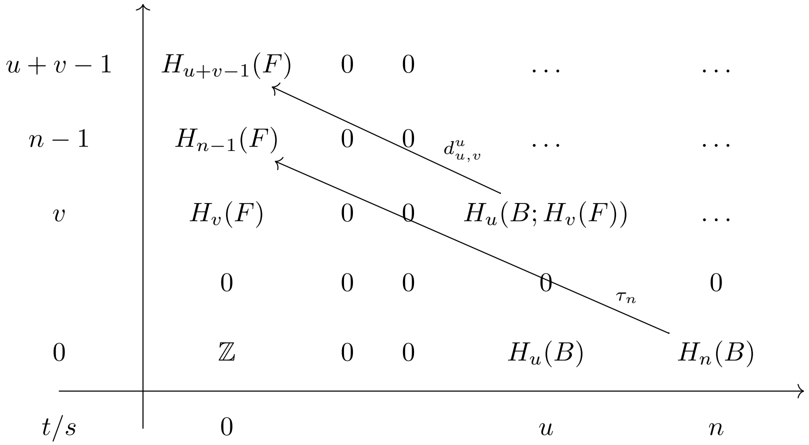
Theorem 4.4.6 ([**Ser51**, Prop. III.5]). *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$. Then there is an exact sequence*

$$\begin{aligned}
H_{u+v-1}(F) \xrightarrow{i_*} H_{u+v-1}(E) \xrightarrow{p_*} H_{u+v-1}(B) \xrightarrow{\tau_{u+v-1}} \dots \\
\dots \xrightarrow{\tau_{n+1}} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\tau_n} H_{n-1}(F) \xrightarrow{i_*} \dots \\
\dots \xrightarrow{p_*} H_2(B) \xrightarrow{\tau_2} H_1(F) \xrightarrow{i_*} H_1(E) \to 0\,.
\end{aligned}$$

Proof. By hypothesis, and the universal coefficient theorem, the Serre spectral sequence

$$E^2_{s,t} = H_s(B; H_t(F)) \Longrightarrow_s H_{s+t}(E)$$

is concentrated in the region where $s \in \{0, u, u+1, \dots\}$ and $t \in \{0, v, v+1, \dots\}$.



Hence the only (nonzero) differentials originating in total degree $n = s + t < u + v$ are the transgressions $\tau_n \colon H_n(B) = E^n_{n,0} \to E^n_{0,n-1} = H_{n-1}(F)$. The first possible differential from total degree $u + v$ is $d^u_{u,v} \colon E^u_{u,v} \to E^u_{0,u+v-1} = H_{u+v-1}(F)$, where $E^u_{u,v}$ is a quotient of $E^2_{u,v} = H_u(B; H_v(F))$. It follows that in total degrees $s + t < u + v$ the E^∞-term is given by

$$E^\infty_{s,t} = \begin{cases} \mathbb{Z} & \text{for } (s,t) = (0,0), \\ \ker(\tau_n) & \text{for } s = n \geq 2 \text{ and } t = 0, \\ \operatorname{cok}(\tau_n) & \text{for } s = 0 \text{ and } t = n-1 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In total degree $1 \leq n \leq u + v - 1$ we therefore have

$$F_0 H_n(E) = \cdots = F_{n-1} H_n(E) = \operatorname{cok}(\tau_{n+1})$$

and a short exact sequence

$$0 \to F_{n-1} H_n(E) \longrightarrow H_n(E) \longrightarrow \ker(\tau_n) \to 0\,.$$

This gives an exact sequence

$$H_{n+1}(B) \xrightarrow{\tau_{n+1}} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{\tau_n} H_{n-1}(F)$$

for each $n \leq u + v - 2$. When $n = u + v - 1$ the target of τ_{n+1} is a quotient of $H_n(F)$, but i_* nonetheless maps $H_n(F)$ onto its cokernel. Splicing these together we obtain Serre's exact sequence. $\square$

Serre's sequence agrees with the long exact homology sequence of the pair (E, F), in the stated range of degrees. The following reformulation is dual to a form of the homotopy excision theorem cf. [**Hat02**, Prop. 4.28].

PROPOSITION 4.4.7. *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$. Then*

$$q_* \colon H_n(E, F) \longrightarrow H_n(B, b_0)$$

is an isomorphism for $n \leq u + v - 1$ and is surjective for $n = u + v$.

Proof. There is a relative Serre spectral sequence

$$E^2_{s,t} = H_s(B, b_0; H_t(F)) \Longrightarrow_s H_{s+t}(E, F)$$

obtained by omitting the edge $s = 0$. The differential in lowest possible total degree is

$$d^{v+1} \colon E^{v+1}_{u+v+1,0} \longrightarrow E^{v+1}_{u,v} ,$$

so when $n \le u + v$ the edge homomorphism

$$q_* \colon H_n(E, F) \longrightarrow E^\infty_{n,0} = H_n(B)$$

is an isomorphism for $n < u + v$ and a surjection when $n = u + v$. □

Corollary 4.4.8. *If $i_* \colon H_n(F) \to H_n(E)$ is an isomorphism for $n < k$ and surjective for $n = k$, then $H_s(B) = 0$ for $0 < s \le k$.*

Proof. We apply the proposition with $v = 1$. By the long exact homology sequence for (E, F) we have $H_n(E, F) = 0$ for $n \le k$. Hence, if $H_s(B) = 0$ for $0 < s < u$ then $0 = H_u(E, F) \cong H_u(B)$ as long as $u \le k$. By induction on u it follows that $H_s(B) = 0$ for $0 < s \le k$. □

We can also compare Serre's sequence with the long exact homology sequence of the pair (Mp, E), where $Mp \simeq B$ is the mapping cylinder of $p \colon E \to B$. See Hall [**Hal65**] or Clapp [**Cla81**] for the fact that $q \colon Mp \to B$ is a Hurewicz fibration. ((ETC: Maybe a relative kind of fibration is needed.))

Proposition 4.4.9. *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. Suppose that $H_s(B) = 0$ for $0 < s < u$ and that $H_t(F) = 0$ for $0 < t < v$. Then*

$$\tilde{H}_{n-1}(F) \cong H_n(CF, F) \xrightarrow{i_*} H_n(Mp, E)$$

is an isomorphism for $n < u + v$ and is surjective for $n = u + v$.

Proof. There is a relative Serre spectral sequence

$$E^2_{s,t} = H_s(B; H_t(CF, F)) \Longrightarrow_s H_{s+t}(Mp, E)$$

obtained by omitting the edge $t = 0$ and increasing the fiber degrees by 1. The differential in lowest possible total degree is

$$d^u \colon E^u_{u,v+1} \longrightarrow E^u_{0,u+v} ,$$

so when $n \le u + v$ the edge homomorphism

$$i \colon H_n(CF, F) \longrightarrow E^\infty_{0,n} \cong H_n(Mp, E)$$

is an isomorphism for $n < u + v$ and a surjection for $n = u + v$. □

Corollary 4.4.10. *If $p_* \colon H_n(E) \to H_n(B)$ is an isomorphism for $n < k$ and surjective for $n = k$, then $H_t(F) = 0$ for $0 < t < k$.*

Proof. We apply the proposition with $u = 2$. By the long exact homology sequence for (Mp, E), and the equivalence $Mp \simeq B$, we have $H_n(Mp, E) = 0$ for $n \le k$. Hence, if $H_t(F) = 0$ for $0 < t < v$ then $H_v(F) \cong H_{v+1}(CF, F) \cong H_{v+1}(Mp, E)$ vanishes as long as $v + 1 \le k$. By induction on v it follows that $H_t(F) = 0$ for $0 < t < k$. □

4.5. Theorems of Hurewicz and Freudenthal

We can deduce absolute and relative Hurewicz theorems, as well as Freudenthal's suspension theorem, from Serre's exact sequence. Spectral sequences thus give an alternative approach to these results, as opposed to the homotopy excision theorem with its geometric proof, which was used to deduce these results in [**Hat02**, §4.2].

DEFINITION 4.5.1. Let $s_n \in \tilde{H}_n(S^n)$ be a chosen generator, and let X be any based space. The *(absolute) Hurewicz homomorphism*

$$h_n \colon \pi_n(X) \longrightarrow \tilde{H}_n(X)$$

is given by

$$[f] \longmapsto f_*(s_n)\,.$$

The elements in the image of h_n are said to be *spherical.*

Let $d_{n+1} \in H_{n+1}(D^{n+1}, S^n)$ be a chosen generator, and let (X, A) be any pair of based spaces. The *relative Hurewicz homomorphism*

$$h_{n+1} \colon \pi_{n+1}(X, A) \longrightarrow H_{n+1}(X, A)$$

is given by

$$[f] \longmapsto f_*(d_{n+1})\,.$$

REMARK 4.5.2. With a specified suspension isomorphism

$$\sigma \colon \tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$$

we can demand that $S^{n+1} = \Sigma S^n$ and $\sigma(s_n) = s_{n+1}$. Then h_n and h_{n+1} are compatible with Freudenthal's suspension $E \colon \pi_n(X) \to \pi_{n+1}(\Sigma X)$ and the isomorphism above. We can also demand that

$$\partial \colon H_{n+1}(D^{n+1}, S^n) \longrightarrow \tilde{H}_n(S^n)$$

maps d_{n+1} to s_n, in which case the relative h_{n+1} and the absolute h_n are compatible with the connecting homomorphisms $\partial \colon \pi_{n+1}(X, A) \to \pi_n(A)$ and $\partial \colon H_{n+1}(X, A) \to \tilde{H}_n(A)$.

First, we have the absolute Hurewicz theorem for 1-connected CW complexes. We refer to [**Hat02**, Thm. 4.32] for a Hurewicz theorem for general spaces.

THEOREM 4.5.3. *Let X be an $(n-1)$-connected CW complex, with $n \geq 2$. Then*

$$h_n \colon \pi_n(X) \xrightarrow{\cong} H_n(X)$$

is an isomorphism.

PROOF. We prove this by induction on n. By a theorem of Milnor [**Mil59**, Cor. 3], ΩX has the homotopy type of a $(n-2)$-connected CW complex, so we may inductively assume that $h_{n-1} \colon \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is an isomorphism. To start the induction, for $n = 2$, we appeal to Poincaré's result [**Hat02**, Thm. 2A.1] that $h_1 \colon \pi_1(\Omega X) \to H_1(\Omega X)$ is an isomorphism, where $\pi_1(\Omega X) = \pi_2(X)$ is already abelian.

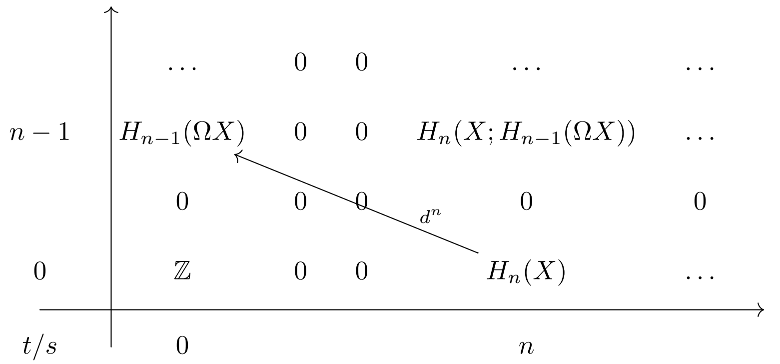
Consider the homotopy fiber sequence

$$\Omega X \longrightarrow PX \longrightarrow X$$

and the associated commutative diagram

$$\begin{tikzcd}
\pi_{n-1}(\Omega X) \arrow[d, "h_{n-1}"'] & \pi_n(PX,\Omega X) \arrow[l, "\partial"', "\cong"] \arrow[r, "q_*", "\cong"'] \arrow[d] & \pi_n(X) \arrow[d, "h_n"] \\
H_{n-1}(\Omega X) & H_n(PX,\Omega X) \arrow[l, "\partial"', "\cong"] \arrow[r, "q_*"] & H_n(X)\,.
\end{tikzcd}$$

The isomorphisms in the upper row follow from the long exact homotopy sequences of a pair and of a fibration, together with the fact that PX is contractible. The isomorphism in the lower row follows from the long exact homology sequence of the pair $(PX, \Omega X)$, and the fact just mentioned. By our inductive hypothesis, the left hand homomorphism h_{n-1} is an isomorphism.



By the Serre spectral sequence, or the exact sequence deduced from it, the homomorphism $q_*\colon H_n(PX,\Omega X) \to H_n(X)$ is an isomorphism, since $n \le n + (n-1) - 1$ for $n \ge 2$. Hence $h_n\colon \pi_n(X) \to H_n(X)$ is an isomorphism. $\square$

COROLLARY 4.5.4. *If X is a 1-connected CW complex with $H_m(X) = 0$ for $0 < m < n$ then X is $(n-1)$-connected.*

PROOF. For $m < n$, suppose we have proved that X is $(m-1)$-connected. Then $h_m\colon \pi_m(X) \to H_m(X)$ is an isomorphism, so the assumption that $H_m(X) = 0$ implies that $\pi_m(X) = 0$. Hence X is m-connected. Continue inductively, until $m = n - 1$. $\square$

Second, we have a relative Hurewicz theorem for maps of 1-connected CW complexes. We refer to [**Hat02**, Thm. 4.37] for a relative Hurewicz theorem for 0-connected spaces.

DEFINITION 4.5.5. A map $f\colon X \to Y$ of 0-connected spaces is *n-connected* if $f_*\colon \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$. Replacing f by the inclusion $X \subset Mf$ into the mapping cylinder of f, and considering the long exact homotopy sequence

$$\dots \to \pi_m(X) \longrightarrow \pi_m(Mf) \longrightarrow \pi_m(Mf, X) \xrightarrow{\partial} \pi_{m-1}(X) \to \dots$$

of the pair (Mf, X), we see that f is n-connected if and only if $\pi_m(Mf, X) = 0$ for each $m \le n$.

DEFINITION 4.5.6. For any map $f\colon X \to Y$ let $p\colon Ef = X \times_Y Y^I \to Y$ be the associated *path space fibration*. There is a homotopy equivalence $X \to Ef$, compatible with the two maps to Y. The fiber $p^{-1}(y_0) = Ff = X \times_Y PY$ of this fibration is the *homotopy fiber* of f at $y_0 \in Y$. The Serre spectral sequence

$$E^2_{s,t} = H_s(Y; H_t(Ff)) \Longrightarrow_s H_{s+t}(Ef)$$

for

$$Ff \longrightarrow Ef \xrightarrow{p} Y$$

can be rewritten in the form

$$E^2_{s,t} = H_s(Y; H_t(Ff)) \Longrightarrow_s H_{s+t}(X)\,,$$

in which case we think of it as being associated to the homotopy fiber sequence

$$Ff \longrightarrow X \xrightarrow{f} Y\,.$$

Milnor [**Mil59**, Thm. 3] proved that if X and Y are homotopy equivalent to CW complexes, then so is Ff.

THEOREM 4.5.7. *Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $\pi_m(Mf, X) = 0$ for $m \le n$, where $Mf \simeq Y$ is the mapping cylinder of f. Then*

$$h_{n+1}\colon \pi_{n+1}(Mf, X) \xrightarrow{\cong} H_{n+1}(Mf, X)$$

is an isomorphism.

PROOF. There is only something to prove for $n \ge 1$. Using a path space fibration we may replace $f\colon X \to Y$ with a homotopy equivalent Hurewicz fibration $p\colon E \to B$, with B a CW complex. Its fiber $F = p^{-1}(b_0)$ is then the homotopy fiber of f, and $\pi_m(Mp, E) = 0$ for $m \le n$. In the commutative diagram

$$\begin{tikzcd}
\pi_n(F) \arrow[d, "h_n"', "\cong"] & \pi_{n+1}(CF, F) \arrow[l, "\partial"', "\cong"] \arrow[r, "i_*", "\cong"'] \arrow[d, "h_{n+1}"] & \pi_{n+1}(Mp, E) \arrow[d, "h_{n+1}"] \\
H_n(F) & H_{n+1}(CF, F) \arrow[l, "\partial"', "\cong"] \arrow[r, "i_*", "\cong"'] & H_{n+1}(Mp, E)
\end{tikzcd}$$

the upper row consists of isomorphisms, because F is equivalent to the homotopy fiber of the inclusion $E \subset Mp$. Likewise, $\pi_{m-1}(F) \cong \pi_m(Mp, E) = 0$ for $m \le n$, so F is $(n-1)$-connected. If $n = 1$, then $\pi_1(F)$ is a quotient of $\pi_2(B)$, since $\pi_1(E) = 0$, so $\pi_1(F)$ is abelian. The absolute Hurewicz theorem for F thus tells us that the left hand h_n is an isomorphism. The lower row consists of isomorphisms by Proposition 4.4.9, applied to $F \to E \to B$ with $u = 2$ and $v = n$. Hence the right hand h_{n+1} is an isomorphism. □

COROLLARY 4.5.8. *Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $f_*\colon H_m(X) \to H_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$. Then $f_*\colon \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$.*

PROOF. An equivalent statement is the following: Let $f\colon X \to Y$ be a map of 1-connected CW complexes, and suppose that $H_m(Mf, X) = 0$ for $m \le n$. Then $\pi_m(Mf, X) = 0$ for $m \le n$.

For $m \le n$, suppose we have proved that f is $(m-1)$-connected. Then $h_m \colon \pi_m(Mf, X) \to H_m(Mf, X)$ is an isomorphism by Theorem 4.5.7, so the assumption that $H_m(Mf, X) = 0$ implies that $\pi_m(Mf, X) = 0$. Hence f is m-connected. Continue inductively, until $m = n$. □

Third, we turn to Freudenthal's suspension homomorphism

$$E \colon \pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X).$$

DEFINITION 4.5.9. We define the *cone* and *suspension* of a based space X to be $CX = I \wedge X$ and $\Sigma X = S^1 \wedge X \cong CX/X = I/\partial I \wedge X$, respectively. We write $[t, x]$ for the image of $(t, x) \in I \times X$ under the quotent map to CX or $CX/X \cong \Sigma X$.

LEMMA 4.5.10. *Let $\eta \colon X \to \Omega\Sigma X$ map x to the loop $s \mapsto [s, x]$, and let $\bar\eta \colon CX \to P\Sigma X$ map $[t, x]$ to the path $s \mapsto [st, x]$. Then the diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\eta"'] & CX \arrow[r] \arrow[d, "\bar\eta"'] & \Sigma X \arrow[d, "="] \\
\Omega\Sigma X \arrow[r] & P\Sigma X \arrow[r, "p"] & \Sigma X
\end{tikzcd}$$

commutes.

PROOF. Direct from the definitions. □

PROPOSITION 4.5.11. *Let X be a $(k-1)$-connected CW complex. Then*

$$\eta_* \colon H_n(X) \xrightarrow{\cong} H_n(\Omega\Sigma X)$$

is an isomorphism for $n \le 2k-1$.

PROOF. There is only something to prove for $k \ge 1$. By the previous lemma we have a commutative diagram

$$\begin{tikzcd}
H_n(X) \arrow[d, "\eta_*"'] & H_{n+1}(CX, X) \arrow[l, "\partial"', "\cong"] \arrow[r, "\cong"] \arrow[d, "\bar\eta_*"'] & H_{n+1}(\Sigma X) \arrow[d, "="] \\
H_n(\Omega\Sigma X) & H_{n+1}(P\Sigma X, \Omega\Sigma X) \arrow[l, "\partial"', "\cong"] \arrow[r, "q_*"] & H_{n+1}(\Sigma X).
\end{tikzcd}$$

Note that ΣX is k-connected and $\Omega\Sigma X$ is $(k-1)$-connected, so q_* in the lower row is an isomorphism for $n+1 \le 2k$ by Proposition 4.4.7. □

This gives a proof of Freudenthal's suspension theorem, cf. [**Hat02**, Cor. 4.24].

THEOREM 4.5.12. *Let X be a $(k-1)$-connected CW complex. Then $\eta \colon X \to \Omega\Sigma X$ is $(2k-1)$-connected, meaning that*

$$\eta_* \colon \pi_n(X) \longrightarrow \pi_n(\Omega\Sigma X)$$

and

$$E \colon \pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

are isomorphisms for $n < 2k-1$ and surjective for $n = 2k-1$.

PROOF. When $k = 1$ we use that X and $\Omega\Sigma X$ are 0-connected, so that in the commutative diagram

$$\begin{tikzcd} \pi_1(X) \arrow[r, "\eta_*"] \arrow[d, two heads, "h_1"'] & \pi_1(\Omega\Sigma X) \arrow[d, "\cong"', "h_1"] \\ H_1(X) \arrow[r, "\eta_*", "\cong"'] & H_1(\Omega\Sigma X) \end{tikzcd}$$

the vertical maps are the abelianization homomorphisms, which is surjective for X and an isomorphism for $\Omega\Sigma X$, since $\pi_1(\Omega\Sigma X) \cong \pi_2(\Sigma X)$ is commutative. The lower homomorphism η_* is an isomorphism by the proposition above, hence the upper homomorphism η_* is surjective.

For $k \geq 2$ we use that X is 1-connected to deduce that ΣX is 2-connected and $\Omega\Sigma X$ is 1-connected. Hence Corollary 4.5.8 and Proposition 4.5.11 imply that $\eta\colon X \to \Omega\Sigma X$ is $(2k-1)$-connected. The suspension homomorphism E corresponds to η_* under the isomorphism $\pi_{n+1}(\Sigma X) \cong \pi_n(\Omega\Sigma X)$. $\square$

4.6. Finite generation and finiteness

((ETC: Discuss some results from [**Ser51**, Ch. V].))

DEFINITION 4.6.1. An abelian group G is *finitely generated* if there exists a surjective homomorphism

$$\mathbb{Z}^k \longrightarrow G$$

for some finite k. In this case,

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_s$$

is isomorphic to a finite direct sum of cyclic groups, i.e., groups of the form $\mathbb{Z}$ or $\mathbb{Z}/m$, where $m \geq 2$. Here r is the dimension of $G \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space, which we call the *rank* of the group G.

DEFINITION 4.6.2. A space X has *homology of finite type* if each group $H_n(X)$ is finitely generated.

A 1-connected space X has *homotopy of finite type* if each homotopy group $\pi_n(X)$ is finitely generated. In this case we also say that X has *finite type*.

((ETC: What should homotopy of finite type mean for spaces with nontrivial fundamental group? With multiple path components?))

We will show that a 1-connected space has homology of finite type if and only if it has (homotopy of) finite type. This applies, for instance, to $X = S^n$ for $n \geq 2$. The following is a special case of [**Whi78**, Thm. XIII.7.11].

THEOREM 4.6.3. *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. If two of the following conditions hold, then so does the third.*

1. *$H_t(F)$ is finitely generated for each t.*
2. *$H_n(E)$ is finitely generated for each n.*
3. *$H_s(B)$ is finitely generated for each s.*

PROOF. There are three cases, which we treat in sequence.

(1) If F and B have homology of finite type, then each group

$$E^2_{s,n-s} = H_s(B; H_{n-s}(F))$$

is finitely generated, by the universal coefficient theorem. Hence so is each subquotient

$$E^\infty_{s,n-s} \cong F_sH_n(E)/F_{s-1}H_n(E)\,.$$

It follows by induction on $s \ge 0$ that each $F_sH_n(E)$ is finitely generated. When $s = n$, this equals $H_n(E)$.

(2) If F and E have homology of finite type, then we must show that B has homology of finite type. Let $n \ge 2$ and assume by induction that $H_s(B)$ is finitely generated for $s < n$. Then

$$E^2_{s,t} = H_s(B; H_t(F))$$

is finitely generated for each $s < n$ and t, hence so is each subquotient $E^r_{s,t}$ in this region. Since $H_n(E)$ is finitely generated, so is its quotient

$$E^\infty_{n,0} \cong H_n(E)/F_{n-1}H_n(E)\,.$$

We prove by descending induction on r that $E^r_{n,0}$ is finitely generated. This is clear for $r = n+1$, since $E^{n+1}_{n,0} = E^\infty_{n,0}$. Suppose that $E^{r+1}_{n,0}$ is finitely generated, where $r \ge 2$. We have an exact sequence

$$0 \to E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d^r_{n,0}} E^r_{n-r,r-1}\,.$$

Here $E^r_{n-r,r-1}$ is one of the subquotients we have argued must be finitely generated, hence its subgroup $\mathrm{im}(d^r_{n,0})$ is also finitely generated. We have assumed inductively that $E^{r+1}_{n,0}$ is finitely generated, so this extension proves that $E^r_{n,0}$ is finitely generated. Hence

$$E^2_{n,0} = H_n(B; H_0(F)) \cong H_n(B)$$

is finitely generated, as we wanted to prove.

(3) If E and B have homology of finite type, then we must prove that F has homology of finite type. Let $n \ge 1$ and assume by induction that $H_t(F)$ is finitely generated for $t < n$. Then

$$E^2_{s,t} = H_s(B; H_t(F))$$

is finitely generated for each s and $t < n$, hence so is each subquotient $E^r_{s,t}$ in this region. Since $H_n(E)$ is finitely generated, so is its subgroup

$$E^\infty_{0,n} \cong F_0H_n(E)\,.$$

We prove by descending induction on r that $E^r_{0,n}$ is finitely generated. This is clear for $r = n+2$, since $E^{n+2}_{0,n} = E^\infty_{0,n}$. Suppose that $E^{r+1}_{0,n}$ is finitely generated, where $r \ge 2$. We have an exact sequence

$$E^r_{r,n-r+1} \xrightarrow{d^r_{r,n-r+1}} E^r_{n,0} \longrightarrow E^{r+1}_{n,0} \to 0\,.$$

Here $E^r_{r,n-r+1}$ is one of the subquotients we have argued must be finitely generated, hence its quotient group $\mathrm{im}(d^r_{r,n-r+1})$ must also be finitely generated. We have assumed inductively that $E^{r+1}_{n,0}$ is finitely generated, so this extension proves that $E^r_{n,0}$ is finitely generated. Hence

$$E^2_{n,0} = H_0(B; H_n(F)) \cong H_n(F)$$

is finitely generated, as we wanted to prove. □

THEOREM 4.6.4. *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. If two of the following conditions hold, then so does the third.*

(1) *$\tilde{H}_t(F)$ is finite for each t.*
(2) *$\tilde{H}_n(E)$ is finite for each n.*
(3) *$\tilde{H}_s(B)$ is finite for each s.*

SKETCH PROOF. In the proof of Theorem 4.6.3 replace "finitely generated" by "finite", making allowance for the fact that $H_0(X) = \mathbb{Z}$ for each of the spaces in question. □

((ETC: Serre classes.))

DEFINITION 4.6.5. Let G be a discrete group, and $n \geq 0$. An *Eilenberg–MacLane space* of type (G, n) is a CW complex $K(G, n)$ such that

$$\pi_i K(G, n) \cong \begin{cases} G & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

When viewed as a discrete space, the group G is of type $(G, 0)$.

Eilenberg–MacLane spaces of type $(G, 1)$ can be constructed by giving a presentation of G in terms of generators and relations, and building a 0-reduced CW complex X with one 1-cell for each generator and one 2-cell realizing each relation, so that $\pi_1(X) \cong G$. One then attaches k-cells for $k \geq 3$ to kill the higher homotopy groups.

For $n \geq 2$ and G abelian an Eilenberg–MacLane space of type (G, n) can be constructed from a presentation of G by building an $(n-1)$-reduced CW complex X with one n-cell for each generator and one $(n+1)$-cell for each relation, so that $\pi_n(X) \cong H_n(X) \cong G$. One then attaches k-cells for $k \geq n+2$ to kill the higher homotopy groups.

It follows by an obstruction theory argument that any two Eilenberg–MacLane spaces of the same type (G, n) are homotopy equivalent, by a map that induces the identity $G = G$ on π_n. Hence there is an equivalence

$$K(G, n-1) \simeq \Omega K(G, n)$$

whenever $K(G, n)$ is defined, and there are homotopy fiber sequences

$$G \longrightarrow PK(G, 1) \xrightarrow{p} K(G, 1)$$

for any group G, and

$$K(G, n-1) \longrightarrow PK(G, n) \xrightarrow{p} K(G, n)$$

for any abelian group G and $n \geq 1$. In particular, for any universal G-bundle

$$G \longrightarrow EG \xrightarrow{p} BG \simeq K(G, 1)$$

the classifying space BG is an Eilenberg–MacLane space of type $(G, 1)$. As noted in Remark 4.3.20, its (co-)homology groups are the group (co-)homology groups of G, which admit a purely algebraic description in terms of Tor and Ext over the ring $\mathbb{Z}[G]$.

PROPOSITION 4.6.6. *Let G be a finitely generated abelian group. Then each homology group*

$$H_i(BG) = H_i(K(G, 1))$$

is finitely generated. If G is finite, then each reduced homology group $\tilde{H}_i(BG)$ is finite.

PROOF. We can write G as a finite product

$$G \cong \mathbb{Z} \times \dots \times \mathbb{Z} \times C_{m_1} \times \dots C_{m_s}$$

of cyclic groups, and there is then a homotopy equivalence

$$BG \simeq B\mathbb{Z} \times \dots \times B\mathbb{Z} \times BC_{m_1} \times \dots \times BC_{m_s},$$

since both sides are Eilenberg–MacLane spaces of type $(G, 1)$. Here $B\mathbb{Z} \simeq S^1$ has the homotopy type of the circle, and $BC_m \simeq S^\infty/C_m$ has the homotopy type of an infinite lens space, i.e., the orbit space for the free action by $C_m \subset U(1)$ on the contractible space $S^\infty = S(\mathbb{C}^\infty)$. Both S^1 and S^∞/C_m admit CW structures with finitely many cells in each dimension, cf. [**Hat02**, Ex. 2.43], hence have homology of finite type. More precisely,

$$H_i(S^1) \cong \begin{cases} \mathbb{Z} & \text{for } i \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(BC_m) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/m & \text{for } i \geq 1 \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

By the Künneth theorem (or Serre spectral sequence for the product fibration), it follows that the finite product BG has homology of finite type. If G is finite, so that $r = 0$, it also follows that the reduced homology groups of BG are finite. □

We can now prove two corollaries from [**Ser51**, §VI.2].

PROPOSITION 4.6.7. *Let G be a finitely generated abelian group, and let $n \geq 1$. Then each homology group*

$$H_i(K(G, n))$$

is finitely generated.

PROOF. This was proved in the previous proposition for $n = 1$. The cases $n \geq 2$ follow by induction, by Theorem 4.6.3 applied to the homotopy fiber sequence

$$K(G, n-1) \longrightarrow PK(G, n) \xrightarrow{p} K(G, n),$$

where we know that $F \simeq K(G, n-1)$ and $E = PK(G, n) \simeq *$ have homology of finite type, while $B = K(G, n)$ is 1-connected. □

PROPOSITION 4.6.8. *Let G be a finite abelian group, and let $n \geq 1$. Then each reduced homology group*

$$\tilde{H}_i(K(G, n))$$

is finite.

PROOF. In the previous proof, replace "finitely generated" by "finite" and replace the reference to Theorem 4.6.3 with Theorem 4.6.4. □

Recall how Postnikov sections and Whitehead covers can be constructed by the method of killing homotopy groups.

LEMMA 4.6.9. *Let X be a 0-connected CW complex, and let $n \geq 0$. There is a homotopy fiber sequence*

$$\tau_{>n}X \xrightarrow{i} X \xrightarrow{p} \tau_{\leq n}X$$

where

$$p_*\colon \pi_m(X) \longrightarrow \pi_m(\tau_{\leq n}X)$$

is an isomorphism for $m \leq n$ and $\pi_m(\tau_{\leq n}X) = 0$ for $m > n$. Equivalently,

$$i_*\colon \pi_m(\tau_{>n}X) \longrightarrow \pi_m(X)$$

is an isomorphism for $m > n$ and $\pi_m(\tau_{>n}X) = 0$ for $m \leq n$.

PROOF. We inductively obtain $\tau_{\leq n}X$ from X by attaching $(k+1)$-cells to kill π_k of the previous stage, for each $k \geq n+1$. If X was $(n-1)$-connected, the result is a $K(G,n)$ with $G = \pi_n(X)$. We let $\tau_{>n}X$ be the homotopy fiber of the map $p\colon X \to \tau_{\leq n}X$. □

DEFINITION 4.6.10. We call $\tau_{\leq n}X = \tau_{<n+1}X$ the *n-th Postnikov section* of X, and refer to $\tau_{>n}X = \tau_{\geq n+1}X$ as the *n-connected cover* of X.

REMARK 4.6.11. There are equivalences

$$\tau_{\leq n}(\tau_{\geq n}X) \simeq K(\pi_n(X), n) \simeq \tau_{\geq n}(\tau_{\leq n}X)$$

obtained by passing to the n-th Postnikov section and the $(n-1)$-connected cover, in either order.

THEOREM 4.6.12. *Let X be a 1-connected space. Then X has homology of finite type if and only if it has (homotopy of) finite type.*

PROOF. We may assume that X is a CW complex, and prove the two implications in order.

(1) Suppose that X is 1-connected with homology of finite type. Let $n \geq 2$ and suppose, by induction, that the $(n-1)$-connected cover $\tau_{\geq n}X$ has homology of finite type. Then

$$\pi_n(X) \cong \pi_n(\tau_{\geq n}X) \cong H_n(\tau_{\geq n}X)$$

is finitely generated, so $K(\pi_n(X), n)$ has homology of finite type. By Theorem 4.6.3 applied to the homotopy fiber sequence

$$\tau_{>n}X \longrightarrow \tau_{\geq n}X \longrightarrow K(\pi_n(X), n)$$

it follows that $\tau_{>n}X$ has homology of finite type, completing the inductive step. In the course of the proof, we also showed that $\pi_n(X)$ is finitely generated, for each $n \geq 2$, so X has (homotopy of) finite type.

(2) Suppose that X is 1-connected (with homotopy) of finite type. Let $n \geq 2$ and consider the map $p\colon X \to \tau_{\leq n}X$ to the n-th Postnikov section. It induces an isomorphism on π_m for $m \leq n$, and a surjection for $m = n+1$, hence is $(n+1)$-connected. By the relative Hurewicz theorem 4.5.7, it follows that $p_*\colon H_n(X) \to H_n(\tau_{\leq n}X)$ is an isomorphism. It therefore suffices to prove that $\tau_{\leq n}X$ has homology of finite type. This follows by a finite induction from Theorem 4.6.3 applied to the homotopy fiber sequences

$$K(\pi_m(X), m) \longrightarrow \tau_{\leq m}X \longrightarrow \tau_{<m}X\,,$$

since each space $K(\pi_m(X), m)$ has homology of finite type by Proposition 4.6.7. □

COROLLARY 4.6.13. *Each group $\pi_i(S^n)$ is finitely generated.*

THEOREM 4.6.14. *Let X be a 1-connected space. Then $\tilde{H}_n(X)$ is finite for each n if and only if $\pi_n(X)$ is finite for each n.*

PROOF. In the proof of Theorem 4.6.12, replace “finitely generated” by “finite” and replace the reference to Theorem 4.6.3 with Theorem 4.6.4. $\square$

Using the multiplicative structure in the cohomology Serre spectral sequence, we will make the following calculation. See Corollary 6.6.3.

THEOREM 4.6.15 (Serre). *Let $n \geq 1$ be odd. Then*

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Granting this, we can make the following deductions.

COROLLARY 4.6.16. *Let $n \geq 1$ be odd, and let $f \colon S^n \to K(\mathbb{Z}, n)$ represent a generator of $\pi_n K(\mathbb{Z}, n) \cong \mathbb{Z}$. Then*

(1) $H_0(K(\mathbb{Z}, n)) \cong \mathbb{Z}$.
(2) $H_i(K(\mathbb{Z}, n)) = 0$ *for* $0 < i < n$.
(3) $f_* \colon H_n(S^n) \to H_n(K(\mathbb{Z}, n))$ *is an isomorphism.*
(4) $H_i(K(\mathbb{Z}, n))$ *is finite for each* $i > n$.

PROOF. Cases (1), (2) and (3) follow from the Hurewicz theorem. Case (4) follows from Theorems 4.6.12 and 4.6.15, since a finitely generated abelian group of rank 0 is finite. $\square$

THEOREM 4.6.17 ([**Ser51**, Prop. V.3]). *Let $n \geq 1$ be odd. Then $\pi_i(S^n)$ is finite for each $i > n$.*

PROOF. The case $n = 1$ is well known, so we assume $n \geq 3$. Replace the map $f \colon S^n \to K(\mathbb{Z}, n)$ by an equivalent Hurewicz fibration $p \colon E \to B$ with fiber F. There is then a homotopy fiber sequence

$$F \longrightarrow S^n \xrightarrow{f} K(\mathbb{Z}, n),$$

where $F = \tau_{>n} S^n$ is the n-connected cover of S^n. In particular, $\tilde{H}_t(F) = 0$ for $t \leq n$.

$$
\begin{tikzcd}[row sep=small, column sep=small]
v & H_v(F) & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
n+1 & \cdots & 0 & 0 & \cdots \arrow[uulll] & \cdots & \cdots & \cdots & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & \cdots & \cdots & \cdots & \cdots \arrow[uuuuuulllllll] \\
t/s & 0 & & & n & & & &
\end{tikzcd}
$$

We claim that $H_t(F)$ is finite for each $t > n$. If this is not the case, there is a minimal $v > n$ such that $H_v(F)$ is infinite. Consider the Serre spectral sequence

$$E^2_{s,t} = H_s(K(\mathbb{Z}, n); H_t(F)) \Longrightarrow_s H_{s+t}(S^n)$$

with $E^2_{0,v} \cong H_v(F)$. By assumption, each group $E^2_{s,t}$ is finite for $t < v$, except when $(s, t) = (0, 0)$ or $(n, 0)$. Each differential

$$d^r_{r,v-r+1} \colon E^r_{r,v-r+1} \longrightarrow E^r_{0,v}$$

therefore maps from a finite group. It follows by a finite induction that $E^\infty_{0,v}$ is infinite. Since this group maps injectively to $H_v(S^n) = 0$, we have a contradiction.

By Theorem 4.6.14, it follows that $\pi_t(F)$ is finite for each $t > n$. The conclusion then follows from the isomorphisms $\pi_t(F) \cong \pi_t(S^n)$, valid for this range of values of t. □

THEOREM 4.6.18 ([**Ser51**, Cor. V.2]). *Let $n \geq 2$ be even. Then $\pi_i(S^n)$ is finite for each $i > n$, except for $i = 2n - 1$, and $\pi_{2n-1}(S^n)$ is the direct sum of $\mathbb{Z}$ and a finite group.*

In other words, $\pi_{2n-1}(S^n)$ is finitely generated of rank 1.

PROOF. The Puppe sequence for $f \colon S^n \to K(\mathbb{Z}, n)$ extends to the left, to define a homotopy fiber sequence

$$K(\mathbb{Z}, n-1) \longrightarrow F \longrightarrow S^n ,$$

where $F = \tau_{>n} S^n$ is the n-connected cover of S^n. The associated Serre spectral sequence

$$E^2_{s,t} = H_s(S^n; H_t(K(\mathbb{Z}, n-1))) \Longrightarrow_s H_{s+t}(F)$$

is concentrated in the two columns $s \in \{0, n\}$. The entries with $t \in \{0, n-1\}$ are isomorphic to $\mathbb{Z}$, the entries with $0 < t < n-1$ are trivial, and the entries with

$t \geq n$ are finite.

$$
\begin{tikzcd}
n & \cdots & 0 & 0 & \cdots & 0 \\
n-1 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll, "d^n"'] & 0 \\
t/s & 0 & & & n &
\end{tikzcd}
$$

Since the abutment is n-connected, the differential

$$d^n_{n,0} \colon H_n(S^n) \longrightarrow H_{n-1}(K(\mathbb{Z}, n-1))$$

is an isomorphism. Hence the E^∞-term is $\mathbb{Z}$ in bidegrees $(0,0)$ and $(n, n-1)$, finite in bidegrees $(0,t)$ and (n,t) for $t \geq n$, and trivial otherwise. It follows that $H_i(F)$ is finite for each $i > n$, except for $i = 2n-1$, and $H_{2n-1}(F)$ the direct sum of $\mathbb{Z}$ and a finite group.

By the universal coefficient theorem, $H^{2n-1}(F)$ is the direct sum of $\mathbb{Z}$ and a finite group. Using the Eilenberg–MacLane representability theorem for cohomology, see Theorem 7.1.2, there is a map $f' \colon F \longrightarrow K(\mathbb{Z}, 2n-1)$ representing an element of infinite order in $H^{2n-1}(F)$, so that

$$f'_* \colon H_{2n-1}(F) \longrightarrow H_{2n-1}(K(\mathbb{Z}, 2n-1)) \cong \mathbb{Z}$$

has finite kernel and cokernel. (We may arrange that the cokernel is trivial.) Note that $2n-1$ is odd, so Corollary 4.6.16 applies to $H_*(K(\mathbb{Z}, 2n-1))$.

Let F' be the homotopy fiber of f', so that we have a homotopy fiber sequence

$$F' \longrightarrow F \xrightarrow{f'} K(\mathbb{Z}, 2n-1)\,.$$

Note that F' is at least n-connected.

$$
\begin{tikzcd}[row sep=small, column sep=small]
v & H_v(F') & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & 0 & 0 & \cdots \arrow[uulll] & \cdots & \cdots & \cdots & \cdots \\
n+1 & \cdots & 0 & 0 & \cdots & \cdots \arrow[uuullll] & \cdots & \cdots & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & \cdots & \cdots & \cdots & \cdots \arrow[uuuuuulllllll] \\
t/s & 0 & & & 2n-1 & & & &
\end{tikzcd}
$$

We claim that $H_t(F')$ is finite for each $t > n$. If this is not the case, there is a minimal $v > n$ such that $H_v(F')$ is infinite. Consider the Serre spectral sequence

$$E^2_{s,t} = H_s(K(\mathbb{Z}, 2n-1); H_t(F')) \Longrightarrow_s H_{s+t}(F)$$

with $E^2_{0,v} \cong H_v(F')$. Each differential

$$d^r_{r,v-r+1} \colon E^r_{r,v-r+1} \longrightarrow E^r_{0,v}$$

maps from a finite group, except if $v = 2n-2$ and $r = 2n-1$. In the exceptional case, the subgroup $E^\infty_{2n-1,0}$ of $E^2_{2n-1,0} \cong H_{2n-1}(K(\mathbb{Z}, 2n-1))$ equals the image of the edge homomorphism f'_*, which has finite index, so also $d^{2n-1}_{2n-1,0}$ must have finite image. It follows that the quotient $E^\infty_{0,v}$ of $E^2_{0,v} \cong H_v(F')$ must be infinite. Since $E^\infty_{0,v}$ is isomorphic to the image of $H_v(F') \to H_v(F)$, which is contained in the kernel of $H_v(F) \to H_v(K(\mathbb{Z}, 2n-1))$, this contradicts the calculation of $H_*(F)$ and the fact that f'_* has finite kernel.

By Theorem 4.6.14, it follows that $\pi_t(F')$ is finite for each $t > n$. This implies that $\pi_t(F)$ is finite for each $t > n$, except for $t = 2n-1$, and that $\pi_{2n-1}(F)$ is the direct sum of $\mathbb{Z}$ and a finite group. The conclusion then follows from the isomorphism $\pi_t(F) \cong \pi_t(S^n)$, valid for $t > n$. $\square$

CHAPTER 5

Multiplicative Spectral Sequences

The cohomology groups of a space X come equipped with a cup product, derived from the diagonal map $\Delta\colon X \to X \times X$, which make $H^*(X)$ a graded commutative ring. The corresponding "coring" structure in $H_*(X)$ is less familiar, and requires flatness hypotheses to be dealt with in purely algebraic terms. We will see that some spectral sequences converging to $H^*(X)$ respect the cup product structure in a suitable manner, and this turns out to be a powerful calculational tool. In particular, this ring structure is what Leray referred to when calling the objects he studied "anneau spectral", or "spectral rings".

Since the first examples of spectral sequences with multiplicative structure arise from cohomology, we first discuss cohomologically graded spectral sequences. This amounts to the usual convention of writing a graded abelian group G_* as a cograded abelian group G^*, where $G^s = G_{-s}$. If (C_*, ∂) is a chain complex then (C^*, δ) is the cochain complex with $\delta\colon C^s \to C^{s+1}$ given by $\partial\colon C_{-s} \to C_{-s-1}$. The r-th term of a spectral sequence will therefore be written in cohomological notation as $E_r^{s,t} = E^r_{-s,-t}$.

Thereafter we discuss pairings of spectral sequences, and ring spectral sequences. These can be seen to arise from pairings of exact couples, but a more useful formalism is a richer structure called a Cartan–Eilenberg system. Each Cartan–Eilenberg system gives rise to an exact couple and a spectral sequence, and a pairing of Cartan–Eilenberg systems gives rise to a pairing of (exact couples and) spectral sequences.

This applies, in particular, to the cohomological Serre spectral sequence of a fibration $F \to E \to B$, and the resulting ring structure implies a close relationship between the graded commutative cohomology rings $H^*(F)$, $H^*(E)$ and $H^*(B)$. ((ETC: Make applications.))

When we come to the Adams spectral sequence, we will also see that a pairing of spectra gives rise to a pairing of Adams spectral sequences, so that for a ring spectrum representing a multiplicative cohomology theory, the Adams spectral sequence can converge to the graded coefficient ring of that cohomology theory.

5.1. Cohomological grading

Definition 5.1.1. A *cohomologically bigraded* abelian group $A^{*,*}$ is a doubly-indexed sequence

$$A^{*,*} = (A^{s,t})_{s,t}$$

of abelian groups, where $s, t \in \mathbb{Z}$. A morphism $f\colon A^{*,*} \to B^{*,*}$ of (cohomological) bidegree (u, v) is a sequence of homomorphisms

$$f^{s,t}\colon A^{s,t} \longrightarrow B^{s+u,t+v}\,.$$

A morphism $d\colon E^{*,*} \to E^{*,*}$ is a *differential* if $dd = 0$.

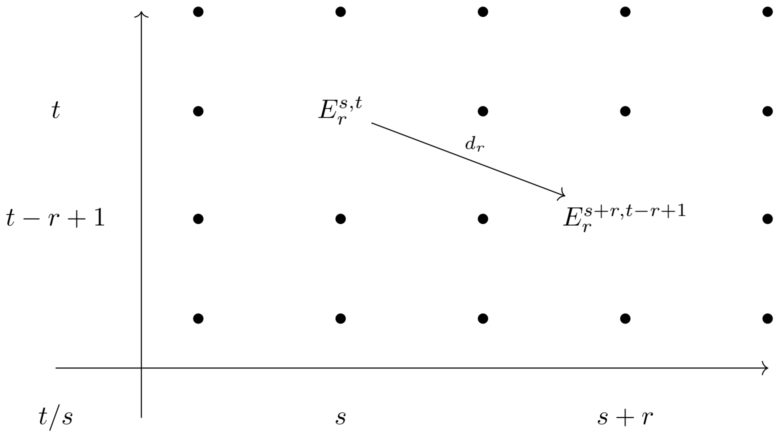
Definition 5.1.2. Let $p \in \mathbb{Z}$. A *cohomological E_p-spectral sequence* $(E_r, d_r)_{r \geq p}$ is a sequence of bigraded abelian groups $E_r = E_r^{*,*}$ and differentials

$$d_r \colon E_r^{*,*} \longrightarrow E_r^{*,*}$$

of bidegree $(r, 1-r)$, together with isomorphisms

$$H(E_r, d_r) \cong E_{r+1}$$

of bigraded abelian groups, for all $r \geq p$.



We call E_r the E_r-term and d_r the d_r-differential. In $E_r^{s,t}$ we call s the filtration degree, t the complementary degree and $s+t$ the total degree. (We could say "codegree", but this gets cumbersome.) Note that

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$$

increases the filtration degree by r and increases the total degree by 1. Hence

$$H^{s,t}(E_r, d_r) = \frac{\ker(d_r^{s,t})}{\operatorname{im}(d_r^{s-r,t+r-1})}$$

is the cohomology at the center of the diagram

$$E_r^{s-r,t+r-1} \xrightarrow{d_r^{s-r,t+r-1}} E_r^{s,t} \xrightarrow{d_r^{s,t}} E_r^{s+r,t-r+1} .$$

Each homological spectral sequence $(E^r, d^r)_{r \geq p}$ can be viewed as a cohomological spectral sequence $(E_r, d_r)_{r \geq p}$ with

$$E_r^{s,t} = E^r_{-s,-t}$$

and

$$d_r^{s,t} = d^r_{-s,-t}$$

for all $r \geq p$ and $s, t \in \mathbb{Z}$. Note that the sign of r is not reversed. Conversely, each cohomological spectral sequence can be viewed as a homological spectral sequence.

A morphism $\phi \colon E \to {}'E$ of cohomological E_p-spectral sequences is a sequence of degree-preserving morphisms

$$\phi_r \colon E_r \longrightarrow {}'E_r$$

for each $r \geq p$, such that $\phi_r d_r = {}'d_r \phi_r$, and such that the induced morphism $\phi_r^* \colon H(E_r, d_r) \to H({}'E_r, {}'d_r)$ corresponds to $\phi_{r+1} \colon E_{r+1} \to {}'E_{r+1}$.

Definition 5.1.3. A cohomological *unrolled exact couple* $(A, E) = (A^s, E^s)_s$ is a diagram of the form

$$\begin{tikzcd} \dots & A^{s-1} \arrow[l] \arrow[d, "\beta_{s-1}"] & A^s \arrow[l, "\alpha_{s-1}"'] \arrow[d, "\beta_s"] & A^{s+1} \arrow[l, "\alpha_s"'] \arrow[d, "\beta_{s+1}"] & A^{s+2} \arrow[l, "\alpha_{s+1}"'] & \dots \arrow[l] \\ & E^{s-1} \arrow[ur, "\gamma_{s-1}"'] & E^s \arrow[ur, "\gamma_s"'] & E^{s+1} \arrow[ur, "\gamma_s+1"'] & & \end{tikzcd},$$

in which each triangle forms a long exact sequence

$$\dots \to A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\beta_s} E^s \xrightarrow{\gamma_s} A^{s+1} \to \dots .$$

Here each A^s and E^s is a cohomologically graded abelian group, and α_s, β_s and γ_s are graded morphisms of graded abelian groups.

((ETC: It might be better to write α^s in place of α_s, but we also want to write α^{r-1} for the iterated map, which could then be confusing.))

Each (homological) exact couple $(A_s, E_s)_s$ can be viewed as a cohomological exact couple $(A^s, E^s)_s$ with $A^s = A_{-s}$ and $E^s = E_{-s}$, and vice versa. The following diagram sits inside the cohomological unrolled exact couple.

$$\begin{tikzcd} A^{s-r+1} & A^s \arrow[l, "\alpha^{r-1}"'] \arrow[d, "\beta_s"'] & A^{s+1} \arrow[l, "\alpha_s"'] & A^{s+r} \arrow[l, "\alpha^{r-1}"'] \\ & E^s \arrow[ur, "\gamma_s"'] & & \end{tikzcd}$$

Definition 5.1.4. For $r \geq 1$ and $s \in \mathbb{Z}$ let

$$Z_r^s = \gamma_s^{-1} \operatorname{im}(\alpha^{r-1} \colon A^{s+r} \to A^{s+1})$$
$$B_r^s = \beta_s \ker(\alpha^{r-1} \colon A^s \to A^{s+r-1})$$

be the *r-th cocycle* and *coboundary* groups in filtration s. Let $Z_\infty^s = \bigcap_r Z_r^s$ and $B_\infty^s = \bigcup_r B_r^s$, and let

$$E_r^s = Z_r^s / B_r^s$$

for all $1 \leq r \leq \infty$. There are inclusions

$$0 = B_1^s \subset \dots \subset B_r^s \subset \dots \subset B_\infty^s \subset \ker(\gamma_s) \subset Z_\infty^s \subset \dots \subset Z_r^s \subset \dots \subset Z_1^s = E^s ,$$

a differential

$$\begin{aligned} d_r^s \colon E_r^s &\longrightarrow E_r^{s+r} \\ [x] &\longmapsto [\beta_{s+r}(y)] \end{aligned}$$

where $\gamma_s(x) = \alpha^{r-1}(y)$, and isomorphisms $H^s(E_r, d_r) \cong E_{r+1}^s$. This defines the spectral sequence associated to the exact couple.

Definition 5.1.5. We give

$$A^{-\infty} = \operatorname*{colim}_s A^s$$

the decreasing filtration

$$A^{-\infty} \supset \dots \supset F^s A^{-\infty} \supset F^{s+1} A^{-\infty} \supset \dots$$

with

$$F^s A^{-\infty} = \operatorname{im}(A^s \longrightarrow A^{-\infty}) .$$

DEFINITION 5.1.6. We say that the exact couple $(A^s, E^s)_s$ is *degreewise discrete* if each $\alpha_s \colon A^{s+1} \to A^s$ preserves the total degree, and for each n there is an integer $b(n)$ such that $(A^s)^n = 0$ for $s > b(n)$. ((ETC: This is where "bounded below" could get confusing.))

PROPOSITION 5.1.7. *(1) There is an injective homomorphism*

$$\zeta \colon \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \longrightarrow E^s_\infty ,$$

which is an isomorphism if $Z^s_\infty = \ker(\gamma_s)$.

(2) If the exact couple is degreewise discrete, then ζ is an isomorphism and the spectral sequence

$$E_r^{s,t} \Longrightarrow_s (A^{-\infty})^{s+t}$$

converges.

DEFINITION 5.1.8. A *decreasing filtration* of a cochain complex $C^* = (C^*, \delta)$ is a sequence of subcomplexes

$$C^* \supset \dots \supset F^s C^* \supset F^{s+1} C^* \supset \dots .$$

We refer to the grading n of $C^* = (C^n)_n$ and $F^s C^* = (F^s C^n)_n$ as the *total degree*, and to s as the *filtration degree*. We set $n = s + t$, where t is the *complementary degree*. For each s there is a short exact sequence of cochain complexes

$$0 \to F^{s+1} C^* \longrightarrow F^s C^* \longrightarrow \frac{F^s C^*}{F^{s+1} C^*} \to 0 .$$

We call $(F^s C^* / F^{s+1} C^*)_s = (F^s C^n / F^{s+1} C^n)_{s,n}$ the *associated (bi-)graded* abelian group of the filtration. The filtration is *exhaustive* if

$$C^* = \bigcup_s F^s C^* .$$

It is *degreewise discrete* if for each n there is a finite $b = b(n)$ such that $F^{b+1} C^n = 0$.

DEFINITION 5.1.9. The exact couple associated to a filtered cochain complex $(F^s C^*)_s$ is the diagram

$$\begin{tikzcd}
\dots & H^*(F^s C^*) \arrow[l] \arrow[d, "\beta_s"'] & H^*(F^{s+1} C^*) \arrow[l, "\alpha_s"'] & \dots \arrow[l] \\
& H^*(F^s C^* / F^{s+1} C^*) \arrow[ur, dashed, "\gamma_s"'] & &
\end{tikzcd}$$

where

$$(A^s)^* = H^*(F^s C^*)$$
$$(E^s)^* = H^*(F^s C^* / F^{s+1} C^*) .$$

Here α_s and β_s preserve the total degree n, while γ_s increases it by 1. The bigrading is given by

$$A^{s,t} = H^{s+t}(F^s C^*)$$
$$E^{s,t} = H^{s+t}(F^s C^* / F^{s+1} C^*) .$$

The associated spectral sequence has

$$E_1^{s,t} = H^{s+t}(F^sC^*/F^{s+1}C^*),$$

and

$$d_1^{s,t} = \beta_{s+1}\gamma_s \colon E_1^{s,t} \longrightarrow E_1^{s+1,t}$$

equals the connecting homomorphism in the long exact cohomology sequence associated to the extension

$$0 \to F^{s+1}C^*/F^{s+2}C^* \longrightarrow F^sC^*/F^{s+2}C^* \longrightarrow F^sC^*/F^{s+1}C^* \to 0$$

of cochain complexes.

DEFINITION 5.1.10. Given a filtration $(F^sC^*)_s$ of a cochain complex $C^* = (C^*, \delta)$, let

$$F^sH^*(C^*) = \mathrm{im}(H^*(F^sC^*) \to H^*(C^*)).$$

This defines a decreasing filtration

$$\cdots \supset F^sH^*(C^*) \supset F^{s+1}H^*(C^*) \supset \dots$$

of the graded abelian group $H^*(C^*)$.

DEFINITION 5.1.11. Let $(F^sG^*)_s$ be a decreasing filtration of a graded abelian group $G^* = (G^n)_n$. Suppose that the filtration is exhaustive and degreewise discrete. A cohomological spectral sequence $(E_r, d_r)_{r\ge p}$ *converges* to G^*, written

$$E_r^{s,t} \Longrightarrow_s G^{s+t},$$

if there are isomorphisms

$$E_\infty^{s,t} \cong \frac{F^sG^{s+t}}{F^{s+1}G^{s+t}}$$

for all (s,t).

PROPOSITION 5.1.12. *If $(F^sC^*)_s$ exhausts $C^* = (C^*, \delta)$ and is degreewise discrete, then the spectral sequence*

$$E_1^{s,t} = H^{s+t}(F^sC^*/F^{s+1}C^*) \Longrightarrow_s H^{s+t}(C^*)$$

converges to $H^*(C^*)$ with the decreasing filtration $(F^sH^*(C^*))_s$.

5.2. Cohomology of spaces

Given a sequence of spaces

$$Y \longrightarrow \dots \longrightarrow Y_{s-1} \xrightarrow{f_s} Y_s \longrightarrow \dots \longrightarrow$$

it is a nontrivial hypothesis on the maps f_s that the induced diagram

$$C_*(Y) \longrightarrow \dots \longrightarrow C_*(Y_{s-1}) \longrightarrow C_*(Y_s) \longrightarrow \dots \longrightarrow$$

consists of surjective homomorphisms. This would, however, suffice to ensure that the dual diagram

$$C^*(Y) \supset \cdots \supset C^*(Y_{s-1}) \supset C^*(Y_s) \supset \dots$$

defines a decreasing filtration of $C^*(Y)$, when suitably indexed. A more convenient framework is given by working with relative cochain complexes.

DEFINITION 5.2.1. Let

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

be an increasing filtration of a space X. The singular cochain complex $C^*(X)$ then has the decreasing filtration

$$C^*(X) \supset \cdots \supset C^*(X, X_{s-1}) \supset C^*(X, X_s) \supset \dots$$

with

$$F^s C^*(X) = C^*(X, X_{s-1})$$
$$\frac{F^s C^*(X)}{F^{s+1} C^*(X)} \cong C^*(X_s, X_{s-1})\,.$$

If $X_{a-1} = \emptyset$ for some a, then $C^*(X) = C^*(X, X_{a-1})$ and the filtration is exhaustive. If $X_b = X$ for some b then $C^*(X, X_b) = 0$ and the filtration is (degreewise) discrete.

REMARK 5.2.2. Note the index shift in the definition of $F^s C^*(X)$, which gives the convenient form

$$0 \to C^*(X, X_s) \longrightarrow C^*(X, X_{s-1}) \longrightarrow C^*(X_s, X_{s-1}) \to 0$$

for the short exact sequence defining the associated graded of the filtration. The hypothesis that $C^n(X, X_{s-1})$ vanishes for sufficiently large s (possibly depending on n) is often not realistic. However, recall from Proposition 2.5.11 that for convergence we only need that the exact couple is degreewise discrete, i.e., that $H^n(X, X_{s-1}) = 0$ for s sufficiently large, and this is satisfied in many cases.

PROPOSITION 5.2.3. *Let $(X_s)_s$ be a filtration of X. There is a cohomological spectral sequence*

$$E_1^{s,t} = H^{s+t}(X_s, X_{s-1}) \Longrightarrow_s H^{s+t}(X)$$

with $d_1 \colon E_1^{s,t} \to E_1^{s+1,t}$ equal to the connecting homomorphism in the long exact cohomology sequence of the triple (X_{s+1}, X_s, X_{s-1}).

If $X_{a-1} = \emptyset$ for some a, and $H^n(X, X_{s-1}) = 0$ for all $s \geq b(n)$, for some $b(n)$ depending on n, then the spectral sequence converges to $H^{s+t}(X)$, with the filtration

$$F^s H^*(X) = \mathrm{im}(H^*(X, X_{s-1}) \to H^*(X)) = \ker(H^*(X) \to H^*(X_{s-1}))\,.$$

PROOF. This is the spectral sequence associated to the exact couple associated to the decreasing filtration of $C^*(X)$ given by $F^s C^*(X) = C^*(X, X_{s-1})$. The additional hypotheses ensure that the exact couple is discrete and that $F^a A^{-\infty} = A^{-\infty}$ is exhaustively filtered. The second expression for $F^s H^*(X)$ is clear from the long exact cohomology sequence of (X, X_{s-1}). $\square$

$$\begin{array}{c|cc}
 & \vdots & \vdots \\
t & H^t(A) \xrightarrow{\delta^t} & H^{1+t}(X,A) \\
 & \vdots & \vdots \\
1 & H^1(A) \xrightarrow{\delta^1} & H^2(X,A) \\
0 & H^0(A) \xrightarrow{\delta^0} & H^1(X,A) \\
\hline
t/s & 0 & 1
\end{array}$$

For a pair of spaces (X, A), we can view the long exact cohomology sequence

$$\cdots \to H^n(X,A) \longrightarrow H^n(X) \longrightarrow H^n(A) \xrightarrow{\delta^n} H^{n+1}(X,A) \to \dots$$

as a cohomological two-column spectral sequence, with

$$E_1^{s,t} = \begin{cases} H^t(A) & \text{for } s = 0, \\ H^{1+t}(X,A) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_1^{0,t} = \delta^t \colon H^t(A) \longrightarrow H^{1+t}(X,A)\,.$$

This corresponds to the bounded filtration with $X_{-1} = \emptyset$, $X_0 = A$ and $X_1 = X$. The (E_1, d_1)-term is shown above. This leads to the following $E_2 = E_\infty$-term.

$$\begin{array}{c|cc}
 & \vdots & \vdots \\
t & \ker(\delta^t) & \operatorname{cok}(\delta^t) \\
 & \vdots & \vdots \\
1 & \ker(\delta^1) & \operatorname{cok}(\delta^1) \\
0 & \ker(\delta^0) & \operatorname{cok}(\delta^0) \\
\hline
t/s & 0 & 1
\end{array}$$

The groups $E_\infty^{s,n-s}$ give the associated graded of the decreasing filtration

$$H^*(X) = F^0H^n(X) \supset F^1H^n(X) \supset 0\,,$$

with $F^1H^n(X) = \mathrm{im}(H^n(X,A) \to H^n(X)) = \ker(H^n(X) \to H^n(A))$. Hence $F^1H^n(X) \cong E_\infty^{1,n-1}$, and there is a short exact sequence

$$0 \to F^1H^n(X) \longrightarrow H^n(X) \longrightarrow E_\infty^{0,n} \to 0\,.$$

This is the same extension as that obtained from the long exact cohomology sequence, namely

$$0 \to \mathrm{cok}(\delta^{n-1}) \longrightarrow H^n(X) \longrightarrow \ker(\delta^n) \to 0\,.$$

Note that in the cohomological spectral sequence the differentials map from the left to the right, while the filtration inclusions $F^{s+1}H^n(X) \subset F^sH^n(X)$ map from the right to the left, when we view them as placed in total degree n, with filtration quotients identified with the components of the E_∞-term.

PROPOSITION 5.2.4. *Let X be a CW complex, equipped with the skeleton filtration. The associated cohomology spectral sequence*

$$E_1^{s,t} = H^{s+t}(X^{(s)}, X^{(s-1)}) \Longrightarrow_s H^{s+t}(X)$$

has (E_1, d_1)-term equal to the cellular cocomplex $(C^*_{CW}(X), \partial)$, and E_2-term equal to the cellular cohomology $H^*_{CW}(X)$, both of which are concentrated on the line $t = 0$. It collapses at $E_2 = E_\infty$, and converges to $H^*(X)$. Hence $H^*_{CW}(X) \cong H^*(X)$.

PROOF. The d_1-differential equals the connecting homomorphism in the long exact cohomology sequence of the triple $(X^{(s+1)}, X^{(s)}, X^{(s-1)})$, by naturality with respect to the vertical map

$$\begin{tikzcd}
0 \arrow[r] & C^*(X, X^{(s)}) \arrow[r] \arrow[d] & C^*(X, X^{(s-1)}) \arrow[r] \arrow[d] & C^*(X^{(s)}, X^{(s-1)}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C^*(X^{(s+1)}, X^{(s)}) \arrow[r] & C^*(X^{(s+1)}, X^{(s-1)}) \arrow[r] & C^*(X^{(s)}, X^{(s-1)}) \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of cochain complexes. Convergence follows from $X^{(-1)} = \emptyset$ and $H^n(X, X^{(s-1)}) = 0$ for all $s > n$, which we can deduce from $H_n(X, X^{(s-1)}) = 0$ for $s > n$ using the universal coefficient theorem. □

DEFINITION 5.2.5. A *(generalized) cohomology theory M* on the category of CW pairs is a contravariant functor assigning to each CW pair (X, A) a graded abelian group

$$M^*(X, A) = (M^n(X, A))_n\,,$$

and a natural transformation

$$\delta\colon M^*(A) \longrightarrow M^{*+1}(X, A)$$

of degree $+1$, such that

(1) Exactness: the sequence

$$\dots \to M^*(X, A) \xrightarrow{j^*} M^*(X) \xrightarrow{i^*} M^*(A) \xrightarrow{\delta} M^{*+1}(X, A) \to \dots$$

is long exact.

(2) Homotopy invariance: if $f \simeq g\colon (X, A) \to (Y, B)$ are homotopic, then $f^* = g^*$.

(3) Excision: if $X = A \cup B$ is a union of subcomplexes, then the inclusion induces an isomorphism

$$M^*(X, A) \xrightarrow{\cong} M^*(B, A \cap B)\,.$$

(4) Additivity: the canonical map

$$M^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha M^*(X_\alpha)$$

is an isomorphism.

The coefficient groups of a cohomology theory M is the (cohomologically) graded abelian group

$$M^* = (M^n(\text{point}))_n\,.$$

There are isomorphisms

$$M^{s+t}(D^s, \partial D^s) \cong \tilde{M}^{s+t}(S^s) \cong M^t$$

for all $s \geq 0$, $t \in \mathbb{Z}$.

Let X be a CW complex. Applying M^* to the triples $(X, X^{(s)}, X^{(s-1)})$ we obtain the exact couple

$$\begin{tikzcd}[column sep=small]
\dots & M^*(X, X^{(s-1)}) \arrow[l] \arrow[d, "\beta_s"'] & & M^*(X, X^{(s)}) \arrow[ll, "\alpha_s"'] & \dots \arrow[l] \\
& M^*(X^{(s)}, X^{(s-1)}) \arrow[urr, dashed, "\gamma_s"'] & & &
\end{tikzcd} \tag{5.1}$$

with

$$\begin{aligned}
A^{s,t} &= M^{s+t}(X, X^{(s-1)}) \\
E^{s,t} &= M^{s+t}(X^{(s)}, X^{(s-1)}) \cong C^s_{CW}(X; M^t)\,,
\end{aligned}$$

where $d_1^s \colon E_1^{s,t} \to E_1^{s+1,t}$ corresponds to

$$\delta^s \colon C^s_{CW}(X; M^t) \longrightarrow C^{s+1}_{CW}(X; M^t)\,.$$

Hence

$$E_2^{s,t} = H^s_{CW}(X; M^t) \cong H^s(X; M^t)\,.$$

Since $X^{(-1)} = \emptyset$, the map

$$M^*(X) = A^{0,*} \xrightarrow{\cong} A^{-\infty,*}$$

is an isomorphism, and $M^*(X)$ is exhaustively filtered by the graded subgroups

$$F^s M^*(X) = \operatorname{im}(M^*(X, X^{(s-1)}) \to M^*(X)) = \ker(M^*(X) \to M^*(X^{(s-1)}))\,.$$

Definition 5.2.6. The spectral sequence

$$E_2^{s,t} = H^s(X; M^t) \Longrightarrow_s M^{s+t}(X)$$

associated to the exact couple (5.1) is the *Atiyah–Hirzebruch spectral sequence* for X and the cohomology theory M.

For now we only prove convergence when X is a finite-dimensional CW complex or M^* is bounded below, postponing the general case until we have discussed sequential limits and derived limits.

Proposition 5.2.7. *If X is finite-dimensional, then the filtration $(F^s M^*(X))_s$ is bounded and the Atiyah–Hirzebruch spectral sequence converges to $M^*(X)$.*

Proof. By hypothesis, there is a b such that $X = X^{(b)}$. For all $s > b$ we then have $A^s = M^*(X, X^{(s-1)}) = 0$ and $F^s A^{-\infty} = 0$. This spectral sequence is concentrated in the columns $0 \leq s \leq b$. □

Proposition 5.2.8. *If M^* is bounded below, then the filtration $(F^s M^*(X))_s$ is degreewise bounded and the Atiyah–Hirzebruch spectral sequence converges to $M^*(X)$.*

Proof. By hypothesis, there is an integer a such that $M^n = 0$ for all $n < a$. Then $M^n(D^s, \partial D^s) = 0$ for all $s > n - a$. Fix an n, and let $b \ge n - a$. Then $M^n(X^{(s)}, X^{(b)}) = 0$ and $M^{n-1}(X^{(s)}, X^{(b)}) = 0$ for all $s > b$. Let $Y = X/X^{(b)}$, so that $Y^{(s)} = X^{(s)}/X^{(b)}$. There is a homotopy cofiber sequence

$$\bigvee_{s>b} \Sigma_+ Y^{(s)} \xrightarrow{1-\alpha} \bigvee_{s>b} \Sigma_+ Y^{(s)} \longrightarrow \Sigma_+ T$$

where $T \simeq Y$ is the mapping telescope of $(Y^{(s)})_{s>b}$. The associated long exact sequence in reduced M-cohomology has the form

$$\dots \to \prod_{s>b} \tilde{M}^{n-1}(Y^{(s)}) \xrightarrow{\delta} \tilde{M}^n(Y) \longrightarrow \prod_{s>b} \tilde{M}^n(Y^{(s)}) \xrightarrow{1-\alpha} \dots$$

which proves that $\tilde{M}^n(Y) = M^n(X, X^{(b)}) = 0$. For all $s > n - a$ we then have $(A^s)^n = M^n(X, X^{(s-1)}) = 0$ and $(F^s A^{-\infty})^n = 0$. This spectral sequence is concentrated in the region $s \ge 0$ and $t \ge a$. □

We get an *Eilenberg–Steenrod uniqueness theorem* for cohomology. ((ETC: Also discuss compatibility of connecting homomorphisms?))

Theorem 5.2.9. *Let G be an abelian group and let M be a cohomology theory with coefficient groups $M^0 = G$ and $M^t = 0$ for $t \ne 0$. Then M is naturally isomorphic to HG, so that*

$$M^n(X) \cong H^n(X; G)$$

for all n.

Proof. The coefficients M^* are bounded below (and above). The Atiyah–Hirzebruch spectral sequence of X for M has E_2-term

$$E_2^{s,t} = \begin{cases} H^s(X; G) & \text{for } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since this is concentrated on the line $t = 0$, the d_r-differentials for $r \ge 2$ must vanish, so that $E_2 = E_\infty$ is concentrated on the line $t = 0$. Since $E_\infty^{n,0}$ is the only group in total degree n, the extension problems are very easy, and we conclude that $M^n(X) \cong E_\infty^{n,0} \cong H^n(X; G)$ for each n. □

((Alternatively, one can consider the exact couple with $A^s = M^*(X^{(s-1)})$ and convergence to the limit $A^\infty = \lim_s A^s$. This is less convenient for pairings.))

5.3. Cohomological Serre spectral sequence

Definition 5.3.1. Let $p \colon E \to B$ be a Hurewicz fibration, with B a CW complex. Let $E_s = p^{-1}(B^{(s)})$. The *(cohomological) Serre spectral sequence* of $p \colon E \to B$ is the spectral sequence

$$E_1^{s,t}(p) = H^{s+t}(E_s, E_{s-1}) \Longrightarrow H^{s+t}(E)$$

associated to the exact couple

$$\begin{tikzcd}
\dots & H^*(E, E_{s-1}) \arrow[l] \arrow[d, "\beta_s"'] & H^*(E, E_s) \arrow[l, "\alpha_s"'] & \dots \arrow[l] \\
& H^*(E_s, E_{s-1}) \arrow[ur, dashed, "\gamma_s"'] & &
\end{tikzcd}$$

with $A^{s,t} = H^{s+t}(E, E_{s-1})$ and $E^{s,t} = H^{s+t}(E_s, E_{s-1})$.

Proposition 5.3.2 ([**Whi78**, XIII.4.6*]). *There are natural isomorphisms*

$$H^{s+t}(E_s, E_{s-1}) \cong C^s_{CW}(B; \mathscr{H}^t(F)),$$

where $\mathscr{H}^t(F)$ denotes the local coefficient system on B given by $H^t(F_b)$ at $b \in B$, with $F_b = p^{-1}(b)$. If B is 1-connected, with base point b_0, then this equals the cellular s-cochains $C^s_{CW}(B; H^t(F))$ with coefficients in the abelian group $H^t(F)$, with $F = p^{-1}(b_0)$.

Sketch proof. We use the notation from the homological case, Proposition 4.2.3. By excision and additivity we have isomorphisms

$$H^{s+t}(E_s, E_{s-1}) \cong H^{s+t}(\coprod_\alpha \Phi^*_\alpha E, \coprod_\alpha \phi^*_\alpha E) \cong \prod_\alpha H^{s+t}(\Phi^*_\alpha E, \phi^*_\alpha E).$$

By fiber homotopy triviality of $\Phi^*_\alpha E \to D^s_\alpha$ we have isomorphisms

$$H^s(D^s_\alpha, \partial D^s_\alpha) \otimes H^t(F_{b_\alpha}) \cong H^{s+t}((D^s_\alpha, \partial D^s_\alpha) \times F_{b_\alpha}) \cong H^{s+t}(\Phi^*_\alpha E, \phi^*_\alpha E).$$

Fixing an isomorphism

$$C^s_{CW}(B; \mathscr{H}^t(F)) \cong \prod_\alpha H^s(D^s_\alpha, \partial D^s_\alpha) \otimes H^t(F_{b_\alpha}) \tag{5.2}$$

we obtain the stated E_1-term. □

Proposition 5.3.3. *$(A^s)^n = H^n(E, E_{s-1}) = 0$ for $s > n$ and $(A^s)^n = H^n(E)$ for $s \le 0$, so the exact couple $(A^s, E^s)_s$ is degreewise bounded, and the Serre spectral sequence is concentrated in the first quadrant and converges to $H^*(E)$.*

Sketch proof. Since $\mathscr{H}^t(F)$ is trivial for $t < 0$ we have $H^n(E_s, E_{s-1}) = 0$ for $s > n$, which implies that $H^n(E_u, E_{s-1}) = 0$ for all $u \ge s > n$. A mapping telescope argument then shows that $H^n(E, E_{s-1}) = 0$, as claimed. □

Proposition 5.3.4 ([**Whi78**, XIII.4.8*]). *The diagram*

$$\begin{tikzcd}
H^{s+t}(E_s, E_{s-1}) \arrow[r, "d_1^{s,t}"] \arrow[d, "\cong"'] & H^{s+t+1}(E_{s+1}, E_s) \arrow[d, "\cong"] \\
C^s_{CW}(B; \mathscr{H}^t(F)) \arrow[r, "\delta^s"] & C^{s+1}_{CW}(B; \mathscr{H}^t(F))
\end{tikzcd}$$

commutes.

Remark 5.3.5. Whitehead states this with $(-1)^s \delta^s$ in place of δ^s. To give a correct statement one must make the sign conventions more precise than we have done above. Working with cubes instead of discs, let us fix generators

$$\gamma_1 \in H_1(I, \partial I) \qquad \text{and} \qquad g_1 \in H^1(I, \partial I)$$

such that $\langle g_1, \gamma_1 \rangle = 1$. Using the cross products

$$H_s(I^s, \partial I^s) \otimes H_u(I^u, \partial I^u) \xrightarrow{\times} H_{s+u}(I^{s+u}, \partial I^{s+u})$$
$$H^s(I^s, \partial I^s) \otimes H^u(I^u, \partial I^u) \xrightarrow{\times} H^{s+u}(I^{s+u}, \partial I^{s+u})$$

we can define generators

$$\gamma_s = \gamma_1 \times \cdots \times \gamma_1 \in H_s(I^s, \partial I^s)$$
$$g_s = g_1 \times \cdots \times g_1 \in H^s(I^s, \partial I^s)$$

such that $\gamma_s \times \gamma_u = \gamma_{s+u}$ and $g_s \times g_u = g_{s+u}$. In view of the graded commutation rule

$$\langle g_s \times g_u, \gamma_s \times \gamma_u \rangle = (-1)^{su} \langle g_s, \gamma_s \rangle \langle g_u, \gamma_u \rangle$$

it follows that

$$\langle g_s, \gamma_s \rangle = (-1)^{s(s-1)/2} = \begin{cases} +1 & \text{for } s \equiv 0, 1 \mod 4, \\ -1 & \text{for } s \equiv 2, 3 \mod 4. \end{cases}$$

It therefore seems best to specify (5.2) so that a sequence

$$(g_{s,\alpha} \otimes f_\alpha)_\alpha \in \prod_\alpha H^s(I^s_\alpha, \partial I^s_\alpha) \otimes H^t(F_{b_\alpha})$$

is identified with the cellular cochain

$$\gamma_{s,\alpha} \longmapsto (-1)^{s(s-1)/2} f_\alpha \in C^s_{CW}(B; \mathscr{H}^t(F))$$

where $\gamma_{s,\alpha} \in H_s(I^s_\alpha, \partial I^s_\alpha) \subset C_s^{CW}(B)$. It seems that Whitehead [**Whi78**, p. 630] instead specifies this isomorphism without the sign $(-1)^{s(s-1)/2}$, mapping $\gamma_{s,\alpha}$ to f_α, which leads to the extra sign $(-1)^s$ in the proposition above.

THEOREM 5.3.6. *The Serre spectral sequence*

$$E_2^{s,t}(p) \Longrightarrow_s H^{s+t}(E)$$

for $F \to E \xrightarrow{p} B$ has E_2-term

$$E_2^{s,t}(p) = H^s(B; \mathscr{H}^t(F)).$$

If B is 1-connected, this simplifies to

$$E_2^{s,t}(p) = H^s(B; H^t(F)).$$

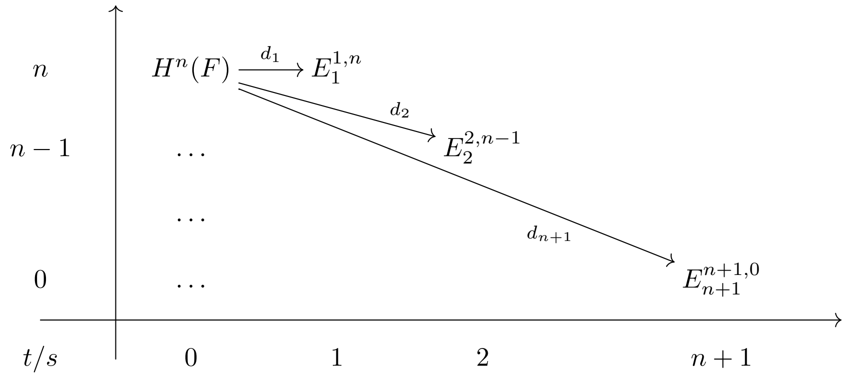
We suppose that B is 0-reduced, with $i \colon F \to E$ the inclusion of the fiber of $p \colon E \to B$ over the base point $b_0 \in B$.

PROPOSITION 5.3.7. *The edge homomorphism $i^* \colon H^n(E) \to H^n(F)$ factors as the surjection*

$$H^n(E) \twoheadrightarrow E_\infty^{0,n}$$

followed by the inclusion

$$E_\infty^{0,n} \rightarrowtail E_1^{0,n} \cong H^n(F).$$



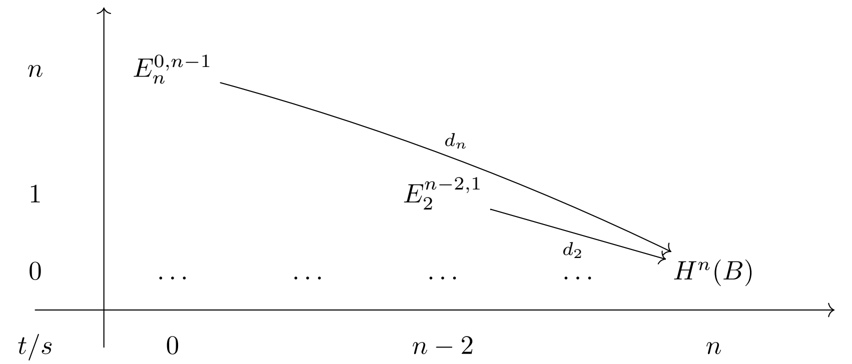
We also suppose that F is 0-connected.

PROPOSITION 5.3.8. *The edge homomorphism $p^*\colon H^n(B) \to H^n(E)$ factors as the surjection*

$$H^n(B) \cong E_2^{n,0} \twoheadrightarrow E_\infty^{n,0}$$

followed by the inclusion

$$E_\infty^{n,0} \rightarrowtail H^n(E)\,.$$



DEFINITION 5.3.9. The additive relation

$$(q^*)^{-1}\delta\colon H^{n-1}(F) \xrightarrow{\delta} H^n(E,F) \xleftarrow{q^*} H^n(B,b_0)\,,$$

sending x with $\delta(x) = q^*(y)$ to the class of y, defines a homomorphism

$$\tau^n\colon \delta^{-1}\operatorname{im}(q^*) \longrightarrow H^n(B,b_0)/\ker(q^*)$$

called the *cohomology transgression*. The elements of $\delta^{-1}\operatorname{im}(q^*)$, on which τ^n are defined, are said to be *transgressive*.

PROPOSITION 5.3.10. *The transgression τ^n corresponds to the differential*

$$d_n^{0,n-1}\colon E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

under isomorphisms $E_n^{0,n-1} \cong \delta^{-1}\operatorname{im}(q^*)$ and $E_n^{n,0} \cong H^n(B,b_0)/\ker(q^*)$.

THEOREM 5.3.11. *Let $F \to E \to B$ be a Hurewicz fibration, with B a 1-connected CW complex and F a 0-connected space. Suppose that $H^s(B) = 0$ for $0 < s < u$ and that $H^t(F) = 0$ for $0 < t < v$. Then there is an exact sequence*

$$\begin{aligned} 0 \to H^1(E) \xrightarrow{i^*} H^1(F) \xrightarrow{\tau^2} H^2(B) \xrightarrow{p^*} \dots \\ \dots \xrightarrow{i^*} H^{n-1}(F) \xrightarrow{\tau^n} H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{i^*} H^n(F) \xrightarrow{\tau^{n+1}} \dots \\ \dots \xrightarrow{\tau^{u+v-1}} H^{u+v-1}(B) \xrightarrow{p^*} H^{u+v-1}(E) \xrightarrow{i^*} H^{u+v-1}(F)\,. \end{aligned}$$

$$\begin{tikzcd}
u+v-1 & H^{u+v-1}(F) \arrow[ddrrr, "d_u^{0,u+v-1}"] & 0 & 0 & \dots & \dots \\
n-1 & H^{n-1}(F) \arrow[dddrrrr, "\tau^n"] & 0 & 0 & \dots & \dots \\
v & H^v(F) & 0 & 0 & H^u(B; H^v(F)) & \dots \\
 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & H^u(B) & H^n(B) \\
t/s & 0 & & & u & n
\end{tikzcd}$$

5.4. Pairings of spectral sequences

Suppose that B is 1-connected, or that $\mathscr{H}^*(F) = H^*(F)$ is a constant coefficient system. The cup products for B and F induce a pairing

$$H^s(B; H^t(F)) \otimes H^u(B; H^v(F)) \longrightarrow H^{s+u}(B; H^{t+v}(F))\,.$$

Its relationship via the Serre spectral sequence

$$E_2^{*,*} = H^*(B; H^*(F)) \Longrightarrow H^*(E)$$

to the cup product

$$\cup\colon H^n(E) \otimes H^m(E) \longrightarrow H^{n+m}(E)$$

can be expressed in terms of a pairing of spectral sequences

$$E_r^{s,t} \otimes E_r^{u,v} \longrightarrow E_r^{s+u,t+v}\,,$$

making $(E_r^{*,*})_r$ a ring spectral sequence. Such a pairing reduces the problem of calculating the d_r-differentials in $E_r^{*,*}$ to finding their values on classes that generate $E_r^{*,*}$ under this product, i.e., the ring indecomposables. This is often a significant reduction compared to the task of finding the values on classes that generate $E_r^{*,*}$ as a bigraded abelian group.

We formulate the following definition for a pairing of two spectral sequences to a third, but often all three of these are the same spectral sequence.

Definition 5.4.1. Let $(E_r, d_r)_{r\ge p}$, $({}'E_r, {}'d_r)_{r\ge p}$ and $({}''E_r, {}''d_r)_{r\ge p}$ be (cohomologically indexed) E_p-spectral sequences. A *pairing of spectral sequences*

$$\mu_r\colon ({}'E_r, {}''E_r) \longrightarrow E_r$$

is a sequence of pairings

$$\mu_r\colon {}'E_r^{*,*} \otimes {}''E_r^{*,*} \longrightarrow E_r^{*,*}$$

for $r \ge p$, taking ${}'E_r^{s,t} \otimes {}''E_r^{u,v}$ to $E_r^{s+u,t+v}$ for all (s,t) and (u,v), such that

(1) the Leibniz rule

$$d_r(\mu_r(x \otimes y)) = \mu_r({}'d_r(x) \otimes y) + (-1)^{s+t}\mu_r(x \otimes {}''d_r(y))$$

holds (in $E_r^{s+u+r,t+v-r+1}$) for all $x \in {}'E_r^{s,t}$ and $y \in {}''E_r^{u,v}$, and

(2) the induced pairing

$$\begin{aligned} H(\mu_r)\colon H({}'E_r, {}'d_r) \otimes H({}''E_r, {}''d_r) &\longrightarrow H(E_r, d_r) \\ [x] \otimes [y] &\longmapsto [\mu_r(x \otimes y)] \end{aligned}$$

corresponds to $\mu_{r+1}\colon {}'E_{r+1} \otimes {}''E_{r+1} \to E_{r+1}$ under the isomorphisms $H({}'E_r, {}'d_r) \cong {}'E_{r+1}$, $H({}''E_r, {}''d_r) \cong {}''E_{r+1}$ and $H(E_r, d_r) \cong E_{r+1}$.

Remark 5.4.2. The tensor product ${}'E_r^{*,*} \otimes {}''E_r^{*,*}$ of two bigraded abelian groups is itself bigraded, with the group

$$\bigoplus_{s+u=\sigma} \bigoplus_{t+v=\tau} {}'E_r^{s,t} \otimes {}''E_r^{u,v}$$

in bidegree (σ, τ). We thus assume that μ_r preserves this bigrading.

The second condition implies that μ_r determines μ_{r+1}, so a pairing of E_p-spectral sequences is specified by the initial pairing μ_p. However, not every pairing of bigraded abelian groups will satisfy the Leibniz rule, and inductively induce pairings of later E_r-terms that also satisfy the Leibniz rule, so being part of a pairing $(\mu_r)_{r\ge p}$ of spectral sequences is a significant additional hypothesis on μ_p.

Remark 5.4.3. Writing $x \cdot y$ for $\mu_r(x \otimes y)$, $|x| = s + t$ for the total degree of x, and omitting primes, the Leibniz rule takes the form

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y)\,.$$

In diagrammatic form the diagonal composite

$$\begin{tikzcd}
{}'E_r^{s,*} \otimes {}''E_r^{u,*} \arrow[rr, "{}'d_r\otimes 1"] \arrow[dr, "\mu_r"] \arrow[dd, "1\otimes {}''d_r"'] & & {}'E_r^{s+r,*} \otimes {}''E_r^{u,*} \arrow[dd, "\mu_r"] \\
& E_r^{s+u,*} \arrow[dr, "d_r"] & \\
{}'E_r^{s,*} \otimes {}''E_r^{u+r,*} \arrow[rr, "\mu_r"] & & E_r^{s+u+r,*}
\end{tikzcd}$$

equals the sum of the two peripheral composites, under the assumption that we define ${}'d_r \otimes 1$ and $1 \otimes {}''d_r$ so that

$$\begin{aligned} ({}'d_r \otimes 1)(x \otimes y) &= {}'d^r(x) \otimes y \\ (1 \otimes {}''d_r)(x \otimes y) &= (-1)^{|x|} x \otimes {}''d_r(y)\,. \end{aligned}$$

This is in line with the general convention that

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)\,,$$

for bigraded homomorphisms f and g, and bigraded elements x and y, where $|g|$ denotes the total degree of g. In the case at hand, $|1| = 0$ while $|''d_r| = 1$. Note that this convention requires access to the internal grading of each object $'E_r^s$, $''E_r^u$ and E_r^{s+u}, or at least to the action by $(-1)^t$ on an element of internal degree t.

REMARK 5.4.4. The sum $D_r = {'d_r} \otimes 1 + 1 \otimes {''d_r}$ defines a differential on $'E_r \otimes {''E_r}$, of bidegree $(r, 1-r)$. This does not in general make

$$('E_r \otimes {''E_r}, D_r)_{r \geq p}$$

a spectral sequence, because the cross product

$$'E_{r+1} \otimes {''E_{r+1}} \cong H('E_r, {'d_r}) \otimes H(''E_r, {''d_r}) \xrightarrow{\times} H('E_r \otimes {''E_r}, D_r)$$

is not in general an isomorphism. However, in situations where this is an isomorphism, a spectral sequence pairing $\mu_r \colon ('E_r, {''E_r}) \to E_r$ is the same as a spectral sequence morphism $\mu_r \colon {'E_r} \otimes {''E_r} \to E_r$. This happens if $\mathrm{Tor}('E_r, {''E_r}) = 0$ for each r, e.g. if each $'E_r$ or $''E_r$ is torsion-free.

REMARK 5.4.5. A pairing

$$\mu^r \colon ('E^r, {''E^r}) \longrightarrow E^r$$

of homologically indexed spectral sequences is defined in the same way, via the identification $E^r_{s,t} = E_r^{-s-t}$. The signs $(-1)^{s+t} = (-1)^{|x|} = (-1)^{-s-t}$ in the Leibniz rule then match up, independently of whether we view x as an element in the cohomological or the homological spectral sequence.

DEFINITION 5.4.6. A *ring spectral sequence* is a spectral sequence $(E_r, d_r)_{r \geq p}$ equipped with a unital and associative pairing

$$\mu_r \colon (E_r, E_r) \longrightarrow E_r\,.$$

Here unitality means that there is an infinite cycle $1 \in E_p^{0,0}$ (with image $1 \in E_r^{0,0}$) such that $\mu_r(1 \otimes x) = x = \mu_r(x \otimes 1)$ for all $x \in E_r$, while associativity means that the diagram

$$\begin{tikzcd}
E_r^{*,*} \otimes E_r^{*,*} \otimes E_r^{*,*} \arrow[r, "\mu_r \otimes 1"] \arrow[d, "1 \otimes \mu_r"'] & E_r^{*,*} \otimes E_r^{*,*} \arrow[d, "\mu_r"] \\
E_r^{*,*} \otimes E_r^{*,*} \arrow[r, "\mu_r"] & E_r^{*,*}
\end{tikzcd}$$

commutes, for each $r \geq p$. It is *commutative* if the diagram

$$\begin{tikzcd}
E_r^{*,*} \otimes E_r^{*,*} \arrow[rr, "\tau"] \arrow[dr, "\mu_r"'] & & E_r^{*,*} \otimes E_r^{*,*} \arrow[dl, "\mu_r"] \\
& E_r^{*,*} &
\end{tikzcd}$$

commutes for each $r \geq p$, where

$$\tau(x \otimes y) = (-1)^{(s+t)(u+v)} y \otimes x$$

for $x \in E_r^{s,t}$ and $y \in E_r^{u,v}$.

REMARK 5.4.7. Writing $x \cdot y$ for $\mu_r(x \otimes y)$, the unitality, associativity and commutativity conditions ask that

$$\begin{aligned} 1 \cdot x = x &= x \cdot 1 \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ x \cdot y &= (-1)^{|x||y|} y \cdot x \,, \end{aligned}$$

where $|x| = s + t$ and $|y| = u + v$ denote the total degrees of x and y. Note that in the commutative case the Leibniz rule expressions for $d_r(x \cdot y)$ and $d_r((-1)^{|x||y|} y \cdot x)$ are equal.

DEFINITION 5.4.8. Let $'\phi_r \colon {'E_r} \to {'\bar{E}_r}$, $''\phi_r \colon {''E_r} \to {''\bar{E}_r}$ and $\phi_r \colon E_r \to \bar{E}_r$ be morphisms of spectral sequences, and let $\mu_r \colon ({'E_r}, {''E_r}) \to E_r$ and $\bar{\mu}_r \colon ({'\bar{E}_r}, {''\bar{E}_r}) \to \bar{E}_r$ be pairings. Then the morphisms are *compatible* with the pairings if each diagram

$$\begin{tikzcd} {'E_r^{*,*}} \otimes {''E_r^{*,*}} \arrow[r, "\mu_r"] \arrow[d, "{'\phi_r} \otimes {''\phi_r}"'] & E_r^{*,*} \arrow[d, "\phi_r"] \\ {'\bar{E}_r^{*,*}} \otimes {''\bar{E}_r^{*,*}} \arrow[r, "\bar{\mu}_r"] & \bar{E}_r^{*,*} \end{tikzcd}$$

commutes, for $r \ge p$. A *ring morphism*

$$\phi_r \colon (E_r, d_r, \mu_r) \longrightarrow ({'E_r}, {'d_r}, {'\mu_r})$$

of ring spectral sequences is a morphism $\phi_r \colon E_r \to {'E_r}$ of spectral sequences that is compatible with the pairings μ_r and $'\mu_r$, and which satisfies $\phi_p(1) = 1$.

DEFINITION 5.4.9. Let $(E_r, d_r, \mu_r)_{r \ge p}$ be a ring spectral sequence. A *left module spectral sequence* over it is a spectral sequence $({''E_r}, {''d_r})$ with a unital and associative pairing

$$\lambda_r \colon (E_r, {''E_r}) \longrightarrow {''E_r} \,.$$

A *right module spectral sequence* is a spectral sequence $({'E_r}, {'d_r})$ with a unital and associative pairing

$$\rho_r \colon ({'E_r}, E_r) \longrightarrow {'E_r} \,.$$

Suppose that $(E_r, d_r, \mu_r)_{r \ge p}$ is commutative. An *algebra spectral sequence* over it is a ring spectral sequence $({'E_r}, {'d_r}, {'\mu_r})_{r \ge p}$ with a ring morphism $\eta_r \colon E_r \to {'E_r}$ such that E_r is central in $'E_r$. This means that the diagram

$$\begin{tikzcd} E_r^{*,*} \otimes {'E_r^{*,*}} \arrow[rr, "\tau"] \arrow[d, "\eta_r \otimes 1"'] & & {'E_r^{*,*}} \otimes E_r^{*,*} \arrow[d, "1 \otimes \eta_r"] \\ {'E_r^{*,*}} \otimes {'E_r^{*,*}} \arrow[dr, "{'\mu_r}"'] & & {'E_r^{*,*}} \otimes {'E_r^{*,*}} \arrow[dl, "{'\mu_r}"] \\ & {'E_r^{*,*}} & \end{tikzcd}$$

commutes, for each $r \ge p$. If $'\mu_r$ is commutative, then η_r is automatically central.

DEFINITION 5.4.10. Let $\Lambda^{*,*}$ be a bigraded ring. We can view it as a ring spectral sequence with $E_r^{*,*} = \Lambda^{*,*}$, for each $r \ge p$, with all differentials zero.

A *left* $\Lambda^{*,*}$*-module spectral sequence* is then a spectral sequence $({}''E_r, d_r)_{r\geq p}$ with each ${}''E_r^{*,*}$ a left $\Lambda^{*,*}$-module and each d_r a $\Lambda^{*,*}$-linear homomorphism. This means that

$$d_r(\lambda \cdot y) = (-1)^{|\lambda|}\lambda \cdot d_r(y)$$

for each $\lambda \in \Lambda^{*,*}$ and $y \in {}''E_r^{*,*}$. Here $|\lambda|$ denotes the total degree.

Likewise, a *right* $\Lambda^{*,*}$*-module spectral sequence* is a spectral sequence $(E'_r, d_r)_{r\geq p}$ with each ${}'E_r^{*,*}$ a right $\Lambda^{*,*}$-module and each d_r a $\Lambda^{*,*}$-linear homomorphism. This means that

$$d_r(x \cdot \lambda) = d_r(x) \cdot \lambda$$

for each $\lambda \in \Lambda^{*,*}$ and $x \in {}'E_r^{*,*}$.

A pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ is $\Lambda^{*,*}$-bilinear if $\mu_r(x \cdot \lambda \otimes y) = \mu_r(x \otimes \lambda \cdot y)$ for all $x \in {}'E_r^{*,*}$, $\lambda \in \Lambda^{*,*}$, $y \in {}''E_r^{*,*}$ and $r \geq p$. We can then uniquely factor μ_r through the tensor product over $\Lambda^{*,*}$, i.e., over the coequalizer in the following diagram.

$$\begin{tikzcd}
{}'E_r^{*,*} \otimes \Lambda^{*,*} \otimes {}''E_r^{*,*} \arrow[r, shift left, "1\otimes\lambda_r"] \arrow[r, shift right, "\rho_r\otimes 1"'] & {}'E_r^{*,*} \otimes {}''E_r^{*,*} \arrow[r, "\pi"] \arrow[dr, "\mu_r"'] & {}'E_r^{*,*} \otimes_{\Lambda^{*,*}} {}''E_r^{*,*} \arrow[d] \\
& & E_r^{*,*}
\end{tikzcd}$$

REMARK 5.4.11. Usually, $\Lambda^{*,*} = \Lambda^*$ is concentrated in filtration degree $s = 0$, so that the total degree equals the internal grading of Λ^*. This happens for the Atiyah–Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(X; M^*) \Longrightarrow M^*(X)$$

for a multiplicative cohomology theory M, with $\Lambda^* = M^*$.

Even more frequently, $\Lambda^* = \Lambda$ is an ungraded ring, concentrated in internal degree $t = 0$, hence in bidegree $(s, t) = (0, 0)$. This happens for the Λ-coefficient cohomology spectral sequence

$$E_1^{s,*} = H^*(X_s, X_{s-1}; \Lambda) \Longrightarrow_s H^*(X; \Lambda)$$

for a filtered space. In this case, $|\lambda| = 0$, so left Λ-linearity has the usual meaning.

REMARK 5.4.12. The sum $D_r = {}'d_r \otimes 1 + 1 \otimes {}''d_r$ defines a differential on ${}'E_r \otimes_{\Lambda^{*,*}} {}''E_r$, of bidegree $(r, 1-r)$. This does not in general make

$$({}'E_r \otimes_{\Lambda^{*,*}} {}''E_r, D_r)_{r\geq p}$$

a spectral sequence, because the cross product

$${}'E_{r+1} \otimes_{\Lambda^{*,*}} {}''E_{r+1} \cong H({}'E_r, {}'d_r) \otimes_{\Lambda^{*,*}} H({}''E_r, {}''d_r) \xrightarrow{\times} H({}'E_r \otimes_{\Lambda^{*,*}} {}''E_r, D_r)$$

is not in general an isomorphism. However, in situations where this is an isomorphism, a $\Lambda^{*,*}$-bilinear spectral sequence pairing $\mu_r \colon ({}'E_r, {}''E_r) \to E_r$ is the same as a spectral sequence morphism $\mu_r \colon {}'E_r \otimes_{\Lambda^{*,*}} {}''E_r \to E_r$. By the Künneth theorem [**ML63**, Thm. V.10.1], this is always the case if $\Lambda^{*,*}$ is a bigraded field, e.g. if Λ^* is a graded field, or Λ is a field in the usual sense.

DEFINITION 5.4.13. Suppose that $\Lambda^{*,*}$ is (bigraded) commutative. A $\Lambda^{*,*}$-*algebra spectral sequence* is a ring spectral sequence $(E_r, d_r, \mu_r)_{r\geq p}$ such that E_r is a $\Lambda^{*,*}$-algebra, $d_r(\lambda \cdot 1) = 0$ for each $\lambda \in \Lambda^{*,*}$, and the isomorphism $E_{r+1} \cong H(E_r, d_r)$

is $\Lambda^{*,*}$-linear, for each $r \geq p$. The ring pairing μ_r then factors uniquely through the coequalizer structure morphism π as a $\Lambda^{*,*}$-linear morphism

$$E_r^{*,*} \otimes_{\Lambda^{*,*}} E_r^{*,*} \longrightarrow E_r^{*,*} .$$

A pairing of spectral sequences induces a pairing of E_∞-terms. Recall the r-th (co-)cycle groups and r-th (co-)boundary groups from Lemma 2.3.1, here in cohomological indexing.

LEMMA 5.4.14. *Let $\mu_r \colon ('E_r, ''E_r) \to E_r$ be a pairing of E_p-spectral sequences. Then*

$$\mu_p \colon 'E_p^s \otimes ''E_p^u \longrightarrow E_p^{s+u}$$

restricts to pairings

$$\mu_p \colon 'Z_r^s \otimes ''Z_r^u \longrightarrow Z_r^{s+u}$$
$$\mu_p \colon 'B_r^s \otimes ''Z_r^u \longrightarrow B_r^{s+u}$$
$$\mu_p \colon 'Z_r^s \otimes ''B_r^u \longrightarrow B_r^{s+u}$$

for all $p \leq r \leq \infty$ and (s,u), making the following diagram with exact rows commute.

$$\begin{tikzcd}
'B_r^s \otimes ''Z_r^u \oplus 'Z_r^s \otimes ''B_r^u \arrow[r] \arrow[d, "\mu_p \oplus \mu_p"'] & 'Z_r^s \otimes ''Z_r^u \arrow[r] \arrow[d, "\mu_p"'] & 'E_r^s \otimes ''E_r^u \arrow[r] \arrow[d, "\mu_r"] & 0 \\
B_r^{s+u} \arrow[r, tail] & Z_r^{s+u} \arrow[r] & E_r^{s+u} \arrow[r] & 0
\end{tikzcd}$$

(Exactness of the upper row follows from right exactness of the tensor product.)

PROOF. The cases $p \leq r < \infty$ are proved by induction on r. The case $r = p$ is clear, since $'Z_p^s = 'E_p^s$ and $'B_p^s = 0$, etc. Suppose the results holds for some $r \geq p$. If $x \in 'Z_{r+1}^s$ and $y \in ''Z_{r+1}^u$ then $'d_r([x]) = 0$ and $''d_r([y]) = 0$ so

$$\begin{aligned} d_r([\mu_p(x \otimes y)]) &= d_r(\mu_r([x] \otimes [y])) \\ &= \mu_r('d_r([x]) \otimes [y]) + (-1)^{|x|} \mu_r([x] \otimes ''d_r([y])) = 0 \end{aligned}$$

by the Leibniz rule, which implies $\mu_p(x \otimes y) \in Z_{r+1}^{s+u} \subset Z_r^{s+u}$, and this defines $\mu_p \colon 'Z_{r+1}^s \otimes ''Z_{r+1}^u \to Z_{r+1}^{s+u}$.

If $x \in 'B_{r+1}^s$ and $y \in ''Z_{r+1}^u$ then $[x] = 'd_r([z])$ and $''d_r([y]) = 0$, for some $z \in 'Z_r^{s-r}$, so

$$\begin{aligned} d_r([\mu_p(z \otimes y)]) &= d_r(\mu_r([z] \otimes [y])) \\ &= \mu_r('d_r([z]) \otimes [y]) + (-1)^{|z|} \mu_r([z] \otimes ''d_r([y])) = \mu_r('d_r([z]) \otimes [y]) \end{aligned}$$

by the Leibniz rule, which implies $[\mu_p(x \otimes y)] = \mu_r([x] \otimes [y]) = \mu_r('d_r([z]) \otimes [y]) \in \mathrm{im}(d_r)$, so that $\mu_p(x \otimes y) \in B_{r+1}^{s+u}$, and this defines $\mu_p \colon 'B_{r+1}^s \otimes ''Z_{r+1}^u \to B_{r+1}^{s+u}$.

The case $x \in 'Z_{r+1}^s$ and $y \in ''B_{r+1}^u$ is very similar.

The case $r = \infty$, defining the pairing

$$\mu_\infty \colon 'E_\infty^s \otimes ''E_\infty^u \longrightarrow E_\infty^{s+u} ,$$

follows by passage to (co-)limits. If $x \in 'Z_\infty^s \subset 'Z_r^s$ and $y \in ''Z_\infty^u \subset ''Z_r^u$ then $\mu_p(x \otimes y) \in Z_r^{s+u}$ for all r, hence $\mu_p(x \otimes y) \in Z_\infty^{s+u}$. If $x \in 'B_\infty^s$ and $y \in ''Z_\infty^u \subset ''Z_r^u$ then $x \in 'B_r^s$ for some r, so $\mu_p(x \otimes y) \in B_r^{s+u} \subset B_\infty^{s+u}$. The case $x \in 'Z_\infty^s$ and $y \in ''B_\infty^u$ is, again, very similar. □

We also formulate the definition of a pairing of filtrations from two such to a third, but often all three filtrations are the same.

DEFINITION 5.4.15. Let $(F^s{}'G^*)_s$, $(F^s{}''G^*)_s$ and $(F^sG^*)_s$ be (decreasing) filtrations of the graded abelian groups $'G^*$, $''G^*$ and G^*, respectively. A bilinear pairing

$$\nu\colon {}'G^* \otimes {}''G^* \longrightarrow G^*$$

is *filtration-preserving* if

$$\nu(F^s{}'G^* \otimes F^u{}''G^*) \subset F^{s+u}G^*$$

for each (s,u). More precisely, this means that $\nu(x \otimes y) \in F^{s+u}G^*$ whenever $x \in F^s{}'G^*$ and $y \in F^u{}''G^*$. Let

$$\nu^{s,u}\colon F^s{}'G^* \otimes F^u{}''G^* \longrightarrow F^{s+u}G^*$$

be the lift of ν, making the diagram

$$\begin{tikzcd}
F^s{}'G^* \otimes F^u{}''G^* \arrow[r] \arrow[d, "\nu^{s,u}"'] & {}'G^* \otimes {}''G^* \arrow[d, "\nu"] \\
F^{s+u}G^* \arrow[r, tail] & G^*
\end{tikzcd}$$

commute. (In general, the upper horizontal arrow need not be injective.) There are then uniquely defined homomorphisms

$$\bar{\nu}^{s,u}\colon \frac{F^s{}'G^*}{F^{s+1}{}'G^*} \otimes \frac{F^u{}''G^*}{F^{u+1}{}''G^*} \longrightarrow \frac{F^{s+u}G^*}{F^{s+u+1}G^*}$$

making the following diagram with exact rows commute.

(5.3)

$$\begin{tikzcd}
\begin{matrix} F^{s+1}{}'G^* \otimes F^u{}''G^* \\ \oplus \\ F^s{}'G^* \otimes F^{u+1}{}''G^* \end{matrix} \arrow[r] \arrow[d, "\nu^{s+1,u} \oplus \nu^{s,u+1}"] & F^s{}'G^* \otimes F^u{}''G^* \arrow[r] \arrow[d, "\nu^{s,u}"'] & \frac{F^s{}'G^*}{F^{s+1}{}'G^*} \otimes \frac{F^u{}''G^*}{F^{u+1}{}''G^*} \arrow[r] \arrow[d, "\bar{\nu}^{s,u}"] & 0 \\
F^{s+u+1}G^* \arrow[r, tail] & F^{s+u}G^* \arrow[r] & \frac{F^{s+u}G^*}{F^{s+u+1}G^*} \arrow[r] & 0
\end{tikzcd}$$

(Exactness of the upper row follows from right exactness of the tensor product.)

DEFINITION 5.4.16. Suppose that $'E_r$ converges to $'G^*$, $''E_r$ converges to $''G^*$ and E_r converges to G^*. A spectral sequence pairing $\mu_r\colon ({}'E_r, {}''E_r) \to E_r$ *converges* to a filtration-preserving pairing $\nu\colon {}'G^* \otimes {}''G^* \to G^*$ if the diagram

$$\begin{tikzcd}
\frac{F^s{}'G^*}{F^{s+1}{}'G^*} \otimes \frac{F^u{}''G^*}{F^{u+1}{}''G^*} \arrow[r, "\bar{\nu}^{s,u}"] \arrow[d, leftrightarrow, "\cong"'] & \frac{F^{s+u}G^*}{F^{s+u+1}G^*} \arrow[d, leftrightarrow, "\cong"] \\
{}'E^s_\infty \otimes {}''E^u_\infty \arrow[r, "\mu_\infty"] & E^{s+u}_\infty
\end{tikzcd}$$

commutes for all (s,u).

REMARK 5.4.17. Suppose that the filtrations of $'G^*$, $''G^*$ and G^* are exhaustive and degreewise discrete. Convergence of $(\mu_r)_r$ to ν then lets us recover $\nu\colon {}'G^* \otimes {}''G^* \to G^*$ up to *filtration shifts*. More explicitly, in total degrees n and m we assume that there are integers $a'(n)$ and $a''(m)$ such that $F^s{}'G^n = 0$ for $s > a'(n)$ and $F^u{}''G^m = 0$ for $u > a''(m)$. This forms the basis for a descending induction on (s, u), where we may suppose that

$$\nu^{s+1,u}\colon F^{s+1}{}'G^n \otimes F^u{}''G^m \longrightarrow F^{s+u+1}G^{n+m}$$

$$\nu^{s,u+1}\colon F^s{}'G^n \otimes F^{u+1}{}''G^m \longrightarrow F^{s+u+1}G^{n+m}$$

have been determined. Assuming that we have determined

$$\mu_\infty\colon {}'E_\infty^{s,n-s} \otimes {}''E_\infty^{u,m-u} \longrightarrow E_\infty^{s+u,n+m-s-u}\,,$$

which is identified with

$$\bar\nu^{s,u}\colon \frac{F^s{}'G^n}{F^{s+1}{}'G^n} \otimes \frac{F^u{}''G^m}{F^{u+1}{}''G^m} \longrightarrow \frac{F^{s+u}G^{n+m}}{F^{s+u+1}G^{n+m}}\,,$$

we can use diagram (5.3) to determine

$$\nu^{s,u}\colon F^s{}'G^n \otimes F^u{}''G^m \longrightarrow F^{s+u}G^{n+m}\,,$$

up to some indeterminacy. More precisely, any two possible choices of $\nu^{s,u}$ differ by a composite of the form

$$\begin{aligned} F^s{}'G^n \otimes F^u{}''G^m \longrightarrow & \frac{F^s{}'G^n}{F^{s+1}{}'G^n} \otimes \frac{F^u{}''G^m}{F^{u+1}{}''G^m} \\ & \xrightarrow{f} F^{s+u+1}G^{n+m} \longrightarrow F^{s+u}G^{n+m}\,, \end{aligned}$$

where f is any homomorphism. Having determined

$$F^s{}'G^n \otimes F^u{}''G^m \xrightarrow{\nu^{s,u}} F^{s+u}G^{n+m} \longrightarrow G^{n+m}$$

for all finite s and u, we can then pass to colimits to obtain $\nu\colon {}'G^n \otimes {}''G^m \to G^{n+m}$, since

$$\operatorname*{colim}_s \operatorname*{colim}_u F^s{}'G^n \otimes F^u{}''G^m \xrightarrow{\cong} {}'G^n \otimes {}''G^m$$

is an isomorphism, by the commutation of sequential colimits with tensor products. ((ETC: Reference for this algebraic fact?))

((ETC: Convolution product?))

5.5. Pairings of exact couples

Given (cohomological) exact couples $({}'A, {}'E)$, $({}''A, {}''E)$ and (A, E), and a pairing $\mu\colon {}'E \otimes {}''E \to E$, Massey [**Mas54**, §3, §8] defined conditions that are essentially equivalent to saying that $\mu = \mu_1$ is part of a pairing $(\mu_r)_r$ of the associated spectral sequences. These conditions are often not easy to check directly, but in [**Mas54**, §7, §9], Massey asserts that they can be verified in the case of a filtered differential graded ring. In essence, the argument uses that these exact couples arise from Cartan–Eilenberg systems, and the pairing arises from a pairing of Cartan–Eilenberg systems. We shall therefore concentrate on this approach to pairings of spectral sequences.

CHAPTER 6

Cartan–Eilenberg systems

6.1. Cohomological Cartan–Eilenberg systems

Recall from Section 3.5 the definition of a homological Cartan–Eilenberg system (H_*, η, ∂). Since our main examples of multiplicative spectral sequences are cohomologically graded, we now reformulate the definition in these terms, conforming with [**CE56**, §XV.7].

Definition 6.1.1. A (cohomological) *finite Cartan–Eilenberg system* (H^*, η, δ) consists of graded abelian groups

$$H^*(i,j)$$

for all integers $i \le j$, structure morphisms preserving degree

$$\eta\colon H^*(i',j') \longrightarrow H^*(i,j)$$

for all integers $i \le j$, $i' \le j'$ with $i \le i'$ and $j \le j'$, and connecting homomorphisms

$$\delta\colon H^*(i,j) \longrightarrow H^{*+1}(j,k)$$

for all integers $i \le j \le k$. These must satisfy

(1) Functoriality: $\eta\colon H^*(i,j) \to H^*(i,j)$ equals the identity, and

$$\eta \circ \eta\colon H^*(i'',j'') \to H^*(i',j') \to H^*(i,j)$$

equals $\eta\colon H^*(i'',j'') \to H^*(i,j)$ for all integers $i \le j$, $i' \le j'$ and $i'' \le j''$ with $i \le i' \le i''$ and $j \le j' \le j''$.

(2) Naturality: The diagrams

$$\begin{tikzcd} H^*(i',j') \arrow[r, "\delta"] \arrow[d, "\eta"'] & H^*(j',k') \arrow[d, "\eta"] \\ H^*(i,j) \arrow[r, "\delta"] & H^*(j,k) \end{tikzcd}$$

commutes, for all integers $i \le j \le k$ and $i' \le j' \le k'$ with $i \le i'$, $j \le j'$ and $k \le k'$.

(3) Exactness: The sequence

$$\dots \xrightarrow{\delta} H^*(j,k) \xrightarrow{\eta} H^*(i,k) \xrightarrow{\eta} H^*(i,j) \xrightarrow{\delta} H^{*+1}(j,k) \xrightarrow{\eta} \dots$$

is exact, for all integers $i \le j \le k$.

Definition 6.1.2. By an *extended integer* we mean an element of

$$\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}\,,$$

linearly ordered with $-\infty$ minimal and ∞ maximal.

DEFINITION 6.1.3. An *extended Cartan–Eilenberg system* (H^*, η, δ) is defined as a finite Cartan–Eilenberg system, except that all references to "integers" are replaced with "extended integers", and subject to the following additional condition.

(4) Colimit: For each extended integer j the canonical homomorphism

$$\operatorname*{colim}_i H^*(i,j) \xrightarrow{\cong} H^*(-\infty, j)$$

is an isomorphism.

EXAMPLE 6.1.4. A homological Cartan–Eilenberg system (H_*, η, ∂) gives rise to a cohomological Cartan–Eilenberg system (H^*, η, δ), and vice versa, by setting

$$H^n(i,j) = H_{-n}(-j,-i)$$

for all (extended) integers $i \le j$. The structure homomorphism

$$\eta\colon H^n(i',j') \longrightarrow H^n(i,j)$$

equals $\eta\colon H_{-n}(-j',-i') \to H_{-n}(-j,-i)$, while the connecting homomorphism

$$\delta\colon H^n(i,j) \longrightarrow H^{n+1}(j,k)$$

equals $\partial\colon H_{-n}(-j,-i) \to H_{-n-1}(-k,-j)$.

EXAMPLE 6.1.5. Let $(F^sC^*)_s$ be a decreasing filtration of a cochain complex C^*. The associated finite Cartan–Eilenberg system is given by

$$H^*(i,j) = H^*(F^iC^*/F^jC^*)$$

for integers $i \le j$, and $\eta\colon H^*(i',j') \to H^*(i,j)$ is induced by the chain map $F^{i'}C^*/F^{j'}C^* \to F^iC^*/F^jC^*$. The connecting homomorphism associated to the short exact sequence

$$0 \to F^jC^*/F^kC^* \longrightarrow F^iC^*/F^kC^* \longrightarrow F^iC^*/F^jC^* \to 0$$

defines $\delta\colon H^*(i,j) \to H^{*+1}(j,k)$. Suppose also that the filtration exhausts C^*. Letting $F^{-\infty}C^* = C^*$ and $F^\infty C^* = 0$, the same expressions define an extended Cartan–Eilenberg system with $H^*(s,\infty) = H^*(F^sC^*)$ and $H^*(-\infty,\infty) = H^*(C^*)$.

EXAMPLE 6.1.6. Let $(X_s)_s$ be an increasing filtration of a space X, so that $F^sC^*(X) = C^*(X, X_{s-1})$ defines a decreasing filtration of $C^*(X)$. The associated finite Cartan–Eilenberg system is given by

$$H^*(i,j) = H^*(F^iC^*(X)/F^jC^*(X)) = H^*(X_{j-1}, X_{i-1})$$

for integers $i \le j$, and $\eta\colon H^*(i',j') \to H^*(i,j)$ is induced by the inclusion of (X_{j-1}, X_{i-1}) into $(X_{j'-1}, X_{i'-1})$. The morphism $\delta\colon H^*(i,j) \to H^{*+1}(j,k)$ equals the connecting homomorphism $\delta\colon H^*(X_{j-1}, X_{i-1}) \to H^{*+1}(X_{k-1}, X_{j-1})$ in the long exact cohomology sequence of the triple $(X_{k-1}, X_{j-1}, X_{i-1})$. Suppose also that $X_{a-1} = \emptyset$ for some finite a, so that $F^aC^*(X) = C^*(X)$. Letting $X_{-\infty} = \emptyset$ and $X_\infty = X$ the same expressions define an extended Cartan–Eilenberg system with $H^*(s,\infty) = H^*(X, X_{s-1})$ and $H^*(-\infty,\infty) = H^*(X)$.

REMARK 6.1.7. It follows from exactness that $H^*(j,j) = 0$ for each j. We can visualize a cohomological extended Cartan–Eilenberg system as a triangular diagram in the extended (i,j)-plane, with a connecting homomorphism $\delta\colon H^*(i,j) \to$

$H^{*+1}(j,k)$ for each rectangle with corners at (i,j), (i,k), (j,j) and (j,k). The colimit condition specifies the left hand column in terms of the rest of the diagram.

$$
\begin{tikzcd}
\infty & H^*(-\infty,\infty) \arrow[d] & \dots \arrow[l] \arrow[d] & H^*(i,\infty) \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & H^*(j,\infty) \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] \\
 & \dots \arrow[d] & \dots \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] \\
k & H^*(-\infty,k) \arrow[d] & \dots \arrow[l] \arrow[d] & H^*(i,k) \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & H^*(j,k) \arrow[l] \arrow[d] & 0 \arrow[l] \\
j & H^*(-\infty,j) \arrow[d] \arrow[uuurrrr, dashed, "\delta"] \arrow[urrrr, dashed, "\delta"] & \dots \arrow[l] \arrow[d] & H^*(i,j) \arrow[l] \arrow[d] \arrow[uuurr, dashed, "\delta"] \arrow[urr, dashed, "\delta"'] & \dots \arrow[l] \arrow[d] & 0 \arrow[l] & \\
 & \dots \arrow[d] & \dots \arrow[l] \arrow[d] & \dots \arrow[l] \arrow[d] & 0 \arrow[l] & & \\
 & \dots & \dots \arrow[l] & 0 \arrow[l] & & & \\
 & & & & & & \\
 & -\infty & & i & & j &
\end{tikzcd}
$$

As in the homological case there are two exact couples associated to an extended Cartan–Eilenberg system, generating the same spectral sequence, and we concentrate on the one given by the top row and the superdiagonal.

Definition 6.1.8. To each cohomological extended Cartan–Eilenberg system (H^*,η,δ) we associate the (top) cohomological exact couple $(A^s,E^s)_s$ given by the diagram

$$
\begin{tikzcd}
\dots & H^*(s,\infty) \arrow[l, "\eta"'] \arrow[d, "\eta"'] & H^*(s+1,\infty) \arrow[l, "\eta"'] & \dots \arrow[l, "\eta"'] \\
 & H^*(s,s+1) \arrow[ur, dashed, "\delta"'] & \dots &
\end{tikzcd}
$$

where

$$
\begin{aligned}
(A^s)^* &= H^*(s,\infty) \\
(E^s)^* &= H^*(s,s+1)
\end{aligned}
$$

with α_s and β_s given by η, while γ_s is given by δ.

The spectral sequence $(E_r,d_r)_{r\geq 1}$ associated to (H^*,η,δ) is the spectral sequence associated to the exact couple $(A^s,E^s)_s$.

The following two propositions and four lemmas are given by reindexing the corresponding results from Section 3.5.

PROPOSITION 6.1.9. *In the spectral sequence $(E_r, d_r)_{r\geq 1}$ associated to an extended Cartan–Eilenberg system (H^*, η, δ) we have*

$$\begin{aligned} Z_r^s &= \delta^{-1}\operatorname{im}(\eta\colon H^{*+1}(s+r,\infty) \to H^{*+1}(s+1,\infty)) \\ &= \ker(\delta\colon H^*(s,s+1) \to H^{*+1}(s+1,s+r)) \\ &= \operatorname{im}(\eta\colon H^*(s,s+r) \to H^*(s,s+1)) \end{aligned}$$

and

$$\begin{aligned} B_r^s &= \eta\ker(\eta\colon H^*(s,\infty) \to H^*(s-r+1,\infty)) \\ &= \operatorname{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,s+1)) \\ &= \ker(\eta\colon H^*(s,s+1) \to H^*(s-r+1,s+1)) \end{aligned}$$

so that η induces an isomorphism

$$E_r^s \xrightarrow{\cong} \operatorname{im}(\eta\colon H^*(s,s+r) \to H^*(s-r+1,s+1))\,.$$

The d_r-differential is given by

$$\begin{aligned} d_r^s\colon E_r^s &\longrightarrow E_r^{s+r} \\ [x] &\longmapsto [\delta(z)] \end{aligned}$$

where $z \in H^*(s,s+r)$, $x = \eta(z) \in H^*(s,s+1)$ and $\delta(z) \in H^{*+1}(s+r,s+r+1)$.

PROOF. For the r-th cocycles,

$$\begin{aligned} &\delta^{-1}\operatorname{im}(\eta\colon H^{*+1}(s+r,\infty) \to H^{*+1}(s+1,\infty)) \\ &\qquad = \delta^{-1}\ker(\eta\colon H^{*+1}(s+1,\infty) \to H^{*+1}(s+1,s+r)) \\ &\qquad\qquad = \ker(\delta\colon H^*(s,s+1) \to H^{*+1}(s+1,s+r)) \end{aligned}$$

by exactness and naturality.

$$\begin{tikzcd}
 & H^{*+1}(s+1,\infty) \arrow[d, "\eta"] & H^{*+1}(s+r,\infty) \arrow[l, "\eta"'] \\
H^*(s,s+r) \arrow[d, "\eta"'] & H^{*+1}(s+1,s+r) & 0 \\
H^*(s,s+1) \arrow[uur, "\delta"] \arrow[ur, "\delta"'] & 0 &
\end{tikzcd}$$

For the r-th coboundaries,

$$\begin{aligned} &\eta\ker(\eta\colon H^*(s,\infty) \to H^*(s-r+1,\infty)) \\ &\qquad = \eta\operatorname{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,\infty)) \\ &\qquad\qquad = \operatorname{im}(\delta\colon H^{*-1}(s-r+1,s) \to H^*(s,s+1)) \end{aligned}$$

for the same reasons.

$$\begin{tikzcd}
H^*(s-r+1,\infty) & H^*(s,\infty) \arrow[l, "\eta"'] \arrow[d, "\eta"] \\
H^*(s-r+1,s+1) & H^*(s,s+1) \arrow[l, "\eta"] \\
H^{*-1}(s-r+1,s) \arrow[uur, "\delta"] \arrow[ur, "\delta"'] & 0
\end{tikzcd}$$

Considering the composition $\eta'' \circ \eta'$ (where the primes only serve to keep the two homomorphisms apart),

$$\begin{tikzcd}
 & & H^*(s,\infty) \arrow[d, "\eta"'] & \\
 & & H^*(s,s+r) \arrow[d, "\eta'"'] & H^{*+1}(s+1,s+r) \\
H^*(-\infty,s+1) & H^*(s-r+1,s+1) \arrow[l, "\eta"'] & H^*(s,s+1) \arrow[l, "\eta''"'] \arrow[ur, "\delta"'] & 0 \\
 & H^{*-1}(s-r+1,s) \arrow[ur, "\delta"'] & 0 &
\end{tikzcd}$$

the isomorphism

$$\eta'' \colon H^*(s,s+1)/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'')$$

restricts to the asserted isomorphism

$$E_r^s = Z_r^s/B_r^s = \operatorname{im}(\eta')/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta'' \circ \eta')\,.$$

Note that we already know that $B_r^s = \ker(\eta'') \subset \operatorname{im}(\eta') = Z_r^s$, so that $\operatorname{im}(\eta') \cap \ker(\eta'') = \ker(\eta'')$.

If $x = \eta(z) \in Z_r^s \subset H^*(s,s+1)$ with $z \in H^*(s,s+r)$, then $\delta(x) = \eta(y) \in H^{*+1}(s+1,\infty)$ with $y = \delta(z) \in H^{*+1}(s+r,\infty)$, by naturality. Hence $\eta(y) = \delta(z) \in H^{*+1}(s+r,s+r+1)$, also by naturality. Thus $d_r^s([x]) = [\eta(y)] = [\delta(z)]$.

$$\begin{tikzcd}
 & H^{*+1}(s+1,\infty) & H^{*+1}(s+r,\infty) \arrow[l, "\eta"'] \arrow[d, "\eta"] \\
 & & H^{*+1}(s+r,s+r+1) \\
H^*(s,s+r) \arrow[d, "\eta"'] \arrow[uurr, "\delta"] \arrow[urr, "\delta"'] & & \\
H^*(s,s+1) \arrow[uuur, "\delta"] & 0 &
\end{tikzcd}$$

$\square$

LEMMA 6.1.10. *The colimit*

$$G^* = H^*(-\infty, \infty) \cong \operatorname*{colim}_s H^*(s, \infty)$$

is exhaustively filtered by

$$F^s G^* = \operatorname{im}(\eta\colon H^*(s, \infty) \to H^*(-\infty, \infty)) .$$

LEMMA 6.1.11. *Consider an extended* (H^*, η, δ) *such that*

$$\dots \xleftarrow{\eta} H^*(s, \infty) \xleftarrow{\eta} H^*(s+1, \infty) \xleftarrow{\eta} \dots$$

is degreewise discrete. Then

$$\begin{aligned} Z^s_\infty &= \ker(\delta\colon H^*(s, s+1) \to H^{*+1}(s+1, \infty)) \\ &= \operatorname{im}(\eta\colon H^*(s, \infty) \to H^*(s, s+1)) \end{aligned}$$

and the filtration $(F^s G^*)_s$ *is degreewise discrete.*

PROOF. If $H^{n+1}(i, \infty) = 0$ for $i > b = b(n+1)$ then

$$\ker(\delta\colon H^n(s, s+1) \to H^{n+1}(s+1, \infty)) = \ker(\delta\colon H^n(s, s+1) \to H^{n+1}(s+1, s+r))$$

for all $s + r > b$, i.e., for all $r > b - s$, so $(Z^s_\infty)^n$ equals this common value of $(Z^s_r)^n$. □

LEMMA 6.1.12. *Consider any extended* (H^*, η, δ). *Then*

$$\begin{aligned} B^s_\infty &= \operatorname{im}(\delta\colon H^{*-1}(-\infty, s) \to H^*(s, s+1)) \\ &= \ker(\eta\colon H^*(s, s+1) \to H^*(-\infty, s+1)) . \end{aligned}$$

PROOF. The union $B^s_\infty \cong \operatorname{colim}_r B^s_r$ equals

$$\operatorname*{colim}_r \ker(\eta\colon H^*(s, s+1) \to H^*(s-r+1, s+1)) \cong \ker(\eta\colon H^*(s, s+1) \to H^*(-\infty, s+1))$$

since $H^*(-\infty, s+1) \cong \operatorname{colim}_r H^*(s-r+1, s+1)$. □

LEMMA 6.1.13. *Consider any extended* (H^*, η, δ). *There is a preferred isomorphism*

$$\frac{\operatorname{im}(\eta\colon H^*(s, \infty) \to H^*(s, s+1))}{\ker(\eta\colon H^*(s, s+1) \to H^*(-\infty, s+1))} \cong \frac{F^s G^*}{F^{s+1} G^*}$$

for each $s \in \mathbb{Z}$.

PROPOSITION 6.1.14. *Let* (H^*, η, δ) *be an extended cohomological Cartan–Eilenberg system, with associated spectral sequence* $(E_r, d_r)_{r\ge1}$ *and filtered target* $G^* = H^*(-\infty, \infty)$.

(1) There is always a preferred injective homomorphism

$$\frac{F^s G^*}{F^{s+1} G^*} \overset{\zeta}{\rightarrowtail} E^{s,*}_\infty ,$$

which is an isomorphism if $Z^s_\infty = \operatorname{im}(\eta\colon H^*(s, \infty) \to H^*(s, s+1))$.

(2) In particular, if the sequence

$$\dots \xleftarrow{\eta} H^*(s, \infty) \xleftarrow{\eta} H^*(s+1, \infty) \xleftarrow{\eta} \dots$$

is degreewise discrete, then ζ *is an isomorphism and the spectral sequence*

$$E^{*,*}_r \Longrightarrow G^*$$

converges.

Sketch proof. Consider the following diagram, with $G^* = H^*(-\infty, \infty)$.

$$\begin{tikzcd}
G^* & & H^*(s,\infty) \arrow[ll, "i_s"'] \arrow[d, "\beta_s"'] & H^*(s+1,\infty) \arrow[l, "\alpha_s"'] \\
& & H^*(s,s+1) \arrow[ur, dashed, "\gamma_s"'] &
\end{tikzcd}$$

The maps i_s and β_s induce isomorphisms

$$\frac{F^sG^*}{F^{s+1}G^*} \xleftarrow{\cong} \frac{H^*(s,\infty)}{\operatorname{im}(\alpha_s)+\ker(i_s)} \xrightarrow{\cong} \frac{\ker(\gamma_s)}{\beta_s\ker(i_s)}$$

The inclusion $\ker(\gamma_s) \subset Z^s_\infty$ and identity $\beta_s \ker(i_s) = B^s_\infty$ then give the inclusion

$$\frac{\ker(\gamma_s)}{\beta_s\ker(i_s)} \subset \frac{Z^s_\infty}{B^s_\infty} = E^s_\infty\,.$$

□

6.2. Pairings of Cartan–Eilenberg systems

We can now define a pairing of cohomological Cartan–Eilenberg systems, following Douady's presentation [**Dou58**] in the Cartan seminar.

Definition 6.2.1. Let $('H^*, \eta, \delta)$, $(''H^*, \eta, \delta)$ and (H^*, η, δ) be finite cohomological Cartan–Eilenberg systems. A *pairing* $\mu\colon ('H^*, ''H^*) \to H^*$ *of finite Cartan–Eilenberg systems* is a collection of degree-preserving homomorphisms

$$\mu_r\colon 'H^*(s,s+r)\otimes ''H^*(u,u+r) \longrightarrow H^*(s+u,s+u+r)$$

for $r \ge 1$ and $s, u \in \mathbb{Z}$. These are required to satisfy the following two conditions.

(SPP I) Each square

$$\begin{tikzcd}
'H^*(s',s'+r')\otimes ''H^*(u',u'+r') \arrow[r, "\mu_{r'}"] \arrow[d, "\eta\otimes\eta"'] & H^*(s'+u',s'+u'+r') \arrow[d, "\eta"] \\
'H^*(s,s+r)\otimes ''H^*(u,u+r) \arrow[r, "\mu_r"] & H^*(s+u,s+u+r)
\end{tikzcd}$$

commutes, for $r \ge 1$, $r' \ge 1$, $s \le s'$, $u \le u'$, $s+r \le s'+r'$ and $u+r \le u'+r'$.

(SPP II) In each (non-commutative) diagram

$$\begin{tikzcd}
'H^*(s,s+r)\otimes ''H^*(u,u+r) \arrow[rr, "\eta\otimes\delta"] \arrow[dr, "\mu_r"] \arrow[dd, "\delta\otimes\eta"'] & & 'H^*(s,s+1)\otimes ''H^*(u+r,u+r+1) \arrow[dd, "\mu_1"] \\
& H^*(s+u,s+u+r) \arrow[dr, "\delta"] & \\
'H^*(s+r,s+r+1)\otimes ''H^*(u,u+1) \arrow[rr, "\mu_1"] & & H^*(s+u+r,s+u+r+1)
\end{tikzcd}$$

with $r \ge 1$ and $s, u \in \mathbb{Z}$, the diagonal composite equals the sum of the two outer composites:

$$\delta\mu_r = \mu_1(\delta\otimes\eta) + \mu_1(\eta\otimes\delta)\,.$$

REMARK 6.2.2. In terms of elements $x \in {}'H^*(s, s+r)$ and $y \in {}''H^*(u, u+r)$, the spectral pairing condition (SPP II) asks that

$$\delta(x \cdot y) = \delta(x) \cdot \eta(y) + (-1)^{|x|}\eta(x) \cdot \delta(y)\,,$$

where we write $\cdot$ for the pairings μ_r and μ_1, and $|x|$ equals the total degree of x. In other words, $|x| = n$ if $x \in {}'H^n(s, s+r)$. This follows from how $\delta \otimes \eta$ and $\eta \otimes \delta$ are defined to act on $x \otimes y$, since η has degree 0 and δ has degree 1.

THEOREM 6.2.3 ([**Dou58**, Thm. II A]). *A pairing $\mu\colon ({}'H^*, {}''H^*) \to H^*$ of finite Cartan–Eilenberg systems induces a pairing $\mu_r\colon ({}'E_r, {}''E_r) \to E_r$ of the associated spectral sequences, with*

$$\mu_1 \colon {}'E_1^s \otimes {}''E_1^u \longrightarrow E_1^{s+u}$$

equal to

$$\mu_1 \colon {}'H^*(s, s+1) \otimes {}''H^*(u, u+1) \longrightarrow H^*(s+u, s+u+1)\,.$$

REMARK 6.2.4. Recalling Definition 5.4.1, this part of Douady's theorem asserts that

$$\mu_r \colon {}'E_r^s \otimes {}''E_r^u \longrightarrow E_r^{s+u}$$

for each $r \ge 1$ satisfies the Leibniz rule

$$\begin{aligned} d_r\mu_r &= \mu_r({}'d_r \otimes 1) + \mu_r(1 \otimes {}''d_r) \\ d_r(x \cdot y) &= {}'d_r(x) \cdot y + (-1)^{|x|} x \cdot {}''d_r(y) \end{aligned}$$

for $x \in {}'E_r$ and $y \in {}''E_r$, and that μ_{r+1} is induced by μ_r in the sense that

$$\mu_{r+1}([x] \otimes [y]) = [\mu_r(x \otimes y)]$$

in $H(E_r, d_r) \cong E_{r+1}$, where ${}'d_r(x) = 0$ and ${}''d_r(y) = 0$.

((ETC: Douady writes (in French): "The demonstration consists of a long series of verifications that the reader (if there is one) will do himself if he wishes."))

((ETC: As noted by Sebastian Goette (MathOverflow 2016), less than (SPP I) is needed for this result.))

PROOF. We prove this by induction on $r \ge 1$, using the diagram below.

$$\begin{tikzcd}
{}'H^*(s,s+1) \otimes {}''H^*(u,u+1) \arrow[dr, "\mu_1"] & & \\
& H^*(s+u,s+u+1) & \\
{}'H^*(s,s+r) \otimes {}''H^*(u,u+r) \arrow[uu, "\eta\otimes\eta"] \arrow[rr, "\eta\otimes\delta"] \arrow[dr, "\mu_r"] \arrow[dd, "\delta\otimes\eta"'] & & {}'H^*(s,s+1) \otimes {}''H^*(u+r,u+r+1) \arrow[dd, "\mu_1"] \\
& H^*(s+u,s+u+r) \arrow[uu, "\eta"] \arrow[dr, "\delta"] & \\
{}'H^*(s+r,s+r+1) \otimes {}''H^*(u,u+1) \arrow[rr, "\mu_1"] & & H^*(s+u+r,s+u+r+1)
\end{tikzcd}$$

Classes $[x] \in {}'E_r^s$ and $[y] \in {}''E_r^u$ are represented by r-th cocycles

$$x = \eta(z) \in {}'Z_r^s \subset {}'H^*(s, s+1)$$
$$y = \eta(w) \in {}''Z_r^u \subset {}''H^*(u, u+1)\,,$$

with $z \in {}'H^*(s, s+r)$ and $w \in {}''H^*(u, u+r)$. Then $\mu_r([x] \otimes [y]) \in E_r^{s+u}$ is the class of

$$\mu_1(x \otimes y) \in Z_r^{s+u} \subset H^*(s+u, s+u+1)\,,$$

which we can write as $\eta(\mu_r(z \otimes w))$ with $\mu_r(z \otimes w) \in H^*(s+u, s+u+r)$. Hence we can calculate $d_r(\mu_r([x] \otimes [y])) \in E_r^{s+u+r}$ as the class of

$$\delta(\mu_r(z \otimes w)) \in Z_r^{s+u+r} \subset H^*(s+u+r, s+u+r+1)\,.$$

This equals the sum of

$$\mu_1(\delta \otimes \eta)(z \otimes w) = \mu_1(\delta(z) \otimes y)$$

and

$$\mu_1(\eta \otimes \delta)(z \otimes w) = (-1)^{|z|}\mu_1(x \otimes \delta(w))\,,$$

where $|z| = |[x]|$. Here $\delta(z) \in {}'H^*(s+r, s+r+1)$ represents ${}'d_r([x])$, so $\mu_1(\delta(z) \otimes y)$ represents $\mu_r({}'d_r([x]) \otimes [y]) \in E_r^{s+u+r}$. Similarly, $\delta(w) \in {}''H^*(u, u+r)$ represents ${}''d_r([y])$, so $\mu_1(x \otimes \delta(w))$ represents $\mu_r([x] \otimes {}''d_r([y])) \in E_r^{s+u+r}$. Hence $d_r(\mu_r([x] \otimes [y]))$ equals the sum

$$\mu_r({}'d_r([x]) \otimes [y]) + (-1)^{|[x]|}\mu_r([x] \otimes {}''d_r([y])) \in E_r^{s+u+r}\,,$$

as claimed.

Having proved that μ_1 restricts to define μ_r on E_r-classes for each $r \geq 1$, it follows that μ_r induces μ_{r+1} upon passage to homology with respect to d_r, since both are calculated from μ_1. □

((Note that (SPP II) relates triangulated structure to monoidal structure, and that the precise interaction between these notions is not well axiomatized [**May01**]. Higher category theory may give a cleaner presentation of this interaction.))

Definition 6.2.5. Let $({}'H^*, \eta, \delta)$, $({}''H^*, \eta, \delta)$ and (H^*, η, δ) be extended cohomological Cartan–Eilenberg systems. A *pairing* $\mu\colon ({}'H^*, {}''H^*) \to H^*$ *of extended Cartan–Eilenberg systems* is a pairing (μ_r) of the underlying finite Cartan–Eilenberg systems, together with degree-preserving homomorphisms

$$\mu_\infty\colon {}'H^*(s, \infty) \otimes {}''H^*(u, \infty) \longrightarrow H^*(s+u, \infty)$$

for $s, u \in \mathbb{Z}$, satisfying the following additional condition.

(SPP III) The squares

$$\begin{tikzcd}
{}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] \arrow[d, "\eta\otimes\eta"'] & H^*(s+u,\infty) \arrow[d, "\eta"] \\
{}'H^*(s,s+r) \otimes {}''H^*(u,u+r) \arrow[r, "\mu_r"] & H^*(s+u,s+u+r)
\end{tikzcd}$$

and

$$\begin{tikzcd}
{}'H^*(s',\infty) \otimes {}''H^*(u',\infty) \arrow[r, "\mu_\infty"] \arrow[d, "\eta\otimes\eta"'] & H^*(s'+u',\infty) \arrow[d, "\eta"] \\
{}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] & H^*(s+u,\infty)
\end{tikzcd}$$

commute, for $r \geq 1$, $s \leq s'$ and $u \leq u'$.

In other words, condition (SPP III) extends (SPP I) to the case $r' = \infty$ and $1 \leq r \leq \infty$, where $s + \infty$, $u + \infty$ and $s + u + \infty$ are interpreted as ∞.

LEMMA 6.2.6. *Given a pairing* $\mu \colon ({}'H^*, {}''H^*) \to H^*$ *of extended Cartan–Eilenberg systems, with filtered target groups*

$$\begin{aligned} {}'G^* &= {}'H^*(-\infty, \infty) \\ {}''G^* &= {}''H^*(-\infty, \infty) \\ G^* &= H^*(-\infty, \infty) \end{aligned}$$

there is a unique filtration-preserving pairing

$$\nu \colon {}'G^* \otimes {}''G^* \longrightarrow G^*$$

making the diagrams

$$\begin{tikzcd} {}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] \arrow[d, two heads] & H^*(s+u,\infty) \arrow[d, two heads] \\ F^s{}'G^* \otimes F^u{}''G^* \arrow[r, "\nu^{s,u}"] \arrow[d] & F^{s+u}G^* \arrow[d, tail] \\ {}'G^* \otimes {}''G^* \arrow[r, "\nu"] & G^* \end{tikzcd}$$

commute for all $s, u \in \mathbb{Z}$.

PROOF. The isomorphisms $\operatorname{colim}_s {}'H^*(s,\infty) \cong {}'G^*$ and $\operatorname{colim}_u {}''H^*(u,\infty) \cong {}''G^*$ induce an isomorphism

$$\operatorname*{colim}_{s,u} {}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \xrightarrow{\cong} {}'G^* \otimes {}''G^*$$

so ν is the canonical map induced by the composites

$${}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \xrightarrow{\mu_\infty} H^*(s+u,\infty) \longrightarrow G^* ,$$

which are compatible by the second part of (SPP III). This makes the outer rectangle commute. The tensor product of the defining surjections ${}'H^*(s,\infty) \to F^s{}'G^*$ and ${}''H^*(u,\infty) \to F^u{}''G^*$ gives the surjection ${}'H^*(s,\infty) \otimes {}''H^*(u,\infty) \to F^s{}'G^* \otimes F^u{}''G^*$ in the left hand column, whose kernel maps to zero in $F^{s+u}G^* \subset G^*$. Hence there is a unique homomorphism $\nu^{s,u}$ making the upper square commute. It follows that the lower square commutes, by the stated surjectivity. □

PROPOSITION 6.2.7. *Let* $({}'H^*, \eta, \delta)$, $({}''H^*, \eta, \delta)$ *and* (H^*, η, δ) *be extended Cartan–Eilenberg systems with associated spectral sequences* $({}'E_r, {}'d_r)$, $({}''E_r, {}''d_r)$ *and* (E_r, d_r) *converging to* ${}'G^*$, ${}''G^*$ *and* G^*, *respectively. Let*

$$\mu \colon ({}'H^*, {}''H^*) \longrightarrow H^*$$

be a pairing of extended Cartan–Eilenberg systems. Then the associated spectral sequence pairing

$$\mu_r \colon ({}'E_r, {}''E_r) \longrightarrow E^r$$

converges to the filtration-preserving pairing

$$\nu \colon {}'G^* \otimes {}''G^* \longrightarrow G^* .$$

Proof. We show that the lower square in the diagram

$$
\begin{tikzcd}
{}'H^*(s,\infty)\otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] \arrow[d, two heads] & H^*(s+u,\infty) \arrow[d, two heads] \\
F^{s}{}'G^*\otimes F^{u}{}''G^* \arrow[r, "\nu^{s,u}"] \arrow[d, two heads] & F^{s+u}G^* \arrow[d, two heads] \\
\dfrac{F^{s}{}'G^*}{F^{s+1}{}'G^*}\otimes\dfrac{F^{u}{}''G^*}{F^{u+1}{}''G^*} \arrow[r, "\bar\nu^{s,u}"] \arrow[d, "\zeta\otimes\zeta"'] & \dfrac{F^{s+u}G^*}{F^{s+u+1}G^*} \arrow[d, tail, "\zeta"] \\
{}'E^s_\infty\otimes {}''E^u_\infty \arrow[r, "\mu_\infty"] & E^{s+u}_\infty
\end{tikzcd}
$$

commutes, where each ζ is given as in the sketch proof of Proposition 6.1.14. The upper and middle squares commute by the definition of $\nu^{s,u}$ and $\bar\nu^{s,u}$, respectively. By the surjectivity of the upper and middle left hand maps, it suffices to prove that the outer rectangle commutes. In view of the construction of ζ, the outer rectangle can instead be factored as follows.

$$
\begin{tikzcd}
{}'H^*(s,\infty)\otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] \arrow[d] & H^*(s+u,\infty) \arrow[d] \\
{}'Z^s_\infty\otimes {}''Z^u_\infty \arrow[r, "\mu_1|"] \arrow[d, two heads] & Z^{s+u}_\infty \arrow[d, two heads] \\
{}'E^s_\infty\otimes {}''E^u_\infty \arrow[r, "\mu_\infty"] & E^{s+u}_\infty
\end{tikzcd}
$$

Here the lower square defines μ_∞ in terms of the restricted pairing $\mu_1|$, and the upper square is part of the following commutative diagram.

$$
\begin{tikzcd}
{}'H^*(s,\infty)\otimes {}''H^*(u,\infty) \arrow[r, "\mu_\infty"] \arrow[d] & H^*(s+u,\infty) \arrow[d] \\
{}'Z^s_\infty\otimes {}''Z^u_\infty \arrow[r, "\mu_1|"] \arrow[d, tail] & Z^{s+u}_\infty \arrow[d, tail] \\
{}'H^*(s,s+1)\otimes {}''H^*(u,u+1) \arrow[r, "\mu_1"] & H^*(s+u,s+u+1)
\end{tikzcd}
$$

□

Remark 6.2.8. In the presence of (SPP I), condition (SPP II) follows from the stronger condition below.

(SPP II+) In each (non-commutative) diagram

$$\begin{tikzcd}
{'H^*(s,s+r) \otimes {''H^*(u,u+r)}} \arrow[r, "1\otimes\delta"] \arrow[d, "\delta\otimes 1"'] \arrow[dr, "\mu_r"] & {'H^*(s,s+r) \otimes {''H^*(u+r,u+2r)}} \arrow[d, "\mu_r"] \\
 & H^*(s+u,s+u+r) \arrow[d, "\delta"] \\
{'H^*(s+r,s+2r) \otimes {''H^*(u,u+r)}} \arrow[r, "\mu_r"] & H^*(s+u+r,s+u+2r)
\end{tikzcd}$$

with $r \geq 1$ and $s, u \in \mathbb{Z}$, the diagonal composite equals the sum of the two outer composites:

$$\delta\mu_r = \mu_r(\delta \otimes 1) + \mu_r(1 \otimes \delta)\,.$$

((ETC: This appears in Neisendorfer's Memoir [**Nei80**].))

6.3. Filtered differential graded rings

Many multiplicative Cartan–Eilenberg systems, with associated multiplicative spectral sequences, arise from filtered differential graded rings.

DEFINITION 6.3.1. The *tensor product* of two cochain complexes $('C^*, '\delta)$ and $(''C^*, ''\delta)$ is the total complex

$$C^* = {'C^*} \otimes {''C^*}$$

with

$$C^k = \bigoplus_{i+j=k} {'C^i} \otimes {''C^j}\,,$$

equipped with the differential $\delta = {'\delta} \otimes 1 + 1 \otimes {''\delta}$, given by

$$\delta(x \otimes y) = {'\delta(x)} \otimes y + (-1)^{|x|} x \otimes {''\delta(y)}\,,$$

where $|x| = i$ is the total degree of $x \in {'C^i}$. We note that $\delta\delta = 0$, so that (C^*, δ) is a cochain complex.

The *unit cochain complex* is $\mathbb{Z}$, concentrated in degree 0.

The *twist isomorphism*

$$\tau\colon {'C^*} \otimes {''C^*} \xrightarrow{\cong} {''C^*} \otimes {'C^*}$$

is the chain isomorphism given by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x\,.$$

LEMMA 6.3.2. *The tensor product, unit complex and twist isomorphism define a symmetric monoidal structure on the category of cochain complexes.*

PROOF. This means that the tensor product is associative, unital and commutative, up to coherent isomorphisms. The associativity isomorphism

$$({'C^*} \otimes {''C^*}) \otimes {'''C^*} \cong {'C^*} \otimes ({''C^*} \otimes {'''C^*})$$

maps $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$.

The unitality isomorphisms

$$\mathbb{Z} \otimes C^* \cong C^* \cong C^* \otimes \mathbb{Z}$$

identify $1 \otimes x$, x and $x \otimes 1$.

The commutativity isomorphism is given by the twist isomorphism.

The required coherence diagrams are listed in [**ML71**, §VII.1 and §VII.7]. □

The tensor product lets us define pairings $'C^* \otimes {}''C^* \to C^*$ of two cochain complexes to a third. We concentrate on the case when the three cochain complexes are the same.

Definition 6.3.3. A *differential graded ring* is a cochain complex (C^*, δ) equipped with a unital and associative cochain homomorphism

$$\mu \colon C^* \otimes C^* \longrightarrow C^* .$$

In other words, μ makes (C^*, δ) a monoid in the monoidal category of cochain complexes.

More explicitly, μ maps $x \otimes y \in C^n \otimes C^m$ to $\mu(x \otimes y) = x \cdot y \in C^{n+m}$ and satisfies the Leibniz rule

$$\delta(x \cdot y) = \delta(x) \cdot y + (-1)^{|x|} x \cdot \delta(y) .$$

Furthermore, there is a cocycle $1 \in C^0$ with $x \cdot 1 = x = 1 \cdot x$ for all x, and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and z. In categorical terms, associativity and unitality ask that the diagrams

$$\begin{tikzcd}
(C^* \otimes C^*) \otimes C^* \arrow[rr, "\cong"] \arrow[d, "\mu\otimes 1"'] & & C^* \otimes (C^* \otimes C^*) \arrow[d, "1\otimes\mu"] \\
C^* \otimes C^* \arrow[r, "\mu"] & C^* & C^* \otimes C^* \arrow[l, "\mu"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{Z} \otimes C^* \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & C^* \otimes C^* \arrow[d, "\mu"] & C^* \otimes \mathbb{Z} \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\
& C^* &
\end{tikzcd}$$

commute, where $\eta \colon \mathbb{Z} \to C^*$ maps $1 \in \mathbb{Z}$ to $1 \in C^*$.

Example 6.3.4. The singular cochains $C^*(X)$ on a space X form a differential graded ring, with respect to the cup product

$$\cup \colon C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$$

given by the Alexander–Whitney formula.

Lemma 6.3.5. *The cohomology $H^*(C^*)$ of a differential graded ring (C^*, δ, μ) is a graded ring.*

Proof. For cocycles $x \in C^n$ and $y \in C^m$ the product of their cohomology classes $[x] \in H^n(C^*)$ and $[y] \in H^m(C^*)$ is the cohomology class

$$[x] \cdot [y] = [x \cdot y] \in H^{n+m}(C^*)$$

of the product $x \cdot y = \mu(x \cdot y)$. This is a cocycle by the Leibniz rule, and its cohomology class only depends on the cohomology classes of x and y, by further applications of the Leibniz rule. □

REMARK 6.3.6. If C^* is a complex of Λ-modules for some commutative ring Λ, and μ is Λ-bilinear, we say that C^* is a *differential graded* Λ*-algebra*, often abbreviated to a "DG algebra". The cohomology $H^*(C^*)$ is then a graded Λ-algebra. The further abbreviation "DGA" can be confusing in this context, since a "DGA algebra" means a "differential graded augmented algebra", in the terminology from the Cartan seminar. We will discuss augmentations later, in the context of Hopf algebras.

REMARK 6.3.7. There is more structure in the cohomology of a differential graded ring than this graded ring structure, including a variety of Massey products. If $a = [x]$, $b = [y]$ and $c = [z]$ satisfy $a \cdot b = 0$ and $b \cdot c = 0$ in $H^*(C^*)$, then we can write $x \cdot y = \delta(u)$ and $y \cdot z = \delta(v)$, for some cochains u and v. The expression

$$w = u \cdot z - (-1)^{|x|} x \cdot v$$

then defines a cocycle, since

$$\delta(w) = \delta(u) \cdot z - x \cdot \delta(v) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0\,.$$

Its cohomology class

$$[w] = [u \cdot z - (-1)^{|x|} x \cdot v] \in \langle a, b, c\rangle$$

then defines an element in the Massey product $\langle a, b, c\rangle \subset H^n(C^*)$, where $n = |a| + |b| + |c| - 1$. Different choices of cobounding classes u and v may give different classes $[w]$, and the Massey product equals the set of all possible such values. ((ETC: This is not the most standard sign convention.)) ((ETC: Give reference to Borromean rings example?))

DEFINITION 6.3.8. A differential graded ring (C^*, δ, μ) is *commutative* if the diagram

$$\begin{tikzcd} C^* \otimes C^* \arrow[rr, "\tau", "\cong"'] \arrow[dr, "\mu"'] & & C^* \otimes C^* \arrow[dl, "\mu"] \\ & C^* & \end{tikzcd}$$

commutes, i.e., if $x \cdot y = (-1)^{|x||y|} y \cdot x$ for all $x, y \in C^*$.

REMARK 6.3.9. The cohomology of a commutative differential graded ring is a (graded) commutative ring, but there are natural examples of non-commutative differential graded rings, such as the cochains $C^*(X)$ on a space X, whose cohomology is nonetheless (graded) commutative. There are more flexible notions of commutativity up to chain homotopy, and higher chain homotopies, that are often more appropriate. An E_∞ *differential graded ring* satisfies "homotopy everything" conditions. These lead to the construction of power operations in the cohomology of these differential graded rings, or algebras, of which the Steenrod operations in mod p cohomology are prime examples.

We can also consider pairings of two filtered cochain complexes to a third. Again, we concentrate on the case when the three filtered cochain complexes are the same.

DEFINITION 6.3.10. A *filtered differential graded ring* is a cochain complex (C^*, δ) equipped with a decreasing filtration $(F^sC^*)_s$ and an associative and unital cochain morphism $\mu\colon C^* \otimes C^* \longrightarrow C^*$, such that the product preserves the filtration. In other words, the image of the composite

$$F^sC^* \otimes F^uC^* \longrightarrow C^* \otimes C^* \xrightarrow{\mu} C^*$$

is contained in $F^{s+u}C^*$, for all $s, u \in \mathbb{Z}$.

LEMMA 6.3.11. *Let C^* be a filtered differential graded ring. There is a unique chain map $\mu^{s,u}$ making the diagram*

$$\begin{tikzcd} F^sC^* \otimes F^uC^* \arrow[r, "\mu^{s,u}"] \arrow[d] & F^{s+u}C^* \arrow[d] \\ C^* \otimes C^* \arrow[r, "\mu"] & C^* \end{tikzcd}$$

commute, for each pair (s, u). These induce a unique chain map μ_r making the diagram

$$\begin{tikzcd} F^sC^* \otimes F^uC^* \arrow[r, "\mu^{s,u}"] \arrow[d, two heads] & F^{s+u}C^* \arrow[d, two heads] \\ \dfrac{F^sC^*}{F^{s+r}C^*} \otimes \dfrac{F^uC^*}{F^{u+r}C^*} \arrow[r, "\mu_r"] & \dfrac{F^{s+u}C^*}{F^{s+u+r}C^*} \end{tikzcd}$$

commute, for all $r \geq 1$, s and u.

PROOF. Both $\mu^{s+r,u}$ and $\mu^{s,u+r}$ take values in $F^{s+u+r}C^*$. □

A pairing of filtered cochain complexes induces a pairing of finite Cartan–Eilenberg systems and the associated spectral sequences. Most of the following result is given in [**Mas54**, §7, §9], with proofs left to the reader.

PROPOSITION 6.3.12. *Let C^* be a filtered differential graded ring, with associated finite Cartan–Eilenberg system*

$$H^*(i, j) = H^*(F^iC^*/F^jC^*)$$

for integers $i \leq j$. The pairing μ induces a pairing

$$\mu_r\colon H^*(s, s+r) \otimes H^*(u, u+r) \longrightarrow H^*(s+u, s+u+r)$$

of finite Cartan–Eilenberg systems, and a pairing

$$\mu_r\colon E_r^s \otimes E_r^u \longrightarrow E_r^{s+u}$$

of the associated spectral sequences, making $(E_r, d_r)_{r\geq1}$ a ring spectral sequence. The E_1-term is given by

$$E_1^{s,t} = H^{s+t}(F^sC^*/F^{s+1}C^*)$$

and the E_1-pairing

$$\mu_1\colon H^*(F^sC^*/F^{s+1}C^*) \otimes H^*(F^uC^*/F^{u+1}C^*) \longrightarrow H^*(F^{s+u}C^*/F^{s+u+1}C^*)$$

is given by

$$\mu_1\colon [\pi(\tilde{x})] \otimes [\pi(\tilde{y})] \longmapsto [\pi\mu^{s,u}(\tilde{x} \otimes \tilde{y})]\,,$$

where $\pi\colon F^sC^* \to F^sC^*/F^{s+1}C^*$, etc.

If the filtration $(F^sC^*)_s$ exhausts C^*, then (μ_r) and

$$\mu_\infty\colon H^*(s, \infty) \otimes H^*(u, \infty) \longrightarrow H^*(s+u, \infty)$$

define a pairing of extended Cartan–Eilenberg systems, with $H^*(s,\infty) = H^*(F^sC^*)$. The pairing of spectral sequences converges to the filtration-preserving pairing

$$\mu\colon H^*(C^*) \otimes H^*(C^*) \longrightarrow H^*(C^*)\,,$$

where $G^n = H^n(C^*)$ is exhaustively filtered by $F^sG^n = \operatorname{im}(H^n(F^sC^*) \to H^n(C^*))$, for $s \in \mathbb{Z}$.

Proof. The chain homomorphism

$$\mu_r\colon F^sC^*/F^{s+r}C^* \otimes F^uC^*/F^{u+r}C^* \longrightarrow F^{s+u}C^*/F^{s+u+r}C^*$$

and the cohomology cross product induce the finite Cartan–Eilenberg system pairing

$$\begin{aligned}\mu_r\colon H^*(F^sC^*/F^{s+r}C^*) &\otimes H^*(F^uC^*/F^{u+r}C^*)\\ &\xrightarrow{\times} H^*(F^sC^*/F^{s+r}C^* \otimes F^uC^*/F^{u+r}C^*)\\ &\xrightarrow{\mu_{r*}} H^*(F^{s+u}C^*/F^{s+u+r}C^*)\,.\end{aligned}$$

In the extended case we set $F^\infty C^* = 0$ and $F^{-\infty}C^* = C^*$, and the chain homomorphism $\mu^{s,u}$ induces

$$\mu_\infty\colon H^*(F^sC^*) \otimes H^*(F^uC^*) \xrightarrow{\times} H^*(F^sC^* \otimes F^uC^*) \xrightarrow{\mu^{s,u}_*} H^*(F^{s+u}C^*)\,.$$

We must confirm conditions (SPP I) and (SPP II) in the finite case, and condition (SPP III) in the extended case.

The diagram

$$\begin{tikzcd} F^{s'}C^* \otimes F^{u'}C^* \arrow[r, "\mu^{s',u'}"] \arrow[d] & F^{s'+u'}C^* \arrow[d, tail] \\ F^sC^* \otimes F^uC^* \arrow[r, "\mu^{s,u}"] & F^{s+u}C^* \end{tikzcd}$$

of cochain complexes commutes, for $s \le s'$ and $u \le u'$, and induces a commutative diagram

$$\begin{tikzcd} \dfrac{F^{s'}C^*}{F^{s'+r'}C^*} \otimes \dfrac{F^{u'}C^*}{F^{u'+r'}C^*} \arrow[r, "\mu_{r'}"] \arrow[d] & \dfrac{F^{s'+u'}C^*}{F^{s'+u'+r}C^*} \arrow[d, tail] \\ \dfrac{F^sC^*}{F^{s+r}C^*} \otimes \dfrac{F^uC^*}{F^{u+r}C^*} \arrow[r, "\mu_r"] & \dfrac{F^{s+u}C^*}{F^{s+u+r}C^*} \end{tikzcd}$$

of quotient complexes, for $r \ge 1$, $r' \ge 1$, $s+r \le s'+r'$ and $u+r \le u'+r'$. Passing to cohomology, we obtain the square required to commute in (SPP I).

Let $\tilde{x} \in F^sC^*$ and $\tilde{y} \in F^uC^*$ lift cocycles $x \in F^sC^*/F^{s+r}C^*$ and $y \in F^uC^*/F^{u+r}C^*$, representing classes $[x] \in H^*(s,s+r)$ and $[y] \in H^*(u,u+r)$. Note that $\delta(\tilde{x}) \in F^{s+r}C^{*+1}$ and $\delta(\tilde{y}) \in F^{u+r}C^{*+1}$. The product

$$\tilde{z} = \mu^{s,u}(\tilde{x} \otimes \tilde{y}) \in F^{s+u}C^*$$

then lifts

$$z = \mu_r(x \otimes y) \in \frac{F^{s+u}C^*}{F^{s+u+r}C^*}$$

representing $[z] = \mu_r([x] \otimes [y]) \in H^*(s+u, s+u+r)$. Its image
$$\delta([z]) = \delta\mu_r([x] \otimes [y]) \in H^{*+1}(s+u+r, s+u+r+1)$$
under the connecting homomorphism is then given by the class $[\pi\delta(\tilde{z})]$ of the image of the coboundary
$$\delta(\tilde{z}) = \delta\mu^{s,u}(\tilde{x} \otimes \tilde{y}) \in F^{s+u+r}C^{*+1}$$
under the projection $\pi\colon F^{s+u+r}C^{*+1} \to F^{s+u+r}C^{*+1}/F^{s+u+r+1}C^{*+1}$. By the Leibniz rule,
$$\delta\mu(\tilde{x} \otimes \tilde{y}) = \mu(\delta(\tilde{x}) \otimes \tilde{y}) + (-1)^{|\tilde{x}|}\mu(\tilde{x} \otimes \delta(\tilde{y}))$$
in C^*, so $[\pi\delta(\tilde{z})]$ equals the sum of
$$[\pi\mu^{s+r,u}(\delta(\tilde{x}) \otimes \tilde{y})] = [\mu_1(\pi\delta(\tilde{x}) \otimes \pi(\tilde{y}))] = \mu_1(\delta([x]) \otimes \eta([y]))$$
and $(-1)^{|\tilde{x}|} = (-1)^{|x|} = (-1)^{|[x]|}$ times
$$[\pi\mu^{s,u+r}(\tilde{x} \otimes \delta(\tilde{y}))] = [\mu_1(\pi(\tilde{x}) \otimes \pi\delta(\tilde{y}))] = \mu_1(\eta([x]) \otimes \delta([y]))\,.$$
This proves that $\delta\mu_r = \mu_1(\delta \otimes \eta) + \mu_1(\eta \otimes \delta)$ when evaluated on any $[x] \otimes [y]$, as demanded by (SPP II).

Letting $F^\infty C^* = 0$, the proof of (SPP I) extends as stated to the cases with $r' = \infty$ and $r \geq 1$ or $r = \infty$, where we interpret $n + \infty$ as ∞ for all integers n, and this proves (SPP III). □

REMARK 6.3.13. If we replace π with the canonical projection $\pi\colon F^sC^* \to F^sC^*/F^{s+r}C^*$, so that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$, then the above proof of (SPP II) proves the stronger form (SPP II+) from Remark 6.2.8.

6.4. Multiplicative Serre spectral sequence

REMARK 6.4.1. We return to the situation of a fiber sequence $F \to E \xrightarrow{p} B$. Serre's original construction [**Ser51**] of his spectral sequence used singular cubes $\sigma\colon I^n \to E$ to define a cubical chain complex $(A_*(E), \partial)$ (say) with homology calculating $H_*(E)$, which could be increasingly filtered by saying that σ lies in $F_sA_*(E)$ if $p\sigma\colon I^n \to E \to B$ factors through the projection $I^n \to I^s$ to the s first coordinates. Dually, the cubical cochain complex $(A^*(E), \delta)$ calculating $H^*(E)$ is decreasingly filtered by saying that a cochain lies in $F^sA^*(E)$ if it vanishes on chains of filtration $\leq s-1$.

There is a cup product making $A^*(E)$ a differential graded ring, and the decreasing filtration $(F^sA^*(E))_s$ respects the product, making $A^*(E)$ a filtered differential graded ring. Hence the associated spectral sequence
$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E)\,,$$
which is the cohomology Serre spectral sequence for $p\colon E \to B$, is a ring spectral sequence. The pairings of E_1- and E_2-terms are given in terms of the cup products in $A^*(B)$, $H^*(B)$ and $H^*(F)$, and the spectral sequence pairing converges to the cup product in $H^*(E)$.

Instead of working with cubical chains and cochains, we will filter the singular cochain complex $C^*(E)$ by the subcomplexes $F^sC^*(E) = C^*(E, E_{s-1})$. These are not strictly respected by the cochain level cup product, because the cross product of two cochains vanishing on E_{s-1} and E_{u-1} will vanish on all chains in $E_{s-1} \times E$ and in $E \times E_{u-1}$, but usually not on all chains in $E_{s-1} \times E \cup E \times E_{u-1}$. Hence $C^*(E)$ is not a filtered differential graded ring, and we must give a different proof of the multiplicativity of the cohomology Serre spectral sequence. For this we will adapt [**Whi78**, §XIII.8], making use of excision isomorphisms and the formalism of pairings of Cartan–Eilenberg systems.

DEFINITION 6.4.2. Let $p\colon E \to B$ be a fibration, with B a CW complex. Let $E_s = p^{-1}(B^{(s)})$, with $E_s = \emptyset$ for $-\infty \le s < 0$ and $E_\infty = E$. Define a cohomological extended Cartan–Eilenberg system $H^* = H^*(p)$ by

$$H^*(i,j) = H^*(E_{j-1}, E_{i-1})$$

for $-\infty \le i \le j \le \infty$, with $\delta\colon H^*(i,j) \to H^{*+1}(j,k)$ equal to the connecting homomorphism

$$\delta\colon H^*(E_{j-1}, E_{i-1}) \longrightarrow H^{*+1}(E_{k-1}, E_{j-1}) .$$

The associated spectral sequence is the cohomological Serre spectral sequence

$$E_r^{s,t} = E_r^{s,t}(p) \Longrightarrow_s H^{s+t}(E)$$

with

$$E_1^{s,t} \cong C^s_{CW}(B; \mathscr{H}^t(F)) \qquad \text{and} \qquad E_2^{s,t} \cong H^s(B; \mathscr{H}^t(F)) .$$

PROPOSITION 6.4.3. *Let $p'\colon E' \to B'$ and $p''\colon E'' \to B''$ be fibrations, with B' and B'' CW complexes. Let $p' \times p''\colon E' \times E'' \to B' \times B''$ be the product fibration. There is a natural pairing of extended Cartan–Eilenberg systems*

$$\mu\colon (H^*(p'), H^*(p'')) \longrightarrow H^*(p' \times p'')$$

with components

$$\begin{aligned}\mu_r\colon H^*(E'_{s+r-1}, E'_{s-1}) &\otimes H^*(E''_{u+r-1}, E''_{u-1}) \\ &\xrightarrow{\times} H^*(E'_{s+r-1} \times E''_{u+r-1}, E'_{s-1} \times E''_{u+r-1} \cup E'_{s+r-1} \times E''_{u-1}) \\ &\qquad\longrightarrow H^*((E' \times E'')_{s+u+r-1}, (E' \times E'')_{s+u-1})\end{aligned}$$

and

$$\begin{aligned}\mu_\infty\colon H^*(E', E'_{s-1}) &\otimes H^*(E'', E''_{u-1}) \\ &\xrightarrow{\times} H^*(E' \times E'', E'_{s-1} \times E'' \cup E' \times E''_{u-1}) \\ &\qquad\longrightarrow H^*(E' \times E'', (E' \times E'')_{s+u-1}) .\end{aligned}$$

PROOF. To simplify the notation a little we restrict to the case where $p' = p'' = p\colon E \to B$, but the general case is easily recovered by working with p' in the first factor and p'' in the second factor of each product.

The product $B \times B$ has the CW structure with k-skeleton

$$(B \times B)^{(k)} = \bigcup_{i+j=k} B^{(i)} \times B^{(j)} .$$

We lift the skeleton filtration along $p \times p$ to define the filtration on $E \times E$ with

$$(E \times E)_k = \bigcup_{i+j=k} E_i \times E_j .$$

We then have inclusions

$$(B \times B)^{(s+u-1)} \subset B^{(s-1)} \times B \cup B \times B^{(u-1)}$$

and

$$(E \times E)_{s+u-1} \subset E_{s-1} \times E \cup E \times E_{u-1}$$

of subspaces of $B \times B$ and $E \times E$, respectively. This defines

$$\begin{aligned}\mu_\infty \colon H^*(E, E_{s-1}) \otimes H^*(E, E_{u-1}) &\xrightarrow{\times} H^*(E \times E, E_{s-1} \times E \cup E \times E_{u-1}) \\ &\longrightarrow H^*(E \times E, (E \times E)_{s+u-1})\end{aligned}$$

as the composite of the cohomology cross product and the (now) evident restriction map. The definition of μ_r for finite $r \geq 1$ is a little more elaborate. The subcomplexes

$$B^{(s+r-1)} \times B^{(u+r-1)}$$

and

$$(B \times B)^\wedge_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} B^{(i)} \times B^{(j)}$$

of $B \times B$ have intersection

$$B^{(s-1)} \times B^{(u+r-1)} \cup B^{(s+r-1)} \times B^{(u-1)}$$

and union

$$B^{(s+r-1)} \times B^{(u+r-1)} \cup (B \times B)^{(s+u+r-1)} .$$

Note that $(B \times B)^{(s+u-1)} \subset (B \times B)^\wedge_{s,u,r}$. Likewise, the subspaces

$$E_{s+r-1} \times E_{u+r-1}$$

and

$$(E \times E)^\wedge_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} E_i \times E_j$$

of $E \times E$ have intersection

$$E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1}$$

and union

$$E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1} .$$

Furthermore, $(E \times E)_{s+u-1} \subset (E \times E)^\wedge_{s,u,r}$. See Figure 6.1. Hence there is an excision isomorphism

$$\begin{aligned}&H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^\wedge_{s,u,r}) \\ &\qquad \xrightarrow{\cong} H^*(E_{s+r-1} \times E_{u+r-1}, E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1}) ,\end{aligned}$$

and a restriction homomorphism

$$\begin{aligned}&H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^\wedge_{s,u,r}) \\ &\qquad \longrightarrow H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) .\end{aligned}$$

The pairing μ_r equals the composite

$$\begin{aligned}&H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \\ &\qquad \xrightarrow{\times} H^*(E_{s+r-1} \times E_{u+r-1}, E_{s-1} \times E_{u+r-1} \cup E_{s+r-1} \times E_{u-1}) \\ &\qquad \xleftarrow{\cong} H^*(E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1}, (E \times E)^\wedge_{s,u,r}) \\ &\qquad \longrightarrow H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) .\end{aligned}$$

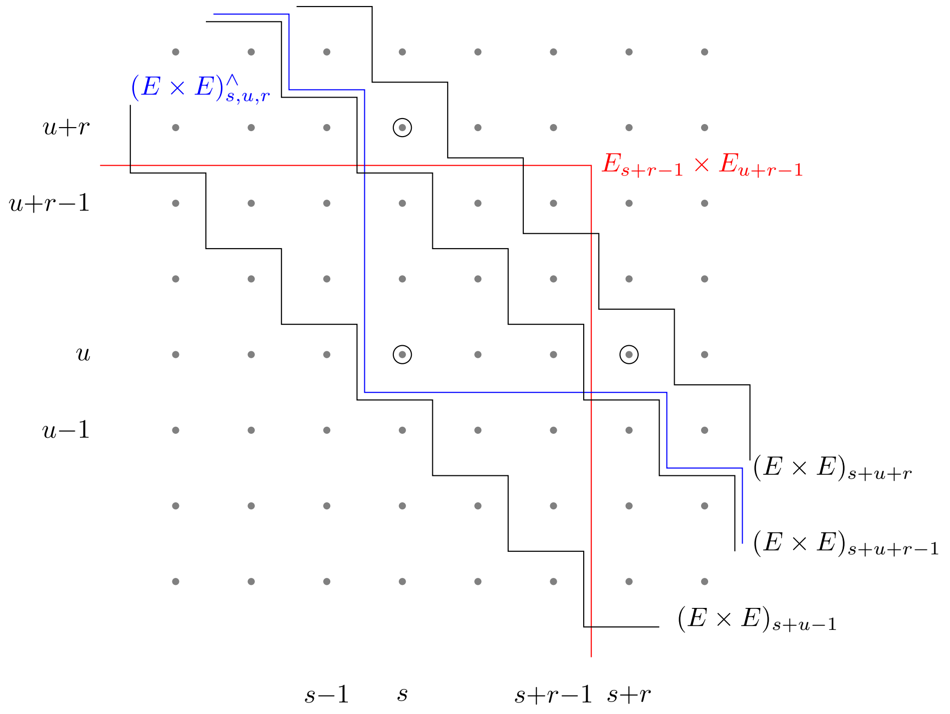


FIGURE 6.1. Subspaces of $E \times E$

Condition (SPP I) follows by naturality of the three homomorphisms composing to μ_r with respect to the inclusions

$$\begin{aligned}
E_{s-1} &\subset E_{s'-1} \\
E_{s+r-1} &\subset E_{s'+r'-1} \\
E_{u-1} &\subset E_{u'-1} \\
E_{u+r-1} &\subset E_{u'+r'-1} \\
(E \times E)_{s+u-1} &\subset (E \times E)_{s'+u'-1} \\
(E \times E)_{s+u+r-1} &\subset (E \times E)_{s'+u'+r'-1} \\
(E \times E)^{\wedge}_{s,u,r} &\subset (E \times E)^{\wedge}_{s',u',r'}
\end{aligned}$$

for $s \le s'$, $u \le u'$, $s + r \le s' + r'$ and $u + r \le u' + r'$. Only the last one requires comment: The inclusion

$$(E \times E)^{\wedge}_{s,u,r} = \bigcup_{\substack{i+j=s+u+r-1 \\ i<s \text{ or } j<u}} E_i \times E_j \subset \bigcup_{\substack{i'+j'=s'+u'+r'-1 \\ i'<s' \text{ or } j'<u'}} E_{i'} \times E_{j'} = (E \times E)^{\wedge}_{s',u',r'}$$

holds since if $i < s$ and $i + j = s + u + r - 1$ then $E_i \times E_j \subset E_i \times E_{j'}$ with $i < s'$ and $i + j' = s' + u' + r' - 1$, and similarly if $j < u \le u'$.

Condition (SPP III) holds in the same way, setting $r' = \infty$, and noting that the excision isomorphism in the definition of μ_r is the identity map of

$$H^*(E \times E, E_{s-1} \times E \cup E \times E_{u-1})$$

when $r = \infty$.

To verify condition (SPP II) we consider the composite

$$\begin{aligned} & H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \\ & \qquad \xrightarrow{\mu_r} H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \\ & \qquad \xrightarrow{\delta} H^{*+1}((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}) \\ & \qquad \cong \prod_{i+j=s+u+r} H^{*+1}(E_i \times E_j, E_{i-1} \times E_j \cup E_i \times E_{j-1}), \end{aligned}$$

where the final isomorphism follows from excision. We claim that (1) the component with $(i,j) = (s+r, u)$ equals

$$\begin{aligned} & H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \\ & \qquad \xrightarrow{\delta\otimes\eta} H^{*+1}(E_{s+r}, E_{s+r-1}) \otimes H^*(E_u, E_{u-1}) \\ & \qquad \xrightarrow{\times} H^{*+1}(E_{s+r} \times E_u, E_{s+r-1} \times E_u \cup E_{s+r} \times E_{u-1}), \end{aligned}$$

(2) the component with $(i,j) = (s, u+r)$ equals

$$\begin{aligned} & H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \\ & \qquad \xrightarrow{\eta\otimes\delta} H^{*+1}(E_s, E_{s-1}) \otimes H^*(E_{u+r}, E_{u+r-1}) \\ & \qquad \xrightarrow{\times} H^{*+1}(E_s \times E_{u+r}, E_{s-1} \times E_{u+r} \cup E_s \times E_{u+r-1}), \end{aligned}$$

and (3) the remaining components are zero. This implies the relation

$$\delta\mu_r = \mu_1(\delta \otimes \eta) + \mu_1(\eta \otimes \delta).$$

((ETC: Elaborate?))

For the first claim we use the commutative diagram in Figure 6.2, with the following abbreviations.

$$\begin{aligned} X &= E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r} \\ Y &= E_{s+r-1} \times E_{u+r-1} \cup (E \times E)_{s+u+r-1} \\ Z &= E_{s-1} \times E_u \cup E_{s+r} \times E_{u-1} \end{aligned}$$

The two quadrangles containing $H^{*+1}(X, Y)$ commute by the naturality of δ with respect to the maps of triples

$$((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \subset (X, Y, (E \times E)^{\wedge}_{s,u,r})$$

and

$$(E_{s+r} \times E_u, E_{s+r-1} \times E_u \cup E_{s+r} \times E_{u-1}, Z) \subset (X, Y, (E \times E)^{\wedge}_{s,u,r}).$$

((ETC: Elaborate?))

The second claim follows from a similar diagram.

For the third claim we assume $i + j = s + u + r$ with $i \notin \{s, s+r\}$, so that $j \notin \{u, u+r\}$, and use the abbreviations

$$\begin{aligned} V &= E_{s-1} \times E \cup E \times E_{u-1} \\ W &= E_{s-1} \times E \cup E_{s+r-1} \times E_{u+r-1} \cup E \times E_{u-1} \end{aligned}$$

and the following commutative diagram.

$$\begin{tikzcd}
H^*((E_{s+r-1}, E_{s-1}) \times (E_{u+r-1}, E_{u-1})) & \\
H^*(Y, (E \times E)^{\wedge}_{s,u,r}) \arrow[u, "\cong"] \arrow[d] & H^*(W, V) \arrow[l, "\cong"'] \arrow[ul, "\cong"'] \arrow[dl] \arrow[d, "\delta"] \\
H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \arrow[d, "\delta"'] \arrow[dr, "\delta"] & H^{*+1}(W, W) \arrow[d] \\
H^{*+1}((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}) \arrow[r] \arrow[d] & H^{*+1}((E \times E)_{s+u+r} \cap W, (E \times E)_{s+u+r-1}) \arrow[dl] \\
H^{*+1}((E_i, E_{i-1}) \times (E_j, E_{j-1})) &
\end{tikzcd}$$

The quadrangle commutes by naturality of δ with respect to the map of triples

$$((E \times E)_{s+u+r} \cap W, (E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \subset (W, W, V)\,.$$

Since $H^{*+1}(W, W)$ is trivial, it follows that the left hand vertical composite is zero. $\square$

$$
\begin{tikzcd}
H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_{u+r-1}, E_{u-1}) \arrow[r, "1\otimes\eta"] \arrow[d, "\times"'] & H^*(E_{s+r-1}, E_{s-1}) \otimes H^*(E_u, E_{u-1}) \arrow[d, "\times"] \arrow[dddr, bend left, "\delta\otimes 1"] & \\
H^*((E_{s+r-1}, E_{s-1}) \times (E_{u+r-1}, E_{u-1})) \arrow[r] & H^*((E_{s+r-1}, E_{s-1}) \times (E_u, E_{u-1})) & \\
H^*(Y, (E \times E)^{\wedge}_{s,u,r}) \arrow[u, "\cong"] \arrow[r] \arrow[d] \arrow[dr, "\delta"] & H^*(E_{s+r-1} \times E_u \cup E_{s+r} \times E_{u-1}, Z) \arrow[u, "\cong"'] \arrow[dd, bend left, "\delta"] & \\
H^*((E \times E)_{s+u+r-1}, (E \times E)_{s+u-1}) \arrow[d, "\delta"'] & H^{*+1}(X, Y) \arrow[dl] \arrow[d] & H^{*+1}(E_{s+r}, E_{s+r-1}) \otimes H^*(E_u, E_{u-1}) \arrow[dl, "\times"] \\
H^{*+1}((E \times E)_{s+u+r}, (E \times E)_{s+u+r-1}) \arrow[r] & H^{*+1}((E_{s+r}, E_{s+r-1}) \times (E_u, E_{u-1})) &
\end{tikzcd}
$$

FIGURE 6.2. The component $(i, j) = (s + r, u)$ of $\delta\mu_r$

By Theorem 6.2.3 and Proposition 6.2.7 the pairing

$$\mu\colon (H^*(p'), H^*(p'')) \longrightarrow H^*(p' \times p'')$$

of extended Cartan–Eilenberg systems induces a pairing

$$(\mu_r\colon (E_r(p'), E_r(p'')) \to E_r(p' \times p''))$$

of the associated cohomological Serre spectral sequences, converging to a filtration-preserving pairing

$$\nu\colon H^*(E') \otimes H^*(E'') \longrightarrow H^*(E' \times E'')$$

of their abutments. We now make these pairing explicit. Recall the isomorphism $E_1^{s,t} = H^{s+t}(E_s, E_{s-1}) \cong C^s_{CW}(B; \mathscr{H}^t(F))$ from Proposition 5.3.2.

PROPOSITION 6.4.4. *The pairing of E_1-terms*

$$\begin{aligned} E_1^{s,t}(p') \otimes E_1^{u,v}(p'') &= H^{s+t}(E'_s, E'_{s-1}) \otimes H^{u+v}(E''_u, E''_{u-1}) \\ &\xrightarrow{\mu_1} H^{s+u+t+v}((E' \times E'')_{s+u}, (E' \times E'')_{s+u-1}) = E_1^{s+u,t+v}(p' \times p'') \end{aligned}$$

corresponds to $(-1)^{tu}$ times the cross product

$$C^s_{CW}(B'; \mathscr{H}^t(F)) \otimes C^u_{CW}(B''; \mathscr{H}^v(F')) \xrightarrow{\times} C^{s+u}_{CW}(B' \times B''; \mathscr{H}^{t+v}(F \times F')) .$$

SKETCH PROOF. To simplify the notation, we again assume $p' = p'' = p$. The cohomology cross products

$$\begin{aligned} H^s(B^{(s)}, B^{(s-1)}; \mathscr{H}^t(F)) \otimes H^u(B^{(u)}, B^{(u-1)}; \mathscr{H}^v(F)) \\ \xrightarrow{\times} H^{s+u}(B^{(s+u)}, B^{(s+u-1)}; \mathscr{H}^t(F) \otimes \mathscr{H}^v(F)) \end{aligned}$$

and

$$\mathscr{H}^t(F) \otimes \mathscr{H}^v(F) \xrightarrow{\times} \mathscr{H}^{t+v}(F \times F)$$

then combine to define the cross product of the proposition. The sign $(-1)^{tu}$ arises from the factor

$$\begin{aligned} H^{s+t}((I^s_\alpha, \partial I^s_\alpha) \times F_{b_\alpha}) \otimes H^{u+v}((I^u_\beta, \partial I^u_\beta) \times F_{b_\beta}) \\ \longrightarrow H^{s+u+t+v}((I^{s+u}_{\alpha,\beta}, \partial I^{s+u}_{\alpha,\beta}) \times F_{b_\alpha} \times F_{b_\alpha}) \end{aligned}$$

of the pairing μ_1, which sends $(g_{s,\alpha} \times f_\alpha) \otimes (g_{u,\beta} \times f_\beta)$ to $(-1)^{tu} g_{s+u,\alpha,\beta} \times f_\alpha \times f_\beta$, where $t = |f_\alpha|$. The cross product does not account for the grading of f_α, hence is missing this sign. □

LEMMA 6.4.5. *The pairing of E_2-terms*

$$\mu_2\colon E_2^{s,t}(p') \otimes E_2^{u,v}(p') \longrightarrow E_2^{s+u,t+v}(p' \times p'')$$

corresponds to $(-1)^{tu}$ times the cohomology cross product

$$H^s(B'; \mathscr{H}^t(F')) \otimes H^u(B''; \mathscr{H}^v(F'')) \xrightarrow{\times} H^{s+u}(B' \times B''; \mathscr{H}^{t+v}(F' \times F'')) .$$

PROOF. We obtain μ_2 from μ_1 by passing to cohomology with respect to the d_1-differentials. □

LEMMA 6.4.6. *The filtration-preserving pairing*

$$\nu\colon H^*(E') \otimes H^*(E'') \longrightarrow H^*(E' \times E'')$$

equals the cohomology cross product.

Proof. By definition,

$$\begin{aligned}\mu_\infty \colon H^*(E', E'_{s-1}) \otimes H^*(E'', E''_{u-1}) &\longrightarrow H^*(E' \times E'', E'_{s-1} \times E'' \cup E' \times E''_{u-1}) \\ &\longrightarrow H^*(E' \times E'', (E' \times E'')_{s+u-1})\end{aligned}$$

is given by the relative cohomology cross product followed by restriction. Passing to the colimit for $s \to -\infty$ and $u \to -\infty$ gives ν, and this colimit is achieved already for $s = u = 0$. □

To pass from the external cross product to the internal cup product, we assume $p' = p'' = p \colon E \to B$ and pull back along a filtration-preserving approximation $D \colon E \to E \times E$ to the diagonal map $\Delta \colon E \to E \times E$.

Proposition 6.4.7. *Let B be a CW complex based at a 0-cell b_0, let $p \colon E \to B$ be a (Hurewicz) fibration, and let $F = p^{-1}(b_0)$ be its fiber. There is a homotopy*

$$\bar{H} \colon I \times B \longrightarrow B \times B$$

with $\bar{H}(t, b_0) = (b_0, b_0)$ for all t, from the diagonal map $\Delta \colon B \to B \times B$ to a cellular map $\bar{D} \colon B \to B \times B$. It admits a lift

$$H \colon I \times E \longrightarrow E \times E$$

with $(p \times p)H = \bar{H}(1 \times p)$, from the diagonal map $\Delta \colon E \to E \times E$ to a filtration-preserving map $D \colon E \to E \times E$. This restricts to a homotopy

$$\tilde{H} \colon I \times F \longrightarrow F \times F$$

from the diagonal map $\Delta \colon F \to F \times F$ to a map $\tilde{D} \colon F \to F \times F$.

Proof. By cellular approximation, the map $\Delta \colon B \to B \times B$ is homotopic to a cellular map $\bar{D} \colon B \to B \times B$, and we may assume that the homotopy $\bar{H}$ is stationary on $\{b_0\}$, since Δ is already cellular on that subspace.

The diagonal map $\Delta \colon E \to E \times E$ lifts $\Delta p \colon E \to B \times B$, so by the homotopy lifting property for $p \times p$ we have a homotopy $H \colon I \times E \to E \times E$ from Δ to a map $D \colon E \to E \times E$ with $(p \times p)D = \bar{D}p$.

$$\begin{tikzcd}
E \arrow[rr, "\Delta"] \arrow[d, "i_0"'] & & E \times E \arrow[d, "p \times p"] \\
I \times E \arrow[r, "1 \times p"'] \arrow[urr, dotted, "H"] & I \times B \arrow[r, "\bar{H}"'] & B \times B
\end{tikzcd}$$

The restriction $H|I \times F$ then factors through $F \times F \subset E \times E$, giving the required homotopy $\tilde{H}$ from $\Delta \colon F \to F \times F$ to a map $\tilde{D}$. □

Proposition 6.4.8. *The filtration-preserving map $D \colon E \to E \times E$ induces a morphism*

$$D^* \colon H^*(p \times p) \longrightarrow H^*(p)$$

of Cartan–Eilenberg systems and a morphism

$$D_r^* \colon E_r^{*,*}(p \times p) \longrightarrow E_r^{*,*}(p)$$

of cohomological Serre spectral sequences. The homomorphism D_1^* corresponds to the restriction

$$\bar{D}^* \colon C^*_{CW}(B \times B; \mathscr{H}^*(F \times F)) \longrightarrow C^*_{CW}(B; \mathscr{H}^*(F))$$

associated to the cellular map $\bar{D}\colon B \to B \times B$ and the coefficient homomorphism $\tilde{D}^* = \Delta^*\colon \mathscr{H}^*(F \times F) \to \mathscr{H}^*(F)$. The homomorphism D_2^* corresponds to the restriction homomorphism

$$\bar{D}^* = \Delta^*\colon H^*(B \times B; \mathscr{H}^*(F \times F)) \longrightarrow H^*(B; \mathscr{H}^*(F)) .$$

The induced morphisms of filtered target groups is

$$D^* = \Delta^*\colon H^*(p \times p)(-\infty, \infty) = H^*(E \times E) \longrightarrow H^*(E) = H^*(p)(-\infty, \infty) .$$

PROOF. The map of pairs $D\colon (E_{j-1}, E_{i-1}) \to ((E\times E)_{j-1}, (E\times E)_{i-1})$ induces

$$D^*\colon H^*(p\times p)(i,j) = H^*((E\times E)_{j-1}, (E\times E)_{i-1}) \longrightarrow H^*(E_{j-1}, E_{i-1}) = H^*(p)(i,j)$$

for all (extended) integers $i \le j$. The rest follows by chasing the definitions, and using the homotopies $\bar{H}$, $\tilde{H}$ and H to note that $\bar{D}^* = \Delta^*$, $\tilde{D}^* = \Delta^*$ and $D^* = \Delta^*$, once we have passed to cohomology groups. □

THEOREM 6.4.9. *Let $p\colon E \to B$ be a Hurewicz fibration, with B a CW complex. Each choice of filtration-preserving lift $D\colon E \to E \times E$ lifting a (cellular) diagonal approximation $\bar{D}\colon B \to B \times B$ induces a pairing of extended Cartan–Eilenberg systems*

$$D^*\mu\colon (H^*(p), H^*(p)) \longrightarrow H^*(p)$$

and of cohomological Serre spectral sequences

$$D^*\mu_r\colon (E_r^{*,*}(p), E_r^{*,*}(p)) \longrightarrow E_r^{*,*}(p) .$$

The pairing of E_1-terms

$$E_1^{s,t}(p) \otimes E_1^{u,v}(p) \longrightarrow E_1^{s+u,t+v}(p)$$

corresponds to $(-1)^{tu}$ times the cochain cup product

$$C^s_{CW}(B; \mathscr{H}^t(F)) \otimes C^u_{CW}(B; \mathscr{H}^v(F)) \xrightarrow{\cup} C^{s+u}_{CW}(B; \mathscr{H}^{t+v}(F))$$

associated to $\bar{D}$. The pairing of E_2-term,

$$E_2^{s,t}(p) \otimes E_2^{u,v}(p) \longrightarrow E_2^{s+u,t+v}(p)$$

corresponds to $(-1)^{tu}$ times the cohomology cup product

$$H^s(B; \mathscr{H}^t(F)) \otimes H^u(B; \mathscr{H}^v(F)) \xrightarrow{\cup} H^{s+u}(B; \mathscr{H}^{t+v}(F)) ,$$

and is independent of the choice of D and $\bar{D}$. This pairing of spectral sequences converges to the cup product pairing

$$H^*(E) \otimes H^*(E) \xrightarrow{\cup} H^*(E)$$

in the cohomology of the total space.

PROOF. This follows by composing the external pairing μ from Proposition 6.4.3 with the morphism D^* from Proposition 6.4.8. The composites

$$\mathscr{H}^t(F) \otimes \mathscr{H}^v(F) \xrightarrow{\times} \mathscr{H}^{t+v}(F \times F) \xrightarrow{\tilde{D}^*} \mathscr{H}^{t+v}(F)$$

$$H^s(B) \otimes H^u(B) \xrightarrow{\times} H^{s+u}(B \times B) \xrightarrow{\bar{D}^*} H^{s+u}(B)$$

$$H^*(E) \otimes H^*(E) \xrightarrow{\times} H^*(E \times E) \xrightarrow{D^*} H^*(E)$$

are equal to the respective cup products, in view of the homotopies $\tilde{H}\colon \Delta \simeq \tilde{D}$, $\bar{H}\colon \Delta \simeq \bar{D}$ and $H\colon \Delta \simeq D$. □

6.5. The cohomological Wang and Gysin sequences

THEOREM 6.5.1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence, with $B \simeq S^u$ a 1-connected CW complex. There is a long exact sequence*

$$\dots \to H^{n-1}(F) \xrightarrow{\delta} H^{n-u}(F) \xrightarrow{i^!} H^n(E) \xrightarrow{i^*} H^n(F) \xrightarrow{\delta} H^{n-u+1}(F) \to \dots$$

where i^* is a ring homomorphism and

$$\delta(x \cup y) = \delta(x) \cup y + (-1)^{|x|(u-1)} x \cup \delta(y)\,.$$

PROOF. The Serre spectral sequence

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow_s H^{s+t}(E)$$

is a ring spectral sequence with $E_2 = E_u$ and $E_{u+1} = E_\infty$. Setting $H^*(B) = \mathbb{Z}\{1, g_u\}$ we can write $d_u(1 \otimes x) = g_u \otimes \delta(x)$ with $\delta\colon H^t(F) \to H^{t-u+1}(F)$. The Leibniz rule

$$d_u(1 \otimes x \cup y) = d_u(1 \otimes x) \cup (1 \otimes y) + (-1)^{|x|}(1 \otimes x) \cup d_u(1 \otimes y)$$

translates to the given derivation rule for δ. □

Recall the divided power algebra $\Gamma(x) = \mathbb{Z}\{\gamma_i(x) \mid i \geq 0\}$ with $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_i(x) \cdot \gamma_j(x) = (i,j)\gamma_{i+j}(x)$, graded so that $|\gamma_i(x)| = i|x|$. Here $(i,j) = (i+j)!/i!j!$ is the binomial coefficient. Let $\Lambda(x) = \mathbb{Z}\{1, x\}$ denote the exterior algebra on x, with $x^2 = 0$. Usually $|x|$ is even in the divided power case, and odd in the exterior case.

THEOREM 6.5.2. *Let $u \geq 2$. If u is odd, then*

$$H^*(\Omega S^u) \cong \Gamma(x)$$

with $|x| = u - 1$. If u is even, then

$$H^*(\Omega S^u) \cong \Lambda(x) \otimes \Gamma(y)$$

with $|x| = u - 1$ and $|y| = 2(u-1)$.

PROOF. The Wang sequence for $\Omega S^u \to PS^u \to S^u$, with PS^u contractible, reduces to isomorphisms

$$\delta\colon \tilde{H}^n(F) \xrightarrow{\cong} H^{n-u+1}(F)\,.$$

Suppose first that $u \geq 3$ is odd. Let $\gamma_0(x) = 1$ and inductively set $\gamma_i(x) \in H^{i(u-1)}(\Omega S^u)$ for $i \geq 1$ so that $\delta(\gamma_i(x)) = \gamma_{i-1}(x)$. By induction on i and j,

$$\delta(\gamma_i(x) \cup \gamma_j(x)) = \gamma_{i-1}(x) \cup \gamma_j(x) + \gamma_i(x) \cup \gamma_{j-1}(x)$$

equals $(i-1,j) + (i,j-1) = (i,j)$ times

$$\delta(\gamma_{i+j}(x)) = \gamma_{i+j-1}(x)\,.$$

This proves that $\gamma_i(x) \cup \gamma_j(x) = (i,j)\gamma_{i+j}(x)$.

Next suppose that $u \geq 2$ is even. Fix $x \in H^{u-1}(\Omega S^u)$ so that $\delta(x) = 1$. By graded commutativity, $x^2 = 0$. Let $\gamma_0(y) = 1$ and inductively set $\gamma_i(y) \in H^{2i(u-1)}(\Omega S^u)$ for $i \geq 1$ so that $\delta(\gamma_i(y)) = x\gamma_{i-1}(y)$. Then $\delta(x\gamma_i(y)) = 1 \cup \gamma_i(y) - x \cup x\gamma_{i-1}(y) = \gamma_i(y)$, so $\gamma_i(y)$ generates $H^{2i(u-1)}(\Omega S^u)$ while $x\gamma_i(y)$ generates $H^{(2i+1)(u-1)}(\Omega S^u)$. By induction on i and j,

$$\delta(\gamma_i(y) \cup \gamma_j(y)) = x\gamma_{i-1}(y) \cup \gamma_j(y) + \gamma_i(y) \cup x\gamma_{j-1}(y)$$

equals $(i-1,j)+(i,j-1)=(i,j)$ times

$$\delta(\gamma_{i+j}(y)) = x\gamma_{i+j-1}(y)\,.$$

Hence $\gamma_i(y)\cup\gamma_j(y) = (i,j)\gamma_{i+j}(y)$. $\square$

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \cdots \\
3(u-1) & \mathbb{Z} \arrow[rrrdd, "d_u"] & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 \\
2(u-1) & \mathbb{Z} \arrow[rrrdd, "d_u"] & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 \\
u-1 & \mathbb{Z} \arrow[rrrdd, "d_u"] & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
t/s & 0 & & & u
\end{tikzcd}$$

THEOREM 6.5.3. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence, with $F \simeq S^v$ and B a 1-connected CW complex. There is a long exact sequence*

$$\cdots \to H^{n-v-1}(B) \xrightarrow{e\cup} H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{p^!} H^{n-v}(B) \xrightarrow{e\cup} H^{n+1}(B) \to \dots$$

where p^* is a ring homomorphism and $e = \delta(1) \in H^{v+1}(B)$ is the Euler class *of the (oriented spherical) fibration.*

PROOF. The Serre spectral sequence

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow_s H^{s+t}(E)$$

is a ring spectral sequence with $E_2 = E_{v+1}$ and $E_{v+2} = E_\infty$. Setting $H^*(F) = \mathbb{Z}\{1, g_v\}$ we can write $d_{v+1}(x\otimes g_v) = \delta(x)\otimes 1$ with $\delta\colon H^{s-v-1}(B)\to H^s(B)$. The Leibniz rule

$$d_{v+1}((1\otimes g_v)\cup(x\otimes 1)) = d_{v+1}(1\otimes g_v)\cup(x\otimes 1) + (-1)^v(1\otimes g_v)\cup d_{v+1}(x\otimes 1)$$

translates to $\delta(x) = (-1)^{v|x|}e\cup x$, since $d_{v+1}(x\otimes 1)=0$ lies in a trivial group. We can replace δ with $x\mapsto e\cup x$ without affecting the exactness of the sequence. $\square$

$$
\begin{tikzcd}
v & H^0(B) \arrow[ddrrr, "d_{v+1}"] & H^{s-v-1}(B) \arrow[ddrrr, "d_{v+1}"] & H^{s-v}(B) \arrow[ddrrr, "d_{v+1}"] & \cdots & \cdots & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H^0(B) & H^{s-v-1}(B) & \cdots & H^{v+1}(B) & H^s(B) & H^{s+1}(B) \\
t/s & 0 & s-v-1 & \cdots & v+1 & s & s+1
\end{tikzcd}
$$

REMARK 6.5.4. The Euler class vanishes if p admits a section $s\colon B \to E$. If B is a closed, oriented $(v+1)$-manifold with fundamental class $[B] \in H_{v+1}(B)$, and $E = S(TB) \to B$ is the unit sphere bundle in the tangent bundle $TB \to B$, then the Euler class $e \in H^{v+1}(B)$ evaluates on $[B]$ to the Euler characteristic of B:

$$\langle e, [B] \rangle = \chi(B)\,.$$

See [**MS74**, Cor. 11.12]. In particular, the Euler characteristic vanishes if B admits an everywhere nonzero vector field.

REMARK 6.5.5. Let $U(k)$ denote the rank k *unitary group*. It acts freely on the contractible Stiefel space

$$V_k(\mathbb{C}^\infty) = \{(v_1, \dots, v_k) \mid v_i^* v_j = \delta_{i,j}\}$$

of unitary k-frames in $\mathbb{C}^\infty = \bigcup_n \mathbb{C}^n$, with orbit space the Grassmannian

$$Gr_k(\mathbb{C}^\infty) = \{V \subset \mathbb{C}^\infty \mid \dim_{\mathbb{C}}(V) = k\}$$

of k-dimensional complex linear subspaces of $\mathbb{C}^\infty$. The principal $U(k)$-bundle

$$U(k) \longrightarrow V_k(\mathbb{C}^\infty) \longrightarrow Gr_k(\mathbb{C}^\infty)$$

is thus universal, and $Gr_k(\mathbb{C}^\infty) \simeq BU(k)$ is a model for the *classifying space* of $U(k)$. We get natural bijections

$$\mathrm{Vect}_k^{\mathbb{C}}(B) \cong \mathrm{Bun}_{U(k)}(B) \cong [B, BU(k)] \cong [B, Gr_k(\mathbb{C}^\infty)]$$

for all CW complexes B. Here $\mathrm{Vect}_k^{\mathbb{C}}(B)$ denotes the set of isomorphism classes of rank k complex vector bundles $E \to B$.

When $k = 1$, we have $V_1(\mathbb{C}^\infty) = S(\mathbb{C}^\infty) \cong S^\infty$ and $Gr_1(\mathbb{C}^\infty) \cong \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so

$$[B, BU(1)] \cong [B, \mathbb{C}P^\infty] \cong [B, K(\mathbb{Z}, 2)] \cong H^2(B)$$

by the Eilenberg–MacLane representability theorem. The class $c_1(L) \in H^2(B)$ corresponding to a complex line bundle $L \to B$ is called the *first Chern class* of L, and classifies L up to isomorphism.

When $k \geq 2$, the space $BU(k) \simeq Gr_k(\mathbb{C}^\infty)$ is not an Eilenberg–MacLane space, so $[B, BU(k)]$ is not naturally identified with a cohomology group of B. However, each cohomology class $c \in H^n(BU(k))$ pulls back along the classifying map $f\colon B \to BU(k)$ of any $\mathbb{C}^k$-bundle $E \to B$ to define a class $c(E) = f^*(c) \in H^n(B)$. This class $c(E)$ depends naturally on $E \to B$, and is called a *characteristic class*. To determine all characteristic classes for complex vector bundles, we calculate $H^*(BU(k))$.

THEOREM 6.5.6. *For each $k \geq 0$ there are isomorphisms*

$$H^*(BU(k)) \cong \mathbb{Z}[c_1, \dots, c_k]$$

with $|c_i| = 2i$. The Gysin sequence associated to the fiber sequence

$$U(k)/U(k-1) \longrightarrow V_k(\mathbb{C}^\infty)/U(k-1) \xrightarrow{p} V_k(\mathbb{C}^\infty)/U(k)\,,$$

with

$$\begin{aligned} F &= U(k)/U(k-1) \cong S^{2k-1} \\ E &= V_k(\mathbb{C}^\infty)/U(k-1) \simeq BU(k-1) \\ B &= V_k(\mathbb{C}^\infty)/U(k) = Gr_k(\mathbb{C}^\infty) \simeq BU(k)\,, \end{aligned}$$

breaks up into short exact sequences

$$0 \to H^{*-2k}(BU(k)) \xrightarrow{c_k \cup} H^*(BU(k)) \xrightarrow{p^*} H^*(BU(k-1)) \to 0\,.$$

Here $p^*(c_i) = c_i$ for $1 \leq i < k$, while $c_k \in H^{2k}(BU(k))$ is the Euler class of $p\colon E \to B$.

PROOF. We proceed by induction on k, hence assume that

$$H^*(BU(k-1)) = \mathbb{Z}[c_1, \dots, c_{k-1}]$$

where $c_i \in H^{2i}(BU(k-1))$ has been specified for $1 \leq i \leq k-1$. We use the fiber sequence $F \to E \to B$, defined as above. Here $U(k)$ acts transitively on $S(\mathbb{C}^k) = S^{2k-1}$, with stabilizer $U(k-1)$, which gives the identification $U(k)/U(k-1) \cong S^{2k-1}$. The restricted $U(k-1)$-action on $V_k(\mathbb{C}^\infty)$ makes $V_k(\mathbb{C}^\infty) \to V_k(\mathbb{C}^\infty)/U(k-1) = E$ a universal principal $U(k-1)$-bundle, so that $E \simeq BU(k-1)$. (One can define an explicit equivalence $E \simeq Gr_{k-1}(\mathbb{C}^\infty)$.)

Since $H^*(BU(k-1))$ is trivial in odd degrees, the Gysin sequence for $F \to E \to B$ breaks up into exact sequences

$$\begin{aligned} 0 \to H^{n-2k}(BU(k)) \xrightarrow{e\cup} H^n(BU(k)) \xrightarrow{p^*} H^n(BU(k-1)) \\ \xrightarrow{p^!} H^{n-2k+1}(BU(k)) \xrightarrow{e\cup} H^{n+1}(BU(k)) \to 0\,, \end{aligned}$$

one for each even integer n. It follows by induction on n that $H^{n+1}(BU(k)) = 0$ for $n+1$ odd, so the Gysin sequence really breaks up into short exact sequences, and $H^*(BU(k))$ is concentrated in even degrees. Moreover, $p^*\colon H^n(BU(k)) \to H^n(BU(k-1))$ is an isomorphism for $n < 2k$, so we can uniquely define $c_i \in H^{2i}(BU(k))$ for $1 \leq i < k$ by the condition $p^*(c_i) = c_i \in H^{2i}(BU(k-1))$. Finally, we set $c_k = e \in H^{2k}(BU(k))$ to be the Euler class of this spherical fibration, so that

$$d_{2k}(1 \otimes g_{2k-1}) = c_k \otimes 1$$

in the cohomological Serre spectral sequence. To show that the resulting ring homomorphism

$$h\colon \mathbb{Z}[c_1, \dots, c_k] \longrightarrow H^*(BU(k))$$

is an isomorphism, we use induction on the degree $*$ and the following vertical map of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{2k}\mathbb{Z}[c_1,\dots,c_k] \arrow[r, "c_k\cdot"] \arrow[d, "\Sigma^{2k}h"'] & \mathbb{Z}[c_1,\dots,c_k] \arrow[r] \arrow[d, "h"'] & \mathbb{Z}[c_1,\dots,c_{k-1}] \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H^{*-2k}(BU(k)) \arrow[r, "c_k\cup"] & H^*(BU(k)) \arrow[r, "p^*"] & H^*(BU(k-1)) \arrow[r] & 0
\end{tikzcd}$$

□

REMARK 6.5.7. We call $c_i \in H^{2i}(BU(k))$ the *i-th Chern class*. For each $\mathbb{C}^k$-bundle $E \to B$ with classifying map $f\colon B \to BU(k)$, we call $c_i(E) = f^*(c_i) \in H^{2i}(B)$ the i-th Chern class of the bundle. The Chern classes $c_i(E)$ determine the ring homomorphism

$$\begin{aligned} f^*\colon H^*(BU(k)) &\longrightarrow H^*(B) \\ c_i &\longmapsto c_i(E)\,. \end{aligned}$$

This is generally less information than the isomorphism class of the vector bundle, i.e., the homotopy class of $f\colon B \to BU(k)$, but characteristic classes often provide conveniently accessible cohomological invariants of this less accessible homotopical datum.

((ETC: Whitney sum and Cartan formula. Functorial construction gives $Bi \simeq p\colon BU(k-1) \to BU(k)$, where $i\colon U(k-1) \to U(k)$ is the inclusion. Stable classes in $H^*(BU) = \mathbb{Z}[c_k \mid k \geq 1]$.))

((ETC: $H^*(BO(k);\mathbb{F}_2) = \mathbb{F}_2[w_1,\dots,w_k]$ for $w_i \in H^i(BO(k);\mathbb{F}_2)$. Thom's formula $Sq^i(U) = \Phi(w_i)$.))

6.6. Rational cohomology of integral Eilenberg–MacLane spaces

Let $n \geq 1$. Recall that $K(\mathbb{Z},n)$ is a $(n-1)$-connected CW complex, with $\pi_n K(\mathbb{Z},n) \cong \mathbb{Z}$ and $\pi_i K(\mathbb{Z},n) = 0$ for $i \neq n$. Each homology group $H_i(K(\mathbb{Z},n))$ is finitely generated of rank equal to the dimension of

$$H_i(K(\mathbb{Z},n)) \otimes \mathbb{Q} \xrightarrow{\cong} H_i(K(\mathbb{Z},n);\mathbb{Q})$$

over $\mathbb{Q}$. The evaluation pairing induces an isomorphism

$$H^i(K(\mathbb{Z},n))/(\text{torsion}) \xrightarrow{\cong} \mathrm{Hom}(H_i(K(\mathbb{Z},n))/(\text{torsion}),\mathbb{Z})\,.$$

DEFINITION 6.6.1. For $n \geq 1$ let the *universal class*

$$u_n \in H^n(K(\mathbb{Z},n)) \cong \mathrm{Hom}(H_n(K(\mathbb{Z},n)),\mathbb{Z})$$

correspond to the inverse Hurewicz isomorphism

$$h_n^{-1}\colon H_n(K(\mathbb{Z},n)) \xrightarrow{\cong} \pi_n(K(\mathbb{Z},n)) \cong \mathbb{Z}\,.$$

((ETC: Many authors write ι_n for this universal class.))

THEOREM 6.6.2. *Let $n \geq 1$. If n is odd then*

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_n) = \mathbb{Q}\{1,u_n\}$$

with $u_n^2 = 0$. If n is even then

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong \mathbb{Q}[u_n] = \mathbb{Q}\{1,u_n,u_n^2,\dots\}\,.$$

Finite type and the universal coefficient theorem imply the following consequence, which proves Theorem 4.6.15.

COROLLARY 6.6.3. *Let $n \geq 1$. If n is odd then*

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } i \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even then

$$H_i(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } 0 \leq i \equiv 0 \mod n, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF OF THEOREM. When $n = 1$, the cohomology of $K(\mathbb{Z}, 1) \simeq S^1$ is well-known to be exterior on $g_1 = u_1$ in degree 1.

Suppose that the theorem holds for an odd $n \geq 1$. We use the cohomology Serre spectral sequence with rational coefficients

$$E_2^{s,t} = H^s(K(\mathbb{Z}, n+1); H^t(K(\mathbb{Z}, n); \mathbb{Q})) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}, n+1); \mathbb{Q})$$

for the homotopy fiber sequence

$$K(\mathbb{Z}, n) \longrightarrow PK(\mathbb{Z}, n+1) \xrightarrow{p} K(\mathbb{Z}, n+1)$$

This is isomorphic to the integral spectral sequence tensored with $\mathbb{Q}$, which is still a spectral sequence since $\mathbb{Q}$ is torsion-free, hence flat, so that tensoring with it is exact. Since $K(\mathbb{Z}, n+1)$ has finite type, we have an isomorphism

$$H^*(K(\mathbb{Z}, n+1); \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(K(\mathbb{Z}, n); \mathbb{Q}) \\ \xrightarrow{\cong} E_2^{*,*} = H^*(K(\mathbb{Z}, n+1); H^*(K(\mathbb{Z}, n); \mathbb{Q})) .$$

Since $PK(\mathbb{Z}, n+1)$ is contractible, the abutment is $\mathbb{Q}$ in total degree 0. The E_2-term is concentrated in the two rows $t = 0$ and $t = n$, so

$$d_{n+1} \colon H^n(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{\cong} H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q})$$

must be an isomorphism. More precisely, this transgressive differential is an integral isomorphism mapping u_n to

$$d_{n+1}(u_n) = u_{n+1} ,$$

by compatibility of the Hurewicz homomorphisms with coboundaries and pullbacks.

(If one does not wish to check this, it suffices to know that $d_{n+1}(u_n)$ is a rational unit times u_{n+1}, in which case we calculate below that $d_{n+1}(u_{n+1}^j \cup u_n)$ is a unit times u_{n+1}^{j+1}, hence generates the same $\mathbb{Q}$-vector space as the latter class.)

$$\begin{tikzcd}
n & \mathbb{Q} \arrow[rrrrddd, "d_{n+1}"] & 0 & 0 & 0 & \mathbb{Q} \arrow[rrrrddd, "d_{n+1}"] & 0 & 0 & 0 & \mathbb{Q} \arrow[rrrrddd, "d_{n+1}"] & 0 & 0 & 0 & \dots \\
 & 0 & & & & 0 & & & & 0 & & & & \\
 & 0 & & & & 0 & & & & 0 & & & & \\
0 & \mathbb{Q} & 0 & 0 & 0 & \mathbb{Q} & 0 & 0 & 0 & \mathbb{Q} & 0 & 0 & 0 & \dots \\
t/s & 0 & & & & n+1 & & & & 2(n+1) & & & &
\end{tikzcd}$$

We now proceed as for the Gysin sequence. Suppose inductively for a $j \geq 0$ that

$$H^i(K(\mathbb{Z}, n+1); \mathbb{Q}) = \begin{cases} \mathbb{Q}\{u_{n+1}^j\} & \text{for } i = j(n+1), \\ 0 & \text{for } j(n+1) < i < (j+1)(n+1). \end{cases}$$

Then

$$d_{n+1} \colon E_2^{i,n} \longrightarrow E_2^{i+n+1,0}$$

must be an isomorphism, for each $j(n+1) \leq i < (j+1)(n+1)$. Since

$$d_{n+1}(u_{n+1}^j \cup u_n) = u_{n+1}^j \cup d_{n+1}(u_n) = u_{n+1}^{j+1}$$

must generate $H^{(j+1)(n+1)}(K(\mathbb{Z}, n+1); \mathbb{Q})$, the inductive claim also holds for $j+1$. This proves the theorem for $n+1$ even.

Next, suppose that the theorem holds for an even $n \geq 2$. We use the same Serre spectral sequence as above, but now the E_2-term is concentrated in the rows $0 \leq t \equiv 0 \mod n$. Again the transgressive differential

$$d_{n+1} \colon H^n(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{\cong} H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q})$$

maps u_n to (a unit times) u_{n+1}.

$$\begin{tikzcd}
 & \dots \arrow[rrrdd, "d_{n+1}"] & & & \dots & & \dots \\
 & 0 & & & 0 & & \\
2n & \mathbb{Q} \arrow[rrrdd, "d_{n+1}"] & 0 & 0 & \mathbb{Q} & 0 & \dots \\
 & 0 & & & 0 & & \\
n & \mathbb{Q} \arrow[rrrdd, "d_{n+1}"] & 0 & 0 & \mathbb{Q} & 0 & \dots \\
 & 0 & & & 0 & & \\
0 & \mathbb{Q} & 0 & 0 & \mathbb{Q} & 0 & \dots \\
t/s & 0 & & & n+1 & & u
\end{tikzcd}$$

It follows from the Leibniz rule that

$$d_{n+1}(u_n^j) = j u_{n+1} \cup u_n^{j-1} \tag{6.1}$$

for all $j \geq 1$. Since we are working with rational coefficients, $ju_{n+1} \cup u_n^{j-1}$ generates $E_2^{n+1,(j-1)n}$, so that

$$E_{n+2}^{s,t} = \begin{cases} \mathbb{Q} & \text{for } (s,t) = (0,0), \\ 0 & \text{otherwise, for } s \leq n+1. \end{cases}$$

It remains to confirm that $H^i(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$ for all $i > n+1$. Let $u > n+1$ and suppose, inductively, that $H^i(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$ for $n+1 < i < u$. Then $E_2^{u,0} \cong H^u(K(\mathbb{Z}, n); \mathbb{Q})$, and we must have $E_\infty^{u,0} = 0$ since the abutment is trivial in total degree n. The final differential

$$d_u \colon E_u^{0,u-1} \longrightarrow E_u^{u,0}$$

is trivial, because $E_u^{0,u-1} \subset E_{n+2}^{0,u-1} = 0$. Furthermore,

$$d_{u-n-1} \colon E_{u-n-1}^{n+1,u-n-2} \longrightarrow E_{u-n-1}^{u,0}$$

with $u-n-1 \geq 2$ must also be zero, because $E_{u-n-1}^{n+1,u-n-2}$ is trivial if $0 < u-n-2 < n$ or if $u-n-1 \geq n+2$. When $u = 2(n+1)$ the differential

$$d_{n+1} \colon E_{n+1}^{n+1,n} \longrightarrow E_{n+1}^{2(n+1),0}$$

must be zero because the source is generated by $d_{n+1}(u_n^2) = 2u_{n+1} \cup u_n$ and $d_{n+1} d_{n+1} = 0$. Hence we can only have $E_\infty^{u,0} = 0$ of $E_2^{u,0} = 0$, i.e., if $H^u(K(\mathbb{Z}, n+1); \mathbb{Q}) = 0$. This confirms the claim by induction on n, and proves the theorem for $n+1$ odd. □

Remark 6.6.4. For $n \geq 2$ even, the use of the Leibniz rule to calculate $d_{n+1} \colon E_{n+1}^{0,jn} \to E_{n+1}^{n+1,(j-1)n}$ relies essentially on knowing the cup product structure of $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ and the fact that the Serre spectral sequence differential d_{n+1} is a derivation. Furthermore, the presence of the coefficient j in (6.1) means that this argument does not work integrally, since j is usually not an integral unit.

6.7. First p-torsion in $\pi_*(S^3)$

The 2-connected cover of S^2 sits in the Puppe fiber sequence

$$K(\mathbb{Z}, 1) \longrightarrow \tau_{\geq 3} S^2 \longrightarrow S^2 \xrightarrow{g_2} K(\mathbb{Z}, 2) .$$

Since $\Omega K(\mathbb{Z}, 2) \simeq K(\mathbb{Z}, 1) \simeq S^1$ we can recognize this as the Hopf fiber sequence

$$S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2$$

and its classifying map $g_2 \colon S^2 \to BS^1 \simeq \mathbb{C}P^\infty$.

The 3-connected cover of S^3 is less familiar. We have a Puppe fiber sequence

$$K(\mathbb{Z}, 2) \longrightarrow \tau_{\geq 4} S^3 \longrightarrow S^3 \xrightarrow{g_3} K(\mathbb{Z}, 3) .$$

The cohomology of $\Omega K(\mathbb{Z}, 3) \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ is well known, and allows the following calculation.

PROPOSITION 6.7.1. *The Serre spectral sequence*

$$E_2^{s,t} = H^s(S^3; H^t(K(\mathbb{Z},2))) \Longrightarrow_s H^{s+t}(\tau_{\geq 4} S^3)$$

has E_2-term

$$E_2^{*,*} \cong H^*(S^3) \otimes H^*(\mathbb{C}P^\infty) = \Lambda(g_3) \otimes \mathbb{Z}[y]$$

with $g_3 \in H^3(S^3)$ and $y = u_2 \in H^2(\mathbb{C}P^\infty)$, and nonzero differentials

$$d_3(y^j) = j g_3 y^{j-1}$$

for all $j \geq 1$. Hence

$$H^i(\tau_{\geq 4} S^3) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/j & \text{for } i = 2j+1 \geq 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(\tau_{\geq 4} S^3) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/j & \text{for } i = 2j \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The natural homomorphism

$$H^*(S^3) \otimes H^*(\mathbb{C}P^\infty) \xrightarrow{\cong} H^*(S^3; H^*(\mathbb{C}P^\infty))$$

is an isomorphism. The E_2-term thus appears as below.

$$\begin{tikzcd}
 & \cdots \arrow[ddrrr, "d_3"] & & & \cdots \\
 & 0 & & & 0 \\
4 & \mathbb{Z} \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z} \\
 & 0 & & & 0 \\
2 & \mathbb{Z} \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z} \\
 & 0 & & & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
t/s & 0 & & & 3
\end{tikzcd}$$

Since $\tau_{\geq 4} S^3$ is 3-connected, the differential $d_3 \colon \mathbb{Z}\{y\} = E_3^{0,2} \to E_3^{3,0} = \mathbb{Z}\{g_3\}$ is an isomorphism. With the right choice of identifications, this implies that

$$d_3(y) = g_3 \,.$$

The Leibniz rule thus implies

$$d_3(y^j) = j g_3 y^{j-1}$$

for all $j \geq 0$. This leaves the following $E_4 = E_\infty$-term, with gy^{j-1} generating a copy of $\mathbb{Z}/j$ in bidegree $(3, 2(j-1))$, for each $j \geq 2$.

$$
\begin{array}{c|cccc}
 & 0 & & & \cdots \\
 & 0 & & & 0 \\
4 & 0 & 0 & 0 & \mathbb{Z}/3 \\
 & 0 & & & 0 \\
2 & 0 & 0 & 0 & \mathbb{Z}/2 \\
 & 0 & & & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0 \\
\hline
t/s & 0 & & & 3
\end{array}
$$

This calculates $H^*(\tau_{\geq 4}S^3)$, and our finite type result and the universal coefficient theorem then determine $H_*(\tau_{\geq 4}S^3)$. □

COROLLARY 6.7.2. *$\pi_4(S^3) \cong \mathbb{Z}/2$ is generated by $E\eta$.*

PROOF. We have $\pi_4(\tau_{\geq 4}S^3) \cong H_4(\tau_{\geq 4}S^3) \cong \mathbb{Z}/2$ by the Hurewicz theorem, and $\pi_4(\tau_{\geq 4}S^3) \cong \pi_4(S^3)$ by the long exact sequence in homotopy for the fiber sequence defining $\tau_{\geq 4}S^3$. We also know that $E\colon \pi_3(S^2) \to \pi_4(S^3)$ is surjective, by Freudenthal's stability theorem, so $E\eta$ must generate $\pi_4(S^3)$. □

Let p be a prime. Further arguments, with the Serre class of finite abelian groups of order prime to p, shows that

$$\pi_i(S^3) \cong \pi_i(\tau_{\geq i}S^3) \cong H_i(\tau_{\geq i}S^3)$$

for $3 < i \leq 2p$ maps to

$$H_i(\tau_{\geq 4}S^3)$$

by a homomorphism with kernel and cokernel finite groups of order prime to p. Hence the p-Sylow subgroup of $\pi_i(S^3)$ is trivial for $3 < i < 2p$, and is isomorphic to $\mathbb{Z}/p$ for $i = 2p$. A map representing the first p-torsion in $\pi_*(S^3)$ is often denoted $\alpha_1\colon S^{2p} \longrightarrow S^3$.

6.8. Cohomology of $K(\mathbb{Z}/2, 2)$

To proceed to calculate $\pi_5(S^3) \cong \pi_5(\tau_{\geq 5}S^3)$ we might study $H_*(\tau_{\geq 5}S^3)$ using the Puppe fiber sequence

$$K(\mathbb{Z}/2, 3) \longrightarrow \tau_{\geq 5}S^3 \longrightarrow \tau_{\geq 4}S^3 \longrightarrow K(\mathbb{Z}/2, 4)$$

and the Serre spectral sequence

$$E_2^{*,*} = H^*(\tau_{\geq 4}S^3; H^*(K(\mathbb{Z}/2, 3))) \Longrightarrow H^*(\tau_{\geq 5}S^3)\,.$$

For this, we would need to know $H^*(K(\mathbb{Z}/2, 3))$, which we might hope to deduce from $H^*(K(\mathbb{Z}/2, 2))$ using the loop–path fibration

$$K(\mathbb{Z}/2, 2) \longrightarrow PK(\mathbb{Z}/2, 3) \longrightarrow K(\mathbb{Z}/2, 3)\,.$$

To get started with this, we might first deduce $H^*(K(\mathbb{Z}/2,2))$ from the loop–path fibration

$$K(\mathbb{Z}/2,1) \longrightarrow PK(\mathbb{Z}/2,2) \longrightarrow K(\mathbb{Z}/2,2)\,,$$

where the cohomology of $K(\mathbb{Z}/2,1) \simeq \mathbb{R}P^\infty$ is well known. However, in the cohomological Serre spectral sequence with integral coefficients

$$E_2^{s,t} = H^s(K(\mathbb{Z}/2,2); H^t(\mathbb{R}P^\infty)) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}/2,2))$$

there are more classes in the E_2-term than those that arise as products of classes on the axes:

$$H^s(K(\mathbb{Z}/2,2)) \otimes H^t(\mathbb{R}P^\infty) \longrightarrow H^s(K(\mathbb{Z}/2,2); H^t(K(\mathbb{Z}/2,1)))\,,$$

due to the presence of Tor-terms. Hence it is more convenient to make the calculation with coefficients in the field $\mathbb{F}_2$, and thereafter to use Bockstein arguments ((ETC: see later)) to recover the integral information.

Here $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[a]$ with $a = u_1 \in H^1(\mathbb{R}P^\infty;\mathbb{F}_2)$, and the cohomological Serre spectral sequence with $\mathbb{F}_2$-coefficients has the form

$$E_2^{s,t} = H^s(K(\mathbb{Z}/2,2); H^t(\mathbb{R}P^\infty;\mathbb{F}_2)) \Longrightarrow_s H^{s+t}(PK(\mathbb{Z}/2,2);\mathbb{F}_2)$$

with

$$H^s(K(\mathbb{Z}/2,2);\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^t(\mathbb{R}P^\infty;\mathbb{F}_2) \xrightarrow{\cong} E_2^{s,t}\,.$$

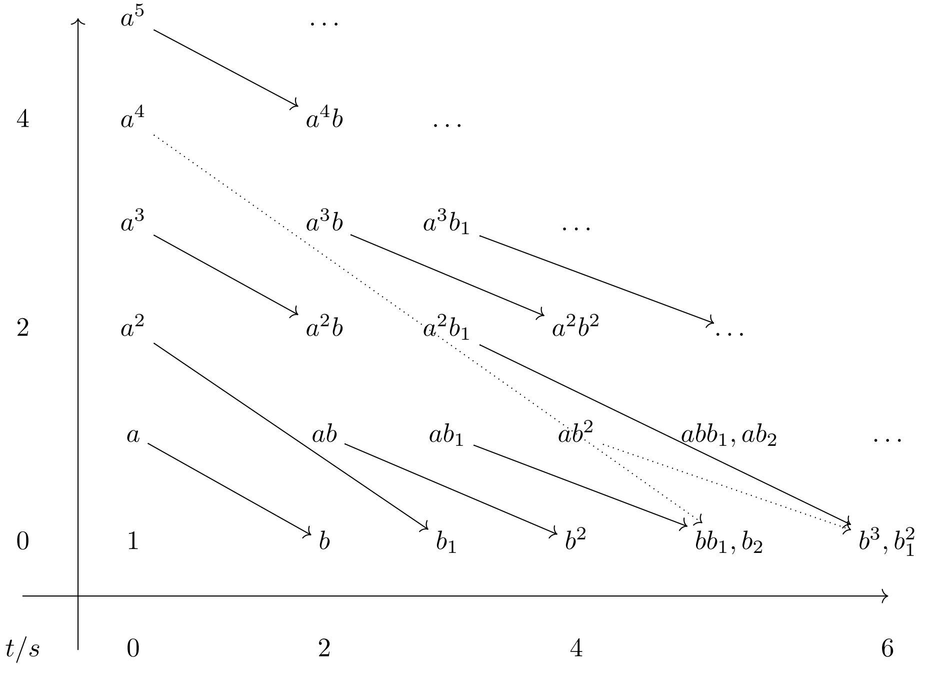
As usual, the abutment $H^*(PK(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2$ is known to vanish in positive degrees, and we seek to use this to determine the cohomology of the base. Clearly $K(\mathbb{Z}/2,2)$ is 1-connected, and $d_2(a) = b$ with b generating $H^2(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2$. Since $d_2(a^2) = ba - ab = 0$, we must have $d_3(a^2) = b_1$ for some nonzero $b_1 \in H^3(K(\mathbb{Z}/2,2);\mathbb{F}_2)$. Furthermore, $d_2(ab) = b^2$ must be nonzero, and $d_2(ab_1) = bb_1$ must be nonzero. Since $d_3(a^4) = b_1a^2 + a^2b_1 = 0$ and $d_2(a^2b_1) = 0$ we must have $d_3(a^2b_1) = b_1^2$ nonzero. At this point we must decide whether $d_2(ab^2) = b^3$ is nonzero in $H^6(K(\mathbb{Z}/2,2);\mathbb{F}_2)$, so that $d_5(a^4) = b_2$ is nonzero in $H^5(K(\mathbb{Z}/2,2);\mathbb{F}_2)$, or if $b^3 = 0$ and $d_4(a^4) = ab^2$.

In fact, the former is the case. We can see this using the map $f\colon K(\mathbb{Z},2) \to K(\mathbb{Z}/2,2)$ inducing the surjection $\pi_2(f)\colon \mathbb{Z} \to \mathbb{Z}/2$. Here $f^*(b) = y$. Since $y^3 \neq 0$ in $H^6(K(\mathbb{Z},2);\mathbb{F}_2)$, it follows that $b^3 \neq 0$, so that $d_5(a^4) = b_2$ for some nonzero $b_2 \in H^6(K(\mathbb{Z}/2,1);\mathbb{F}_2)$. The reader can continue this argument, up to total degree 8, where one must decide whether $b^2b_1^2$ and bb_1b_2 are linearly independent in $H^{10}(K(\mathbb{Z}/2,2);\mathbb{F}_2)$, in which case $d_9(a^8) = b_3$ for a nonzero $b_3 \in H^9(K(\mathbb{Z}/2,2);\mathbb{F}_2)$, or if $d_8(a^8)$ is a nonzero linear combination of abb_1^2 and ab_1b_2.

Again, some external information in addition to the multiplicative structure of the spectral sequence is needed. In the next chapter we discuss the natural cohomology operations

$$Sq^i\colon H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2)$$

introduced by Steenrod, which were used by Serre [**Ser53**] to calculate the mod 2 cohomology of Eilenberg–MacLane spaces. Similar results for mod p cohomology, with p an odd prime, are due to Cartan [**Car54**].


a^5 . . .
4 a^4 a^4b . . .
a^3 a^3b a^3b_1 . . .
2 a^2 a^2b a^2b_1 a^2b^2 . . .
a ab ab_1 ab^2 abb_1, ab_2 . . .
0 1 b b_1 b^2 bb_1, b_2 b^3, b_1^2
t/s 0 2 4 6

CHAPTER 7

The Steenrod algebra

7.1. Cohomology operations

Eilenberg and MacLane proved a representability theorem for cohomology.

DEFINITION 7.1.1. For $n \geq 1$ and G any abelian group let the *universal class*

$$u_n \in H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)),G)$$

correspond to the inverse Hurewicz isomorphism

$$h_n^{-1} \colon H_n(K(G,n)) \xrightarrow{\cong} \pi_n(K(G,n)) \cong G\,.$$

For $n = 0$, with $K(G,0) = G$, we let $u_0 \in \tilde{H}^0(K(G,0);G)$ be the class of the 0-cocycle that takes $g \in K(G,0)$ to $g \in G$.

Recall that $[X,Y]$ denotes the based homotopy classes of base-point preserving maps from a CW complex X to a space Y.

THEOREM 7.1.2 (Eilenberg–MacLane, [**Hat02**, Thm. 4.57]). *There is a natural isomorphism*

$$\begin{aligned} [X, K(G,n)] &\xrightarrow{\cong} \tilde{H}^n(X;G) \\ [f] &\longmapsto f^*(u_n) \end{aligned}$$

for all based CW complexes X.

SKETCH PROOF. Fix a homotopy equivalence

$$\tilde{\sigma} \colon K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$$

inducing the identity homomorphism $G \cong \pi_n(K(G,n)) \cong \pi_n(\Omega K(G,n+1)) \cong \pi_{n+1}(K(G,n+1)) \cong G$, and let

$$\sigma \colon \Sigma K(G,n) \longrightarrow K(G,n+1)$$

be the adjoint map. We define a generalized cohomology theory M on CW pairs (X,A) by

$$M^n(X,A) = [X/A, K(G,n)]\,,$$

with $\delta \colon M^n(A) \longrightarrow M^{n+1}(X,A)$ sending the homotopy class of $f \colon A \to K(G,n)$ to the homotopy class of the composite

$$X/A \simeq X \cup CA \longrightarrow \Sigma A \xrightarrow{\Sigma f} \Sigma K(G,n) \xrightarrow{\sigma} K(G,n+1)\,.$$

Here $\sigma \circ \Sigma f \colon \Sigma A \to K(G,n+1)$ can also be described as the left adjoint of $\tilde{\sigma} f \colon A \to K(G,n+1)$. The abelian group structure on $M^n(X,A)$, and the additivity of δ, can be deduced from the fact that $K(G,n) \simeq \Omega^2 K(G,n+2)$ is a double loop space. The coexactness of the Puppe cofiber sequence

$$A \longrightarrow X \longrightarrow X \cup CA \longrightarrow \Sigma A \longrightarrow \dots$$

proves exactness, while homotopy invariance, excision and additivity are straightforward.

The coefficients groups of this cohomology theory are $M^t = M^t(\text{point}) = [S^0, K(G,t)]$, which equals G for $t = 0$ and 0 for $t \neq 0$. Hence the hypotheses of the Eilenberg–Steenrod uniqueness theorem (see Theorem 5.2.9) are satisfied, and $M^*(X,A) \cong H^*(X,A;G)$. For based CW complexes X we deduce that there is a natural isomorphism

$$[X, K(G,n)] = M^n(X, \{x_0\}) \cong H^n(X, \{x_0\}; G) \cong \tilde{H}^n(X;G)\,.$$

By the Yoneda lemma, the isomorphism must be induced by the class

$$y_n \in \tilde{H}^n(K(G,n);G)$$

that corresponds to the identity map of $X = K(G,n)$, and more careful check of definitions shows that $y_n = u_n$ is the universal class. □

A cohomology operation is a natural transformation between (possibly generalized) cohomology groups. We concentrate on the case of ordinary cohomology theories.

Definition 7.1.3. A *cohomology operation* of type $(G,n;G',n')$ is a natural transformation

$$\theta_X \colon \tilde{H}^n(X;G) \longrightarrow \tilde{H}^{n'}(X;G')\,.$$

Here $X \mapsto H^n(X;G)$ and $X \mapsto H^{n'}(X;G')$ are viewed as functors from CW complexes to sets, so each θ_X is a function, not necessarily a homomorphism. The sum (or difference) of two cohomology operations of type $(G,n;G',n')$ is another cohomology operation of the same type, so the set of such cohomology operations is an abelian group.

Lemma 7.1.4. *The abelian group of cohomology operations of type* $(G,n;G',n')$ *is isomorphic to*

$$[K(G,n), K(G',n')] \cong \tilde{H}^{n'}(K(G,n);G')\,.$$

Proof. This is the Yoneda lemma classifying natural transformations from a represented functor. A map $\theta\colon K(G,n) \to K(G',n')$ corresponds to the natural transformation θ with components θ_X taking the homotopy class of $f\colon X \to K(G,n)$ to the homotopy class of $\theta f\colon X \to K(G',n')$. Conversely, the natural transformation θ corresponds to the homotopy class of a map $\theta\colon K(G,n) \to K(G',n')$ representing $\theta_{K(G,n)}(u_n)$ in $\tilde{H}^{n'}(K(G,n);G')$. □

Computing the cohomology of $K(G,n)$ is thus equivalent to determining the cohomology operations from $H^n(X;G)$. By the Hurewicz theorem, there are only nontrivial cohomology operations of type $(G,n;G',n')$ when $n' \geq n$.

Example 7.1.5. For $k \geq 1$ and R a commutative ring, let the *k-th power operation*

$$\xi^k = \xi^k_X \colon H^n(X;R) \longrightarrow H^{kn}(X;R)$$

be the cohomology operation of type $(R,n;R,kn)$ given by

$$\xi^k(x) = x^k = x \cup \cdots \cup x$$

(with k copies of x). This operation is additive if $k = p$ is a prime and $p = 0$ in R. ((ETC: Is there a standard notation?))

7.2. Steenrod operations

Let p be a prime. Steenrod [**Ste47**], [**Ste52**], [**Ste53**] introduced cohomology operations in mod p cohomology, i.e., cohomology with coefficients in the field $\mathbb{F}_p = \mathbb{Z}/p$, which in a sense generate all other such cohomology operations. These are "reduced power operations", meaning that they are linked to the p-th power operation

$$\xi^p \colon H^n(X;\mathbb{F}_p) \longrightarrow H^{pn}(X;\mathbb{F}_p)\,,$$

but generally land in $H^{n'}(X;\mathbb{F}_p)$ with $n \le n' \le pn$. See Steenrod–Epstein [**Ste62**], May [**May70**] and Hatcher [**Hat02**, §4.L] for more detailed expositions.

We start with $p = 2$, when the reduced power operations are called *reduced squaring operations*, or *Steenrod squares*. The following theorem characterizes these, and can be taken as the basis for an axiomatic development of the theory.

THEOREM 7.2.1 ([**Ste62**, §I.1]). *There are natural transformations*

$$Sq^i \colon \tilde{H}^n(X;\mathbb{F}_2) \longrightarrow \tilde{H}^{n+i}(X;\mathbb{F}_2)$$

for all $i \ge 0$ and $n \ge 0$. These satisfy

(1) $Sq^0(x) = x$ *for all* x*;*
(2) $Sq^n(x) = x \cup x$ *for* $n = |x|$*;*
(3) $Sq^i(x) = 0$ *for* $i > |x|$*;*
(4)
$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)\,.$$

Note that Sq^i increases cohomological degree by i. By the first three items, the only "new" operations are the $Sq^i(x)$ for $0 < i < n$. The fourth item is the *Cartan formula* from [**Car50**].

PROOF OF THEOREM. To define the $Sq^i(x)$ for $x \in \tilde{H}^n(X;\mathbb{F}_2)$ represented by the homotopy class of a map $f \colon X \to K(\mathbb{F}_2, n)$, we will construct maps

$$\mathbb{R}P^\infty_+ \wedge X \xrightarrow{1\wedge f} \mathbb{R}P^\infty_+ \wedge K_n \xrightarrow{1\wedge\Delta} S^\infty_+ \wedge_{C_2} K_n \wedge K_n \xrightarrow{\theta} K_{2n}\,.$$

Here $\mathbb{R}P^\infty = S^\infty/C_2$ and we write $K_n = K(\mathbb{F}_2, n)$ and $K_{2n} = K(\mathbb{F}_2, 2n)$ to simplify the notation. ((ETC: Maybe $H_n = K(\mathbb{F}_2, n)$ is better, since this is the n-th space in the Eilenberg–MacLane spectrum $H = H\mathbb{F}_2$.)) The homotopy class of the composite represents an element

$$y = [\theta(1\wedge\Delta)(1\wedge f)] \in \tilde{H}^{2n}(\mathbb{R}P^\infty_+ \wedge X;\mathbb{F}_2)\,.$$

By the Künneth theorem,

$$\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge X;\mathbb{F}_2) \cong H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \otimes \tilde{H}^*(X;\mathbb{F}_2)$$

where $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[a]$ with $|a| = 1$. Hence we can write

$$y = \sum_{i=0}^n a^{n-i} \otimes Sq^i(x)$$

for a unique sequence of elements $Sq^i(x) \in \tilde{H}^{n+i}(X;\mathbb{F}_2)$. This defines the (potentially) nonzero $Sq^i(x)$.

To explain θ, we must first introduce the *quadratic construction*

$$D_2(X) = S^\infty_+ \wedge_{C_2} X \wedge X\,,$$

also denoted $Q(X)$ [**Kah69**, §1] and ΛX [**Hat02**, p. 503]. Our notation (including the structure maps δ and β that appear below) conforms with that used for *extended powers* in [**BMMS86**, §I.2]. Here $C_2 = \{e, t\}$ is the group of order 2, with unit element e. It acts freely from the right on the unit sphere $S^\infty = S(\mathbb{R}^\infty)$, with $v \cdot t = -v$ for each unit vector v, and the orbit space is $S^\infty/C_2 = \mathbb{R}P^\infty$. For a based CW complex X the group C_2 acts from the left on the smash product

$$X \wedge X = \frac{X \times X}{X \vee X}$$

by the twist isomorphism $\tau\colon X \wedge X \longrightarrow X \wedge X$, with $t \cdot (x \wedge y) = y \wedge x$. The quadratic construction is the *balanced product*

$$S^\infty_+ \wedge_{C_2} X \wedge X = (S^\infty_+ \wedge X \wedge X)/(\sim)$$

where $\sim$ denotes the relation

$$(-v, x \wedge y) = (v \cdot t, x \wedge y) \sim (v, t \cdot (x \wedge y)) = (v, y \wedge x)$$

for $v \in S^\infty$, $x \in X$ and $y \in Y$. Let $S^i = S(\mathbb{R}^{i+1}) \subset S^\infty$. The action of C_2 respects this subspace, so we can filter $D_2(X)$ by the subspaces

$$\dots \subset D_2^{i-1}(X) \subset D_2^i(X) = S^i_+ \wedge_{C_2} X \wedge X \subset \dots \subset D_2(X)\,.$$

There are homeomorphisms $X \wedge X \cong S^0_+ \wedge_{C_2} X \wedge X = D_2^0(X)$ and

$$I_+ \wedge X \wedge X/(\sim) \cong S^1_+ \wedge_{C_2} X \wedge X = D_2^1(X)$$

where $(0, x \wedge y) \sim (1, y \wedge x)$ at the left hand side. Hence there is a long exact cohomology sequence

$$\dots \to \tilde H^{*-1}(X \wedge X; \mathbb{F}_2) \xrightarrow{\delta} \tilde H^*(D_2^1(X); \mathbb{F}_2) \longrightarrow \tilde H^*(X \wedge X; \mathbb{F}_2) \xrightarrow{1-\tau} H^*(X \wedge X; \mathbb{F}_2) \to \dots\,.$$

We now specialize to the case $X = K_n = K(\mathbb{F}_2, n)$ and degree $* = 2n$. By the Künneth theorem, $K_n \wedge K_n$ is $(2n-1)$-connected, and

$$\tilde H^{2n}(K_n \wedge K_n; \mathbb{F}_2) = \mathbb{F}_2\{u_n \wedge u_n\}$$

where $u_n \in \tilde H^n(K_n; \mathbb{F}_2)$ is the universal class. Furthermore,

$$(1-\tau)(u_n \wedge u_n) = u_n \wedge u_n - (-1)^{n^2} u_n \wedge u_n = 0\,,$$

since we are working with $\mathbb{F}_2$-coefficients, so $\theta_0 = u_n \wedge u_n$ admits a unique extension $\theta_1 \in \tilde H^{2n}(D_2^1(K_n); \mathbb{F}_2)$. Moreover, $D_2^1(K_n) \to D_2(K_n)$ is $(2n+1)$-connected (it amounts to adding cells of dimension $\ge 2n+2$), so the restriction homomorphism

$$\tilde H^{2n}(D_2(K_n); \mathbb{F}_2) \xrightarrow{\cong} \tilde H^{2n}(D_2^1(K_n); \mathbb{F}_2)$$

is an isomorphism, and θ_1 admits a unique extension $\theta \in \tilde H^{2n}(D_2(K_n); \mathbb{F}_2)$. It is represented by a map

$$\theta\colon D_2(K_n) = S^\infty_+ \wedge_{C_2} K_n \wedge K_n \longrightarrow K_{2n}$$

whose restriction

$$\theta_0\colon D_2^0(K_n) \cong K_n \wedge K_n \longrightarrow K_{2n}$$

represents the smash (= reduced cross) product $\wedge\colon \tilde H^n(X; \mathbb{F}_2) \otimes \tilde H^n(Y; \mathbb{F}_2) \to \tilde H^{2n}(X \wedge Y; \mathbb{F}_2)$.

The (reduced) diagonal map $\Delta\colon X \to X \wedge X$ satisfies $t \cdot \Delta(x) = \Delta(x) = x \wedge x$, hence induces a map

$$1 \wedge \Delta\colon \mathbb{R}P^\infty_+ \wedge X \longrightarrow S^\infty_+ \wedge_{C_2} X \wedge X = D_2(X)$$

sending $([v], x)$ to $[v \wedge x \wedge x]$, for $v \in S^\infty$ and $x \in X$. Its restriction to $v \in S^0 \subset S^\infty$ is identified with the diagonal map

$$\Delta \colon X \cong \mathbb{R}P^0_+ \wedge X \longrightarrow D^0_2(X) \cong X \wedge X\,.$$

Given a class $x \in \tilde{H}^n(X; \mathbb{F}_2)$, represented by a map $f \colon X \to K_n$, we can form the following commutative diagram.

$$\begin{tikzcd}
X \arrow[rrr, "\Delta"] \arrow[dr] \arrow[dd, "f"'] & & & X \wedge X \arrow[dr] \arrow[dd, "f \wedge f"] & \\
& \mathbb{R}P^\infty_+ \wedge X \arrow[rrr, "1 \wedge \Delta"] \arrow[dd, "1 \wedge f"] & & & S^\infty_+ \wedge_{C_2} X \wedge X \arrow[dd, "1 \wedge f \wedge f"] \\
K_n \arrow[rrr, "\Delta"] \arrow[dr] & & & K_n \wedge K_n \arrow[dr] \arrow[dd, "\theta_0"] & \\
& \mathbb{R}P^\infty_+ \wedge K_n \arrow[rrr, "1 \wedge \Delta"] & & & S^\infty_+ \wedge_{C_2} K_n \wedge K_n \arrow[dd, "\theta"] \\
& & & K_{2n} \arrow[dr, "="] & \\
& & & & K_{2n}
\end{tikzcd}$$

The composite $\theta(1 \wedge \Delta)(1 \wedge f) = \theta(1 \wedge f \wedge f)(1 \wedge \Delta) \colon \mathbb{R}P^\infty_+ \wedge X \to K_{2n}$ defines the cohomology class we write as

$$\sum_{i=0}^{n} a^{n-i} \otimes Sq^i(x) \in H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes \tilde{H}^*(X; \mathbb{F}_2) \cong \tilde{H}^*(\mathbb{R}P^\infty_+ \wedge X; \mathbb{F}_2)\,.$$

Its restriction to $\tilde{H}^*(X; \mathbb{F}_2)$, corresponding to $i = n$, is the pullback along Δ of $x \wedge x \in \tilde{H}^{2n}(X \wedge X; \mathbb{F}_2)$, represented by $\theta_0(f \wedge f)$, which equals $x^2 = x \cup x \in \tilde{H}^{2n}(X; \mathbb{F}_2)$. This defines the natural transformations Sq^i, satisfying conditions (2) and (3) in the theorem.

(In the universal case, $(1 \wedge \Delta)^* \theta^* u_{2n} = \sum_i a^{n-i} \otimes Sq^i(u_n)$ in $\tilde{H}^{2n}(\mathbb{R}P^\infty_+ \wedge K_n; \mathbb{F}_2)$.)

The Cartan formula (4) can be deduced from the following diagram.

$$\begin{tikzcd}
D_2(K_n \wedge K_m) \arrow[r, "D_2(\wedge)"] \arrow[d, "\delta"'] & D_2(K_{n+m}) \arrow[dd, "\theta"] \\
D_2(K_n) \wedge D_2(K_m) \arrow[d, "\theta \wedge \theta"'] & \\
K_{2n} \wedge K_{2m} \arrow[r, "\wedge"] & K_{2(n+m)}
\end{tikzcd}$$

It commutes up to homotopy, as can be verified by comparing the two composites after restriction to $(K_n \wedge K_m) \wedge (K_n \wedge K_n) = D^0_2(K_n \wedge K_m)$. If $f \colon X \to K_n$ and

$g\colon Y \to K_m$ represent $x \in \tilde{H}^n(X;\mathbb{F}_2)$ and $y \in \tilde{H}^m(Y;\mathbb{F}_2)$, respectively, then the composite

$$\mathbb{R}P^\infty_+ \wedge X \wedge Y \xrightarrow{1\wedge\Delta} D_2(X\wedge Y) \xrightarrow{D_2(f\wedge g)} D_2(K_n \wedge K_m) \longrightarrow K_{2(n+m)}$$

can be expanded in two ways, to yield the identity

$$\sum_{k=0}^{n+m} a^{n+m-k} \otimes Sq^k(x\wedge y) = \sum_{i=0}^{n}\sum_{j=0}^{m} a^{n-i}\cup a^{m-j} \otimes Sq^i(x)\cup Sq^j(y)\,.$$

Comparing terms gives the Cartan formula.

By naturality, the Cartan formula also holds for relative and unreduced cohomology, as well as for the external smash product and cross product pairings. For example,

$$Sq^k(x\wedge y) = \sum_{i+j=k} Sq^i(x)\wedge Sq^j(y)$$

in $\tilde{H}^*(X\wedge Y;\mathbb{F}_2)$.

Property (1), that Sq^0 equals the identity operation, is not obvious. The statement for $n=1$ follows by naturality from the case $x = u_1 \in H^1(K_1;\mathbb{F}_2)$, which is an assertion about the composite

$$\mathbb{R}P^\infty_+ \wedge K_1 \xrightarrow{1\wedge\Delta} S^\infty_+ \wedge_{C_2} K_1 \wedge K_1 \xrightarrow{\theta} K_2\,.$$

By naturality with respect to $g_1\colon S^1 \to K_1$, it suffices to check that

$$\mathbb{R}P^1_+ \wedge S^1 \xrightarrow{1\wedge\Delta} S^1_+ \wedge_{C_2} S^1 \wedge S^1$$

induces the nonzero homomorphism (an isomorphism) in $H^2(-;\mathbb{F}_2)$, which can be seen from an explicit cellular model. See [**Hat02**, p. 505].

This shows that $Sq^0(g_1) = g_1$ in $\tilde{H}^*(S^1;\mathbb{F}_2)$. When combined with the Cartan formula for $\Sigma X = S^1 \wedge X$, it follows that each reduced squaring operation commutes with the suspension isomorphisms

$$\sigma\colon \tilde{H}^n(X;\mathbb{F}_2) \xrightarrow{\cong} \tilde{H}^{n+1}(\Sigma X;\mathbb{F}_2)$$

given by $\sigma(x) = g_1 \wedge x$, since $Sq^i(g_1\wedge x) = Sq^0(g_1)\wedge Sq^i(x) = g_1 \wedge Sq^i(x)$. It then follows, by naturality with respect to $X\cup CA \to \Sigma A$, that each Sq^i commutes with the connecting homomorphisms

$$\delta\colon H^n(A;\mathbb{F}_2) \longrightarrow H^{n+1}(X,A;\mathbb{F}_2)\,.$$

It also follows that each Sq^i is additive, i.e., is an $\mathbb{F}_2$-linear homomorphism.

Finally, to verify that $Sq^0(x) = x$ for $x \in H^n(X;\mathbb{F}_2)$ it suffices, by naturality, to check the case $x = u_n \in H^n(K_n;\mathbb{F}_2)$, and since $g_n\colon S^n \to K_n$ induces an isomorphism $g_n^*\colon H^n(K_n;\mathbb{F}_2) \to H^n(S^n;\mathbb{F}_2)$, it suffices to treat the case $x = g_n \in H^n(S^n;\mathbb{F}_2)$. This now follows from the case $x = g_1 \in H^1(S^1;\mathbb{F}_2)$, by commutation of Sq^0 with the suspension isomorphism. □

The operation Sq^1 had also been previously considered.

Definition 7.2.2. Let

$$0 \to G' \longrightarrow G \longrightarrow G'' \to 0$$

be a short exact sequence of abelian groups. The induced short exact sequences

$$0 \to C_*(X; G') \longrightarrow C_*(X; G) \longrightarrow C_*(X; G'') \to 0$$
$$0 \to C^*(X; G') \longrightarrow C^*(X; G) \longrightarrow C^*(X; G'') \to 0$$

of chain and cochain complexes induce long exact sequences in homology and cohomology, with connecting homomorphisms

$$\beta\colon H_n(X; G'') \longrightarrow H_{n-1}(X; G')$$
$$\beta\colon H^n(X; G'') \longrightarrow H^{n+1}(X; G')$$

called the *homology and cohomology Bockstein homomorphisms* associated to the extension $0 \to G' \to G \to G'' \to 0$. These are natural in X, so the cohomology Bockstein is a cohomology operation of type $(G'', n; G', n+1)$.

LEMMA 7.2.3. *Let $0 \to G' \to G_1 \to G'' \to 0$ and $0 \to G'' \to G_2 \to G''' \to 0$ be extensions of abelian groups. Then the composite Bockstein homomorphisms*

$$H_n(X; G''') \xrightarrow{\beta_2} H_{n-1}(X; G'') \xrightarrow{\beta_1} H_{n-2}(X; G')$$
$$H^n(X; G''') \xrightarrow{\beta_2} H^{n+1}(X; G'') \xrightarrow{\beta_1} H^{n+2}(X; G')$$

are both zero.

PROOF. There exists a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G' \arrow[r, "="] \arrow[d] & G' \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G''' \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & G'' \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G''' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. (In this situation, we say that the Yoneda composite of $G' \to G_1 \to G''$ and $G'' \to G_2 \to G'''$ is trivial. Compare [**ML63**, Lem. XII.5.3].) ((ETC: Return to Yoneda composition in Ext later.)) Then the homology Bockstein β_2 for $G'' \to G_2 \to G'''$ factors as

$$H_n(X; G''') \xrightarrow{\beta} H_{n-1}(X; G_1) \xrightarrow{j} H_{n-1}(X; G'') ,$$

and the composite

$$H_{n-1}(X; G_1) \xrightarrow{j} H_{n-1}(X; G'') \xrightarrow{\beta_1} H_{n-2}(X; G')$$

is zero. The cohomology proof is essentially the same. □

PROPOSITION 7.2.4. *$Sq^1 = \beta\colon H^n(X; \mathbb{F}_2) \to H^{n+1}(X; \mathbb{F}_2)$ equals the cohomology Bockstein for the extension $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. In particular, $Sq^1 Sq^1 = \beta\beta = 0$.*

PROOF. By naturality it suffices that $Sq^1(u_n) = \beta(u_n) \in H^{n+1}(K_n;\mathbb{F}_2)$ for $u_n \in H^n(K_n;\mathbb{F}_2)$. Consider the Moore space $M_n = S^n \cup_2 e^{n+1}$, which admits an $(n+1)$-connected map $f\colon M_n \to K_n$. Since $f^*\colon H^{n+1}(K_n;\mathbb{F}_2) \to H^{n+1}(M_n;\mathbb{F}_2)$ is an isomorphism, it suffices to check that $Sq^1(a) = \beta(a)$ for $a = [f]$. Since Sq^1 and β both commute with suspension isomorphisms, it suffices to verify this when $n = 1$ and $M_1 = S^1 \cup_2 e^2 \cong \mathbb{R}P^2$. Here $Sq^1(a) = a^2$ generates $H^2(\mathbb{R}P^2;\mathbb{F}_2)$, and a direct calculation with $\tilde{H}^*(\mathbb{R}P^2;\mathbb{Z}/4)$ shows that $\beta(a) = a^2$.

The composite $\beta\beta$ is trivial, by the previous lemma with $G' = G'' = G''' = \mathbb{Z}/2$, $G_1 = G_2 = \mathbb{Z}/4$ and $G = \mathbb{Z}/8$. □

LEMMA 7.2.5. *The Steenrod squares on the powers of any $a \in H^1(X;\mathbb{F}_2)$ are given by*

$$Sq^i(a^j) = \binom{j}{i} a^{i+j}\,.$$

The binomial coefficient can be read mod 2, since the expression takes place in $H^*(X;\mathbb{F}_2)$. Hence Lucas' theorem (Lemma 7.3.3 below) is helpful.

PROOF. Let the inhomogeneous sum $Sq(x) = \sum_i Sq^i(x) \in \bigoplus_n H^n(X;\mathbb{F}_2)$ denote the *total squaring operation* on x. The Cartan formula then reads

$$Sq(xy) = Sq(x)Sq(y)$$

and $Sq(a) = a + a^2 = a(1+a)$ in $H^*(X;\mathbb{F}_2)$. Hence

$$Sq(a^j) = Sq(a)^j = (a+a^2)^j = a^j(1+a)^j$$

so that $Sq^i(a^j) = a^j \cdot \binom{j}{i} a^i = \binom{j}{i} a^{i+j}$ for $0 \le i \le j$, and $Sq^i(a^j) = 0$ otherwise. □

Here is the analogue of Theorem 7.2.1 for odd primes p.

THEOREM 7.2.6 ([**Ste62**, §VI.1]). *Let p be an odd prime. There are natural transformations*

$$P^i\colon \tilde{H}^n(X;\mathbb{F}_p) \longrightarrow \tilde{H}^{n+2i(p-1)}(X;\mathbb{F}_p)$$

for all $i \ge 0$ and $n \ge 0$. These satisfy

(1) $P^0(x) = x$ *for all* x;
(2) $P^i(x) = x^p$ *for* $|x| = 2i$;
(3) $P^i(x) = 0$ *for* $|x| < 2i$;
(4)

$$P^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup P^j(y)\,.$$

The fourth item is the Cartan formula. The "new" operations are P^i for $0 < i < |x|/2$. We call the Bockstein operation

$$\beta\colon H^n(X;\mathbb{F}_p) \longrightarrow H^{n+1}(X;\mathbb{F}_p)$$

associated to the extension $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p$ the *mod p Bockstein*. Note that P^i increases cohomological degree by $2i(p-1)$, while β increases it by 1.

LEMMA 7.2.7. $\beta\beta = 0$.

PROOF. This follows from Lemma 7.2.3, using the diagram with $G' = G'' = G''' = \mathbb{Z}/p$, $G_1 = G_2 = \mathbb{Z}/p^2$ and $G = \mathbb{Z}/p^3$. □

LEMMA 7.2.8.

$$\beta(x \cup y) = \beta(x) \cup y + (-1)^{|x|} x \cup \beta(y) .$$

PROOF. This follows from the Leibniz rule for the coboundary in $C^*(X; \mathbb{Z}/p^2)$ acting on a cochain cup product. □

7.3. The Adem relations

Let $Sq^i Sq^j$ denote the composite operation

$$\tilde{H}^n(X; \mathbb{F}_2) \xrightarrow{Sq^j} \tilde{H}^{n+j}(X; \mathbb{F}_2) \xrightarrow{Sq^i} \tilde{H}^{n+i+j}(X; \mathbb{F}_2) .$$

These satisfy the *Adem relations.*

THEOREM 7.3.1 ([**Ade52**], [**Ste62**, §I.1]). *The identity*

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

holds, for $i < 2j$.

Again, the binomial coefficients can be read mod 2. The summation limits can be omitted, given the convention that $\binom{n}{k} = 0$ for $k < 0$ and $k > n$. The Adem relations in degrees $* \le 11$ are listed in Figure 7.1. In particular,

$$Sq^1 Sq^{2j} = Sq^{2j+1} \quad , \quad Sq^1 Sq^{2j+1} = 0 \quad \text{and} \quad Sq^{2j+1} Sq^{j+1} = 0$$

for all $j \ge 0$.

SKETCH PROOF. We consider the universal case of $Sq^i Sq^j(x)$ for $x = u_n$ in $H^n(X; \mathbb{F}_2)$ with $X = K_n$, and apply the quadratic construction twice.

$$\begin{tikzcd}
\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \wedge K_n \arrow[r, "1\wedge 1\wedge\Delta"] \arrow[dd] & \mathbb{R}P^\infty_+ \wedge D_2(K_n) \arrow[r, "1\wedge\theta"] \arrow[d, "1\wedge\Delta"] & \mathbb{R}P^\infty_+ \wedge K_{2n} \arrow[d, "1\wedge\Delta"] \\
 & D_2(D_2(K_n)) \arrow[r, "D_2(\theta)"] \arrow[d, "\beta"] & D_2(K_{2n}) \arrow[d, "\theta"] \\
B\Sigma_{4+} \wedge K_n \arrow[r, "1\wedge\Delta"] & D_4(K_n) \arrow[r, "\theta'"] & K_{4n}
\end{tikzcd}$$

Here

$$D_2(D_2(X)) = S^\infty_+ \wedge_{C_2} (S^\infty_+ \wedge_{C_2} X^{\wedge 2})^{\wedge 2} \cong (S^\infty \times (S^\infty)^2)_+ \wedge_{C_2 \ltimes (C_2)^2} X^{\wedge 4} ,$$

where $C_2 \ltimes (C_2)^2$ denotes the semi-direct product. In the upper part of the diagram,

$$(1 \wedge \Delta)^* \theta^*(u_{4n}) = \sum_k a^{2n-k} \otimes Sq^k(u_{2n})$$

$$
\begin{array}{ll}
Sq^1Sq^1 = 0 & Sq^1Sq^8 = Sq^9 \\
Sq^1Sq^2 = Sq^3 & Sq^2Sq^7 = Sq^9 + Sq^8Sq^1 \\
Sq^1Sq^3 = 0 & Sq^3Sq^6 = 0 \\
Sq^2Sq^2 = Sq^3Sq^1 & Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 \\
Sq^1Sq^4 = Sq^5 & Sq^5Sq^4 = Sq^7Sq^2 \\
Sq^2Sq^3 = Sq^5 + Sq^4Sq^1 & Sq^1Sq^9 = 0 \\
Sq^3Sq^2 = 0 & Sq^2Sq^8 = Sq^{10} + Sq^9Sq^1 \\
Sq^1Sq^5 = 0 & Sq^3Sq^7 = Sq^9Sq^1 \\
Sq^2Sq^4 = Sq^6 + Sq^5Sq^1 & Sq^4Sq^6 = Sq^{10} + Sq^8Sq^2 \\
Sq^3Sq^3 = Sq^5Sq^1 & Sq^5Sq^5 = Sq^9Sq^1 \\
Sq^1Sq^6 = Sq^7 & Sq^6Sq^4 = Sq^7Sq^3 \\
Sq^2Sq^5 = Sq^6Sq^1 & Sq^1Sq^{10} = Sq^{11} \\
Sq^3Sq^4 = Sq^7 & Sq^2Sq^9 = Sq^{10}Sq^1 \\
Sq^4Sq^3 = Sq^5Sq^2 & Sq^3Sq^8 = Sq^{11} \\
Sq^1Sq^7 = 0 & Sq^4Sq^7 = Sq^{11} + Sq^9Sq^2 \\
Sq^2Sq^6 = Sq^7Sq^1 & Sq^5Sq^6 = Sq^{11} + Sq^9Sq^2 \\
Sq^3Sq^5 = Sq^7Sq^1 & Sq^6Sq^5 = Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2 & Sq^7Sq^4 = 0 \\
Sq^5Sq^3 = 0 &
\end{array}
$$

FIGURE 7.1. The Adem relations in degrees $* \leq 11$

in $\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge K_{2n}; \mathbb{F}_2) \cong \mathbb{F}_2[a] \otimes \tilde{H}^*(K; \mathbb{F}_2)$, which maps to

$$
\begin{aligned}
z = (1 \wedge 1 \wedge \Delta)^*(1 \wedge \theta)^*(\sum_k a^{2n-k} \otimes Sq^k(u_{2n}))& \\
&= \sum_k a^{2n-k} \otimes (1 \wedge \Delta)^* \theta^*(Sq^k(u_{2n})) \\
&= \sum_k a^{2n-k} \otimes Sq^k((1 \wedge \Delta)^* \theta^*(u_{2n})) \\
&= \sum_k a^{2n-k} \otimes Sq^k(\sum_\ell b^{n-\ell} \otimes Sq^\ell(u_n)) \\
&= \sum_{i,j} a^{2n-i-j} \otimes \sum_\ell Sq^i(b^{n-\ell}) \otimes Sq^j(Sq^\ell(u_n)) \\
&= \sum_{i,j,\ell} \binom{n-\ell}{i} a^{2n-i-j} \otimes b^{n+i-\ell} \otimes Sq^j Sq^\ell(u_n)
\end{aligned}
$$

in $\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \wedge K_n; \mathbb{F}_2) \cong \mathbb{F}_2[a] \otimes \mathbb{F}_2[b] \otimes \tilde{H}^*(K_n; \mathbb{F}_2)$.

We claim that z is invariant under the twist map $\tau \wedge 1$ that interchanges the two copies of $\mathbb{R}P^\infty_+$. This implies an identity among the composite operations $Sq^j Sq^\ell(u_n)$, for varying j and ℓ, from which the Adem relations can be extracted with some effort. See [**Ste62**, p. 119] or [**Hat02**, p. 508].

To prove the claim, we use the extended power

$$D_4(X) = E\Sigma_{4+} \wedge_{\Sigma_4} (X \wedge X \wedge X \wedge X)\,,$$

where Σ_4 denotes the symmetric group on four letters and $p\colon E\Sigma_4 \to B\Sigma_4$ is a universal principal Σ_4-bundle. The group Σ_4 acts freely from the right on $E\Sigma_4$, and acts from the left on $X^{\wedge 4} = X\wedge X\wedge X\wedge X$ by permuting the factors. When $X = K_n$ the map $\theta'_0\colon K_n^{\wedge 4} \to K_{4n}$ representing the fourfold smash product extends, uniquely up to homotopy, to a map $\theta'\colon D_4(K_n) \to K_{4n}$. An inclusion $G = C_2 \ltimes (C_2 \times C_2) \subset \Sigma_4$ induces $\beta\colon D_2(D_2(X)) \to D_4(X)$, so that $\theta'\beta \simeq \theta D_2(\theta)$. The diagonal map $\Delta\colon K_n \to K_n^{\wedge 4}$ is Σ_4-equivariant, and leads to the map $1 \wedge \Delta\colon B\Sigma_{4+} \wedge K_n \to D_4(K_n)$. The inclusion $1 \times \Delta\colon H = C_2 \times C_2 \subset C_2 \ltimes (C_2 \times C_2) = G \subset \Sigma_4$ now induces $\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \cong B(C_2 \times C_2)_+ \to B\Sigma_{4+}$ and the left hand vertical map, making the whole diagram commute up to homotopy. Hence z can also be calculated as the pullback of $(1\wedge\Delta)^*(\theta')^*(u_{4n}) \in H^*(B\Sigma_4;\mathbb{F}_2)\otimes\tilde{H}^*(K_n;\mathbb{F}_2)$. There is an inner automorphism of Σ_4 that maps $H = C_2\times C_2$ to itself by the twist map τ. Since inner automorphisms induce the identity map on group cohomology, i.e., on $H^*(B\Sigma_4;\mathbb{F}_2)$, the claim that z is invariant under τ follows. □

The reduced power operations P^i and the mod p Bockstein β satisfy the following Adem relations.

THEOREM 7.3.2 ([**Ade53**], [**Ste62**, §VI.1]). *Let p be an odd prime. If $i < pj$ then*

$$P^i P^j = \sum_{k=0}^{[i/p]} (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk} P^{i+j-k} P^k\,.$$

If $i \le pj$ then

$$P^i \beta P^j = \sum_{k=0}^{[i/p]} (-1)^{i+k} \binom{(p-1)(j-k)}{i-pk} \beta P^{i+j-k} P^k$$
$$- \sum_{k=0}^{[(i-1)/p]} (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk-1} P^{i+j-k} \beta P^k\,.$$

In each case the summation limits can be omitted, given the convention that $\binom{n}{k} = 0$ for $k < 0$ and $k > n$. The first few odd-primary Adem relations (for $j = 1$) are

$$P^i P^1 = (-1)^i \binom{p-2}{i} P^{i+1}$$

for $i < p$, which implies $(P^1)^p = 0$,

$$P^i \beta P^1 = (-1)^i \binom{p-1}{i} \beta P^{i+1} - (-1)^i \binom{p-2}{i-1} P^{i+1}\beta$$

for $i < p$, and

$$P^p \beta P^1 = \beta P^p P^1$$

(for $i = p$).

By Lucas' theorem, binomial coefficents mod p can be conveniently calculated from base p expansions. See [**Ste62**, Lem. 2.6] or [**Hat02**, Lem. 3C.6] for a proof.

LEMMA 7.3.3 (Lucas). *Let p be a prime, and write $n = \sum_i n_i p^i$ and $k = \sum_i k_i p^i$ with $n_i, k_i \in \{0, 1, \dots, p-1\}$. Then*

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \mod p\,.$$

Here $\binom{n_i}{k_i} = 0$ if $k_i > n_i$. For $p = 2$, this reduces the calcuation of $\binom{n}{k}$ to the cases $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and $\binom{0}{1} = 0$. Hence $\binom{n}{k} \equiv 0 \mod 2$ if and only if there is an i such that $n_i = 0$ and $k_i = 1$, i.e., there is a 1 below a 0 when n and k are written in base 2.

7.4. The Steenrod algebra

DEFINITION 7.4.1. The *mod 2 Steenrod algebra* is the (unital and associative) graded $\mathbb{F}_2$-algebra $A = \mathscr{A}(2)$ generated by the symbols Sq^i for $i \ge 0$, subject to the Adem relations

$$Sq^i Sq^j = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

for $i < 2j$, and $Sq^0 = 1$.

For each odd prime p, the *mod p Steenrod algebra* is the $\mathbb{F}_p$-algebra $A = \mathscr{A}(p)$ generated by the symbols P^i for $i \ge 0$ and β, subject to the Adem relations, $P^0 = 1$ and $\beta\beta = 0$.

LEMMA 7.4.2. *Let p be any prime. For each space X the mod p cohomology $H^*(X; \mathbb{F}_p)$ is naturally a graded left A-module, where $A = \mathscr{A}(p)$.*

PROOF. For $p = 2$, each symbol Sq^i in A acts on $H^*(X; \mathbb{F}_2)$ as the Steenrod operation of the same name. This defines a left action by A, since the Steenrod operations satisfy the Adem relations and Sq^0 acts as the identity.

The proof for odd p is essentially the same. □

DEFINITION 7.4.3. Let $I = (i_1, i_2, \dots, i_\ell)$ be any finite sequence of positive integers. We call $\ell = \ell(I)$ the *length* of I, write

$$|I| = \sum_{s=1}^{\ell} i_s$$

for the *degree* of I, and say that I is *admissible* if

$$i_s \ge 2i_{s+1}$$

for each $1 \le s < \ell$. Let

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdot \ldots \cdot Sq^{i_\ell}$$

denote the product in A, as well as the corresponding composite of Steenrod operations. The empty sequence $I = ()$ is admissible of length 0, and $Sq^{()} = 1$ equals the identity.

We also refer to $\ell(I)$ and $|I|$ as the length and (cohomological) degree of Sq^I, respectively, and say that Sq^I is admissible when I is admissible.

(0) 1
(1) Sq^1
(2) Sq^2
(3) Sq^3, Sq^2Sq^1
(4) Sq^4, Sq^3Sq^1
(5) Sq^5, Sq^4Sq^1
(6) Sq^6, Sq^5Sq^1, Sq^4Sq^2
(7) Sq^7, Sq^6Sq^1, Sq^5Sq^2, $Sq^4Sq^2Sq^1$
(8) Sq^8, Sq^7Sq^1, Sq^6Sq^2, $Sq^5Sq^2Sq^1$
(9) Sq^9, Sq^8Sq^1, Sq^7Sq^2, $Sq^6Sq^2Sq^1$, Sq^6Sq^3
(10) Sq^{10}, Sq^9Sq^1, Sq^8Sq^2, $Sq^7Sq^2Sq^1$, Sq^7Sq^3, $Sq^6Sq^3Sq^1$
(11) Sq^{11}, $Sq^{10}Sq^1$, Sq^9Sq^2, $Sq^8Sq^2Sq^1$, Sq^8Sq^3, $Sq^7Sq^3Sq^1$

FIGURE 7.2. The admissible monomials in degrees $* \le 11$

THEOREM 7.4.4 ([**Ste62**, Thm. I.3.1]). *The admissible monomials Sq^I form a vector space basis for $A = \mathscr{A}(2)$.*

SKETCH PROOF. The monomials Sq^I clearly generate A. If I is not admissible, meaning that $i_s < 2i_{s+1}$ for some s, then we can rewrite Sq^I by means of the Adem relation for $Sq^{i_s}Sq^{i_{s+1}}$. This replaces I with sequences of lower *moment* $\sum_{s=1}^{\ell} si_s$, so the process eventually halts. This proves that the admissible monomials generate A.

To prove that the admissible monomials form a basis, recall the action

$$Sq^i(a^j) = \binom{j}{i} a^{i+j}$$

of the Steenrod operations on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a]$. By the Cartan formula, this determines the action of Sq^I on

$$H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a_1, \dots, a_n] ,$$

where the product contains n copies of $\mathbb{R}P^\infty$. A proof by induction on n shows that the elements

$$Sq^I(a_1 \cdot \dots \cdot a_n) \in \mathbb{F}_2[a_1, \dots, a_n]$$

for I admissible of degree $|I| \le n$ are linearly independent. Since n can be chosen to be arbitrarily large, this proves that the admissible Sq^I are linearly independent. □

The basis of admissible monomials for A in degrees $* \le 11$ is listed in Figure 7.2.

DEFINITION 7.4.5. Let the *augmentation* $\epsilon \colon A \to \mathbb{F}_2$ be the graded ring homomorphism given by $\epsilon(1) = 1$. Its kernel is the *augmentation ideal*

$$I(A) = \ker(\epsilon)$$

which equals the positive degree part of A. The classes in the image $I(A)^2 \subset I(A)$ of the pairing

$$I(A) \otimes I(A) \subset A \otimes A \overset{\cdot}{\longrightarrow} A$$

are said to be *decomposable*, and the quotient

$$Q(A) = I(A)/I(A)^2$$

is the graded vector space of *(algebra) indecomposables* of A.

THEOREM 7.4.6 ([**Ade52**, Thm. 1.5], [**Ste62**, Thm. 4.3]). *The operation Sq^k is decomposable if and only if k is not a power of 2. Hence*

$$Sq^1, Sq^2, Sq^4, \dots, Sq^{2^i}, \dots$$

generate A as an algebra, and

$$Q(A) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}\,.$$

PROOF. If k is not a power of 2, we can write $k = i + 2^\ell$ with $0 < i < 2^\ell$. The Adem relation

$$Sq^i Sq^{2^\ell} = \binom{2^\ell - 1}{i} Sq^{i+2^\ell} + \text{(decomposable terms)}$$

and the case $\binom{2^\ell-1}{i} = 1$ of Lucas' theorem show that $Sq^k = Sq^{i+2^\ell}$ is decomposable.

Conversely, to see that Sq^k is not decomposable for $k = 2^\ell$, consider the A-module action on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a]$. From

$$Sq^i(a^{2^\ell}) = \begin{cases} a^{2^\ell} & \text{for } i = 0, \\ a^{2^{\ell+1}} & \text{for } i = 2^\ell, \\ 0 & \text{otherwise} \end{cases}$$

we see that any operation of degree $0 < * < 2^\ell$ acts trivially on a^{2^ℓ}. Hence any decomposable operation of degree 2^ℓ must also map a^{2^ℓ} to zero. Since Sq^{2^ℓ} instead maps a^{2^ℓ} to $a^{2^{\ell+1}}$, it cannot be decomposable. □

PROPOSITION 7.4.7. *If X is a space with $H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]$ or $H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{h+1})$ with $h \geq 2$, and $|x| = n$, then n is a power of 2.*

PROOF. Since $H^{n+i}(X; \mathbb{F}_2) = 0$ for $0 < i < n$ the operation $Sq^n(x)$ must be trivial if Sq^n is decomposable. Since $Sq^n(x) = x^2$ is assumed to be nontrivial, it must instead be the case that Sq^n is indecomposable. □

PROPOSITION 7.4.8. *If $f\colon S^{2n-1} \to S^n$ has odd Hopf invariant, then n is a power of 2.*

PROOF. If f has odd Hopf invariant, then its mapping cone $Cf = S^n \cup_f e^{2n}$ is a space with $H^*(Cf; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$ with $|x| = n$. □

Let p be any odd prime.

DEFINITION 7.4.9. Let $I = (\epsilon_1, i_1, \dots, \epsilon_\ell, i_\ell, \epsilon_{\ell+1})$ be a sequence of integers with $\ell \geq 0$, each $\epsilon_s \in \{0, 1\}$, and each $i_s \geq 1$. Let

$$P^I = \beta^{\epsilon_1} P^{i_1} \cdot \dots \cdot \beta^{\epsilon_\ell} P^{i_\ell} \beta^{\epsilon_{\ell+1}}$$

be the product in $A = \mathscr{A}(p)$, as well as the corresponding composite of Bockstein and Steenrod operations. Here $\beta^0 = 1$ and $\beta^1 = \beta$. We say that I is *admissible* if

$$i_s \geq \epsilon_{s+1} + p i_{s+1}$$

for each $1 \leq s < \ell$, and write

$$|I| = \epsilon_1 + 2i_1(p-1) + \dots + \epsilon_\ell + 2i_\ell(p-1) + \epsilon_{\ell+1}$$

to denote the degree of P^I.

THEOREM 7.4.10 ([**Ste62**, Thm. VI.2.5]). *The admissible monomials P^I form a vector space basis for $A = \mathscr{A}(p)$.*

THEOREM 7.4.11 ([**Ste62**, Thm. VI.2.7]). *The operation P^k is decomposable if and only if k is not a power of p. Hence*

$$\beta, P^1, P^p, \dots, P^{p^i}, \dots$$

generate A as an algebra, and

$$Q(A) \cong \mathbb{F}_p\{\beta, P^{p^i} \mid i \geq 0\}.$$

When $p = 3$, the operation P^i has degree $4i$.

PROPOSITION 7.4.12. *If X is a space with $H^*(X;\mathbb{F}_3) \cong \mathbb{F}_3[x]$ or $H^*(X;\mathbb{F}_3) \cong \mathbb{F}_3[x]/(x^{h+1})$ with $h \geq 3$, and $|x| = n$ is a power of 2, then $n \in \{2,4\}$.*

PROOF. If $n = 1$, then $x^2 = -x^2$ by graded commutativity, which contradicts $2x^2 \neq 0$ in $H^*(X;\mathbb{F}_3)$. Hence $n = 2j$ is even, β acts trivially, and $P^j(x) = x^3 \neq 0$. If P^j is decomposable, then $j = 2k$ must be even and $P^k(x) = \pm x^2 \neq 0$, with P^k indecomposable. In the latter case, k is a power of 3 and $4k = n$, so $k = 1$ and $n = 4$. Otherwise, j is a power of 3, so $j = 1$ and $n = 2$. □

THEOREM 7.4.13. *If X is a space of finite type with $H^*(X) \cong \mathbb{Z}[x]$ or $H^*(X) \cong \mathbb{Z}[x]/(x^{h+1})$ with $h \geq 3$, then $n = |x|$ is 2 or 4. If $H^*(X) \cong \mathbb{Z}[x]/(x^3)$ then $n = 2^i \geq 2$ is a power of 2.*

PROOF. The finite type assumption ensures that $H^*(X;\mathbb{F}_p) \cong H^*(X) \otimes \mathbb{F}_p$. Suppose that $H^*(X) \cong \mathbb{Z}[x]$ or $\mathbb{Z}[x]/(x^{h+1})$ with $h \geq 2$. By graded commutativity, $n = |x|$ is even. Proposition 7.4.7 implies that n is a power of 2. If $h \geq 3$, then Proposition 7.4.12 implies that $n \in \{2,4\}$. □

REMARK 7.4.14. The complex and quaternionic projective spaces $\mathbb{C}P^\infty$, $\mathbb{C}P^h$, $\mathbb{H}P^\infty$ and $\mathbb{H}P^h$ show that $\mathbb{Z}[x]$ and $\mathbb{Z}[x]/(x^{h+1})$ with $|x| = n$ are realized as the integral cohomology of spaces for $n \in \{2,4\}$ and any $h \geq 0$. The octonionic projective plane $\mathbb{O}P^2 = S^8 \cup_\sigma e^{16}$ realizes the case $n = 8$ and $h = 2$, but there is no space $\mathbb{O}P^3$ realizing the case $n = 8$ and $h = 3$.

The question remains whether $\mathbb{Z}[x]/(x^3)$ can be realized as the cohomology of a space when $|x| = n = 2^i$ with $i \geq 4$. This is equivalent to the *Hopf invariant one problem*, of deciding whether there exists a map $f\colon S^{2n-1} \to S^n$ with $H^*(Cf) \cong \mathbb{Z}[x]/(x^3)$, which was famously decided in the negative for all $i \geq 4$ by Adams [**Ada60**]. (The case $i = 4$ was excluded earlier by Toda.) We will see later that Adams' result corresponds to nonzero differentials in the Adams spectral sequence for the sphere spectrum.

7.5. Cohomology of Eilenberg–MacLane spaces

Using Steenrod operations, we can resolve the question from Section 6.8 about the mod 2 cohomology Serre spectral sequence for the loop–path fibration of $K(\mathbb{Z}/2,2)$.

LEMMA 7.5.1. *Let p be any prime. The mod p cohomology transgression*

$$d_n^{0,n-1}\colon E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

commutes with the Steenrod operations in $H^*(F;\mathbb{F}_p)$ and $H^*(B;\mathbb{F}_p)$.

PROOF. Recall that $\tau^n = d_n^{0,n-1}$ is given by the additive relation

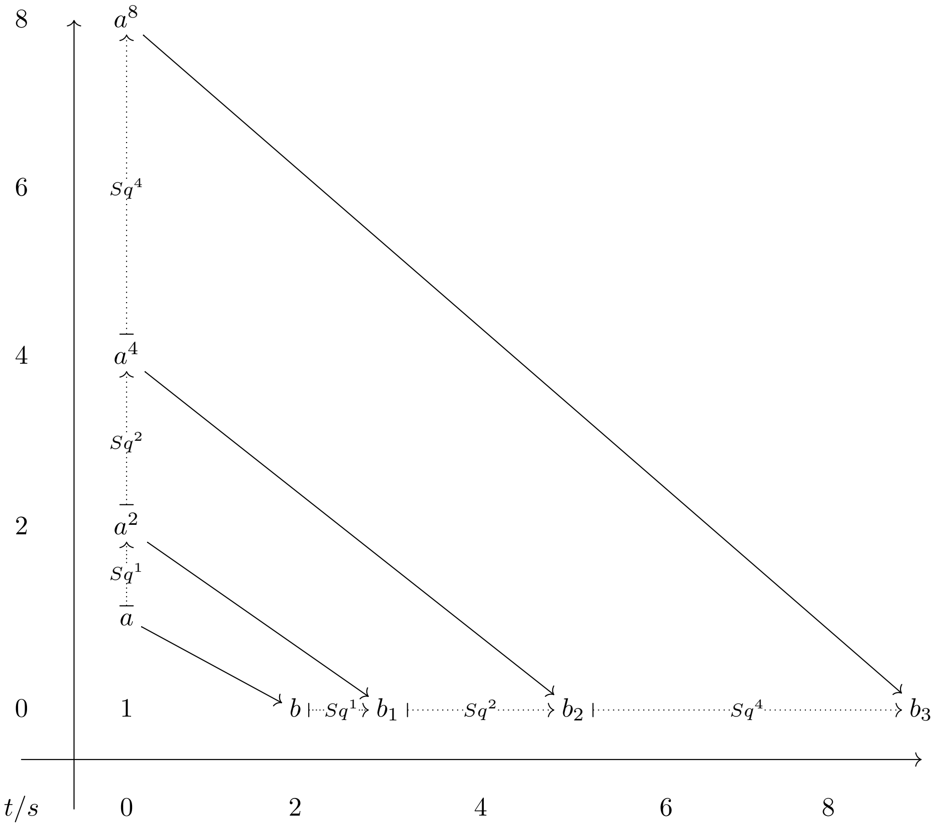
$$(q^*)^{-1}\delta\colon H^{n-1}(F;\mathbb{F}_p) \xrightarrow{\delta} H^n(E,F;\mathbb{F}_p) \xleftarrow{q^*} H^n(B,b_0;\mathbb{F}_p).$$

Any cohomology operation commutes with q^*, and the Steenrod operations commute with δ. Hence if $\tau^n(x) = y$ then $\tau^{n+i}(Sq^i(x)) = Sq^i(y)$ in the $p = 2$ case, since $\delta(Sq^i(x)) = Sq^i(\delta(x)) = Sq^i(q^*(y)) = q^*(Sq^i(y))$, and similarly for odd p. $\square$

Definition 7.5.2. For $p = 2$ and $i \geq 1$ let

$$M_i = (2^{i-1}, 2^{i-2}, \dots, 2, 1) .$$

It is the unique admissible sequence of length i and degree $2^i - 1$.



Proposition 7.5.3.

$$H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[b, b_1, b_2, \dots]$$

with $b = u_2 \in H^2(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ *and* $b_i = Sq^{M_i}(b) \in H^{2^i+1}(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ *for each* $i \geq 1$. *The Serre spectral sequence*

$$\begin{aligned} E_2^{*,*} &\cong H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \otimes H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \\ &\cong \mathbb{F}_2[b, b_1, b_2, \dots] \otimes \mathbb{F}_2[a] \\ &\Longrightarrow H^*(PK(\mathbb{Z}/2, 2); \mathbb{F}_2) = \mathbb{F}_2 \end{aligned}$$

has transgressive differentials $d_2(a) = b$ *and*

$$d_{2^i+1}(a^{2^i}) = b_i$$

for each $i \geq 1$.

Sketch proof. By induction on i, we have $Sq^{M_i}(a) = a^{2^i}$, for each $i \geq 1$. Hence each a^{2^i} is transgressive, with $d_{2^i+1}(a^{2^i}) = d_{2^i+1}(Sq^{M_i}(a)) = Sq^{M_i}(d_2(a)) = Sq^{M_i}(b) = b_i$. It follows by an induction on $u \geq 0$, using a theorem of Borel, that the $\mathbb{F}_2$-algebra homomorphism

$$\mathbb{F}_2[b, b_i \mid i \geq 1] \otimes \mathbb{F}_2[a] \longrightarrow H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \otimes \mathbb{F}_2[a] \cong E_2^{*,*}$$

is an isomorphism in base degrees $s \leq u$. □

This was generalized by Serre to calculate $H^*(K(G, n); \mathbb{F}_2)$ for all finitely generated abelian G. The role of the collection $\{M_i\}_i$ is replaced by a condition on the excess of an admissible sequence.

Definition 7.5.4. If $I = (i_1, \dots, i_\ell)$ is an admissible sequence, so that $i_s \geq 2i_{s+1}$ for each $1 \leq s < \ell$, we define its *excess* to be

$$e(I) = (i_1 - 2i_2) + \cdots + (i_{\ell-1} - 2i_\ell) + i_\ell = i_1 - i_2 - \cdots - i_\ell = 2i_1 - |I| .$$

This is a non-negative integer. The only admissible sequence with $e(I) = 0$ is $I = ()$, and the only admissible sequences with $e(I) = 1$ are the M_i for $i \geq 1$.

Theorem 7.5.5 ([**Ser53**, Thm. 2]). *Suppose $n \geq 1$. Then*

$$H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \mathbb{F}_2[Sq^I(u_n) \mid e(I) < n] .$$

In words: the mod 2 cohomology algebra of $K(\mathbb{Z}/2, n)$ is the polynomial algebra generated by the classes $Sq^I(u_n)$, where $u_n \in H^n(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ is the universal class, and where I ranges over all admissible sequences of excess less than n. Serre's result includes the following stable range calculation.

Corollary 7.5.6. *The homomorphism*

$$\begin{aligned} \Sigma^n A &\longrightarrow \tilde{H}^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \\ \Sigma^n Sq^I &\longmapsto Sq^I(u_n) \end{aligned}$$

is an isomorphism in degrees $* \leq 2n$, i.e., for $|I| \leq n$.

Proof. Each admissible I of degree $|I| \leq n$ has excess $e(I) < n$, except for $I = (n)$, and $Sq^n(u_n) = u_n^2$. Hence the $Sq^I(u_n)$ with I admissible of degree $|I| \leq n$ range over the algebra generators of $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ in degrees $* \leq 2n$, together with the unique decomposable monomial in that range of degrees. □

Let $\bar{u}_n \in H^n(K(\mathbb{Z}, n); \mathbb{F}_2)$ denote the unique nonzero class, given by reduction modulo 2 of the universal class in $H^n(K(\mathbb{Z}, n); \mathbb{Z})$. Note that $\beta(\bar{u}_n) = 0$, so that $Sq^1(\bar{u}_n) = 0$. Let i_ℓ denote the last entry in an admissible sequence $I = (i_1, \dots, i_\ell)$.

Theorem 7.5.7 ([**Ser53**, Thm. 3]). *Suppose $n \geq 2$. Then*

$$H^*(K(\mathbb{Z}, n); \mathbb{F}_2) \cong \mathbb{F}_2[Sq^I(\bar{u}_n) \mid e(I) < n, i_\ell > 1] .$$

In words: the mod 2 cohomology algebra of $K(\mathbb{Z}, n)$ is the polynomial algebra generated by the classes $Sq^I(\bar{u}_n)$, where $I = (i_1, \dots, i_\ell)$ ranges over all admissible sequences of excess less than n, except those of length $\ell \geq 1$ with final term $i_\ell = 1$. When $n = 2$, only the empty sequence satisfies these conditions, so that $H^*(K(\mathbb{Z}, 2); \mathbb{F}_2) \cong \mathbb{F}_2[\bar{u}_n]$, as we already know. Serre's result implies the following stable range statement.

COROLLARY 7.5.8. *Let $n \ge 2$. The homomorphism*

$$\Sigma^n A/ASq^1 \longrightarrow \tilde{H}^*(K(\mathbb{Z},n);\mathbb{F}_2)$$
$$\Sigma^n Sq^I \longmapsto Sq^I(\bar{u}_n)$$

is an isomorphism in degrees $* \le 2n$, i.e., for $|I| \le n$.

PROOF. By ASq^1 we mean the left ideal in A generated by Sq^1. In view of the relation $Sq^1Sq^1 = 0$, it has a basis consisting of the admissible Sq^I with $I = (i_1, \dots, i_\ell)$ where $i_\ell = 1$. Hence the $Sq^I(\bar{u}_n)$ with I admissible of degree $|I| \le n$ and $i_\ell > 1$ (if $\ell \ge 1$) range over the algebra generators of $H^*(K(\mathbb{Z},n);\mathbb{F}_2)$ in degrees $* \le 2n$, together with the unique decomposable monomial, $Sq^n(\bar{u}_n) = \bar{u}_n^2$, in that range of degrees. □

EXAMPLE 7.5.9. The exact Serre sequence

$$0 \to H^n(K(\mathbb{Z}/2,n);\mathbb{F}_2) \xrightarrow{i^*} H^n(K(\mathbb{Z},n);\mathbb{F}_2) \xrightarrow{\tau^{n+1}} H^{n+1}(K(\mathbb{Z},n+1);\mathbb{F}_2) \xrightarrow{p^*} \dots$$
$$\dots \xrightarrow{\tau^{2n}} H^{2n}(K(\mathbb{Z},n+1);\mathbb{F}_2) \xrightarrow{p^*} H^{2n}(K(\mathbb{Z}/2,n);\mathbb{F}_2) \xrightarrow{i^*} H^{2n}(K(\mathbb{Z},n);\mathbb{F}_2)$$

associated to the homotopy fiber sequence

$$K(\mathbb{Z},n) \xrightarrow{i} K(\mathbb{Z}/2,n) \xrightarrow{p} K(\mathbb{Z},n+1)$$

satisfies $i^*(u_n) = \bar{u}_n$, so that $i^*(Sq^I(u_n)) = Sq^I(\bar{u}_n)$, by naturality. Hence i^* is surjective, and $\tau^m = 0$ for $n < m \le 2n$, It follows that $p^*(\bar{u}_{n+1}) = Sq^1 u_n$, since this is the only nonzero class in its degree, so that $p^*(Sq^I\bar{u}_{n+1}) = Sq^I Sq^1 u_n$. In particular, the Serre sequence splits up into the short exact sequences

$$0 \to \Sigma^{n+1} A/ASq^1 \xrightarrow{p^*} \Sigma^n A \xrightarrow{i^*} \Sigma^n A/ASq^1 \to 0$$

in degrees $n \le * \le 2n$. Here $p^*(\Sigma^{n+1}Sq^I) = \Sigma^n Sq^I Sq^1$, while $i^*(\Sigma^n Sq^I) = \Sigma^n Sq^I$ mod ASq^1. ((ETC: We will encounter the A-module extension

$$0 \to \Sigma A/ASq^1 \longrightarrow A \longrightarrow A/ASq^1 \to 0$$

later in the context of the Adams spectral sequence.))

The analogous results for odd primes p were obtained by Cartan [**Car54**]. Let

$$I = (\epsilon_1, i_1, \dots, \epsilon_\ell, i_\ell, \epsilon_{\ell+1}),$$

with $\ell \ge 0$, $\epsilon_s \in \{0,1\}$ and $i_s \ge 1$, for each $1 \le s \le \ell$. Let

$$a_s = \epsilon_s + 2(p-1)i_s$$

be the degree of $\beta^{\epsilon_s} P^{i_s}$, with $a_{\ell+1} = \epsilon_{\ell+1}$. The admissibility condition, that $i_s \ge \epsilon_{s+1} + p i_{s+1}$, is equivalent to the condition $a_s \ge p a_{s+1}$. Hence

$$(a_1 - pa_2) + \dots + (a_\ell - pa_{\ell+1}) + a_{\ell+1} = a_1 - (p-1)a_2 - \dots - (p-1)a_{\ell+1}$$

is non-negative. We can write this as $\epsilon_1 + (p-1)e(I)$, where

$$e(I) = 2i_1 - a_2 - \dots - a_{\ell+1}\,.$$

defines the p-primary excess of I.

DEFINITION 7.5.10. For $I = (\epsilon_1, i_1, \dots, \epsilon_\ell, i_\ell, \epsilon_{\ell+1})$ admissible, let

$$\begin{aligned} e(I) &= 2i_1 - a_2 - \dots - a_{\ell+1} \\ &= 2i_1 - (\epsilon_2 + 2(p-1)i_2) - \dots - \epsilon_{\ell+1} \end{aligned}$$

be the *excess* of I.

This agrees with Kraines [**Kra71**] and Tamanoi [**Tam99**], but differs from the convention of May [**May70**, Not. 10.1(b)], who adds ϵ_1 to the above definition of $e(I)$.

DEFINITION 7.5.11. Let $S_{\mathbb{F}_p}(x_i \mid i)$ denote the free graded commutative $\mathbb{F}_p$-algebra (= symmetric algebra) on a set of generators x_i, i.e., the tensor product of a polynomial algebra $\mathbb{F}_p[x_i]$ for each x_i of even degree and an exterior algebra $\Lambda_{\mathbb{F}_p}(x_i)$ for each x_i of odd degree.

THEOREM 7.5.12 ([**Car54**, Thm. 6]). *Let p be an odd prime, and $n \ge 1$. Then*

$$H^*(K(\mathbb{Z}/p, n); \mathbb{F}_p) \cong S_{\mathbb{F}_p}(P^I(u_n) \mid e(I) < n)$$

is the symmetric algebra generated by the classes $P^I(u_n)$, where

$$u_n \in H^n(K(\mathbb{Z}/p, n); \mathbb{F}_p)$$

is the universal class, and where $I = (\epsilon_1, i_1, \dots, \epsilon_\ell, i_\ell, \epsilon_{\ell+1})$ ranges over all admissible sequences of excess less than n.

SKETCH PROOF. Cartan's condition $pa_k < (p-1)(n + a_0 + \dots + a_k)$ translates to $pa_1 < (p-1)(n + a_1 + \dots + a_{\ell+1})$ in our notation, and is equivalent to $e(I) < n$. See also [**May70**, Thm. 10.3], where $e(I)$ corresponds to our $\epsilon_1 + e(I)$, and the condition " $_1 = 1$" should be read as "$\epsilon_1 = 1$". □

COROLLARY 7.5.13. *The homomorphism*

$$\begin{aligned} \Sigma^n A &\longrightarrow \tilde{H}^*(K(\mathbb{Z}/p, n); \mathbb{F}_p) \\ \Sigma^n P^I &\longmapsto P^I(u_n) \end{aligned}$$

is an isomorphism in degrees $* < 2n$, i.e., for $|I| < n$.

PROOF. Each admissible I of degree $|I| < n$ has excess $e(I) < n$. The decomposable classes in $H^*(K(\mathbb{Z}/p, n); \mathbb{F}_p)$ lie in degrees $* \ge 2n$. □

THEOREM 7.5.14 ([**Car54**, Thm. 6]). *Let p be an odd prime, and $n \ge 1$. Then*

$$H^*(K(\mathbb{Z}, n); \mathbb{F}_p) \cong S_{\mathbb{F}_p}(P^I(\bar{u}_n) \mid e(I) < n, \epsilon_{\ell+1} = 0)$$

is the symmetric algebra generated by the classes $P^I(\bar{u}_n)$, where

$$\bar{u}_n \in H^n(K(\mathbb{Z}, n); \mathbb{F}_p)$$

is the mod p reduction of the universal class, and where $I = (\epsilon_1, i_1, \dots, \epsilon_\ell, i_\ell, \epsilon_{\ell+1})$ ranges over all admissible sequences of excess less than n and with $\epsilon_{\ell+1} = 0$.

COROLLARY 7.5.15. *The homomorphism*

$$\begin{aligned} \Sigma^n A/A\beta &\longrightarrow \tilde{H}^*(K(\mathbb{Z}, n); \mathbb{F}_p) \\ \Sigma^n P^I &\longmapsto P^I(\bar{u}_n) \end{aligned}$$

is an isomorphism in degrees $* < 2n$, i.e., for $|I| < n$.

PROOF. Each admissible P^I of degree $|I| < n$ (and not ending with β) has excess $e(I) < n$. The decomposable classes in $H^*(K(\mathbb{Z}/p, n); \mathbb{F}_p)$ lie in degrees $* \geq 2n$. □

7.6. Stable cohomology operations

The Steenrod operations Sq^I and P^I are stable, in the following sense.

DEFINITION 7.6.1. A *stable cohomology operation* $\theta = (\theta_k)_k$ *of type* $(G; G', n)$ is a sequence of cohomology operations θ_k of type $(G, k; G', n+k)$ such that each diagram

$$\begin{tikzcd}
\tilde{H}^k(X;G) \arrow[r, "\theta_k"] \arrow[d, "\sigma"', "\cong"] & \tilde{H}^{n+k}(X;G') \arrow[d, "\cong"', "\sigma"] \\
\tilde{H}^{k+1}(\Sigma X;G) \arrow[r, "\theta_{k+1}"] & \tilde{H}^{n+k+1}(\Sigma X;G')
\end{tikzcd}$$

commutes, where σ denotes the suspension isomorphism.

((ETC: Is there a sign $(-1)^n$ needed? Is $\Sigma X = S^1 \wedge X$ or $X \wedge S^1$?))

DEFINITION 7.6.2. The *cohomology suspension*

$$\omega \colon \tilde{H}^{m+1}(Y; G') \longrightarrow \tilde{H}^m(\Omega Y; G')$$

maps the homotopy class of $f \colon Y \to K(G', m+1)$ to the homotopy class of $\Omega f \colon \Omega Y \to \Omega K(G', m+1) \simeq K(G', m)$.

REMARK 7.6.3. The standard notation for the cohomology suspension is σ, not ω, but for this argument is seems clearer to reserve $\tilde{\sigma}$ to denote the equivalence $K(G,k) \simeq \Omega K(G, k+1)$ and the suspension isomorphism represented by it.

LEMMA 7.6.4. *A sequence* $(\theta_k)_k$ *of cohomology operations of type* $(G, k; G', n+k)$ *is stable if and only if* $\omega(\theta_{k+1}) = \theta_k$ *for each* k*, where*

$$\omega \colon \tilde{H}^{n+k+1}(K(G, k+1); G') \longrightarrow \tilde{H}^{n+k}(K(G,k); G')$$

is the cohomology suspension.

PROOF. By the Eilenberg–MacLane representability theorem, $\theta = (\theta_k)_k$ is stable if and only if each diagram

$$\begin{tikzcd}
K(G,k) \arrow[r, "\theta_k"] \arrow[d, "\tilde{\sigma}"', "\simeq"] & K(G', n+k) \arrow[d, "\simeq"', "\tilde{\sigma}"] \\
\Omega K(G, k+1) \arrow[r, "\Omega\theta_{k+1}"] & \Omega K(G', n+k+1)
\end{tikzcd}$$

commutes up to homotopy. This is equivalent to the condition that ω maps the cohomology class represented by θ_{k+1} to the cohomology class represented by θ_k. □

In other words, the abelian group of stable cohomology operations of type $(G; G', n)$ is isomorphic to the sequential limit

$$\lim_k \tilde{H}^{n+k}(K(G,k); G')$$

of the diagram

$$(7.1) \qquad \dots \xrightarrow{\omega} \tilde{H}^{n+k+1}(K(G,k+1);G') \xrightarrow{\omega} \tilde{H}^{n+k}(K(G,k);G') \xrightarrow{\omega} \dots .$$

The composite of a stable operation of type $(G;G',n)$ followed by a stable operation of type $(G';G'',m)$ is a stable operation of type $(G;G'',n+m)$, so the collection of all stable cohomology operations of type $(G;G,n)$ for $n \in \mathbb{Z}$ forms a graded (usually non-commutative) ring. When $G = \mathbb{F}_p$, this ring is the mod p Steenrod algebra, as we can now deduce from the calculations of Serre and Cartan.

Proposition 7.6.5. *Let p be any prime, and let $A^n \subset A = \mathscr{A}(p)$ denote the degree n part of the mod p Steenrod algebra. The homomorphism*

$$\begin{aligned} A^n &\xrightarrow{\cong} \lim_k \tilde{H}^{n+k}(K(\mathbb{F}_p,k);\mathbb{F}_p) \\ \theta &\longmapsto (\theta(u_k))_k \end{aligned}$$

is an isomorphism. Hence A is isomorphic to the graded ring of stable cohomology operations of type $(\mathbb{F}_p;\mathbb{F}_p,n)$ for arbitrary n.

Proof. The homomorphisms

$$\begin{aligned} \Sigma^k A^n &\longrightarrow \tilde{H}^{n+k}(K(\mathbb{F}_p,k);\mathbb{F}_p) \\ \Sigma^k\theta &\longmapsto \theta(u_k) \end{aligned}$$

are compatible with the cohomology suspensions ω, and are isomorphisms for $k > n$. Hence they combine to map A^n isomorphically to the group of compatible sequences $(\theta_k)_k$. In particular, each morphism ω in (7.1) is an isomorphism, for $k > n$. It is clear that the product in A corresponds to the composition of (stable) cohomology operations. □

((ETC: In terms of spectra, $A \cong H^*(H)$. Here $H^n(H) \cong \lim_k \tilde{H}^{n+k}(K(\mathbb{F}_p,k);\mathbb{F}_p)$ because $\mathrm{Rlim}_k \tilde{H}^{n+k-1}(K(\mathbb{F}_p,k);\mathbb{F}_p) = 0$. Dually, $A_* \cong H_*(H)$ with $H_n(H) \cong \mathrm{colim}_k \tilde{H}_{n+k}(K(\mathbb{F}_p,k);\mathbb{F}_p)$.))

7.7. Hopf algebras

Let $A = \mathscr{A}(2)$. The mod 2 cohomology of any space $H^*(X;\mathbb{F}_2)$, is naturally an A-module and a commutative $\mathbb{F}_2$-algebra, satisfying the Cartan formula

$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$

and the instability condition $Sq^i(x) = 0$ for $i > |x|$. Following Milnor [**Mil58**, Lem. 1], there is an algebra homomorphism

$$\begin{aligned} \psi\colon A &\longrightarrow A \otimes A \\ Sq^k &\longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j , \end{aligned}$$

and each $A \otimes A$-module can be viewed as an A-module by restriction along ψ. The Cartan formula then says that the cup product

$$H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \xrightarrow{\cup} H^*(X;\mathbb{F}_2)$$

is an A-module homomorphism, where the A-module structure in the source is obtained by restriction in this way. We also say that $H^*(X;\mathbb{F}_2)$ is a A-module algebra. Completely similar results apply at odd primes p. The coproduct ψ makes A a cocommutative Hopf algebra, and we shall now review this algebraic structure. The paper [**MM65**] by Milnor and Moore is a standard reference.

DEFINITION 7.7.1. Let R be a commutative ring, which will be a field $\mathbb{F}_p$ in our main applications. For R-modules L and M we write $L \otimes M = L \otimes_R M$ for the tensor product over R and $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M, N)$ for the R-linear homomorphisms. If L, M and N are (homologically) graded, then

$$(L \otimes M)_k = \bigoplus_{i+j=k} L_i \otimes M_j$$

and

$$\mathrm{Hom}(M, N)_i = \prod_{i+j=k} \mathrm{Hom}(M_j, N_k)\,.$$

The twist isomorphism

$$\tau \colon L \otimes M \longrightarrow M \otimes L$$

maps $x \otimes y$ to $(-1)^{ij} y \otimes x$, for $x \in L_i$ and $y \in N_j$. There is a natural isomorphism

$$\mathrm{Hom}(L \otimes M, N) \cong \mathrm{Hom}(L, \mathrm{Hom}(M, N))$$

taking $f \colon L \otimes M \to N$ to $g \colon L \to \mathrm{Hom}(M, N)$, with $f(x \otimes y) = g(x)(y)$. Here f is left adjoint to g and g is right adjoint to f. The natural evaluation homomorphism (= adjunction counit)

$$\epsilon \colon \mathrm{Hom}(M, N) \otimes M \longrightarrow N$$

is left adjoint to the identity on $\mathrm{Hom}(M, N)$, and the natural homomorphism (= adjunction unit)

$$\eta \colon L \longrightarrow \mathrm{Hom}(M, L \otimes M)$$

is right adjoint to the identity on $L \otimes M$. We say that (graded) R-modules form a *closed symmetric monoidal category*, cf. [**ML63**, §VII.7].

DEFINITION 7.7.2. A (graded) *R-algebra* is a (graded) R-module A with a product $\phi \colon A \otimes A \to A$ and a unit $\eta \colon R \to A$ such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1\otimes\phi"] \arrow[d, "\phi\otimes 1"'] & A \otimes A \arrow[d, "\phi"] \\
A \otimes A \arrow[r, "\phi"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\phi"] & A \otimes R \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

commute. It is *commutative* if the diagram

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & A \otimes A \arrow[dl, "\phi"] \\
& A &
\end{tikzcd}$$

commutes.

DEFINITION 7.7.3. The ring R is the initial R-algebra. The product $\phi\colon R\otimes R \to R$ is the canonical isomorphism and the unit $\eta\colon R \to R$ is the identity.

The tensor product of two R-algebras A and B is the R-algebra $A \otimes B$ with product given by the composite

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\phi\otimes\phi} A \otimes B$$

and unit

$$R \cong R \otimes R \xrightarrow{\eta\otimes\eta} A \otimes B\,.$$

In the full subcategory of commutative R-algebras, the tensor product is the categorical sum.

DEFINITION 7.7.4. An R-algebra (A, ϕ, η) is *augmented* if it comes equipped with an algebra morphism $\epsilon\colon A \to R$. Let

$$I(A) = \ker(\epsilon\colon A \to R)$$

be the *augmentation ideal*, and let the R-module of *indecomposables* $Q(A)$ be the cokernel

$$I(A) \otimes I(A) \xrightarrow{\phi} I(A) \xrightarrow{\pi} Q(A) \to 0$$

of the restricted product. A subset $S \subset I(A)$ that generates A as an R-algebra will map to a subset $\pi(S) \subset Q(A)$ that generates $Q(A)$ as an R-module, and the converse often holds. The elements in $I(A)^2 = \phi(I(A) \otimes I(A))$ are said to be (algebra) *decomposable*, and an element $x \in I(A)$ with $\pi(x) \neq 0$ is (algebra) *indecomposable*.

((ETC: If $A = R[[x]]$ is a formal power series algebra, with $\epsilon(x) = 0$, then $Q(A) \cong R\{x\}$, but x does not generate A algebraically.))

DEFINITION 7.7.5. A *left A-module* is a (graded) R-module M with a pairing $\lambda\colon A \otimes M \to M$ such that

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "1\otimes\lambda"] \arrow[d, "\phi\otimes 1"'] & A \otimes M \arrow[d, "\lambda"] \\
A \otimes M \arrow[r, "\lambda"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\lambda"] \\
& M
\end{tikzcd}$$

commute.

A *right A-module* is a (graded) R-module L with a pairing $\rho\colon L \otimes A \to L$ such that

$$\begin{tikzcd}
L \otimes A \otimes A \arrow[r, "\rho\otimes 1"] \arrow[d, "1\otimes\phi"'] & L \otimes A \arrow[d, "\rho"] \\
L \otimes A \arrow[r, "\rho"] & L
\end{tikzcd}$$

and

$$\begin{tikzcd} L \otimes A \arrow[d, "\rho"'] & L \otimes R \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ L & \end{tikzcd}$$

commute.

Given a right A-module L and a left A-module M, the tensor product $L \otimes_A M$ is the coequalizer

$$L \otimes A \otimes M \overset{1\otimes\lambda}{\underset{\rho\otimes 1}{\rightrightarrows}} L \otimes M \xrightarrow{\pi} L \otimes_A M$$

where $1 \otimes \lambda$ and $\rho \otimes 1$ are given by the left and right action maps, respectively.

Given two left A-modules M and N, the R-module of A-linear homomorphisms $\mathrm{Hom}_A(M, N)$ is the equalizer

$$\mathrm{Hom}_A(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \overset{\lambda^*}{\underset{\lambda_*}{\rightrightarrows}} \mathrm{Hom}(A \otimes M, N)\,,$$

where $\lambda^*(f) = f\lambda\colon A \otimes M \to N$ and $\lambda_*(f) = \lambda(1 \otimes f)\colon A \otimes M \to N$ for $f\colon M \to N$.

EXAMPLE 7.7.6. Let G be a topological group, with multiplication $m\colon G \times G \to G$. The Pontryagin product

$$\phi\colon H_*(G; R) \otimes H_*(G; R) \xrightarrow{\times} H_*(G \times G; R) \xrightarrow{m_*} H_*(G; R)$$

and the homomorphisms $\eta\colon R \to H_*(G; R)$ and $\epsilon\colon H_*(G; R) \to R$ induced by $\{e\} \subset G$ and $G \to \{e\}$ make $H_*(G; R)$ an augmented R-algebra. Likewise, if X is a topological space with a left G-action, then $M = H_*(X; R)$ is a left $H_*(G; R)$-module.

Dually, for any space X the cup product

$$\cup\colon H^*(X; R) \otimes H^*(X; R) \xrightarrow{\times} H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X; R)$$

and the homomorphism $\eta\colon R \to H^*(X; R)$ induced by $X \to \{x_0\}$ make $H^*(X; R)$ a (graded) commutative R-algebra. A choice of base point $x_0 \in X$ determines an augmentation $\epsilon\colon H^*(X; R) \to R$, induced by $\{x_0\} \subset X$.

EXAMPLE 7.7.7. If V is an R-module, then the left action

$$\lambda\colon A \otimes A \otimes V \xrightarrow{\phi\otimes 1} A \otimes V$$

makes $A \otimes V$ a left A-module, known as an *extended* A-module. There is a natural isomorphism

$$\mathrm{Hom}_A(A \otimes V, N) \cong \mathrm{Hom}(V, UN)\,,$$

where N is any A-module and UN its underlying R-module. Hence the extended A-module functor $V \mapsto A \otimes V$ is left adjoint to the forgetful functor U from left A-modules to R-modules.

If R is a field, then the extended A-modules are the same as the free A-modules, all of which are projective. If, moreover, A is a connected R-algebra then each projective A-module is free, by a theorem of Kaplansky [**Mar83**, Prop. 11.2].

The dual theory of coalgebras and comodules is developed in [**MM65**] and [**EM66**].

DEFINITION 7.7.8. A (graded) *R-coalgebra* is a (graded) R-module C with a coproduct $\psi\colon C \to C \otimes C$ and a counit $\epsilon\colon C \to R$ such that

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\psi"'] & C \otimes C \arrow[d, "\psi\otimes 1"] \\ C \otimes C \arrow[r, "1\otimes\psi"] & C \otimes C \otimes C \end{tikzcd}$$

and

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] & C \otimes R \end{tikzcd}$$

commute. It is *cocommutative* if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

commutes.

REMARK 7.7.9. We can write

$$\psi(x) = \sum_\alpha x'_\alpha \otimes x''_\alpha$$

for suitable $x'_\alpha, x''_\alpha \in C$. Then

$$\sum_{\alpha,\beta} (x'_\alpha)'_\beta \otimes (x'_\alpha)''_\beta \otimes x''_\alpha = \sum_{\alpha,\beta} x'_\alpha \otimes (x''_\alpha)'_\beta \otimes (x''_\alpha)''_\beta$$

by coassociativity, and

$$\sum_\alpha \epsilon(x'_\alpha)x''_\alpha = x = \sum_\alpha x'_\alpha \epsilon(x''_\alpha)$$

by counitality. Cocommutativity asks that

$$\sum_\alpha x'_\alpha \otimes x''_\alpha = \sum_\alpha (-1)^{|x'_\alpha||x''_\alpha|} x''_\alpha \otimes x'_\alpha \,.$$

We often omit the summation indices in these formulas, and write

$$\psi(x) = \sum x' \otimes x''$$
$$\sum (x')' \otimes (x')'' \otimes x'' = \sum x' \otimes (x'')' \otimes (x'')''$$
$$\sum \epsilon(x')x'' = x = \sum x' \epsilon(x'')$$
$$\sum x' \otimes x'' = \sum (-1)^{|x'||x''|} x'' \otimes x' \,.$$

DEFINITION 7.7.10. The ring R is the terminal R-coalgebra. The coproduct $\psi\colon R \to R \otimes R$ is the inverse of the canonical isomorphism and the counit $\epsilon\colon R \to R$ is the identity.

The tensor product of two R-coalgebras C and D is the R-coalgebra $C \otimes D$ with coproduct given by the composite

$$C \otimes D \xrightarrow{\psi\otimes\psi} C \otimes C \otimes D \otimes D \xrightarrow{1\otimes\tau\otimes 1} C \otimes D \otimes C \otimes D$$

and counit

$$C \otimes D \xrightarrow{\epsilon\otimes\epsilon} R \otimes R \cong R\,.$$

In the full subcategory of cocommutative R-coalgebras, the tensor product is the categorical product.

Definition 7.7.11. An R-coalgebra (C, ψ, ϵ) is *coaugmented* if it comes equipped with a coalgebra morphism $\eta\colon R \to C$. Let

$$J(C) = \operatorname{cok}(\eta\colon R \to C)$$

be the *coaugmentation coideal*, ((ETC: also known as the *unit coideal*)) and let the R-module of *primitives* $P(C)$ be the kernel

$$0 \to P(C) \xrightarrow{\iota} J(C) \xrightarrow{\psi} J(C) \otimes J(C)$$

of the corestricted coproduct. In terms of elements,

$$P(C) \cong \{x \in C \mid \psi(x) = x \otimes 1 + 1 \otimes x\}\,,$$

and an element $x \in C$ with $\psi(x) = x \otimes 1 + 1 \otimes x$ is said to be (coalgebra) *primitive*. ((ETC: Also define “imprimitive”?))

Remark 7.7.12. In the coaugmented case, we can write

$$\psi(x) = x \otimes 1 + \sum_\alpha x'_\alpha \otimes x''_\alpha + 1 \otimes x$$

for $x \in I(C) = \ker(\epsilon) \cong J(C)$, with $x'_\alpha, x''_\alpha \in I(C)$, and this often gets abbreviated to

$$\psi(x) = x \otimes 1 + \sum x' \otimes x'' + 1 \otimes x\,.$$

Definition 7.7.13. A *left C-comodule* is a (graded) R-module M with a coaction $\nu\colon M \to C \otimes M$ such that

$$\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[d, "\nu"'] & C \otimes M \arrow[d, "\psi\otimes 1"] \\
C \otimes M \arrow[r, "1\otimes\nu"] & C \otimes C \otimes M
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & M \arrow[dl, "\cong"'] \arrow[d, "\nu"] \\
R \otimes M & C \otimes M \arrow[l, "\epsilon\otimes 1"]
\end{tikzcd}$$

commute.

A *right C-comodule* is a (graded) R-module L with a coaction $\sigma\colon L \to L \otimes C$ ((ETC: Not a standard notation.)) such that

$$\begin{tikzcd}
L \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & L \otimes C \arrow[d, "1\otimes\psi"] \\
L \otimes C \arrow[r, "\sigma\otimes 1"] & L \otimes C \otimes C
\end{tikzcd}$$

and

$$\begin{tikzcd} L \arrow[d, "\sigma"'] \arrow[dr, "\cong"] & \\ L \otimes C \arrow[r, "1\otimes\epsilon"'] & L \otimes R \end{tikzcd}$$

commute.

Given a right C-comodule L and a left C-comodule M, the *cotensor product* $L \,\square_C\, M$ is the equalizer

$$L \,\square_C\, M \xrightarrow{\iota} L \otimes M \overset{1\otimes\nu}{\underset{\sigma\otimes 1}{\rightrightarrows}} L \otimes C \otimes M$$

where $1 \otimes \nu$ and $\sigma \otimes 1$ are given by the left and right coaction maps, respectively.

Given two left C-comodules M and N, the R-module of C-colinear homomorphisms $\mathrm{Hom}_C(M, N)$ is the equalizer

$$\mathrm{Hom}_C(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \overset{\nu^*}{\underset{\nu_*}{\rightrightarrows}} \mathrm{Hom}(M, C \otimes N)\,,$$

where $\nu^*(f) = (1 \otimes f)\nu \colon M \to C \otimes N$ and $\nu_*(f) = \nu f \colon M \to C \otimes N$ for $f \colon M \to N$.

Remark 7.7.14. Note that we write $\mathrm{Hom}_B(M, N)$ to denote the B-module homomorphisms $f \colon M \to N$ when B is an algebra and M and N are B-modules, and to denote the B-comodule homomorphisms $f \colon M \to N$ when B is a coalgebra and M and N are B-comodules. This will also apply to the derived functors $\mathrm{Ext}^s_B(M, N)$. We may say "module Ext" or "comodule Ext" to distinguish the two cases.

Example 7.7.15. Let G be a topological group, with multiplication $m \colon G \times G \to G$. Suppose that $H^*(G; R)$ is finitely generated and projective over R in each degree, so that the cross product

$$H^*(G; R) \otimes H^*(G; R) \xrightarrow{\times} H^*(G \times G; R)$$

is an isomorphism. (Recall that $\otimes = \otimes_R$.) Then the Pontryagin coproduct

$$\psi \colon H^*(G; R) \xrightarrow{m^*} H^*(G \times G; R) \xrightarrow{\times^{-1}} H^*(G; R) \otimes H^*(G; R)$$

and the homomorphisms $\epsilon \colon H^*(G; R) \to R$ and $\eta \colon R \to H^*(G; R)$ induced by $\{e\} \subset G$ and $G \to \{e\}$ make $H^*(G; R)$ a coaugmented R-coalgebra. Likewise, if X is a topological space with a left G-action, then $M = H^*(X; R)$ is a left $H^*(G; R)$-comodule. (The hypothesis on G ensures that

$$H^*(G; R) \otimes H^*(X; R) \xrightarrow{\times} H^*(G \times X; R)$$

is also an isomorphism.)

Dually, for any space X with $H_*(X; R)$ flat over R in each degree, the diagonal coproduct

$$H_*(X; R) \xrightarrow{\Delta_*} H_*(X \times X; R) \xrightarrow{\times^{-1}} H_*(X; R) \otimes H_*(X; R)$$

and the homomorphism $\epsilon \colon H_*(X; R) \to R$ induced by $X \to \{x_0\}$ make $H_*(X; R)$ a (graded) cocommutative R-coalgebra. A choice of base point $x_0 \in X$ determines a coaugmentation $\eta \colon R \to H_*(X; R)$, induced by $\{x_0\} \subset X$.

EXAMPLE 7.7.16. If V is an R-module, then the left coaction

$$\nu\colon C \otimes V \xrightarrow{\psi\otimes 1} C \otimes C \otimes V$$

makes $C \otimes V$ a left C-comodule, known as an *extended* C-comodule. There is a natural isomorphism

$$\mathrm{Hom}(UM, V) \cong \mathrm{Hom}_C(M, C \otimes V)\,,$$

where M is any C-comodule and UM its underlying R-module. Hence the extended C-comodule functor $V \mapsto C \otimes V$ is right adjoint to the forgetful functor U from left C-comodules to R-modules.

If R is a field, then every extended C-comodule is injective, and each injective C-comodule is a retract of an extended C-comodule. ((ETC: If, moreover, C is connected, is every injective C-comodule actually extended?))

DEFINITION 7.7.17. A (graded) *R-bialgebra* is a (graded) R-module B with a product $\phi\colon B \otimes B \to B$, unit $\eta\colon R \to B$, coproduct $\psi\colon B \to B \otimes B$ and counit $\epsilon\colon B \to R$ such that

(1) (B, ϕ, η) is an R-algebra,
(2) (B, ψ, ϵ) is an R-coalgebra, and
(3) ψ and ϵ are R-algebra homomorphisms.

LEMMA 7.7.18. *The following are equivalent:*

- *ψ and ϵ are R-algebra homomorphisms.*
- *ϕ and η are R-coalgebra homomorphisms.*

PROOF. The conditions that ψ and ϵ are R-algebra homomorphisms ask that the diagrams

$$\begin{tikzcd}
B \otimes B \arrow[r, "\psi\otimes\psi"] \arrow[dd, "\phi"'] & B \otimes B \otimes B \otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
 & B \otimes B \otimes B \otimes B \arrow[d, "\phi\otimes\phi"] \\
B \arrow[r, "\psi"] & B \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \arrow[r, "\cong"] \arrow[d, "\eta"'] & R \otimes R \arrow[d, "\eta\otimes\eta"] \\
B \arrow[r, "\psi"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\phi"'] & R \otimes R \arrow[d, "\cong"] \\
B \arrow[r, "\epsilon"] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "="] \arrow[d, "\eta"'] & R \arrow[d, "="] \\
B \arrow[r, "\epsilon"] & R
\end{tikzcd}$$

commute. These are also the conditions that ϕ and η are R-coalgebra homomorphisms. □

DEFINITION 7.7.19. There are natural homomorphisms

$$P(B) \rightarrowtail J(B) \xleftarrow{\cong} I(B) \twoheadrightarrow Q(B)$$

for each bialgebra B. If $P(B) \to Q(B)$ is surjective, then we say that B is *primitively generated*.

This terminology is most appropriate when a set of module generators for $Q(B)$ also generates B as an algebra.

DEFINITION 7.7.20. A *Hopf algebra* over R is an R-bialgebra B equipped with an R-linear conjugation $\chi\colon B \to B$ such that

$$\begin{tikzcd}
B \arrow[r, "\psi"] \arrow[d, "\psi"'] \arrow[dr, "\epsilon"] & B \otimes B \arrow[dr, "1\otimes\chi"] & \\
B \otimes B \arrow[dr, "\chi\otimes 1"'] & R \arrow[dr, "\eta"] & B \otimes B \arrow[d, "\phi"] \\
& B \otimes B \arrow[r, "\phi"'] & B
\end{tikzcd}$$

commutes.

With the notation $\psi(b) = \sum b' \otimes b''$, we can write the condition as follows:

$$\sum b' \cdot \chi(b'') = \eta\epsilon(b) = \sum \chi(b') \cdot b'' .$$

LEMMA 7.7.21. *A bialgebra admits at most one conjugation.*

Hence being a Hopf algebra is a property, not a structure, for bialgebras.

LEMMA 7.7.22. *The conjugation $\chi\colon B \to B$ is an anti-homomorphism of algebras, and an anti-homomorphism of coalgebras, so that $\chi\phi = \phi\tau(\chi \otimes \chi)$ and $\psi\chi = (\chi \otimes \chi)\tau\psi$.*

LEMMA 7.7.23. *Let B be a commutative or cocommutative Hopf algebra. Then $\chi^2 = 1$, so $\chi = \chi^{-1}\colon B \to B$.*

See [**MM65**, §8] or [**DNR01**, §4.2] for proofs. The following examples are closely related to those first studied by Heinz Hopf [**Hop41**].

EXAMPLE 7.7.24. Let G be a topological group. Suppose that $H_*(G;R)$ is flat over R in each degree, so that the unit $\eta\colon R \to H_*(G;R)$, Pontryagin product

$$\phi\colon H_*(G;R) \otimes H_*(G;R) \longrightarrow H_*(G;R) ,$$

counit $\epsilon\colon H_*(G;R) \to R$ and diagonal coproduct

$$\psi\colon H_*(G;R) \longrightarrow H_*(G;R) \otimes H_*(G;R)$$

make $H_*(G;R)$ an R-bialgebra. The inverse map $i\colon G \to G$ induces the conjugation

$$\chi = i_*\colon H_*(G;R) \longrightarrow H_*(G;R)$$

making $H_*(G;R)$ a cocommutative Hopf algebra over R.

Suppose instead that $H^*(G;R)$ is finitely generated and projective over R in each degree, so that the unit $\eta\colon R \to H^*(G;R)$, cup product

$$\phi\colon H^*(G;R) \otimes H^*(G;R) \longrightarrow H^*(G;R) ,$$

counit $\epsilon\colon H^*(G;R) \to R$ and Pontryagin coproduct

$$\psi\colon H^*(G;R) \longrightarrow H^*(G;R) \otimes H^*(G;R)$$

make $H^*(G;R)$ an R-bialgebra. The inverse map $i\colon G \to G$ induces the conjugation

$$\chi = i^*\colon H^*(G;R) \longrightarrow H^*(G;R)$$

making $H^*(G;R)$ a commutative Hopf algebra over R.

Definition 7.7.25. Let B be a Hopf algebra over R. For left B-modules L and M we give the tensor product $L \otimes M$ the "diagonal" B-module structure with left action $\lambda\colon B \otimes L \otimes M \to L \otimes M$ given by the composition

$$B \otimes L \otimes M \xrightarrow{\psi} B \otimes B \otimes L \otimes M \xrightarrow{1\otimes\tau\otimes 1} B \otimes L \otimes B \otimes M \xrightarrow{\lambda\otimes\lambda} L \otimes M\,.$$

For left B-modules M and N we give $\operatorname{Hom}(M,N)$ the "conjugate" B-module structure with left action $\lambda\colon B \otimes \operatorname{Hom}(M,N) \to \operatorname{Hom}(M,N)$ given by the right adjoint of the composition

$$\begin{aligned} B \otimes \operatorname{Hom}(M,N) \otimes M &\xrightarrow{\psi\otimes 1\otimes 1} B \otimes B \otimes \operatorname{Hom}(M,N) \otimes M \\ &\xrightarrow{1\otimes\tau\otimes 1} B \otimes \operatorname{Hom}(M,N) \otimes B \otimes M \xrightarrow{1\otimes 1\otimes\chi\otimes 1} B \otimes \operatorname{Hom}(M,N) \otimes B \otimes M \\ &\xrightarrow{1\otimes 1\otimes\lambda} B \otimes \operatorname{Hom}(M,N) \otimes M \xrightarrow{1\otimes\epsilon} B \otimes N \xrightarrow{\lambda} N\,. \end{aligned}$$

There is a natural isomorphism

$$\operatorname{Hom}_B(L \otimes M, N) \cong \operatorname{Hom}_B(L, \operatorname{Hom}(M,N))\,,$$

so that $f\colon L\otimes M \to N$ is B-linear if and only if its right adjoint $g\colon L \to \operatorname{Hom}(M,N)$ is B-linear.

If B is cocommutative, then the twist isomorphism $\tau\colon L \otimes M \to M \otimes L$ is B-linear, and the left B-modules form a closed symmetric monoidal category.

((ETC: Margolis [**Mar83**, §12.1] writes $L \wedge M$ for this tensor product of B-modules.))

Example 7.7.26. The left B-action on the functional dual $DM = \operatorname{Hom}(M,R)$ of a left B-module M is adjoint to the composition

$$B \otimes DM \otimes M \xrightarrow{\tau\otimes 1} DM \otimes B \otimes M \xrightarrow{1\otimes\chi\otimes 1} DM \otimes B \otimes M \xrightarrow{1\otimes\lambda} DM \otimes M \xrightarrow{\epsilon} R\,.$$

Remark 7.7.27. For $b \in B$ with $\psi(b) = \sum b' \otimes b''$, $\ell \in L$ and $m \in M$ we have

$$b \cdot (\ell \otimes m) = \sum (-1)^{|b''||\ell|} b' \cdot \ell \otimes b'' \cdot m\,.$$

For $f \in \operatorname{Hom}(M,N)$ we have

$$(b \cdot f)(m) = \sum (-1)^{|b''||f|} b' \cdot f(\chi(b'') \cdot m)\,.$$

In particular, for $b \in B$ and $f \in DM = \operatorname{Hom}(M,R)$, we have

$$(b \cdot f)(m) = (-1)^{|b||f|} f(\chi(b) \cdot m)\,.$$

Definition 7.7.28. Let B be a Hopf algebra over R. For left B-comodules L and M we give the tensor product $L \otimes M$ the "codiagonal" B-comodule structure with left coaction $\nu\colon L \otimes M \to B \otimes L \otimes M$ given by the composition

$$L \otimes M \xrightarrow{\nu\otimes\nu} B \otimes L \otimes B \otimes M \xrightarrow{1\otimes\tau\otimes 1} B \otimes B \otimes L \otimes M \xrightarrow{\phi\otimes 1\otimes 1} B \otimes L \otimes M\,.$$

If B is commutative, then the twist isomorphism $\tau\colon L \otimes M \to M \otimes L$ is B-colinear, and the left B-comodules form a symmetric monoidal category.

((ETC: We might write $L \wedge M$ for this tensor product of C-comodules.))

REMARK 7.7.29. For left B-comodules M and N we cannot generally give the R-module $\mathrm{Hom}(M,N)$ a natural "coconjugate" B-comodule structure such that $f\colon L\otimes M\to N$ is B-colinear if and only if its right adjoint $g\colon L\to \mathrm{Hom}(M,N)$ is B-colinear. If $M=\mathrm{colim}_i M_i$ and $\nu_i\colon \mathrm{Hom}(M_i,N)\to B\otimes \mathrm{Hom}(M_i,N)$ is a suitable coaction, then $\lim_i \nu_i\colon \mathrm{Hom}(M,N)\to \lim_i B\otimes \mathrm{Hom}(M_i,N)$ will not generally factor through $B\otimes \lim_i \mathrm{Hom}(M_i,N)\cong B\otimes \mathrm{Hom}(M,N)$.

When B is flat as an R-module there is, however, a different internal function object $F(M,N)$ with a natural B-comodule structure, and a natural isomorphism

$$\mathrm{Hom}_B(L\otimes M,N)\cong \mathrm{Hom}_B(L,F(M,N))$$

so that $f\colon L\otimes M\to N$ is B-colinear if and only if $g\colon L\to F(M,N)$ is B-colinear. See Hovey's paper [**Hov04**, Thm. 1.3.1] for a construction, which satisfies $F(M,B\otimes V)\cong B\otimes \mathrm{Hom}(M,V)$ when $N=B\otimes V$ is a coextended B-comodule. Here V is any left R-module. There is a natural homomorphism $F(M,N)\to \mathrm{Hom}(M,N)$, which is injective if M is finitely generated over R, and an isomorphism if M is finitely presented over R, cf. [**Hov04**, Prop. 1.3.2]. We can think of $F(M,N)$ as the elements of $\mathrm{Hom}(M,N)$ with algebraic B-coaction.

A second approach [**Boa82**] is to consider B-comodules as a subcategory of B^*-modules, where B^* is the (non-commutative) ring of (right) R-module homomorphisms $B\to R$. A third approach is to consider $\mathrm{Hom}(M,N)$ as a "completed" B-comodule, with coaction $\mathrm{Hom}(M,N)\to B\,\widehat{\otimes}\,\mathrm{Hom}(M,N)$ landing in a completed tensor product.

For a module M over a Hopf algebra B, the extended B-module $B\otimes UM$ and the diagonal B-module $B\otimes M$ are not equal, but isomorphic. We call this the *untwisting isomorphism*, but the name may not be standard.

PROPOSITION 7.7.30 ([**ABP69**, Thm. 3.1], [**LMSM86**, Lem. II.4.8]). *Let B be a Hopf algebra and M a left B-module. The composite*

$$B\otimes M \xrightarrow{\psi\otimes 1} B\otimes B\otimes M \xrightarrow{1\otimes\lambda} B\otimes M\,,$$

mapping $b\otimes m$ to $\sum b'\otimes b''m$, defines an isomorphism

$$B\otimes UM \xrightarrow{\cong} B\otimes M$$

from the extended B-module on UM to the tensor product of B and M with the diagonal B-action.

PROOF. The inverse isomorphism is given by the composite

$$B\otimes M \xrightarrow{\psi\otimes 1} B\otimes B\otimes M \xrightarrow{1\otimes\chi\otimes 1} B\otimes B\otimes M \xrightarrow{1\otimes\lambda} B\otimes M$$

mapping $b\otimes m$ to $\sum b'\otimes \chi(b'')m$. □

PROPOSITION 7.7.31 ([**BMMS86**, pp. 92–93]). *Let B be a Hopf algebra and M a left B-comodule. The composite*

$$B\otimes M \xrightarrow{1\otimes\nu} B\otimes B\otimes M \xrightarrow{\phi\otimes 1} B\otimes M$$

defines an isomorphism

$$B\otimes M \xrightarrow{\cong} B\otimes UM$$

from the tensor product of B and M with the diagonal B-coaction to the extended B-comodule on UM.

PROOF. The inverse isomorphism is given by the composite

$$B \otimes M \xrightarrow{1\otimes\nu} B \otimes B \otimes M \xrightarrow{1\otimes\chi\otimes 1} B \otimes B \otimes M \xrightarrow{\phi\otimes 1} B \otimes M\,.$$

□

We now turn to the behavior of these algebraic notions under dualization.

DEFINITION 7.7.32. Let $DM = \mathrm{Hom}(M, R)$ denote the *functional dual* of a (graded) R-module M.

((ETC: Adapting Dold–Puppe [**DP80**, §1] and Lewis–May–Steinberger [**LMSM86**, §III.1], M is said to be (strongly dualizable, finite or) *dualizable* if the canonical homomorphism

$$M \otimes DM \cong \mathrm{Hom}(R, M) \otimes \mathrm{Hom}(M, R) \xrightarrow{\circ} \mathrm{Hom}(M, M)$$

is an isomorphism. Equivalently: M is finitely generated projective over R.))

LEMMA 7.7.33. *Let M be a graded R-module. If M is bounded below then DM is bounded above, while if M is bounded above then DM is bounded below. If M is finitely generated and projective over R in each degree, then DM is also finitely generated and projective over R in each degree, and the canonical homomorphism*

$$\rho\colon M \longrightarrow DDM$$

is an isomorphism.

LEMMA 7.7.34. *Let L and M be graded R-modules. If L and M are both bounded below (or both are bounded above, or one of them is bounded above and below), and L (or M) is finitely generated projective over R in each degree, then the canonical homomorphism*

$$\otimes\colon DL \otimes DM \longrightarrow D(L \otimes M)$$

is an isomorphism. Here $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \cdot g(y)$ for $f \in DL$, $g \in DM$, $x \in L$ and $y \in M$.

LEMMA 7.7.35. *Let A be a graded R-algebra that is bounded below (or bounded above) and finitely generated projective over R in each degree. Then DA with the coproduct*

$$\psi\colon DA \xrightarrow{D\phi} D(A \otimes A) \xrightarrow{\otimes^{-1}} DA \otimes DA$$

and counit

$$\epsilon\colon DA \xrightarrow{D\eta} DR \cong R$$

is a graded R-coalgebra.

Conversely, if C is a graded R-coalgebra, then DC with the product

$$\phi\colon DC \otimes DC \xrightarrow{\otimes} D(C \otimes C) \xrightarrow{D\psi} DC$$

and the unit

$$\eta\colon R \cong DR \xrightarrow{D\epsilon} DC$$

is a graded R-algebra.

LEMMA 7.7.36. *Let A be an augmented graded R-algebra that is bounded below (or bounded above) and finitely generated projective over R in each degree. Then DA is coaugmented by*

$$\eta\colon R \cong DR \xrightarrow{D\epsilon} DA\,,$$

and the isomorphism $J(DA) \cong DI(A)$ restricts to an isomorphism

$$P(DA) \cong DQ(A)\,.$$

Conversely, if C is a coaugmented graded R-coalgebra, then DC is augmented by

$$\epsilon\colon DC \xrightarrow{D\eta} DR \cong R\,,$$

and the isomorphism $I(DC) \cong DJ(C)$ induces a homomorphism

$$Q(DC) \longrightarrow DP(C)\,.$$

If R is a field, then this is a surjection. If, furthermore, C is bounded below (or bounded above) and finitely generated over the field R in each degree, then this is an isomorphism.

PROOF.

$$\begin{tikzcd}
I(DC)\otimes I(DC) \arrow[r, "\phi"] \arrow[d, "\cong"'] & I(DC) \arrow[r, "\pi"] \arrow[dd, "\cong"'] & Q(DC) \arrow[r] \arrow[dd] & 0 \\
DJ(C)\otimes DJ(C) \arrow[d, "\otimes"'] & & & \\
D(J(C)\otimes J(C)) \arrow[r, "D\psi"] & DJ(C) \arrow[r, "D\iota"] & DP(C) \arrow[r] & 0
\end{tikzcd}$$

□

LEMMA 7.7.37. *Let M be a left A-module, with A and M both bounded below (or both bounded above, or A bounded above and below), and with A finitely generated projective over R in each degree. Then DM with the left coaction*

$$\nu\colon DM \xrightarrow{D\lambda} D(A\otimes M) \xrightarrow{\otimes^{-1}} DA\otimes DM$$

is a left DA-comodule. ((ETC: Likewise for right A-modules.))

Conversely, if C is a graded R-coalgebra and M is a left C-comodule, then DM with the left action

$$\lambda\colon DC\otimes DM \xrightarrow{\otimes} D(C\otimes M) \xrightarrow{D\nu} DM$$

is a left DC-module.

LEMMA 7.7.38. *Let L and M be right and left A-modules, respectively, with L, M and A all bounded below (or all bounded above, or two of them bounded above and below), and with A finitely generated projective over R in each degree. Then the isomorphism $DL\otimes DM \cong D(L\otimes M)$ restricts to an isomorphism*

$$DL\,\square_{DA}\,DM \cong D(L\otimes_A M)\,.$$

Lemma 7.7.39. *Let M and N be left A-modules, with M, N and A all bounded below (or all bounded above, or A bounded above and below), and with A finitely generated projective over R in each degree. Then $f \mapsto Df$ defines a homomorphism*

$$D\colon \operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_{DA}(DN,DM)\,.$$

If, furthermore, M and N are finitely generated projective over R in each degree, then D is an isomorphism.

Conversely, if M and N are left C-comodules, then $f \mapsto Df$ defines a homomorphism

$$D\colon \operatorname{Hom}_C(M,N) \longrightarrow \operatorname{Hom}_{DC}(DN,DM)\,.$$

If M, N and C are all bounded below (or all bounded above, or C is bounded above and below), and they are all finitely generated projective over R in each degree, then D is an isomorphism.

((ETC: Are there further simplifications when R is a field, so that $\rho\colon M \to DDM$ and $\rho\colon N \to DDN$ are injective?))

Proposition 7.7.40. *Let B be a graded R-bialgebra that is bounded below (or bounded above) and finitely generated projective over R in each degree. Then DB with the product*

$$\phi\colon DB \otimes DB \xrightarrow{\otimes} D(B \otimes B) \xrightarrow{D\psi} DB\,,$$

unit

$$\eta\ : R \cong DR \xrightarrow{D\epsilon} DB\,,$$

coproduct

$$\psi\colon DB \xrightarrow{D\phi} D(B \otimes B) \xrightarrow{\otimes^{-1}} DB \otimes DB$$

and counit

$$\epsilon\colon DB \xrightarrow{D\eta} DR \cong R$$

is a graded R-bialgebra. If B is commutative (resp. cocommutative), then DB is cocommutative (resp. commutative). If B is a Hopf algebra, then DB is a Hopf algebra with conjugation

$$\chi\colon DB \xrightarrow{D\chi} DB\,.$$

Example 7.7.41. Let $R = \mathbb{Z}$. There is a bicommutative Hopf algebra $B = \mathbb{Z}[\xi]$, with underlying algebra the polynomial ring on one generator ξ in nonzero even degree. The product is given by $\phi(\xi^i \otimes \xi^j) = \xi^{i+j}$. For degree reasons, the coproduct on ξ can only be $\psi(\xi) = \xi \otimes 1 + 1 \otimes \xi$, which implies that

$$\psi(\xi^k) = \sum_{i+j=k} (i,j)\, \xi^i \otimes \xi^j$$

by the binomial theorem. The conjugation satisfies $\chi(\xi) = -\xi$. The coalgebra primitives and algebra indecomposables of B are

$$\mathbb{Z}\{\xi\} \cong P(B) \xrightarrow{\cong} Q(B) \cong \mathbb{Z}\{\xi\}\,,$$

so B is primitively generated.

The dual Hopf algebra $DB = \Gamma(x)$ has underlying algebra the divided power ring on one generator x in a nonzero even degree. Here $\Gamma(x) = \mathbb{Z}\{\gamma_k(x) \mid k \geq$

$0\}$ with $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_k(x)$ dual to ξ^k. The product is given by $\phi(\gamma_i(x) \otimes \gamma_j(x)) = (i,j)\,\gamma_{i+j}(x)$, and the coproduct is given by

$$\psi(\gamma_k(x)) = \sum_{i+j=k} \gamma_i(x) \otimes \gamma_j(x)\,.$$

The conjugation satisfies $\chi(\gamma_k(x)) = (-1)^k \gamma_k(x)$. The coalgebra primitives of DB are

$$P(DB) = \mathbb{Z}\{x\}$$

while the algebra indecomposables are

$$Q(DB) \cong \mathbb{Z}\{x\} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}/p\{\gamma_{p^n}(x) \mid n \geq 1\}\,.$$

This uses the number-theoretic fact that

$$\gcd\{\binom{k}{i} \mid 0 < i < k\} = \begin{cases} p & \text{if } k = p^n \text{ with } n \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

((ETC: Reference?)) In other words, $\gamma_k(x)$ is indecomposable if and only if $k = p^n$ is a prime power, and in this case $p\gamma_k(x)$ is decomposable.

The general theory ensures that

$$\mathbb{Z}\{x\} = P(DB) \cong DQ(B) \cong D(\mathbb{Z}\{\xi\})$$

while in this example, the homomorphism

$$\mathbb{Z}\{x\} \oplus \bigoplus_{p,n} \mathbb{Z}/p\{\gamma_{p^n}(x)\} \cong Q(DB) \longrightarrow DP(B) = D(\mathbb{Z}\{\xi\})$$

is not an isomorphism.

REMARK 7.7.42. For $|\xi| = u - 1 \geq 2$, this example is homologically realized by $B \cong H_*(\Omega S^u)$ with $DB \cong H^*(\Omega S^u)$, and ΩS^u is equivalent as an A_∞ space (in particular, as a homotopy associative H-space) to a topological group G. The problem of realizing B cohomologically is more subtle, and was discussed in Remark 7.4.14.

((ETC: Return to structure theorems.))

7.8. The dual Steenrod algebra

Milnor proved that the Cartan formula for the Steenrod operations implies that the mod p Steenrod algebra is a Hopf algebra, for each prime p.

THEOREM 7.8.1 ([**Mil58**, Lem. 1], [**Ste62**, Thm. II.1.1, Thm. VI.2.10]). *Let $A = \mathscr{A}(p)$ be the mod p Steenrod algebra. The assignments*

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p = 2$, and

$$\beta \longmapsto \beta \otimes 1 + 1 \otimes \beta$$

$$P^k \longmapsto \sum_{i+j=k} P^i \otimes P^j$$

for p odd, extend uniquely to ring homomorphisms

$$\psi \colon A \longrightarrow A \otimes A$$

so that

$$\theta(x \cup y) = \sum (-1)^{|\theta''||x|} \theta'(x) \cup \theta''(y)$$

for each $\theta \in A$, $x, y \in H^*(X; \mathbb{F}_p)$ *and* $\psi(\theta) = \sum \theta' \otimes \theta'' \in A \otimes A$.

Sketch proof. Following Milnor, let R be the set of $\theta \in A$ for which there exists an element $\rho \in A \otimes A$ such that

$$\theta\phi = \phi\rho \colon H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X; \mathbb{F}_p)$$

for all spaces X. Then R is closed under sum and product in A, and contains the Sq^k for $p = 2$ and β and the P^k for p odd, hence is equal to the whole of A.

To prove uniqueness of ρ, evaluate $\theta\phi$ on $H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ for a space X that faithfully detects the action by A in a large range of degrees. If $|\theta| = n$, one can let $X = K(\mathbb{Z}/p, n)$ or $X = K(\mathbb{Z}/p, 1)^n$.

Letting $\psi(\theta) = \rho$ then defines the ring homomorphism ψ. □

Remark 7.8.2. The admissible basis shows that A is concentrated in non-negative cohomological degrees, and is finite-dimensional over $\mathbb{F}_p$ in each degree. Moreover, $\mathbb{F}_p\{1\}$ equals the degree 0 part of A, so we say that A is a *connected* algebra. This implies that there is a unique augmentation $\epsilon \colon A \to \mathbb{F}_p$.

Theorem 7.8.3 ([**Mil58**, Thm. 1], [**Ste62**, Thm. II.1.2, Thm. VI.2.11]). *The Steenrod algebra A, with the coproduct $\psi \colon A \to A \otimes A$ and the augmentation $\epsilon \colon A \to \mathbb{F}_p$, is a cocommutative Hopf algebra over $\mathbb{F}_p$.*

Proof. The known formulas for $\psi(Sq^k)$, $\psi(\beta)$ and $\psi(P^k)$ imply that ψ is coassociative and counital. The existence of the conjugation χ follows from the fact that A is connected [**MM65**, Def. 8.4]. It satisfies

$$\sum_{i+j=k} Sq^i \chi(Sq^j) = 0$$

for $k \geq 1$, $\chi(\beta) = -\beta$, and

$$\sum_{i+j=k} P^i \chi(P^j) = 0$$

for $k \geq 1$. □

Definition 7.8.4. For each prime p, let the (mod p) *dual Steenrod algebra* $A_* = DA = \mathrm{Hom}(A, \mathbb{F}_p)$ be the function dual of the mod p Steenrod algebra.

Corollary 7.8.5 ([**Mil58**, Cor. 1]). *The dual Steenrod algebra A_* is a commutative Hopf algebra over $\mathbb{F}_p$.*

Remark 7.8.6. The finite type results for A imply that A_* is concentrated in non-negative homological degrees, and is finite-dimensional over $\mathbb{F}_p$ in each degree. Hence $DA_* \cong A$. Moreover, $\mathbb{F}_p\{1\}$ equals the degree 0 part of A_*, so A_* is connected.

Milnor determined the structure of A_* as an algebra, with product dual to the coproduct $\psi \colon A \to A \otimes A$, as well as its coproduct, dual to the product $\phi \colon A \otimes A \to A$. We will now see his results can be proved.

Let X be any space, and p any prime. For brevity we set $H_*(X) = H_*(X; \mathbb{F}_p)$ and $H^*(X) = H^*(X; \mathbb{F}_p)$. There are natural left and right A-module and A^*-comodule structures on $H_*(X)$ and $H^*(X)$, for a total of eight combinations, as explained by Boardman in his paper [**Boa82**]. Four of these were discussed by Milnor in [**Mil58**], and we review these below. The remaining four are then obtained by use of the conjugation $\chi \colon A \to A$, or its dual.

First, the cup product

$$\cup \colon H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

and the Steenrod operations

$$\lambda \colon A \otimes H^*(X) \longrightarrow H^*(X)$$

naturally give $H^*(X)$ the structure of a left A-module algebra. This means that the diagrams

$$\begin{tikzcd}
A \otimes A \otimes H^*(X) \arrow[r, "1\otimes\lambda"] \arrow[d, "\phi\otimes 1"'] & A \otimes H^*(X) \arrow[d, "\lambda"] \\
A \otimes H^*(X) \arrow[r, "\lambda"] & H^*(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes H^*(X) \otimes H^*(X) \arrow[r, "1\otimes\cup"] \arrow[d, "\psi\otimes 1"'] & A \otimes H^*(X) \arrow[d, "\lambda"] \\
A \otimes A \otimes H^*(X) \otimes H^*(X) \arrow[d, "1\otimes\tau\otimes 1"'] & H^*(X) \\
A \otimes H^*(X) \otimes A \otimes H^*(X) \arrow[r, "\lambda\otimes\lambda"] & H^*(X) \otimes H^*(X) \arrow[u, "\cup"']
\end{tikzcd}$$

commute, together with unitality conditions (which we omit to display). Furthermore, $H^*(X)$ is commutative, in the graded sense.

Example 7.8.7. The Cartan formula tells us that the cohomology cross product pairing

$$\times \colon H^*(X) \otimes H^*(Y) \longrightarrow H^*(X \times Y)$$

is A-linear, where A acts diagonally on the left and by the standard action on the right. When X or Y is of finite type mod p, so that $\times$ is an isomorphism, this shows that the diagonal A-action on the tensor product models the Cartesian product of spaces, to the eyes of mod p cohomology. Likewise, it models the smash product of spaces to the eyes of reduced mod p cohomology.

Second, applying $\mathrm{Hom}(-, \mathbb{F}_p)$ to the left A-module action λ defines a homomorphism

$$\mathrm{Hom}(\lambda, 1) \colon \mathrm{Hom}(H^*(X), \mathbb{F}_p) \longrightarrow \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_p) .$$

When $H_*(X)$ has finite type, there are natural isomorphisms

$$H_*(X) \xrightarrow{\cong} \mathrm{Hom}(H^*(X), \mathbb{F}_p)$$

$$A_* \otimes H_*(X) \xrightarrow{\cong} \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_p)$$

and the composite

$$H_*(X) \cong \mathrm{Hom}(H^*(X), \mathbb{F}_p) \longrightarrow \mathrm{Hom}(A \otimes H^*(X), \mathbb{F}_p) \cong A_* \otimes H_*(X)$$

defines a natural left A_*-coaction

$$\nu\colon H_*(X) \longrightarrow A_* \otimes H_*(X)\,.$$

Using CW approximation and commutation of homology with strongly filtered colimits, one can show that this coaction is well-defined and natural for all spaces X, not just those with mod p homology of finite type. ((ETC: We can give a more direct construction when we have presented homology in terms of the Eilenberg–MacLane spectrum.)) The cup product is dual to the homomorphism

$$\Delta_*\colon H_*(X) \longrightarrow H_*(X \times X) \cong H_*(X) \otimes H_*(X)$$

induced by the diagonal map $\Delta\colon X \to X \times X$. It follows that the diagrams

$$\begin{tikzcd}
H_*(X) \arrow[r, "\nu"] \arrow[d, "\nu"'] & A_* \otimes H_*(X) \arrow[d, "\psi\otimes 1"] \\
A_* \otimes H_*(X) \arrow[r, "1\otimes\nu"] & A_* \otimes A_* \otimes H_*(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_*(X) \otimes H_*(X) \arrow[r, "\nu\otimes\nu"] & A_* \otimes H_*(X) \otimes A_* \otimes H_*(X) \arrow[d, "1\otimes\tau\otimes 1"] \\
H_*(X) \arrow[u, "\Delta_*"] \arrow[d, "\nu"'] & A_* \otimes A_* \otimes H_*(X) \otimes H_*(X) \arrow[d, "\phi\otimes 1\otimes 1"] \\
A_* \otimes H_*(X) \arrow[r, "1\otimes\Delta_*"] & A_* \otimes H_*(X) \otimes H_*(X)
\end{tikzcd}$$

commute. Hence $H_*(X)$ is naturally a left A_*-comodule coalgebra. Furthermore, $H_*(X)$ is cocommutative, in the graded sense.

EXAMPLE 7.8.8. A dualized Cartan formula tells us that the homology cross product pairing

$$\times\colon H_*(X) \otimes H_*(Y) \xrightarrow{\cong} H_*(X \times Y)$$

is A_*-colinear, where A_* coacts diagonally on the left and by the standard coaction on the right. This shows that the diagonal A_*-coaction on the tensor product models the Cartesian product of spaces, to the eyes of mod p homology. Likewise, it models the smash product of spaces to the eyes of reduced mod p homology.

Third, we can give $H_*(X)$ the structure of a right A-module, with action

$$\rho\colon H_*(X) \otimes A \longrightarrow H_*(X)$$

taking $\xi \in H_n(X)$ and $\theta \in A^k$ to $\rho(\xi \otimes \theta) = \xi \cdot \theta \in H_{n-k}(X)$. Here $\xi \cdot \theta$ is characterized by the condition

$$(-1)^{|\theta|}\langle \theta \cdot x, \xi\rangle = \langle x, \xi \cdot \theta\rangle$$

for each $x \in H^*(X)$, where $\theta \cdot x = \lambda(\theta \otimes x) = \theta(x)$. In other words,

$$\begin{aligned} \theta\cdot\,\colon H^*(X) &\longrightarrow H^*(X) \\ x &\mapsto \theta \cdot x \end{aligned}$$

corresponds to the dual of the homomorphism

$$\begin{aligned} \cdot\,\theta \colon H_*(X) &\longrightarrow H_*(X) \\ \xi &\longmapsto \xi \cdot \theta \end{aligned}$$

under the identification $H^*(X) \cong \mathrm{Hom}(H_*(X), \mathbb{F}_p)$, with appropriate signs. ((ETC: This is the sign convention from [**Ada69**, p. 76].)) It is traditional to write

$$\begin{aligned} Sq^I_*(\xi) &= \xi \cdot Sq^I \\ P^I_*(\xi) &= \xi \cdot P^I \end{aligned}$$

for these right actions, but one should beware that this means that

$$\begin{aligned} Sq^J_* Sq^I_* &= Sq^{IJ}_* \\ P^J_* P^I_* &= P^{IJ}_* \,, \end{aligned}$$

where IJ denotes the concatenation of I and J. Direct calculation ((ETC: Maybe spell this out?)) then shows that the diagrams

$$\begin{tikzcd} H_*(X) \otimes A \otimes A \arrow[r, "\rho\otimes 1"] \arrow[d, "1\otimes\phi"'] & H_*(X) \otimes A \arrow[d, "\rho"] \\ H_*(X) \otimes A \arrow[r, "\rho"] & H_*(X) \end{tikzcd}$$

and

$$\begin{tikzcd} H_*(X) \otimes A \arrow[rr, "\rho"] \arrow[d, "\Delta_*\otimes 1"'] & & H_*(X) \arrow[d, "\Delta_*"] \\ H_*(X) \otimes H_*(X) \otimes A \arrow[d, "1\otimes 1\otimes\psi"'] & & H_*(X) \otimes H_*(X) \\ H_*(X) \otimes H_*(X) \otimes A \otimes A \arrow[rr, "1\otimes\tau\otimes 1"] & & H_*(X) \otimes A \otimes H_*(X) \otimes A \arrow[u, "\rho\otimes\rho"'] \end{tikzcd}$$

commute, so that $H_*(X)$ is a (cocommutative) right A-module coalgebra.

Fourth, applying $\mathrm{Hom}(-, \mathbb{F}_p)$ to the right A-module action ρ defines a homomorphism

$$\mathrm{Hom}(\rho, 1) \colon \mathrm{Hom}(H_*(X), \mathbb{F}_p) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_p) \,.$$

The natural homomorphism

$$H^*(X) \otimes A_* \cong \mathrm{Hom}(H_*(X), \mathbb{F}_p) \otimes \mathrm{Hom}(A, \mathbb{F}_p) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_p)$$

is an isomorphism if $H^*(X)$ is bounded above, in which case the composite

$$H^*(X) \cong \mathrm{Hom}(H_*(X), \mathbb{F}_p) \longrightarrow \mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_p) \cong H^*(X) \otimes A_*$$

defines a natural right A_*-coaction

$$\lambda^* \colon H^*(X) \longrightarrow H^*(X) \otimes A_* \,.$$

(The notation λ^* is the one used by Milnor in [**Mil58**, §4].) In general, there is an isomorphism

$$\mathrm{Hom}(H_*(X) \otimes A, \mathbb{F}_p) \cong H^*(X) \mathbin{\widehat{\otimes}} A_* \,,$$

where the right hand side denotes the *completed tensor product* with

$$\prod_n H^{n+k}(X) \otimes A_n$$

in cohomological degree k. We then have a completed right A_*-coaction

$$\lambda^* \colon H^*(X) \longrightarrow H^*(X) \mathbin{\widehat{\otimes}} A_* .$$

The diagrams

$$\begin{tikzcd}
H^*(X) \arrow[r, "\lambda^*"] \arrow[d, "\lambda^*"'] & H^*(X) \mathbin{\widehat{\otimes}} A_* \arrow[d, "1\otimes\psi"] \\
H^*(X) \mathbin{\widehat{\otimes}} A_* \arrow[r, "\lambda^*\otimes 1"] & H^*(X) \mathbin{\widehat{\otimes}} A_* \mathbin{\widehat{\otimes}} A_*
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^*(X) \mathbin{\widehat{\otimes}} A_* \mathbin{\widehat{\otimes}} H^*(X) \mathbin{\widehat{\otimes}} A_* \arrow[r, "1\otimes\tau\otimes 1"] & H^*(X) \mathbin{\widehat{\otimes}} H^*(X) \mathbin{\widehat{\otimes}} A_* \mathbin{\widehat{\otimes}} A_* \arrow[d, "1\otimes 1\otimes\phi"] \\
H^*(X) \mathbin{\widehat{\otimes}} H^*(X) \arrow[u, "\lambda^*\otimes\lambda^*"] \arrow[d, "\cup"'] & H^*(X) \mathbin{\widehat{\otimes}} H^*(X) \mathbin{\widehat{\otimes}} A_* \arrow[d, "\cup\otimes 1"] \\
H^*(X) \arrow[r, "\lambda^*"] & H^*(X) \mathbin{\widehat{\otimes}} A_*
\end{tikzcd}$$

commute. Hence λ^* is an algebra homomorphism, and $H^*(X)$ is a (commutative) completed right A_*-comodule algebra.

Recall the admissible sequences $M_i = (2^{i-1}, \dots, 4, 2, 1)$ for $i \ge 1$. We set $M_0 = ()$. Recall also that $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$ and

$$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[a] ,$$

with a in degree 1 corresponding to the universal class u_1 in mod 2 cohomology. We let $\alpha_j \in H_j(\mathbb{R}P^\infty; \mathbb{F}_2)$ be dual to a^j, so that $H_*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2\{\alpha_j \mid j \ge 0\}$.

Lemma 7.8.9.

$$Sq^I(a) = \begin{cases} a^{2^i} & \text{if } I = M_i,\ i \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

for I admissible.

Proof. This follows by induction on the length of I, using the formula

$$Sq^k(a^{2^i}) = \binom{2^i}{k} a^{k+2^i} = \begin{cases} a^{2^i} & \text{for } k = 0, \\ a^{2^{i+1}} & \text{for } k = 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

□

Definition 7.8.10. For $i \ge 1$ let the *Milnor generator*

$$\xi_i \in A_{2^i - 1}$$

be characterized by

$$\langle Sq^I, \xi_i \rangle = \begin{cases} 1 & \text{for } I = M_i, \\ 0 & \text{otherwise,} \end{cases}$$

for each admissible I of degree $2^i - 1$. Furthermore, let $\xi_0 = 1$.

REMARK 7.8.11. Milnor actually writes ζ_i for this class in A_{2^i-1}. Other authors instead write ζ_i for the conjugate $\chi(\xi_i)$ of this class, which can be confusing. Another notation for the conjugate is $\bar{\xi}_i$.

LEMMA 7.8.12. *The homomorphism*

$$\tilde{H}_j(\mathbb{R}P^\infty;\mathbb{F}_2) \longrightarrow \operatorname*{colim}_n \tilde{H}_{j-1+n}(K(\mathbb{Z}/2,n);\mathbb{F}_2) \cong A_{j-1}$$

with Hom-*dual*

$$A^{j-1} \cong \lim_n \tilde{H}^{j-1+n}(K(\mathbb{Z}/2,n);\mathbb{F}_2) \longrightarrow \tilde{H}^j(\mathbb{R}P^\infty;\mathbb{F}_2)$$

is given by

$$\alpha_j \longmapsto \begin{cases} \xi_i & \text{for } j = 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The homomorphism

$$\begin{aligned} A^{j-1} &\longrightarrow \tilde{H}^j(\mathbb{R}P^\infty;\mathbb{F}_2) \\ \theta &\longmapsto \theta(a) \end{aligned}$$

maps Sq^{M_i} to a^j for $i \geq 0$ and $j = 2^i$ and sends the remaining admissible Sq^I to zero. Hence the dual homomorphism $\tilde{H}_j(\mathbb{R}P^\infty;\mathbb{F}_2) \to A_{j-1}$ maps α_j to ξ_i for $j = 2^i$ with $i \geq 0$, and to zero for the remaining j. □

Since A is cocommutative, A_* is a commutative $\mathbb{F}_2$-algebra, and in fact it has a particularly simple structure.

THEOREM 7.8.13 ([**Mil58**, Thm. 2, App. 1]). *There is an algebra isomorphism*

$$A_* \cong \mathbb{F}_2[\xi_i \mid i \geq 1]\,,$$

with $|\xi_i| = 2^i - 1$.

SKETCH PROOF. The monomials

$$\xi^R = \xi_1^{r_1}\xi_2^{r_2} \cdot \dots \cdot \xi_\ell^{r_\ell}$$

where $R = (r_1, r_2, \dots, r_\ell, 0, \dots)$ ranges over all finite length sequences of non-negative integers, form a basis for $\mathbb{F}_2[\xi_i \mid i \geq 1]$, which maps to A_*. Milnor checks [**Mil58**, Lem. 8] that in each degree n, a matrix with entries

$$\langle Sq^I, \xi^R\rangle \in \mathbb{F}_2$$

is lower triangular with no zeros on the diagonal, hence is invertible, where I ranges over the admissible sequences of degree n and R ranges over the sequences of degree $\sum_i (2^i - 1) r_i$ equal to n. Since these Sq^I form a basis for A^n, it follows that these monomials ξ^R form a basis for A_n. □

PROPOSITION 7.8.14.

$$\lambda^*(a) = \sum_{i\geq 0} a^{2^i} \otimes \xi_i$$

in $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \mathbin{\widehat{\otimes}} A_*$.

PROOF. The right A-module action

$$H_j(\mathbb{R}P^\infty;\mathbb{F}_2)\otimes A^{j-1}\longrightarrow H_1(\mathbb{R}P^\infty;\mathbb{F}_2)$$

is zero unless $j=2^i$, in which case

$$\rho(\alpha_{2^i}\otimes Sq^I)=\begin{cases}\alpha_1 & \text{if } I=M_i,\\ 0 & \text{otherwise,}\end{cases}$$

for I admissible of degree 2^i-1. Dually, the right A^*-coaction

$$H^1(\mathbb{R}P^\infty;\mathbb{F}_2)\longrightarrow H^j(\mathbb{R}P^\infty;\mathbb{F}_2)\otimes A_{j-1}$$

is zero unless $j=2^i$, in which case it maps a to $a^{2^i}\otimes\xi_i$. Collecting terms for all j, we obtain the stated formula for $\lambda^*(a)$. □

Since A is non-commutative, A_* is not cocommutative. The coproduct for A_* encodes much the same information as the Adem relations do for A, but the following formula is often easier to work with for theoretical purposes.

THEOREM 7.8.15 ([**Mil58**, Thm. 3, App. 1]). *The coproduct $\psi\colon A_*\to A_*\otimes A_*$ is given by*

$$\psi(\xi_k)=\sum_{i+j=k}\xi_i^{2^j}\otimes\xi_j\,,$$

where $\xi_0=1$.

PROOF. The completed right A_*-coaction λ^* is multiplicative, hence satisfies

$$\lambda^*(a^{2^j})=\lambda^*(a)^{2^j}=\Big(\sum_{i\ge0}a^{2^i}\otimes\xi_i\Big)^{2^j}=\sum_{i\ge0}a^{2^{i+j}}\otimes\xi_i^{2^j}\,.$$

It is also coassociative, so that

$$\begin{aligned}(\lambda^*\otimes1)(\lambda^*(a))&=(\lambda^*\otimes1)(\sum_{j\ge0}a^{2^j}\otimes\xi_j)\\&=\sum_{j\ge0}\lambda^*(a^{2^j})\otimes\xi_j=\sum_{i\ge0}\sum_{j\ge0}a^{2^{i+j}}\otimes\xi_i^{2^j}\otimes\xi_j\end{aligned}$$

is equal to

$$(1\otimes\psi)(\lambda^*(a))=(1\otimes\psi)(\sum_{k\ge0}a^{2^k}\otimes\xi_k)=\sum_{k\ge0}a^{2^k}\otimes\psi(\xi_k)$$

as an element in $H^*(\mathbb{R}P^\infty;\mathbb{F}_2)\,\widehat{\otimes}\,A_*\,\widehat{\otimes}\,A_*$. Comparing coefficients of a^{2^k} gives the stated formula for $\psi(\xi_k)$, for each $k\ge0$. □

((ETC: The indecomposable quotient $Q(A)=\mathbb{F}_2\{Sq^{2^i}\mid i\ge0\}$ is dual to the primitives $P(A_*)=\mathbb{F}_2\{\xi_1^{2^i}\mid i\ge0\}$. Furthermore, the indecomposable quotient $Q(A_*)=\mathbb{F}_2\{\xi_i\mid i\ge1\}$ is dual to the primitives $P(A)=\mathbb{F}_2\{Q_j\mid j\ge0\}$, with Q_j in degree $2^{j+1}-1$ dual to ξ_{j+1}. Here $Q_0=\beta$ and $Q_j=[Sq^{2^j},Q_{j-1}]=Sq^{2^j}Q_{j-1}+Q_{j-1}Sq^{2^j}$ for $j\ge1$.))

For odd primes p, we set

$$P^{M_i}=P^{p^{i-1}}P^{p^{i-2}}\dots P^pP^1$$

for $i \geq 1$, and $P^{M_0} = 1$. There is an equivalence $K(\mathbb{Z}/p, 1) \simeq BC_p = S^\infty/C_p$, with

$$H^*(BC_p; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(a) \otimes \mathbb{F}_p[b]\,,$$

with a in degree 1 corresponding to the universal class u_1, and $b = \beta(a)$ in degree 2. Let α_{2j} and α_{2j+1} be dual to b^j and ab^j, respectively, so that $H_*(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p\{\alpha_j \mid j \geq 0\}$.

There is also a map $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2) \to K(\mathbb{Z}/p, 2)$ inducing the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/p$ on π_2, and

$$H^*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[b]$$

with b in degree 2 corresponding to the reduced universal class $\bar{u}_2$. Let β_j be dual to b^j, in degree $2j$, so that $H_*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p\{\beta_j \mid j \geq 0\}$.

Lemma 7.8.16.

$$P^I(a) = \begin{cases} a & \textit{for } P^I = 1, \\ b^{p^i} & \textit{for } P^I = P^{M_i}\beta \textit{ with } i \geq 0, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$P^I(b) = \begin{cases} b^{p^i} & \textit{for } P^I = P^{M_i} \textit{ with } i \geq 0, \\ 0 & \textit{otherwise,} \end{cases}$$

for I admissible.

Definition 7.8.17. For $i \geq 0$ let the Milnor generator

$$\tau_i \in A_{2p^i-1}$$

be characterized by

$$\langle P^I, \tau_i \rangle = \begin{cases} 1 & \text{for } P^I = P^{M_i}\beta, \\ 0 & \text{otherwise,} \end{cases}$$

for each admissible I of degree $2p^i - 1$. In particular, $\tau_0 = \beta$.

For $i \geq 1$ let the Milnor generator

$$\xi_i \in A_{2p^i-2}$$

be characterized by

$$\langle P^I, \xi_i \rangle = \begin{cases} 1 & \text{for } P^I = P^{M_i}, \\ 0 & \text{otherwise,} \end{cases}$$

for each admissible I of degree $2p^i - 2$. Furthermore, let $\xi_0 = 1$.

((ETC: Is $\langle \beta, \beta \rangle = 1$ with the standard conventions?))

Lemma 7.8.18. *The homomorphism*

$$\tilde{H}_j(BC_p; \mathbb{F}_p) \longrightarrow \operatorname*{colim}_n \tilde{H}_{j-1+n}(K(\mathbb{Z}/p, n); \mathbb{F}_p) \cong A_{j-1}$$

with Hom-*dual*

$$A^{j-1} \cong \lim_n \tilde{H}^{j-1+n}(K(\mathbb{Z}/p, n); \mathbb{F}_p) \longrightarrow \tilde{H}^j(BC_p; \mathbb{F}_p)$$

is given by

$$\alpha_j \longmapsto \begin{cases} \tau_i & \textit{for } j = 2p^i, \\ 0 & \textit{otherwise.} \end{cases}$$

The homomorphism

$$\tilde{H}_{2j}(\mathbb{C}P^\infty;\mathbb{F}_p) \to \tilde{H}_{2j}(K(\mathbb{Z}/p,2);\mathbb{F}_p) \longrightarrow \operatorname*{colim}_n \tilde{H}_{2j-2+n}(K(\mathbb{Z}/p,n);\mathbb{F}_p) \cong A_{2j-2}$$

with Hom-*dual*

$$A^{2j-2} \cong \lim_n \tilde{H}^{2j-2+n}(K(\mathbb{Z}/p,n);\mathbb{F}_p) \longrightarrow \tilde{H}^{2j}(K(\mathbb{Z}/p,2);\mathbb{F}_p) \to \tilde{H}^{2j}(\mathbb{C}P^\infty;\mathbb{F}_p)$$

is given by

$$\beta_j \longmapsto \begin{cases} \xi_i & \text{for } j = p^i, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.8.19 ([**Mil58**, Thm. 2]). *There is an algebra isomorphism*

$$A_* \cong \Lambda_{\mathbb{F}_p}(\tau_i \mid i \geq 0) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]\,,$$

with $|\tau_i| = 2p^i - 1$ *and* $|\xi_i| = 2p^i - 2$.

Proposition 7.8.20.

$$\lambda^*(a) = a \otimes 1 + \sum_{i\geq 0} b^{p^i} \otimes \tau_i$$

$$\lambda^*(b) = \sum_{i\geq 0} b^{p^i} \otimes \xi_i$$

in $H^*(BC_p;\mathbb{F}_p)\,\widehat{\otimes}\,A_*$.

Theorem 7.8.21 ([**Mil58**, Thm. 3]). *The coproduct* $\psi\colon A_* \to A_* \otimes A_*$ *is given by*

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j\,.$$

((ETC: The indecomposable quotient $Q(A) = \mathbb{F}_p\{\beta, P^{p^i} \mid i \geq 0\}$ is dual to the primitives $P(A_*) = \mathbb{F}_p\{\tau_0, \xi_1^{p^i} \mid i \geq 0\}$. Furthermore, the indecomposable quotient $Q(A_*) = \mathbb{F}_p\{\tau_0, \tau_i, \xi_i \mid i \geq 1\}$ is dual to the primitives $P(A) = \mathbb{F}_p\{\beta, Q_i, P_i^0 \mid i \geq 1\}$, with Q_i in degree $2p^i - 1$ dual to τ_i and P_i^0 in degree $2p^2 - 2$ dual to ξ_i. Here $Q_0 = \beta$ and $Q_{i+1} = [P^{p^i}, Q_i] = P^{p^i} Q_i - Q_i P^{p^i}$ for $i \geq 0$.))

((ETC: Milnor basis. Define P_t^s as dual to $\xi_t^{p^s}$?))

CHAPTER 8

Convergence (TO BE WRITTEN)

8.1. Algebraic limits and colimits

((ETC: Sequential colim, lim and Rlim. Six-term exact lim-Rlim sequence. Vanishing of Rlim. Mittag–Leffler condition? Pro-isomorphism?))

8.2. Filtrations, revisited

8.3. Strong convergence

8.4. Conditional convergence

8.5. The Bockstein spectral sequence

((Browder. Torsion, localization, completion.))

8.6. Complex orientations

((ETC: Multiplicative Atiyah–Hirzebruch spectral sequence.))

CHAPTER 9

Stable Homotopy Theory

9.1. Smooth bordism and stable homotopy groups

Lev Pontryagin [**Pon50**] (and earlier?) and René Thom [**Tho54**] developed the close connection between the bordism classification of manifolds and the stable range homotopy groups of certain spaces. See also [**MS74**, §17,§18], [**Sto68**] and [**Rud98**].

9.1.1. Transversality. We can view the k-sphere S^k as the one-point compactification $\mathbb{R}^k \cup \{\infty\}$, based at infinity, or as the quotient space $D^k/\partial D^k$, based at the image of the boundary. Any map $f\colon S^{n+k} \to S^k$ is homotopic to a smooth map with $0 \in \mathbb{R}^k \subset S^k$ as a regular value, i.e., a map that is transverse to 0, and the preimage $M = f^{-1}(0)$ is then a closed smooth n-dimensional submanifold of $\mathbb{R}^{n+k} \subset S^{n+k}$. The stabilization $f \wedge S^1 = f \wedge 1\colon S^{n+k} \wedge S^1 \to S^k \wedge S^1$ then has the same preimage

$$(f \wedge 1)^{-1}(0) \cong f^{-1}(0)\,,$$

but is now realized as a submanifold of $\mathbb{R}^{n+k+1}$. If $F\colon I_+ \wedge S^{n+k} \to S^k$ is a homotopy from f_0 to f_1, with both f_0 and f_1 transverse to 0, then F can be deformed relative to $\partial I_+ \wedge S^{n+k}$ to a smooth map that is transverse to 0. The preimage $W = F^{-1}(0)$ is then a compact smooth $(n+1)$-dimensional submanifold of $I \times \mathbb{R}^{n+k} \subset I_+ \wedge S^{n+k}$, with boundary

$$\partial W \cong M_0 \coprod M_1\,.$$

We call W a *bordism* from $M_0 = f_0^{-1}(0)$ to $M_1 = f_1^{-1}(0)$, and say that M_0 and M_1 are *cobordant*. This defines an equivalence relation, and we write $[M]$ for the bordism class of M. The set of all bordism classes of closed (always smooth) n-manifolds is denoted $\mathscr{N}_n$.

Lemma 9.1.1. *The rule $[f] \mapsto [M]$ with $M = f^{-1}(0)$ defines a homomorphism of graded (commutative) rings*

$$\pi_*(S) \longrightarrow \mathscr{N}_*\,.$$

Proof. We have seen that $f \mapsto f^{-1}(0)$ defines a function $\pi_{n+k}(S^k) \to \mathscr{N}_n$ that is compatible with stabilization, hence factors uniquely through the stable homotopy group

$$\pi_n(S) = \operatorname*{colim}_k \pi_{n+k}(S^k)\,.$$

The disjoint union of manifolds defines a sum

$$+\colon \mathscr{N}_n \times \mathscr{N}_n \longrightarrow \mathscr{N}_n$$

and the Cartesian product of manifolds defines a product

$$\cdot\colon \mathscr{N}_n \times \mathscr{N}_m \longrightarrow \mathscr{N}_{n+m}\,,$$

making $\mathscr{N}_* = (\mathscr{N}_n)_n$ a graded commutative $\mathbb{F}_2$-algebra, known as the *unoriented bordism ring.*

The sum in $\pi_{n+k}(S^k)$ takes $[f]$ and $[g]$ to the class of

$$f + g \colon S^{n+k} \longrightarrow S^{n+k} \vee S^{n+k} \overset{f\vee g}{\longrightarrow} S^k ,$$

so that $(f+g)^{-1}(0) \cong f^{-1}(0) \coprod g^{-1}(0)$. The smash product of $f \colon S^{n+k} \to S^k$ and $g \colon S^{m+\ell} \to S^\ell$ defines a map

$$S^{n+m+k+\ell} \cong S^{n+k} \wedge S^{m+\ell} \overset{f\wedge g}{\longrightarrow} S^k \wedge S^\ell = S^{k+\ell} ,$$

with $(f \wedge g)^{-1}(0) \cong f^{-1}(0) \times g^{-1}(0)$. It follows that the sum and product in $\pi_*(S)$ are mapped to the sum and product in $\mathscr{N}_*$. □

To be useful, this ring homomorphism must be refined, by either restricting the manifolds $M \subset \mathbb{R}^{n+k}$ studied to account for special structure on their normal bundles, which arises from their construction as transverse preimages, or by extending the targets of the maps $f \colon S^{n+k} \to S^k$ to allow for more general normal bundles, or both.

9.1.2. Framed bordism. A smooth embedding $M \subset \mathbb{R}^{n+k}$ induces an embedding of the tangent bundle $\tau \colon TM \to M$ into the trivial bundle $\epsilon^{n+k} \colon M \times \mathbb{R}^{n+k} \to M$, with normal complement the *normal bundle* $\nu \colon NM \to M$. For each $x \in M$, the fiber $N_xM \subset \mathbb{R}^{n+k}$ is the orthogonal complement of $T_xM \subset \mathbb{R}^{n+k}$.

If $M = f^{-1}(0)$ is the preimage of the regular value $0 \in \mathbb{R}^k \subset S^k$, then the derivative $f_* \colon T\mathbb{R}^{n+k}|_M \to T\mathbb{R}^k|_0$ of f along M induces a bundle isomorphism

$$\theta \colon NM \overset{\cong}{\longrightarrow} M \times \mathbb{R}^k .$$

This is a trivialization, or *framing*, of the normal bundle of M. If we replace f with $f \wedge 1$, then the normal bundle of $M \subset \mathbb{R}^{n+k+1}$ is $\nu \oplus \epsilon^1 \colon NM \times \mathbb{R} \to M$, with trivialization $\theta \times \mathbb{R} \colon NM \times \mathbb{R} \cong M \times \mathbb{R}^{k+1}$. We say that θ and $\theta \times \mathbb{R}$ define the same *stable framing*, and that (M, θ) is *stably framed.*

If $F \colon I_+ \wedge S^{n+k} \to S^k$ is a smooth homotopy from f_0 to f_1, all of which are transverse to 0, then the derivative F_* of F along the compact $(n+1)$-manifold $W = F^{-1}(0) \subset I \times \mathbb{R}^{n+k}$ induces a trivialization

$$\Theta \colon NW \overset{\cong}{\longrightarrow} W \times \mathbb{R}^k$$

that restricts to the trivializations θ_0 and θ_1 of the normal bundles of $M_0 = f_0^{-1}(0) \subset \mathbb{R}^{n+k}$ and $M_1 = f_1^{-1}(0) \subset \mathbb{R}^{n+k}$, respectively. We say that M_0 and M_1 are *stably framed cobordant.* This defines an equivalence relation, and we write Ω^{fr}_n for the set of all stably framed bordism classes of stably framed closed n-manifolds.

THEOREM 9.1.2 ([**Pon50**] ((ETC: earlier?))). *The rule* $[f] \mapsto [(M, \theta)]$ *with* $M = f^{-1}(0)$ *and* $\theta \colon NM \cong M \times \mathbb{R}^k$ *defines an isomorphism of graded commutative rings*

$$\pi_*(S) \overset{\cong}{\longrightarrow} \Omega^{fr}_* .$$

SKETCH PROOF. To construct the inverse, consider a stably framed, closed n-manifold M. There exists an embedding $M \subset \mathbb{R}^{n+k}$, with a trivialization $\theta \colon NM \cong M \times \mathbb{R}^k$, and any two such become isotopic if we enlarge k. Choosing a Euclidean metric, we get a homeomorphism

$$D(\theta) \colon D(NM) \overset{\cong}{\longrightarrow} M \times D^k$$

of unit disc bundles over M. Let $S(\theta)\colon S(NM) \cong M \times \partial D^k$ denote its restriction to the unit sphere bundles. We can view M as a subspace of $D(NM)$ by the zero section. By the tubular neighborhood theorem there is an embedding $D(NM) \subset \mathbb{R}^{n+k}$ that extends the inclusion $M \subset \mathbb{R}^{n+k}$, such that the open disc bundle $D(NM) - S(NM) = \operatorname{int} D(NM) \subset \mathbb{R}^{n+k}$ is an open neighborhood of M. We can then form the composite map

$$f\colon S^{n+k} \longrightarrow \frac{S^{n+k}}{S^{n+k} - \operatorname{int} D(NM)} \xleftarrow{\cong} \frac{D(NM)}{S(NM)} \xrightarrow{\cong} \frac{M \times D^k}{M \times \partial D^k} \longrightarrow \frac{D^k}{\partial D^k} \cong S^k\,.$$

It has $0 \in D^k \to S^k$ as a regular value, with preimage $f^{-1}(0) \cong M \times \{0\} \cong M$, which is normally framed by θ. Hence the stable class of $[f] \in \pi_{n+k}(S^k)$ in $\pi_n(S)$ maps to the stably framed bordism class of (M, θ), and these are mutually inverse correspondences. □

Pontryagin used this construction, and the classification of stably framed closed surfaces, to prove that $\pi_2(S) \cong \mathbb{Z}/2$, generated by the stable class η^2 of the composite

$$\eta \circ E\eta\colon S^4 \longrightarrow S^3 \longrightarrow S^2\,.$$

This $\mathbb{Z}/2$ detects the Arf invariant of a quadratic form that refines the bilinear intersection form on $H_1(-;\mathbb{F}_2)$ of the framed surface. In particular, not every framed closed surface is framed cobordant to a sphere. Pontryagin thereby rectified an earlier mistake he had made (in 1938) concerning this problem.

Similar work shows that the stable homotopy classes $\nu^2 \in \pi_6(S)$ and $\sigma^2 \in \pi_{14}(S)$, where ν and σ are the stable classes of the Hopf fibrations $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$, correspond to 6- and 14-dimensional framed manifolds, respectively, that are not framed cobordant to homotopy spheres. Work by Kervaire–Milnor [**KM63**] addressed the question whether each framed n-manifold can be modified, by a process now called "surgery", so as to be framed cobordant to a homotopy sphere. This is can always be done unless $n = 4m - 2$, in which case there is a possible obstruction in $\mathbb{Z}/2$, known as the Kervaire invariant of the framed bordism class, given by the Arf invariant of a quadratic form on the middle homology $H_{2m-1}(-;\mathbb{F}_2)$ of the manifold. Browder [**Bro69**] showed that the Kervaire invariant vanishes for each n not of the form $2(2^j - 1)$. The *Kervaire invariant one* problem then asks: For which $n = 2(2^j - 1)$ does there exist a class $\theta_j \in \pi_n(S) \cong \Omega_n^{fr}$ with nontrivial Arf–Kervaire invariant? The squared Hopf fibration examples show that such classes exists for $j \in \{1, 2, 3\}$. Mahowald–Tangora [**MT67**] showed that $\theta_4 \in \pi_{30}(S)$ exists, and Barratt–Jones–Mahowald [**BJM84**] proved that $\theta_5 \in \pi_{62}(S)$ exists, by hard calculations with the mod 2 Adams spectral sequence for the sphere spectrum. The next problem, concerning the existence of $\theta_6 \in \pi_{126}(S)$ lies outside our current computational range. It was a great surprise when Hopkins–Hill–Ravenel [**HHR16**] proved, using an equivariant form of complex bordism, that θ_j does not exist for any $j \geq 7$. The case $j = 6$ remains open.

9.1.3. Unoriented bordism. For a general smooth embedding $M \subset \mathbb{R}^{n+k}$ there need not exist a (stable) trivialization θ of the normal bundle $\nu\colon NM \to M$. However, there exists a Gauss map

$$g\colon M \longrightarrow Gr_k(\mathbb{R}^{n+k}) \subset Gr_k(\mathbb{R}^\infty) \simeq BO(k)$$

to the Grassmann manifold of k-dimensional real subspaces of $\mathbb{R}^{n+k}$, given by $g(x) = N_xM \subset \mathbb{R}^{n+k}$ for all $x \in M$.

By including $\mathbb{R}^{n+k}$ in $\mathbb{R}^\infty \oplus \mathbb{R}^k \cong \mathbb{R}^\infty$ we can continue this map to the Grassmannian of k-dimensional subspaces of $\mathbb{R}^\infty$, which is a classifying space for principal $O(k)$-bundles. The universal principal $O(k)$-bundle

$$O(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow Gr_k(\mathbb{R}^\infty)\,,$$

where $V_k(\mathbb{R}^\infty)$ is the contractible Stiefel space of orthonormal k-frames in $\mathbb{R}^\infty$, has an associated "tautological" $\mathbb{R}^k$-bundle $\gamma^k \colon E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$, whose fiber over $V \in Gr_k(\mathbb{R}^\infty)$ is the k-dimensional vector space $V \subset \mathbb{R}^\infty$.

((ETC: Slightly better to let $V_k(\mathbb{R}^\infty)$ and $Gr_k(\mathbb{R}^\infty)$ be the spaces of k-frames and k-dimensional subspaces of $\mathbb{R}^\infty \oplus \mathbb{R}^k$, rather than of $\mathbb{R}^\infty$, with stabilizations $V_k(\mathbb{R}^\infty) \to V_{k+1}(\mathbb{R}^\infty)$ and $Gr_k(\mathbb{R}^\infty) \to Gr_{k+1}(\mathbb{R}^\infty)$ sending $(v_1, \dots, v_k)$ and V to $(v_1, \dots, v_k, e_{k+1})$ and $V \oplus \mathbb{R}$, respectively.))

The identity maps on the N_xM, for $x \in M$, define a bundle map

$$\begin{tikzcd}
NM \arrow[r, "\hat{g}"] \arrow[d] & E(\gamma^k) \arrow[d] \\
M \arrow[r, "g"] & Gr_k(\mathbb{R}^\infty)
\end{tikzcd}$$

covering the Gauss map. Equivalently, there is an isomorphism $\nu \cong g^*(\gamma^k)$, expressing the normal bundle of M as the pullback along g of the tautological bundle over $Gr_k(\mathbb{R}^\infty)$.

Definition 9.1.3. For a Euclidean vector bundle $\xi \colon E(\xi) \to B$, with unit disc bundle $D(\xi) \to B$ and unit sphere bundle $S(\xi) \to B$, let the *Thom complex* be the quotient space

$$Th(\xi) = D(\xi)/S(\xi)\,.$$

In particular, let $Th(\gamma^k)$ denote the Thom complex of the tautological $\mathbb{R}^k$-bundle $\gamma^k \colon E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$.

((ETC: Another common notation for the Thom complex is $M(\xi)$.))

If ξ is associated to the principal $O(k)$-bundle $p \colon P \to B$, then

$$E(\xi) \cong P \times_{O(k)} \mathbb{R}^k\,,$$

so that

$$Th(\xi) \cong \frac{P \times_{O(k)} D^k}{P \times_{O(k)} \partial D^k} \cong P_+ \wedge_{O(k)} D^k/\partial D^k \cong P_+ \wedge_{O(k)} S^k\,.$$

In particular, $Th(\gamma^k) \simeq MO(k) = EO(k)_+ \wedge_{O(k)} S^k$. If B is a compact Hausdorff space, then $Th(\xi) \cong E(\xi) \cup \{\infty\}$ can be characterized as the one-point compactification of the total space $E(\xi)$. In general, $Th(\xi)$ is the quotient of the fiberwise one-point compactification $P \times_{O(k)} S^k$ of $E(\xi)$ by the section $P \times_{O(k)} \{\infty\} \cong B$ at infinity.

Lemma 9.1.4. *The Thom complex is functorial, and there is a natural homeomorphism $Th(\xi \oplus \epsilon^1) \cong Th(\xi) \wedge S^1$.*

PROOF. A bundle map $\xi \to \eta$ induces maps $D(\xi) \to D(\eta)$, $S(\xi) \to S(\eta)$ and $Th(\xi) \to Th(\eta)$, so the Thom complex is functorial.

The Whitney sum bundle $\xi \oplus \epsilon^1$ has total space $E(\xi \oplus \epsilon^1) \cong E(\xi) \times \mathbb{R}$, so $D(\xi \oplus \epsilon^1) \cong D(\xi) \times D^1$ and $S(\xi \oplus \epsilon^1) \cong S(\xi) \times D^1 \cup D(\xi) \times \partial D^1$. Hence

$$Th(\xi \oplus \epsilon^1) \cong \frac{D(\xi) \times D^1}{S(\xi) \times D^1 \cup D(\xi) \times \partial D^1} \cong Th(\xi) \wedge S^1 .$$

□

Returning to the context of the normal bundle $NM \to M$ and the Gauss map $g\colon M \to Gr_k(\mathbb{R}^\infty)$, we can now use the bundle map $\hat{g}\colon NM \to E(\gamma^k)$ to form the *Pontryagin–Thom construction*

$$f\colon S^{n+k} \longrightarrow \frac{S^{n+k}}{S^{n+k} - \operatorname{int} D(NM)} \xleftarrow{\cong} \frac{D(NM)}{S(NM)} = Th(\nu) \xrightarrow{\hat{g}} Th(\gamma^k) \simeq MO(k) ,$$

representing a homotopy class

$$[f] \in \pi_{n+k}(Th(\gamma^k)) \cong \pi_{n+k}(MO(k)) .$$

In general, two embeddings $M \to \mathbb{R}^{n+k}$ and $M \to \mathbb{R}^{n+\ell}$ become isotopic if we increase k and ℓ to a sufficiently large common value, and isotopic embeddings induce homotopic Pontryagin–Thom maps f. Furthermore, replacing $M \subset \mathbb{R}^{n+k}$ with $M \subset \mathbb{R}^{n+k+1}$ has the effect of replacing $f\colon S^{n+k} \to Th(\gamma^k)$ with the composite

$$S^{n+k+1} \cong S^{n+k} \wedge S^1 \xrightarrow{f \wedge 1} Th(\gamma^k) \wedge S^1 \xrightarrow{\sigma} Th(\gamma^{k+1}) .$$

Here

$$\sigma\colon Th(\gamma^k) \wedge S^1 \cong Th(\gamma^k \oplus \epsilon^1) \longrightarrow Th(\gamma^{k+1})$$

is the map of Thom complexes induced by the bundle map

$$\begin{array}{ccc} E(\gamma^k \oplus \epsilon^1) & \longrightarrow & E(\gamma^{k+1}) \\ \downarrow & & \downarrow \\ Gr_k(\mathbb{R}^\infty) & \rightarrowtail & Gr_{k+1}(\mathbb{R}^\infty) \end{array}$$

covering the inclusion taking $V \subset \mathbb{R}^\infty$ to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$. Hence, to the closed n-manifold M we can associate a well-defined class in

$$\operatorname*{colim}_k \pi_{n+k}(Th(\gamma^k)) \cong \operatorname*{colim}_k \pi_{n+k}(MO(k)) = \pi_n(MO) .$$

Conversely, given $f\colon S^{n+k} \to Th(\gamma^k) \simeq MO(k)$ we can deform f to be transverse to the zero section $Gr_k(\mathbb{R}^\infty) \subset Th(\gamma^k)$, in which case the preimage

$$M = f^{-1}(Gr_k(\mathbb{R}^\infty))$$

is a smooth and closed submanifold of S^{n+k} of codimension k, i.e., a closed n-manifold.

THEOREM 9.1.5 ([**Tho54**, Thm. IV.8]). *The rule $[f] \mapsto [M]$ defines an isomorphism of graded commutative rings*

$$\pi_*(MO) \cong \mathscr{N}_* .$$

More generally, we are lead to study sequences of spaces $(M_k)_k$ and their homotopy groups $\pi_{n+k}(M_k)$ in a "stable" range of degrees n that grows to infinity with k. We can compare these groups for different k if we are given maps $\sigma\colon M_k \wedge S^1 \to M_{k+1}$, inducing homomorphisms

$$\pi_{n+k}(M_k) \xrightarrow{E} \pi_{n+k+1}(M_k \wedge S^1) \xrightarrow{\sigma_*} \pi_{n+k+1}(M_{k+1}) .$$

The stable range groups are then given by the sequential colimit

$$\pi_n(M) = \operatorname*{colim}_k \pi_{n+k}(M_k) .$$

These objects $M = (M_k, \sigma)_k$ are the (sequential) spectra of algebraic topology, and a key feature of stable homotopy theory is to view a spectrum M as an object that gives an undivided presentation of the sequence of abelian groups $\pi_*(M) = (\pi_n(M))_n$.

Proceeding from the theorem above, and knowledge of the cohomology

$$H^*(BO; \mathbb{F}_2) \cong \lim_k H^*(BO(k); \mathbb{F}_2) \cong \mathbb{F}_2[w_i \mid i \geq 1]$$

with $|w_i| = i$, as a module over the mod 2 Steenrod algebra A, Thom went on to calculate the cohomology $H^*(MO; \mathbb{F}_2)$ as an A-module, finding it to be free on specific generators. Using a trivial case of the Adams spectral sequence, this led to the following conclusion.

Theorem 9.1.6 ([**Tho54**, Thm. IV.12]).

$$\pi_*(MO) \cong \mathbb{F}_2[x_n \mid n \neq 2^i - 1] = \mathbb{F}_2[x_2, x_4, x_5, \dots]$$

is the graded polynomial ring over $\mathbb{F}_2$ on one generator x_n in each positive degree n not of the form $2^i - 1$.

Example 9.1.7. $\mathscr{N}_3 \cong \pi_3(MO) \cong 0$, so each closed 3-manifold is the boundary $M \cong \partial W$ of a compact 4-manifold.

9.1.4. Oriented bordism. One may consider other kinds of bordism, usually corresponding to conditions on the stable normal bundle that are intermediate between being trivialized (as for framed bordism) and satisfying no further requirements (as for unoriented bordism). ((ETC: It is also possible to consider bordism for more general topological manifolds, or piecewise-linear (PL) manifolds, in which case the normal vector bundles are replaced by the weaker notion of a *microbundle*, cf. [**Mil64**] and [**MM79**], and transversality is not as easily achieved as in the smooth case.))

In the case of an oriented (closed, smooth) manifold M, each tangent space T_xM comes with a choice of orientation, which determines an orientation of each normal space N_xM. Hence there is a Gauss map

$$g\colon M \longrightarrow \widetilde{Gr}_k(\mathbb{R}^{n+k}) \subset \widetilde{Gr}_k(\mathbb{R}^\infty) \simeq BSO(k)$$

to the oriented Grassmann manifold of oriented k-dimensional subspaces of $\mathbb{R}^{n+k}$, which is a double covering of $Gr_k(\mathbb{R}^{n+k})$. The universal principal $SO(k)$-bundle

$$SO(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow \widetilde{Gr}_k(\mathbb{R}^\infty)$$

shows that $\widetilde{Gr}_k(\mathbb{R}^\infty) \simeq BSO(k)$, and there is a tautological oriented $\mathbb{R}^k$-bundle $\tilde{\gamma}^k\colon E(\tilde{\gamma}^k) \to \widetilde{Gr}_k(\mathbb{R}^\infty)$ with Thom complex

$$Th(\tilde{\gamma}^k) \simeq ESO(k)_+ \wedge_{SO(k)} S^k = MSO(k) .$$

The pull-back of $\tilde{\gamma}^{k+1}$ along $\widetilde{Gr}_k(\mathbb{R}^\infty) \subset \widetilde{Gr}_{k+1}(\mathbb{R}^\infty)$ is $\tilde{\gamma}^k \oplus \epsilon^1$, so there is a map

$$\sigma \colon Th(\tilde{\gamma}^k) \wedge S^1 \longrightarrow Th(\tilde{\gamma}^{k+1}) .$$

To each oriented n-manifold M the Pontryagin–Thom construction

$$f \colon S^{n+k} \longrightarrow \frac{S^{n+k}}{S^{n+k} - \operatorname{int} D(NM)} \cong Th(\nu) \xrightarrow{\hat{g}} Th(\tilde{\gamma}^k) \simeq MSO(k)$$

then determines a well-defined class in

$$\operatorname*{colim}_k \pi_{n+k}(Th(\tilde{\gamma}^k)) \cong \operatorname*{colim}_k \pi_{n+k}(MSO(k)) = \pi_n(MSO) .$$

Let Ω_n denote the group of oriented bordism classes of oriented (compact, smooth) n-manifolds. Disjoint union and Cartesian product of oriented manifolds gives $\Omega_* = (\Omega_n)_n$ the structure of a graded commutative ring, called the *oriented bordism ring.*

THEOREM 9.1.8 ([**Tho54**, Thm. IV.8]). *The rule $[f] \mapsto [M]$ defines an isomorphism of graded commutative rings*

$$\pi_*(MSO) \cong \Omega_* .$$

From knowledge of the rational cohomology

$$H^*(BSO; \mathbb{Q}) \cong \lim_k H^*(BSO(k); \mathbb{Q}) \cong \mathbb{Q}[p_i \mid i \geq 1]$$

with $|p_i| = 4i$, and the dual rational homology algebra, Thom could calculate $\pi_*(MSO)$ rationally.

THEOREM 9.1.9 ([**Tho54**, Thm. IV.17]).

$$\pi_*(MSO) \otimes \mathbb{Q} \cong \mathbb{Q}[y_i \mid i \geq 1]$$

with $|y_i| = 4i$. One may take y_i to be the oriented bordism class of $\mathbb{C}P^{2i}$.

The integral structure of $\pi_*(MSO) \cong \Omega_*$ was determined by Wall [**Wal60**].

EXAMPLE 9.1.10. $\Omega_3 \cong \pi_3(MSO) \cong 0$, so each closed oriented 3-manifold is the boundary $M \cong \partial W$ of a compact oriented 4-manifold. This special case had previously been shown by Rohlin [**Roh51**].

9.1.5. Complex bordism.

9.1.6. Bordism (homology) theories. ((ETC: The Steenrod (realization) problem. Thom, Sullivan.))

9.2. Sequential spectra

Building on the work of Lima [**Lim58**] and Boardman [**Vog70**], Adams' Chicago lectures from 1971 [**Ada74**, Part III] gave a construction of the stable (homotopy) category as a (closed) symmetric monoidal category, based on an underlying category of sequential CW spectra without precise monoidal properties. Around 1995 several categories of spectra with closed symmetric monoidal properties were discovered. We focus on "orthogonal spectra", and use the paper [**MMSS01**] by Mandell, May, Schwede and Shipley to give a parallel development of sequential and orthogonal spectra.

We work in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces and basepoint preserving maps.

DEFINITION 9.2.1. A *sequential spectrum* M is a sequence of spaces M_k for $k \ge 0$ and structure maps

$$\sigma\colon M_k \wedge S^1 \longrightarrow M_{k+1} .$$

A *map of sequential spectra* $f\colon M \to N$ is a sequence of maps $f_k\colon M_k \to N_k$ such that the diagram

$$\begin{tikzcd} M_k \wedge S^1 \arrow[r, "\sigma"] \arrow[d, "f_k \wedge 1"'] & M_{k+1} \arrow[d, "f_{k+1}"] \\ N_k \wedge S^1 \arrow[r, "\sigma"] & N_{k+1} \end{tikzcd}$$

commutes, for each $k \ge 0$. Let $Sp^{\mathbb{N}}$ denote the category of sequential spectra.

The superscript $\mathbb{N}$ refers to the indexing by integers $k \ge 0$, which will be modified in the section on orthogonal spectra.

DEFINITION 9.2.2. The *graded homotopy groups* $\pi_*(M)$ of a sequential spectrum M are given in degree n by the colimit

$$\pi_n(M) = \operatorname*{colim}_k \pi_{n+k}(M_k)$$

of the sequence of homomorphisms

$$\dots \longrightarrow \pi_{n+k}(M_k) \longrightarrow \pi_{n+k+1}(M_{k+1}) \longrightarrow \dots ,$$

for $n+k \ge 2$, each mapping the homotopy class of $x\colon S^{n+k} \to M_k$ to the homotopy class of the composite

$$\sigma(x \wedge 1)\colon S^{n+k+1} \cong S^{n+k} \wedge S^1 \xrightarrow{x\wedge 1} M_k \wedge S^1 \xrightarrow{\sigma} M_{k+1} .$$

Any map $f\colon M \to N$ induces compatible homomorphisms $\pi_{n+k}(f_k)\colon \pi_{n+k}(M_k) \to \pi_{n+k}(N_k)$ with colimit $\pi_n(f)\colon \pi_n(M) \to \pi_n(N)$, for all integers n, making π_* a functor from sequential spectra to graded abelian groups.

We often write f_* for $\pi_n(f)$ or $\pi_*(f)$.

DEFINITION 9.2.3. A map $f\colon M \to N$ is a *stable equivalence* if the induced homomorphism $f_*\colon \pi_*(M) \to \pi_*(N)$ is an isomorphism. We may then write $f\colon M \xrightarrow{\sim} N$ or $M \sim N$. The stable equivalences form a subcategory $\mathscr{W} \subset Sp^{\mathbb{N}}$, which properly contains the homotopy equivalences.

((ETC: Might prefer to write $\simeq$ for stable equivalence, since homotopy equivalence is rarely relevant. An alternative is to go to the stable category and write $\cong$, but for some purposes it will be necessary to stay at the point-set spectrum level.))

DEFINITION 9.2.4. A *localization* $\mathscr{C}[\mathscr{W}^{-1}]$ of a category $\mathscr{C}$ at a subcategory $\mathscr{W}$ is a category with a functor $\iota\colon \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ mapping each morphism in $\mathscr{W}$ to an isomorphism in $\mathscr{C}[\mathscr{W}^{-1}]$, such that for any functor $F\colon \mathscr{C} \to \mathscr{D}$ mapping each morphism in $\mathscr{W}$ to an isomorphism in $\mathscr{D}$ there is a unique functor $\bar{F}\colon \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ such that $F = \bar{F} \circ \iota$.

$$\begin{tikzcd} \mathscr{W} \arrow[r, tail] & \mathscr{C} \arrow[r, "\iota"] \arrow[d, "F"'] & \mathscr{C}[\mathscr{W}^{-1}] \arrow[dl, dotted, "\bar{F}"] \\ & \mathscr{D} & \end{tikzcd}$$

The localization $\mathscr{C}[\mathscr{W}^{-1}]$, if it exists, is well-defined up to unique isomorphism as a category under $\mathscr{C}$. However, in general there can be set-theoretical hindrances to the existence of a localization, since we require our categories to have a set, instead of a proper class, of morphisms between any two objects. Quillen's theory of (closed) model categories provides one approach to showing that a localization exists, and this will allow us to make sense of the following definition.

DEFINITION 9.2.5. The *stable category* $\mathrm{Ho}(Sp^{\mathbb{N}})$ is the localization

$$Sp^{\mathbb{N}} \xrightarrow{\iota} Sp^{\mathbb{N}}[\mathscr{W}^{-1}] = \mathrm{Ho}(Sp^{\mathbb{N}})$$

of the category $Sp^{\mathbb{N}}$ of sequential spectra with respect to the subcategory $\mathscr{W}$ of stable equivalences. Let

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{N}})(M, N)$$

denote the set of morphisms in the stable category from M to N.

We get a factorization of functors

$$\pi_*\colon Sp^{\mathbb{N}} \xrightarrow{\iota} \mathrm{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\bar{\pi}_*} grAb$$

from a stable model category via a triangulated category to a graded abelian category. We shall later give an equivalent definition of the stable category as a localization $\mathrm{Ho}(Sp^{\mathbb{O}})$ of a category $Sp^{\mathbb{O}}$ of orthogonal spectra. The latter stable model category has better (closed symmetric) monoidal properties than $Sp^{\mathbb{N}}$, compatible with the "tensor triangulated" structure on the stable category and the "abelian monoidal" structure on graded abelian groups. Both homotopy categories are constructed using Quillen model structures in a paper by Mandell, May, Schwede and Shipley [**MMSS01**].

DEFINITION 9.2.6 ([**MMSS01**, Def. 1.3, Ex. 4.1]). For each $\ell \geq 0$ let

$$Ev_\ell\colon Sp^{\mathbb{N}} \longrightarrow \mathscr{T}$$

be the (level ℓ) *evaluation functor* mapping $M = (M_k, \sigma)_k$ to M_ℓ. Let the (level ℓ) *free functor*

$$F_\ell\colon \mathscr{T} \longrightarrow Sp^{\mathbb{N}}$$

be its left adjoint, so that there is a natural bijection

$$Sp^{\mathbb{N}}(F_\ell X, N) \cong \mathscr{T}(X, Ev_\ell(N)) .$$

Explicitly,

$$(F_\ell X)_k = \begin{cases} X \wedge S^{k-\ell} & \text{for } k \geq \ell, \\ * & \text{otherwise.} \end{cases}$$

The structure maps $\sigma\colon (F_\ell X)_k \wedge S^1 \to (F_\ell X)_{k+1}$ are the identities when $k \geq \ell$, and the base point inclusion otherwise.

In particular, $F_0 X = \Sigma^\infty X$ is the *suspension spectrum* of X, with $(\Sigma^\infty X)_k = X \wedge S^k$ for each $k \geq 0$. For each integer n we define the *n-sphere spectrum* S^n by

$$S^n = \begin{cases} F_0 S^n & \text{for } n \geq 0, \\ F_{-n} S^0 & \text{for } n < 0, \end{cases}$$

so that $(S^n)_k = S^{n+k}$ for $n + k \geq 0$ and $*$ otherwise. In particular, $S^0 = S$ with $S_k = S^k$ for each $k \geq 0$ is the *sphere spectrum*.

Definition 9.2.7. A spectrum M is $(n-1)$-*connected*, or n-*connective*, if $\pi_*(M) = 0$ for all $* < n$. It is *bounded below* if it is n-connective for some integer n. We abbreviate (-1)-connected (or 0-connective) to *connective*.

Example 9.2.8. The homotopy groups

$$\pi_n(S) = \operatorname*{colim}_k \pi_{n+k}(S^k)$$

of the sphere spectrum are the *stable homotopy groups of spheres*, also known as the *stable stems*. By the Hurewicz theorem they are trivial for $n < 0$, and isomorphic to $\mathbb{Z}$ for $n = 0$, so the sphere spectrum is connective. The determination of $\pi_n(S)$ for $n > 0$ is an ongoing field of study.

Definition 9.2.9 ([**MMSS01**, Def. 6.2, Def. 5.4, Thm. 6.5]). Let I be the set of inclusions $i\colon S^{n-1}_+ \to D^n_+$ for $n \ge 0$, where $S^{-1} = \emptyset$. Let $FI = F^{\mathbb{N}} I$ be the set of maps of sequential spectra $F_\ell i\colon F_\ell S^{n-1}_+ \to F_\ell D^n_+$ for $\ell \ge 0$ and $n \ge 0$.

A map $i\colon M \to N$ of sequential spectra is a *relative cell spectrum* (= relative FI-cell complex) if N is the colimit of a sequence of maps

$$M = N(0) \longrightarrow \dots \longrightarrow N(j) \longrightarrow N(j+1) \longrightarrow \dots \longrightarrow N$$

where each $N(j) \to N(j+1)$ is obtained by cobase change

$$\begin{tikzcd} \bigvee_\alpha S(\alpha) \arrow[r] \arrow[d, "\phi"'] & \bigvee_\alpha D(\alpha) \arrow[d, "\Phi"] \\ N(j) \arrow[r] & N(j+1) \end{tikzcd}$$

from a sum of maps $S(\alpha) \to D(\alpha)$ in FI. We say that N is a *cell spectrum* (= FI-cell complex) if $* \to N$ is a relative cell spectrum.

A map $i\colon M \to N$ in $Sp^{\mathbb{N}}$ is a *Quillen cofibration* (= q-cofibration) if it is a retract of a relative cell spectrum $i'\colon M' \to N'$, meaning that there is a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"'] & M' \arrow[r] \arrow[d, "i'"] & M \arrow[d, "i"] \\ N \arrow[r] & N' \arrow[r] & N \end{tikzcd}$$

where the horizontal composites are the identity maps.

We say that N is *Quillen cofibrant* (= q-cofibrant) if $* \to N$ is a Quillen cofibration. Any retract of a cell spectrum is Quillen cofibrant. ((ETC: Converse?)) If $q\colon M^c \xrightarrow{\sim} M$ is a stable equivalence, and M^c is Quillen cofibrant, then we say that M^c is a *cofibrant replacement* for M.

Remark 9.2.10. Cobase changes and colimits are created levelwise, so for a cell spectrum N each space N_k is a cell complex, and $\sigma\colon N_k \wedge S^1 \to N_{k+1}$ is the inclusion of a subcomplex. ((ETC: Converse? Will the cell filtrations $(N(j)_k)_j$ and $(N(j)_{k+1})_j$ be compatible?))

Example 9.2.11. Cofibrant replacements can be constructed by CW approximation. ((ETC: Check details. The cellular and skeletal filtrations are generally different.))

Definition 9.2.12. Let M be a sequential spectrum. The *adjoint structure map*

$$\tilde{\sigma}\colon M_k \longrightarrow \Omega M_{k+1}$$

is the right adjoint, for the loop–suspension adjunction, of the structure map $\sigma\colon M_k \wedge S^1 \to M_{k+1}$.

The homomorphism $\pi_{n+k}(M_k) \to \pi_{n+k+1}(M_{k+1})$ in the definition of $\pi_n(M)$ can be reexpressed as mapping the homotopy class of $x\colon S^{n+k} \to M_k$ to the homotopy class of the left adjoint $S^{n+k+1} \cong S^{n+k} \wedge S^1 \to M_{k+1}$ of the composite

$$S^{n+k} \xrightarrow{x} M_k \xrightarrow{\tilde{\sigma}} \Omega M_{k+1}\,.$$

Definition 9.2.13 ([**MMSS01**, Def. 9.4]). A commutative square of based spaces

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in which p and q are Serre fibrations, is a *weak homotopy pullback* if the induced map $D \to A \times_B E$ is a weak homotopy equivalence or, equivalently, if $g\colon q^{-1}(a) \to p^{-1}(f(a))$ is a weak homotopy equivalence for each $a \in A$.

Definition 9.2.14 ([**MMSS01**, Prop. 9.5]). A map $p\colon M \to N$ of sequential spectra is a *stable fibration* (= q-fibration) if and only if $p_k\colon M_k \to N_k$ is a Serre fibration and the diagram

$$\begin{tikzcd} M_k \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_k"'] & \Omega M_{k+1} \arrow[d, "\Omega p_{k+1}"] \\ N_k \arrow[r, "\tilde{\sigma}"] & \Omega N_{k+1} \end{tikzcd}$$

is a weak homotopy pullback, for each $k \ge 0$.

We say that M is *stably fibrant* (= q-fibrant) if $M \to *$ is a stable fibration. If $j\colon N \xrightarrow{\sim} N^f$ is a stable equivalence, and N^f is stably fibrant, then we say that N^f is a *fibrant replacement* for N.

Lemma 9.2.15. *M is stably fibrant if and only if it is an* Ω-spectrum*, i.e., if each adjoint structure map*

$$\tilde{\sigma}\colon M_k \xrightarrow{\simeq} \Omega M_{k+1}$$

is a weak homotopy equvalence.

Proof. Each map $M_k \to *$ is a Serre fibration. □

Example 9.2.16. A fibrant replacement $M \sim M^f$ can be constructed by setting M^f_k equal to the mapping telescope (or homotopy colimit) of the sequence of maps

$$M_k \xrightarrow{\tilde{\sigma}} \Omega M_{k+1} \xrightarrow{\Omega\tilde{\sigma}} \Omega^2 M_{k+2} \xrightarrow{\Omega^2\tilde{\sigma}} \Omega^3 M_{k+3} \longrightarrow \dots\,.$$

((ETC: Check details regarding adjoint structure maps for M^f, why they are weak homotopy equivalences, and why $M \to M^f$ is a stable equivalence.))

See Quillen [**Qui67**], Dwyer–Spalinski [**DS95**] or Hovey [**Hov99**, §1.1] for more detailed introductions to model category theory.

DEFINITION 9.2.17. A *model category* $\mathscr{C}$ is a category with all small limits and colimits, together with a model structure. A *model structure* on $\mathscr{C}$ is three subcategories, of *weak equivalences*, *cofibrations* and *fibrations*, with the following properties.

(1) If $f\colon L \to M$ and $g\colon M \to N$ are composable morphisms, and two of f, g and $gf\colon L \to N$ are weak equvalences, then so is the third of these.
(2) If $f\colon M \to N$ is a retract of $f'\colon M' \to N'$, and f' is a weak equivalence, cofibration or fibration, then f is a weak equivalence, cofibration or fibration, respectively.
(3) If $i\colon K \to L$ is a cofibration, $p\colon M \to N$ is a fibration, the square diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, tail, "i"'] & M \arrow[d, two heads, "p"] \\ L \arrow[r] \arrow[ur, dotted] & N \end{tikzcd}$$

commutes, and i or p is a weak equivalence, then there exists a map $L \to M$ making both triangles commute.
(4) Each map $K \to N$ admits factorizations

$$K \xrightarrow{j} M \xrightarrow{p} N \quad \text{and} \quad K \xrightarrow{i} L \xrightarrow{q} N$$

where j and q are weak equivalences, i and j are cofibrations, and p and q are fibrations.

THEOREM 9.2.18. *The category $Sp^{\mathbb{N}}$ of sequential spectra is a ((ETC: compactly generated, proper, topological)) model category with respect to the classes of stable equivalences, Quillen cofibrations and stable fibrations.*

We refer to [**MMSS01**, Thm. 9.2] for the proof.

((ETC: The compact spaces S^{n-1}_+ and D^n_+ admit the small object argument, by a variant of Lemma 3.2.5. Are transfinite composites required for the relative cell spectra?))

REMARK 9.2.19. Any two of the subcategories in a model structure determine the third. For example, the Quillen cofibrations can be characterized as those maps $i\colon K \to L$ that have the left lifting property, as in (3) above, with respect to all acyclic stable fibrations (= acyclic q-fibrations), i.e., the maps $p\colon M \to N$ that are stable equivalence and stable fibrations.

DEFINITION 9.2.20. For each sequential spectrum M and space X we define $X \wedge M$, $M \wedge X$ and $\mathrm{Map}(X, M)$ to be the sequential spectra with k-th spaces $X \wedge M_k$, $M_k \wedge X$ and $\mathrm{Map}(X, M_k)$, respectively, and with structure maps

$$X \wedge M_k \wedge S^1 \xrightarrow{1\wedge\sigma} X \wedge M_{k+1}\,,$$

$$M_k \wedge X \wedge S^1 \xrightarrow{1\wedge\tau} M_k \wedge S^1 \wedge X \xrightarrow{1\wedge\sigma} M_{k+1} \wedge X$$

and

$$\mathrm{Map}(X, M_k) \wedge S^1 \longrightarrow \mathrm{Map}(X, M_k \wedge S^1) \xrightarrow{\mathrm{Map}(1,\sigma)} \mathrm{Map}(X, M_{k+1})\,.$$

Equivalently, the adjoint structure maps for $\mathrm{Map}(X, M)$ are

$$\mathrm{Map}(X, M_k) \xrightarrow{\mathrm{Map}(1,\tilde\sigma)} \mathrm{Map}(X, \Omega M_{k+1}) \cong \Omega\,\mathrm{Map}(X, M_{k+1})\,.$$

In particular, let $I_+ \wedge M$, $CM = I \wedge M$, $\Sigma M = S^1 \wedge M$, $\mathrm{Map}(I_+, M)$, $PM = \mathrm{Map}(I, M)$ and $\Omega M = \mathrm{Map}(S^1, M)$ denote the *cylinder*, *cone*, *suspension*, *free paths*, *paths* and *loops* on M, respectively.

See [**Hov99**, §1.2] for a discussion of cylinder objects and homotopies between maps in a model category.

Lemma 9.2.21. *If M is Quillen cofibrant then $I_+ \wedge M$ is a cylinder object for M, meaning that*

$$M \vee M \cong \partial I_+ \wedge M \xrightarrow{i_0 \vee i_1} I_+ \wedge M \xrightarrow{\simeq} M$$

is a factorization of the fold map $M \vee M \to M$ through a Quillen cofibration followed by a stable equivalence (in fact, a homotopy equivalence).

Two maps $f, g \colon M \to N$ are homotopic, denoted $f \simeq g$, if there exists a map

$$H \colon I_+ \wedge M \longrightarrow N$$

such that $Hi_0 = f$ and $Hi_1 = g$. Homotopic maps induce identical homomorphisms of stable homotopy groups, so any homotopy equivalence is a stable equivalence.

Proposition 9.2.22 ([**Hov99**, Prop. 1.2.8]). *If M is Quillen cofibrant and N is stably fibrant, then a map $f \colon M \to N$ is a stable equivalence if and only if it is a homotopy equivalence.*

This is a formal consequence of the model category structure.

Let $Sp^{\mathbb{N}}_{cf} \subset Sp^{\mathbb{N}}$ denote the full subcategory of simultaneously Quillen cofibrant and stably fibrant spectra, and let $Sp^{\mathbb{N}}_{cf}/{\simeq}$ denote the quotient category with the same objects, but with morphism sets the homotopy classes

$$(Sp^{\mathbb{N}}_{cf}/{\simeq})(M, N) = Sp^{\mathbb{N}}(M, N)/{\simeq}$$

of maps $M \to N$.

Theorem 9.2.23 ([**Hov99**, Thm. 1.2.10]). *The induced functor*

$$Sp^{\mathbb{N}}_{cf}/{\simeq} \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}}) = Sp^{\mathbb{N}}[\mathscr{W}^{-1}]$$

is an equivalence of categories. If $q \colon M^c \xrightarrow{\sim} M$ and $j \colon N \xrightarrow{\sim} N^f$ are cofibrant and fibrant replacements, respectively, then

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{N}})(M, N) \cong Sp^{\mathbb{N}}(M^c, N^f)/{\simeq} .$$

In particular, the stable category $\mathrm{Ho}(Sp^{\mathbb{N}}) = Sp^{\mathbb{N}}[\mathscr{W}^{-1}]$ exists as a category (with sets, not proper classes, of morphisms). Furthermore, we can identify the morphisms sets $[M, N]$ by replacing M and N with stably equivalent spectra M^c and N^f, with M^c Quillen cofibrant and N^f stably fibrant, and then calculating the set of homotopy classes of maps $M^c \to N^f$.

Corollary 9.2.24. *For each integer n there is a natural isomorphism*

$$[S^n, M] \cong \pi_n(M) .$$

Proof. Any fibrant replacement (and these exist, by the model category structure) $j \colon M \xrightarrow{\sim} M^f$ induces isomorphisms $j_* \colon \pi_*(M) \cong \pi_*(M^f)$ and $j_* \colon [S^n, M] \cong [S^n, M^f]$, so we may assume that M is stably fibrant, i.e., an Ω-spectrum. Furthermore, S^n is a cell spectrum, hence Quillen cofibrant, so

$$[S^n, M] \cong Sp^{\mathbb{N}}(S^n, M)/{\simeq}$$

is given by the homotopy classes of spectrum maps $x\colon S^n \to M$.

When $n \geq 0$, this is the homotopy classes of maps $x_0\colon S^n \to M_0$. Since M is an Ω-spectrum, each homomorphism

$$\pi_n(M_0) \longrightarrow \pi_{n+1}(M_1) \longrightarrow \dots \longrightarrow \pi_n(M)$$

is an isomorphism, so $[S^n, M] \cong \pi_n(M)$. When $n \leq 0$, we are instead considering the homotopy classes of maps $x_{-n}\colon S^0 \to M_{-n}$. Again, each homomorphism

$$\pi_0(M_{-n}) \longrightarrow \pi_1(M_{-n+1}) \longrightarrow \dots \longrightarrow \pi_n(M)$$

is an isomorphism, so $[S^n, M] \cong \pi_n(M)$. □

9.3. Triangulated structure

We now prove that the stable category is, indeed, stable.

LEMMA 9.3.1. *There is a natural isomorphism*

$$\pi_n(\mathrm{Map}(S^1, M)) \cong \pi_{1+n}(M)\,.$$

PROOF. This is the colimit of the isomorphisms

$$\pi_{n+k}(\mathrm{Map}(S^1, M_k)) \cong \pi_{1+n+k}(M_k)\,,$$

matching $S^{n+k} \to \mathrm{Map}(S^1, M_k)$ to its left adjoint $S^1 \wedge S^{n+k} \to M_k$. □

PROPOSITION 9.3.2. *There is a natural isomorphism*

$$E\colon \pi_n(M) \xrightarrow{\cong} \pi_{1+n}(S^1 \wedge M)$$

mapping the class of $x\colon S^{n+k} \to M_k$ *to the class of* $1 \wedge x\colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$.

PROOF. First, suppose that $x\colon S^{n+k} \to M_k$ is such that $1 \wedge x\colon S^{1+n+k} \to S^1 \wedge M_k$ represents zero in $\mathrm{colim}_k\, \pi_{1+n+k}(S^1 \wedge M_k)$. By first increasing k, we may assume that $1 \wedge x\colon S^{1+n+k} \to S^1 \wedge M_k$ is null-homotopic. It follows that

$$x \wedge 1 = \tau(1 \wedge x)\tau\colon S^{n+k} \wedge S^1 \longrightarrow M_k \wedge S^1$$

and $\sigma(x \wedge 1)\colon S^{n+k+1} \to M_{k+1}$ are null-homotopic. Hence $\sigma(x \wedge 1)$ represents the zero class in $\pi_n(M) = \mathrm{colim}_k\, \pi_{n+k}(M_k)$. Since x and $\sigma(x \wedge 1)$ represent the same class in this colimit, E is injective.

$$\begin{tikzcd}
\pi_{n+k}(M_k) \arrow[r] \arrow[d] & \pi_{n+k+1}(M_{k+1}) \arrow[r] \arrow[d] & \pi_{n+k+2}(M_{k+2}) \arrow[r] \arrow[d] & \dots \\
\pi_{1+n+k}(S^1 \wedge M_k) \arrow[r] & \pi_{1+n+k+1}(S^1 \wedge M_{k+1}) \arrow[r] & \pi_{1+n+k+2}(S^1 \wedge M_{k+2}) \arrow[r] & \dots
\end{tikzcd}$$

Second, consider an element in $\mathrm{colim}_k\, \pi_{1+n+k}(S^1 \wedge M_k)$ represented by the homotopy class of a map $y\colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$, as well as by its stabilization $\sigma^2(y \wedge 1 \wedge 1)$. Let

$$x = \tau y \tau \wedge 1\colon S^{n+k} \wedge S^1 \wedge S^1 \longrightarrow M_k \wedge S^1 \wedge S^1\,.$$

Then $1 \wedge x\colon S^1 \wedge S^{n+k} \wedge S^1 \wedge S^1 \to S^1 \wedge M_k \wedge S^1 \wedge S^1$ is homotopic to $y \wedge 1 \wedge 1$, since a cyclic permutation of $S^1 \wedge S^1 \wedge S^1$ is homotopic to the identity. Hence $\sigma^2(y \wedge 1 \wedge 1)$ is homotopic to $(1 \wedge \sigma^2)(1 \wedge x) = 1 \wedge \sigma^2(x)\colon S^{1+n+k+2} \to S^1 \wedge M_{k+2}$. It therefore represents the same class as the image of $\sigma^2(x)\colon S^{n+k+2} \to M_{k+2}$, which proves that E is surjective. □

PROPOSITION 9.3.3. *The adjunction unit*

$$\eta\colon M \longrightarrow \mathrm{Map}(S^1, S^1 \wedge M)$$

is a stable equivalence.

PROOF. The homomorphism η_* factors as the composite

$$\eta_*\colon \pi_n(M) \cong \pi_{1+n}(S^1 \wedge M) \cong \pi_n(\mathrm{Map}(S^1, S^1 \wedge M))$$

of the isomorphisms from the previous two lemmas, hence is an isomorphism for each n. □

PROPOSITION 9.3.4. *The adjunction counit*

$$\epsilon\colon S^1 \wedge \mathrm{Map}(S^1, M) \longrightarrow M$$

is a stable equivalence.

PROOF. The composite

$$\mathrm{Map}(S^1, M) \xrightarrow{\eta_{\mathrm{Map}(S^1,M)}} \mathrm{Map}(S^1, S^1 \wedge \mathrm{Map}(S^1, M)) \xrightarrow{\mathrm{Map}(S^1,\epsilon)} \mathrm{Map}(S^1, M)$$

is the identity [**ML71**, Thm. IV.1.1], and $\pi_n(\eta)$ is an isomorphism for each n, so $\pi_n(\mathrm{Map}(S^1, \epsilon))$ is an isomorphism. Hence $\pi_{1+n}(\epsilon)$ is an isomorphism. □

Here is the promised stability result. It shows that the stable model structure makes $Sp^{\mathbb{N}}$ a *stable model category* in the sense of [**SS03**, Def. 2.1.1], which implies that its homotopy category is *triangulated*.

THEOREM 9.3.5. *The suspension functor $M \mapsto S^1 \wedge M$ is an equivalence of categories*

$$S^1 \wedge -\colon \mathrm{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}})\,.$$

In other words, for all sequential spectra M and N the function

$$S^1 \wedge -\colon [M, N] \xrightarrow{\cong} [S^1 \wedge M, S^1 \wedge N]$$

is a bijection, and each M is stably equivalent to a spectrum of the form $S^1 \wedge N$.

PROOF. For any cell complex X the functor $X \wedge -$ preserves Quillen cofibrations and the functor $\mathrm{Map}(X, -)$ preserves stable fibrations. In particular, $S^1 \wedge -$ preserves Quillen cofibrant objects and $\mathrm{Map}(S^1, -)$ preserves stably fibrant objects. Hence the adjunction

$$Sp^{\mathbb{N}}(S^1 \wedge M^c, N^f) \cong Sp^{\mathbb{N}}(M^c, \mathrm{Map}(S^1, N^f))$$

passes to a natural bijection

$$\begin{aligned}[S^1 \wedge M, N] &\cong Sp^{\mathbb{N}}(S^1 \wedge M^c, N^f)/{\simeq} \\ &\cong Sp^{\mathbb{N}}(M^c, \mathrm{Map}(S^1, N^f))/{\simeq} \cong [M, \mathrm{Map}(S^1, N)]\,.\end{aligned}$$

Replacing N by $S^1 \wedge N$, the composite

$$[M, N] \xrightarrow{S^1 \wedge -} [S^1 \wedge M, S^1 \wedge N] \cong [M, \mathrm{Map}(S^1, S^1 \wedge N)]$$

is induced by the adjunction unit $\eta\colon N \to \mathrm{Map}(S^1, S^1 \wedge N)$. Since this is a stable equivalence, the induced function is a bijection.

The adjunction counit $\epsilon\colon S^1 \wedge \mathrm{Map}(S^1, M) \to M$ exhibits M as being stably equivalent to $S^1 \wedge N$ for $N = \mathrm{Map}(S^1, M)$. □

DEFINITION 9.3.6. For a map $f\colon M \to N$ of spectra let

$$Cf = N \cup CM$$

be the *mapping cone* (= homotopy cofiber) of $f\colon M \to N$. There are canonical maps

$$M \xrightarrow{f} N \xrightarrow{i} Cf \xrightarrow{q} S^1 \wedge M$$

with i and q induced by $M \to CM$ and $N \to *$, respectively. We call this the (homotopy cofiber) *Puppe sequence* generated by f.

PROPOSITION 9.3.7. *The Puppe sequence of $f\colon M \to N$ induces a long exact sequence of stable homotopy groups*

$$\cdots \to \pi_n(M) \xrightarrow{f_*} \pi_n(N) \xrightarrow{i_*} \pi_n(Cf) \xrightarrow{\partial} \pi_{n-1}(M) \to \dots ,$$

where $E \circ \partial = q_*\colon \pi_n(Cf) \to \pi_n(S^1 \wedge M)$.

PROOF. It suffices to prove exactness at $\pi_n(N)$, and $i_* f_* = 0$ is clear.

$$\begin{tikzcd} & S^{n+k} \arrow[r, "i"] \arrow[d, "y"'] & CS^{n+k} \arrow[r, "q"] \arrow[d, "z"'] & S^1 \wedge S^{n+k} \arrow[d, "x'"'] \\ M_k \arrow[r, "f_k"] & N_k \arrow[r, "i_k"] & Cf_k \arrow[r, "q_k"] & S^1 \wedge M_k \end{tikzcd}$$

If $y\colon S^{n+k} \to N_k$ corresponds to a class $[y]$ in $\ker(i_*)$, then we may increase k and assume that $i_k y$ is null-homotopic, hence extends to a map $z\colon CS^{n+k} \to Cf_k$. The induced map of quotients $x'\colon S^1 \wedge S^{n+k} \to S^1 \wedge M_k$ then represents a class $[x']$ in $\pi_{1+n}(S^1 \wedge M)$ which corresponds, under the isomorphism E, to a class $[x] \in \pi_n(M)$ with $f_*([x]) = [y]$. □

DEFINITION 9.3.8. A map $f\colon M \to N$ is *n-connected* if $\pi_*(f)$ is an isomorphism for $* < n$ and surjective for $* = n$, or, equivalently, if Cf is n-connected.

More generally, a Puppe sequence induces long exact sequences

$$\cdots \to [S^1 \wedge M, T] \xrightarrow{q^*} [Cf, T] \xrightarrow{i^*} [N, T] \xrightarrow{f^*} [M, T] \to \dots$$

and

$$\cdots \to [T, M] \xrightarrow{f_*} [T, N] \xrightarrow{i_*} [T, Cf] \xrightarrow{q_*} [T, S^1 \wedge M] \to \dots ,$$

both co- and contravariantly, where T is an arbitrary spectrum. This can be deduced from some basic properties satisfied by the suspension operator $S^1 \wedge -$ and the Puppe sequences, which make the stable category a triangulated category. See Proposition 9.3.15.

REMARK 9.3.9. Puppe [**Pup67**, Stz. 3.5] introduced axioms for a triangulated category (at a 1962 conference in Aarhus), and Verdier [**Ver96**] added the octahedral axiom (in his 1967 PhD thesis). Further references include Beilinson–Bernstein–Deligne [**BBD82**, §1.1], Margolis [**Mar83**, App. 2], Neeman [**Nee01**, Ch. 1], Hovey–Palmieri–Strickland [**HPS97**, App. A.1] and May [**May01**, §2].

DEFINITION 9.3.10 ([**ML71**, §VIII.2]). An *Ab-category* is a category $\mathscr{C}$ in which each morphism set $\mathscr{C}(X, Y)$ is an abelian group and composition is bilinear. An *additive category* is an *Ab*-category with all finite sums and products, such that these are canonically isomorphic. We write 0 for the zero object, i.e., the empty sum and product.

We adopt May's formulation [**May01**, Def. 2.1] of the definition below.

DEFINITION 9.3.11. A *triangulated category* is an additive category $\mathscr{C}$ equipped with an additive functor

$$\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$$

called *suspension*, and a collection Δ of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X\,,$$

called *distinguished triangles*, briefly denoted (f, g, h). We assume that $\Sigma\colon \mathscr{C} \to \mathscr{C}$ is an equivalence of categories. Furthermore, we assume that:

(1) For any object X the triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished, for any morphism $f\colon X \to Y$ there exists a distinguished triangle (f, g, h), and any triangle that is isomorphic to a distinguished triangle is itself distinguished.

(2) If (f, g, h) is distinguished, then so is its rotation

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y\,.$$

(3) Consider the following braid diagram.

$$\begin{tikzcd}
X \arrow[dr, "f"'] \arrow[rr, bend left, "h"] & & Z \arrow[dr, "h'"] \arrow[rr, bend left, "g'"] & & W \arrow[dr, "g''"] \arrow[rr, bend left, "j''"] & & \Sigma U \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"'] & & V \arrow[ur, dotted, "j'"] \arrow[dr, "h''"] & & \Sigma Y \arrow[ur, "\Sigma f'"'] & \\
& & U \arrow[ur, dotted, "j"] \arrow[rr, bend right, "f''"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

Assume that $h = gf$ and $j'' = (\Sigma f')g''$, and that (f, f', f''), (g, g', g'') and (h, h', h'') are distinguished. Then there are maps j and j' such that the diagram commutes and (j, j', j'') is distinguished.

Axiom (3) is the braid form of Verdier's octahedral axiom. These axioms imply the following 3×3 lemma.

LEMMA 9.3.12 ([**BBD82**, Prop. 1.1.11], [**May01**, Lem. 2.6]). *Assume that $jf = f'i$ and the two top rows and two left columns are distinguished in the following* *diagram.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"'] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] \arrow[d, dotted, "k'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[r, dotted, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, dotted, "g''"] \arrow[d, "j''"] & Z'' \arrow[r, dotted, "h''"] \arrow[d, dotted, "k''"] \arrow[dr, phantom, "(-)"] & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

Then there is an object Z'' and maps f'', g'', h'', k, k' and k'' such that the diagram is commutative, except for its bottom right hand square, which commutes up to the sign -1, and all four rows and columns are distinguished.

In particular, the fill-in axiom of Puppe and Verdier follows from those above. We state it as a lemma.

LEMMA 9.3.13 ([**May01**, Lem. 2.2]). *If the rows are distinguished and the left hand square commutes in the following diagram, then there is a morphism k that makes the remaining two squares commute.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

REMARK 9.3.14. The precise axioms for a triangulated category may not be optimal: all natural examples seem to come from a stable model category or a stable ∞-category, which satisfy refined axioms as discussed in [**Nee91**]. In these examples, there is a construction of the fill-in morphism k that is well-defied up to addition of a composite

$$Z \xrightarrow{h} \Sigma X \longrightarrow Y' \xrightarrow{g'} Z',$$

and this ambiguity is generally smaller than that allowed by the lemma above.

PROPOSITION 9.3.15. *For (f, g, h) distinguished and T any object, the sequences*

$$\dots \longrightarrow \mathscr{C}(T, X) \xrightarrow{f_*} \mathscr{C}(T, Y) \xrightarrow{g_*} \mathscr{C}(T, Z) \xrightarrow{h_*} \mathscr{C}(T, \Sigma X) \longrightarrow \dots$$

and

$$\dots \longleftarrow \mathscr{C}(X, T) \xleftarrow{f^*} \mathscr{C}(Y, T) \xleftarrow{g^*} \mathscr{C}(Z, T) \xleftarrow{h^*} \mathscr{C}(\Sigma X, T) \longleftarrow \dots$$

are exact.

PROOF. We show that $\mathrm{im}(f_*) = \ker(g_*)$. Given $i \colon T \to X$ in $\mathscr{C}(T, X)$ we have

$$\begin{tikzcd}
T \arrow[r, "1"] \arrow[d, "i"'] & T \arrow[r] \arrow[d, "j"] & 0 \arrow[r] \arrow[d, dotted, "k"] & \Sigma T \arrow[d, "\Sigma i"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

with $j = fi$, and there is a fill-in map k. Hence $gfi = 0$, so $\mathrm{im}(f_*) \subset \ker(g_*)$.

Conversely, given $j\colon T \to Y$ in $\mathscr{C}(T, Y)$ with $gj = 0$ we have

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "j"] & 0 \arrow[r] \arrow[d] & \Sigma T \arrow[r, "-\Sigma 1"] \arrow[d, dotted, "\Sigma i"] & \Sigma T \arrow[d, "\Sigma j"] \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

and there is a fill-in map Σi. Hence $\Sigma j = \Sigma(fi)$, so $j = fi$, and $\ker(g_*) \subset \mathrm{im}(f_*)$.

The other cases are proved by similar arguments. $\square$

Puppe [**Pup67**, §3] proved the following theorem for a smaller category than $\mathrm{Ho}(Sp^{\mathbb{N}})$, while in its present form the result is due to Boardman (unpublished, but see [**Vog70**] or [**Ada74**]).

THEOREM 9.3.16. *The stable category* $\mathrm{Ho}(Sp^{\mathbb{N}})$ *is triangulated, with suspension functor*

$$\Sigma M = S^1 \wedge M$$

and distinguished triangles the diagram that are isomorphic to Puppe sequences

$$M \xrightarrow{f} N \xrightarrow{i} Cf \xrightarrow{q} S^1 \wedge M\,.$$

PROOF. We first argue that $\mathrm{Ho}(Sp^{\mathbb{N}})$ is an Ab-category. For any L and N there are spectra L' and N' and stable equivalences $L \sim S^2 \wedge L'$ and $N \sim \mathrm{Map}(S^2, N')$. The homotopy commutative cogroup structure on S^2 induces abelian group structures

$$[L, M] \cong [S^2 \wedge L', M] \quad \text{and} \quad [M, N] \cong [M, \mathrm{Map}(S^2, N')]$$

so that the composition pairing

$$\begin{aligned}
[M, N] \times [L, M] \cong [M, \mathrm{Map}(S^2, N')] \times [S^2 \wedge L', M]& \\
\xrightarrow{\circ} [S^2 \wedge L', \mathrm{Map}(S^2, N')] \cong [L, N]&
\end{aligned}$$

is bilinear.

The stable category has all sums and products, defined levelwise. The Ab-category structure implies that the finite sums are canonically isomorphic to the finite products, see [**ML71**, §VIII.2], so that $\mathrm{Ho}(Sp^{\mathbb{N}})$ is additive. In particular,

$$M \vee N \xrightarrow{\sim} M \times N$$

is a stable equivalence, and

$$\pi_*(M) \oplus \pi_*(N) \cong \pi_*(M \vee N) \cong \pi_*(M \times N) \cong \pi_*(M) \times \pi_*(N)\,.$$

The (left) suspension functor is an equivalence by Theorem 9.3.5. Axiom (1) is straightforward, and Axiom (2) follows from the known fact that the Puppe sequence

$$N \xrightarrow{i} Cf \xrightarrow{i'} Ci \xrightarrow{q'} \Sigma N$$

is isomorphic, in the stable category, to

$$N \xrightarrow{i} Cf \xrightarrow{q} \Sigma M \xrightarrow{-\Sigma f} \Sigma N\,.$$

To verify the braid/octahedral axiom, we follow [**May01**, §5]. We may assume that the distinguished triangles (f, f', f''), (g, g', g'') and (h, h', h'') are Puppe cofiber sequences, so that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left, "i(g)"] \arrow[dr, "i(h)"] & & Cg \arrow[rr, bend left, "j''"] \arrow[dr, "q(g)"] & & \Sigma Cf \\
& Y \arrow[ur, "g"] \arrow[dr, "i(f)"'] & & Ch \arrow[ur, "j'"] \arrow[dr, "q(h)"] & & \Sigma Y \arrow[ur, "\Sigma i(f)"'] & \\
& & Cf \arrow[ur, "j"] \arrow[rr, bend right, "q(f)"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

with $j: Y \cup CX \to Z \cup CX$ induced by $g: Y \to Z$ and $j': Z \cup CX \to Z \cup CY$ induced by $Cf: CX \to CY$. It remains to verify that (j, j', j'') is distinguished, which amounts to constructing an explicit (stable) equivalence $Cj \sim Cg$ that is compatible with j' and j''. ((ETC: Is this a homotopy equivalence?)) ((ETC: May gives a further reduction using the model structure.)) $\square$

9.4. Spectral homology and cohomology

The Eilenberg–MacLane representability of ordinary cohomology readily extends to show that any spectrum M represents a generalized cohomology theory. Dually, George Whitehead [**Whi62**, §5] showed how spectra also give rise to generalized homology theories. Since we are working with based spaces these theories will always be reduced, but we do not add tildes to indicate this.

DEFINITION 9.4.1. Let $M = (M_k, \sigma)_k$ be a sequential spectrum. For each space X let

$$M_n(X) = \pi_n(M \wedge X)$$

be equal to $\operatorname{colim}_k \pi_{n+k}(M_k \wedge X)$, and let

$$\sigma: M_n(X) \xrightarrow{\cong} M_{1+n}(S^1 \wedge X)$$

be the composite of the isomorphisms

$$\pi_n(M \wedge X) \xrightarrow{E} \pi_{1+n}(S^1 \wedge M \wedge X) \xrightarrow{(\tau\wedge 1)_*} \pi_{1+n}(M \wedge S^1 \wedge X)\,.$$

For any pair (X, A) let

$$M_n(X, A) = M_n(X \cup CA)$$

and let

$$\partial: M_n(X, A) \longrightarrow M_{n-1}(A)$$

be the composite

$$M_n(X \cup CA) \xrightarrow{q_*} M_n(S^1 \wedge A) \xrightarrow{\sigma^{-1}} M_{n-1}(A)$$

induced by $q: X \cup CA \to S^1 \wedge A$ and the inverse of σ.

Note that if (X, A) is a CW pair, or $A \to X$ is a Hurewicz cofibration, then $X \cup CA \simeq X/A$ so $M_n(X, A) \cong M_n(X/A)$.

PROPOSITION 9.4.2. *The functor* $(X, A) \mapsto M_*(X, A)$ *and the natural transformation* $\partial\colon M_*(X, A) \to M_{*-1}(A)$ *define a (generalized) homology theory for pairs* (X, A). *Its coefficient groups are* $M_* \cong \pi_*(M)$.

PROOF. We check the axioms for CW pairs (X, A). Functoriality and naturality are clear. By Proposition 9.3.7, the Puppe sequence

$$A \wedge M \xrightarrow{f \wedge 1} X \wedge M \xrightarrow{i \wedge 1} Cf \wedge M \xrightarrow{q \wedge 1} S^1 \wedge A \wedge M$$

of $f \wedge 1$, where $f\colon A \to X$ is the inclusion and $Cf = X \cup CA$, induces a long exact sequence of homotopy groups. It is isomorphic to the sequence

$$M \wedge A \xrightarrow{1 \wedge f} M \wedge X \xrightarrow{1 \wedge i} M \wedge Cf \xrightarrow{1 \wedge q} M \wedge S^1 \wedge A$$

by way of the evident twist maps, and this proves exactness. Homotopy invariance and excision are likewise clear.

Additivity for infinite sums requires a bit more effort. Let $(X_\alpha)_{\alpha \in J}$ be a collection of (based) spaces, where J is some indexing set. Letting F range over the filtering (= directed and nonempty) poset of finite subsets of J, we can form the inclusions

$$\bigvee_{\alpha \in F} M_k \wedge X_\alpha \longrightarrow \bigvee_{\alpha \in J} M_k \wedge X_\alpha$$

and consider the canonical homomorphism

$$\operatorname*{colim}_F \pi_{n+k}(\bigvee_{\alpha \in F} M_k \wedge X_\alpha) \longrightarrow \pi_{n+k}(\bigvee_{\alpha \in J} M_k \wedge X_\alpha)\,.$$

We claim that this is an isomorphism, because S^{n+k} and the cylinder $I_+ \wedge S^{n+k}$ are both compact. For this we need that $\bigvee_{\alpha \in J} M_k \wedge X_\alpha$ is strongly filtered by the $\bigvee_{\alpha \in F} M_k \wedge X_\alpha$, in a more general sense than in Definition 3.2.1. See [**Str**, Lem. 3.6] for a proof, which is similar to that of Lemma 3.2.5.

These isomorphisms are compatible for increasing k, and passing to sequential colimits we deduce that

$$\operatorname*{colim}_F M_n(\bigvee_{\alpha \in F} X_\alpha) \xrightarrow{\cong} M_n(\bigvee_{\alpha \in J} X_\alpha)$$

is an isomorphism. By finite additivity,

$$\bigoplus_{\alpha \in F} M_n(X_\alpha) \xrightarrow{\cong} M_n(\bigvee_{\alpha \in F} X_\alpha)$$

and

$$\operatorname*{colim}_F \bigoplus_{\alpha \in F} M_n(X_\alpha) \xrightarrow{\cong} \bigoplus_{\alpha \in J} M_n(X_\alpha)$$

are isomorphisms. Stringing these together, we have confirmed the additivity axiom for the homology theory $X \mapsto M_*(X)$. Finally, the coefficients groups of this theory are $M_* = M_*(S^0) = \pi_*(M \wedge S^0) \cong \pi_*(M)$. $\square$

REMARK 9.4.3. The isomorphisms

$$M_n(X) = \pi_n(M \wedge X) \cong [S^n, M \wedge X]$$

and

$$[S^n, M \wedge X] \cong [S^1 \wedge S^n, S^1 \wedge M \wedge X] \cong [S^{1+n}, M \wedge S^1 \wedge X]$$

express the homology theory M in terms of the triangulated structure of the stable category. We will define the associated cohomology theory so that

$$M^n(X) \cong [S^{-n} \wedge X, M] \cong [\Sigma^\infty X, S^n \wedge M]$$

and

$$[S^{-n} \wedge X, M] \cong [S^{-1-n} \wedge S^1 \wedge X, M] .$$

However, to correctly calculate morphisms to M in the stable category, we need to consider homotopy classes of maps to a fibrant replacement $M \sim M^f$ of f, i.e., to a stably equivalent Ω-spectrum (recall Lemma 9.2.15). The issue is that while there typically is a map

$$\mathrm{Map}(X, M)^f \longrightarrow \mathrm{Map}(X, M^f)$$

from a fibrant replacement of $\mathrm{Map}(X, M)$ to the indicated mapping spectrum, it is not always true that this map is a stable equivalence. This issue stems from the fact that sequential colimits do not generally commute with infinite products. This is not a problem if X is homotopy equivalent to a finite cell complex, as assumed by Whitehead [**Whi62**, (5.10)], but we do not wish to restrict to this case.

Definition 9.4.4. Let $M = (M_k, \sigma)_k$ be a sequential spectrum. Let $M^f = (M^f_k, \sigma)_k$ be a fibrant replacement of M, i.e., an Ω-spectrum with a stable equivalence $M \sim M^f$. For each space X let

$$M^n(X) = \pi_{-n} \mathrm{Map}(X, M^f)$$

be equal to $\mathrm{colim}_k\, \pi_{-n+k} \mathrm{Map}(X, M^f_k)$, and let

$$\sigma \colon M^n(X) \xrightarrow{\cong} M^{1+n}(S^1 \wedge X)$$

be the composite of the isomorphisms

$$\pi_{-n} \mathrm{Map}(X, M^f) \cong \pi_{-1-n} \mathrm{Map}(S^1, \mathrm{Map}(X, M^f)) \cong \pi_{-1-n} \mathrm{Map}(S^1 \wedge X, M^f) .$$

For any pair (X, A) let

$$M^n(X, A) = M^n(X \cup CA)$$

and let

$$\delta \colon M^n(A) \longrightarrow M^{1+n}(X, A)$$

be the composite

$$M^n(A) \xrightarrow{\sigma} M^{1+n}(S^1 \wedge A) \xrightarrow{q^*} M^{1+n}(X \cup CA)$$

induced by the isomorphism σ and $q \colon X \cup CA \to S^1 \wedge A$.

((ETC: (In-)dependence of choice of fibrant replacement.))

Proposition 9.4.5. *The contravariant functor $(X, A) \mapsto M^*(X, A)$ and the natural transformation $\delta \colon M^*(A) \to M^{1+*}(X, A)$ define a (generalized) cohomology theory for pairs (X, A). Its coefficient groups are $M^* = \pi_{-*}(M)$.*

Proof. We check the axioms for CW pairs (X, A). Contravariant functoriality and naturality are clear. For each k the Puppe fiber sequence

$$\cdots \to \Omega \mathrm{Map}(A, M^f_k) \xrightarrow{\mathrm{Map}(q,1)} \mathrm{Map}(X \cup CA, M^f_k)$$
$$\xrightarrow{\mathrm{Map}(i,1)} \mathrm{Map}(X, M^f_k) \xrightarrow{\mathrm{Map}(f,1)} \mathrm{Map}(A, M^f_k)$$

induces a long exact sequence of homotopy groups

$$\cdots \to \pi_{1-n+k} \operatorname{Map}(A, M^f_k) \xrightarrow{\delta} \pi_{-n+k} \operatorname{Map}(X \cup CA, M^f_k) \\ \xrightarrow{i^*} \pi_{-n+k} \operatorname{Map}(X, M^f_k) \xrightarrow{f^*} \pi_{-n+k} \operatorname{Map}(A, M^f_k)$$

for $k > n$ (and some weaker form of exactness for $k = n$). Passing to sequential colimits over k confirms the exactness axiom. Homotopy invariance and excision are straightforward.

Additivity requires that the canonical exchange map

$$M^n(\bigvee_\alpha X_\alpha) \cong \operatorname*{colim}_k \prod_\alpha \pi_{-n+k} \operatorname{Map}(X_\alpha, M^f_k) \\ \xrightarrow{\kappa} \prod_\alpha \operatorname*{colim}_k \pi_{-n+k} \operatorname{Map}(X_\alpha, M^f_k) \cong \prod_\alpha M^n(X_\alpha)$$

is an isomorphism, which holds because M^f is an Ω-spectrum, so that each colimit is achieved at a finite stage, i.e., for any $k \geq n$. The coefficient groups of this cohomology theory are $M^* = M^*(S^0) = \pi_{-*} \operatorname{Map}(S^0, M) \cong \pi_{-*}(M)$. □

REMARK 9.4.6. Each morphism $f \in [M, N]$ in the stable category induces morphisms

$$f_* \colon M_n(X) \cong [S^n, M \wedge X] \longrightarrow [S^n, N \wedge X] \cong N_n(X)$$
$$f_* \colon M^n(X) \cong [S^{-n} \wedge X, M] \longrightarrow [S^{-n} \wedge X, N] \cong N^n(X)$$

of homology and cohomology theories. In particular, we have a functor

$$\operatorname{Ho}(Sp^{\mathbb{N}}) \longrightarrow \text{Cohomology theories}$$

from the stable category to the category of cohomology theories. By Brown's representability theorem [**Bro62**, Thm. I], each cohomology theory $X \mapsto M^*(X)$ arises in this way from some spectrum M, so this functor is essentially surjective. The functor is also full, in the sense that each morphism of cohomology theories comes from a morphism in the stable category, but in general it is not faithful, meaning that there may be nontrivial morphisms $f \in [M, N]$ that induce the zero morphism $f_* \colon M^*(X) \to N^*(X)$ for each space X. These are called *superphantom maps* by Margolis [**Mar83**, p. 81], and their existence shows that the stable category is not equivalent to the category of cohomology theories. The former has the richer structure, and the latter is the quotient category where superphantom maps are ignored. More explicitly, there is a short exact sequence

$$0 \to \operatorname*{Rlim}_k [\Sigma M_k, N_k] \longrightarrow [M, N] \longrightarrow \lim_k [M_k, N_k] \to 0$$

and if $f \colon M \to N$ is such that each $f_k \colon M_k \to N_k$ is null-homotopic, then f induces the zero morphism $f_* \colon M^k(X) \to N^k(X)$ for all X. Hence the superphantom maps from M to N are given by the derived limit group $\operatorname{Rlim}_k [\Sigma M_k, N_k]$. Goodwillie (MathOverflow, 2013) notes that there are nonzero superphantom maps $KU \to \Sigma H\mathbb{Z}$.

EXAMPLE 9.4.7. Let G be an abelian group. The *Eilenberg–MacLane spectrum* HG has k-th space

$$HG_k = K(G, k)$$

an Eilenberg–MacLane space of type (G, k), and adjoint structure maps given by homotopy equivalences

$$\tilde{\sigma}\colon K(G,k) \xrightarrow{\simeq} \Omega K(G,k+1)\,.$$

Hence HG is an Ω-spectrum. Its coefficient groups are $\pi_*(HG) = G$ concentrated in degree 0, and this characterizes HG up to stable equivalence. It represents ordinary homology and cohomology with G-coefficients, so that

$$\tilde{H}_n(X;G) \cong HG_n(X) = \operatorname*{colim}_k \pi_{n+k}(HG_k \wedge X)$$
$$\tilde{H}^n(X;G) \cong HG^n(X) = [X, HG_n]$$

for $n \geq 0$. These groups are trivial for $n < 0$.

EXAMPLE 9.4.8. The sphere spectrum S has k-th space $S_k = S^k$ and structure maps given by the identifications $\sigma\colon S^k \wedge S^1 \cong S^{k+1}$. Its coefficient groups

$$\pi_n(S) = \operatorname*{colim}_k \pi_{n+k}(S^k)$$

are the stable homotopy groups of spheres. There is a fibrant replacement $S \sim S^f$ with

$$S^f_k = Q(S^k) = \operatorname*{colim}_\ell \Omega^\ell S^{k+\ell}$$

for each k, such that each adjoint structure map

$$\tilde{\sigma}\colon Q(S^k) \xrightarrow{\cong} \Omega Q(S^{k+1})$$

is a homeomorphism. The sphere spectrum represents *stable homotopy and cohomotopy*, so that

$$\pi^S_n(X) \cong S_n(X) \cong \operatorname*{colim}_k \pi_{n+k}(X \wedge S^k)$$
$$\pi^n_S(X) \cong S^n(X) \cong \operatorname*{colim}_\ell [X \wedge S^\ell, S^{n+\ell}]\,.$$

The Pontryagin–Thom construction extends to an isomorphism

$$\Omega^{fr}_*(X) \cong \pi^S_*(X_+)$$

where $\Omega^{fr}_n(X)$ is given by framed bordism classes of framed n-manifolds $M^n \to X$, equipped with structure maps to X.

When $X = BG$ is the classifying space of a finite group G, the proven Segal conjecture [**Car84**] implies that

$$\pi^0_S(BG_+) \cong A(G)^\wedge_{I(G)}$$

is the completion of the *Burnside ring* $A(G)$ of G at its augmentation ideal, while

$$\pi^n_S(BG_+) = 0$$

for $n > 0$. The precise statement also determines $\pi^n_S(BG_+) = 0$ for $n < 0$, in terms of stable homotopy groups.

EXAMPLE 9.4.9. The *Thom spectra* MO and MSO have k-th spaces

$$MO_k = Th(\gamma^k) \simeq EO(k)_+ \wedge_{O(k)} S^k$$
$$MSO_k = Th(\tilde{\gamma}^k) \simeq ESO(k)_+ \wedge_{SO(k)} S^k$$

the Thom complexes of the tautological vector bundles $E(\gamma^k) \to Gr_k(\mathbb{R}^\infty)$ and $E(\tilde\gamma^k) \to \widetilde{Gr}_k(\mathbb{R}^\infty)$. The structure maps

$$\sigma\colon Th(\gamma^k) \wedge S^1 \cong Th(\gamma^k \oplus \epsilon^1) \longrightarrow Th(\gamma^{k+1})$$
$$\sigma\colon Th(\tilde\gamma^k) \wedge S^1 \cong Th(\tilde\gamma^k \oplus \epsilon^1) \longrightarrow Th(\tilde\gamma^{k+1})$$

are induced by the vector bundle maps covering the inclusions $Gr_k(\mathbb{R}^\infty) \subset Gr_{k+1}(\mathbb{R}^\infty)$ and $\widetilde{Gr}_k(\mathbb{R}^\infty) \subset \widetilde{Gr}_{k+1}(\mathbb{R}^\infty)$. Their coefficient groups are $\pi_*(MO) \cong \mathscr{N}_*$ and $\pi_*(MSO) \cong \Omega_*$. The associated homology theories are precisely (unoriented) bordism and oriented bordism, so that

$$\mathscr{N}_n(X) \cong MO_n(X_+) = \operatorname*{colim}_k \pi_{n+k}(MO(k) \wedge X_+)$$
$$\Omega_n(X) \cong MSO_n(X_+) = \operatorname*{colim}_k \pi_{n+k}(MSO(k) \wedge X_+)\,.$$

EXAMPLE 9.4.10. There are ((ETC: ring)) spectrum maps

$$\begin{tikzcd} S \arrow[r] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\ & H\mathbb{Z} \arrow[r] & H\mathbb{F}_2 \end{tikzcd}$$

inducing ((ETC: multiplicative)) morphisms

$$\begin{tikzcd} \Omega^{fr}_*(X) \arrow[r] & \Omega_*(X) \arrow[r] \arrow[d] & \mathscr{N}_*(X) \arrow[d] \\ & H_*(X;\mathbb{Z}) \arrow[r] & H_*(X;\mathbb{F}_2) \end{tikzcd}$$

of homology theories. The vertical maps take an oriented (resp. unoriented) bordism class $[f]$ with $f\colon M^n \to X$ to the image $f_*[M]$ of the integral (resp. mod 2) fundamental class of M. The map $MO \to H\mathbb{F}_2$ admits a section, so each mod 2 homology class can be represented by a closed manifold. The map $MSO \to H\mathbb{Z}$ does not admit a section, and not every integral homology class can be represented by a closed oriented manifold. Thom [**Tho54**, Cor III.7, Thm. III.9] showed than for $n \le 6$ every integral homology class can be represented by a closed oriented manifold, but for each $n \ge 7$ there exist integral homology classes that cannot be so represented.

((ETC: Complex bordism. Landweber exact homology theories.))

EXAMPLE 9.4.11. The classification of rank k complex vector bundles

$$\mathrm{Vect}^{\mathbb{C}}_k(X) \cong [X, BU(k)]\,,$$

where $BU(k) \simeq Gr_k(\mathbb{C}^\infty)$, extends to a classification of complex vector bundles of arbitrary rank

$$\mathrm{Vect}^{\mathbb{C}}(X) \cong [X_+, \coprod_{k\ge 0} BU(k)]\,.$$

The Whitney sum $\xi \oplus \eta$ of vector bundles induces a commutative monoid structure on both sides of this bijection, which is induced by a map

$$\coprod_{i\ge 0} BU(i) \times \coprod_{j\ge 0} BU(j) \longrightarrow \coprod_{k\ge 0} BU(k)$$

on the right hand side. We can localize, to make the operation $-\oplus\epsilon^1$ invertible, and obtain an isomorphism

$$KU(X)\cong[X_+,\mathbb{Z}\times BU]$$

where BU is the classifying space for the infinite unitary group $U=\bigcup_k U(k)$. When X is a finite-dimensional CW complex, the left hand side is the *group completion* of $\mathrm{Vect}^{\mathbb{C}}(X)$, also known as the *complex K-group* of X. The (complex) *Bott periodicity theorem* [**Bot59**] asserts that

$$\mathbb{Z}\times BU\simeq\Omega U\,,$$

while

$$U\simeq\Omega(\mathbb{Z}\times BU)$$

is clear from the existence of a principal U-bundle $p\colon EU\to BU$ with contractible total space. Hence

$$\begin{aligned}\mathbb{Z}\times BU&\simeq\Omega^2(\mathbb{Z}\times BU)\\ U&\simeq\Omega^2U\,.\end{aligned}$$

Following Atiyah and Hirzebruch [**AH61**], we can therefore define an Ω-spectrum KU with

$$KU_k=\begin{cases}\mathbb{Z}\times BU & \text{for } k \text{ even,}\\ U & \text{for } k \text{ odd,}\end{cases}$$

having adjoint structure maps given by the two homotopy equivalences above.

Working with real vector bundles, we have the classification

$$\mathrm{Vect}^{\mathbb{R}}(X)\cong[X_+,\coprod_{k\ge0}BO(k)]$$

with $BO(k)\simeq Gr_k(\mathbb{R}^\infty)$, with localization

$$KO(X)\cong[X_+,\mathbb{Z}\times BO]$$

where BO is the classifying space of the infinite orthogonal group $O=\bigcup_k O(k)$. When X is finite-dimensional, this is the *real K-group* of X. The (real) Bott periodicity theorem [**Bot59**] asserts that

$$\begin{aligned}\mathbb{Z}\times BO&\simeq\Omega(U/O)\\ U/O&\simeq\Omega(Sp/U)\\ Sp/U&\simeq\Omega Sp\\ Sp&\simeq\Omega(\mathbb{Z}\times BSp)\\ \mathbb{Z}\times BSp&\simeq\Omega(U/Sp)\\ U/Sp&\simeq\Omega(O/U)\\ O/U&\simeq\Omega O\\ O&\simeq\Omega(\mathbb{Z}\times BO)\,.\end{aligned}$$

Here $Sp=\bigcup_k Sp(k)$ denotes the infinite symplectic group, and the homogeneous spaces are formed using complexification $O\to U$, symplectification $U\to Sp$, (forgetful) complexification $Sp\to U$ and realification $U\to O$. It follows that

$$\mathbb{Z}\times BO\simeq\Omega^8(\mathbb{Z}\times BO)\,.$$

We can therefore define an Ω-spectrum KO with

$$KO_k = \begin{cases} \mathbb{Z} \times BO & \text{for } k \equiv 0 \mod 8, \\ U/O & \text{for } k \equiv 1 \mod 8, \\ Sp/U & \text{for } k \equiv 2 \mod 8, \\ Sp & \text{for } k \equiv 3 \mod 8, \\ \mathbb{Z} \times BSp & \text{for } k \equiv 4 \mod 8, \\ U/Sp & \text{for } k \equiv 5 \mod 8, \\ O/U & \text{for } k \equiv 6 \mod 8, \\ O & \text{for } k \equiv 7 \mod 8, \end{cases}$$

having adjoint structure maps given by the eight homotopy equivalences above. It follows that $KO_k \simeq \Omega^\ell(\mathbb{Z} \times BO)$ where $k + \ell \equiv 0 \mod 8$, and we may assume that $0 \le \ell < 8$.

The associated cohomology theories are complex and real (topological) K-theory, with

$$KU^n(X) \cong [X_+, KU_n]$$
$$KO^n(X) \cong [X_+, KO_n]$$

(unreduced theories) for $n \ge 0$, extended 2- and 8-periodically, respectively, for $n < 0$. The coefficient groups of these theories are

$$\pi_*(KU) = \begin{cases} \mathbb{Z} & \text{for } * \text{ even}, \\ 0 & \text{for } * \text{ odd}, \end{cases}$$

and

$$\pi_*(KO) = \begin{cases} \mathbb{Z} & \text{for } * \equiv 0, 4 \mod 8, \\ \mathbb{Z}/2 & \text{for } * \equiv 1, 2 \mod 8, \\ 0 & \text{for } * \equiv 3, 5, 6, 7 \mod 8. \end{cases}$$

The external tensor product of vector bundles induces pairings

$$KU^n(X) \otimes KU^m(Y) \longrightarrow KU^{n+m}(X \times Y)$$
$$KO^n(X) \otimes KO^m(Y) \longrightarrow KO^{n+m}(X \times Y)$$

turning these coefficient groups into graded rings. Their structures are

$$\pi_*(KU) = \mathbb{Z}[u, u^{-1}]$$

with $|u| = 2$, and

$$\begin{aligned} \pi_*(KO) &= \mathbb{Z}[\eta, A, B, B^{-1}]/(2\eta, \eta^3, \eta A, A^2 - 4B) \\ &= (\dots, \mathbb{Z}\{1\}, \mathbb{Z}/2\{\eta\}, \mathbb{Z}/2\{\eta^2\}, 0, \mathbb{Z}\{A\}, 0, 0, 0, \mathbb{Z}\{B\}, \dots) \end{aligned}$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$. Complexification of vector bundles induces a natural transformation $c\colon KO^n(X) \to KU^n(X)$, and the induced ring homomorphism

$$c\colon \pi_*(KO) \longrightarrow \pi_*(KU)$$

is given by $c(\eta) = 0$, $c(A) = 2u^2$ and $c(B) = u^4$.

When $X = BG$ is the classifying space of a finite group G, Atiyah [**Ati61b**] proved that

$$KU^0(BG) \cong R(G)^\wedge_{I(G)}$$

is the completion of the complex *representation ring* $R(G)$ at its augmentation ideal, while

$$KU^1(BG) = 0\,.$$

Since $KU^n(BG)$ is 2-periodic, this calculates $KU^*(BG)$ in all degrees. The corresponding result for connected compact Lie groups G is due to Atiyah and Hirzebruch [**AH61**], while the result for general compact Lie groups is part of the Atiyah–Segal completion theorem [**AS69**], and motivated the Segal conjecture for stable cohomotopy, mentioned above. The corresponding results for real K-theory are due to Anderson [**And64**]. In particular

$$\begin{aligned} KO^0(BG) &\cong RO(G)^\wedge_{I(G)} \\ KO^1(BG) &\cong 0 \\ &\dots \\ KO^4(BG) &\cong RSp(G)^\wedge_{I(G)} \\ KO^5(BG) &\cong 0 \\ &\dots\,, \end{aligned}$$

where $I(G)$ now denotes the augmentation ideal in the real representation ring $RO(G)$, and $RSp(G)$ is the $RO(G)$-module of quaternionic representations.

REMARK 9.4.12. The expressions in Remark 9.4.3 for $M_*(X)$ and $M^*(X)$ for spaces X, in terms of morphisms in the stable category, suggest that we can extend these homology and cohomology theories over the suspension spectrum functor

$$\begin{aligned} \Sigma^\infty \colon \operatorname{Ho}(\mathscr{T}) &\longrightarrow \operatorname{Ho}(Sp^{\mathbb{N}}) \\ X &\longmapsto \Sigma^\infty X = F_0 X\,, \end{aligned}$$

so as to define the M-homology

$$M_n(X) = \pi_n(M \wedge X) = [S^n, M \wedge X]$$

and M-cohomology

$$M^n(X) = [X, S^n \wedge M]$$

of a *spectrum* X.

For cohomology, this makes sense as stated, and extends the previous definition. In particular, with $H = H\mathbb{F}_p$ the mod p Eilenberg–MacLane spectrum, we see that

$$A^n \cong H^n(H) = [H, S^n \wedge H]$$

recovers the stable cohomology operations of type $(\mathbb{F}_p; \mathbb{F}_p, n)$, i.e., the degree n part of the Steenrod algebra. Hence there is an algebra isomorphism

$$A \cong H^*(H)\,.$$

For homology, we have not yet made sense of $M \wedge X$ when M and X are both sequential spectra. This can be done [**Ada74**, §III.4], but a more satisfactory construction can be given in the context of orthogonal spectra, which we turn to in the next section. This will then lead to the formula

$$A_n \cong H_n(H) = [S^n, H \wedge H]$$

for the degree n part of the dual Steenrod algebra, and there is a Hopf algebra isomorphism

$$A_* \cong H_*(H)\,.$$

9.5. Orthogonal spaces

We now give a different model for the stable category, namely as the homotopy category $\mathrm{Ho}(Sp^{\mathbb{O}}) = Sp^{\mathbb{O}}[\mathscr{W}^{-1}]$ obtained by inverting the stable equivalences in a category $Sp^{\mathbb{O}}$ of orthogonal spectra. The categories $Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$ are not equivalent, but their associated homotopy categories

$$\mathrm{Ho}(Sp^{\mathbb{N}}) \simeq \mathrm{Ho}(Sp^{\mathbb{O}})$$

are, so that we may replace our earlier use of $\mathrm{Ho}(Sp^{\mathbb{N}})$ with $\mathrm{Ho}(Sp^{\mathbb{O}})$. This has the advantage that $Sp^{\mathbb{O}}$ is closed symmetric monoidal, with the (orthogonal) sphere spectrum S as unit object, a symmetric monoidal smash product $L \wedge M$ as monoidal pairing, and a function spectrum $F(M, N)$ as the closed structure. Furthermore, these data induce a closed symmetric monoidal structure on $\mathrm{Ho}(Sp^{\mathbb{O}})$.

Orthogonal spectra were defined in [**May80**], under the name of $\mathscr{I}_*$-prespectra, and orthogonal ring spectra were defined even earlier in [**May77**], under the name of $\mathscr{I}_*$-prefunctors, but the good properties mentioned above first became apparent with the introduction by Jeff Smith of symmetric spectra in 1994 ("Specters of symmetry", unpublished), and the unification of the two ideas in [**MMSS01**].

Following Schwede ((ETC: reference?)) we index orthogonal spectra on a minimal (= skeletal) subcategory of the category of all finite-dimensional inner product spaces and isometries used by Mandell–May–Schwede–Shipley. See also "model structure on orthogonal spectra" on https://ncatlab.org/nlab/show/HomePage and [**HHR16**, Prop. A.12].

Definition 9.5.1. Let $O(k)$ denote the group of orthogonal $k \times k$ matrices. It acts linearly on $\mathbb{R}^k$ and its one-point compactification $S^k = \mathbb{R}^k \cup \{\infty\}$. We consider $O(k) \times O(\ell)$ as a subgroup of $O(k + \ell)$, via the block sum of matrices.

We continue to work in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces and basepoint preserving maps.

Definition 9.5.2. An *orthogonal spectrum* M consists of a left $O(k)$-space M_k and a map

$$\sigma \colon M_k \wedge S^1 \longrightarrow M_{k+1}$$

for each $k \geq 0$, such that the composite

$$\sigma^\ell \colon M_k \wedge S^\ell \xrightarrow{\sigma} M_{k+1} \wedge S^{\ell-1} \xrightarrow{\sigma} \dots \xrightarrow{\sigma} M_{k+\ell-1} \wedge S^1 \xrightarrow{\sigma} M_{k+\ell}$$

is $O(k) \times O(\ell)$-equivariant for every $k, \ell \geq 0$.

To justify this definition, we take a step back and define a closed symmetric monoidal category of orthogonal spaces. The sphere spectrum S is a commutative monoid in this category, and the category of right S-modules becomes the category of orthogonal spectra. This is closed symmetric monoidal because S is commutative. There is an analogous story for sequential spectra, which are the right S-modules in a category of sequential spaces, but in this case S is not commutative, so the module category does not inherit the monoidal structure.

Definition 9.5.3. Let $\mathbb{O}$ be the topological category with objects the integers $k \geq 0$, and with morphism spaces

$$\mathbb{O}(k, \ell) = \begin{cases} O(k) & \text{for } k = \ell, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is symmetric monoidal, with unit object 0 and monoidal pairing

$$+\colon (k,\ell) \longmapsto k+\ell$$
$$+\colon (A,B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A \in O(k)$ and $B \in O(\ell)$. The symmetry isomorphism

$$\tau\colon k+\ell \xrightarrow{\cong} \ell+k$$

equals the block permutation matrix

$$\chi_{k,\ell} = \begin{pmatrix} 0 & I_\ell \\ I_k & 0 \end{pmatrix}$$

where $I_k \in O(k)$ and $I_\ell \in O(\ell)$ are the identity matrices.

The object k may also be viewed as $\mathbb{R}^k$ with the standard inner product, in which case the monoidal pairing is the direct sum of inner product spaces, and the symmetry is the usual twist isomorphism $\mathbb{R}^k \oplus \mathbb{R}^\ell \cong \mathbb{R}^\ell \oplus \mathbb{R}^k$.

Definition 9.5.4. An *orthogonal space* is a continuous functor

$$M\colon \mathbb{O} \longrightarrow \mathscr{T}.$$

A *map* of orthogonal spaces is a continuous natural transformation $f\colon M \to N$. We write $\mathscr{T}^{\mathbb{O}}$ for the topological category of orthogonal spaces.

Explicitly, M maps each $k \ge 0$ to a space $M_k \in \mathscr{T}$, and for each $A \in O(k)$ we have a map $M(A)\colon M_k \to M_k$, which defines a continuous left group action

$$\begin{aligned} \lambda\colon O(k)_+ \wedge M_k &\longrightarrow M_k \\ (A,x) &\longmapsto Ax. \end{aligned}$$

Lemma 9.5.5. *The category $\mathscr{T}^{\mathbb{O}}$ is tensored and cotensored over $\mathscr{T}$, by setting*

$$\begin{aligned} (X \wedge M)_k &= X \wedge M_k \\ (M \wedge X)_k &= M_k \wedge X \\ F(X,M)_k &= \mathrm{Map}(X, M_k) \end{aligned}$$

for $M \in \mathscr{T}^{\mathbb{O}}$ and $X \in \mathscr{T}$, with the evident $O(k)$-actions. There are natural homeomorphisms

$$\mathrm{Map}(X, \mathscr{T}^{\mathbb{O}}(M,N)) \cong \mathscr{T}^{\mathbb{O}}(M \wedge X, N) \cong \mathscr{T}^{\mathbb{O}}(M, F(X,N)).$$

Lemma 9.5.6. *The category $\mathscr{T}^{\mathbb{O}}$ has all small limits and colimits, given for any diagram $\alpha \mapsto M_\alpha$ by*

$$\begin{aligned} (\lim_\alpha M_\alpha)_k &= \lim_\alpha (M_\alpha)_k \\ (\operatorname*{colim}_\alpha M_\alpha)_k &= \operatorname*{colim}_\alpha (M_\alpha)_k, \end{aligned}$$

with the evident $O(k)$-actions.

LEMMA 9.5.7. *The category* $(\mathscr{T}^{\mathbb{O}}, U, \otimes, \mathrm{Hom})$ *of orthogonal spaces is closed symmetric monoidal, with unit object given by* $U_0 = S^0$ *and* $U_k = *$ *for* $k \geq 1$, *with monoidal pairing given by the Day convolution product*

$$(L \otimes M)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j$$

and with closed structure given by

$$\mathrm{Hom}(M, N)_i = \prod_{i+j=k} \mathrm{Map}(M_j, N_k)^{O(j)} .$$

The symmetry $\tau \colon L \otimes M \xrightarrow{\cong} M \otimes L$ *maps*

$$C \wedge x \wedge y \in O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j$$

to

$$C\chi_{j,i} \wedge y \wedge x \in O(k)_+ \wedge_{O(j)\times O(i)} M_j \wedge L_i$$

for $i + j = k$. *There is a natural homeomorphism*

$$\mathscr{T}^{\mathbb{O}}(L \otimes M, N) \cong \mathscr{T}^{\mathbb{O}}(L, \mathrm{Hom}(M, N)) .$$

PROOF. The Day convolution can also be written as the topological colimit

$$(L \otimes M)_k = \operatorname*{colim}_{i,j,i+j\to k} L_i \wedge M_j .$$

where the indexing category is the left fiber $+/k$ of $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ at k.

The symmetry is well defined, because

$$C \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \wedge x \wedge y = C \wedge Ax \wedge By$$

is mapped to

$$C \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \chi_{j,i} \wedge y \wedge x = C\chi_{j,i} \wedge By \wedge Ax .$$

It would not be well-defined without the factor $\chi_{j,i}$.

The adjunction homomorphism for L, M and N can be expanded to

$$\prod_k \mathscr{T}((L \otimes M)_k, N_k)^{O(k)} \cong \prod_{i,j} \mathscr{T}(L_i \wedge M_j, N_{i+j})^{O(i)\times O(j)}$$
$$\cong \prod_{i,j} \mathscr{T}(L_i, \mathrm{Map}(M_j, N_{i+j})^{O(j)})^{O(i)} \cong \prod_i \mathscr{T}(L_i, \mathrm{Hom}(M, N)_i)^{O(i)}$$

□

DEFINITION 9.5.8. The (orthogonal) *sphere spectrum* S has underlying orthogonal space

$$S \colon k \longmapsto S_k = S^k = \mathbb{R}^k \cup \{\infty\}$$

based at ∞, with $A \in O(k)$ acting on S^k by its linear action on $\mathbb{R}^k$.

The following lemma makes the whole theory work.

Lemma 9.5.9. *The sphere spectrum is a commutative monoid in orthogonal spaces, with unit* $\eta\colon U \to S$ *given by the identity* $\eta_0\colon U_0 = S^0 = S_0$, *and with multiplication* $\mu\colon S \otimes S \to S$ *given by the* $O(k)$*-equivariant maps*

$$\mu_k\colon (S\otimes S)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j \longrightarrow S^k = S_k$$

that are left adjoint to the $O(i)\times O(j)$*-equivariant identifications*

$$S^i \wedge S^j \xrightarrow{\cong} S^k$$

for $i+j=k$.

Proof. Associativity and unitality are clear, so the key thing to check is commutativity, which amounts to the commutativity of the diagrams

$$\begin{tikzcd}
\bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j \arrow[rr, "\tau"] \arrow[dr, "\mu_k"'] & & \bigvee_{j+i=k} O(k)_+ \wedge_{O(j)\times O(i)} S^j \wedge S^i \arrow[dl, "\mu_k"] \\
& S^k &
\end{tikzcd}$$

for all $k \ge 0$. This follows from the observation that $I_k \wedge x \wedge y$ for $x \in S^i$ and $y \in S^j$ maps via $\chi_{j,i} \wedge y \wedge x$ to $\chi_{j,i} \cdot (y \wedge x) = x \wedge y$ along the upper and right hand route, and maps directly to $x \wedge y$ along the left hand route. $\square$

Lemma 9.5.10. *For each* $\ell \ge 0$ *the evaluation functor* $Ev_\ell\colon \mathscr{T}^{\mathbb{O}} \longrightarrow \mathscr{T}$ *given by* $Ev_\ell(M) = M_\ell$ *has left adjoint* $G_\ell\colon \mathscr{T} \longrightarrow \mathscr{T}^{\mathbb{O}}$ *given by*

$$G_\ell(X)_k = \begin{cases} O(\ell)_+ \wedge X & \text{for } k=\ell, \\ * & \text{otherwise.}\end{cases}$$

Lemma 9.5.11. *There is a natural isomorphism*

$$G_i(X) \otimes G_j(Y) \cong G_{i+j}(X \wedge Y)$$

for $X, Y \in \mathscr{T}$ *and* $i, j \ge 0$.

9.6. Orthogonal spectra

Definition 9.6.1 ([**MMSS01**, Def. 1.9, Ex. 4.4]). An *orthogonal spectrum* M is a right S-module in orthogonal spaces. A map $f\colon M \to N$ of orthogonal spectra is a map of right S-modules. We write $Sp^{\mathbb{O}}$ for the topological category of orthogonal spectra.

Remark 9.6.2. This agrees with Definition 9.5.2, because the right S-action $\rho\colon M \otimes S \to M$ is given by $O(k)\times O(\ell)$-equivariant maps

$$\rho_{k,\ell}\colon M_k \wedge S^\ell \longrightarrow M_{k+\ell}$$

that satisfy unitality and associativity, and which are therefore determined by the components $\sigma = \rho_{k,1}\colon M_k \wedge S^1 \to M_{k+1}$, for all $k \ge 0$. Conversely, the latter determine the right S-action when the equivariance condition for $\sigma^\ell\colon M_k \wedge S^\ell \to M_{k+\ell}$ is satisfied.

LEMMA 9.6.3. *The category $Sp^{\mathbb{O}}$ is tensored and cotensored over $\mathcal{T}$, by setting*

$$\begin{aligned}
(X \wedge M)_k &= X \wedge M_k \\
(M \wedge X)_k &= M_k \wedge X \\
F(X, M)_k &= \mathrm{Map}(X, M_k)
\end{aligned}$$

for $M \in Sp^{\mathbb{O}}$ and $X \in \mathcal{T}$. There are natural homeomorphisms

$$\mathrm{Map}(X, Sp^{\mathbb{O}}(M, N)) \cong Sp^{\mathbb{O}}(M \wedge X, N) \cong Sp^{\mathbb{O}}(M, F(X, N)) .$$

PROOF. The right S-actions are given by

$$X \wedge M_k \wedge S^\ell \xrightarrow{1 \wedge \sigma^\ell} X \wedge M_{k+\ell}$$

$$M_k \wedge X \wedge S^\ell \xrightarrow{1 \wedge \tau} M_k \wedge S^\ell \wedge X \xrightarrow{\sigma^\ell \wedge 1} M_{k+\ell} \wedge X$$

$$\mathrm{Map}(X, M_k) \wedge S^\ell \xrightarrow{(?)} \mathrm{Map}(X, M_k \wedge S^\ell) \xrightarrow{\mathrm{Map}(1, \sigma^\ell)} \mathrm{Map}(X, M_{k+\ell}) .$$

□

LEMMA 9.6.4. *The category $Sp^{\mathbb{O}}$ has all small limits and colimits, given for any diagram $\alpha \mapsto M_\alpha$ by*

$$\begin{aligned}
(\lim_\alpha M_\alpha)_k &= \lim_\alpha (M_\alpha)_k \\
(\operatorname{colim}_\alpha M_\alpha)_k &= \operatorname{colim}_\alpha (M_\alpha)_k .
\end{aligned}$$

PROOF. The right S-actions are given by

$$\left(\lim_\alpha (M_\alpha)_k\right) \wedge S^\ell \xrightarrow{(?)} \lim_\alpha \left((M_\alpha)_k \wedge S^\ell\right) \xrightarrow{\lim \sigma^\ell} \lim_\alpha (M_\alpha)_{k+\ell}$$

$$\left(\operatorname{colim}_\alpha (M_\alpha)_k\right) \wedge S^\ell \xleftarrow{\cong} \operatorname{colim}_\alpha \left((M_\alpha)_k \wedge S^\ell\right) \xrightarrow{\operatorname{colim} \sigma^\ell} \operatorname{colim}_\alpha (M_\alpha)_{k+\ell} .$$

□

LEMMA 9.6.5. *The forgetful functor $U \colon Sp^{\mathbb{O}} \to \mathcal{T}^{\mathbb{O}}$ has left adjoint $L \mapsto L \otimes S$ and right adjoint $N \mapsto \mathrm{Hom}(S, N)$. The evaluation functor $Ev_\ell \colon Sp^{\mathbb{O}} \to \mathcal{T}$ given by $Ev_\ell(M) = M_\ell$ has left adjoint $F_\ell \colon \mathcal{T} \to Sp^{\mathbb{O}}$ given by*

$$F_\ell(X) = G_\ell(X) \otimes S ,$$

so that

$$F_\ell(X)_k = \begin{cases} O(k)_+ \wedge_{O(k-\ell)} X \wedge S^{k-\ell} & \text{for } k \ge \ell, \\ * & \text{otherwise.} \end{cases}$$

PROOF. The left adjoint of the composite $Ev_\ell U \colon Sp^{\mathbb{O}} \to \mathcal{T}^{\mathbb{O}} \to \mathcal{T}$ is the composite of left adjoints $G_\ell(-) \otimes S \colon \mathcal{T} \to \mathcal{T}^{\mathbb{O}} \to Sp^{\mathbb{O}}$. This evaluates on X to the orthogonal spectrum $F_\ell(X)$ given at level k by

$$F_\ell(X)_k = (G_\ell(X) \otimes S)_k = \bigvee_{i+j=k} O(k)_+ \wedge_{O(i) \times O(j)} G_\ell(X)_i \wedge S^j ,$$

which equals

$$O(k)_+ \wedge_{O(\ell) \times O(k-\ell)} O(\ell)_+ \wedge X \wedge S^{k-\ell} \cong O(k)_+ \wedge_{O(k-\ell)} X \wedge S^{k-\ell}$$

for $k \ge \ell$, and is $*$ for $k < \ell$. □

Definition 9.6.6. The orthogonal *suspension spectrum* $\Sigma^\infty X = F_0 X$ of a space X is given by

$$(\Sigma^\infty X)_k = X \wedge S^k$$

with the standard $O(k)$-action on S^k, for each $k \geq 1$. For each integer n we define the orthogonal *n-sphere spectrum* S^n by

$$S^n = \begin{cases} F_0 S^n & \text{for } n \geq 0, \\ F_{-n} S^0 & \text{for } n < 0, \end{cases}$$

so that

$$(S^n)_k = \begin{cases} S^{n+k} & \text{for } n \geq 0, \\ O(k)_+ \wedge_{O(n+k)} S^{n+k} & \text{for } n < 0 \text{ and } n+k \geq 0, \\ * & \text{for } n+k < 0, \end{cases}$$

with the evident $O(k)$-action.

Definition 9.6.7. Given an orthogonal spectrum $M = (M_k, \sigma)_k$, the *underlying* sequential spectrum $UM = (M_k, \sigma)_k$ is obtained by forgetting the $O(k)$-action on M_k and ignoring the $O(k) \times O(\ell)$-equivariance condition on σ^ℓ, for each $k \geq 0$ and $\ell \geq 0$. Let

$$U \colon Sp^{\mathbb{O}} \longrightarrow Sp^{\mathbb{N}}$$

denote the forgetful functor.

Definition 9.6.8. The *homotopy groups* $\pi_*(M) = (\pi_n(M))_n$ of an orthogonal spectrum M are the homotopy groups of its underlying sequential spectrum:

$$\pi_n(M) = \pi_n(UM) = \operatorname*{colim}_k \pi_{n+k}(M_k)\,.$$

A map $f \colon M \to N$ of orthogonal spectra is a *stable equivalence* if the induced homomorphism $f_* \colon \pi_*(M) \to \pi_*(N)$ is an isomorphism, which is equivalent to asking that the underlying map of sequential spectra is a stable equivalence.

Proposition 9.6.9 ([**MMSS01**, Prop. 3.2]). *The forgetful functor $U \colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ admits a left adjoint, called the* prolongation *functor*

$$P \colon Sp^{\mathbb{N}} \longrightarrow Sp^{\mathbb{O}}\,,$$

which satisfies

$$P(F_\ell^{\mathbb{N}} X) = F_\ell X$$

for each $\ell \geq 0$ and space X, and which commutes with colimits.

Here $F_\ell^{\mathbb{N}}$ denotes the free functor that is left adjoint to $Ev_\ell \colon Sp^{\mathbb{N}} \to \mathscr{T}$, which was simply denoted F_ℓ in Section 9.2.

$$\begin{tikzcd}
\mathscr{T} \arrow[r, shift left, "G_\ell"] \arrow[rr, bend left=50, "F_\ell"] & \mathscr{T}^{\mathbb{O}} \arrow[l, shift left, "Ev_\ell"] \arrow[r, shift left, "-\otimes S"] & Sp^{\mathbb{O}} \arrow[l, shift left, "U"] \arrow[r, shift right, "U"'] \arrow[ll, bend left=50, "Ev_\ell"] \arrow[rr, bend right=50, "\pi_*"'] & Sp^{\mathbb{N}} \arrow[l, shift right, "P"'] \arrow[r, "\pi_*"] & grAb
\end{tikzcd}$$

EXAMPLE 9.6.10. $F_0(S^0) = S$ equals the sphere spectrum, while $F_1(S^1)$ has k-th space

$$F_1(S^1)_k = O(k)_+ \wedge_{O(k-1)} (S^1 \wedge S^{k-1}) \cong Th(\epsilon^1 \oplus \tau_{S^{k-1}})$$

for each $k \geq 1$. The left adjoint of the identity $S^1 = Ev_1(S)$ is a map of orthogonal spectra

$$\lambda\colon F_1(S^1) \longrightarrow S = F_0(S^0)$$

given at levels $k \geq 1$ by the $O(k)$-equivariant extension

$$\lambda_k\colon O(k)_+ \wedge_{O(k-1)} S^k \longrightarrow S^k$$

of the $O(k-1)$-action on $S^1 \wedge S^{k-1} \cong S^k$. This is $2(k-1)$-connected, so λ is a stable equivalence. More generally [**MMSS01**, Lem. 8.6], the left adjoint

$$\lambda^\ell\colon F_{\ell+1}(S^1) \longrightarrow F_\ell(S^0)$$

of the canonical inclusion $S^1 \to O(\ell+1)_+ \wedge_{O(1)} S^1 = F_\ell(S^0)_{\ell+1}$ is a stable equivalence for each $\ell \geq 0$. This is the feature of orthogonal spectra that allows us to define the stable equivalences as the π_*-isomorphisms.

REMARK 9.6.11. In the parallel theory of symmetric spectra, based on the symmetric groups Σ_k in place of the orthogonal groups $O(k)$, the corresponding maps λ^ℓ are not π_*-isomorphisms, but must nonetheless be taken to be stable equivalences, hence invertible in the stable category, to ensure that the stably fibrant objects are the Ω-spectra. By working with orthogonal spectra we do not need to distinguish between stable equivalences and π_*-isomorphisms, which simplifies the exposition.

DEFINITION 9.6.12. The *stable category* $\mathrm{Ho}(Sp^{\mathbb{O}})$ is the localization

$$Sp^{\mathbb{O}} \xrightarrow{\iota} Sp^{\mathbb{O}}[\mathscr{W}^{-1}] = \mathrm{Ho}(Sp^{\mathbb{O}})$$

of the category of orthogonal spectra with respect to the subcategory $\mathscr{W}$ of stable equivalences. For orthogonal spectra M and N, let

$$[M, N] = \mathrm{Ho}(Sp^{\mathbb{O}})(M, N)$$

denote the set of morphisms in the stable category from M to N.

This does not conflict with our earlier usage, because of the following theorem. This is a consequence of a stronger statement, proved in [**MMSS01**, Thm. 10.4], saying that the adjunction (P, U) defines a Quillen equivalence between the stable model structures on $Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$.

THEOREM 9.6.13. *The functor $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ preserves stable equivalences and induces an equivalence of categories*

$$U\colon \mathrm{Ho}(Sp^{\mathbb{O}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}}).$$

We have the following analogue of Definition 9.2.9.

DEFINITION 9.6.14 ([**MMSS01**, Def. 6.2, Def. 5.4, Thm. 6.5]). Let I be the set of inclusions $i\colon S^{n-1}_+ \to D^n_+$ for $n \geq 0$. Let $FI = F^{\mathbb{O}}I$ be the set of maps of orthogonal spectra $F_\ell i\colon F_\ell S^{n-1}_+ \to F_\ell D^n_+$, for $\ell \geq 0$ and $n \geq 0$.

A map $i\colon M \to N$ of orthogonal spectra is a *relative cell spectrum* if N is the colimit of a sequence of maps starting with M, where each map is obtained by cobase change from a sum of maps in FI. A map $i\colon M \to N$ in $Sp^{\mathbb{O}}$ is a *Quillen cofibration* (= q-cofibration) if it is a retract of a relative cell spectrum.

We say that an orthogonal spectrum N is a *cell spectrum* if $* \to N$ is a relative cell spectrum, and that N is *Quillen cofibrant* (= q-cofibrant) if $* \to N$ is a Quillen cofibration. If $q\colon M^c \xrightarrow{\sim} M$ is a stable equivalence of orthogonal spectra, and M^c is Quillen cofibrant, then we say that M^c is a *cofibrant replacement* for M.

We also have the following analogue of Definition 9.2.14.

Definition 9.6.15 ([**MMSS01**, Prop. 9.5]). A map $p\colon M \to N$ of orthogonal spectra is a *stable fibration* (= q-fibration) if and only if the underlying map $Up\colon UM \to UN$ of sequential spectra is a stable fibration, i.e., if $p_k\colon M_k \to N_k$ is a (non-equivariant) Serre fibration and

$$\begin{tikzcd} M_k \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_k"'] & \Omega M_{k+1} \arrow[d, "\Omega p_{k+1}"] \\ N_k \arrow[r, "\tilde{\sigma}"] & \Omega N_{k+1} \end{tikzcd}$$

is a (non-equivariant) weak homotopy pullback, for each $k \geq 0$.

We say that an orthogonal spectrum M is *stably fibrant* (= q-fibrant) if $M \to *$ is a stable fibration. If $j\colon N \xrightarrow{\sim} N^f$ is a stable equivalence of orthogonal spectra, and N^f is stably fibrant, then we say that N^f is a *fibrant replacement* for N.

The prolongation of a (relative) sequential cell spectrum is a (relative) orthogonal cell spectrum, with the same cell filtration. An orthogonal spectrum is stably fibrant if and only if it is an Ω-spectrum, i.e., if each adjoint structure map is a weak homotopy equivalence.

Theorem 9.6.16. *The category $Sp^{\mathbb{O}}$ of orthogonal spectra is a ((ETC: compactly generated, proper, topological)) model category with respect to the classes of stable equivalences, Quillen cofibrations and stable fibrations.*

Again, we refer to [**MMSS01**, Thm. 9.2] for the proof. To compare the stable model structures on $Sp^{\mathbb{N}}$ and $Sp^{\mathbb{O}}$ we follow [**Hov99**, §1.3] and discuss Quillen adjunctions and Quillen equivalences.

Definition 9.6.17 ([**Hov99**, Def. 1.3.1]). Let $\mathscr{C}$ and $\mathscr{D}$ be model categories, and let $F\colon \mathscr{C} \to \mathscr{D}$ be left adjoint to $G\colon \mathscr{D} \to \mathscr{C}$, so that there is a natural bijection

$$\mathscr{D}(F(X), Y) \cong \mathscr{C}(X, G(Y)).$$

We say that the adjoint pair (F, G) is a *Quillen adjunction* if

(1) F preserves cofibrations, and
(2) G preserves fibrations.

This is just one of four equivalent formulations of this definition, because of the following lemma.

Lemma 9.6.18 ([**Hov99**, Lem. 1.3.4]). *F preserves cofibrations if and only if G preserves acyclic fibrations, and G preserves fibrations if and only if F preserves acyclic cofibrations.*

Proof. If Fi is a cofibration in $\mathscr{D}$ for each cofibration i in $\mathscr{C}$, and q is an acyclic fibration in $\mathscr{D}$, then Gq has the right lifting property with respect to each cofibration i, hence is itself an acyclic fibration. The other three cases are similar. □

Definition 9.6.19 ([**Hov99**, Def. 1.3.6]). Given a Quillen adjunction (F, G), let the *total left derived functor* $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ be defined by

$$(LF)(X) = F(X^c)$$

where $X^c \sim X$ is a (functorially defined) cofibrant replacement. Let the *total right derived functor* $RG\colon \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$ be defined by

$$(RG)(Y) = G(Y^f)$$

where $Y \sim Y^f$ is a (functorially defined) fibrant replacement.

Lemma 9.6.20 ([**Hov99**, Lem. 1.3.10]). *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and (F, G) a Quillen adjunction. Then $LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ is left adjoint to $RG\colon \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$, so that (LF, RG) form an adjoint pair.*

Definition 9.6.21 ([**Hov99**, Def. 1.3.12]). A Quillen adjunction (F, G) is called a *Quillen equivalence* when, for each cofibrant X in $\mathscr{C}$ and each fibrant Y in $\mathscr{D}$, a map $f\colon F(X) \to Y$ is a weak equivalence in $\mathscr{D}$ if and only if its right adjoint $g\colon X \to G(Y)$ is a weak equivalence in $\mathscr{C}$.

Proposition 9.6.22 ([**Hov99**, Prop. 1.3.13, Cor. 1.3.16]). *Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ be a Quillen adjunction. The following are equivalent:*

1. *(F, G) is a Quillen equivalence.*
2. *$LF\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ and $RG\colon \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$ are adjoint equivalences of categories.*
3. *G reflects weak equivalences between fibrant objects and, for every cofibrant X in $\mathscr{C}$ the map $\eta\colon X \to G((FX)^f)$ is a weak equivalence.*

Theorem 9.6.23 ([**MMSS01**, Thm. 10.4]). *The adjoint pair (P, U), with $P\colon Sp^{\mathbb{N}} \to Sp^{\mathbb{O}}$ and $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$, is a Quillen equivalence. Hence*

$$LP\colon \mathrm{Ho}(Sp^{\mathbb{N}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{O}})$$

$$RU\colon \mathrm{Ho}(Sp^{\mathbb{O}}) \xrightarrow{\simeq} \mathrm{Ho}(Sp^{\mathbb{N}})$$

are adjoint equivalences of categories.

Sketch proof. $U\colon Sp^{\mathbb{O}} \to Sp^{\mathbb{N}}$ preserves stable equivalences and stable fibrations, so (P, U) is a Quillen adjunction. Furthermore, U reflects stable equivalences and, for every Quillen cofibrant M in $Sp^{\mathbb{N}}$ the map $\eta\colon M \to UPM \sim U((PM)^f)$ is a stable equivalence. This verifies the equivalent conditions of Proposition 9.6.22. □

9.7. Closed symmetric monoidal structure

Since S is commutative, the category of right S-modules is isomorphic to the category of left S-modules. A right S-action $\rho\colon M \otimes S \to M$ determines a left S-action

$$\lambda\colon S \otimes M \xrightarrow{\tau} M \otimes S \xrightarrow{\rho} M$$

and vice versa. Furthermore, we can form the tensor product $L \otimes_S M$ and the function object $\mathrm{Hom}_S(M, N)$ of right S-modules L, M and N, and these remain right

S-modules. These define the *smash product* and *function spectrum*, respectively, for orthogonal spectra.

THEOREM 9.7.1. *The category $Sp^{\mathbb{O}}$ of orthogonal spectra is closed symmetric monoidal, with unit object the sphere spectrum S, monoidal pairing the smash product given by the coequalizer*

$$L \otimes S \otimes M \underset{1\otimes\lambda}{\overset{\rho\otimes 1}{\rightrightarrows}} L \otimes M \xrightarrow{\pi} L \otimes_S M = L \wedge M$$

and with closed structure the function spectrum given by the equalizer

$$F(M, N) = \mathrm{Hom}_S(M, N) \xrightarrow{\iota} \mathrm{Hom}(M, N) \underset{\rho_*}{\overset{\rho^*}{\rightrightarrows}} \mathrm{Hom}(M \otimes S, N)\,.$$

((ETC: Explain $\rho^* = \mathrm{Hom}(\rho, 1)$ and ρ_*.)) The symmetry $\tau \colon L \wedge M \xrightarrow{\cong} M \wedge L$ is induced by $\tau \colon L \otimes M \xrightarrow{\cong} M \otimes L$. There is a natural homeomorphism

$$Sp^{\mathbb{O}}(L \wedge M, N) \cong Sp^{\mathbb{O}}(L, F(M, N))\,.$$

REMARK 9.7.2. The coequalizer defining $(L \wedge M)_k$ can be expanded as follows.

$$\begin{array}{c} \bigvee_{a+b+c=k} O(k)_+ \wedge_{O(a)\times O(b)\times O(c)} L_a \wedge S^b \wedge M_c \\ {\scriptstyle \rho\wedge 1}\Big\downarrow\Big\downarrow{\scriptstyle 1\wedge\lambda} \\ \bigvee_{i+j=k} O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j \\ \Big\downarrow{\scriptstyle \pi} \\ (L \wedge M)_k \end{array}$$

The identifications for $b = 1$ generate the remaining ones, and set the composite

$$L_a \wedge S^1 \wedge M_c \xrightarrow{\sigma\wedge 1} L_{a+1} \wedge M_c \longrightarrow (L \wedge M)_{a+1+c}$$

equal to the composite

$$L_a \wedge S^1 \wedge M_c \xrightarrow{1\wedge\sigma\tau} \{\chi_{c,1}\}_+ \wedge L_a \wedge M_{c+1} \longrightarrow (L \wedge M)_{a+1+c}\,,$$

for all $a \ge 0$ and $c \ge 0$.

The equalizer defining $F(M, N)_i$ can be expanded as below.

$$\begin{array}{c} F(M, N)_i \\ \Big\downarrow{\scriptstyle \iota} \\ \prod_{i+j=k} \mathrm{Map}(M_j, N_k)^{O(j)} \\ {\scriptstyle \rho^*}\Big\downarrow\Big\downarrow{\scriptstyle \rho_*} \\ \prod_{i+a+b=c} \mathrm{Map}(M_a \wedge S^b, N_c)^{O(a)\times O(b)} \end{array}$$

The conditions for $b = 1$ generate the remaining ones, and demand that the composite

$$F(M, N)_i \longrightarrow \mathrm{Map}(M_{a+1}, N_{i+a+1}) \xrightarrow{\mathrm{Map}(\sigma,1)} \mathrm{Map}(M_a \wedge S^1, N_{i+a+1})$$

is equal to the composite

$$F(M,N)_i \longrightarrow \mathrm{Map}(M_a, N_{i+a}) \overset{\mathrm{Map}(-\wedge S^1,\sigma)}{\longrightarrow} \mathrm{Map}(M_a \wedge S^1, N_{i+a+1}) .$$

DEFINITION 9.7.3. For $i+j=k$ let $\iota_{i,j} \colon L_i \wedge M_j \to (L \wedge M)_k$ map $x \wedge y \in L_i \wedge M_j$ to the image of $I_k \wedge x \wedge y \in O(k)_+ \wedge_{O(i)\times O(j)} L_i \wedge M_j \subset (L \otimes M)_k$ under $\pi_k \colon (L \otimes M)_k \to (L \otimes_S M)_k = (L \wedge M)_k$.

((ETC: Give bilinearity diagram for pairings $L \wedge M \to N$?))

The smash product of spectra extends the smash product of spaces, in the following sense.

LEMMA 9.7.4. *There is a natural isomorphism*

$$F_i(X) \wedge F_j(Y) \cong F_{i+j}(X \wedge Y) .$$

for $X, Y \in \mathscr{T}$ and $i, j \geq 0$.

PROOF. $G_i(X) \otimes S \otimes_S G_j(Y) \otimes S \cong G_{i+j}(X \wedge Y) \otimes S$ using Lemma 9.5.11. □

EXAMPLE 9.7.5. The isomorphism

$$S^1 \wedge S^{-1} = F_0 S^1 \wedge F_1 S^0 \cong F_1 S^1$$

followed by the stable equivalence $\lambda \colon F_1 S^1 \to S$ from Example 9.6.10 define a stable equivalence

$$S^1 \wedge S^{-1} \overset{\sim}{\longrightarrow} S .$$

The smash product of spectra also generalizes the smash product of a space with a spectrum.

LEMMA 9.7.6. *For $X \in \mathscr{T}$ and $M \in Sp^{\mathbb{O}}$ there are natural isomorphisms*

$$\Sigma^\infty X \wedge M \cong X \wedge M$$
$$M \wedge \Sigma^\infty X \cong M \wedge X$$
$$F(\Sigma^\infty X, M) \cong F(X, M) .$$

The homotopy group functor π_* is compatible with the smash product of orthogonal spectra and the tensor product of graded abelian groups, in a lax sense.

PROPOSITION 9.7.7. *There is a natural homomorphism*

$$\cdot \colon \pi_*(L) \otimes \pi_*(M) \longrightarrow \pi_*(L \wedge M)$$

and a homomorphism

$$\mathbb{Z} \longrightarrow \pi_*(S)$$

that make $\pi_* \colon Sp^{\mathbb{O}} \longrightarrow grAb$ a lax symmetric monoidal functor.

REMARK 9.7.8. For π_* to be a lax monoidal functor means that the two evident composite pairings

$$\pi_*(L) \otimes \pi_*(M) \otimes \pi_*(N) \longrightarrow \pi_*(L \wedge M \wedge N)$$

are equal (where we have suppressed the associativity isomorphisms), together with two unitality conditions, see [**ML71**, §XI.2]. To be symmetric then means that the square

$$\begin{tikzcd}
\pi_*(L) \otimes \pi_*(M) \arrow[r, "\cdot"] \arrow[d, "\tau"'] & \pi_*(L \wedge M) \arrow[d, "\tau_*"] \\
\pi_*(M) \otimes \pi_*(L) \arrow[r, "\cdot"] & \pi_*(M \wedge L)
\end{tikzcd}$$

commutes. Lax monoidal functors send monoids to monoids, and lax symmetric monoidal functors take commutative monoids to commutative monoids. There is a natural homomorphism

$$\pi_* F(M, N) \longrightarrow \mathrm{Hom}(\pi_*(M), \pi_*(N))$$

that is right adjoint to the composite

$$\pi_* F(M, N) \otimes \pi_*(M) \overset{\cdot}{\longrightarrow} \pi_*(F(M, N) \wedge M) \xrightarrow{\epsilon_*} \pi_*(N)\,,$$

where $\epsilon\colon F(M, N) \wedge M \to N$ is the adjunction counit (= evaluation).

Sketch proof of Proposition 9.7.7. Given $f\colon S^{\ell+i} \to L_i$ and $g\colon S^{m+j} \to M_j$ we form the composite

$$f \cdot g\colon S^{\ell+m+i+j} \xrightarrow{\tau'} S^{\ell+i+m+j} \cong S^{\ell+i} \wedge S^{m+j} \xrightarrow{f \wedge g} L_i \wedge M_j \xrightarrow{\iota_{i,j}} (L \wedge M)_{i+j}\,,$$

where τ' is any map of degree $(-1)^{mi}$. If $m \ge 0$ we can let $\tau' = 1 \wedge \tau_{S^m, S^i} \wedge 1$, but we should also allow $m < 0$ in this construction. Then $[f \cdot g] \in \pi_{\ell+m+i+j}((L \wedge M)_{i+j})$ only depends on $[f]$ and $[g]$. Furthermore, one can check that the stable class of $[f \cdot g]$ in $\pi_{\ell+m}(L \wedge M)$ only depends on the stable classes of $[f]$ in $\pi_\ell(L)$ and of $[g]$ in $\pi_m(M)$, so that we obtain a well-defined pairing

$$\pi_\ell(L) \times \pi_m(M) \longrightarrow \pi_{\ell+m}(L \wedge M)\,.$$

This is bilinear, and hence factors uniquely through the tensor product, as asserted. One can also check that

$$\tau_*([f] \cdot [g]) = (-1)^{\ell m} [g] \cdot [f]$$

for $\ell = |f|$ and $m = |g|$, so that the lax monoidal functor π_* is symmetric. □

Corollary 9.7.9. *The pairing*

$$\cdot\colon \pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S \wedge S) \cong \pi_*(S)$$

makes $\pi_*(S)$ a graded commutative ring. For each orthogonal spectrum M the pairing

$$\cdot\colon \pi_*(M) \otimes \pi_*(S) \longrightarrow \pi_*(M \wedge S) \cong \pi_*(M)$$

makes $\pi_*(M)$ a right $\pi_*(S)$-module. The lax monoidal structure homomorphism factors uniquely through

$$\pi_*(L) \otimes_{\pi_*(S)} \pi_*(M) \longrightarrow \pi_*(L \wedge M)$$

and the closed structure homomorphism factors uniquely through

$$\pi_* F(M, N) \longrightarrow \mathrm{Hom}_{\pi_*(S)}(\pi_*(M), \pi_*(N))\,.$$

Definition 9.7.10. An *orthogonal ring spectrum* is an orthogonal spectrum E equipped with a multiplication $\mu\colon E \wedge E \to E$ and a unit $\eta\colon S \to E$ such that

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

and

$$\begin{tikzcd} S \wedge E \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & E \wedge E \arrow[d, "\mu"] & E \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\cong"] \\ & E & \end{tikzcd}$$

commute. It is *commutative* if the diagram

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

commutes.

DEFINITION 9.7.11. An *orthogonal left E-module spectrum* is an orthogonal spectrum M with a pairing $\lambda\colon E \wedge M \to M$ such that

$$\begin{tikzcd} E \wedge E \wedge M \arrow[r, "1\wedge\lambda"] \arrow[d, "\mu\wedge 1"'] & E \wedge M \arrow[d, "\lambda"] \\ E \wedge M \arrow[r, "\lambda"] & M \end{tikzcd}$$

and

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & E \wedge M \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

commute.

There are similar definitions of right module spectra, bimodule spectra, algebra spectra and commutative algebra spectra.

EXAMPLE 9.7.12. The sphere spectrum S is a commutative orthogonal ring spectrum. Any orthogonal spectrum M is an orthogonal (left and right) S-module spectrum. A (commutative) orthogonal ring spectrum E is a (commutative) S-algebra spectrum.

REMARK 9.7.13. There are also weaker notions, of ring spectra and module spectra *up to homotopy*, for which the (structure maps and) diagrams above are only required to (exist and) commute in the stable category $\mathrm{Ho}(Sp^{\mathbb{O}})$.

LEMMA 9.7.14. *For each ring spectrum E (orthogonal, or up to homotopy), the homotopy groups $\pi_*(E)$ form a graded $\pi_*(S)$-algebra, which is graded commutative if E is commutative. For each left E-module spectrum M (orthogonal, or up to homotopy) the homotopy groups $\pi_*(M)$ form a graded left $\pi_*(E)$-module.*

((ETC: Get spectral sequences

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(E)}_{s,t}(\pi_*(L), \pi_*(M)) \Longrightarrow_s \pi_{s+t}(L \wedge_E M)$$

and

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(E)}(\pi_*(M), \pi_*(N)) \Longrightarrow_s \pi_{-s-t} F_E(M, N)$$

for orthogonal ring spectra E and appropriate E-module L, M and N. See [**EKMM97**, Thm. IV.4.1].))

9.8. Monoidal model structure

DEFINITION 9.8.1. Let $\mathscr{C}$ be a closed symmetric monoidal category, with monoidal pairing $\otimes$. Let $i\colon A \to X$ and $j\colon B \to Y$ be morphisms in $\mathscr{C}$. Their *pushout-product map*

$$i \,\square\, j\colon A \otimes Y \cup_{A\otimes B} X \otimes B \longrightarrow X \otimes Y$$

is the canonical morphism from the pushout to the lower right hand corner in the following commutative square.

$$\begin{tikzcd} A \otimes B \arrow[r, "1\otimes j"] \arrow[d, "i\otimes 1"'] & A \otimes Y \arrow[d, "i\otimes 1"] \\ X \otimes B \arrow[r, "1\otimes j"] & X \otimes Y \end{tikzcd}$$

DEFINITION 9.8.2. Let $\mathscr{C}$ be a closed symmetric monoidal category with a model structure. The *pushout-product axiom* requires that:

- If $i\colon A \to X$ and $j\colon B \to Y$ are cofibrations, then so is their pushout-product $i \,\square\, j$.
- If, furthermore, (i or) j is a weak equivalence, then so is $i \,\square\, j$.

The *unit axiom* requires that:

- The canonical map $q \otimes 1\colon U^c \otimes Y \to U \otimes Y$ is a weak equivalence for each cofibrant Y, where $q\colon U^c \sim U$ is a cofibrant replacement of the unit.

REMARK 9.8.3. The unit axiom is automatically satisfied when the unit object U is cofibrant, which is the case for the stable model structure on $Sp^{\mathbb{O}}$. However, there are other useful model structures on orthogonal spectra, such as the positive stable model structure, for which the unit S is not cofibrant. The positive model structure lifts to a model structure on commutative orthogonal ring spectra, hence is useful for the study of the homotopy theory of commutative algebra spectra. ((ETC: André–Quillen cohomology.))

DEFINITION 9.8.4. A *monoidal model category* $\mathscr{C}$ is a a closed symmetric monoidal category with a model structure satisfying the pushout-product axiom and the unit axiom.

DEFINITION 9.8.5. Let $\mathscr{C}$ be a monoidal model category. The *total left derived pairing*

$$\otimes^L\colon \mathrm{Ho}(\mathscr{C}) \times \mathrm{Ho}(\mathscr{C}) \longrightarrow \mathrm{Ho}(\mathscr{C})$$

maps (X, Y) to $X^c \otimes Y^c$, where $X^c \sim X$ and $Y^c \sim Y$ are cofibrant replacements. The *total right derived closed structure*

$$\mathrm{Hom}^R\colon \mathrm{Ho}(\mathscr{C})^{op} \times \mathrm{Ho}(\mathscr{C}) \longrightarrow \mathrm{Ho}(\mathscr{C})$$

maps (X, Y) to $\mathrm{Hom}(X^c, Y^f)$, where $X^c \sim X$ and $Y \sim Y^f$ are cofibrant and fibrant replacements, respectively.

THEOREM 9.8.6 ([**Hov99**, Thm. 4.3.2]). *Let $\mathscr{C}$ be a monoidal model category. The total left derived pairing $\otimes^L$, unit object U, symmetry τ and total right derived closed structure* Hom^R *define a closed symmetric monoidal structure on* $\mathrm{Ho}(\mathscr{C})$*. In particular, there is an adjunction*

$$\mathrm{Ho}(\mathscr{C})(X \otimes^L Y, Z) \cong \mathrm{Ho}(\mathscr{C})(X, \mathrm{Hom}^R(Y, Z))\,.$$

THEOREM 9.8.7. *The closed symmetric monoidal category $Sp^{\mathbb{O}}$ of orthogonal spectra, with the stable model structure, is a monoidal model category.*

COROLLARY 9.8.8. *The stable category* $\mathrm{Ho}(Sp^{\mathbb{O}})$ *of orthogonal spectra is a closed symmetric monoidal category.*

REMARK 9.8.9. It is traditional to write $L \wedge M$ for the total left derived smash product

$$L \wedge^L M = L^c \wedge M^c$$

and to write $F(M, N)$ for the total right derived function spectrum

$$F^R(M, N) = F(M^c, N^f)\,,$$

omitting the superscripts L and R from the notation. These are the constructions that are homotopically meaningful when the objects L, M and N are only given up to stable equivalence.

DEFINITION 9.8.10. Given spectra M and X, we define

$$M_n(X) = [S^n, M \wedge X]$$

$$\sigma\colon M_n(X) \xrightarrow{\cong} M_{1+n}(S^1 \wedge X)$$

as in Definition 9.4.1, thereby extending the homology theory M over the functor $\Sigma^\infty\colon \mathscr{T} \to Sp^{\mathbb{O}}$ from based spaces to orthogonal spectra.

DEFINITION 9.8.11. Given spectra M and X, we define

$$M^n(X) = [X, S^n \wedge M]$$

$$\sigma\colon M^n(X) \xrightarrow{\cong} M^{1+n}(S^1 \wedge X)$$

as in Definition 9.4.4, thereby extending the cohomology theory M over the functor $\Sigma^\infty\colon \mathscr{T} \to Sp^{\mathbb{O}}$ from based spaces to orthogonal spectra.

SKETCH PROOF OF THEOREM 9.8.7. We must verify the pushout-product axiom. To verify the first part, it suffices to consider pairs of maps $i\colon F_k S^{n-1}_+ \to F_k D^n_+$ and $j\colon F_\ell S^{m-1}_+ \to F_\ell D^m_+$, in the set FI generating the relative cell spectra. The pushout-product map $i \,\square\, j$ then has the form

$$F_{k+\ell}(S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1})_+ \longrightarrow F_{k+\ell}(D^n \times D^m)_+\,,$$

which is a Quillen cofibration.

The second part is proved in [**MMSS01**, Prop. 12.6], and relies on Proposition 9.8.12 below. □

PROPOSITION 9.8.12 ([**MMSS01**, Prop. 12.3]). *For any Quillen cofibrant orthogonal spectrum L, the functor*

$$L \wedge -\colon M \longmapsto L \wedge M$$

preserves stable equivalences.

This is first proved for $L = F_\ell S^n$, from which the general case follows. Informally, the proposition says that Quillen cofibrant orthogonal spectra are flat.

EXAMPLE 9.8.13. The functor

$$\Sigma^{-1} = S^{-1} \wedge -\colon M \longmapsto S^{-1} \wedge M$$

from $\mathrm{Ho}(Sp^{\mathbb{O}})$ to itself defines an inverse equivalence to $\Sigma = S^1 \wedge -\colon M \mapsto S^1 \wedge M$. ((ETC: Is $\lambda \wedge 1\colon F_1(S^1) \wedge M \to M$ a stable equivalence if M is not cofibrant?))

PROPOSITION 9.8.14. *If L is ℓ-connective and M is m-connective, with L (or M) Quillen cofibrant, then $L \wedge M$ is $(\ell+m)$-connective and*

$$\cdot\colon \pi_\ell(L) \otimes \pi_m(M) \xrightarrow{\cong} \pi_{\ell+m}(L \wedge M)$$

is an isomorphism.

SKETCH PROOF. There exists a stable equivalence $M^c \sim M$, where M^c is built from $*$ by attaching n-cells of the form (CS^{n-1}, S^{n-1}) with $n \ge m$, and $L \wedge M^c \sim L \wedge M$. There is also a stable equivalence $L^c \sim L$, where L^c is built from n-cells with $n \ge \ell$, and $L^c \wedge M^c \sim L \wedge M^c$. Here $L^c \wedge M^c$ is built from n-cells with $n \ge \ell + m$, which implies that $L^c \wedge M^c$ is $(\ell+m)$-connective.

A more precise account of the m- and $(m+1)$-cells of M^c, and of the ℓ- and $(\ell+1)$-cells of L^c, shows that the $(m+\ell)$- and $(m+\ell+1)$-cells of $M^c \wedge L^c$ give a presentation of $\pi_{m+\ell}(M^c \wedge L^c)$ as the tensor product $\pi_m(M^c) \otimes \pi_\ell(L^c)$. □

REMARK 9.8.15. Since the stable model structure on $Sp^{\mathbb{O}}$ is both monoidal and stable, the homotopy category $\mathrm{Ho}(Sp^{\mathbb{O}})$ is both closed symmetric monoidal and triangulated, and several compatibility conditions between the latter structures are satisfied. Two of these are given in [**HPS97**, App. A.2] and [**May01**, Def. 4.1]. Let $\Sigma X = S^1 \wedge X$.

(1) The composite

$$\Sigma S^1 = S^1 \wedge S^1 \xrightarrow{\tau} S^1 \wedge S^1 = \Sigma S^1$$

is multiplication by -1.

(2) For each distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and object W in $\mathrm{Ho}(Sp^{\mathbb{O}})$, the following triangles are distinguished.

$$W \wedge X \xrightarrow{1\wedge f} W \wedge Y \xrightarrow{1\wedge g} W \wedge Z \xrightarrow{1\wedge h} \Sigma(W \wedge X)$$

$$X \wedge W \xrightarrow{f\wedge 1} Y \wedge W \xrightarrow{g\wedge 1} Z \wedge W \xrightarrow{h\wedge 1} \Sigma(X \wedge W)$$

$$F(W,X) \xrightarrow{F(1,f)} F(W,Y) \xrightarrow{F(1,g)} F(W,Z) \xrightarrow{F(1,h)} \Sigma F(W,X)$$

$$\Sigma^{-1}F(X,W) \xrightarrow{-F(h,1)} F(Z,W) \xrightarrow{F(g,1)} F(Y,W) \xrightarrow{F(f,1)} F(X,W)$$

In (2) we use fixed identifications $W \wedge \Sigma X \cong \Sigma(W \wedge X)$, $\Sigma X \wedge W \cong \Sigma(X \wedge W)$, $F(W, \Sigma X) \cong \Sigma F(W,X)$ and $F(\Sigma X, W) \cong \Sigma^{-1}F(X,W)$, coming from the closed structure.

REMARK 9.8.16. May [**May01**] gives three more compatibility conditions that are also satisfied, but these are not the full story, as explained by Keller and Neeman [**KN02**]. We will make use of the following Leibniz rule for the connecting homomorphism in homotopy. ((ETC: Does it follow from the conditions of May or Keller–Neeman?))

Let $i\colon A \to X$ and $j\colon B \to Y$ be Quillen cofibrations and let

$$W = A \wedge Y \cup_{A\wedge B} X \wedge B$$

be the pushout in the commutative square

$$\begin{tikzcd}
A \wedge B \arrow[r, "i\wedge 1"] \arrow[d, "1\wedge j"'] & X \wedge B \arrow[d, "1\wedge j"] \\
A \wedge Y \arrow[r, "i\wedge 1"] & X \wedge Y\,.
\end{tikzcd}$$

By the pushout-product axiom the canonical map $i \,\square\, j \colon W \to X \wedge Y$ is a Quillen cofibration, and we have isomorphisms

$$\frac{X \wedge Y}{W} \cong \frac{X}{A} \wedge \frac{Y}{B} \quad \text{and} \quad \frac{W}{A \wedge B} \cong A \wedge \frac{Y}{B} \vee \frac{X}{A} \wedge B\,.$$

PROPOSITION 9.8.17. *Let $x \in \pi_n(X/A)$ and $y \in \pi_m(Y/B)$. Then*

$$\partial(x \cdot y) = \partial x \cdot y + (-1)^n x \cdot \partial y$$

in $\pi_{-1+n+m}(W/(A \wedge B)) \cong \pi_{-1+n+m}(A \wedge (Y/B)) \oplus \pi_{-1+n+m}((X/A) \wedge B)$.

Here $\partial x \in \pi_{-1+n}(A)$ and $\partial y \in \pi_{-1+m}(B)$ are given by the composites

$$\pi_n(X/A) \xleftarrow{\cong} \pi_n(X, A) \xrightarrow{\partial} \pi_{-1+n}(A)$$
$$\pi_m(Y/B) \xleftarrow{\cong} \pi_m(Y, B) \xrightarrow{\partial} \pi_{-1+m}(B)$$

and $\partial(x \cdot y)$ is calculated using the following diagram.

$$\begin{tikzcd}
\pi_n(X, A) \otimes \pi_m(Y, B) \arrow[r, "\cong"] \arrow[d, "\cdot"'] & \pi_n(X/A) \otimes \pi_m(Y/B) \arrow[d, "\cdot"'] \\
\pi_{n+m}(X \wedge Y, W) \arrow[r, "\cong"] \arrow[d, "\partial"'] & \pi_{n+m}(X/A \wedge Y/B) \\
\pi_{-1+n+m}(W, A \wedge B) \arrow[r] & \pi_{-1+n+m}(W/(A \wedge B)) \\
& \pi_{-1+n+m}(A \wedge (Y/B)) \oplus \pi_{-1+n+m}((X/A) \wedge B) \arrow[u, "\cong"]
\end{tikzcd}$$

REMARK 9.8.18. When applied to (cofibrant replacements of) maps of the form $F(X', A) \to F(X', X)$ and $F(X'', B) \to F(X'', X)$, this gives a Leibniz rule for pairings $\cdot\colon [X', -]_* \otimes [X'', -]_* \to [X' \wedge X'', -]_*$ that generalize the pairing $\cdot\colon \pi_*(-) \otimes \pi_*(-) \to \pi_*(-)$.

PROOF. There are canonical isomorphisms

$$\pi_{-1+n}(Fi) \cong \pi_n(X \cup CA) \cong \pi_n(X/A)\,,$$

where Fi denotes the homotopy fiber of $i\colon A \to X$, so an element $x \in \pi_n(X/A)$ can be represented by the homotopy class of a map $\bar{f}\colon S^{n+k} \to (X/A)_k$, of a map $\hat{f}\colon D^{n+k} \cup CS^{-1+n+k} \to X_k \cup CA_k$, or of a pair of maps $(f\colon D^{n+k} \to X_k, \tilde{f}\colon S^{-1+n+k} \to A_k)$ with $f|S^{-1+n+k} = i\tilde{f}$, for k sufficiently large. The image $\partial x \in \pi_{-1+n}(A)$ of x under the connecting homomorphism is the homotopy class of $\tilde{f}$.

$$
\begin{tikzcd}
S^{-1+n} \arrow[r] \arrow[dd, "\tilde f"'] & D^n \arrow[rr] \arrow[dd, "f"'] & & D^n \cup CS^{-1+n} \arrow[r, "\simeq"] \arrow[dl, "\simeq"'] \arrow[dd, "\hat f"'] & \Sigma S^{-1+n} \arrow[dd, "\Sigma\tilde f"'] \\
& & S^n \arrow[dd, "\bar f"] & & \\
A \arrow[r, "i"] & X \arrow[rr] & & X \cup CA \arrow[r] \arrow[dl, "\simeq"'] & \Sigma A \\
& & X/A & &
\end{tikzcd}
$$

Likewise, any class $y \in \pi_m(Y/B)$ can be represented by the homotopy class of a pair of maps $(g\colon D^{m+\ell} \to Y_\ell, \tilde{g}\colon S^{-1+m+\ell} \to B_\ell)$ with $g|S^{-1+m+\ell} = j\tilde{g}$, for ℓ sufficiently large.

The product $x \cdot y \in \pi_{n+m}(X/A \wedge Y/B) \cong \pi_{n+m}((X \wedge Y)/W)$ is then represented by $(-1)^{mk}$ times the pair of maps

$$
\begin{aligned}
f \wedge g\colon D^{n+k} \wedge D^{m+\ell} &\longrightarrow X_k \wedge Y_\ell \xrightarrow{\iota_{k,\ell}} (X \wedge Y)_{k+\ell} \\
\tilde{f} \wedge g \cup f \wedge \tilde{g}\colon S^{-1+n+k} \wedge D^{m+\ell} \cup D^{n+k} \wedge S^{-1+m+\ell} &\longrightarrow W_{k+\ell}\,.
\end{aligned}
$$

Hence $\partial(x \cdot y) \in \pi_{-1+n+m}(W)$ is represented by $(-1)^{mk}$ times $\tilde{f} \wedge g \cup f \wedge \tilde{g}$, and its projection to

$$
\pi_{-1+n+m}(W/(A \wedge B)) \cong \pi_{-1+n-m}(A \wedge (Y/B) \vee (X/A) \wedge B)
$$

is $(-1)^{mk}$ times the homotopy class of the composite

$$
\begin{aligned}
\partial(D^{n+k} \wedge D^{m+\ell}) &= S^{-1+n+k} \wedge D^{m+\ell} \cup D^{n+k} \wedge S^{-1+m+\ell} \\
&\xrightarrow{(+1,(-1)^{n+k})} S^{-1+n+k} \wedge S^{m+\ell} \vee S^{n+k} \wedge S^{-1+m+\ell} \\
&\xrightarrow{\tilde{f}\wedge\bar{g}\vee\bar{f}\wedge\tilde{g}} (A \wedge Y/B \vee B \wedge X/A)_{k+\ell}\,.
\end{aligned}
$$

Here the signs $+1$ and $(-1)^{n+k}$ reflect how the orientation of the boundary of $D^{n+k} \wedge D^{m+\ell}$ behaves under the projections given by collapsing $D^{n+k} \wedge S^{-1+m+\ell}$ and $S^{-1+n+k} \wedge D^{m+\ell}$, respectively.

$$\begin{tikzcd}
S^{-1+n} \wedge S^{-1+m} \arrow[rr, "\tilde f \wedge \tilde g"] \arrow[d] & & A \wedge B \arrow[d] & \\
\partial(D^m \wedge D^n) \arrow[rr, "\tilde f \wedge g \cup f \wedge \tilde g"] \arrow[dd] \arrow[dr, "{(+1,(-1)^n)}"] & & W \arrow[dr] \arrow[dd, dotted] & \\
& S^{-1+n} \wedge S^m \vee S^n \wedge S^{-1+m} \arrow[rr, "\tilde f \wedge \bar g \vee \bar f \wedge \tilde g"] & & A \wedge Y/B \vee X/A \wedge B \\
D^n \wedge D^m \arrow[rr, "f \wedge g"] \arrow[d] & & X \wedge Y \arrow[d] & \\
S^n \wedge S^m \arrow[rr, "\bar f \wedge \bar g"] & & X/A \wedge Y/B &
\end{tikzcd}$$

It follows that the further projection to $\pi_{-1+n+m}(A \wedge (Y/B))$ is $(-1)^{mk}$ times the homotopy class of

$$\tilde f \wedge \bar g \colon S^{-1+n+k} \wedge S^{m+\ell} \longrightarrow (A \wedge Y/B)_{k+\ell}\,,$$

which equals the product $\partial x \cdot y$. Likewise, the projection to $\pi_{-1+n+m}((X/A) \wedge B)$ is $(-1)^{mk}(-1)^{n+k} = (-1)^n(-1)^{(m-1)k}$ times the homotopy class of

$$\bar f \wedge \tilde g \colon S^{n+k} \wedge S^{-1+m+\ell} \longrightarrow (X/A \wedge B)_{k+\ell}\,,$$

which equals the product $(-1)^n x \cdot \partial y$. □

((ETC: Can we give this proof in $Sp^{\mathbb{O}}$? May assume A, X, B and Y are stably fibrant (= Ω-spectra), but then X/A and Y/B will usually not have this property.))

((ETC: Schwede–Shipley monoid axiom from [**SS00**].))

9.9. Multiplicative (co-)homology theories

To exhibit the Eilenberg–MacLane spectra HG as orthogonal spectra, we use a functorial construction of Eilenberg–MacLane spaces due to McCord, who works in the category $\mathscr{T}$ of based, compactly generated, weak Hausdorff spaces.

Definition 9.9.1 ([**McC69**, §5, §6]). Let G be a commutative topological monoid and X a based space. Let

$$B(G, X) = \coprod_{j \ge 0} (G \times X)^j/\sim$$

be the space of formal sums

$$u = \sum_{i=1}^{j} (g_i, x_i)$$

with $g_i \in G$ and $x_i \in X$, subject to the relations

$$(g', x') + (g'', x'') = (g'', x'') + (g', x')$$
$$(g', x) + (g'', x) = (g' + g'', x)$$
$$(g, x_0) = 0\,,$$

where $x_0 \in X$ is the base point.

We think of u as a finite set of points in X, labeled with elements in G, ignoring any label at the base point. We give the image $B_j(G,X) \subset B(G,X)$ of $(G\times X)^j$ the quotient topology, and give $B(G,X) = \bigcup_{j\ge 0} B_j(G,X)$ the (weak) colimit topology. In particular, there is a closed inclusion

$$G_+ \wedge X = B_1(G,X) \longrightarrow B(G,X)\,.$$

The construction is clearly natural in the commutative topological monoid G and the based space X. There is a natural map

$$\rho\colon B(G,X)\wedge Y \longrightarrow B(G,X\wedge Y)$$

mapping $(\sum_i (g_i,x_i))\wedge y$ to $\sum_i (g_i, x_i\wedge y)$, so homotopic maps $X \to X'$ induce homotopic maps $B(G,X)\to B(G,X')$. There is also a natural pairing

$$\iota\colon B(G,X)\wedge B(H,Y) \longrightarrow B(G\otimes H, X\wedge Y)$$

sending $\sum_i (g_i,x_i)\wedge \sum_j (h_j,y_j)$ to $\sum_{i,j}(g_i\otimes h_j)x_i\wedge y_j$.

((ETC: Discuss CW structure on $B(G,X)$ for G discrete, X triangulated.))

THEOREM 9.9.2 ([**McC69**, Thm. 8.8]). *If G is a discrete abelian group, and (X,A) is a based triangulable pair, then*

$$B(G,X) \longrightarrow B(G,X/A)$$

is a numerable principal $B(G,A)$-bundle. In particular, it is a Hurewicz fibration with fiber $B(G,A)$.

COROLLARY 9.9.3. *Let G be a discrete abelian group. Then $B(G,D^n)$ is contractible and $B(G,S^n)$ is a $K(G,n)$-space, for each $n\ge 0$.*

PROOF. The homotopy equivalence $D^n \to *$ induces a homotopy equivalence $B(G,D^n)\simeq B(G,*) = *$. The Hurewicz fibration

$$B(G,S^{n-1}) \longrightarrow B(G,D^n) \longrightarrow B(G,S^n)$$

exhibits $B(G,S^n)$ as a (connected) delooping of $B(G,S^{n-1})$. Since $B(G,S^0)\cong G$ is a $K(G,0)$-space, it follows by induction that $B(G,S^n)$ is a $K(G,n)$-space. □

DEFINITION 9.9.4. For each (discrete) abelian group G let the *Eilenberg–MacLane spectrum HG* be the orthogonal spectrum with

$$(HG)_k = B(G,S^k)$$

having the $O(k)$-action induced by the linear $O(k)$-action on $S^k = \mathbb{R}^k \cup \{\infty\}$, and with structure maps

$$\sigma\colon (HG)_k\wedge S^1 = B(G,S^k)\wedge S^1 \xrightarrow{\rho} B(G,S^k\wedge S^1)\cong (HG)_{k+1}$$

for each $k\ge 0$.

This is an Ω-spectrum with

$$\pi_n(HG) = \begin{cases} G & \text{for } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence there are natural isomorphisms

$$HG_*(X)\cong \tilde H_*(X;G)$$
$$HG^*(X)\cong \tilde H^*(X;G)$$

for all based X of the homotopy type of a CW complex.

((ETC: Is HG Quillen cofibrant?))

DEFINITION 9.9.5. Let $\mu\colon G' \otimes G'' \to G$ be a pairing of (discrete) abelian groups. The induced maps

$$\begin{aligned}\mu\colon (HG')_i \wedge (HG'')_j = B(G', S^i) \wedge B(G'', S^j) \\ \xrightarrow{\iota} B(G' \otimes G'', S^i \wedge S^j) \xrightarrow{\mu} B(G, S^{i+j}) = (HG)_{i+j}\end{aligned}$$

are $O(i) \times O(j)$-equivariant and compatible with the right S-module structures, hence induce a map

$$\mu\colon HG' \wedge HG'' \longrightarrow HG$$

of orthogonal spectra.

EXAMPLE 9.9.6. If $\mu\colon R \otimes R \to R$ is a ring multiplication, then

$$\mu\colon HR \wedge HR \longrightarrow HR$$

makes HR an orthogonal ring spectrum, which is commutative if R is commutative. If $\lambda\colon R \otimes N \to N$ is a left R-module action, then

$$\lambda\colon HR \wedge HN \longrightarrow HN$$

makes HN an orthogonal left HR-module spectrum. In particular, $H\mathbb{Z}$ is a commutative orthogonal ring spectrum, and HG is an orthogonal (left and right) $H\mathbb{Z}$-module spectrum, for each abelian group G.

((ETC: Examples of Thom (ring) spectra.))

DEFINITION 9.9.7. Let $\mu\colon L \wedge M \to N$ be a map of orthogonal spectra. For based spaces or spectra X and Y the *homology smash product* pairing

$$L_*(X) \otimes M_*(Y) \xrightarrow{\wedge} N_*(X \wedge Y)$$

is given by the composition

$$\begin{aligned}[S^\ell, L \wedge X] \otimes [S^m, M \wedge Y] \xrightarrow{\cdot} [S^\ell \wedge S^m, L \wedge X \wedge M \wedge Y] \\ \xrightarrow{1\wedge\tau\wedge 1} [S^{\ell+m}, L \wedge M \wedge X \wedge Y] \xrightarrow{\mu\wedge 1\wedge 1} [S^{\ell+m}, N \wedge X \wedge Y]\end{aligned}$$

while the *cohomology smash product* pairing

$$L^*(X) \otimes M^*(Y) \xrightarrow{\wedge} N^*(X \wedge Y)$$

is given by the composition

$$\begin{aligned}[X, S^\ell \wedge L] \otimes [Y, S^m \wedge M] \xrightarrow{\cdot} [X \wedge Y, S^\ell \wedge L \wedge S^m \wedge M] \\ \xrightarrow{1\wedge\tau\wedge 1} [X \wedge Y, S^\ell \wedge S^m \wedge L \wedge M] \xrightarrow{1\wedge 1\wedge\mu} [X \wedge Y, S^{\ell+m} \wedge N] .\end{aligned}$$

If $X = Y$ are spaces, the *cup product* pairing is the composition

$$\cup\colon L^*(X) \otimes M^*(X) \xrightarrow{\wedge} N^*(X \times X) \xrightarrow{\Delta^*} N^*(X) .$$

((ETC: Discuss pairings of (co-)homology theories, including interaction with connecting homomorphisms. The Leibniz rule. Pairings of Atiyah–Hirzebruch spectral sequences?))

We now follow [**Ada69**, Lec. 3] to discuss Steenrod operations and cooperations for (generalized) E-cohomology and E-homology, but will apply this in the classical case $E = H\mathbb{F}_p$.

Proposition 9.9.8. *Let E be a spectrum. The composition pairing*

$$\phi\colon E^*(E) \otimes E^*(E) \longrightarrow E^*(E)$$

makes $E^*(E)$ a graded ring. For each spectrum Y the composition pairing

$$\lambda\colon E^*(E) \otimes E^*(Y) \longrightarrow E^*(Y)$$

makes $E^*(Y)$ a graded left $E^*(E)$-module. When $E = H = H\mathbb{F}_p$, this algebra equals the mod p Steenrod algebra

$$A \cong H^*(H)\,,$$

and the action

$$\lambda\colon H^*(H) \otimes H^*(Y) \longrightarrow H^*(Y)$$

agrees with the natural left A-action

$$\lambda\colon A \otimes H^*(Y;\mathbb{F}_p) \longrightarrow H^*(Y;\mathbb{F}_p)\,.$$

Proposition 9.9.9. *Let E be a ring spectrum (orthogonal, or up to homotopy). The map*

$$X \wedge E \wedge E \wedge Y \overset{1\wedge\mu\wedge 1}{\longrightarrow} X \wedge E \wedge Y$$

induces a pairing

$$m\colon \pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y) \longrightarrow \pi_*(X \wedge E \wedge Y)$$

for all spectra X and Y. If X and E are such that $\pi_*(X \wedge E)$ is flat as a right $\pi_*(E)$-module then m is an isomorphism for all Y.

Proof. The composite

$$\pi_*(X \wedge E) \otimes \pi_*(E \wedge Y) \overset{\cdot}{\longrightarrow} \pi_*(X \wedge E \wedge E \wedge Y) \overset{1\wedge\mu\wedge 1}{\longrightarrow} \pi_*(X \wedge E \wedge Y)$$

equalizes the two homomorphisms from $\pi_*(X \wedge E) \otimes \pi_*(E) \otimes \pi_*(E \wedge Y)$ by (homotopy) associativity. If $\pi_*(X \wedge E)$ is flat over $\pi_*(E)$, then $\pi_*(X \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge Y)$ and $\pi_*(X \wedge E \wedge Y)$ both define homology theories for CW complexes Y, or cell spectra Y. Since m is an isomorphism for $Y = S$, it follows by induction that it is an isomorphism for all cell spectra Y, hence for all Y when the smash products are interpreted in the total left derived sense. □

Definition 9.9.10. Suppose that E is a commutative ring spectrum (orthogonal, or up to homotopy), and that $E_*E = \pi_*(E \wedge E)$ is flat as a right (or left) module over $E_* = \pi_*(E)$. Let

$$\begin{aligned}
\eta_L = (\eta \wedge 1)_*\colon E_* = \pi_*(S \wedge E) &\longrightarrow \pi_*(E \wedge E) = E_*E\\
\eta_R = (1 \wedge \eta)_*\colon E_* = \pi_*(E \wedge S) &\longrightarrow \pi_*(E \wedge E) = E_*E\\
\phi = \wedge\colon E_*E \otimes E_*E = \pi_*(E \wedge E) \otimes \pi_*(E \wedge E) &\longrightarrow \pi_*(E \wedge E) = E_*E\\
\epsilon = \mu_*\colon E_*E = \pi_*(E \wedge E) &\longrightarrow \pi_*(E) = E_*\\
\chi = \tau_*\colon E_*E = \pi_*(E \wedge E) &\longrightarrow \pi_*(E \wedge E) = E_*E
\end{aligned}$$

denote the *left unit*, *right unit*, *product*, *counit* and *conjugation*. Furthermore, let

$$\psi = (1 \wedge \eta \wedge 1)_*\colon E_*E = \pi_*(E \wedge S \wedge E) \longrightarrow \pi_*(E \wedge E \wedge E) \overset{m}{\cong} E_*E \otimes_{E_*} E_*E$$

define the *coproduct*.

PROPOSITION 9.9.11. *The pair (E_*E, E_*), with the structure maps above, form a graded Hopf algebroid. If $\eta_L = \eta_R$ then E_*E is a graded commutative Hopf algebra over E_*. When $E = H = H\mathbb{F}_p$, this Hopf algebra over $E_* = \mathbb{F}_p$ equals the dual Steenrod algebra*

$$A_* \cong H_*(H)\,.$$

REMARK 9.9.12. The terminology "Hopf algebroid" is due to Haynes Miller, and means that E_* and E_*E are graded commutative rings that corepresent the object set and morphism set of a functor from graded commutative rings to small groupoids, i.e., to small categories in which each morphism is invertible. The homomorphisms η_L and η_R corepresent the target (= codomain) and source (= domain), ϵ corepresents the identity morphism, ψ corepresents composition, and χ expresses the existence of inverses.

DEFINITION 9.9.13. For each space or spectrum Y, let

$$\nu = (1 \wedge \eta \wedge 1)_* \colon E_*(Y) = \pi_*(E \wedge S \wedge Y) \longrightarrow \pi_*(E \wedge E \wedge Y) \overset{m}{\cong} E_*E \otimes_{E_*} E_*(Y)$$

define the E_*E-coaction on $E_*(Y)$.

LEMMA 9.9.14. *The coaction ν makes $E_*(Y)$ a left E_*E-comodule. When $E = H = H\mathbb{F}_p$, the coaction*

$$\nu \colon H_*(Y) \longrightarrow H_*H \otimes_{H_*} H_*(Y)$$

agrees with the natural left A_*-coaction

$$\nu \colon H_*(Y; \mathbb{F}_p) \longrightarrow A_* \otimes H_*(Y; \mathbb{F}_p)\,.$$

Note that this construction does not presume that $H_*(Y; \mathbb{F}_p)$ is of finite type, unlike the discussion in Section 7.8.

((ETC: Maybe discuss universal coefficient theorems and Künneth theorems for spectral (co-)homology theories?))

CHAPTER 10

Homological Algebra (TO BE WRITTEN)

10.1. Tor and Ext

((ETC: Interpretation of Ext^1 as extensions. Yoneda composition?))

10.2. Ext over Hopf algebras

((ETC: Structure theorems for Hopf algebras.))

A connected algebra is automatically augmented by the inverse $\epsilon = \eta^{-1}$ of the unit in degree 0. Dually, a connected coalgebra is automatically coaugmented by the inverse $\eta = \epsilon^{-1}$ of the counit in degree 0.

THEOREM 10.2.1 ([**MM65**, Thm. 4.4]). *Let R be a commutative ring, let A be a connected R-bialgebra, let B be a connected left A-module coalgebra, and give $C = R \otimes_A B$ the induced R-coalgebra structure. Suppose that the composite*

$$i\colon A \cong A \otimes R \xrightarrow{1\otimes\eta} A \otimes B \xrightarrow{\lambda} B$$

is split injective, and that the composite

$$\pi\colon B \cong A \otimes_A B \xrightarrow{\epsilon\otimes 1} R \otimes_A B = C$$

is split surjective, both as homomorphisms of R-modules. Then there exists a homomorphism

$$h\colon B \longrightarrow A \otimes C$$

that is simultaneously an isomorphism of left A-modules and right C-comodules.

COROLLARY 10.2.2 ([**MM65**, Thm. 4.4]). *Let R be a field, let $A \subset B$ be a pair of connected R-bialgebras, and set $C = R \otimes_A B$. Then there exists an isomorphism*

$$h\colon B \xrightarrow{\cong} A \otimes C$$

of left A-modules and right C-comodules. In particular, B is free as a left A-module.

((ETC: Dual statement with $C = R \,\square_A\, B$.))
((ETC: Adams: Sub (Hopf) algebras of the dual Steenrod algebra.))
((ETC: Adams–Margolis: Sub (Hopf) algebras of the Steenrod algebra.))

10.3. Double complexes

10.4. The Cartan–Eilenberg spectral sequence

CHAPTER 11

The Adams Spectral Sequence

((ETC: Double centralizer theorem/problem))

The classical mod p *Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

aims to study the abelian group $[X,Y] = \mathrm{Ho}(Sp^{\mathbb{O}})(X,Y)$ of stable morphisms $f\colon X \to Y$, by means of the A-modules $H^*(X)$ and $H^*(Y)$ and the derived functors of Hom_A, where A denotes the mod p Steenrod algebra and $H = H\mathbb{F}_p$. It was introduced by Adams in [**Ada58**, §3]. The generalization to the study of $[X,Y]$ by means of the E^*E-modules $E^*(X)$ and $E^*(Y)$, for a (homotopy commutative) ring spectrum E, is known as the *Adams–Novikov spectral sequence* (principally for $E = MU$ [**Nov67**] and $E = BP$), or as the *E-based Adams spectral sequence*. There is also a homological formulation

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

of the Adams spectral sequence, in terms of the dual mod p Steenrod algebra A_* and the A_*-comodules $H_*(X)$ and $H_*(Y)$, which is a little more generally applicable than the cohomological version.

11.1. The d-invariant

The *degree* $\deg(f)$ of a map $f\colon M^n \to N^n$ of closed, connected, oriented n-manifolds with fundamental classes $[M]$ and $[N]$ is the integer satisfying $f_*([M]) = \deg(f)[N]$ in $H_n(N;\mathbb{Z}) \cong \mathbb{Z}$. The d-invariant is defined to detect similar information.

Definition 11.1.1. For spectra X and Y, let the (mod p cohomology) *d-invariant* be the homomorphism

$$\begin{aligned} d\colon [X,Y]_* &\longrightarrow \mathrm{Hom}_A^*(H^*(Y), H^*(X)) \\ [f] &\longmapsto f^* \end{aligned}$$

where $[X,Y]_n = [S^n \wedge X, Y]$ denotes the degree n morphisms $X \to Y$ in the stable category, and $\mathrm{Hom}_A^n(M,N) = \mathrm{Hom}_A(M, \Sigma^n N)$ denotes the A-module homomorphisms $M \to N$ of cohomological degree $-n$, for (graded) A-modules M and N. Hence d maps the homotopy class of $f\colon S^n \wedge X \to Y$ to the induced homomorphism $f^*\colon H^*(Y) \to H^*(S^n \wedge X) \cong \Sigma^n H^*(X)$.

Let the (mod p homology) *d-invariant* be the homomorphism

$$\begin{aligned} d\colon [X,Y]_* &\longrightarrow \mathrm{Hom}_{A_*}^*(H_*(X), H_*(Y)) \\ [f] &\longmapsto f_* \end{aligned}$$

where $\mathrm{Hom}_{A_*}^n(M,N) = \mathrm{Hom}_{A_*}(\Sigma^n M, N)$ denotes the A_*-comodule homomorphisms $M \to N$ of homological degree n, for (graded) A_*-comodules M and N.

Hence d maps the homotopy class of $f\colon S^n \wedge X \to Y$ to the induced homomorphism $f_*\colon \Sigma^n H_*(X) \cong H_*(S^n \wedge X) \to H_*(Y)$.

((ETC: Resolve notational inconsistency between graded and ungraded $\mathrm{Hom}_A(M,N)$, $\mathrm{Hom}_A(M,N)_i$ and $\mathrm{Hom}^n_A(M,N)$.))

EXAMPLE 11.1.2. When $X = S$, the cohomology d-invariant specializes to the homomorphism

$$d\colon \pi_*(Y) \longrightarrow \mathrm{Hom}^*_A(H^*(Y), \mathbb{F}_p)\,,$$

while the homology d-invariant specializes to

$$d\colon \pi_*(Y) \longrightarrow \mathrm{Hom}^*_{A_*}(\mathbb{F}_p, H_*(Y))\,.$$

LEMMA 11.1.3. *The cohomology d-invariant is obtained by dualization from the homology d-invariant, in the sense that it equals the composition*

$$[X,Y]_* \xrightarrow{d} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(Y)) \xrightarrow{D} \mathrm{Hom}^*_A(H^*(Y), H^*(X))\,.$$

By Lemma 7.7.39, the dualization homomorphism D is an isomorphism whenever $H_*(X)$ and $H_*(Y)$ are both bounded below and of finite type over $\mathbb{F}_p$. ((ETC: Only $H_*(Y)$ needs to be bounded below and of finite type. Refine the lemma to reflect this.))

The d-invariant is particularly sensitive for maps to spectra of the form $W = H \wedge T$, where T is an arbitrary spectrum. These are the H-*injective* spectra of [**Mil81**, §1], and can be expressed as sums or products of suspensions of Eilenberg–MacLane spectra. ((ETC: Reference for the notion "injective class". Maybe the Eilenberg–Moore memoir?))

LEMMA 11.1.4. *Let $W_* = H_*(T)$. There are isomorphisms*

$$H \wedge T \xleftarrow{\cong} \bigvee_n \Sigma^n H(W_n) \xrightarrow{\cong} \prod_n \Sigma^n H(W_n)$$

in the stable category, each inducing the identity map of W_n on π_n for $n \in \mathbb{Z}$.

PROOF. Choose a basis for $W_n = H_n(T)$ as an $\mathbb{F}_p$-vector space, and represent its elements by morphisms $f_\alpha\colon S^n \to H \wedge T$. Use the product $\mu\colon H \wedge H \to H$ to extend these to morphisms

$$\bar{f}_\alpha = (\mu \wedge 1)(1 \wedge f_\alpha)\colon \Sigma^n H \cong H \wedge S^n \to H \wedge T\,,$$

and form their sum

$$g_n\colon \Sigma^n H(W_n) \cong \bigvee_\alpha \Sigma^n H \longrightarrow H \wedge T\,.$$

The sum

$$g\colon \bigvee_n \Sigma^n H(W_n) \longrightarrow H \wedge T$$

over $n \in \mathbb{Z}$ then induces the isomorphism $g_*\colon W_* \xrightarrow{\cong} H_*(T)$ in homotopy, hence is a stable equivalence. The canonical map

$$\bigvee_n \Sigma^n H(W_n) \longrightarrow \prod_n \Sigma^n H(W_n)$$

induces the identity of W_* on graded homotopy groups, hence is also a stable equivalence. $\square$

PROPOSITION 11.1.5. *In the case $W \cong H \wedge T$, the homological d-invariant*

$$d \colon [X, W]_* \xrightarrow{\cong} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(W))$$

is an isomorphism. If, furthermore, W is bounded below with mod p homology of finite type, then the cohomological d-invariant

$$d \colon [X, W]_* \xrightarrow{\cong} \mathrm{Hom}^*_A(H^*(W), H^*(X))$$

is an isomorphism.

PROOF. By the Künneth theorem, the homology smash product

$$\wedge \colon H_*(H) \otimes H_*(T) \xrightarrow{\cong} H_*(H \wedge T)$$

is an isomorphism. Here $H_*(H) \cong A_*$, and the source has the diagonal A_*-coaction. By the untwisting isomorphism

$$A_* \otimes H_*(T) \cong A_* \otimes UH_*(T)$$

of Proposition 7.7.31, this is isomorphic to the extended A_*-comodule on the underlying graded $\mathbb{F}_p$-vector space of $H_*(T)$. By adjunction, there is an isomorphism

$$\mathrm{Hom}^*_{A_*}(H_*(X), A_* \otimes UH_*(T)) \cong \mathrm{Hom}^*(UH_*(X), UH_*(T)) \,.$$

Omitting the forgetful functor U from the notation, the composite homomorphism

$$[X, H \wedge T]_* \xrightarrow{d} \mathrm{Hom}^*_{A_*}(H_*(X), H_*(H \wedge T)) \cong \mathrm{Hom}^*(H_*(X), H_*(T))$$

defines a morphism of cohomology theories for (spaces or) spectra X, since $H_*(T)$ is automatically injective as a graded $\mathbb{F}_p$-vector space. Moreover, this morphism is an isomorphism for $X = S$. Hence it, and d, is an isomorphism for every spectrum X.

When W is bounded below, the Künneth theorem gives an isomorphism

$$\wedge \colon H^*(H) \otimes H^*(T) \xrightarrow{\cong} H^*(H \wedge T) \,.$$

Here $H^*(H) \cong A$, and the left hand side has the diagonal A-action. By the untwisting isomorphism

$$A \otimes H^*(T) \cong A \otimes UH^*(T)$$

of Proposition 7.7.30, this agrees with the extended A-module on $UH^*(T)$. By adjunction, there is an isomorphism

$$\mathrm{Hom}^*_A(A \otimes UH^*(T), H^*(X)) \cong \mathrm{Hom}^*(UH^*(T), UH^*(X)) \,.$$

The composite homomorphism

$$[X, H \wedge T]_* \xrightarrow{d} \mathrm{Hom}^*_A(H^*(H \wedge T), H^*(X)) \cong \mathrm{Hom}^*(H^*(T), H^*(X))$$

defines a morphism of cohomology theories, since $H^*(T)$ is automatically projective as a graded $\mathbb{F}_p$-vector space. Moreover, it is an isomorphism for $X = S$ precisely when $H_*(T)$ is of finite type, which for W bounded below is equivalent to $H_*(W) \cong A_* \otimes H_*(T)$ being of finite type. $\square$

11.2. Towers of spectra

DEFINITION 11.2.1. By a *tower* $Y_\star$ of (orthogonal) spectra we mean a diagram of the form

$$\dots \longrightarrow Y_{s+1} \xrightarrow{\alpha} Y_s \longrightarrow \dots \longrightarrow Y_1 \xrightarrow{\alpha} Y_0$$

in $Sp^{\mathbb{O}}$. We write

$$Y_{s,r} = C(\alpha^r \colon Y_{s+r} \to Y_s) = Y_s \cup CY_{s+r}$$

for the mapping cone of $\alpha^r \colon Y_{s+r} \to Y_s$, so that we have a homotopy cofiber sequence

$$Y_{s+r} \xrightarrow{\alpha^r} Y_s \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r} \tag{11.1}$$

for each $s \geq 0$ and $r \geq 0$. In particular, when $r = 1$ we have a homotopy cofiber sequence

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1}\,,$$

for each $s \geq 0$. We often display the tower, and the homotopy cofiber sequences for $r = 1$, as follows.

$$\begin{tikzcd}
\dots \arrow[r] & Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[r] \arrow[d, "\beta"] & \dots \arrow[r] & Y_2 \arrow[r, "\alpha"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_0 \arrow[d, "\beta"] \\
& & Y_{s,1} \arrow[ul, dashed, "\gamma"] & & & Y_{1,1} \arrow[ul, dashed, "\gamma"] & Y_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

Here the dashed arrows refer to maps to the suspension of the indicated target, i.e., of degree -1. We may also refer to this as a *resolution* in (orthogonal) spectra of Y_0, and redraw part of the diagram as follows.

$$\dots \leftarrow \Sigma^{s+1} Y_{s+1} \xleftarrow{\gamma} \Sigma^s Y_{s,1} \xleftarrow{\beta} \Sigma^s Y_s \leftarrow \dots \leftarrow \Sigma Y_1 \xleftarrow{\gamma} Y_{0,1} \xleftarrow{\beta} Y_0 \tag{11.2}$$

By a (strict) map of towers $\phi_\star \colon Y_\star \to Z_\star$ we mean a sequence of maps $\phi_s \colon Y_s \to Z_s$ such that each square

$$\begin{tikzcd}
Y_{s+1} \arrow[r, "\alpha"] \arrow[d, "\phi_{s+1}"'] & Y_s \arrow[d, "\phi_s"] \\
Z_{s+1} \arrow[r, "\alpha"] & Z_s
\end{tikzcd}$$

commutes in $Sp^{\mathbb{O}}$. There are then well-defined maps $\phi_{s,r} \colon Y_{s,r} \to Z_{s,r}$ for all $s \geq 0$ and $r \geq 0$, making the diagrams

$$\begin{tikzcd}
Y_{s+r} \arrow[r, "\alpha^r"] \arrow[d, "\phi_{s+r}"'] & Y_s \arrow[r] \arrow[d, "\phi_s"'] & Y_{s,r} \arrow[r] \arrow[d, "\phi_{s,r}"'] & \Sigma Y_{s+r} \arrow[d, "\Sigma\phi_{s+r}"'] \\
Z_{s+r} \arrow[r, "\alpha^r"] & Z_s \arrow[r] & Z_{s,r} \arrow[r] & \Sigma Z_{s+r}
\end{tikzcd}$$

commute.

The category $\mathrm{Tow}(Sp^{\mathbb{O}})$ of towers of spectra is thus the category of functors $\mathbb{N}^{op} \to Sp^{\mathbb{O}}$, where there is a unique morphism $i \to j$ in $\mathbb{N}^{op}$ precisely when $i \geq j$.

REMARK 11.2.2. It might be more consistent to write Y^s in place of Y_s for the terms in a tower, since s behaves as a cohomological index and the tower induces decreasing filtrations, but the use of a subscript is traditional, and when discussing multiplicative structure we will want to use superscripts for (smash) powers of spectra.

One could also consider biinfinite towers of spectra, which are functors $\mathbb{Z}^{op} \to Sp^{\mathbb{O}}$, but this will not be relevant for the Adams spectral sequence, and there are some technical complications regarding model structures that we avoid by concentrating on non-negatively graded towers.

The homotopy spectral sequence associated to a tower of spectra will only depend its image in the stable category, including the distinguished triangles (11.1), which we shall refer to as resolutions.

DEFINITION 11.2.3. By a *resolution* $(Y_\star, Y_{\star,1})$ in the stable category, we mean a diagram of the form

$$\begin{tikzcd}
\dots \arrow[r] & Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[r] \arrow[d, "\beta"] & \dots \arrow[r] & Y_2 \arrow[r, "\alpha"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_0 \arrow[d, "\beta"] \\
& & Y_{s,1} \arrow[ul, dashed, "\gamma"] & & & Y_{1,1} \arrow[ul, dashed, "\gamma"] & Y_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

in $\mathrm{Ho}(Sp^{\mathbb{O}})$, where each triangle

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1}$$

is distinguished. By a (weak) map of resolutions $\phi_\star \colon (Y_\star, Y_{\star,1}) \to (Z_\star, Z_{\star,1})$ we mean sequences of morphisms $\phi_s \colon Y_s \to Z_s$ and $\phi_{s,1} \colon Y_{s,1} \to Z_{s,1}$ in $\mathrm{Ho}(Sp^{\mathbb{O}})$, such that the diagrams

$$\begin{tikzcd}
Y_{s+1} \arrow[r, "\alpha"] \arrow[d, "\phi_{s+1}"'] & Y_s \arrow[r, "\beta"] \arrow[d, "\phi_s"'] & Y_{s,1} \arrow[r, "\gamma"] \arrow[d, "\phi_{s,1}"'] & \Sigma Y_{s+1} \arrow[d, "\Sigma\phi_{s+1}"'] \\
Z_{s+1} \arrow[r, "\alpha"] & Z_s \arrow[r, "\beta"] & Z_{s,1} \arrow[r, "\gamma"] & \Sigma Z_{s+1}
\end{tikzcd}$$

commute in the stable category.

Here is a different view of a map of resolutions.

$$\begin{tikzcd}
\dots \arrow[r] & Y_{s+1} \arrow[rr, "\alpha"] \arrow[dd, "\phi_{s+1}"'] & & Y_s \arrow[r] \arrow[dl, "\beta"] \arrow[dd, "\phi_s"'] & \dots \arrow[r] & Y_2 \arrow[rr, "\alpha"] \arrow[dd, "\phi_2"'] & & Y_1 \arrow[rr, "\alpha"] \arrow[dl, "\beta"] \arrow[dd, "\phi_1"'] & & Y_0 \arrow[dl, "\beta"] \arrow[dd, "\phi_0"'] \\
& & Y_{s,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "\phi_{s,1}"'] & & & & Y_{1,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "\phi_{1,1}"'] & & Y_{0,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "\phi_{0,1}"'] & \\
\dots \arrow[r] & Z_{s+1} \arrow[rr, crossing over, "\alpha"] & & Z_s \arrow[r] \arrow[dl, "\beta"] & \dots \arrow[r] & Z_2 \arrow[rr, crossing over, "\alpha"] & & Z_1 \arrow[rr, crossing over, "\alpha"] \arrow[dl, "\beta"] & & Z_0 \arrow[dl, "\beta"] \\
& & Z_{s,1} \arrow[ul, dashed, "\gamma"] & & & & Z_{1,1} \arrow[ul, dashed, "\gamma"] & & Z_{0,1} \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

REMARK 11.2.4. Each morphism $\alpha \colon Y_{s+1} \to Y_s$ in $\mathrm{Ho}(Sp^{\mathbb{O}})$ can be embedded in a distinguished triangle, as above, but $Y_{s,1}$ is then only determined up to non-canonical isomorphism in the stable category. Hence, in order to associate an exact couple to the tower $Y_\star$, we need to extend it to a resolution $(Y_\star, Y_{\star,1})$ by fixing choices of these distinguished triangles.

((ETC: Why does each resolution in $\mathrm{Ho}(Sp^{\mathbb{O}})$ come (up to isomorphism) from a tower in $Sp^{\mathbb{O}}$?))

Definition 11.2.5. The *homotopy exact couple* (A, E) associated to a spectrum X and a tower $Y_\star$ of orthogonal spectra, or to X and a resolution $(Y_\star, Y_{\star,1})$ in the stable category, is the diagram

$$\begin{tikzcd}
\dots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "\alpha"] & {[X, Y_{s+1}]_*} \arrow[r, "\alpha"] \arrow[d, "\beta"] & {[X, Y_s]_*} \arrow[r, "\alpha"] \arrow[d, "\beta"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "\beta"] & \dots \\
& & {[X, Y_{s+1,1}]_*} \arrow[ul, "\gamma"] & {[X, Y_{s,1}]_*} \arrow[ul, "\gamma"] & {[X, Y_{s-1,1}]_*} \arrow[ul, "\gamma"] & ,
\end{tikzcd}$$

where

$$\dots \to [X, Y_{s+1}]_n \xrightarrow{\alpha} [X, Y_s]_n \xrightarrow{\beta} [X, Y_{s,1}]_n \xrightarrow{\gamma} [X, Y_{s+1}]_{n-1} \to \dots$$

is a long exact sequence for each $s \ge 0$. The bigraded abelian groups A and E are given by

$$\begin{aligned}
A^{s,t} &= [X, Y_s]_{t-s} = [S^{t-s} \wedge X, Y_s] \\
E^{s,t} &= [X, Y_{s,1}]_{t-s} = [S^{t-s} \wedge X, Y_{s,1}] \,.
\end{aligned}$$

The *homotopy spectral sequence* $(E_r, d_r)_{r\ge1}$ associated to X and $Y_\star$ or $(Y_\star, Y_{\star,1})$, is the spectral sequence associated to the homotopy exact couple, with

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} = [S^{t-s} \wedge X, Y_{s,1}]$$

and

$$d_1^{s,t} = \beta\gamma \colon E_1^{s,t} \longrightarrow E_1^{s+1,t}$$

for all $s \ge 0$ and $t \in \mathbb{Z}$. The d_r-differentials

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

then have (s,t)-bidegree $(r, r-1)$, for each $r \ge 1$.

Remark 11.2.6. In view of the isomorphisms

$$\begin{aligned}
A^{s,t} &\cong [\Sigma^t X, \Sigma^s Y_s] \\
E^{s,t} &\cong [\Sigma^t X, \Sigma^s Y_{s,1}]
\end{aligned}$$

the lower part of the homotopy exact couple is obtained by applying $[X, -]_t = [\Sigma^t X, -]$ to the diagram (11.2). We treat the total degree $t - s$ as a homological grading, so that the differentials have total degree -1, which means that the internal degree t is homological and the filtration degree s is cohomological. Since the filtration degree s interacts most directly with the term number r for the spectral sequence, we write E_r^s for the filtration s part of the E_r-term. It is then traditional to write $E_r^{s,t}$ for the internal degree t part of this graded group, even if $(E_r^s)_t$ might have been more consistent.

Definition 11.2.7. The *abutment* of the homotopy exact couple of X and $Y_\star$ is the graded abelian group $[X, Y_0]_*$ with the descending, exhaustive filtration

$$\dots \subset F^{s+1}[X, Y_0]_* \subset F^s[X, Y_0]_* \subset \dots \subset F^0[X, Y_0]_* = [X, Y_0]_*$$

given by

$$F^s[X,Y_0]_* = \mathrm{im}([X,Y_s]_* \xrightarrow{\alpha^s} [X,Y_0]_*)$$

for $s \ge 0$.

EXAMPLE 11.2.8. There are injective homomorphisms

$$\frac{F^s[X,Y_0]_n}{F^{s+1}[X,Y_0]_n} \overset{\zeta}{\rightarrowtail} E_\infty^{s,s+n}$$

for all $s \ge 0$ and $n \in \mathbb{Z}$. If for each n the groups $[X,Y_s]_n$ vanish for all sufficiently large s, then the filtration $(F^s[X,Y_0]_*)_s$ is degreewise discrete, and the homotopy spectral sequence

$$E_r^{s,t} \Longrightarrow_s [X,Y_0]_{t-s}$$

converges (strongly), by Proposition 2.5.11, so that each ζ is an isomorphism.

EXAMPLE 11.2.9. When $X = S$, the homotopy exact couple of $(Y_\star, Y_{\star,1})$ is the diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y_{s+2}) \arrow[r, "\alpha"] & \pi_*(Y_{s+1}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & \pi_*(Y_s) \arrow[r, "\alpha"] \arrow[d, "\beta"] & \pi_*(Y_{s-1}) \arrow[r] \arrow[d, "\beta"] & \dots \\
& & \pi_*(Y_{s+1,1}) \arrow[ul, "\gamma"] & \pi_*(Y_{s,1}) \arrow[ul, "\gamma"] & \pi_*(Y_{s-1,1}) \arrow[ul, "\gamma"] & ,
\end{tikzcd}$$

where

$$\dots \to \pi_n(Y_{s+1}) \xrightarrow{\alpha} \pi_n(Y_s) \xrightarrow{\beta} \pi_n(Y_{s,1}) \xrightarrow{\gamma} \pi_{n-1}(Y_{s+1}) \to \dots$$

is a long exact sequence for each $s \ge 0$. The bigraded abelian groups A and $E = E_1$ are given by

$$A^{s,t} = \pi_{t-s}(Y_s)$$
$$E^{s,t} = E_1^{s,t} = \pi_{t-s}(Y_{s,1})$$

and $d_1^{s,t} = \beta\gamma\colon E_1^{s,t} \to E_1^{s+1,t}$ equals the composite

$$\pi_{t-s}(Y_{s,1}) \xrightarrow{\gamma} \pi_{t-s-1}(Y_{s+1}) \xrightarrow{\beta} \pi_{t-s-1}(Y_{s+1,1})\,.$$

DEFINITION 11.2.10. The *abutment* of the homotopy exact couple of $Y_\star$ is the graded abelian group $\pi_*(Y_0)$ with the descending, exhaustive filtration given by

$$F^s\pi_*(Y_0) = \mathrm{im}(\pi_*(Y_s) \xrightarrow{\alpha^s} \pi_*(Y_0))$$

for $s \ge 0$.

EXAMPLE 11.2.11. There are injective homomorphisms

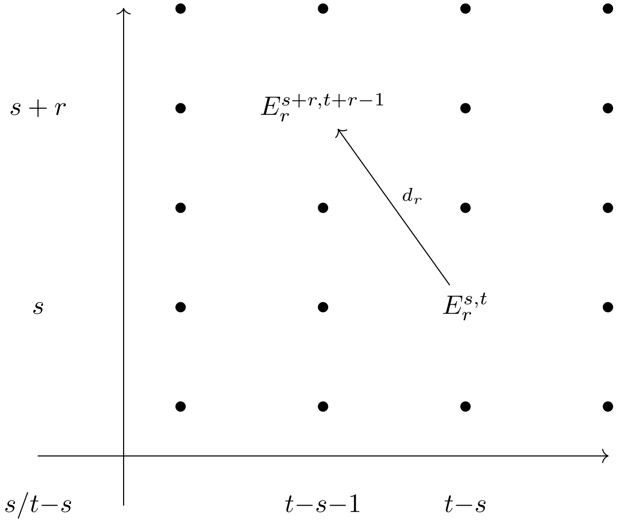
$$\frac{F^s\pi_n(Y_0)}{F^{s+1}\pi_n(Y_0)} \overset{\zeta}{\rightarrowtail} E_\infty^{s,s+n}$$

for all $s \ge 0$ and $n \in \mathbb{Z}$. If the connectivity of the spectra Y_s increases to infinity with s, then the filtration $(F^s\pi_*(Y_0))_s$ is degreewise discrete and the homotopy spectral sequence

$$E_r^{s,t} \Longrightarrow_s \pi_{t-s}(Y_0)$$

converges (strongly), so that each ζ is an isomorphism.

REMARK 11.2.12. The *Adams grading* convention for a homotopy spectral sequence is to use $(t-s,s)$-coordinates, placing each group $E_r^{s,t}$ at the position with horizontal coordinate $t-s$ and vertical coordinate s. The d_r-differentials then have $(t-s,s)$-bigrading $(-1,r)$, mapping one column to the left and r rows up.



When the spectral sequence converges to $[X,Y_0]_*$, the associated graded groups of the filtration $(F^s[X,Y_0]_n)_s$ are given by the groups in the E_∞-term that are located in the column with $t-s=n$. There is then a tower of short exact sequences

$$\begin{tikzcd}
\cdots \arrow[d, tail] & \\
F^{s+1}[X,Y_0]_n \arrow[r, two heads] \arrow[d, tail] & \dfrac{F^{s+1}[X,Y_0]_n}{F^{s+2}[X,Y_0]_n} \cong E_\infty^{s+1,s+1+n} \\
F^{s}[X,Y_0]_n \arrow[r, two heads] \arrow[d, tail] & \dfrac{F^{s}[X,Y_0]_n}{F^{s+1}[X,Y_0]_n} \cong E_\infty^{s,s+n} \\
F^{s-1}[X,Y_0]_n \arrow[r, two heads] \arrow[d, tail] & \dfrac{F^{s-1}[X,Y_0]_n}{F^{s}[X,Y_0]_n} \cong E_\infty^{s-1,s-1+n} \\
\cdots &
\end{tikzcd}$$

mapping down and across, ending with an edge homomorphism

$$[X,Y_0]_n \twoheadrightarrow \frac{[X,Y_0]_n}{F^1[X,Y_0]} \cong E_\infty^{0,n} \rightarrowtail E_1^{0,n} = [X,Y_{0,1}]_n$$

induced by $\beta\colon Y_0 \to Y_{0,1}$.

Remark 11.2.13. We can associate a homological extended Cartan–Eilenberg system (H_*, η, ∂) to a spectrum X and a tower of spectra $Y_\star$, with graded groups

$$H_*(-s-r, -s) = [X, Y_{s,r}]_*$$

for $r \ge 0$. Equivalently,

$$H_*(i,j) = [X, Y_{-j,j-i}]_*$$

for $i \le j$. Alternatively, we can index this as a cohomological extended Cartan–Eilenberg system (H^*, η, δ), with graded groups

$$H^*(s, s+r) = [X, Y_{s,r}]_{-*}$$
$$H^*(i,j) = [X, Y_{i,j-i}]_{-*}\,.$$

A third, homotopical, indexing scheme for the Cartan–Eilenberg system works with the graded groups

$$\pi_*(s, s+r) = [X, Y_{s,r}]_*\,.$$

In each case we interpret Y_s as Y_0 for $-\infty \le s < 0$ and as $*$ for $s = \infty$. The (top) exact couple underlying each of these Cartan–Eilenberg systems is the same as the homotopy exact couple of the tower of spectra, or of its associated resolution in the stable category. ((ETC: Return to this when discussing products and pairings of towers.))

11.3. Adams resolutions

Recall that a spectrum W is H-injective if it has the form $H \wedge T$ for some spectrum T, which means that it is stably equivalent to a wedge sum of suspensions of Eilenberg–MacLane spectra.

Definition 11.3.1. Let Y be an (orthogonal) spectrum. A mod p *Adams resolution* of Y is a resolution

$$\begin{tikzcd}
\dots \arrow[r] & Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[r] \arrow[d, "\beta"] & \dots \arrow[r] & Y_2 \arrow[r, "\alpha"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_0 \arrow[d, "\beta"] \\
 & & Y_{s,1} \arrow[ul, dashed, "\gamma"] & & & Y_{1,1} \arrow[ul, dashed, "\gamma"] & Y_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

in $\mathrm{Ho}(Sp^{\mathbb{O}})$, with a stable equivalence $Y \sim Y_0$, such that

1. (1) $Y_{s,1}$ is H-injective, and
2. (2) $\alpha_* \colon H_*(Y_{s+1}) \to H_*(Y_s)$ is zero,

for each $s \ge 0$. A mod p *Adams tower* for Y is a diagram

$$\dots \longrightarrow Y_{s+1} \xrightarrow{\alpha} Y_s \longrightarrow \dots \longrightarrow Y_1 \xrightarrow{\alpha} Y_0$$

in $Sp^{\mathbb{O}}$, with a stable equivalence $Y \sim Y_0$, such that the associated resolution (with $Y_{s,1} = C(\alpha \colon Y_{s+1} \to Y_s)$) is an Adams resolution.

Remark 11.3.2. In view of the long exact sequences

$$\dots \to H_*(Y_{s+1}) \xrightarrow{\alpha_*} H_*(Y_s) \xrightarrow{\beta_*} H_*(Y_{s,1}) \xrightarrow{\gamma_*} H_{*-1}(Y_{s+1}) \to \dots$$

$$\dots \to H^{*-1}(Y_{s+1}) \xrightarrow{\gamma^*} H^*(Y_{s,1}) \xrightarrow{\beta^*} H^*(Y_s) \xrightarrow{\alpha^*} H^*(Y_{s+1}) \to \dots$$

and the universal coefficient theorem, the condition that α_* is zero is equivalent to each of the following: that β_* is injective, γ_* is surjective, α^* is zero, β^* is surjective or γ^* is injective.

DEFINITION 11.3.3. The mod p *Adams spectral sequence* for $[X, Y]_*$ is the homotopy spectral sequence

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

associated to a mod p Adams resolution $(Y_\star, Y_{\star,1})$ of Y. In the case $X = S$ we write

$$E_1^{s,t}(Y) = \pi_{t-s}(Y_{s,1}) \Longrightarrow_s \pi_{t-s}(Y)$$

for this spectral sequence.

As stated, this depends on a choice of Adams resolution. We now show that Adams resolutions exist, that they are quasi-uniquely defined and natural, and that we can give algebraic descriptions of the E_1- and E_2-terms of the associated homotopy spectral sequences. In particular, the E_2-term will be seen to be independent of the choice of Adams resolution.

DEFINITION 11.3.4. Let $H = H\mathbb{F}_p$, with unit map $h\colon S \to H$ and ring spectrum multiplication $\mu\colon H \wedge H \to H$, and let

$$S \xrightarrow{h} H \xrightarrow{i} \bar{H} \xrightarrow{q} S^1$$

be the Puppe sequence generated by h, with $\bar{H} = Ch = H \cup_h CS$.

Here h induces the stable mod p Hurewicz homomorphism $\pi_*(X) \to H_*(X)$, hence the notation. ((ETC: Alternative notations would be η or ι.))

((ETC: Need $S \to H$ to be a Quillen cofibration, or Y to be Quillen cofibrant, for the smash products defining $\Sigma^s Y_s$ and $\Sigma^s Y_{s,1}$ to be homotopically meaningful. One option is to implicitly work with the derived smash products. This may become an issue when forming convolution products of towers, in order to discuss multiplicative structure.))

DEFINITION 11.3.5. The *canonical Adams resolution of* Y

$$\begin{tikzcd}
\dots \arrow[r] & Y_3 \arrow[r, "\alpha"] & Y_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y \arrow[d, "\beta"] \\
& & H \wedge Y_2 \arrow[ul, dashed, "\gamma"] & H \wedge Y_1 \arrow[ul, dashed, "\gamma"] & H \wedge Y \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

is defined inductively by setting $Y_0 = Y$ and, for $s \geq 0$, letting

$$Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1} \xrightarrow{-\Sigma\alpha} \Sigma Y_s$$

be equal to

$$S \wedge Y_s \xrightarrow{h \wedge 1} H \wedge Y_s \xrightarrow{i \wedge 1} \bar{H} \wedge Y_s \xrightarrow{q \wedge 1} S^1 \wedge Y_s\,.$$

This implicitly defines $\alpha\colon Y_{s+1} \to Y_s$ in $\mathrm{Ho}(Sp^{\mathbb{O}})$, since Σ is an equivalence of categories. Equivalently,

$$\Sigma^s Y_s = \bar{H}^{\wedge s} \wedge Y$$
$$\Sigma^s Y_{s,1} = H \wedge \bar{H}^{\wedge s} \wedge Y$$

for each $s \geq 0$, with β, γ and $-\Sigma\alpha$ induced by h, i and q, respectively.

Note that the canonical Adams resolution of Y equals the canonical Adams resolution

$$\begin{tikzcd}
\dots \arrow[r] & \Sigma^{-3}\bar{H}^{\wedge 3} \arrow[r, "\alpha"] & \Sigma^{-2}\bar{H}^{\wedge 2} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \Sigma^{-1}\bar{H} \arrow[r, "\alpha"] \arrow[d, "\beta"] & S \arrow[d, "\beta"] \\
& & H \wedge \Sigma^{-2}\bar{H}^{\wedge 2} \arrow[ul, dashed, "\gamma"] & H \wedge \Sigma^{-1}\bar{H} \arrow[ul, dashed, "\gamma"] & H \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

of S, smashed with Y.

Lemma 11.3.6. *The canonical Adams resolution $(Y_\star, Y_{\star,1})$ is an Adams resolution of $Y = Y_0$. If Y is bounded below with mod p homology of finite type, then each $Y_{s,1}$ is also bounded below with mod p homology of finite type.*

Proof. Each spectrum $Y_{s,1} = H \wedge Y_s$ is H-injective by construction. Furthermore, each homomorphism

$$\beta_* \colon H_*(Y_s) \longrightarrow H_*(Y_{s,1})$$

is induced by the unit inclusion

$$H \wedge Y_s \cong H \wedge S \wedge Y_s \xrightarrow{1\wedge h\wedge 1} H \wedge H \wedge Y_s\,,$$

which is split by the ring spectrum multiplication

$$H \wedge H \wedge Y_s \xrightarrow{\mu\wedge 1} H \wedge Y_s\,.$$

Hence β_* is (split) injective and $\alpha_* = 0$. (This only uses that $\mu(1 \wedge h) = 1$ in the stable category.)

Note that H and $\bar{H}$ are bounded below, with $H_*(H) \cong A_*$ and $H_*(\bar{H}) \cong J(A_*)$ both being of finite type. It follows from Proposition 9.8.14 that if Y is bounded below, then so is each $Y_{s,1}$. If Y furthermore has mod p homology of finite type, then the Künneth formula

$$H_*(Y_{s,1}) \cong A_* \otimes J(A_*)^{\otimes s} \otimes H_*(Y)$$

shows that each $Y_{s,1}$ also has this property. □

The homological image of an Adams resolution begins as follows.

$$\begin{tikzcd}
\dots & H_*(\Sigma^2 Y_2) \arrow[d, tail, "\beta_*"] & H_*(\Sigma Y_1) \arrow[d, tail, "\beta_*"] & H_*(Y) \arrow[d, tail, "\beta_*"] \\
& H_*(\Sigma^2 Y_{2,1}) \arrow[ul, two heads, "\gamma_*"] & H_*(\Sigma Y_{1,1}) \arrow[ul, two heads, "\gamma_*"] & H_*(Y_{0,1}) \arrow[ul, two heads, "\gamma_*"]
\end{tikzcd}$$

Proposition 11.3.7. *Let X be a spectrum and let $(Y_\star, Y_{\star,1})$ be an Adams resolution of Y. The Adams spectral sequence*

$$E_1^{s,t} = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

satisfies:

(1) *The d-invariant*

$$d\colon E_1^{s,t} \xrightarrow{\cong} \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1}))$$

is an isomorphism.

(2) *The diagram*

$$\begin{tikzcd} E_1^{s,t} \arrow[r, "d", "\cong"'] \arrow[d, "d_1^{s,t}"'] & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1})) \arrow[d, "\mathrm{Hom}(1,\beta_*\gamma_*)"] \\ E_1^{s+1,t} \arrow[r, "d", "\cong"'] & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^{s+1} Y_{s+1,1})) \end{tikzcd}$$

commutes.

(3) *The A_*-comodule complex*

$$\dots \leftarrow H_*(\Sigma^{s+1}Y_{s+1,1}) \xleftarrow{\beta_*\gamma_*} H_*(\Sigma^s Y_{s,1}) \xleftarrow{\beta_*\gamma_*} \dots$$
$$\dots \xleftarrow{\beta_*\gamma_*} H_*(\Sigma Y_{1,1}) \xleftarrow{\beta_*\gamma_*} H_*(Y_{0,1}) \xleftarrow{\beta_*} H_*(Y) \leftarrow 0$$

is exact, and each $H_*(\Sigma^s Y_{s,1})$ is an extended A_*-comodule. Hence this is an injective A_*-comodule resolution of $H_*(Y)$.

Proof. Claim (1) follows from Proposition 11.1.5, using the identification

$$\mathrm{Hom}^{t-s}_{A_*}(H_*(X), H_*(Y_{s,1})) \cong \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1}))\,,$$

since each $\Sigma^s Y_{s,1}$ is H-injective, i.e., has the form $H \wedge T$.

Claim (2) follows from the commutative diagram below, since $d_1^{s,t} = \beta_*\gamma_*$.

$$\begin{tikzcd} E_1^{s,t} \arrow[r, "d", "\cong"'] \arrow[d, "\gamma_*"'] & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^s Y_{s,1})) \arrow[d, "\mathrm{Hom}(1,\gamma_*)"] \\ {[X, \Sigma^{s+1} Y_{s+1}]_t} \arrow[r, "d"] \arrow[d, "\beta_*"'] & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^{s+1} Y_{s+1})) \arrow[d, "\mathrm{Hom}(1,\beta_*)"] \\ E_1^{s+1,t} \arrow[r, "d", "\cong"'] & \mathrm{Hom}^t_{A_*}(H_*(X), H_*(\Sigma^{s+1} Y_{s+1,1})) \end{tikzcd}$$

Claim (3) follows by splicing together the sequences

$$0 \leftarrow H_*(\Sigma^{s+1} Y_{s+1}) \xleftarrow{\gamma_*} H_*(\Sigma^s Y_{s,1}) \xleftarrow{\beta_*} H_*(\Sigma^s Y_s) \leftarrow 0$$

for all $s \ge 0$. These are all short exact, because $\alpha_* = 0$. Since each $\Sigma^s Y_{s,1}$ has the form $H \wedge T$ for some spectrum T, the Künneth formula and untwisting isomorphism show that

$$H_*(\Sigma^s Y_{s,1}) \cong H_*(H) \otimes H_*(T) \cong A_* \otimes H_*(T)$$

is an extended A_*-comodule, for each $s \ge 0$. □

Theorem 11.3.8. *The Adams spectral sequence for $[X, Y]_*$ has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*}(H_*(X), H_*(Y))\,,$$

which only depends on the A_*-comodules $H_*(X)$ and $H_*(Y)$. In the special case $X = S$, we write

$$E_2^{s,t}(Y) = \mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_p, H_*(Y))$$

for this E_2-term.

PROOF. Let $I^s_* = H_*(\Sigma^s Y_{s,1})$, $\delta^s = \beta_*\gamma_* \colon I^s_* \to I^{s+1}_*$ and $\eta = \beta_* \colon H_*(Y) \to I^0_*$. Then

$$\dots \leftarrow I^{s+1}_* \xleftarrow{\delta^s} I^s_* \xleftarrow{\delta^{s-1}} \dots \xleftarrow{\delta^1} I^1_* \xleftarrow{\delta^0} I^0_* \xleftarrow{\eta} H_*(Y) \leftarrow 0$$

is an injective A_*-comodule resolution of $H_*(Y)$, so the cohomology groups of the cochain complex

$$\dots \leftarrow \mathrm{Hom}^t_{A_*}(H_*(X), I^{s+1}_*) \xleftarrow{\mathrm{Hom}(1,\delta^s)} \mathrm{Hom}^t_{A_*}(H_*(X), I^s_*) \xleftarrow{\mathrm{Hom}(1,\delta^{s-1})} \mathrm{Hom}^t_{A_*}(H_*(X), I^{s-1}_*) \leftarrow \dots$$

are by definition the A_*-comodule Ext-groups $\mathrm{Ext}^{s,t}_{A_*}(H_*(X), H_*(Y))$, for all $s \geq 0$ and t. Since this cochain complex is isomorphic to

$$\dots \leftarrow E^{s+1,t}_1 \xleftarrow{d^{s,t}_1} E^{s,t}_1 \xleftarrow{d^{s-1,t}_1} E^{s-1,t}_1 \leftarrow \dots ,$$

these cohomology groups are precisely the components $E^{s,t}_2$ of the Adams spectral sequence E_2-term. □

The cohomological image of an Adams resolution begins as follows.

$$\begin{tikzcd}
\dots \arrow[dr, tail, "\gamma^*"'] & H^*(\Sigma^2 Y_2) \arrow[dr, tail, "\gamma^*"'] & & H^*(\Sigma Y_1) \arrow[dr, tail, "\gamma^*"'] & & H^*(Y) \\
& H^*(\Sigma^2 Y_{2,1}) \arrow[u, two heads, "\beta^*"'] & & H^*(\Sigma Y_{1,1}) \arrow[u, two heads, "\beta^*"'] & & H^*(Y_{0,1}) \arrow[u, two heads, "\beta^*"']
\end{tikzcd}$$

PROPOSITION 11.3.9. *Let X and Y be spectra, and suppose that $(Y_\star, Y_{\star,1})$ is an Adams resolution of Y with each $Y_{s,1}$ bounded below and of finite type mod p. The Adams spectral sequence*

$$E^{s,t}_1 = [X, Y_{s,1}]_{t-s} \Longrightarrow_s [X, Y]_{t-s}$$

satisfies

(1) *The d-invariant*

$$d \colon E^{s,t}_1 \xrightarrow{\cong} \mathrm{Hom}^t_A(H^*(\Sigma^s Y_{s,1}), H^*(X))$$

is an isomorphism.

(2) *The diagram*

$$\begin{tikzcd}
E^{s,t}_1 \arrow[r, "d", "\cong"'] \arrow[d, "d^{s,t}_1"'] & \mathrm{Hom}^t_A(H^*(\Sigma^s Y_{s,1}), H^*(X)) \arrow[d, "\mathrm{Hom}(\gamma^*\beta^*,1)"] \\
E^{s+1,t}_1 \arrow[r, "d", "\cong"'] & \mathrm{Hom}^t_A(H^*(\Sigma^{s+1} Y_{s+1,1}), H^*(X))
\end{tikzcd}$$

commutes.

(3) *The A-module complex*

$$\dots \to H^*(\Sigma^{s+1} Y_{s+1,1}) \xrightarrow{\gamma^*\beta^*} H^*(\Sigma^s Y_{s,1}) \xrightarrow{\gamma^*\beta^*} \dots \xrightarrow{\gamma^*\beta^*} H^*(\Sigma Y_{1,1}) \xrightarrow{\gamma^*\beta^*} H^*(Y_{0,1}) \xrightarrow{\beta^*} H^*(Y) \to 0$$

is exact, and each $H^*(\Sigma^s Y_{s,1})$ is an extended A-module. Hence this is a projective A-module resolution of $H^*(Y)$.

PROOF. Claim (1) follows from Proposition 11.1.5, using the identification

$$\mathrm{Hom}_A^{t-s}(H^*(Y_{s,1}), H^*(X)) \cong \mathrm{Hom}_A^t(H^*(\Sigma^s Y_{s,1}), H^*(X)),$$

since each $\Sigma^s Y_{s,1}$ is H-injective, i.e., has the form $H \wedge T$, and is assumed to be bounded below of finite type mod p.

Claim (2) follows from the commutative diagram below, since $d_1^{s,t} = \beta_* \gamma_*$.

$$\begin{tikzcd}
E_1^{s,t} \arrow[r, "d", "\cong"'] \arrow[d, "\gamma_*"'] & \mathrm{Hom}_A^t(H^*(\Sigma^s Y_{s,1}), H^*(X)) \arrow[d, "\mathrm{Hom}(\gamma^*,1)"] \\
{[X, \Sigma^{s+1} Y_{s+1}]_t} \arrow[r, "d"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_A^t(H^*(\Sigma^{s+1} Y_{s+1}), H^*(X)) \arrow[d, "\mathrm{Hom}(\beta^*,1)"] \\
E_1^{s+1,t} \arrow[r, "d", "\cong"'] & \mathrm{Hom}_A^t(H^*(\Sigma^{s+1} Y_{s+1,1}), H^*(X))
\end{tikzcd}$$

Claim (3) follows by splicing together the sequences

$$0 \to H^*(\Sigma^{s+1} Y_{s+1}) \xrightarrow{\gamma^*} H^*(\Sigma^s Y_{s,1}) \xrightarrow{\beta^*} H^*(\Sigma^s Y_s) \to 0$$

for all $s \geq 0$. These are all short exact, because $\alpha^* = 0$. Since each $\Sigma^s Y_{s,1}$ has the form $H \wedge T$ for some spectrum T, and is bounded below of finite type mod p, the Künneth formula and untwisting isomorphism show that

$$H^*(\Sigma^s Y_{s,1}) \cong H^*(H) \otimes H^*(T) \cong A \otimes H^*(T)$$

is an extended A-module, for each $s \geq 0$. $\square$

THEOREM 11.3.10. *Let X and Y be spectra, with Y bounded below and of finite type mod p. The Adams spectral sequence for $[X, Y]_*$ has E_2-term*

$$E_2^{s,t} \cong \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X)),$$

which only depends on the A-modules $H^*(X)$ and $H^*(Y)$. In the special case $X = S$, we write

$$E_2^{s,t}(Y) = \mathrm{Ext}_A^{s,t}(H^*(Y), \mathbb{F}_p)$$

for this E_2-term.

PROOF. Let $P_s^* = H^*(\Sigma^s Y_{s,1})$, $\partial_s = \gamma^* \beta^* \colon P_s^* \to P_{s-1}^*$ and $\epsilon = \beta^* \colon P_0^* \to H^*(Y)$. Then

$$\cdots \to P_{s+1}^* \xrightarrow{\partial_{s+1}} P_s^* \xrightarrow{\partial_s} \dots \xrightarrow{\partial_2} P_1^* \xrightarrow{\partial_1} P_0^* \xrightarrow{\epsilon} H^*(Y) \to 0$$

is a projective A-module resolution of $H^*(Y)$, so the cohomology groups of the cochain complex

$$\dots \leftarrow \mathrm{Hom}_A^t(P_{s+1}^*, H^*(X)) \xleftarrow{\mathrm{Hom}(\partial_{s+1},1)} \mathrm{Hom}_A^t(P_s^*, H^*(X))$$
$$\xleftarrow{\mathrm{Hom}(\partial_s,1)} \mathrm{Hom}_A^t(P_{s-1}^*, H^*(X)) \leftarrow \dots$$

are by definition the A-module Ext-groups $\mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X))$, for all $s \geq 0$ and t. Since this cochain complex is isomorphic to

$$\dots \leftarrow E_1^{s+1,t} \xleftarrow{d_1^{s,t}} E_1^{s,t} \xleftarrow{d_1^{s-1,t}} E_1^{s-1,t} \leftarrow \dots,$$

these cohomology groups are precisely the components $E_2^{s,t}$ of the Adams spectral sequence E_2-term. □

Algebraic resolutions can quite generally be realized by Adams resolutions.

LEMMA 11.3.11. *Let*

$$\dots \leftarrow I_*^1 \xleftarrow{\delta^0} I_*^0 \xleftarrow{\eta} H_*(Y) \leftarrow 0$$

be an extended A_*-comodule resolution of $H_*(Y)$. Then there exists an Adams resolution $(Y_\star, Y_{\star,1})$ of Y such that the associated injective A_*-comodule resolution

$$\dots \leftarrow H_*(\Sigma Y_{1,1}) \xleftarrow{\delta^0} H_*(Y_{0,1}) \xleftarrow{\eta} H_*(Y_0) \to 0$$

is isomorphic to the resolution above.

PROOF. We set $Y_0 = Y$ and choose H-injective $Y_{s,1}$ such that $H_*(\Sigma^s Y_{s,1}) \cong I_*^s$, for each $s \ge 0$. More explicitly, if

$$I_*^s \cong A_* \otimes V \cong \bigoplus_\alpha \Sigma^{n_\alpha} A_*$$

with $V = \bigoplus_\alpha \Sigma^{n_\alpha} \mathbb{F}_p$ we let $T = \bigvee_\alpha S^{n_\alpha}$ and set $\Sigma^s Y_{s,1} \simeq H \wedge T$. Let $\beta \colon Y_0 \to Y_{0,1}$ correspond to $\eta \colon H_*(Y) \to I_*^0$ under the case $s = 0$ of the isomorphism

$$d \colon [Y_s, Y_{s,1}] \xrightarrow{\cong} \operatorname{Hom}_{A_*}(H_*(\Sigma^s Y_s), I_*^s)\,,$$

and let $\Sigma Y_1 = C\beta$ be its mapping cone, with $\gamma \colon Y_{0,1} \to \Sigma Y_1$ and $\Sigma\alpha \colon \Sigma Y_1 \to \Sigma Y_0$ the canonical maps. Then γ_* realizes the surjection $I_*^0 \to \operatorname{cok}(\eta) \cong \operatorname{im}(\delta^0)$.

Inductively, for $s \ge 1$ let $\beta \colon Y_s \to Y_{s,1}$ correspond under d to the inclusion $\operatorname{im}(\delta^{s-1}) \subset I_*^s$, and let ΣY_{s+1} be its mapping cone, with $\gamma \colon Y_{s,1} \to \Sigma Y_{s+1}$ and $\Sigma\alpha \colon \Sigma Y_{s+1} \to \Sigma Y_s$ the canonical maps. Then γ_* realizes the surjection $I_*^s \to \operatorname{cok}(\delta^{s-1}) \cong \operatorname{im}(\delta^s)$, as required. □

LEMMA 11.3.12. *Let*

$$\dots \to P_1^* \xrightarrow{\partial_1} P_0^* \xrightarrow{\epsilon} H^*(Y) \to 0$$

be a free A-module resolution of $H^*(Y)$, with each P_s^* bounded below and of finite type. Then there exists an Adams resolution $(Y_\star, Y_{\star,1})$ of Y with each $Y_{s,1}$ bounded below and of finite type mod p, such that the associated projective A-module resolution

$$\dots \to H^*(\Sigma Y_{1,1}) \xrightarrow{\partial_1} H^*(Y_{0,1}) \xrightarrow{\epsilon} H^*(Y_0) \to 0$$

is isomorphic to the resolution above.

PROOF. We set $Y_0 = Y$ and choose H-injective $Y_{s,1}$ such that $H^*(\Sigma^s Y_{s,1}) \cong P_s^*$, for each $s \ge 0$. More explicitly, if

$$P_s^* \cong \bigoplus_\alpha \Sigma^{n_\alpha} A \cong \prod_\alpha \Sigma^{n_\alpha} A$$

we let $T = \bigvee_\alpha S^{n_\alpha}$ and set $\Sigma^s Y_{s,1} \simeq H \wedge T$. Let $\beta \colon Y_0 \to Y_{0,1}$ correspond to $\epsilon \colon P_0^* \to H^*(Y)$ under the case $s = 0$ of the isomorphism

$$d \colon [Y_s, Y_{s,1}] \xrightarrow{\cong} \operatorname{Hom}_A(P_s^*, H^*(\Sigma^s Y_s))\,,$$

and let $\Sigma Y_1 = C\beta$ be its mapping cone, with $\gamma \colon Y_{0,1} \to \Sigma Y_1$ and $\Sigma\alpha \colon \Sigma Y_1 \to \Sigma Y_0$ the canonical maps. Then γ^* realizes the inclusion $\operatorname{im}(\partial_1) = \ker(\epsilon) \subset P_0^*$.

Inductively, for $s \ge 1$ let $\beta\colon Y_s \to Y_{s,1}$ correspond under d to the surjection $P^*_s \to \mathrm{im}(\partial_s)$ factoring ∂_s through its image, and let ΣY_{s+1} be its mapping cone, with $\gamma\colon Y_{s,1} \to \Sigma Y_{s+1}$ and $\Sigma\alpha\colon \Sigma Y_{s+1} \to \Sigma Y_s$ the canonical maps. Then γ^* realizes the inclusion $\mathrm{im}(\partial_{s+1}) = \ker(\partial_s) \subset P^*_s$, as required. □

((ETC: Check that each Adams resolution comes, up to isomorphism, from an Adams tower.))

LEMMA 11.3.13. *The Adams spectral sequence edge homomorphism*

$$[X, Y]_n \longrightarrow E^{0,n}_\infty \subset E^{0,n}_2 = \mathrm{Hom}^n_{A_*}(H_*(X), H_*(Y))$$

is equal to the mod p homological d-invariant. If Y is bounded below and of finite type mod p, then the edge homomorphism

$$[X, Y]_n \longrightarrow E^{0,n}_\infty \subset E^{0,n}_2 = \mathrm{Hom}^n_A(H^*(Y), H^*(X))$$

is equal to the mod p cohomological d-invariant.

PROOF. The E_1-edge homomorphism $[X, Y]_* \to [X, Y_{0,1}]_* = E^{0,*}_1$ is induced by $\beta\colon Y \to Y_{0,1}$, and factors through the inclusion $E^{0,*}_2 \subset E^{0,*}_1$ of the kernel of $\beta_*\gamma_*$. The lower row in the commutative diagram

$$\begin{tikzcd}
{[X, \Sigma Y_{1,1}]_*} \arrow[d, "d"', "\cong"] & {[X, Y_{0,1}]_*} \arrow[l, "\beta_*\gamma_*"'] \arrow[d, "d"', "\cong"] & {[X, Y]_*} \arrow[l, "\beta_*"'] \arrow[d, "d"'] & \\
\mathrm{Hom}_{A_*}(H_*(X), I^1_*) & \mathrm{Hom}_{A_*}(H_*(X), I^0_*) \arrow[l, "\delta^0_*"'] & \mathrm{Hom}_{A_*}(H_*(X), H_*(Y)) \arrow[l, "\eta_*"'] & 0 \arrow[l]
\end{tikzcd}$$

is exact, and therefore the E_2-edge homomorphism corresponds under the middle isomorphism d to the right hand homomorphism d. □

DEFINITION 11.3.14. For $f \in [X, Y]_n$ satisfying $d(f) = 0$, then the *mod p Hopf–Steenrod invariant*

$$e(f) \in \mathrm{Ext}^1_{A_*}(H_*(\Sigma^{1+n} X), H_*(Y)) = \mathrm{Ext}^{1,1+n}_{A_*}(H_*(X), H_*(Y))$$

is defined to be the class of the A_*-comodule extension

$$0 \leftarrow H_*(\Sigma^{1+n} X) \xleftarrow{q_*} H_*(Cf) \xleftarrow{i_*} H_*(Y) \leftarrow 0\,.$$

If Y is bounded below and of finite type mod p, then this equals the class

$$e(f) \in \mathrm{Ext}^1_A(H^*(Y), H^*(\Sigma^{1+n} X)) = \mathrm{Ext}^{1,1+n}_A(H^*(Y), H^*(X))$$

of the A-module extension

$$0 \to H^*(\Sigma^{1+n} X) \xrightarrow{q^*} H^*(Cf) \xrightarrow{i^*} H^*(Y) \to 0\,.$$

PROPOSITION 11.3.15. *The Adams spectral sequence near-edge homomorphism*

$$F^1[X, Y]_n \longrightarrow E^{1,1+n}_\infty \subset E^{1,1+n}_2 = \mathrm{Ext}^{1,1+n}_{A_*}(H_*(X), H_*(Y))$$

equals the mod p Hopf–Steenrod invariant, mapping f with $d(f) = 0$ to $e(f)$.

PROOF. A morphism $f \in [X,Y]_n = [\Sigma^n X, Y]$ satisfies $d(f) = 0$ precisely if $\beta f = 0$, in which case there exist morphisms $f_1 \colon \Sigma^n X \to Y_1$ and $Cf \to Y_{0,1}$ making the following diagram commute.

$$
\begin{tikzcd}
\Sigma^n X \arrow[r, "f"] \arrow[d, "f_1"'] & Y \arrow[r, "i"] \arrow[d, "="'] & Cf \arrow[r, "q"] \arrow[d] & \Sigma^{1+n} X \arrow[d, "\Sigma f_1"'] & \\
Y_1 \arrow[r, "\alpha"] & Y_0 \arrow[r, "\beta"] & Y_{0,1} \arrow[r, "\gamma"] & \Sigma Y_1 \arrow[dr, "\Sigma\beta"'] & \\
& & & & \Sigma Y_{1,1}
\end{tikzcd}
$$

Passing to homology, we get a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & H_*(Y) \arrow[r, "i_*"] \arrow[d, "="'] & H_*(Cf) \arrow[r, "q_*"] \arrow[d] & H_*(\Sigma^{1+n} X) \arrow[r] \arrow[d, "\Sigma(\beta f_1)_*"'] & 0 \\
0 \arrow[r] & H_*(Y) \arrow[r, "\eta"] & I^0_* \arrow[r, "\delta^0"] & I^1_* \arrow[r, "\delta^1"] & I^2_*
\end{tikzcd}
$$

of A_*-comodules. Here the (well-defined) cohomology class

$$e(f) \in \mathrm{Ext}^1_{A_*}(H_*(\Sigma^{1+n} X), H_*(Y))$$

of

$$\Sigma(\beta f_1)_* \in \mathrm{Hom}_{A_*}(H_*(\Sigma^{1+n} X), I^1_*)$$

corresponds both to the A_*-comodule extension given by $H_*(Cf)$, and to the class in $E_\infty^{1,1+n} \subset E_2^{1,1+n}$ detecting f in the Adams spectral sequence. $\square$

We can restrict an Adams resolution $Y_\star$ to filtrations $s \geq k$, and obtain an Adams resolution of Y_k.

LEMMA 11.3.16. *Let $(Y_\star, Y_{\star,1})$ an Adams resolution of Y_0, and let $k \geq 0$. Define a resolution $(Z_\star, Z_{\star,1})$ so that*

$$Z_{s+1} \xrightarrow{\alpha} Z_s \xrightarrow{\beta} Z_{s,1} \xrightarrow{\gamma} \Sigma Z_{s+1}$$

is equal to

$$Y_{k+s+1} \xrightarrow{\alpha} Y_{k+s} \xrightarrow{\beta} Y_{k+s,1} \xrightarrow{\gamma} \Sigma Y_{k+s+1}$$

for all $s \geq 0$. Then $(Z_\star, Z_{\star,1})$ is an Adams resolution of Y_k. There is a morphism of spectral sequences

$$(E_r^{s,t}(X,Z), d_r) \longrightarrow (E_r^{k+s,k+t}(X,Y), d_r)$$

from the Adams spectral sequence for X and $Z_0 = Y_k$ to the Adams spectral sequence for X and Y_0, which is an isomorphism for $r = 1$ and $s \geq 0$. In general, it is surjective for $0 \leq s \leq r-2$ and an isomorphism for $s \geq r-1$.

PROOF. This follows from the map of resolutions shown below, and the induced map of exact couples and spectral sequences.

$$
\begin{tikzcd}
\cdots \arrow[r] & Y_{k+2} \arrow[rr, "\alpha"] \arrow[dd, "="'] & & Y_{k+1} \arrow[rr, "\alpha"] \arrow[dl, "\beta"] \arrow[dd, "="'] & & Y_k \arrow[rr, "="] \arrow[dl, "\beta"] \arrow[dd, "="'] & & Y_k \arrow[r] \arrow[dl] \arrow[dd, "\alpha"] & \cdots \\
& & Y_{k+1,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "="'] & & Y_{k,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "="'] & & * \arrow[ul, dashed] \arrow[dd] & & \\
\cdots \arrow[r] & Y_{k+2} \arrow[rr, "\alpha"] & & Y_{k+1} \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & Y_k \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & Y_{k-1} \arrow[r] \arrow[dl, "\beta"] & \cdots \\
& & Y_{k+1,1} \arrow[ul, dashed, "\gamma"] & & Y_{k,1} \arrow[ul, dashed, "\gamma"] & & Y_{k-1,1} \arrow[ul, dashed, "\gamma"] & &
\end{tikzcd}
$$

The r-th cycle groups $Z_r^s = \gamma^{-1} \operatorname{im} \alpha^{r-1}$ for $s \ge k$ are equal for the upper and lower resolutions, while the r-th boundary groups $B_r^s = \beta \ker \alpha^{r-1}$ for $s \ge k$ map isomorphically for $s-(r-1) \ge k$ and injectively in general. Hence the induced map of E_r-terms is an isomorphism for $s \ge k+(r-1)$ and surjective for $k \le s \le k+r-2$. The part $s \ge k$ of the upper resolutions is an Adams resolution, since the maps α and the spectra $Y_{s,1}$ that appear there are also part of the given Adams resolution $(Y_\star, Y_{\star,1})$. Reindexing the spectral sequence associated to the $s \ge k$ part of the upper row gives the stated conclusions. □

((ETC: Given an Adams tower $Y_\star$, the finite tower given by $Y_s/Y_k = Y_{s,k-s}$ at levels $0 \le s \le k$ is generally not an Adams tower, since the induced maps $\alpha\colon Y_{s+1}/Y_k \to Y_s/Y_k$ may not induce zero in mod p homology.))

11.4. Comparison of resolutions

The following hypotheses (1) and (2) are satisfied for Adams resolutions.

PROPOSITION 11.4.1. *Let $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ be resolutions such that*

(1) *$\alpha_*\colon H_*(Y_{s+1}) \to H_*(Y_s)$ is zero and*
(2) *$Z_{s,1}$ is H-injective*

for each $s \ge 0$. Let $\phi_0\colon Y_0 \to Z_0$ be any morphism in $\mathrm{Ho}(Sp^{\mathbb{O}})$. Then there exists a map of resolutions $\phi_\star$ that extends ϕ_0.

Moreover, if $\psi_\star$ is a second map of resolutions extending $\phi_0 = \psi_0$, then $\alpha\phi_s = \alpha\psi_s$ for each $s \ge 1$ and $\phi_s\alpha = \psi_s\alpha$ for each $s \ge 0$.

PROOF. Suppose, by induction, that $\phi_0, \phi_{0,1}, \dots, \phi_{s-1,1}$ and ϕ_s have been compatibly constructed. Consider the diagram below, with horizontal distinguished triangles.

$$
\begin{tikzcd}
Y_{s+1} \arrow[dr, "\alpha"] & & & & \\
& Y_s \arrow[r, "\beta"] \arrow[d, "\phi_s"'] & Y_{s,1} \arrow[r, "\gamma"] \arrow[d, dashed, "\phi_{s,1}"'] & \Sigma Y_{s+1} \arrow[r, "-\Sigma\alpha"] \arrow[d, dashed, "\Sigma\phi_{s+1}"'] & \Sigma Y_s \arrow[d, "\Sigma\phi_s"'] \\
& Z_s \arrow[r, "\beta"] & Z_{s,1} \arrow[r, "\gamma"] & \Sigma Z_{s+1} \arrow[r, "-\Sigma\alpha"] & \Sigma Z_s
\end{tikzcd}
$$

We claim that $\beta\phi_s\alpha\colon Y_{s+1} \to Z_{s,1}$ is zero in the stable category. This follows from Proposition 11.1.5, with $X = Y_{s+1}$ and $W = Z_{s,1}$, since the isomorphism

$$d\colon [Y_{s+1}, Z_{s,1}] \xrightarrow{\cong} \mathrm{Hom}_{A_*}(H_*(Y_{s+1}), H_*(Z_{s,1}))$$

maps $\beta\phi_s\alpha$ to the homomorphism

$$\beta_*\phi_{s*}\alpha_*\colon H_*(Y_{s+1}) \longrightarrow H_*(Z_{s,1})$$

and this is zero because $\alpha_* = 0$. By exactness of the sequence

$$[\Sigma Y_{s+1}, Z_{s,1}] \xrightarrow{\gamma^*} [Y_{s,1}, Z_{s,1}] \xrightarrow{\beta^*} [Y_s, Z_{s,1}] \xrightarrow{\alpha^*} [Y_{s+1}, Z_{s,1}]$$

there exists an extension $\phi_{s,1}\colon Y_{s,1} \to Z_{s,1}$ of $\beta\phi_s$ over β, and by the fill-in axiom for triangulated categories (Lemma 9.3.13) there exists a morphism $\Sigma Y_{s+1} \to \Sigma Z_{s+1}$ making all three squares commute, in $\mathrm{Ho}(Sp^{\mathbb{O}})$. We define ϕ_{s+1} so that $\Sigma\phi_{s+1}$ is the latter morphism. This then completes the inductive step.

Regarding quasi-uniqueness, we have $\alpha\phi_1 = \phi_0\alpha = \psi_0\alpha = \alpha\psi_1$, by assumption. Suppose that $\alpha\phi_s = \alpha\psi_s$ for some $s \ge 1$. Then $\psi_s - \phi_s = \gamma\chi_s$ for some $\chi_s\colon Y_s \to \Sigma^{-1}Z_{s-1,1}$. Hence $\psi_s\alpha - \phi_s\alpha = \gamma\chi_s\alpha = 0$, since $\alpha_* = 0$ and $\Sigma^{-1}Z_{s-1,1}$ is H-injective. It follows that $\alpha\phi_{s+1} = \phi_s\alpha = \psi_s\alpha = \alpha\psi_{s+1}$, and this completes the inductive step.

$$\begin{tikzcd}
Y_{s+1} \arrow[r, "\alpha"] \arrow[d, shift right, "\phi_{s+1}"'] \arrow[d, shift left, "\psi_{s+1}"] & Y_s \arrow[rr, "\alpha"] \arrow[d, shift right, "\phi_s"'] \arrow[d, shift left, "\psi_s"] \arrow[ddr, bend left, "\chi_s"] & & Y_{s-1} \arrow[d, shift right, "\phi_{s-1}"'] \arrow[d, shift left, "\psi_{s-1}"] \\
Z_{s+1} \arrow[r, "\alpha"] & Z_s \arrow[rr, "\alpha"] & & Z_{s-1} \\
& & \Sigma^{-1}Z_{s-1,1} \arrow[ul, "\gamma"] &
\end{tikzcd}$$

□

THEOREM 11.4.2. *Let X and Y be spectra. When viewed as an E_2-spectral sequence, the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow_s [X, Y]_{t-s}$$

does not depend on the choice of Adams resolution for Y. It is contravariantly functorial in X and covariantly functorial in Y.

PROOF. By Proposition 11.4.1, for any morphism $\phi_0\colon Y_0 \to Z_0$ and any two Adams resolutions $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ there is a map $\phi_\star\colon Y_\star \to Z_\star$ of resolutions that extends ϕ_0, and this induces a map

$$\begin{tikzcd}
\dots & H_*(\Sigma Y_{1,1}) \arrow[l, "\delta^1"'] \arrow[d, "\phi_{1,1*}"'] & H_*(Y_{0,1}) \arrow[l, "\delta^0"'] \arrow[d, "\phi_{0,1*}"'] & H_*(Y_0) \arrow[l, "\eta"'] \arrow[d, "\phi_{0*}"'] & 0 \arrow[l] \\
\dots & H_*(\Sigma Z_{1,1}) \arrow[l, "\delta^1"'] & H_*(Z_{0,1}) \arrow[l, "\delta^0"'] & H_*(Z_0) \arrow[l, "\eta"'] & 0 \arrow[l]
\end{tikzcd}$$

of injective A_*-comodule resolutions. When ϕ_0 is the composite of two stable equivalences $Y_0 \sim Y \sim Z_0$ then this chain map is a chain homotopy equivalence, well-defined up to chain homotopy, which induces a canonical isomorphism of Adams E_2-terms.

Contravariant functoriality in X is clear.

For a general morphism $\phi\colon Y \to Z$ we can choose Adams resolutions $(Y_\star, Y_{\star,1})$ for Y and $(Z_\star, Z_{\star,1})$ for Z, and extend $\phi_0\colon Y_0 \sim Y \to Z \sim Z_0$ to $\phi_\star$, as above. The map of resolutions $\phi_\star$ induces a morphism of Adams spectral sequences, which is independent of the choices of resolutions from the E_2-terms and onward. In particular, the homomorphism

$$\phi_*\colon \operatorname{Ext}^{s,t}_{A_*}(H_*(X), H_*(Y)) \longrightarrow \operatorname{Ext}^{s,t}_{A_*}(H_*(X), H_*(Z))$$

is given by the covariant functoriality of A_*-comodule Ext in the second variable. □

THEOREM 11.4.3. *Let X and Y be spectra, with Y bounded below and of finite type mod p. When viewed as an E_2-spectral sequence, the Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_A(H^*(Y), H^*(X)) \Longrightarrow_s [X, Y]_{t-s}$$

does not depend on the choice of Adams resolution for Y. It is contravariantly functorial in X and covariantly functorial in Y.

PROOF. By Proposition 11.4.1, for any morphism $\phi_0\colon Y_0 \to Z_0$ and any two Adams resolutions $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ there is a map $\phi_\star\colon Y_\star \to Z_\star$ of resolutions that extends ϕ_0, and this induces a map

$$\begin{tikzcd}
\dots \arrow[r, "\partial_2"] & H^*(\Sigma Y_{1,1}) \arrow[r, "\partial_1"] & H^*(Y_{0,1}) \arrow[r, "\epsilon"] & H^*(Y_0) \arrow[r] & 0 \\
\dots \arrow[r, "\partial_2"] & H^*(\Sigma Z_{1,1}) \arrow[r, "\partial_1"] \arrow[u, "\phi_{1,1}^*"] & H^*(Z_{0,1}) \arrow[r, "\epsilon"] \arrow[u, "\phi_{1,1}^*"] & H^*(Z_0) \arrow[r] \arrow[u, "\phi^*"] & 0
\end{tikzcd}$$

of projective A-module resolutions. When ϕ_0 is the composite of two stable equivalences $Y_0 \sim Y \sim Z_0$ then this chain map is a chain homotopy equivalence, well-defined up to chain homotopy, which induces a well-defined isomorphism of Adams E_2-terms.

Contravariant functoriality in X is clear.

For a general morphism $\phi\colon Y \to Z$ we can choose Adams resolutions $(Y_\star, Y_{\star,1})$ for Y and $(Z_\star, Z_{\star,1})$ for Z, and extend $\phi_0\colon Y_0 \sim Y \to Z \sim Z_0$ to $\phi_\star$, as above. The map of resolutions $\phi_\star$ induces a morphism of Adams spectral sequences, which is independent of the choices of resolutions from the E_2-terms and onward. In particular, the homomorphism

$$\phi_*\colon \operatorname{Ext}^{s,t}_A(H^*(Z), H^*(X)) \longrightarrow \operatorname{Ext}^{s,t}_A(H^*(Y), H^*(X))$$

is given by the contravariant functoriality of A-module Ext in the first variable. □

Chain maps of algebraic resolutions can quite generally be realized by maps of Adams resolutions.

LEMMA 11.4.4. *Let $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ be resolutions such that*

(1) *$\alpha_*\colon H_*(Y_{s+1}) \to H_*(Y_s)$ is zero and*
(2) *$Z_{s,1}$ is H-injective*

for each $s \ge 0$. Let $\phi_0\colon Y_0 \to Z_0$ be any morphism in $\mathrm{Ho}(Sp^{\mathbb{O}})$, *let $f_0 = (\phi_0)_*$ be the induced homomorphism in homology, and let $f_{*,1} = (f_{s,1})_s$ be a chain map of*

A_*-comodule complexes

$$\begin{tikzcd}
\dots & H_*(\Sigma^2 Y_{2,1}) \arrow[l] \arrow[d, "f_{2,1}"'] & H_*(\Sigma Y_{1,1}) \arrow[l, "\delta^1"'] \arrow[d, "f_{1,1}"'] & H_*(Y_{0,1}) \arrow[l, "\delta^0"'] \arrow[d, "f_{0,1}"'] & H_*(Y_0) \arrow[l, "\eta"'] \arrow[d, "f_0"'] & 0 \arrow[l] \\
\dots & H_*(\Sigma^2 Z_{2,1}) \arrow[l] & H_*(\Sigma Z_{1,1}) \arrow[l, "\delta^1"'] & H_*(Z_{0,1}) \arrow[l, "\delta^0"'] & H_*(Z_0) \arrow[l, "\eta"'] & 0 \arrow[l]
\end{tikzcd}$$

that extends f_0. Then there exists a map $\phi_\star$ of resolutions, with components $\phi_s \colon Y_s \to Z_s$ and $\phi_{s,1} \colon Y_{s,1} \to Z_{s,1}$, such that $(\phi_{s,1})_* = f_{s,1}$ for all $s \geq 0$.

PROOF. The hypotheses (1) and (2) are satisfied for Adams resolutions. They ensure that the top A_*-comodule complex is exact, while the lower complex consists of injective A_*-comodules. Since $Z_{s,1}$ is H-injective there is a unique morphism $\phi_{s,1} \colon Y_{s,1} \to Z_{s,1}$ inducing $f_{s,1}$ in homology, for each $s \geq 0$. In the diagram with distinguished rows

$$\begin{tikzcd}
Y_1 \arrow[r, "\alpha"] \arrow[d, dotted, "\phi_1"'] & Y_0 \arrow[r, "\beta"] \arrow[d, "\phi_0"'] & Y_{0,1} \arrow[r, "\gamma"] \arrow[d, "\phi_{0,1}"'] & \Sigma Y_1 \arrow[d, dotted, "\Sigma\phi_1"'] \\
Z_1 \arrow[r, "\alpha"] & Z_0 \arrow[r, "\beta"] & Z_{0,1} \arrow[r, "\gamma"] & \Sigma Z_1
\end{tikzcd}$$

the central square commutes, because $\phi_{0,1} \circ \beta$ and $\beta \circ \phi_0$ both induce $f_{0,1} \circ \eta = \eta \circ f_0$ in homology, and $Z_{0,1}$ is H-injective. Hence we can fill in the diagram with a morphism $\phi_1 \colon Y_1 \to Z_1$ making the left and right hand squares commute.

Inductively, suppose that we have defined $\phi_0, \dots, \phi_s$ for some $s \geq 1$. In the diagram with distinguished rows

$$\begin{tikzcd}
\Sigma^{-1} Y_{s-1,1} \arrow[dr, "\Sigma^{-1}\gamma"] \arrow[dd, "\Sigma^{-1}\phi_{s-1,1}"'] & & & & \\
& Y_s \arrow[r, "\beta"] \arrow[dd, "\phi_s"'] & Y_{s,1} \arrow[r, "\gamma"] \arrow[dd, "\phi_{s,1}"'] & \Sigma Y_{s+1} \arrow[r, "-\Sigma\alpha"] \arrow[dd, dotted, "\Sigma\phi_{s+1}"'] & \Sigma Y_s \arrow[dd, "\Sigma\phi_s"'] \\
\Sigma^{-1} Z_{s-1,1} \arrow[dr, "\Sigma^{-1}\gamma"'] & & & & \\
& Z_s \arrow[r, "\beta"] & Z_{s,1} \arrow[r, "\gamma"] & \Sigma Z_{s+1} \arrow[r, "-\Sigma\alpha"] & \Sigma Z_s
\end{tikzcd}$$

the composites $\phi_{s,1} \circ \beta \Sigma^{-1}\gamma$ and $\beta \Sigma^{-1}\gamma \circ \Sigma^{-1} f_{s-1,1}$ induce $f_{s,1} \circ \delta^{s-1}$ and $\delta^{s-1} \circ f_{s-1,1}$ in homology, which are equal because the $(f_{s,1})_s$ form a chain map. Furthermore, $\Sigma^{-1}\gamma_* \colon H_*(\Sigma^{-1} Y_{s-1,1}) \to H_*(Y_s)$ is surjective, since $\alpha_* \colon H_*(Y_s) \to H_*(Y_{s-1})$ is zero. Hence the composites $\phi_{s,1} \circ \beta$ and $\beta \circ \phi_s$ induce the same homomorphism in homology, and are therefore equal in $\mathrm{Ho}(Sp^{\mathbb{O}})$ since $Z_{s,1}$ is H-injective. This proves that we can choose a fill-in morphism $\Sigma\phi_{s+1}$ making the two right hand squares commute.

Continuing for all s, we obtain a map $\phi_\star$ of resolutions, as required. □

((ETC: The map of resolutions $\phi_\star$ is at least as unique as in Proposition 11.4.1.))

((ETC: We can choose "good" fill-ins ϕ_s, arising from commuting homotopies. Does this help in realizing maps of Adams resolutions by maps of Adams towers?))

((ETC: Also write this out for cohomology and chain maps of A-module complexes.))

We can compare the Adams spectral sequence for X and Y to the one for S and $F(X,Y)$. These have the same abutment, $[X,Y]_* \cong \pi_* F(X,Y)$.

LEMMA 11.4.5. *(a) If X is finite and $\alpha\colon Y' \to Y$ induces zero in mod p homology, then $F(1,\alpha)\colon F(X,Y') \to F(X,Y)$ induces zero in mod p homology.*

(b) If X is any spectrum and W is H-injective, then $F(X,W)$ is H-injective.

PROOF. (a) Finite spectra X are Spanier–Whitehead dualizable, with a natural stable equivalence $Y \wedge DX \sim F(X,Y)$, so $F(1,\alpha)$ induces the same homomorphism in mod p homology as $\alpha \wedge 1\colon Y' \wedge DX \to Y \wedge DX$.

(b) We may assume that $W = H \wedge T$, which is an H-module spectrum. Then $F(X,W) = F(X, H \wedge T)$ is also an H-module spectrum, which implies that it is H-injective. ((ETC: Elaborate?)) □

LEMMA 11.4.6. *Let X be any spectrum, let $(Y_\star, Y_{\star,1})$ be an Adams resolution of Y, and let $(F(X,Y)_\star, F(X,Y)_{\star,1})$ be an Adams resolution of $F(X,Y)$. Then $(F(X,Y_\star), F(X,Y_{\star,1}))$ is a resolution of $F(X,Y)$ such that $F(X,Y_{s,1})$ is H-injective for each s, and there exists a morphism of resolutions*

$$\theta_\star\colon (F(X,Y)_\star, F(X,Y)_{\star,1}) \longrightarrow (F(X,Y_\star), F(X,Y_{\star,1}))$$

with $\theta_0\colon F(X,Y)_0 \sim F(X,Y_0)$ a stable equivalence.

PROOF. This follows from Lemma 11.4.5 and Proposition 11.4.1. ((ETC: Discuss uniqueness?)) □

PROPOSITION 11.4.7. *For spectra X and Y there is a natural morphism θ from the Adams spectral sequence for S and $F(X,Y)$ to the one for X and Y, given at the E_2-terms by a homomorphism*

$$\theta_*\colon \operatorname{Ext}_{A_*}(\mathbb{F}_p, H_*(F(X,Y))) \longrightarrow \operatorname{Ext}_{A_*}(H_*(X), H_*(Y))\,.$$

If X is finite, then $H_*(F(X,Y)) \cong \operatorname{Hom}(H_*(X), H_*(Y))$ and

$$\theta_*\colon \operatorname{Ext}_{A_*}(\mathbb{F}_p, \operatorname{Hom}(H_*(X), H_*(Y))) \longrightarrow \operatorname{Ext}_{A_*}(H_*(X), H_*(Y))$$

is an isomorphism.

PROOF. The morphism of spectral sequences is induced by the morphism of resolutions from Lemma 11.4.6. The exact couple obtained by applying π_* to $(F(X,Y_\star), F(X,Y_{\star,1}))$ is isomorphic to the exact couple obtained by applying $[X,-]_*$ to $(Y_\star, Y_{\star,1})$, which generates the Adams spectral sequence for X and Y.

If X is finite, then $(F(X,Y_\star), F(X,Y_{\star,1}))$ is itself an Adams resolution, by Lemma 11.4.5, so the morphism of resolutions induces a chain homotopy equivalence of E_1-terms, and an isomorphism of E_2-terms. Furthermore, the canonical map $Y \wedge DX \to F(X,Y)$ is a stable equivalence, so that $\operatorname{Hom}(H_*(X), H_*(Y)) \cong H_*(Y) \otimes DH_*(X) \cong H_*(Y) \otimes H_*(DX) \cong H_*(Y \wedge DX) \cong H_*(F(X,Y))$. □

((ETC: For general X, $(\operatorname{Hom}(H_*(X), I_*^s))_s$ might not define an (injective) A_*-comodule resolution, due to the difficulty with the coconjugate A_*-coaction discussed in Remark 7.7.29. Do we need to assume that $F(X,Y)/p$ is bounded below of finite type in order to write the left hand E_2-term as $\operatorname{Ext}_A(H^*(F(X,Y)), \mathbb{F}_p)$?))

DEFINITION 11.4.8. For any Adams resolution $(Y_\star, Y_{\star,1})$ of Y, let

$$Y_\infty = \operatorname*{holim}_s Y_s$$

be the sequential homotopy limit of the underlying tower

$$\cdots \to Y_{s+1} \xrightarrow{\alpha} Y_s \to \cdots \to Y_0 ,$$

and write $\alpha^\infty \colon Y_\infty \to Y_0 \simeq Y$ for the evident map.

This homotopy limit, or *microscope*, can be defined as the homotopy equalizer of two maps

$$\prod_s Y_s \underset{\alpha}{\overset{1}{\rightrightarrows}} \prod_s Y_s ,$$

where 1 denotes the identity map and α is the product of the maps $\alpha \colon Y_{s+1} \to Y_s$ for $s \geq 0$, together with the trivial map $Y_0 \to *$. There is a natural short exact lim-Rlim sequence

$$0 \to \operatorname*{Rlim}_s \pi_{n+1}(Y_s) \longrightarrow \pi_n(\operatorname*{holim}_s Y_s) \longrightarrow \lim_s \pi_n(Y_s) \to 0$$

for each n. Hence $Y_\infty \sim *$ if and only if $\lim_s \pi_*(Y_s) = 0$ and $\operatorname{Rlim}_s \pi_*(Y_s) = 0$.

The Bousfield *H-nilpotent completion* $Y_H^\wedge$ of Y is defined so that there is a homotopy cofiber sequence

$$Y_\infty \xrightarrow{\alpha^\infty} Y \longrightarrow Y_H^\wedge \longrightarrow \Sigma Y_\infty ,$$

and $Y_\infty \sim *$ if and only if $Y \to Y_H^\wedge$ is a stable equivalence.

PROPOSITION 11.4.9. *The stable homotopy type of $Y_\infty = \operatorname{holim}_s Y_s$ does not depend on the choice of Adams resolution $(Y_\star, Y_{\star,1})$.*

PROOF. Let $(Y_\star, Y_{\star,1})$ and $(Z_\star, Z_{\star,1})$ be Adams resolutions of $Y_0 \sim Y \sim Z_0$. By Proposition 11.4.1 we have maps of resolutions $\phi_\star \colon Y_\star \to Z_\star$ and $\psi_\star \colon Z_\star \to Y_\star$, such that $\psi_s\phi_s\alpha = \alpha \colon Y_{s+1} \to Y_s$ and $\phi_s\psi_s\alpha = \alpha \colon Z_{s+1} \to Z_s$ in the stable category, for all $s \geq 0$. It follows that

$$(\pi_*(\phi_s))_s \colon (\pi_*(Y_s))_s \longrightarrow (\pi_*(Z_s))_s$$

and

$$(\pi_*(\psi_s))_s \colon (\pi_*(Z_s))_s \longrightarrow (\pi_*(Y_s))_s$$

are mutually inverse pro-isomorphisms [**AM69**], [**BK72**, §III.2] of towers, and hence induce isomorphisms

$$\phi_* \colon \lim_s \pi_*(Y_s) \xrightarrow{\cong} \lim_s \pi_*(Z_s)$$

$$\phi_* \colon \operatorname*{Rlim}_s \pi_*(Y_s) \xrightarrow{\cong} \operatorname*{Rlim}_s \pi_*(Z_s) .$$

((ETC: The claim for $\lim_s$ is easy. Can we also prove the claim for Rlim_s without reference to the pro-category?)) The map

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Rlim}_s \pi_{n+1}(Y_s) \arrow[r] \arrow[d, "\phi_*"'] & \pi_n(Y_\infty) \arrow[r] \arrow[d, "\phi_*"'] & \lim_s \pi_n(Y_s) \arrow[r] \arrow[d, "\phi_*"'] & 0 \\
0 \arrow[r] & \operatorname{Rlim}_s \pi_{n+1}(Z_s) \arrow[r] & \pi_n(Z_\infty) \arrow[r] & \lim_s \pi_n(Z_s) \arrow[r] & 0
\end{tikzcd}$$

of lim-Rlim short exact sequences then implies that

$$\phi_* \colon \pi_*(Y_\infty) \xrightarrow{\cong} \pi_*(Z_\infty)$$

is an isomorphism, so that Y_∞ and Z_∞ are stably equivalent. □

((ETC: Can we prove directly from the definition of the microscope that $\psi_\star\phi_\star$ and 1 induce homotopic maps $\operatorname{holim}_s Y_s \to \operatorname{holim}_s Y_s$, and vice versa?))

((ETC: We now summarize the results of the convergence theory for spectral sequences that might have been developed in more detail in Chapter 8.))

DEFINITION 11.4.10 ([**Boa99**, Def. 5.1]). For any exact couple (A, E), let

$$\begin{aligned} A^{-\infty} &= \operatorname*{colim}_s A^s \\ A^{\infty} &= \lim_s A^s \\ RA^{\infty} &= \operatorname*{Rlim}_s A^s . \end{aligned}$$

We say that (A, E) *converges conditionally* to the colimit $A^{-\infty}$ if $A^\infty = 0$ and $RA^\infty = 0$ are both trivial.

If $E^s = 0$ for all $s < 0$, as is the case for each homotopy exact couple associated to an (Adams) resolution, then $A^0 \cong A^{-1} \cong \ldots \cong A^{-\infty}$.

LEMMA 11.4.11. *Let $(Y_\star, Y_{\star,1})$ be an Adams resolution of Y. The homotopy exact couple of X and Y, with $A^{s,*} = [X, Y_s]_*$ and $E^{s,*} = [X, Y_{s,1}]_*$, converges conditionally to $[X, Y]_*$ if and only if $[X, Y_\infty]_* = 0$. This holds for every X if (and only if) $Y_\infty \sim *$.*

PROOF. This follows from the short exact sequence

$$0 \to \operatorname*{Rlim}_s [X, Y_s]_{n+1} \longrightarrow [X, \operatorname*{holim}_s Y_s]_n \longrightarrow \lim_s [X, Y_s]_n \to 0 .$$

□

DEFINITION 11.4.12. For any spectral sequence (E_r, d_r), let

$$RE_\infty = \operatorname*{Rlim}_r Z_r$$

denote the *right derived E_∞-term*, where

$$\cdots \subset Z_{r+1} \subset Z_r \subset \cdots \subset Z_1 = E_1 .$$

is the descending chain of r-th order cycles.

REMARK 11.4.13. If $E_r^s = 0$ for $s < 0$, then $E_{r+1}^s \subset E_r^s$ for all $r > s$, and

$$\operatorname*{Rlim}_r Z_r^s \xrightarrow{\cong} R\lim_r E_r^s ,$$

which partially justifies the notation RE_∞ (rather than RZ_∞). Consider a bidegree (s, t). If (E_r, d_r) stabilizes in that bidegree (so that $E_r^{s,t} = E_\infty^{s,t}$ for all sufficiently large r), then $RE_\infty^{s,t} = 0$. This is always the case of $E_r^{s,t}$ is finite for some r. Hence if (E_r, d_r) stabilizes in each bidegree, then $RE_\infty = 0$. ((ETC: More generally, it suffices that $(E_r^{s,t})_r$ satisfies the Mittag–Leffler condition in each bidegree.))

DEFINITION 11.4.14. A filtration

$$\cdots \subset F^{s+1}G \subset F^sG \subset \cdots \subset G$$

of (graded) abelian groups is *Hausdorff* if

$$\lim_s F^sG = 0$$

and it is *complete* if

$$\operatorname*{Rlim}_s F^sG = 0 .$$

LEMMA 11.4.15. *A filtration $(F^sG)_s$ is Hausdorff and complete if and only if the canonical map*

$$G \xrightarrow{\cong} \lim_s \frac{G}{F^sG}$$

is an isomorphism.

DEFINITION 11.4.16. A spectral sequence (E_r, d_r) *converges strongly* to a filtration $(F^sG)_s$ of a (graded) abelian group G if there are isomorphisms

$$\zeta \colon \frac{F^sG}{F^{s+1}G} \xrightarrow{\cong} E^s_\infty$$

for each s, and the filtration is exhaustive, Hausdorff and complete.

If the spectral sequence arises from an exact couple, we always assume that the isomorphism ζ is the preferred homomorphism from Proposition 2.5.11. Strong convergence, together with solutions to all of the finite extension problems

$$0 \to E^s_\infty \longrightarrow \frac{F^aG}{F^{s+1}G} \longrightarrow \frac{F^aG}{F^sG} \to 0$$

is precisely sufficient to reconstruct the (graded) abelian group G by passage to algebraic colimits and limits.

LEMMA 11.4.17. *If $(F^sG)_s$ is complete Hausdorff and exhaustive, then there are isomorphisms*

$$\operatorname*{colim}_a \lim_s \frac{F^aG}{F^sG} \cong G \cong \lim_s \operatorname*{colim}_a \frac{F^aG}{F^sG}.$$

THEOREM 11.4.18 ([**Boa99**, Thm. 7.3]). *Let (A, E) be an exact couple with $E^s = 0$ for $s < 0$, so that $A^0 \cong A^{-\infty}$. Any two of the following conditions implies the third.*

(1) *The exact couple converges conditionally to the colimit A^0.*
(2) *$RE_\infty = 0$.*
(3) *The spectral sequence converges strongly to A^0, with the filtration $F^sA^0 = \operatorname{im}(\alpha^s \colon A^s \to A^0)$.*

Hence, for a conditionally convergent Adams spectral sequence, the vanishing of RE_∞ is equivalent to strong convergence.

11.5. The Adams filtration

DEFINITION 11.5.1. The abutment of the Adams spectral sequence for X and Y with Adams resolution $(Y_\star, Y_{\star,1})$, is $[X, Y]_*$, with the decreasing, exhaustive filtration given by

$$F^s[X, Y]_* = \operatorname{im}(\alpha^s \colon [X, Y_s]_* \to [X, Y]_*)\,.$$

We call this the *Adams filtration* of $[X, Y]_*$. We say that the elements of $F^s[X, Y]_*$ have Adams filtration $\geq s$, and that the elements of $F^s[X, Y]_* \setminus F^{s+1}[X, Y]_*$ have Adams filtration exactly s.

LEMMA 11.5.2. *The Adams filtration is independent of the choice of Adams resolution.*

Proof. For any other choice of Adams resolution $(Z_\star, Z_{\star,1})$ we have a map of resolutions $\phi_*\colon Y_\star \to Z_\star$ making the diagram

$$\begin{tikzcd} Y_s \arrow[r, "\alpha^s"] \arrow[d, "\phi_s"'] & Y \arrow[d, "="] \\ Z_s \arrow[r, "\alpha^s"] & Y \end{tikzcd}$$

commute, so

$$\operatorname{im}(\alpha^s\colon [X, Y_s]_* \to [X, Y]_*) \subset \operatorname{im}(\alpha^s\colon [X, Z_s]_* \to [X, Y]_*)\,.$$

Reversing the roles of the two resolutions gives the opposite inclusion. Hence the two image filtrations agree. □

The Adams filtration can be characterized in terms of maps that induce zero in mod p (co-)homology.

Proposition 11.5.3. *A morphism $f \in [X, Y]_n$ has Adams filtration $\ge s$ if and only if it can be factored as a composite $f_1 \circ \cdots \circ f_s$ of s morphisms*

$$\Sigma^n X = X_s \xrightarrow{f_s} X_{s-1} \xrightarrow{f_{s-1}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = Y\,,$$

each of which (for $1 \le i \le s$) induces the zero homomorphism $f_{i*}\colon H_*(X_i) \to H_*(X_{i-1})$ in mod p homology.

Proof. If $f = \alpha^s g$ with $g\colon \Sigma^n X \to Y_s$, then f admits the factorization

$$\Sigma^n X = X_s \xrightarrow{\alpha g} Y_{s-1} \xrightarrow{\alpha} \dots \xrightarrow{\alpha} Y_1 \xrightarrow{\alpha} Y_0 = Y$$

where $(\alpha g)_* = 0$ and $\alpha_* = 0$ (in mod p homology) in each case.

Conversely, if $f = f_1 \circ \cdots \circ f_{s+1}$ with $f_{i*} = 0$ for each i, then we may inductively assume that $f_1 \circ \cdots \circ f_s\colon X_s \to Y$ factors as

$$f_1 \circ \cdots \circ f_s = \alpha^s \circ g$$

for some $g\colon X_s \to Y_s$.

$$\begin{tikzcd} X_{s+1} \arrow[r, "f_{s+1}"] \arrow[d, dotted, "g'"'] & X_s \arrow[d, "g"] \arrow[dr, bend left, "f_1\circ\cdots\circ f_s"] & \\ Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[r, "\alpha^s"] \arrow[d, "\beta"] & Y \\ & Y_{s,1} & \end{tikzcd}$$

Then $gf_{s+1}\colon X_{s+1} \to Y_s$ followed by β induces zero in homology, and has target the H-injective spectrum $Y_{s,1}$, hence is null-homotopic. By exactness of the sequence

$$[X_{s+1}, Y_{s+1}] \xrightarrow{\alpha_*} [X_{s+1}, Y_s] \xrightarrow{\beta_*} [X_{s+1}, Y_{s,1}]$$

it follows that $gf_{s+1} = \alpha g'$ for some $g'\colon X_{s+1} \to Y_{s+1}$, which proves that f has Adams filtration $\ge s+1$. □

By the universal coefficient theorem, each condition $f_{i*} = 0$ is equivalent to the condition that $f_i^*\colon H^*(X_{i-1}) \to H^*(X_i)$ is the zero homomorphism.

((ETC: The convergence theory of Chapter 8 gives the following conclusion.))

DEFINITION 11.5.4. Let $(S^1/p^v)_{v\geq 1}$ be the tower of Moore spaces given by the Puppe sequences

$$\begin{tikzcd}
\vdots \arrow[d, "p"'] & \vdots \arrow[d, "="'] & \vdots \arrow[d, "r"'] & \vdots \arrow[d, "p"'] \\
S^1 \arrow[r, "p^3"] \arrow[d, "p"'] & S^1 \arrow[r, "i"] \arrow[d, "="'] & S^1/p^3 \arrow[r, "q"] \arrow[d, "r"'] & S^2 \arrow[d, "p"'] \\
S^1 \arrow[r, "p^2"] \arrow[d, "p"'] & S^1 \arrow[r, "i"] \arrow[d, "="'] & S^1/p^2 \arrow[r, "q"] \arrow[d, "r"'] & S^2 \arrow[d, "p"'] \\
S^1 \arrow[r, "p"] & S^1 \arrow[r, "i"] & S^1/p \arrow[r, "q"] & S^2
\end{tikzcd}$$

and let $(S/p^v)_{v\geq 1}$ be its desuspension, with $S/p^v = F_1 S^1/p^v$.

The *p-completion* of a spectrum Y is the sequential homotopy limit

$$Y_p^\wedge = \operatorname*{holim}_v Y \wedge S/p^v$$

of the tower

$$\dots \longrightarrow Y \wedge S/p^3 \xrightarrow{1\wedge r} Y \wedge S/p^2 \xrightarrow{1\wedge r} Y \wedge S/p\,.$$

Let $\kappa\colon Y \to Y_p^\wedge$ denote the *completion map*, induced by the compatible maps $i\colon S \to S/p^v$. We use the abbreviation

$$Y/p^v = Y \wedge S/p^v$$

for the homotopy cofiber of $p^v\colon Y \to Y$. There is a distinguished triangle

$$Y/p \xrightarrow{e} Y/p^{v+1} \xrightarrow{r} Y/p^v \xrightarrow{\beta_v} \Sigma Y/p$$

for each v, where β_v is the *v-th order Bockstein map.*

((ETC: Each morphism $r\colon S/p^{v+1} \to S/p^v$ is uniquely determined as a fill-in map for p odd, but there is some ambiguity for $p = 2$. This definition of p-completion is not obviously multiplicative. Is the more intrinsic construction given by Bousfield localization needed? Relate Bockstein maps to Bockstein homomorphisms.))

DEFINITION 11.5.5. For an abelian group G, let

$$G_p^\wedge = \lim_v G/p^v$$

denote its *p-completion.* In particular, let $\mathbb{Z}_p = \mathbb{Z}_p^\wedge$ denote the ring of *p-adic integers.* We say that G is *p-complete* if the canonical homomorphism

$$\kappa\colon G \longrightarrow G_p^\wedge$$

is an isomorphism. If G is finite, then κ is the surjection mapping all torsion of order prime to p to zero, which maps the p-Sylow subgroup of G isomorphically to $G_p^\wedge$.

LEMMA 11.5.6. *If Y has finite type, then there are natural isomorphisms*

$$\pi_*(Y_p^\wedge) \xleftrightarrow{\cong} \pi_*(Y)_p^\wedge = \lim_v \pi_*(Y)/p^v \xleftarrow{\cong} \pi_*(Y) \otimes \mathbb{Z}_p\,.$$

If, furthermore, $\pi_*(Y)$ is p-complete in each degree, then $\kappa\colon Y \to Y_p^\wedge$ is a stable equivalence.

((ETC: How can finite type be weakened? Check if the first isomorphism holds if each $\pi_n(Y)$ only has p-power torsion of a bounded order.))

((ETC: Also suffices that Y has finite p-local type, i.e., that each $\pi_n(Y)$ is a finitely generated $\mathbb{Z}_{(p)}$-module.))

((ETC: Discuss multiplicative properties of p-completion. How good is the map $X_p^\wedge \wedge Y_p^\wedge \to (X \wedge Y)_p^\wedge$?))

Proof. Let ${}_{p^v}G = \ker(p^v\colon G \to G)$. The tower of universal coefficient short exact sequences

$$0 \to \pi_n(Y)/p^v \longrightarrow \pi_n(Y/p^v) \longrightarrow {}_{p^v}\pi_{n-1}(Y) \to 0$$

induces an exact sequence

$$0 \to \pi_n(Y)_p^\wedge \longrightarrow \lim_v \pi_n(Y/p^v) \longrightarrow \lim_v {}_{p^v}\pi_{n-1}(Y)\,,$$

where the right hand limit is trivial because $\pi_{n-1}(Y)$ is finitely generated. Hence the left hand arrow is an isomorphism.

In the Milnor short exact sequence

$$0 \to \operatorname{Rlim}_v \pi_{n+1}(Y/p^v) \longrightarrow \pi_n(Y_p^\wedge) \longrightarrow \lim_v \pi_n(Y/p^v) \to 0$$

each group $\pi_{n+1}(Y/p^v)$ is finite, because $\pi_n(Y)$ and $\pi_{n+1}(Y)$ are finitely generated, so the Rlim term vanishes and the right hand arrow is an isomorphism.

For any finitely generated abelian group G the canonical map

$$G \otimes \mathbb{Z}_p \longrightarrow \lim_v G \otimes \mathbb{Z}/p^v \cong \lim_v G/p^v$$

is an isomorphism, since this holds for each cyclic group G. Note that the left hand side commutes with sums, the right hand side commutes with products, and finite sums and finite products agree. $\square$

Proposition 11.5.7. *There are stable equivalences*

$$\kappa\colon Y/p \xrightarrow{\sim} (Y/p)_p^\wedge$$
$$\kappa/p\colon Y/p \xrightarrow{\sim} (Y_p^\wedge)/p$$

and an isomorphism

$$\kappa_*\colon H_*(Y) \xrightarrow{\cong} H_*(Y_p^\wedge)$$

in mod p homology (and cohomology).

Proof. There is a homotopy (co-)fiber sequence

$$F(S[1/p], Y) \longrightarrow Y \xrightarrow{\kappa} Y_p^\wedge$$

where $S[1/p]$ is the homotopy colimit ($=$ telescope) of the sequence

$$S \xrightarrow{p} S \xrightarrow{p} S \xrightarrow{p} S \to \dots .$$

((ETC: Do we need Spanier–Whitehead duality to prove this?)) Since $p\colon S[1/p] \to S[1/p]$ is a stable equivalence, it follows that $F(S[1/p], Y/p) \simeq F(S[1/p], Y)/p \simeq *$, so that $\kappa\colon Y/p \to (Y/p)_p^\wedge$ and $\kappa/p\colon Y/p \to (Y_p^\wedge)/p$ are stable equivalences. Applying integral homology to the second of these, and noting that $H\mathbb{Z} \wedge S/p \simeq H$, we deduce that $\kappa_*\colon H_*(Y) \to H_*(Y_p^\wedge)$ is an isomorphism. $\square$

DEFINITION 11.5.8. Let

$$S \xrightarrow{h} H\mathbb{Z} \xrightarrow{i} \overline{H\mathbb{Z}} \xrightarrow{q} S^1$$

be the Puppe sequence generated by the unit map $h\colon S \to H\mathbb{Z}$ of the integral Eilenberg–MacLane ring spectrum. Note that h is 1-connected (= 2-connective), hence so is $\overline{H\mathbb{Z}}$. For each spectrum Y let

$$\begin{tikzcd}
\dots \arrow[r] & Y'_3 \arrow[r, "\alpha"] & Y'_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y'_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y'_0 \arrow[d, "\beta"] \\
& & Y'_{2,1} \arrow[ul, dashed, "\gamma"] & Y'_{1,1} \arrow[ul, dashed, "\gamma"] & Y'_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

be the *canonical $H\mathbb{Z}$-Adams resolution* of Y, with $Y'_0 = Y$ and

$$Y'_s \xrightarrow{\beta} Y'_{s,1} \xrightarrow{\gamma} Y'_{s+1} \xrightarrow{-\Sigma\alpha} S^1 \wedge Y'_s$$

equal to

$$S \wedge Y'_s \xrightarrow{h\wedge 1} H\mathbb{Z} \wedge Y'_s \xrightarrow{i\wedge 1} \overline{H\mathbb{Z}} \wedge Y'_s \xrightarrow{q\wedge 1} S^1 \wedge Y'_s$$

so that

$$\Sigma^s Y'_s = \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$$
$$\Sigma^s Y'_{s,1} = H\mathbb{Z} \wedge \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$$

for all $s \geq 0$.

Note that $(Y'_\star, Y'_{\star,1})$ is generally not a mod p Adams resolution, since the spectra $Y'_{s,1}$ are not of the form $H \wedge T$.

PROPOSITION 11.5.9. *Let Y be any spectrum. The canonical $H\mathbb{Z}$-Adams resolution $((Y/p)'_\star, (Y/p)'_{\star,1})$ of Y/p is a mod p Adams resolution. If Y/p is ℓ-connective, then $(Y/p)'_s$ is $(s+\ell)$-connective for each $s \geq 0$, so the homotopy exact couple*

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*((Y/p)'_2) \arrow[r, "\alpha"] & \pi_*((Y/p)'_1) \arrow[r, "\alpha"] \arrow[d, "\beta"] & \pi_*(Y/p) \arrow[d, "\beta"] \\
& & \pi_*((Y/p)'_{1,1}) \arrow[ul, dashed, "\gamma"] & \pi_*((Y/p)'_{0,1}) \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

is degreewise discrete, the Adams E_1-term is concentrated in the region $t-s \geq s+\ell$, and

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(Y/p)) \Longrightarrow_s \pi_{t-s}(Y/p)$$

is strongly convergent.

PROOF. Each spectrum

$$\Sigma^s (Y/p)'_{s,1} = H\mathbb{Z} \wedge \overline{H\mathbb{Z}}^{\wedge s} \wedge Y/p$$

has the form $H \wedge T$ with $T = \overline{H\mathbb{Z}}^{\wedge s} \wedge Y$, in view of the stable equivalence $H\mathbb{Z} \wedge S/p \simeq H$. Furthermore, each homomorphism

$$\beta_*\colon H_*((Y/p)'_s) \longrightarrow H_*((Y/p)'_{s,1})$$

is induced by the unit inclusion

$$H \wedge (Y/p)'_s \cong H \wedge S \wedge (Y/p)'_s \xrightarrow{1\wedge h\wedge 1} H \wedge H\mathbb{Z} \wedge (Y/p)'_s\,,$$

which is split by the right module action

$$H \wedge H\mathbb{Z} \wedge (Y/p)'_s \xrightarrow{\rho \wedge 1} H \wedge (Y/p)'_s$$

of $H\mathbb{Z}$ upon H. ((ETC: Other arguments are also possible.))

Suppose that Y/p is ℓ-connective. Since $\overline{H\mathbb{Z}}$ is 2-connective, the smash products

$$\Sigma^s (Y/p)'_s = (\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$$
$$\Sigma^s (Y/p)'_{s,1} = H\mathbb{Z} \wedge (\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$$

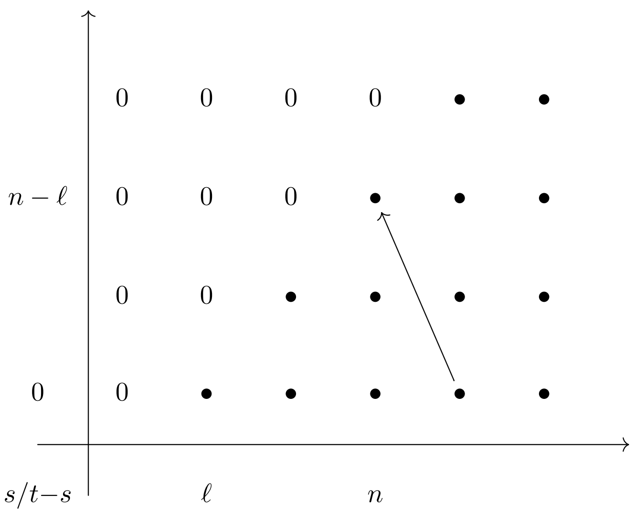
are $(2s+\ell)$-connective, by Proposition 9.8.14. Hence $A^{s,t} = \pi_{t-s}((Y/p)'_s)$ and $E^{s,t} = \pi_{t-s}((Y/p)'_{s,1})$ are trivial for $t-s < s+\ell$, which implies that the terms of the Adams spectral sequence are concentrated on and below the line $t-s = s+\ell$ in the $(t-s,s)$-plane. Moreover, by Proposition 2.5.11 the Adams spectral sequence converges (strongly) to a degreewise discrete filtration of $\pi_*(Y/p)$. In particular, there are canonical isomorphisms

$$E_\infty^{s,t} \cong \frac{F^s \pi_{t-s}(Y/p)}{F^{s+1}\pi_{t-s}(Y/p)}$$

for all $s \ge 0$ and t, where

$$0 = F^{n-\ell+1}\pi_n(Y/p) \subset F^{n-\ell}\pi_n(Y/p) \subset \cdots \subset F^1\pi_n(Y/p) \subset \pi_n(Y/p)$$

for all $n \ge \ell$. □



COROLLARY 11.5.10. *If Y/p is bounded below, then $(Y/p)_\infty \sim *$*

PROOF. We can calculate $(Y/p)_\infty$ using the canonical $H\mathbb{Z}$-Adams resolution of Y/p. If Y/p is ℓ-connective, then $\pi_n((Y/p)'_s) = 0$ for $n < s+\ell$, so $\lim_s \pi_n((Y/p)'_s) = 0$ and $\mathrm{Rlim}_s\, \pi_{n+1}((Y/p)'_s) = 0$, which together imply that $\pi_n((Y/p)_\infty) = 0$ for all n. □

THEOREM 11.5.11. *If Y/p is bounded below, then the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y_p^\wedge)) \Longrightarrow_s [X, Y_p^\wedge]_{t-s}$$

for X and $Y_p^\wedge$ is conditionally convergent (to the achieved colimit).

$$
\begin{tikzcd}
\dots \arrow[r] & Y_2 \arrow[rr, "\alpha"] \arrow[dd, "\kappa"] & & Y_1 \arrow[rr, "\alpha"] \arrow[dl, "\beta"] \arrow[dd, "\kappa"] & & Y_0 \arrow[dl, "\beta"] \arrow[dd, "\kappa"] \\
& & Y_{1,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "\kappa"] & & Y_{0,1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "\kappa"] & \\
\dots \arrow[r] & (Y_2)^\wedge_p \arrow[rr, "\alpha"] \arrow[dd] & & (Y_1)^\wedge_p \arrow[rr, "\alpha"] \arrow[dl, "\beta"] \arrow[dd] & & (Y_0)^\wedge_p \arrow[dl, "\beta"] \arrow[dd] \\
& & (Y_{1,1})^\wedge_p \arrow[ul, dashed, "\gamma"] \arrow[dd] & & (Y_{0,1})^\wedge_p \arrow[ul, dashed, "\gamma"] \arrow[dd] & \\
& \dots \arrow[dd] & & \dots \arrow[dd] & & \dots \arrow[dd] \\
& & \dots \arrow[dd] & & \dots \arrow[dd] & \\
\dots \arrow[r] & Y_2/p^{v+1} \arrow[rr, "\alpha"] \arrow[dd, "r"] & & Y_1/p^{v+1} \arrow[rr, "\alpha"] \arrow[dl, "\beta"] \arrow[dd, "r"] & & Y_0/p^{v+1} \arrow[dl, "\beta"] \arrow[dd, "r"] \\
& & Y_{1,1}/p^{v+1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "r"] & & Y_{0,1}/p^{v+1} \arrow[ul, dashed, "\gamma"] \arrow[dd, "r"] & \\
\dots \arrow[r] & Y_2/p^v \arrow[rr, "\alpha"] & & Y_1/p^v \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & Y_0/p^v \arrow[dl, "\beta"] \\
& & Y_{1,1}/p^v \arrow[ul, dashed, "\gamma"] & & Y_{0,1}/p^v \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}
$$

FIGURE 11.1. Tower of Adams resolutions

PROOF. The smash product of a fixed Adams resolution of S with the tower

$$Y \to \dots \to Y/p^{v+1} \xrightarrow{r} Y/p^v \to \dots$$

gives a tower of Adams resolutions, as in Figure 11.1. The homotopy limit over v of the lower part of the diagram gives a resolution $((Y_\star)^\wedge_p, (Y_{\star,1})^\wedge_p)$, which we claim is also an Adams resolution.

Each H-injective $Y_{s,1}$ has the form $H \wedge T \simeq (H\mathbb{Z} \wedge T)/p$, which implies that $\kappa\colon Y_{s,1} \to (Y_{s,1})^\wedge_p$ is a stable equivalence by Proposition 11.5.7. Hence $(Y_{s,1})^\wedge_p$ is H-injective. By the same proposition, the completion homomorphisms κ_* in the commutative square

$$
\begin{tikzcd}
H_*(Y_{s+1}) \arrow[r, "\alpha_*"] \arrow[d, "\kappa_*"', "\cong"] & H_*(Y_s) \arrow[d, "\kappa_*"', "\cong"] \\
H_*((Y_{s+1})^\wedge_p) \arrow[r, "\alpha_*"] & H_*((Y_s)^\wedge_p)
\end{tikzcd}
$$

are isomorphisms, so the vanishing of the upper α_* implies the vanishing of the lower α_*. This confirms the claim.

We shall prove that

$$\operatorname*{holim}_s (Y_s)^\wedge_p \sim * ,$$

so that the homotopy exact couple for X and $Y^\wedge_p$ (hence also the associated Adams spectral sequence) is conditionally convergent.

First, since $(Y_\star/p, Y_{\star,1}/p)$ is an Adams resolution of Y/p, and Y/p is bounded below, we know that

$$\operatorname*{holim}_s Y_s/p \sim (Y/p)_\infty \sim *$$

by Proposition 11.4.9 and Corollary 11.5.10. Second, we have homotopy cofiber sequences

$$\operatorname*{holim}_s Y_s/p \xrightarrow{e} \operatorname*{holim}_s Y_s/p^{v+1} \xrightarrow{r} \operatorname*{holim}_s Y_s/p^v \xrightarrow{\beta_v} \operatorname*{holim}_s \Sigma Y_s/p$$

for all $v \ge 1$, so

$$\operatorname*{holim}_s Y_s/p^v \sim *$$

in each case, by induction on v. This implies that

$$\operatorname*{holim}_s (Y_s)^\wedge_p = \operatorname*{holim}_s \operatorname*{holim}_v Y_s/p^v \sim \operatorname*{holim}_v \operatorname*{holim}_s Y_s/p^v \sim * ,$$

by the interchange rule for homotopy limits. □

THEOREM 11.5.12. *Let X and Y be spectra, with Y/p bounded below. The Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_{A_*}^{s,t}(H_*(X), H_*(Y^\wedge_p)) \Longrightarrow_s [X, Y^\wedge_p]_{t-s}$$

is strongly convergent if and only if $RE_\infty = 0$. In this case, there are isomorphisms

$$\frac{F^s[X, Y^\wedge_p]_n}{F^{s+1}[X, Y^\wedge_p]_n} \cong E_\infty^{s,s+n}$$

$$[X, Y^\wedge_p]_n \cong \lim_s \frac{[X, Y^\wedge_p]_n}{F^s[X, Y^\wedge_p]_n}$$

for all $s \ge 0$ and n.

PROOF. This is a special case of Boardman's Theorem 11.4.18. □

REMARK 11.5.13. Suppose that Y/p is bounded below. The condition $RE_\infty = 0$ holds if the spectral sequence terms $E_r^{s,t}$ stabilize in each bidegree, which in turn holds if $E_r^{s,t}$ is eventually finite in each bidegree. In particular, this holds if $E_2^{s,t}$ is finite in each bidegree, and this holds if $H_*(X)$ is bounded above and finite in each degree and $H_*(Y)$ is (bounded below and) finite in each degree. For example, it suffices for strong convergence that X is finite and Y/p is bounded below and of finite type.

((ETC: If Y/p is bounded below and of finite type, then each term I_*^s in the canonical injective A_*-comodule resolution of $H_*(Y)$ is of finite type. If, furthermore, $H_*(X)$ is bounded above and of finite type, then $\operatorname{Hom}_{A_*}^t(H_*(X), I_*^s)$ is finite for each bidegree (s, t), hence so is its subquotient $E_2^{s,t}$.))

The special case $X = S$ is worth emphasizing. Recall that $\pi_*(Y^\wedge_p) \cong \pi_*(Y)^\wedge_p \cong \pi_*(Y) \otimes \mathbb{Z}_p$ if Y has finite type.

THEOREM 11.5.14. *Let Y/p be bounded below of finite type. The mod p Adams spectral sequence*

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(Y)) \\ &= \mathrm{Ext}_A^{s,t}(H^*(Y), \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(Y_p^\wedge) \end{aligned}$$

is strongly convergent, meaning that there are isomorphisms

$$\frac{F^s \pi_n(Y_p^\wedge)}{F^{s+1}\pi_n(Y_p^\wedge)} \cong E_\infty^{s,s+n} \qquad \text{and} \qquad \pi_n(Y_p^\wedge) \cong \lim_s \frac{\pi_n(Y_p^\wedge)}{F^s\pi_n(Y_p^\wedge)}$$

for all $s \ge 0$ and n.

11.6. Ext over the Steenrod algebra

Suppose that Y/p is bounded below and of finite type. To calculate the Adams E_2-term

$$E_2 = \mathrm{Ext}_A(H^*(Y), \mathbb{F}_p)$$

we consider a free, hence projective, A-module resolution

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

of $H^*(Y)$. The group $E_2^{s,t}$ is then given by the cohomology in degree s of the cochain complex

$$\ldots \leftarrow \mathrm{Hom}_A^t(P_2, \mathbb{F}_p) \xleftarrow{\delta^1} \mathrm{Hom}_A^t(P_1, \mathbb{F}_p) \xleftarrow{\delta^0} \mathrm{Hom}_A^t(P_0, \mathbb{F}_p) \leftarrow 0$$

with $\delta^s = \mathrm{Hom}(\partial_{s+1}, 1)$ for each $s \ge 0$. The passage to cohomology takes no effort if the resolution is minimal, in the following sense.

DEFINITION 11.6.1. Let $I(A) \subset A$ denote the augmentation ideal. A resolution (P_*, ∂) of an A-module M is *minimal* if $\partial_{s+1}(P_{s+1}) \subset I(A)P_s$ for each $s \ge 0$.

LEMMA 11.6.2. *If (P_*, ∂) is minimal, then $\delta^s = 0$ for each $s \ge 0$, so that*

$$\mathrm{Ext}_A^{s,t}(M, \mathbb{F}_p) = \mathrm{Hom}_A^t(P_s, \mathbb{F}_p)$$

for all $s \ge 0$ and t.

PROOF. Any A-module homomorphism $f \colon P_s \to \Sigma^t \mathbb{F}_p$ maps $I(A)P_s$ to zero, so $\delta^s(f) = \pm f\partial_{s+1} \colon P_{s+1} \to \Sigma^t\mathbb{F}_p$ will be zero when the resolution is minimal. □

LEMMA 11.6.3. *Each bounded below A-module M admits a minimal resolution (P_*, ∂). If M has finite type, then so does each P_s.*

PROOF. Choose an $\mathbb{F}_p$-linear section to the projection $M \to \mathbb{F}_p \otimes_A M$, and let

$$\epsilon \colon P_0 = A \otimes (\mathbb{F}_p \otimes_A M) \longrightarrow M$$

be left adjoint to this section, where P_0 is the free A-module induced up from $\mathbb{F}_p \otimes_A M$. Then $1 \otimes \epsilon \colon \mathbb{F}_p \otimes_A P_0 \to \mathbb{F}_p \otimes_A M$ is an isomorphism, and ϵ is surjective, since $\mathbb{F}_p \otimes_A \mathrm{cok}(\epsilon) = 0$ and $\mathrm{cok}(\epsilon)$ is bounded below.

Inductively, for $s \ge 0$ let $Z_s = \ker(\partial_s)$, which must be interpreted as $\ker(\epsilon)$ when $s = 0$. Choose a section to $Z_s \to \mathbb{F}_p \otimes_A Z_s$, and let

$$\tilde{\partial}_{s+1} \colon P_{s+1} = A \otimes (\mathbb{F}_p \otimes_A Z_s) \longrightarrow Z_s$$

be left adjoint to the section. Then $1 \otimes \tilde{\partial}_{s+1} \colon \mathbb{F}_p \otimes_A P_{s+1} \to \mathbb{F}_p \otimes_A Z_s$ is an isomorphism, and $\tilde{\partial}_{s+1}$ is surjective. Let $\partial_{s+1} \colon P_{s+1} \to P_s$ be its composite with the inclusion $Z_s \subset P_s$.

The condition that $1 \otimes \tilde{\partial}_s$ is an isomorphism, interpreted as $1 \otimes \epsilon$ for $s = 0$, is equivalent to the condition that $\partial_{s+1}(P_{s+1}) \subset I(A)P_s$, as can be seen by chasing the following diagram with exact rows. The middle vertical surjection has kernel $I(A)P_s$.

$$\begin{tikzcd}
 & P_{s+1} \arrow[d, two heads, "\tilde{\partial}_{s+1}"'] \arrow[dr, "\partial_{s+1}"] & & & \\
0 \arrow[r] & Z_s \arrow[r] \arrow[d, two heads] & P_s \arrow[r, "\tilde{\partial}_s"] \arrow[d, two heads] & Z_{s-1} \arrow[r] \arrow[d, two heads] & 0 \\
 & \mathbb{F}_p \otimes_A Z_s \arrow[r] & \mathbb{F}_p \otimes_A P_s \arrow[r, "1\otimes\tilde{\partial}_s"] & \mathbb{F}_p \otimes_A Z_{s-1} \arrow[r] & 0
\end{tikzcd}$$

If M has finite type, then P_0 is finitely generated and free over A, hence it and Z_0 are of finite type. Inductively, if Z_s is of finite type for $s \geq 0$, then so are P_{s+1} and Z_{s+1}. □

((ETC: Uniqueness up to isomorphism of minimal resolutions.))

For any finitely presented A-module M, at the prime $p = 2$, Bruner's program `ext` calculates a minimal resolution (P_*, ∂) of M, in a finite range of bidegrees $s \leq s_{\max}$ and $t \leq t_{\max}$. In essence, it calculates $Z_s = \ker(\partial_s)$ and chooses a minimal generating set for this A-module, which is then a basis for P_{s+1}.

In cohomological (= filtration) degree $s \geq 0$, we write

$$P_s = A\{s_0^*, s_1^*, \dots, s_g^*, \dots\}$$

for the free A-module P_s, so that s_g^* denotes the g-th generator in degree s, counting from $g = 0$. In concrete cases we substitute numbers for s and g in this notation, leading to expressions such as 0_0^*, 1_4^* or 5_{13}^*. The program records the internal degree t of each generator s_g^*. Furthermore, it records the boundary homomorphism $\partial_{s+1} \colon P_{s+1} \to P_s$ by giving its value on each basis element in P_{s+1} as an A-linear combination

$$\sum_g \theta_g s_g^*$$

in P_s, where the $\theta_g \in A$. By minimality,

$$\operatorname{Ext}_A^{s,*}(M, \mathbb{F}_2) = \operatorname{Hom}_A(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{s_0, s_1, \dots, s_g, \dots\},$$

where $s_g \colon P_s \to \mathbb{F}_2$ denotes the dual of s_g^*. In other words, s_g takes the value 1 on s_g^*, and 0 on the other A-module basis elements of P_s. In the concrete cases above, we write 0_0, 1_4 and 5_{13} for these elements in $\operatorname{Ext}_A(M, \mathbb{F}_2)$. The cohomological degree of s_g is thus s, while its internal (homological, or homotopical) degree t is equal to the internal (cohomological) of s_g^*.

Example 11.6.4. We consider the case $Y = S$ and $M = \mathbb{F}_2$. A quick (15 second) machine calculation with $s_{\max} = 12$ and $t_{\max} = 28$ suffices to compute

$$E_2^{*,*}(S) = \operatorname{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{0_0\} \oplus \mathbb{F}_2\{s_g \mid s \geq 1, g \geq 0\}$$

in the range $0 \leq s \leq 12$ and $0 \leq t \leq 28$. This includes the rectangular region $0 \leq s \leq 12$ and $0 \leq t - s \leq 16$ in the $(t - s, s)$-plane shown in Figure 11.2. A

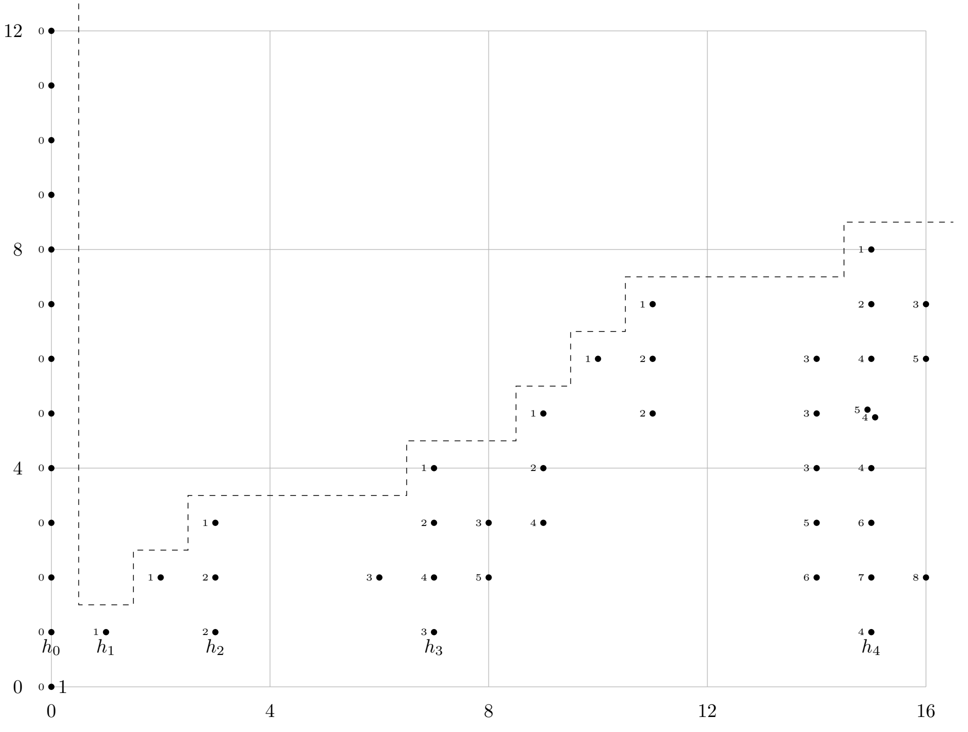


FIGURE 11.2. Vector space basis for $E_2^{s,t}(S) = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$, for $0 \le t - s \le 16$ and $0 \le s \le 12$

filled circle labeled "g" in bidegree $(t-s, s)$ represents the Ext-generator s_g, dual to the A-module generator s_g^* in the minimal resolution, both of which have internal degree t. In this range, most groups $E_2^{s,t}$ have dimension 0 or 1 as $\mathbb{F}_2$-vector spaces, but in bidegree $(t-s, s) = (15, 5)$, corresponding to $(s, t) = (5, 20)$, there are two generators 5_4 and 5_5, which means that

$$E_2^{5,20}(S) = \mathrm{Ext}_A^{5,20}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{5_4, 5_5\}$$

is 2-dimensional. The program `ext` makes a deterministic choice of basis for this $\mathbb{F}_2$-vector space, but other methods of calculation might lead to a different choice of basis, so care is needed when comparing different approaches. Table 11.1 gives the minimal resolution calculated by `ext` in this range.

Table 11.1: Minimal free A-module resolution (P_*, ∂) of $\mathbb{F}_2$ with $P_s = A\{s_0^*, s_1^*, \dots\}$, for $s \le 12$ and $t - s \le 16$

$t-s$	s	x	$\partial(x)$
0	0	0_0^*	1
0	1	1_0^*	$Sq^1(0_0^*)$
1	1	1_1^*	$Sq^2(0_0^*)$
3	1	1_2^*	$Sq^4(0_0^*)$
7	1	1_3^*	$Sq^8(0_0^*)$
15	1	1_4^*	$Sq^{16}(0_0^*)$
0	2	2_0^*	$Sq^1(1_0^*)$
2	2	2_1^*	$Sq^3(1_0^*) + Sq^2(1_1^*)$
3	2	2_2^*	$Sq^4(1_0^*) + Sq(0,1)(1_1^*) + Sq^1(1_2^*)$
6	2	2_3^*	$Sq^7(1_0^*) + Sq^6(1_1^*) + Sq^4(1_2^*)$
7	2	2_4^*	$(Sq^8 + Sq(2,2)(1_0^*) + (Sq^7 + Sq(4,1) + Sq(0,0,1)(1_1^*) + Sq^1(1_3^*)$
8	2	2_5^*	$(Sq^9 + Sq(3,2)(1_0^*) + (Sq^8 + Sq(5,1))(1_1^*) + Sq(0,2)(1_2^*) + Sq^2(1_3^*)$
14	2	2_6^*	$Sq^{15}(1_0^*) + Sq^{14}(1_1^*) + Sq^{12}(1_2^*) + Sq^8(1_3^*)$
15	2	2_7^*	$(Sq^{16} + Sq(10,2) + Sq(7,3) + Sq(4,4) + Sq(2,0,2))(1_0^*)$
			$+ (Sq(12,1) + Sq(3,4) + Sq(0,5) + Sq(8,0,1) + Sq(0,0,0,1))(1_1^*) + Sq^{13}(1_2^*) + Sq^1(1_4^*)$
16	2	2_8^*	$(Sq^{17} + Sq(11,2) + Sq(3,0,2))(1_0^*) + (Sq^{16} + Sq(4,4) + Sq(1,5))(1_1^*)$
			$+ (Sq^{14} + Sq(8,2) + Sq(0,0,2))(1_2^*) + Sq^2(1_4^*)$

Table 11.1: Minimal free A-module resolution (P_*, ∂) of $\mathbb{F}_2$, with $P_s = A\{s_0^*, s_1^*, \dots\}$, for $s \le 12$ and $t - s \le 16$ (cont.)

$t-s$	s	x	$\partial(x)$
0	3	3_0^*	$Sq^1(2_0^*)$
3	3	3_1^*	$Sq^4(2_0^*) + Sq^2(2_1^*) + Sq^1(2_2^*)$
7	3	3_2^*	$(Sq^8 + Sq(2,2))(2_0^*) + (Sq^6 + Sq(0,2))(2_1^*) + Sq^1(2_4^*)$
8	3	3_3^*	$(Sq^9 + Sq(3,2))(2_0^*) + Sq(0,0,1)(2_1^*) + Sq^6(2_2^*) + (Sq^3 + Sq(0,1))(2_3^*)$
9	3	3_4^*	$Sq^{10}(2_0^*) + (Sq^8 + Sq(1,0,1))(2_1^*) + Sq^4(2_3^*) + Sq^3(2_4^*) + Sq^2(2_5^*)$
14	3	3_5^*	$(Sq(9,2) + Sq(6,3))(2_0^*) + Sq(7,2)(2_1^*) + Sq(0,4)(2_2^*) + (Sq^9 + Sq(0,3))(2_3^*)$
			$+ (Sq^8 + Sq(2,2))(2_4^*) + (Sq^7 + Sq(4,1) + Sq(0,0,1))(2_5^*) + Sq^1(2_6^*)$
15	3	3_6^*	$(Sq^{16} + Sq(10,2) + Sq(7,3) + Sq(4,4) + Sq(1,5) + Sq(2,0,2))(2_0^*)$
			$+ (Sq^{14} + Sq(11,1) + Sq(8,2) + Sq(2,4) + Sq(7,0,1) + Sq(4,1,1) + Sq(0,0,2))(2_1^*) + Sq^1(2_7^*)$
0	4	4_0^*	$Sq^1(3_0^*)$
7	4	4_1^*	$Sq^8(3_0^*) + (Sq^5 + Sq(2,1))(3_1^*) + Sq^1(3_2^*)$
9	4	4_2^*	$(Sq^{10} + Sq(4,2))(3_0^*) + (Sq^7 + Sq(1,2) + Sq(0,0,1))(3_1^*) + Sq^2(3_3^*)$
14	4	4_3^*	$(Sq^{15} + Sq(9,2) + Sq(6,3) + Sq(0,5))(3_0^*) + (Sq(9,1) + Sq(6,2))(3_1^*)$
			$+ (Sq^7 + Sq(4,1) + Sq(0,0,1))(3_3^*) + Sq(3,1)(3_4^*)$
15	4	4_4^*	$(Sq^{16} + Sq(7,3) + Sq(4,4) + Sq(1,5))(3_0^*)$
			$+ (Sq^{13} + Sq(10,1) + Sq(7,2) + Sq(4,3) + Sq(1,4) + Sq(6,0,1) + Sq(0,2,1))(3_1^*) + Sq^1(3_6^*)$
0	5	5_0^*	$Sq^1(4_0^*)$
9	5	5_1^*	$Sq^{10}(4_0^*) + (Sq^3 + Sq(0,1))(4_1^*)$
11	5	5_2^*	$Sq^{12}(4_0^*) + (Sq^5 + Sq(2,1))(4_1^*) + Sq^3(4_2^*)$

Table 11.1: Minimal free A-module resolution (P_*, ∂) of $\mathbb{F}_2$, with $P_s = A\{s_0^*, s_1^*, \dots\}$, for $s \leq 12$ and $t - s \leq 16$ (cont.)

$t-s$	s	x	$\partial(x)$
14	5	5_3^*	$(Sq(9,2)+Sq(6,3)+Sq(3,4))(4_0^*)+Sq(5,1)(4_1^*)+(Sq^6+Sq(0,2))(4_2^*)+Sq^1(4_3^*)$
15	5	5_4^*	$(Sq(10,2)+Sq(4,4))(4_0^*)+(Sq^9+Sq(6,1))(4_1^*)+Sq(0,0,1)(4_2^*)+Sq^2(4_3^*)$
15	5	5_5^*	$Sq^{16}(4_0^*)+(Sq^9+Sq(6,1))(4_1^*)+Sq^1(4_4^*)$
0	6	6_0^*	$Sq^1(5_0^*)$
10	6	6_1^*	$Sq^{11}(5_0^*)+Sq^2(5_1^*)$
11	6	6_2^*	$Sq^{12}(5_0^*)+Sq(0,1)(5_1^*)+Sq^1(5_2^*)$
14	6	6_3^*	$Sq^{15}(5_0^*)+Sq^6(5_1^*)+Sq^4(5_2^*)+Sq^1(5_3^*)$
15	6	6_4^*	$(Sq^{16}+Sq(10,2))(5_0^*)+(Sq^7+Sq(4,1)+Sq(0,0,1))(5_1^*)+Sq^1(5_5^*)$
16	6	6_5^*	$Sq(11,2)(5_0^*)+(Sq^8+Sq(5,1))(5_1^*)+Sq(0,2)(5_2^*)+Sq^3(5_3^*)+Sq^2(5_4^*)$
0	7	7_0^*	$Sq^1(6_0^*)$
11	7	7_1^*	$Sq^{12}(6_0^*)+Sq^2(6_1^*)+Sq^1(6_2^*)$
15	7	7_2^*	$(Sq^{16}+Sq(10,2))(6_0^*)+(Sq^6+Sq(0,2))(6_1^*)+Sq^1(6_4^*)$
16	7	7_3^*	$(Sq^{17}+Sq(11,2))(6_0^*)+Sq(0,0,1)(6_1^*)+Sq^6(6_2^*)+(Sq^3+Sq(0,1))(6_3^*)$
0	8	8_0^*	$Sq^1(7_0^*)$
15	8	8_1^*	$Sq^{16}(7_0^*)+(Sq^5+Sq(2,1))(7_1^*)+Sq^1(7_2^*)$
0	9	9_0^*	$Sq^1(8_0^*)$
0	10	10_0^*	$Sq^1(9_0^*)$
0	11	11_0^*	$Sq^1(10_0^*)$
0	12	12_0^*	$Sq^1(11_0^*)$

The minimal resolution starts

$$\cdots \to A\{2^*_g \mid g \ge 0\} \xrightarrow{\partial_2} A\{1^*_i \mid i \ge 0\} \xrightarrow{\partial_1} A\{0^*_0\} \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

with $\epsilon(0^*_0) = 1$ and

$$\partial_1(1^*_i) = Sq^{2^i}\, 0^*_0$$

for each $i \ge 0$. This way $\mathrm{im}(\partial_1) = I(A) = \ker(\epsilon)$, which is minimally generated as an A-module by the Sq^{2^i} for $i \ge 0$. Less obviously,

$$\begin{aligned}
\partial_2(2^*_0) &= Sq^1\, 1^*_0 \\
\partial_2(2^*_1) &= Sq^3\, 1^*_0 + Sq^2\, 1^*_1 \\
\partial_2(2^*_2) &= Sq^4\, 1^*_0 + Q_1\, 1^*_1 + Sq^1\, 1^*_2 \,,
\end{aligned}$$

which correspond to the Adem relations $Sq^1 Sq^1 = 0$, $Sq^3 Sq^1 + Sq^2 Sq^2 = 0$ and $Sq^4 Sq^1 + Q_1 Sq^2 + Sq^1 Sq^4 = 0$, respectively. Compare Figure 7.1. Here $Q_1 = Sq^3 + Sq^2 Sq^1 = Sq(0,1)$ is the Milnor primitive, dual to ξ_2 in the Milnor basis for A_*.

Definition 11.6.5. For an A_*-comodule M_*, with coaction $\nu\colon M_* \to A_* \otimes M_*$, let

$$P_{A_*}(M_*) = \{x \in M_* \mid \nu(x) = 1 \otimes x\}$$

be the subspace of A_*-*comodule primitives.*

For an A-module M, let

$$Q_{A_*}(M) = \mathbb{F}_p \otimes_A M$$

be the quotient space of A-*module indecomposables.*

These should not be confused with the (coalgebra) primitives $P(C)$ of a coaugmented coalgebra and the (algebra) indecomposables $Q(A)$ of an augmented algebra.

Lemma 11.6.6. *For any A_*-comodule M_*, there are natural isomorphisms*

$$\mathrm{Ext}^{0,*}_{A_*}(\mathbb{F}_p, M_*) \cong \mathbb{F}_p \,\square_{A_*}\, M_* \cong P_{A_*}(M_*)$$

and

$$\mathrm{Ext}^{0,*}_{A}(M, \mathbb{F}_p) \cong \mathrm{Hom}_A(M, \mathbb{F}_p) \cong \mathrm{Hom}(Q_A(M), \mathbb{F}_p)\,.$$

In particular,

$$\mathrm{Ext}^{0,*}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}^{0,*}_{A}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{1\}\,.$$

Lemma 11.6.7. *There are natural isomorphisms*

$$\mathrm{Ext}^{1,*}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}^{1,*}_{A}(\mathbb{F}_p, \mathbb{F}_p) \cong P(A_*) \cong \mathrm{Hom}(Q(A), \mathbb{F}_p)$$

where

$$P(A_*) = \mathbb{F}_2\{\xi_1^{2^i} \mid i \ge 0\}$$

for $p = 2$ and

$$P(A_*) = \mathbb{F}_p\{\tau_0, \xi_1^{p^i} \mid i \ge 0\}$$

for p odd.

((ETC: Reference for the following notations? Presumably h_i refers to the Hopf–Steenrod invariants.))

DEFINITION 11.6.8. For $p = 2$ let
$$h_i \in \mathrm{Ext}_A^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$$
denote the class of $\xi_1^{2^i}$, dual to $Sq^{2^i} \in Q(A)$, for each $i \geq 0$.

For p odd, let
$$a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{F}_p, \mathbb{F}_p)$$
denote the class of τ_0, dual to $\beta \in Q(A)$, and let
$$h_i \in \mathrm{Ext}_A^{1,2(p-1)p^i}(\mathbb{F}_p, \mathbb{F}_p)$$
denote the class of $\xi_1^{p^i}$, dual to $P^{p^i} \in Q(A)$, for each $i \geq 0$.

EXAMPLE 11.6.9. In the s_g-notation of `ext`, the generator in $E_2^{0,0}(S)$ is $1 = 0_0$, while the generator in $E_2^{1,2^i}(S)$ is $h_i = 1_i$ for each $i \geq 0$. These classes are labeled in Figure 11.2.

The calculation shows that $E_2^{s,t}(S)$ appears to vanish above a line of slope $1/2$ in the $(t-s, s)$-plane, except for $t-s=0$. This is indeed the case, as was proved by Adams, and confirms that there are no other classes in $E_\infty^{s,t}(S)$ for $0 < t-s \leq 16$ than the ones shown in Figure 11.2.

THEOREM 11.6.10 ([**Ada66**, Thm. 1.1]). *For $p = 2$, the groups $E_2^{s,t}(S)$ are trivial for*
$$0 < t - s < \begin{cases} 2s-1 & \text{for } s \equiv 0 \mod 4, \\ 2s+1 & \text{for } s \equiv 1 \mod 4, \\ 2s+2 & \text{for } s \equiv 2 \mod 4, \\ 2s+3 & \text{for } s \equiv 3 \mod 4. \end{cases}$$

Adams' proof uses the structure of A as a union of finite sub Hopf algebras $A(n)$, and some initial calculations.

DEFINITION 11.6.11. For $p = 2$, let
$$A(n) = \langle Sq^1, Sq^2, \dots, Sq^{2^n} \rangle$$
be the subalgebra of A generated by the Sq^j with $j \leq 2^n$, or, equivalently, by the Sq^{2^i} with $i \leq n$.

For p odd, let
$$A(n) = \langle \beta, P^1, \dots, P^{p^{n-1}} \rangle$$
be the subalgebra of A generated by β and the P^{p^i} with $i < n$.

In each case, $A(n)$ is a finite sub Hopf algebra of A.

EXAMPLE 11.6.12. ((ETC: Assumes the Q_i and $E(n)$ have been defined earlier.)) Let $p = 2$. Then $A(0) = E(0) = E(Sq^1)$, while $E(1) = E(Sq^1, Q_1) \subset A(1) = \langle Sq^1, Sq^2 \rangle$ and $E(2) = E(Sq^1, Q_1, Q_2) \subset A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$.

EXAMPLE 11.6.13. Recall that the r-th Adams differential
$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$
has $(t-s, s)$-bidegree $(-1, r)$. The first possibly nonzero Adams differentials for S are the following.

(1) $d_{s-1}(h_1) \in \{0, s_0\}$ for $s \geq 3$;

(2) $d_2(2_5) \in \{0, 4_1\}$;
(3) $d_2(h_4) \in \{0, 3_5\}$.

Since this spectral sequence converges to $\pi_*(S_2^\wedge) \cong \pi_*(S)_2^\wedge$, and we know that $\pi_1(S) = \mathbb{Z}/2\{\eta\} \neq 0$, it follows that $1_1 = h_1$ must survive to E_∞ and detect $\eta \colon S^1 \to S$. Hence each class $s_0 \in E_2^{s,s}$ also survives to E_∞. We shall see that it detects 2^s, so that the groups $E_\infty^{s,s}(S) \cong \mathbb{F}_2\{s_0\}$ give the associated graded of the 2-adic filtration on $\pi_0(S)_2^\wedge \cong \mathbb{Z}_2$:

$$\cdots \subset 2^{s+1}\mathbb{Z}_2 \subset 2^s\mathbb{Z}_2 \subset \cdots \subset 2\mathbb{Z}_2 \subset \mathbb{Z}_2 .$$

It also follows that $\pi_2(S)_2^\wedge \cong \mathbb{Z}/2$, with a generator detected by 2_1, and that $\pi_3(S)_2^\wedge$ has order $2^3 = 8$. However, the group structure of $\pi_3(S)_2^\wedge$ remains to be determined. Moreover, $\pi_4(S)_2^\wedge = 0$ and $\pi_5(S)_2^\wedge = 0$, since the E_2- and E_∞-terms contain only trivial groups in these total degrees. Furthermore, $\pi_6(S)_2^\wedge \cong \mathbb{Z}/2$, with a generator detected by 2_3.

If $d_2(2_5) = 0$, which turns out to be the case, then $\pi_7(S)_2^\wedge$ has order $2^4 = 16$ and $\pi_8(S)_2^\wedge$ has order $2^2 = 4$. If, on the other hand, $d_2(2_5) = 4_1$ were nonzero, then $\pi_7(S)_2^\wedge$ would have order $2^3 = 8$ and $\pi_8(S)_2^\wedge \cong \mathbb{Z}/2$. To decide between these two cases we must calculate this Adams d_2-differential.

Continuing, $\pi_9(S)_2^\wedge$ has order $2^3 = 8$, $\pi_{10}(S)_2^\wedge = \mathbb{Z}/2$, $\pi_{11}(S)_2^\wedge$ has order $2^3 = 8$, $\pi_{12}(S)_2^\wedge = 0$ and $\pi_{13}(S)_2^\wedge = 0$. We can also see that $\pi_{14}(S)_2^\wedge$ has order dividing $2^5 = 32$, but here there is room for many differentials from topological degree 15.

To proceed, we will use that the (commutative, orthogonal) ring spectrum structure on S makes the associated Adams spectral sequence a (commutative) algebra spectral sequence. This severely limits the possible differential patterns that can be present in the spectral sequence.

11.7. Monoidal structure

For spectra X', X'', Y' and Y'', with smash products $X = X' \wedge X''$ and $Y = Y' \wedge Y''$ there are Adams spectral sequences

$$\begin{aligned} {}'E_2 &= \mathrm{Ext}_{A_*}(H_*(X'), H_*(Y')) \Longrightarrow [X', Y']_* \\ {}''E_2 &= \mathrm{Ext}_{A_*}(H_*(X''), H_*(Y'')) \Longrightarrow [X'', Y'']_* \\ E_2 &= \mathrm{Ext}_{A_*}(H_*(X), H_*(Y)) \Longrightarrow [X, Y]_* . \end{aligned}$$

The smash product of morphisms induces a pairing

$$\wedge \colon [X', Y']_n \otimes [X'', Y'']_m \longrightarrow [X, Y]_{n+m}$$

that takes $f \colon \Sigma^n X' \to Y'$ and $g \colon \Sigma^m X'' \to Y''$ to the composite

$$\Sigma^{n+m} X = S^n \wedge S^m \wedge X' \wedge X'' \xrightarrow{1\wedge\tau\wedge 1} S^n \wedge X' \wedge S^m \wedge X'' \xrightarrow{f \wedge g} Y' \wedge Y'' = Y .$$

It preserves the Adams filtrations, in the sense that $F^s[X', Y']_* \otimes F^u[X'', Y'']_*$ is mapped into $F^{s+u}[X, Y]_*$, since if $f = f_1 \circ \cdots \circ f_s$ and $g = g_1 \circ \cdots \circ g_u$, with $H_*(f_i) = 0$ and $H_*(g_j) = 0$ in each case, then $f \wedge g$ is the composite of $s + u$ maps of the form $f_i \wedge 1$ and $1 \wedge g_j$, each of which induces zero in mod p homology.

((ETC: The following would go in the unwritten chapter on homological algebra.))

Definition 11.7.1. Recall that for Hopf algebras, the tensor product of two (co-)modules is again a (co-)module, using the diagonal (co-)action. Since A_* is a Hopf algebra, there is an internal product

$$\wedge\colon \operatorname{Ext}_{A_*}(M', N') \otimes \operatorname{Ext}_{A_*}(M'', N'') \longrightarrow \operatorname{Ext}_{A_*}(M' \otimes M'', N' \otimes N'')$$

given by choosing injective A_*-comodule resolutions $({}'I^s_*, \delta)_s$ and $({}''I^u_*, \delta)_u$ of N' and N'', respectively, and forming their tensor product $(I^\sigma_*, \delta)_\sigma$ with

$$I^\sigma_* = \bigoplus_{s+u=\sigma} {}'I^s_* \otimes {}''I^u_*$$

and $\delta = \delta \otimes 1 + 1 \otimes \delta$, which is an injective A_*-comodule resolution of $N' \otimes N''$ (by the untwisting isomorphism of Proposition 7.7.31). Given s- and u-cocycles

$$f\colon M' \longrightarrow {}'I^s_* \qquad \text{and} \qquad g\colon M'' \longrightarrow {}''I^u_*$$

the internal product of the cohomology classes $[f]$ and $[g]$ is the class of the composite $(s+u)$-cocycle

$$M' \otimes M'' \xrightarrow{f \otimes g} {}'I^s_* \otimes {}''I^u_* \subset I^{s+u}_* .$$

If we have given A_*-comodule homomorphisms $M \to M' \otimes M''$ and $N' \otimes N'' \to N$ then we can further internalize the product to obtain a pairing

$$\wedge\colon \operatorname{Ext}_{A_*}(M', N') \otimes \operatorname{Ext}_{A_*}(M'', N'') \longrightarrow \operatorname{Ext}_{A_*}(M, N) .$$

If M is an A_*-comodule coalgebra and N is an A_*-comodule algebra, this makes $\operatorname{Ext}_{A_*}(M, N)$ an $\mathbb{F}_p$-algebra.

Definition 11.7.2. Dually, since A is a Hopf algebra there is an internal product

$$\wedge\colon \operatorname{Ext}_{A}(M', N') \otimes \operatorname{Ext}_{A}(M'', N'') \longrightarrow \operatorname{Ext}_{A}(M' \otimes M'', N' \otimes N'')$$

given by choosing projective A-module resolutions $({}'P^*_s, \partial)_s$ and $({}''P^*_u, \partial)_u$ of M' and M'', respectively, and forming their tensor product $(P^*_\sigma, \partial)_\sigma$ with

$$P^*_\sigma = \bigoplus_{s+u=\sigma} {}'P^*_s \otimes {}''P^*_u$$

and $\partial = \partial \otimes 1 + 1 \otimes \partial$, which is a projective A-module resolution of $M' \otimes M''$ (by the untwisting isomorphism of Proposition 7.7.30). Given s- and u-cocycles

$$f\colon {}'P^*_s \longrightarrow N' \qquad \text{and} \qquad g\colon {}''P^*_u \longrightarrow N''$$

the internal product of the cohomology classes $[f]$ and $[g]$ is the class of the composite $(s+u)$-cocycle

$$P^*_\sigma \to {}'P^*_s \otimes {}''P^*_u \xrightarrow{f \otimes g} N' \otimes N'' .$$

If we have given A-module homomorphisms $M \to M' \otimes M''$ and $N' \otimes N'' \to N$ then we can further internalize the product to obtain a pairing

$$\wedge\colon \operatorname{Ext}_{A}(M', N') \otimes \operatorname{Ext}_{A}(M'', N'') \longrightarrow \operatorname{Ext}_{A}(M, N) .$$

If M is an A-module coalgebra and N is an A-module algebra, this makes $\operatorname{Ext}_A(M, N)$ an $\mathbb{F}_p$-algebra. See [**ML63**, §VIII.4].

THEOREM 11.7.3. *(a) For spectra X', X'', Y' and Y'', with $X = X' \wedge X''$ and $Y = Y' \wedge Y''$, there is a natural pairing*

$$\wedge_r \colon ('E_r, ''E_r) \longrightarrow E_r$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\wedge \colon [X', Y']_* \otimes [X'', Y'']_* \longrightarrow [X, Y]_*$$

mapping $f \otimes g$ to $f \wedge g$.

(b) The pairing of E_2-terms

$$\wedge_2 \colon \mathrm{Ext}_{A_*}(H_*(X'), H_*(Y')) \otimes \mathrm{Ext}_{A_*}(H_*(X''), H_*(Y'')) \longrightarrow \mathrm{Ext}_{A_*}(H_*(X), H_*(Y))$$

is the internal product.

(c) If Y'/p and Y''/p are bounded below of finite type, then the E_2-pairing

$$\wedge_2 \colon \mathrm{Ext}_A(H^*(Y'), H^*(X')) \otimes \mathrm{Ext}_A(H^*(Y''), H^*(X'')) \longrightarrow \mathrm{Ext}_A(H^*(Y), H^*(X))$$

is the internal product (followed by the pairing $\mu \colon H^*(X') \otimes H^*(X'') \to H^*(X)$).

The special case $X' = X'' = X = S$ is interesting enough to spell out explicitly. We also concentrate on this case in the proof.

COROLLARY 11.7.4. *There is a natural pairing*

$$\wedge_r \colon (E_r(Y'), E_r(Y'')) \longrightarrow E_r(Y' \wedge Y'')$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\cdot \colon \pi_*(Y') \otimes \pi_*(Y'') \longrightarrow \pi_*(Y' \wedge Y'') \,.$$

The pairing of E_2-terms is the internal product

$$\wedge \colon \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y')) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y'')) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y)) \,.$$

If Y'/p and Y''/p are bounded below of finite type, then this equals the internal product

$$\wedge \colon \mathrm{Ext}_A(H^*(Y'), \mathbb{F}_p) \otimes \mathrm{Ext}_A(H^*(Y''), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(H^*(Y), \mathbb{F}_p) \,.$$

When combined with naturality, with respect to a multiplication $\mu \colon E \wedge E \to E$ or an action $\lambda \colon E \wedge M \to M$, we obtain the following consequences. In particular, $E_r(S)$ is a (graded commutative) algebra spectral sequence, and each Adams spectral sequence $E_r(Y)$ is a (right) $E_r(S)$-module spectral sequence.

COROLLARY 11.7.5. *If E is a ring spectrum (up to homotopy) with multiplication $\mu \colon E \wedge E \to E$, then there is a pairing*

$$\mu_r \colon (E_r(E), E_r(E)) \longrightarrow E_r(E)$$

of Adams spectral sequences making $E_r(E)$ an algebra spectral sequence, with abutment the filtration-preserving graded ring product given by the composition

$$\pi_*(E) \otimes \pi_*(E) \xrightarrow{\cdot} \pi_*(E \wedge E) \xrightarrow{\mu_*} \pi_*(E) \,.$$

The pairing of E_2-terms is the internal product

$$\mu_* \wedge \colon \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \,.$$

If E/p is bounded below of finite type, then this equals the internal product

$$\mu_* \wedge \colon \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \otimes \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(H^*(E), \mathbb{F}_p) \,.$$

COROLLARY 11.7.6. *If M is an E-module ring spectrum (up to homotopy) with action $\lambda\colon E \wedge M \to M$, then there is a pairing*

$$\lambda_r\colon (E_r(E), E_r(M)) \longrightarrow E_r(M)$$

of Adams spectral sequences making $E_r(M)$ an $E_r(E)$-module spectral sequence, with abutment the filtration-preserving module action given by the composition

$$\pi_*(E) \otimes \pi_*(M) \xrightarrow{\cdot} \pi_*(E \wedge M) \xrightarrow{\lambda_*} \pi_*(M)\,.$$

The pairing of E_2-terms is the internal product

$$\lambda_*\wedge\colon \operatorname{Ext}_{A_*}(\mathbb{F}_p, H_*(E)) \otimes \operatorname{Ext}_{A_*}(\mathbb{F}_p, H_*(M)) \longrightarrow \operatorname{Ext}_{A_*}(\mathbb{F}_p, H_*(M))\,.$$

If E/p and M/p are bounded below of finite type, then this equals the internal product

$$\lambda_*\wedge\colon \operatorname{Ext}_A(H^*(E), \mathbb{F}_p) \otimes \operatorname{Ext}_A(H^*(M), \mathbb{F}_p) \longrightarrow \operatorname{Ext}_A(H^*(M), \mathbb{F}_p)\,.$$

REMARK 11.7.7. We will obtain the pairings of Adams spectral sequences from pairings of Cartan–Eilenberg systems. To construct these we assume that Y' and Y'' admit mod p Adams towers $Y'_\star$ and $Y''_\star$, and form their convolution product $Y_\star$. To ensure that $Y_\star$ is a mod p Adams tower for Y, we assume that the towers $Y'_\star$ and $Y''_\star$ are cofibrant in a projective model structure on towers in orthogonal spectra.

DEFINITION 11.7.8. To each spectrum X and tower of spectra $Y_\star$ we associate the *homotopical extended Cartan–Eilenberg system* (π_*, η, ∂) with

$$\pi_*(s, s+r) = [X, Y_{s,r}]_*$$

for all s and $r \ge 0$. Here we interpret Y_s as Y_0 for all $-\infty \le s < 0$ and as $*$ for $s = \infty$. As usual, we let $Y_{s,r} = C(\alpha^r\colon Y_{s+r} \to Y_s)$ denote the mapping cone of α^r. When $r = \infty$, this is interpreted as $Y_{s,\infty} = C(* \to Y_s) \cong Y_s$. The structure morphism

$$\eta\colon \pi_*(s', s'+r') \longrightarrow \pi_*(s, s+r)$$

for extended integers $s' \ge s$ and $s'+r' \ge s+r$ is induced by the natural map $Y_{s',r'} \to Y_{s,r}$ of mapping cones, which appears in the following commutative diagram.

$$\begin{tikzcd}
Y_{s'+r'} \arrow[r, "\alpha^{r'}"] \arrow[d] & Y_{s'} \arrow[r, "i(\alpha^{r'})"] \arrow[d] & Y_{s',r'} \arrow[r, "q(\alpha^{r'})"] \arrow[d] & \Sigma Y_{s'+r'} \arrow[d] \\
Y_{s+r} \arrow[r, "\alpha^{r}"] & Y_{s} \arrow[r, "i(\alpha^{r})"] & Y_{s,r} \arrow[r, "q(\alpha^{r})"] & \Sigma Y_{s+r}
\end{tikzcd}$$

The connecting homomorphism

$$\partial\colon \pi_*(s, s+r) \longrightarrow \pi_{*-1}(s+r, s+r+q)$$

is induced by the composition $j'' = \Sigma i(\alpha^q) \circ q(\alpha^r)$ in the commutative braid diagram below.

$$\begin{tikzcd}
Y_{s+r+q} \arrow[rr, bend left, "\alpha^{r+q}"] \arrow[dr, "\alpha^q"'] & & Y_s \arrow[rr, bend left, "i(\alpha^r)"] \arrow[dr, "i(\alpha^{r+q})"] & & Y_{s,r} \arrow[rr, bend left, "j''"] \arrow[dr, "q(\alpha^r)"] & & \Sigma Y_{s+r,q} \\
& Y_{s+r} \arrow[ur, "\alpha^r"] \arrow[dr, "i(\alpha^q)"'] & & Y_{s,r+q} \arrow[ur, "j'"] \arrow[dr, "q(\alpha^{r+q})"] & & \Sigma Y_{s+r} \arrow[ur, "\Sigma i(\alpha^q)"'] & \\
& & Y_{s+r,q} \arrow[ur, "j"] \arrow[rr, bend right, "q(\alpha^q)"'] & & \Sigma Y_{s+r+q} \arrow[ur, "\Sigma\alpha^q"'] & &
\end{tikzcd}$$

DEFINITION 11.7.9. Let $\mathrm{Tow}(Sp^{\mathbb{O}})$ be the category of functors $Y_\star \colon \mathbb{N} \to Sp^{\mathbb{O}}$, i.e., *towers of orthogonal spectra*. A morphism $\phi_\star \colon Y_\star \to Z_\star$ is a natural transformation, i.e., a strict map of towers.

The evaluation functor $Ev_s \colon \mathrm{Tow}(Sp^{\mathbb{O}}) \to Sp^{\mathbb{O}}$ mapping $Z_\star$ to Z_s has a left adjoint $F_s \colon Sp^{\mathbb{O}} \to \mathrm{Tow}(Sp^{\mathbb{O}})$, mapping Y to the tower

$$\dots \longrightarrow * \longrightarrow Y \xrightarrow{=} Y \xrightarrow{=} \dots \xrightarrow{=} Y$$

given by

$$F_s(Y)_u = \begin{cases} Y & \text{for } u \le s, \\ * & \text{for } u > s, \end{cases}$$

with identity structure maps as indicated. Let FFI be the set of tower morphisms

$$F_s F_\ell i \colon F_s(F_\ell S^{n-1}_+) \longrightarrow F_s(F_\ell D^n_+)$$

for $s \ge 0$, $\ell \ge 0$ and $n \ge 0$, obtained by applying the functors F_s for $s \ge 0$ to the set FI of generating cofibrations for the stable model structure on orthogonal spectra. These all have the form

$$\begin{tikzcd}
\dots \arrow[r] & * \arrow[r] \arrow[d] & F_\ell S^{n-1}_+ \arrow[r, equal] \arrow[d] & F_\ell S^{n-1}_+ \arrow[r, equal] \arrow[d] & \dots \arrow[r, equal] & F_\ell S^{n-1}_+ \arrow[d] \\
\dots \arrow[r] & * \arrow[r] & F_\ell D^n_+ \arrow[r, equal] & F_\ell D^n_+ \arrow[r, equal] & \dots \arrow[r, equal] & F_\ell D^n_+
\end{tikzcd}$$

for suitable s, ℓ and n.

Forming a pushout of the type

$$\begin{tikzcd}
F_s F_\ell S^{n-1}_+ \arrow[r, "i"] \arrow[d, "\phi"'] & F_s F_\ell D^n_+ \arrow[d, "\Phi"] \\
Y_\star \arrow[r] & Z_\star \,,
\end{tikzcd}$$

with $i \in FFI$, thus has the effect of extending a tower $Y_\star$ by freely adjoining an n-cell at level ℓ to Y_u for all $0 \le u \le s$, making no change to Y_u for $u > s$.

DEFINITION 11.7.10. We say that $\phi_\star \colon Y_\star \to Z_\star$ is a *projective stable equivalence* if each component $\phi_s \colon Y_s \to Z_s$ is a stable equivalence of orthogonal spectra, and that $\phi_\star$ is a *projective stable fibration* if each ϕ_s is a stable fibration.

We say that ϕ_* is a *relative cell tower* if $Z_\star$ is the colimit of a sequence

$$Y_\star = Z_\star(0) \longrightarrow \dots \longrightarrow Z_\star(j) \longrightarrow Z_\star(j+1) \longrightarrow \dots \longrightarrow Z_\star$$

where each $Z_\star(j) \longrightarrow Z_\star(j+1)$ is obtained by cobase change along a sum of morphisms in FFI. We say that ϕ_* is a *projective Quillen cofibration* if it is a retract of a relative cell tower.

As usual, $Y_\star$ is a *cell tower* if $* \to Y_\star$ is a relative cell tower, and $Y_\star$ is *projectively cofibrant* of $* \to Y_\star$ is a projective Quillen cofibration, which amounts to the condition that $Y_\star$ is a retract of a cell tower.

THEOREM 11.7.11. *The projective stable equivalences, projective Quillen cofibrations and projective stable fibrations define a model structure on the category* $\mathrm{Tow}(Sp^{\mathbb{O}})$ *of towers of orthogonal spectra.*

We refer to Hirschhorn's book [**Hir03**, Thm. 11.6.1] for the proof. We call this the *projective model structure* on towers.

((ETC: The compact spaces S^{n-1}_+ and D^n_+ admit the small object argument, by a variant of Lemma 3.2.5. Are transfinite composites required for the relative cell towers?))

REMARK 11.7.12. The category of towers of orthogonal spectra also admits other model structures, such as the *Reedy model structure*, which coincides with the *injective model structure*. ((ETC: May be helpful for realizing (weak) maps of Adams resolutions as coming from (strict) maps of Adams towers.)) The projective model structure has the advantage that it is monoidal, which allows us to discuss pairings of towers in a homotopy-invariant manner.

LEMMA 11.7.13. *The projectively cofibrant towers are the retracts of the cell towers $Y_\star$, which are towers*

$$\dots \longrightarrow Y_{s+1} \xrightarrow{\alpha} Y_s \longrightarrow Y_1 \xrightarrow{\alpha} Y_0$$

of orthogonal spectra, where Y_0 is a cell spectrum, each $\alpha \colon Y_{s+1} \to Y_s$ is the inclusion of a cell subspectrum, and each cell in Y_0 is only present in Y_u for $u \le s$, where the bound s depends on the cell, but is finite.

The projectively fibrant towers $Z_\star$ are those for which Z_s is stably fibrant, i.e., an Ω-spectrum, for each $s \ge 0$.

REMARK 11.7.14. Implicit in the model structure is that each tower $Y_\star$ admits a cofibrant replacement $q \colon Y^c_\star \xrightarrow{\sim} Y_\star$, where $Y^c_\star$ is a (retract of a) cell tower. This is a strong form of the classical assumption that Y^c_0 is a CW spectrum and each $Y^c_{s+1} \subset Y^c_s$ is the inclusion of a CW subspectrum.

LEMMA 11.7.15. *If $Y_\star$ is a projectively cofibrant tower of orthogonal spectra, then each collapse map*

$$Y_{s,r} = C(\alpha^r \colon Y_{s+r} \to Y_s) \xrightarrow{\sim} Y_s/Y_{s+r}$$

is a homotopy equivalence. Hence the Cartan–Eilenberg system (π_*, η, ∂) satisfies

$$\pi_*(s, s+r) \cong [X, Y_s/Y_{s+r}]_*$$

and the connecting homomorphism ∂ is that of the ((ETC: strict)) cofiber sequence

$$Y_{s+r}/Y_{s+r+q} \longrightarrow Y_s/Y_{s+r+q} \longrightarrow Y_s/Y_{s+r} \,.$$

DEFINITION 11.7.16. A (strict) *pairing of towers* $\nu\colon (Y'_\star, Y''_\star) \to Y_\star$ is a collection of morphisms

$$\nu_{s,u}\colon Y'_s \wedge Y''_u \longrightarrow Y_{s+u}$$

in $Sp^{\mathbb{O}}$ making the squares

$$\begin{tikzcd}
Y'_{s+1} \wedge Y''_u \arrow[r, "\alpha\wedge 1"] \arrow[d, "\nu_{s+1,u}"'] & Y'_s \wedge Y''_u \arrow[d, "\nu_{s,u}"'] & Y'_s \wedge Y''_{u+1} \arrow[l, "1\wedge\alpha"'] \arrow[d, "\nu_{s,u+1}"] \\
Y_{s+1+u} \arrow[r, "\alpha"] & Y_{s+u} & Y_{s+u+1} \arrow[l, "\alpha"']
\end{tikzcd}$$

commute for all $s, u \ge 0$.

Equivalently, ν is a morphism of *bitowers*, i.e., functors $\mathbb{N}^2 \to Sp^{\mathbb{O}}$, from the external smash product

$$Y'_\star \wedge Y''_\star\colon (s,u) \longmapsto Y'_s \wedge Y''_u$$

to

$$Y \circ +\colon (s,u) \longmapsto Y_{s+u} \,.$$

The functor $Y \mapsto Y \circ +$ from towers to bitowers admits a left adjoint, which defines the *convolution product* of two towers.

DEFINITION 11.7.17. The *convolution product* of two towers $Y'_\star$ and $Y''_\star$ is the left Kan extension $(Y' \wedge Y'')_\star\colon \mathbb{N} \to Sp^{\mathbb{O}}$ defined by

$$(Y' \wedge Y'')_\sigma = \operatorname*{colim}_{s+u\ge\sigma} Y'_s \wedge Y''_u \,,$$

with the canonical map $\alpha\colon (Y' \wedge Y'')_{\sigma+1} \to (Y' \wedge Y'')_\sigma$.

By cofinality [**ML71**, Thm. IX.3.1], the colimit can be calculated over the smaller diagram below.

$$\begin{tikzcd}
 & & & Y'_\sigma \wedge Y''_1 \arrow[r, "1\wedge\alpha"] \arrow[d, "\alpha\wedge 1"] & Y'_\sigma \wedge Y''_0 \\
 & & Y'_{\sigma-1} \wedge Y''_2 \arrow[r, "1\wedge\alpha"] \arrow[d] & Y'_{\sigma-1} \wedge Y''_1 & \\
 & Y'_2 \wedge Y''_{\sigma-1} \arrow[r] \arrow[d, "\alpha\wedge 1"] & \dots & & \\
Y'_1 \wedge Y''_\sigma \arrow[r, "1\wedge\alpha"] \arrow[d, "\alpha\wedge 1"] & Y'_1 \wedge Y''_{\sigma-1} & & & \\
Y'_0 \wedge Y''_\sigma & & & &
\end{tikzcd}$$

(Strict) pairings $\nu\colon (Y'_\star, Y''_\star) \to Y_\star$ thus correspond to maps of towers $\bar\nu_\star\colon (Y' \wedge Y'')_\star \to Y_\star$, and vice versa.

LEMMA 11.7.18. *If $Y'_\star$ and $Y''_\star$ are cell towers, then $(Y' \wedge Y'')_\star$ is also a cell tower with*

$$(Y' \wedge Y'')_\sigma = \bigcup_{s+u=\sigma} Y'_s \wedge Y''_u \,,$$

having one cell of type $F_{s+u}F_{k+\ell}D^{n+m}_+$ for each pair of cells in $Y'_\star$ and $Y''_\star$ of types $F_sF_kD^n_+$ and $F_uF_\ell D^m_+$, respectively.

More generally, if $Y'_\star$ and $Y''_\star$ are projectively cofibrant towers then $(Y' \wedge Y'')_\star$ is also projectively cofibrant.

PROOF. ((ETC: Explain by induction over the cell attachments in $Y'_\star$ and in $Y''_\star$.)) □

LEMMA 11.7.19. *The convolution product makes* $\mathrm{Tow}(Sp^{\mathbb{O}})$ *a symmetric monoidal category, with unit object the tower $U_\star$ with $U_0 = S$ and $U_s = *$ for $s \ge 1$.*

The convolution product with $Y''_\star$ admits a right adjoint, $Y_\star \mapsto F(Y'', Y)_\star$, which defines a closed structure on $\mathrm{Tow}(Sp^{\mathbb{O}})$.

DEFINITION 11.7.20. For $Y''_\star, Y_\star \in \mathrm{Tow}(Sp^{\mathbb{O}})$, let

$$F(Y'', Y)_s = \lim_{s+u \ge \sigma} F(Y''_u, Y_{s+u}) \,,$$

with the canonical map $\alpha \colon F(Y'', Y)_{s+1} \to F(Y'', Y)_s$.

This limit can be calculated over the smaller (but infinite) diagram below.

$$\begin{tikzcd}
\cdots & & \\
F(Y''_2, Y_{s+2}) \arrow[u] \arrow[r, "\alpha_*"] & F(Y''_2, Y_{s+1}) & \\
& F(Y''_1, Y_{s+1}) \arrow[u, "\alpha_*"] \arrow[r, "\alpha_*"] & F(Y''_1, Y_s) \\
& & F(Y''_0, Y_s) \arrow[u, "\alpha^*"]
\end{tikzcd}$$

PROPOSITION 11.7.21. *The category* $\mathrm{Tow}(Sp^{\mathbb{O}})$ *of towers in orthogonal spectra, with respect to the closed symmetric monoidal convolution product and the projective model structure, is monoidal.*

SKETCH PROOF. To verify the first part of the pushout-product axiom, it suffices to consider the cases where $i \colon A_\star \to X_\star$ and $j \colon B_\star \to Y_\star$ are of the form $F_sF_kS^{n-1}_+ \to F_sF_kD^n_+$ and $F_uF_\ell S^{m-1}_+ \to F_uF_\ell D^m_+$, respectively. In this case $i \,\square\, j$ has the form

$$F_{s+u}F_{k+\ell}(S^{n-1} \times D^m \cup D^n \times S^{m-1})_+ \longrightarrow F_{s+u}F_{k+\ell}(D^m \times D^n)_+ \,,$$

hence is a cofibration. The proof of the second part is similar. ((ETC: The second part involves the generating acyclic (= trivial) cofibrations $j \colon F_uF_\ell D^m_+ \to F_uF_\ell(D^m \times I)_+$. Here $i \,\square\, j$ is freely induced by the acyclic (= trivial) cofibration $S^{n-1} \times D^m \times I \cup D^n \times D^m \times \{0\} \to D^n \times D^m \times I$.))

The unit tower $U_\star$ is cofibrant, so the unit axiom is trivially satisfied. □

DEFINITION 11.7.22. Let $Y'_\star$ and $Y''_\star$ be projectively cofibrant towers, and let $Y_\star$ be any tower. ((ETC: Should we assume that $Y_\star$ is (projectively) fibrant? Might build cofibrant replacement for $Y'_\star$ and $Y''_\star$ into the definition.)) A *weak pairing* $\nu\colon (Y'_\star, Y''_\star) \to Y_\star$ is a weak map of resolutions

$$\bar\nu_\star\colon ((Y' \wedge Y'')_\star, (Y' \wedge Y'')_{\star,1}) \longrightarrow (Y_\star, Y_{\star,1}),$$

i.e., collections of morphisms $\bar\nu_\sigma\colon (Y' \wedge Y'')_\sigma \to Y_\sigma$ and $\bar\nu_{\sigma,1}\colon (Y' \wedge Y'')_{\sigma,1} \to Y_{\sigma,1}$ making the diagrams

$$\begin{tikzcd}
(Y' \wedge Y'')_{\sigma+1} \arrow[r, "\alpha"] \arrow[d, "\bar\nu_{\sigma+1}"'] & (Y' \wedge Y'')_\sigma \arrow[r, "\beta"] \arrow[d, "\bar\nu_\sigma"'] & (Y' \wedge Y'')_{\sigma,1} \arrow[r, "\gamma"] \arrow[d, "\bar\nu_{\sigma,1}"'] & \Sigma(Y' \wedge Y'')_{\sigma+1} \arrow[d, "\Sigma\bar\nu_{\sigma+1}"'] \\
Y_{\sigma+1} \arrow[r, "\alpha"] & Y_\sigma \arrow[r, "\beta"] & Y_{\sigma,1} \arrow[r, "\gamma"] & \Sigma Y_{\sigma+1}
\end{tikzcd}$$

commute in $\mathrm{Ho}(Sp^{\mathbb{O}})$.

REMARK 11.7.23. A weak pairing $\nu\colon (Y'_\star, Y''_\star) \to Y_\star$ determines morphisms

$$\nu_{s,u}\colon Y'_s \wedge Y''_u \xrightarrow{\iota_{s,u}} (Y' \wedge Y'')_{s+u} \xrightarrow{\bar\nu_{s+u}} Y_{s+u}$$

in $\mathrm{Ho}(Sp^{\mathbb{O}})$, such that the squares

$$\begin{tikzcd}
Y'_{s+1} \wedge Y''_u \arrow[r, "\alpha\wedge 1"] \arrow[d, "\nu_{s+1,u}"'] & Y'_s \wedge Y''_u \arrow[d, "\nu_{s,u}"'] & Y'_s \wedge Y''_{u+1} \arrow[l, "1\wedge\alpha"'] \arrow[d, "\nu_{s,u+1}"] \\
Y_{s+1+u} \arrow[r, "\alpha"] & Y_{s+u} & Y_{s+u+1} \arrow[l, "\alpha"']
\end{tikzcd}$$

commute in $\mathrm{Ho}(Sp^{\mathbb{O}})$ for all $s, u \ge 0$. However, there is more information in the choice of a weak pairing than what is given by these morphisms in the stable category, since the morphism $\bar\nu_\sigma$ depends on more than its restrictions $\nu_{s,u}$ for $s + u = \sigma$. A weak pairing can be defined in terms of the 2-category of spectra, spectrum maps and homotopy classes of commuting homotopies, but not at the level of the 1-category $\mathrm{Ho}(Sp^{\mathbb{O}})$.

Let $Y'_\star$ and $Y''_\star$ be projectively cofibrant towers of orthogonal spectra, with convolution product $Y_\star = (Y' \wedge Y'')_\star$, which is also projectively cofibrant by Lemma 11.7.18. Hence the collapse maps

$$\begin{aligned}
Y'_{s,r} = Y'_s \cup CY'_{s+r} &\xrightarrow{\sim} Y'_s/Y'_{s+r} \\
Y''_{s,r} = Y''_s \cup CY''_{s+r} &\xrightarrow{\sim} Y''_s/Y''_{s+r} \\
Y_{s,r} = Y_s \cup CY_{s+r} &\xrightarrow{\sim} Y_s/Y_{s+r}
\end{aligned}$$

are stable equivalences. We have (homotopical, extended) Cartan–Eilenberg systems (π'_*, η, ∂), $(\pi''_*, \eta, \partial)$ and (π_*, η, ∂), with

$$\begin{aligned}
\pi'_n(s, s+r) = \pi_n(Y'_{s,r}) &\cong \pi_n(Y'_s/Y'_{s+r}) \\
\pi''_n(s, s+r) = \pi_n(Y''_{s,r}) &\cong \pi_n(Y''_s/Y''_{s+r}) \\
\pi_n(s, s+r) = \pi_n(Y_{s,r}) &\cong \pi_n(Y''_s/Y''_{s+r})
\end{aligned}$$

and

$$\begin{aligned}\pi'_n(s,\infty) &= \pi_n(Y''_s)\\ \pi''_n(s,\infty) &= \pi_n(Y''_s)\\ \pi_n(s,\infty) &= \pi_n(Y_s)\,.\end{aligned}$$

The left hand commutative square

$$\begin{tikzcd}
Y'_{s+r} \wedge Y''_u \vee Y'_s \wedge Y''_{u+r} \arrow[r, "\alpha^r\wedge 1\vee 1\wedge\alpha^r"] \arrow[d, "\iota_{s+r,u}\vee\iota_{s,u+r}"'] & Y'_s \wedge Y''_u \arrow[r, "\pi\wedge\pi"] \arrow[d, "\iota_{s,u}"] & Y'_s/Y'_{s+r} \wedge Y''_u/Y''_{u+r} \arrow[d, "\bar\iota_{s,u,r}"] \\
Y_{s+u+r} \arrow[r, "\alpha^r"] & Y_{s+u} \arrow[r, "\pi"] & Y_{s+u}/Y_{s+u+r}
\end{tikzcd}$$

induces the right hand vertical map $\bar\iota_{s,u,r}$ of horizontal (strict) cofibers.

((ETC: It is less obvious how to consistently pick maps $Y'_{s,r} \wedge Y''_{u,r} \to Y_{s+u,r}$ from the smash product of mapping cones to a mapping cone. Maybe use the minimum of the two cone coordinates?))

Proposition 11.7.24. *Let $Y'_\star$ and $Y''_\star$ be projectively cofibrant towers, with convolution product $Y_\star$. There is a natural pairing $\mu\colon (\pi'_\star, \pi''_\star) \to \pi_\star$ of Cartan–Eilenberg systems, given by the homomorphisms*

$$\mu_r\colon \pi_n(Y'_s/Y'_{s+r}) \otimes \pi_m(Y''_u/Y''_{u+r}) \xrightarrow{\ \cdot\ } \pi_{n+m}(Y'_s/Y'_{s+r} \wedge Y''_u/Y''_{u+r}) \xrightarrow{\bar\iota_{s,u,r*}} \pi_{n+m}(Y_{s+u}/Y_{s+u+r})$$

and

$$\mu_\infty\colon \pi_n(Y'_s) \otimes \pi_m(Y''_u) \xrightarrow{\ \cdot\ } \pi_{n+m}(Y'_s \wedge Y''_u) \xrightarrow{\iota_{s,u*}} \pi_{n+m}(Y_{s+u})$$

Proof. We must verify conditions (SPP I), (SPP II) and (SPP III) from Definitions 6.2.1 and 6.2.5.

Condition (SPP I) follows directly from naturality of the lax monoidal pairing $\cdot\colon \pi_*(M) \otimes \pi_*(N) \to \pi_*(M \wedge N)$, in view of the commutative diagrams

$$\begin{tikzcd}
Y'_{s'}/Y'_{s'+r'} \wedge Y''_{u'}/Y''_{u'+r'} \arrow[r, "\bar\iota_{s',u',r'}"] \arrow[d] & Y_{s'+u'}/Y_{s'+u'+r'} \arrow[d] \\
Y'_s/Y'_{s+r} \wedge Y''_u/Y''_{u+r} \arrow[r, "\bar\iota_{s,u,r}"] & Y_{s+u}/Y_{s+u+r}
\end{tikzcd}$$

for $s \le s'$, $u \le u'$, $s+r \le s'+r'$ and $u+r \le u'+r'$, where the vertical maps are induced by $\alpha^{s'-s}\colon Y'_{s'} \to Y'_s$, $\alpha^{u'-u}\colon Y''_{u'} \to Y''_u$ and $\alpha^{s'+u'-s-u}\colon Y_{s'+u'} \to Y_{s+u}$ by passage to quotients.

Condition (SPP III) likewise follows from the commutative diagrams

$$\begin{tikzcd}
Y'_{s'} \wedge Y''_{u'} \arrow[r, "\iota_{s',u'}"] \arrow[d] & Y_{s'+u'} \arrow[d] \\
Y'_s/Y'_{s+r} \wedge Y''_u/Y''_{u+r} \arrow[r, "\bar\iota_{s,u,r}"] & Y_{s+u}/Y_{s+u+r}
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y'_{s'} \wedge Y''_{u'} \arrow[r, "\iota_{s',u'}"] \arrow[d] & Y_{s'+u'} \arrow[d] \\
Y'_s \wedge Y''_u \arrow[r, "\iota_{s,u}"] & Y_{s+u} \,.
\end{tikzcd}$$

Condition (SPP II) is more delicate. We confirm it in its strengthened form (SPP II+) from Remark 6.2.8, concerning the following diagram.

$$\begin{tikzcd}
\pi_n(Y'_s/Y'_{s+r}) \otimes \pi_m(Y''_u/Y''_{u+r}) \arrow[rr, "1\otimes\partial"] \arrow[dr, "\mu_r"] \arrow[dd, "\partial\otimes 1"'] & & \pi_n(Y'_s/Y'_{s+r}) \otimes \pi_{m-1}(Y''_{u+r}/Y''_{u+2r}) \arrow[dd, "\mu_r"] \\
& \pi_{n+m}(Y_{s+u}/Y_{s+u+r}) \arrow[dr, "\partial"] & \\
\pi_{n-1}(Y'_{s+r}/Y'_{s+2r}) \otimes \pi_m(Y''_u/Y''_{u+r}) \arrow[rr, "\mu_r"] & & \pi_{n+m-1}(Y_{s+u+r}/Y_{s+u+2r})
\end{tikzcd}$$

To simplify the notation, set

$$\begin{aligned}
A &= Y'_{s+r}/Y'_{s+2r} \\
X &= Y'_s/Y'_{s+2r} \\
B &= Y''_{u+r}/Y''_{u+2r} \\
Y &= Y''_u/Y''_{u+2r}
\end{aligned}$$

and $W = A \wedge Y \cup X \wedge B \subset X \wedge Y$, so that $X/A \wedge Y/B \cong (X \wedge Y)/W$ and $W/(A \wedge B) \cong A \wedge Y/B \vee X/A \vee B$. In the diagram

$$\begin{tikzcd}
\pi_n(X/A) \otimes \pi_m(Y/B) \arrow[rr, "1\otimes\partial"] \arrow[dr, "\cdot"] \arrow[dd, "\partial\otimes 1"'] & & \pi_n(X/A) \otimes \pi_{m-1}(B) \arrow[dd, "\cdot"] \\
& \pi_{n+m}(X/A \wedge Y/B) \arrow[dr, "\partial"] & \\
\pi_{n-1}(A) \otimes \pi_m(Y/B) \arrow[rr, "\cdot"] & & \pi_{n+m-1}(A \wedge Y/B \vee X/A \vee B)
\end{tikzcd}$$

we then have the identity

$$\partial(x \cdot y) = \partial(x) \cdot y + (-1)^n x \cdot \partial(y) \tag{11.3}$$

in $\pi_{n+m-1}(A \wedge Y/B \vee X/A \vee B)$ for all $x \in \pi_n(X/A)$ and $y \in \pi_m(Y/B)$, by Proposition 9.8.17.

Condition (SPP II+) now follows from the strictly commutative diagram

$$\begin{tikzcd}
A \wedge B \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\bar\iota_{s+r,u,r} \cup \bar\iota_{s,u+r,r}"] & X \wedge Y \arrow[d, "\bar\iota_{s,u,2r}"] \\
* \arrow[r] & Y_{s+u+r}/Y_{s+u+2r} \arrow[r] & Y_{s+u}/Y_{s+u+2r}
\end{tikzcd}$$

which implies that

$$
\begin{tikzcd}
\pi_{n+m}((X \wedge Y)/W) \arrow[r, "\partial"] \arrow[d, "\bar\iota_{s,u,r*}"'] & \pi_{n+m-1}(W/(A \wedge B)) \arrow[d, "\bar\iota_{s+r,u,r*} \oplus \bar\iota_{s,u+r,r*}"] \\
\pi_{n+m}(Y_{s+u}/Y_{s+u+r}) \arrow[r, "\partial"] & \pi_{n+m-1}(Y_{s+u+r}/Y_{s+u+2r})
\end{tikzcd}
\tag{11.4}
$$

also commutes. Applying $\bar\iota_{s+r,u,r*} \oplus \bar\iota_{s,u+r,r*}$ to (11.3), we conclude that

$$\partial\mu_r(x \otimes y) = \mu_r(\partial(x) \otimes y) + (-1)^n \mu_r(x \otimes \partial(y))$$

in $\pi_{n+m-1}(Y_{s+u+r}/Y_{s+u+2r})$, as required. □

REMARK 11.7.25. To confirm the Leibniz rule (SPP II+), it is critical that the diagram (11.4) commutes, which we deduce from the strict commutativity of the preceding diagram. ((ETC: At this point it is not sufficient to work only in the stable category. Explain!))

PROPOSITION 11.7.26. *Let $Y'_\star$ and $Y''_\star$ be projectively cofibrant Adams towers. Then their convolution $Y_\star$ is a projectively cofibrant Adams tower, with filtration quotients*

$$Y_\sigma/Y_{\sigma+1} \cong \bigvee_{s+u=\sigma} Y'_s/Y'_{s+1} \wedge Y''_u/Y''_{u+1} .$$

The injective A_*-comodule resolution

$$\dots \leftarrow H_*(\Sigma^\sigma Y_{\sigma,1}) \leftarrow \dots \leftarrow H_*(\Sigma Y_{1,1}) \overset{\beta_*\gamma_*}{\longleftarrow} H_*(Y_{0,1}) \leftarrow 0 \tag{11.5}$$

of $H_*(Y_0)$ is the tensor product of the injective A_*-comodule resolutions

$$\dots \leftarrow H_*(\Sigma^s Y'_{s,1}) \leftarrow \dots \leftarrow H_*(\Sigma Y'_{1,1}) \overset{\beta_*\gamma_*}{\longleftarrow} H_*(Y'_{0,1}) \leftarrow 0 \tag{11.6}$$

and

$$\dots \leftarrow H_*(\Sigma^u Y''_{u,1}) \leftarrow \dots \leftarrow H_*(\Sigma Y''_{1,1}) \overset{\beta_*\gamma_*}{\longleftarrow} H_*(Y''_{0,1}) \leftarrow 0 \tag{11.7}$$

of $H_*(Y'_0)$ and $H_*(Y''_0)$, respectively.

PROOF. We view each $\alpha\colon Y'_{s+1} \to Y'_s$ and $\alpha\colon Y''_{u+1} \to Y''_u$ as the inclusion of a subspectrum, so that

$$Y_\sigma = \bigcup_{s+u=\sigma} Y'_s \wedge Y''_u$$

for each $\sigma \ge 0$. In particular, $Y_0 = Y'_0 \wedge Y''_0$. The tower $Y_\star$ is projectively cofibrant by Lemma 11.7.18, so each map $\alpha\colon Y_{\sigma+1} \to Y_\sigma$ is a Quillen cofibration. The inclusions $\iota_{s,u}\colon Y'_s \wedge Y''_u \to Y_{s+u}$ then combine to the stated isomorphism

$$\bigvee_{s+u=\sigma} Y'_s/Y'_{s+1} \wedge Y''_u/Y''_{u+1} \xrightarrow{\cong} Y_\sigma/Y_{\sigma+1} .$$

Since $Y'_s/Y'_{s+1} \simeq Y'_{s,1}$ is H-injective, hence of the form $H \wedge T$, it follows that $Y'_s/Y'_{s+1} \wedge Y''_u/Y''_{u+1}$ is equivalent to $H \wedge T \wedge Y''_u/Y''_{u+1}$, and is therefore also H-injective. This implies that $Y_{\sigma,1} \simeq Y_\sigma/Y_{\sigma+1}$ is H-injective, for each $\sigma \ge 0$.

To prove that $\alpha_*\colon H_*(Y_{\sigma+1}) \to H_*(Y_\sigma)$ is zero, we first show that the cochain complex (11.5) is the tensor product

$$(I^\sigma_*, \delta)_\sigma \cong ('I^s_*, \delta)_s \otimes (''I^u_*, \delta)_u$$

of the cochain complexes (11.6) and (11.7). Since the latter resolve $H_*(Y_0')$ and $H_*(Y_0'')$, respectively, it follows from the algebraic Künneth formula that the former resolves $H_*(Y_0') \otimes H_*(Y_0'') \cong H_*(Y_0)$, which then implies that $\alpha_* = 0$ for each $\sigma \geq 0$.

For each σ, an isomorphism

$$\bigoplus_{s+u=\sigma} {}'I^s_* \otimes {}''I^u_* \cong I^\sigma_*$$

is given by the composition

$$\begin{aligned}
\bigoplus_{s+u=\sigma} H_*(\Sigma^s Y'_{s,1}) \otimes H_*(\Sigma^u Y''_{u,1}) &\cong \bigoplus_{s+u=\sigma} \Sigma^s H_*(Y'_s/Y'_{s+1}) \otimes \Sigma^u H_*(Y''_u/Y''_{u+1}) \\
\cong \Sigma^\sigma \bigoplus_{s+u=\sigma} H_*(Y'_s/Y'_{s+1} \wedge Y''_u/Y''_{u+1}) &\cong \Sigma^\sigma H_*(\bigvee_{s+u=\sigma} Y'_s/Y'_{s+1} \wedge Y''_u/Y''_{u+1}) \\
&\cong \Sigma^\sigma H_*(Y_\sigma/Y_{\sigma+1}) \cong \Sigma^\sigma H_*(Y_{\sigma,1}) \,.
\end{aligned}$$

We claim that $\delta^s \otimes 1 + 1 \otimes \delta^u$ on the (s,u)-summand at the left hand side corresponds to δ^σ on the right hand side. To prove this, we use the notation for $r = 1$ from the proof of (SPP II+) in Proposition 11.7.24, so that we have Quillen cofibrations

$$\begin{aligned}
i \colon A = Y'_{s+1}/Y'_{s+2} &\longrightarrow Y'_s/Y'_{s+2} = X \\
j \colon B = Y''_{u+1}/Y'_{u+2} &\longrightarrow Y''_u/Y'_{u+2} = Y \\
i \,\square\, j \colon W = A \wedge Y \cup X \wedge B &\longrightarrow X \wedge Y
\end{aligned}$$

and the identity

$$\partial(x \otimes y) = \partial(x) \otimes y + (-1)^n x \otimes \partial(y)$$

holds in $H_{n+m-1}(\Sigma^\sigma(W/(A \wedge B)))$ for $x \in H_n(\Sigma^s(X/A)) \cong H_n(\Sigma^s Y'_{s,1}) = {}'I^s_n$ and $y \in H_m(\Sigma^u(Y/B)) \cong H_m(\Sigma^u Y''_{u,1}) = {}''I^u_m$. (This homology Leibniz rule can be deduced from Proposition 9.8.17 by applying $H \wedge -$ to each spectrum, and replacing $\wedge$ with $\wedge_H$.) When combined with the commutative diagram

$$\begin{tikzcd}
H_{n+m}(\Sigma^\sigma((X \wedge Y)/W)) \arrow[r, "\partial"] \arrow[d, "\bar\iota_{s,u,1*}"'] & H_{n+m-1}(\Sigma^\sigma(W/(A \wedge B))) \arrow[d, "\bar\iota_{s+1,u,1*} \oplus \bar\iota_{s,u+1,1*}"] \\
H_{n+m}(\Sigma^\sigma(Y_\sigma/Y_{\sigma+1})) \arrow[r, "\delta^\sigma"] & H_{n+m-1}(\Sigma^\sigma(Y_{\sigma+1}/Y_{\sigma+2}))
\end{tikzcd}$$

this proves that

$$\delta^\sigma(x \otimes y) = \delta^s(x) \otimes y + (-1)^n x \otimes \delta^u(y)$$

in $I^{\sigma+1}_{n+m-1}$, as desired.

As already outlined, we can now deduce that

$$H^*(I^\sigma_*, \delta) \cong H^*({}'I^s_*, \delta) \otimes H^*({}'I^u_*, \delta)$$

is $H_*(Y_0') \otimes H_*(Y_0'') \cong H_*(Y_0)$ concentrated in cohomological degree 0.

$$\begin{tikzcd}
H_*(\Sigma^{s+1} Y_{s+1}) \arrow[d, "\beta_*"] & H_*(\Sigma^s Y_s) \arrow[d, "\beta_*"] & \dots & H_*(\Sigma Y_1) \arrow[d, "\beta_*"] & H_*(Y_0) \arrow[d, "\beta_*"] \\
I^{s+1}_* & I^s_* \arrow[l, "\delta^s"] \arrow[ul, "\gamma_*"'] & I^{s-1}_* \arrow[l, "\delta^{s-1}"] \arrow[ul, "\gamma_*"'] & I^1_* & I^0_* \arrow[l, "\delta^0"] \arrow[ul, "\gamma_*"']
\end{tikzcd}$$

Hence

$$\eta = \beta_* \colon H_*(Y_0) \longrightarrow I^0_*$$

is injective, with implies that $\alpha_* \colon H_*(Y_1) \to H_*(Y_0)$ is zero and $\gamma_* \colon I^0_* \to H_*(\Sigma Y_1)$ is surjective, with kernel the image of β_*, which also equals the kernel of

$$\delta^0 = \beta_*\gamma_* \colon I^0_* \longrightarrow I^1_* \,.$$

Suppose inductively, for $s \ge 1$, that $\gamma_* \colon I^{s-1}_* \to H_*(\Sigma^s Y_s)$ is surjective, with $\ker(\gamma_*)$ equal to the kernel of

$$\delta^{s-1} = \beta_*\gamma_* \colon I^{s-1}_* \longrightarrow I^s_* \,.$$

Then $\beta_* \colon H_*(\Sigma^s Y_s) \to I^s_*$ must be injective, since a nonzero elements in its kernel would make $\ker(\delta^{s-1})$ strictly larger than $\ker(\gamma_*)$. From the long exact sequence we deduce that $\alpha_* \colon H_*(\Sigma^s Y_{s+1}) \to H_*(\Sigma^s Y_s)$ is zero, and that $\gamma_* \colon I^s_* \to H_*(\Sigma^{s+1} Y_{s+1})$ is surjective. Furthermore,

$$\ker(\gamma_*) = \mathrm{im}(\beta_*) = \mathrm{im}(\beta_*\gamma_*) = \mathrm{im}(\delta^{s-1}) = \ker(\delta^s) \,,$$

by the assumed exactness at I^s_*. This completes the inductive step, and shows that $\alpha_* = 0$ in all cases. Hence $Y_\star$ is, indeed, an Adams tower. □

PROPOSITION 11.7.27. *Let $Y'_\star$ and $Y''_\star$ be projectively cofibrant Adams towers, with convolution product $Y_\star$. The pairing*

$$\mu_1 \colon \pi_*(Y'_{s,1}) \otimes \pi_*(Y''_{u,1}) \xrightarrow{\cdot} \pi_*(Y'_{s,1} \wedge Y''_{u,1}) \xrightarrow{\bar\iota_{s,u,1*}} \pi_*(Y_{s+u,1})$$

of E_1-terms corresponds under the d-isomorphisms to the pairing

$$\mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(Y'_{s,1})) \otimes \mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(Y''_{u,1})) \longrightarrow \mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(Y_{s+u,1}))$$

induced by applying $\mathrm{Hom}_{A_*}(\mathbb{F}_p, -) = P_{A_*}(-)$ to the pairing ${}'I^s_* \otimes {}''I^u_* \to I^{s+u}_*$. Hence, the pairing of E_2-terms

$$\mu_2 \colon E_2(Y') \otimes E_2(Y'') \longrightarrow E_2(Y)$$

is the internal product.

PROOF. This follows from the commutative diagram

$$\begin{tikzcd}
\pi_*(Y'_{s,1}) \otimes \pi_*(Y''_{u,1}) \arrow[r, "\cdot"] \arrow[d, "h\otimes h"] & \pi_*(Y'_{s,1} \wedge Y''_{u,1}) \arrow[r, "\bar\iota_{s,u,1*}"] \arrow[d, "h"] & \pi_*(Y_{s+u,1}) \arrow[d, "h"] \\
H_*(Y'_{s,1}) \otimes \pi_*(Y''_{u,1}) \arrow[r, "\cong"] & H_*(Y'_{s,1} \wedge Y''_{u,1}) \arrow[r, "\bar\iota_{s,u,1*}"] & H_*(Y_{s+u,1})
\end{tikzcd}$$

where the right hand mod p Hurewicz homomorphism factors as

$$h \colon \pi_*(Y_{\sigma,1}) \underset{\cong}{\xrightarrow{d}} \mathrm{Hom}_{A_*}(\mathbb{F}_p, H_*(Y_{\sigma,1})) \rightarrowtail H_*(Y_{\sigma,1}) \,,$$

and similarly in the other columns. □

PROOF OF THEOREM 11.7.3. Letting $Y'_\star$ and $Y''_\star$ be projectively cofibrant mod p Adams resolutions of Y' and Y'', their convolution product $Y_\star = (Y' \wedge Y'')_\star$ is a projectively cofibrant mod p Adams resolution of $Y = Y' \wedge Y''$. The pairing of Cartan–Eilenberg systems in Propositions 11.7.24 and 11.7.27 then gives the asserted pairings, in the case $X' = X'' = X = S$. For the general case, one replaces $\pi_*(Y')$ with $[X', Y']_*$, and so on, relying on the appropriate generalization of the homotopy Leibniz rule from Proposition 9.8.17. □

((ETC: See Remark 9.8.18 for the generalization. One might also compare the Adams spectral sequence for X and Y to the one for (S and) $F(X,Y)$, but these are (probably) not generally the same.))

We return to the situation where $(Y' \wedge Y'')_\star$ maps to $Y_\star$ by a map of towers, or by a (weak) map of associated resolutions, but $Y_\star$ is not necessarily equal to the convolution product $(Y' \wedge Y'')_\star$.

PROPOSITION 11.7.28. *Let $\nu\colon (Y'_\star, Y''_\star) \to Y_\star$ be a (strict or weak) pairing of towers.*

(a) There is a natural pairing of homotopy spectral sequences

$$\nu_r\colon E_r(Y') \otimes E_r(Y'') \longrightarrow E_r(Y)$$

with abutment the pairing

$$\nu_{0,0*}\colon \pi_*(Y'_0) \otimes \pi_*(Y''_0) \longrightarrow \pi_*(Y_0)\,.$$

(b) If $Y'_\star$, $Y''_\star$ and $Y_\star$ are mod p Adams towers, then the pairing of Adams E_2-terms

$$\nu_2\colon\ \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y'_0)) \otimes \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y''_0)) \longrightarrow \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Y_0))$$

is the internal product.

(c) If, furthermore, Y'_0/p, Y''_0/p and Y_0/p are bounded below of finite type, then the pairing of Adams E_2-terms

$$\nu_2\colon\ \mathrm{Ext}_A(H^*(Y'_0), \mathbb{F}_p) \otimes \mathrm{Ext}_A(H^*(Y''_0), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(H_*(Y_0), \mathbb{F}_p)$$

is the internal product.

PROOF. The pairing of spectral sequences is the composite of the pairing of spectral sequences

$$\iota_r\colon E_r(Y') \otimes E_r(Y'') \longrightarrow E_r(Y' \wedge Y'')$$

associated to the convolution product $(Y' \wedge Y'')_\star$, followed by the morphism of spectral sequences

$$\bar\nu_r\colon E_r(Y' \wedge Y'') \longrightarrow E_r(Y)$$

associated to the (strict) map $\bar\nu_\star\colon (Y' \wedge Y'')_\star \to Y_\star$ of towers of spectra, or the (weak) map $\bar\nu_\star$ of the associated resolutions in the stable category. The result then follows from Propositions 11.7.24 and 11.7.27, and functoriality of the homotopy spectral sequence of Definition 11.2.5 for (strict) maps of towers or (weak) morphisms of resolutions. □

11.8. Composition pairings

For spectra X, Y and Z the composition of morphisms defines a pairing

$$\circ\colon [Y,Z]_n \otimes [X,Y]_m \longrightarrow [X,Z]_{n+m}$$

that takes $g\colon \Sigma^n Y \to Z$ and $f\colon \Sigma^m X \to Y$ to the composite

$$g \circ \Sigma^n f\colon \Sigma^{n+m} X = \Sigma^n \Sigma^m X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{g} Z\,.$$

It preserves Adams filtrations, in the sense that $F^s[Y,Z]_* \otimes F^u[X,Y]_*$ is mapped into $F^{s+u}[X,Z]_*$, since the combined composite of s and u maps, each of which induces zero in mod p homology, is obviously a composite of $s+u$ such maps.

For any algebra A and (left) A-modules L, M and N there is a natural Yoneda composition product

$$\circ\colon \operatorname{Ext}^s_A(M,N) \otimes \operatorname{Ext}^u_A(L,M) \longrightarrow \operatorname{Ext}^{s+u}_A(L,N)\,.$$

DEFINITION 11.8.1. Let

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \to 0$$

and

$$\cdots \to Q_u \xrightarrow{\partial_u} Q_{u-1} \to \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} L \to 0$$

be projective A-module resolutions. Given cocycles

$$g\colon P_s \longrightarrow N \quad \text{and} \quad f\colon Q_u \longrightarrow M$$

choose a chain map $f_*\colon Q_{*+u} \to P_*$ of degree $-u$ lifting f.

$$\begin{tikzcd}
\dots \arrow[r] & Q_{s+u} \arrow[r] \arrow[d, "f_s"'] & \dots \arrow[r] & Q_u \arrow[r] \arrow[d, "f_0"'] \arrow[dr, "f"] & \dots \arrow[r] & Q_0 \arrow[r] & L \\
\dots \arrow[r] & P_s \arrow[r] \arrow[d, "g"'] & \dots \arrow[r] & P_0 \arrow[r] & M & & \\
& N & & & & &
\end{tikzcd}$$

The composite $g \circ f_s$ is a cocycle, and its cohomology class

$$[g] \circ [f] = [g \circ f_s] \in \operatorname{Ext}^{s+u}_A(L,N)$$

defines the composition product.

((ETC: Maybe it would be more consistent to write f_{s+u} in place of $f_s\colon Q_{s+u} \to P_s$. A chain map of odd degree anticommutes with the boundaries, since we suspend (spectra and) chain complexes on the left.))

The comodule case is similar.

DEFINITION 11.8.2. There is a composition product

$$\circ\colon \operatorname{Ext}^s_C(M,N) \otimes \operatorname{Ext}^u_C(L,M) \longrightarrow \operatorname{Ext}^{s+u}_C(L,N)\,.$$

for any coalgebra C and (left) C-comodules L, M and N, defined for cocycles

$$g\colon M \longrightarrow I^s \quad \text{and} \quad f\colon L \longrightarrow J^u$$

by extending g to a chain map $g^*\colon J^* \to I^{s+*}$ of codegree s and using the diagram

$$\begin{tikzcd}
& L \arrow[d, "f"'] & & & & & \\
\dots & J^u \arrow[l] \arrow[d, "g^u"'] & \dots \arrow[l] & J^0 \arrow[l] \arrow[d, "g^0"'] & M \arrow[l] \arrow[dl, "g"] & & \\
\dots & I^{s+u} \arrow[l] & \dots \arrow[l] & I^s \arrow[l] & \dots \arrow[l] & I^0 \arrow[l] & N \arrow[l]
\end{tikzcd}$$

to form

$$[g] \circ [f] = [g^u \circ f] \in \operatorname{Ext}^{s+u}_C(L,N)\,.$$

In the case of modules over a Hopf algebra B, the interior and composition products are related as follows.

PROPOSITION 11.8.3 ([**Yon58**, Prop. 1]). *For*

$$x' \in \mathrm{Ext}_B^{s'}(M', N') \qquad y' \in \mathrm{Ext}_B^{u'}(L', M')$$
$$x'' \in \mathrm{Ext}_B^{s''}(M'', N'') \qquad y'' \in \mathrm{Ext}_B^{u''}(L'', M'')$$

the identity

$$(x' \circ y') \wedge (x'' \circ y'') = (-1)^{s''u'}(x' \wedge x'') \circ (y' \wedge y'')$$

holds in $\mathrm{Ext}_B^{s'+u'+s''+u''}(L' \otimes L'', N' \otimes N'')$.

PROOF. Let

$$\cdots \to P'_{s'} \xrightarrow{\partial_{s'}} P'_{s'-1} \to \cdots \to P'_1 \xrightarrow{\partial_1} P'_0 \xrightarrow{\epsilon} M' \to 0$$
$$\cdots \to Q'_{u'} \xrightarrow{\partial_{u'}} Q'_{u'-1} \to \cdots \to Q'_1 \xrightarrow{\partial_1} Q'_0 \xrightarrow{\epsilon} L' \to 0$$
$$\cdots \to P''_{s''} \xrightarrow{\partial_{s''}} P''_{s''-1} \to \cdots \to P''_1 \xrightarrow{\partial_1} P''_0 \xrightarrow{\epsilon} M'' \to 0$$
$$\cdots \to Q''_{u''} \xrightarrow{\partial_{u''}} Q''_{u''-1} \to \cdots \to Q''_1 \xrightarrow{\partial_1} Q''_0 \xrightarrow{\epsilon} L'' \to 0$$

be projective B-module resolutions, and choose cocycles

$$g' \colon P'_{s'} \longrightarrow N'$$
$$f' \colon Q'_{u'} \longrightarrow M'$$
$$g'' \colon P''_{s''} \longrightarrow N''$$
$$f'' \colon Q''_{u''} \longrightarrow M''$$

representing x', y', x'' and y'', respectively. Lift f' and f'' to chain maps

$$f'_* \colon Q'_{*+u'} \longrightarrow P'_*$$
$$f''_* \colon Q''_{*+u''} \longrightarrow P''_*$$

of degrees $-u'$ and $-u''$. Then $(P' \otimes P'')_* = P'_* \otimes P''_* \to M' \otimes M''$ and $(Q' \otimes Q'')_* = Q'_* \otimes Q''_* \to L' \otimes L''$ are projective B-module resolutions, with the diagonal B-action, and

$$f'_* \otimes f''_* \colon (Q' \otimes Q'')_{*+u'+u''} = Q'_{*+u'} \otimes Q''_{*+u''} \longrightarrow P'_* \otimes P''_* = (P' \otimes P'')_*$$

is a chain map $(f' \otimes f'')_*$ of degree $(-u' - u'')$, lifting

$$(Q' \otimes Q'')_{u'+u''} \to Q'_{u'} \otimes Q''_{u''} \xrightarrow{f' \otimes f''} M' \otimes M'' .$$

The class of the composite

$$(g' \otimes g'') \circ (f' \otimes f'')_{s'+s''} = (g' \otimes g'') \circ (f'_{s'} \otimes f''_{s''})$$

then defines $(x' \wedge x'') \circ (y' \wedge y'')$, and equals $(-1)^{s''u'}$ times the class of the composite

$$(g' \circ f'_{s'}) \otimes (g'' \circ f''_{s''}) ,$$

which defines $(x' \circ y') \wedge (x'' \circ y'')$. □

COROLLARY 11.8.4. *Let B a Hopf algebra over k. For $x \in \mathrm{Ext}_B^s(k, N)$ and $y \in \mathrm{Ext}_B^u(L, k)$ the identity*

$$x \wedge y = (x \wedge 1) \circ (1 \wedge y) = x \circ y$$

holds in $\mathrm{Ext}_B^{s+u}(k \otimes L, N \otimes k) \cong \mathrm{Ext}_B^{s+u}(L, N)$, *and the identity*

$$(-1)^{su} y \wedge x = (1 \wedge x) \circ (y \wedge 1) = x \circ y$$

holds in $\mathrm{Ext}_B^{u+s}(L \otimes k, k \otimes N) \cong \mathrm{Ext}_B^{u+s}(L, N)$. *In particular, the interior and composition products*

$$\mathrm{Ext}_B^s(k,k) \otimes \mathrm{Ext}_B^u(k,k) \longrightarrow \mathrm{Ext}_B^{s+u}(k,k)$$

agree, and make $\mathrm{Ext}_B^*(k,k)$ *a graded commutative* k*-algebra.*

PROOF. Apply Proposition 11.8.3 with $x'' = y' = 1 \in \mathrm{Ext}_B^0(k,k)$ in the first case, and with $x' = y'' = 1 \in \mathrm{Ext}_B^0(k,k)$ in the second case. □

((ETC: The argument does not seem to assume that B is (co-)commutative, since the twist isomorphism for tensor products of B-modules does not play a role. This agrees with [**Yon58**, Prop. 5].))

((ETC: Similar results for interior product and composition product in comodule Ext over a Hopf algebra.))

For spectra X, Y and Z consider the Adams spectral sequences

$$\begin{aligned} {}'E_2 &= \mathrm{Ext}_A(H_*(Y), H_*(Z)) \Longrightarrow [Y,Z]_* \\ {}''E_2 &= \mathrm{Ext}_A(H_*(X), H_*(Y)) \Longrightarrow [X,Y]_* \\ E_2 &= \mathrm{Ext}_A(H_*(X), H_*(Z)) \Longrightarrow [X,Z]_* \,. \end{aligned}$$

The interaction between the composition product in Ext and the composition in the stable category was determined by Michael Moss.

THEOREM 11.8.5 ([**Mos68**, Thm. 2.1]). *(a) There is a natural pairing*

$$\circ_r \colon ({}'E_r, {}''E_r) \longrightarrow E_r$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\circ \colon [Y,Z]_* \otimes [X,Y]_* \longrightarrow [X,Z]_*$$

mapping $g \otimes f$ *to* $g \circ \Sigma^{|g|} f$.

(b) The pairing of E_2*-terms*

$$\circ_2 \colon \mathrm{Ext}_{A_*}(H_*(Y), H_*(Z)) \otimes \mathrm{Ext}_{A_*}(H_*(X), H_*(Y)) \longrightarrow \mathrm{Ext}_{A_*}(H_*(X), H_*(Z))$$

is the composition product.

(c) If Y/p *and* Z/p *are bounded below of finite type, then the* E_2*-pairing*

$$\circ_2 \colon \mathrm{Ext}_A(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_A(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_A(H^*(Z), H^*(X))$$

is the twisted composition product, mapping $y \otimes x$ *to* $(-1)^{|x||y|} x \circ y$, *where* $|x| = v - u$ *and* $|y| = t - s$ *for* $x \in {}''E_2^{u,v}$ *and* $y \in {}'E_2^{s,t}$.

REMARK 11.8.6. When $Y = S$, this theorem of Moss can be deduced from that for the smash product pairing, since the two pairings

$$[S,Z]_n \otimes [X,S]_m \longrightarrow [X,Z]_{n+m} \cong [S \wedge X, Z \wedge S]_{n+m}$$

mapping $g \otimes f$ to

$$g \circ \Sigma^n f \colon \Sigma^{n+m} X \longrightarrow S^n \longrightarrow Z$$

and

$$g \wedge f \colon S^n \wedge \Sigma^m X \longrightarrow Z \wedge S$$

are equal, and the two pairings

$$\circ, \wedge \colon \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(Z)) \otimes \mathrm{Ext}_{A_*}(H_*(X), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{A_*}(H_*(X), H_*(Z))$$

also agree.

COROLLARY 11.8.7. *(a) There is a natural pairing*

$$\circ_r \colon (E_r(S), E_r(S)) \longrightarrow E_r(S)$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\circ \colon \pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S)$$

mapping $g \otimes f$ *to* $g \circ \Sigma^{|g|} f$*, which equals the smash product* $g \wedge f$*.*

(b) The pairing of E_2*-terms*

$$\circ_2 \colon \operatorname{Ext}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes \operatorname{Ext}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_{A_*}(\mathbb{F}_p, \mathbb{F}_p)$$

is the composition product, which equals the (graded commutative) internal product.

(c) The E_2*-pairing*

$$\circ_2 \colon \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) \otimes \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$$

is the twisted composition product, mapping $y \otimes x$ *to* $(-1)^{|x||y|} x \circ y$*, where* $|x| = v - u$ *and* $|y| = t - s$ *for* $x \in {}''E_2^{u,v}(S)$ *and* $y \in {}'E_2^{s,t}(S)$*, which equals the (graded commutative) internal product.*

((ETC: What can we deduce from

$$\pi_* F(Y, Z) \otimes \pi_* F(X, Y) \longrightarrow \pi_*(F(Y, Z) \wedge F(X, Y)) \longrightarrow \pi_* F(X, Z)$$

in the cases where the Adams spectral sequence for X and Y agrees with the one for S and $F(X, Y)$, and so on?))

11.9. Products in Ext over the Steenrod algebra

In the case $X = Y = S$, Theorem 11.7.3 or its corollaries shows that the mod p Adams spectral sequence for the sphere spectrum is a graded commutative algebra spectral sequence

$$E_2(S)^{s,t} = \operatorname{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(S)^\wedge_p$$

with differentials

$$d_r^{s,t} \colon E_r^{s,t}(S) \longrightarrow E_r^{s+r, t+r-1}(S)\,.$$

The multiplication on the E_2-term is given by the internal product

$$\wedge \colon \operatorname{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \otimes \operatorname{Ext}_A^{u,v}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_A^{s+u, t+v}(\mathbb{F}_p, \mathbb{F}_p)\,,$$

and converges to the smash product pairing

$$\wedge \colon \pi_n(S)^\wedge_p \otimes \pi_m(S)^\wedge_p \longrightarrow \pi_{n+m}(S)^\wedge_p$$

that gives the graded commutative ring structure on $\pi_*(S)^\wedge_p$. Yoneda's Proposition 11.8.3 shows that the internal product pairing is equal to the composition product in Ext, and that the smash product pairing is equal to the composition product in $\pi_*(S)^\wedge_p$.

For $p = 2$, Bruner's program `ext` can calculate the Yoneda (composition) products in Ext, by lifting cocycles to chain maps and evaluating their composites. The computation of products

$$h_i \colon \operatorname{Ext}_A^{s,t}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_A^{s+1, t+2^i}(M, \mathbb{F}_2)$$

with the Hopf–Steenrod classes h_i is particularly simple, and can be read off from the boundary homomorphism

$$\partial_{s+1} \colon P_{s+1} \longrightarrow P_s$$

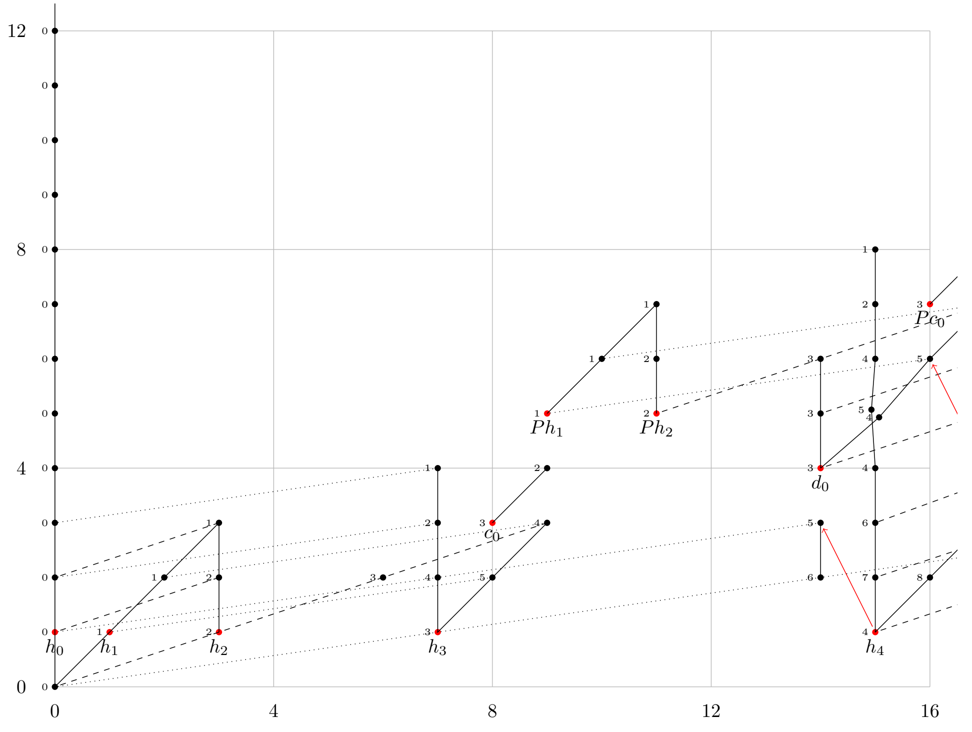


FIGURE 11.3. $E_2^{s,t}(S) = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ with h_0-, h_1-, h_2- and h_3-multiplications and d_2-differentials, for $0 \le t-s \le 16$ and $0 \le s \le 12$

in a minimal resolution for M. ((ETC: Explain.)) In the case $M = \mathbb{F}_2$, the multiplications by h_i for $0 \le i \le 3$ in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ are shown in Figure 11.3, with the following graphical conventions.

- Each nonzero multiplication by $h_0 \in E_2^{1,1}(S)$ is shown by a line connecting x in bidegree $(t-s, s)$ to h_0x in bidegree $(t-s, s+1)$, i.e., by a vertical line of unit length.
- Each nonzero multiplication by $h_1 \in E_2^{1,2}(S)$ is shown by a line connecting x in bidegree $(t-s, s)$ to h_1x in bidegree $(t-s+1, s+1)$, i.e., by a line of slope $+1$.
- Each nonzero multiplication by $h_2 \in E_2^{1,4}(S)$ is shown by a dashed line connecting x in bidegree $(t-s, s)$ to h_2x in bidegree $(t-s+3, s+1)$, i.e., by a dashed line of slope $+1/3$.
- Each nonzero multiplication by $h_3 \in E_2^{1,8}(S)$ is shown by a dotted line connecting x in bidegree $(t-s, s)$ to h_3x in bidegree $(t-s+7, s+1)$, i.e., by a dotted line of slope $+1/7$.

LEMMA 11.9.1. *In the range $t-s \le 16$, the $\mathbb{F}_2$-algebra $E_2^{*,*}(S)$ is generated by the following classes.*

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_4	Pc_0
$t-s$	0	1	3	7	8	9	11	14	15	16
s	1	1	1	1	3	5	5	4	1	7

The relation $c_0^2 = h_1^2 d_0$ holds.

PROOF. The h_i-multiplications can be read off from the minimal resolution (P_*, ∂) of $\mathbb{F}_2$ calculated by `ext`, and is visible in Figure 11.3. The classes h_i in filtration $s=1$ must be algebra indecomposable for filtration degree reasons. The only other basis elements that are not h_i-multiplies are the classes denoted c_0, d_0, Ph_1, Ph_2 and Pc_0, and these must then be algebra decomposable for topological degree reasons, since these all lie in degrees $t-s \ge 8$.

To calculate $c_0^2 = c_0 \cdot c_0$, we instead call on `ext` to lift the cocycle $f = 3_3 \colon P_3 \to \Sigma^{11}\mathbb{F}_2$ to a chain map $f_* \colon P_{*+3} \to \Sigma^{11} P_*$, and then to evaluate the composite

$$P_6 \xrightarrow{f_3} \Sigma^{11} P_3 \xrightarrow{f} \Sigma^{22}\mathbb{F}_2 \,.$$

This turns out to map 6_5^* to 1, hence equals the cocycle 6_5, which we have already seen represents $h_1^2 d_0$. □

REMARK 11.9.2. The prefix P refers to the periodicity operator from [**Ada66**, Thm. 1.2], and the notations $c_0, d_0, \dots$ stem from computations in the range $t-s \le 70$ made by May (unpublished) and Tangora [**Tan70**]. In his work on the Hopf invariant one problem, Adams showed that there are no algebra indecomposables in filtration $s=2$ of $E_2^{*,*}(S) = \mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, and determined the multiplicative relations satisfied by the generators h_i in filtrations $s \le 3$.

THEOREM 11.9.3 ([**Ada60**, Thm. 2.5.1]). *The relations*

$$\begin{aligned} h_i h_{i+1} &= 0 \\ h_i^2 h_{i+2} &= h_{i+1}^3 \\ h_i h_{i+2}^2 &= 0 \end{aligned}$$

hold in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, for each $i \ge 0$. The algebra homomorphism

$$\frac{\mathbb{F}_2[h_i \mid i \ge 0]}{(h_i h_{i+1}, h_i^2 h_{i+2} + h_{i+1}^3, h_i h_{i+2}^2)} \longrightarrow \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism in filtration degrees $s \le 2$, and is injective in degree $s = 3$.

More explicitly,

$$\begin{aligned} &\mathrm{Ext}_A^{0,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{1\} \\ &\mathrm{Ext}_A^{1,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{h_i \mid i \ge 0\} \\ &\mathrm{Ext}_A^{2,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{h_i h_j \mid 0 \le i \le j-2\} \oplus \mathbb{F}_2\{h_j^2 \mid j \ge 0\} \end{aligned}$$

and if we omit the generators $h_i h_{i+1} h_k$, $h_i h_j h_{j+1}$, $h_i h_i h_{i+2}$ and $h_i h_{i+2} h_{i+2}$ from

$$\mathbb{F}_2\{h_i h_j h_k \mid i \le j \le k\}$$

then the remainder maps injectively to $\mathrm{Ext}_A^{3,*}(\mathbb{F}_2, \mathbb{F}_2)$. The class c_0 (which is part of a family of related classes c_i for $i \ge 0$) shows that surjectivity fails for $s = 3$.

((ETC: References for the relations satisfied for $s \ge 4$?))

11.10. Adams differentials for the sphere spectrum

In view of the Leibniz rule

$$d_2(xy) = d_2(x)y + xd_2(y)$$

in $E_2(S)$, the d_2-differential is determined by its values on a set of algebra generators for this E_2-term. In the range $t-s \le 16$, it thus suffices to determine $d_2(x)$ for the x in the table in Lemma 11.9.1, which are marked in red in Figure 11.3.

PROPOSITION 11.10.1. *In the range $t-s \le 16$, the d_2-differential on the algebra generators is given as follows.*

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_4	Pc_0
$d_2(x)$	0	0	0	0	0	0	0	0	$h_0h_3^2$	0

PROOF. The d_2-differentials on h_0, h_2, h_3, c_0, Ph_1, Ph_2, d_0 and Pc_0 land in trivial groups, hence are zero.

The relation $h_0h_1 = 0$ and the Leibniz rule imply that $0 \cdot h_1 + h_0 \cdot d_2(h_1) = d_2(0) = 0$, so that $h_0d_2(h_1) = 0$. Since $h_0 \cdot h_0^3 = h_0^4 \neq 0$, it follows that $d_2(h_1) \neq h_0^3$, and $d_2(h_1) = 0$ is the only possibility.

The final case, of $d_2(h_4)$, deserves to be stated as a separate theorem. □

THEOREM 11.10.2 ([**Ada58**, p. 184]). $d_2(h_4) = h_0h_3^2$.

PROOF. The class $h_0 \in E_2^{1,1}(S)$ detects the homotopy class $2 \in \pi_0(S)_2^\wedge$. ((ETC: Explain?)) The class $h_3 \in E_2^{1,8}(S)$ must survive to $E_\infty(S)$ since $d_r(h_3)$ lies in a trivial group for all $r \ge 2$. Hence it detects a homotopy class $\sigma \in \pi_7(S)_2^\wedge$. By multiplicativity of the Adams spectral sequence for S, it follows that $2\sigma^2 = 2{\cdot}\sigma{\cdot}\sigma$ is detected by $h_0h_3^2 = h_0 \cdot h_3 \cdot h_3$ in $F^3\pi_*(S)_2^\wedge/F^4\pi_*(S)_2^\wedge \cong E_\infty^{3,*}$. However, by the graded commutativity of $\pi_*(S)_2^\wedge$, we have

$$\sigma \cdot \sigma = -\sigma \cdot \sigma \,,$$

since $|\sigma| = 7$ is odd. Thus $2\sigma^2 = 0$, which implies that $h_0h_3^2 = 0$ in $E_\infty(S)$. This can only happen because $h_0h_3^2 \in E_2(S)$ is the boundary of a differential, and $d_2(h_4) = h_0h_3^2$ is the only possibility. □

This recovers a result of Toda, first proved by secondary composition methods.

COROLLARY 11.10.3 ([**Tod55**]). *There is no stable map $S^{15} \to S$ of Hopf–Steenrod invariant one. Hence there is no map $S^{31} \to S^{16}$ of Hopf invariant one, no H-space structure on S^{15}, and no division algebra structure on $\mathbb{R}^{16}$.*

PROOF. Such a map would be detected by h_4, which would have to survive to the E_∞-term, but the nonzero differential $d_2(h_4) = h_0h_3^2$ shows that this is not the case. □

REMARK 11.10.4. It follows that all d_2-differentials are h_i-linear for $0 \le i \le 3$. In particular, the class of the cocycle 2_5 factors as h_1h_3, so that $d_2(h_1h_3) = 0 \cdot h_3 + h_1 \cdot 0 = 0$. This resolves one differential that was left open in Example 11.6.13.

Passing to cohomology with respect to the d_2-differential, we can calculate $E_3(S)$ in our range, and determine its algebra indecomposables.

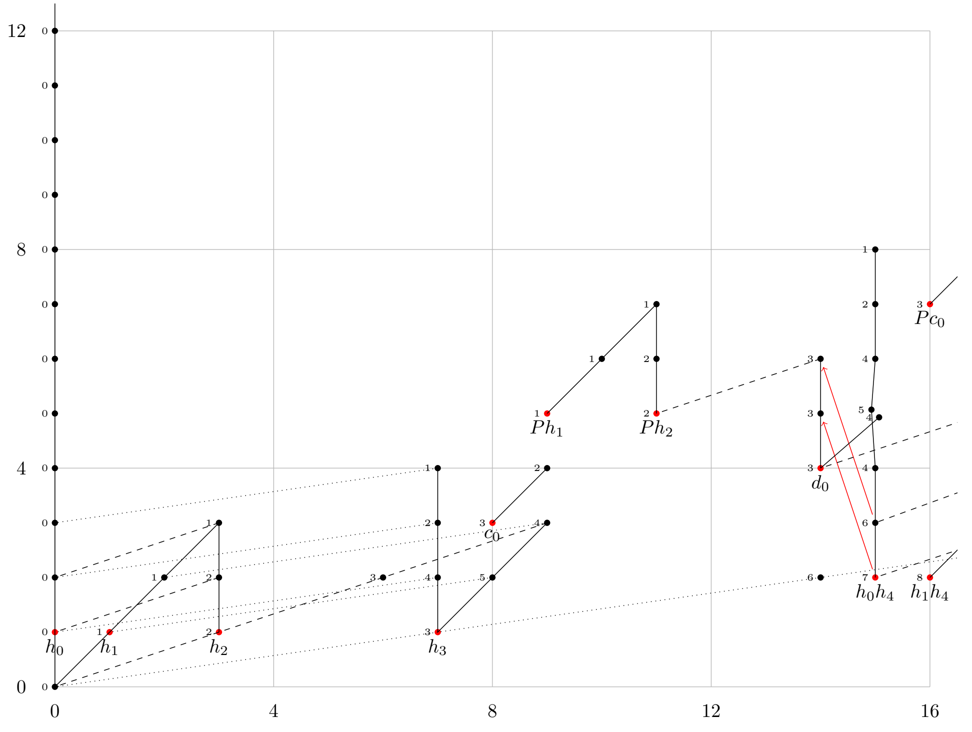


FIGURE 11.4. $E_3^{s,t}(S)$ with h_0-, h_1-, h_2- and h_3-multiplications, and d_3-differentials, for $0 \le t-s \le 16$ and $0 \le s \le 12$

LEMMA 11.10.5. *For $t-s \le 16$, the $\mathbb{F}_2$-algebra $E_3^{*,*}(S)$ is generated by the following classes.*

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_0h_4	h_1h_4	Pc_0
$t-s$	0	1	3	7	8	9	11	14	15	16	16
s	1	1	1	1	3	5	5	4	2	2	7

The h_i-multiplications are visible in Figure 11.4, and the remaining products in this range are zero.

Note that h_0h_4 and h_1h_4 were decomposable on $E_2(S)$, but are indecomposable in $E_3(S)$.

PROPOSITION 11.10.6. *In the range $t-s \le 16$, the d_3-differential on the algebra generators is given as follows.*

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	h_0h_4	h_1h_4	Pc_0
$d_3(x)$	0	0	0	0	0	0	0	0	h_0d_0	0	0

PROOF. The d_3-differentials on h_0, h_2, h_3, c_0, Ph_1, Ph_2, d_0 and Pc_0 land in trivial groups, hence are zero. In particular, d_3 commutes with multiplication by any of these elements.

The differential on h_1 vanishes by h_0-linearity, since

$$h_0 d_3(h_1) = d_3(h_0 h_1) = d_3(0) = 0\,,$$

while $h_0 h_0^4 \neq 0$, so $d_3(h_1) \neq h_0^4$.

By h_0-linearity, $d_3(h_1 h_4)$ is h_0-torsion, hence lies in $\{0, h_1 d_0\}$. By calculating $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ in a larger range, we can show that $d_0 \cdot h_1 h_4 = 0$, while $d_0 \cdot h_1 d_0 = h_1 d_0^2 = 9_9 \neq 0$ in $E_2^{9,9+29}(S)$. Moreover, we claim that $h_1 d_0^2$ remains nonzero in $E_3(S)$. This follows from $d_2(k) \neq 0$, which implies $d_2(h_0 k) \neq 0$, $d_2(r) = 0$ and $d_2(h_0 r) = 0$. Hence

$$d_0 \cdot d_3(h_1 h_4) = d_3(d_0 \cdot h_1 h_4) = d_3(0) = 0$$

and $d_0 \cdot h_1 d_0 \neq 0$ in $E_3(S)$ imply that $d_3(h_1 h_4) \neq h_1 d_0$. The only remaining possibility is $d_3(h_1 h_4) = 0$. (This can also be deduced from the strict commutativity of the product on S, using the quadratic construction on σ to obtain a map $\Sigma^7 P_7^\infty = D_2(S^7) \to D_2(S) \to S$.)

The final case, $d_3(h_0 h_4) = h_0 d_0$, deserves a separate theorem. □

THEOREM 11.10.7. $d_3(h_0 h_4) = h_0 d_0$.

PROOF. ((ETC: This can be proved by comparison with the Adams spectral sequence for $C\sigma$, or using the split surjectivity (Adams conjecture) of the Adams e-invariant $e\colon \pi_{15}(S)^\wedge_2 \to \pi_{15}(j)^\wedge_2 \cong \mathbb{Z}/32$ based on real K-theory.)) □

The Leibniz rule for d_3 implies that $d_3(h_0^2 h_4) = h_0^2 d_0$, as indicated in Figure 11.4. Passing to cohomology with respect to the d_3-differential, we can calculate $E_4(S)$ in our range, and determine its algebra indecomposables.

LEMMA 11.10.8. *For $t - s \leq 16$, the $\mathbb{F}_2$-algebra $E_4^{*,*}(S)$ is generated by the following classes.*

x	h_0	h_1	h_2	h_3	c_0	Ph_1	Ph_2	d_0	$h_0^3 h_4$	$h_1 h_4$	Pc_0
$t-s$	0	1	3	7	8	9	11	14	15	16	16
s	1	1	1	1	3	5	5	4	4	2	7

The h_i-multiplications are visible in Figure 11.5, and the remaining products in this range are zero.

PROPOSITION 11.10.9. *All d_r-differentials for $r \geq 4$ are zero in the range $t-s \leq 16$. Hence $E_4(S) = E_\infty(S)$ in this range.*

PROOF. This is clear for all of the algebra generators other than h_1 and $h_1 h_4$. We see that $d_r(h_1) = 0$ in each case by h_0-linearity, since $h_0^{r+1} \neq 0$ in $E_r(S)$ by induction. Likewise, $d_r(h_1 h_4) = 0$ for $r \in \{4, 5\}$ by h_0-linearity. The only remaining case is $d_6(h_1 h_4) \in \{0, h_0^7 h_4\}$. ((ETC: This can be deduced by Maunder's theorem, or by the construction of a homotopy class η^* detected by $h_1 h_4$, using the quadratic construction $D_2(S^7)$.)) □

11.11. Homotopy of the sphere spectrum

We adopt the following notations from Toda's book [**Tod62**], see Figure 11.6.

DEFINITION 11.11.1.

- Let $\eta \in \pi_1(S)$ be the stable class of the complex Hopf fibration, detected by $h_1 \in E_\infty(S)$ in bidegree $(t-s, s) = (1, 1)$.

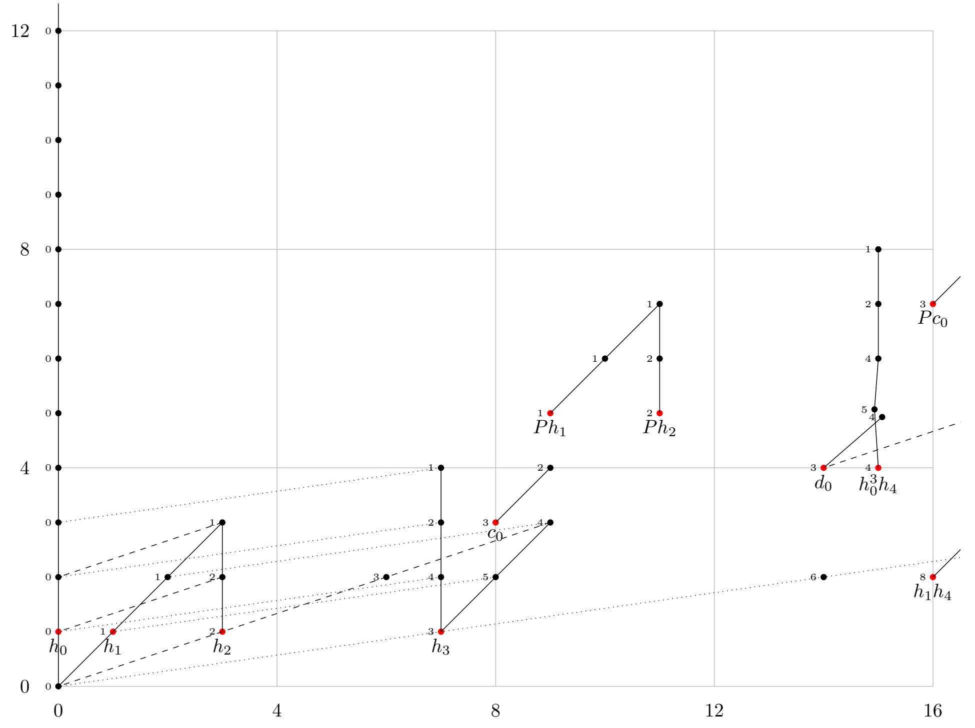


FIGURE 11.5. $E_4^{s,t}(S) = E_\infty^{s,t}(S)$ with h_0-, h_1-, h_2- and h_3-multiplications, for $0 \le t-s \le 16$ and $0 \le s \le 12$

- Let $\nu \in \pi_3(S)$ be the stable class of the quaternionic Hopf fibration, detected by $h_2 \in E_\infty(S)$ in bidegree $(t-s,s) = (3,1)$.
- Let $\sigma \in \pi_7(S)$ be the stable class of the octonionic Hopf fibration, detected by $h_3 \in E_\infty(S)$ in bidegree $(t-s,s) = (7,1)$.
- Let $\epsilon \in \pi_8(S)^\wedge_2$ be the unique homotopy class detected by $c_0 \in E_\infty(S)$ in bidegree $(t-s,s) = (8,3)$.
- Let $\mu \in \pi_9(S)^\wedge_2$ be the unique homotopy class detected by $Ph_1 \in E_\infty(S)$ in bidegree $(t-s,s) = (9,5)$.
- Let $\zeta \in \pi_{11}(S)^\wedge_2$ be detected by $Ph_2 \in E_\infty(S)$ in bidegree $(t-s,s) = (11,5)$. This determines ζ up to an odd multiple. (A definite choice can be made using the J-homomorphism.)
- Let $\kappa \in \pi_{14}(S)^\wedge_2$ be the unique homotopy class detected by $d_0 \in E_\infty(S)$ in bidegree $(t-s,s) = (14,4)$.
- Let $\rho \in \pi_{15}(S)^\wedge_2$ be detected by $h_0^3h_4 \in E_\infty(S)$ in bidegree $(t-s,s) = (15,4)$. This determines ρ up to an odd multiple, modulo $\eta\kappa$. (A definite choice can be made using the J-homomorphism.)
- Let $\eta^* \in \pi_{16}(S)^\wedge_2$ be detected by $h_1h_4 \in E_\infty(S)$ in bidegree $(t-s,s) = (16,2)$. This determines η^* modulo $\eta\rho$. (A definite choice can be made using the Adams e-invariant.)

Let E be a ring spectrum and M an E-module spectrum, and suppose that the Adams spectral sequences $E_r(E)$ and $E_r(M)$ converge to $\pi_*(E)$ and $\pi_*(M)$,

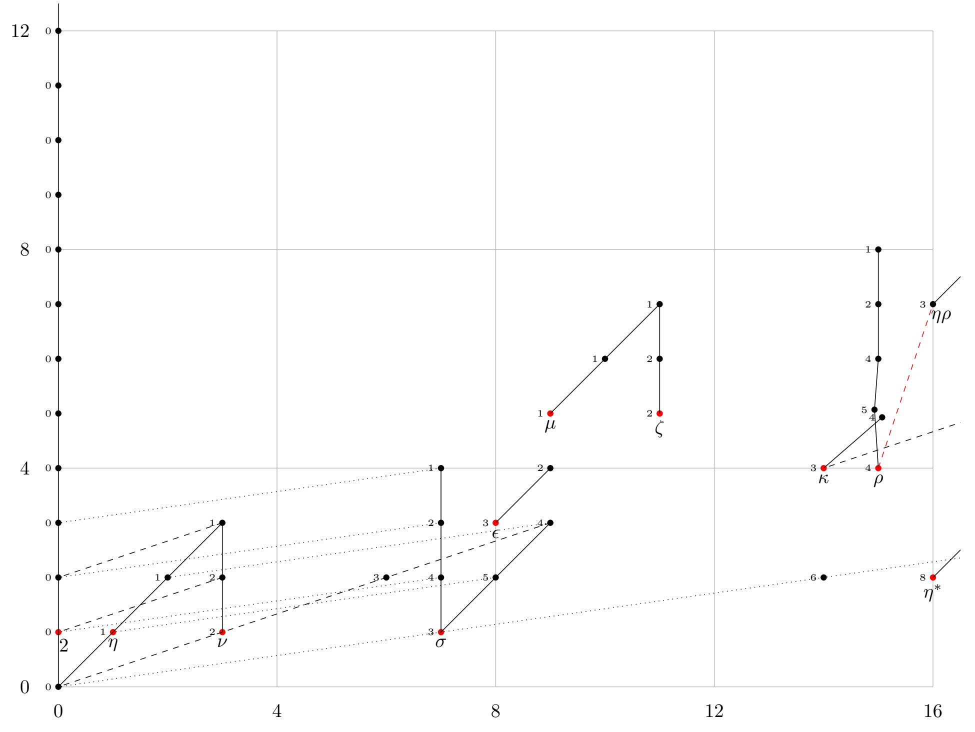


FIGURE 11.6. The associated graded of $\pi_n(S)$ for $0 \le n \le 16$

respectively. For instance, we may consider any spectrum Y as an S-module, so that $E_r(Y)$ is an $E_r(S)$-module spectral sequence.

DEFINITION 11.11.2. Let $\alpha \in \pi_*(E)$ be detected by $a \in E_\infty(E)$, and consider nonzero classes b and $c \in E_\infty(M)$. We say that there is an *α-extension* from b to c if there exists a $\beta \in \pi_*(M)$ such that β is detected by b and $\alpha\beta$ is detected by c, and such that there is no class $\beta' \in \pi_*(M)$ of higher Adams filtration than β for which $\alpha\beta'$ is detected by c. This is a *hidden α-extension* if $ab = 0$.

((ETC: In this definition, c should be viewed as being defined modulo the classes (in the same bidegree) detecting products $\alpha\beta'$ with β' of higher Adams filtration than β.))

(ETC: Generalize α-extensions to maps $f\colon X \to Y$, comparing the filtrations $f_*(F^s\pi_*(X))$ and $F^u\pi_*(Y)$ to form the bifiltration $\Phi^{s,u} = f_*(F^s\pi_*(X)) \cap F^u\pi_*(Y)$ of $\pi_*(Y)$. Consider $\Phi^{s,u}/(\Phi^{s+1,u} + \Phi^{s,u+1})$.))

PROPOSITION 11.11.3. *$\eta\rho \in \pi_{16}(S)^\wedge_2$ is detected by $Pc_0 \in E_\infty(S)$ in bidegree $(t-s,s) = (16,7)$, while $\eta^2\kappa = 0$. Hence there is a hidden η-extension from $h_0^3h_4$ to Pc_0.*

PROOF. ((ETC: This can be deduced using the e-invariant to the image-of-J spectrum, or perhaps by a comparison with the Adams spectral sequence for $C\eta$.)) □

DEFINITION 11.11.4. When a spectral sequence (E_r, d_r) converges to G, and $a \in E^s_\infty$ is a nonzero class, we write $\{a\} \subset G$ for the set of $\alpha \in G$ that are detected by a. This is the coset of $F^{s+1}G$ in F^sG that corresponds to a under the isomorphism $F^sG/F^{s+1}G \cong E^s_\infty$. When $F^{s+1}G = 0$ in the total degree of a, this is a single element and we write $\alpha = \{a\}$.

We can now summarize these initial findings about the graded commutative ring $\pi_*(S)^\wedge_2$, in degrees $* \le 16$. We write $\mathbb{Z}/n\{\alpha\}$ for the cyclic group of order n generated by a class α.

THEOREM 11.11.5.

(0) $\pi_0(S)^\wedge_2 \cong \mathbb{Z}_2$;
$2^s \in \{h_0^s\}$ *for* $s \ge 0$.

(1) $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2\{\eta\}$;
$\eta = \{h_1\}$.

(2) $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2\{\eta^2\}$;
$\eta^2 = \{h_1^2\}$.

(3) $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/8\{\nu\}$;
$\nu \in \{h_2\}$, $2\nu \in \{h_0h_2\}$, $4\nu = \{h_0^2h_2\}$;
$\eta^3 = 4\nu$.

(4) $\pi_4(S)^\wedge_2 = 0$.

(5) $\pi_5(S)^\wedge_2 = 0$.

(6) $\pi_6(S)^\wedge_2 = \mathbb{Z}/2\{\nu^2\}$;
$\nu^2 = \{h_2^2\}$.

(7) $\pi_7(S)^\wedge_2 = \mathbb{Z}/16\{\sigma\}$;
$\sigma \in \{h_3\}$, $2\sigma \in \{h_0h_3\}$, $4\sigma \in \{h_0^2h_3\}$, $8\sigma = \{h_0^3h_3\}$.

(8) $\pi_8(S)^\wedge_2 = \mathbb{Z}/2\{\epsilon\} \oplus \mathbb{Z}/2\{\eta\sigma\}$;
$\eta\sigma \in \{h_1h_3\}$, $\epsilon = \{c_0\}$.

(9) $\pi_9(S)^\wedge_2 = \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta\epsilon\} \oplus \mathbb{Z}/2\{\eta^2\sigma\}$;
$\eta^2\sigma \in \{h_1^2h_3\}$, $\eta\epsilon \in \{h_1c_0\}$, $\mu = \{Ph_1\}$;
$\nu^3 = \eta\epsilon + \eta^2\sigma$.

(10) $\pi_{10}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\mu\}$;
$\eta\mu = \{h_1Ph_1\}$;
$\eta^2\epsilon = 0$, $\nu\sigma = 0$.

(11) $\pi_{11}(S)^\wedge_2 = \mathbb{Z}/8\{\zeta\}$;
$\zeta \in \{Ph_2\}$, $2\zeta \in \{h_0Ph_2\}$, $4\zeta = \{h_0^2Ph_2\}$;
$\eta^2\mu = 4\zeta$, $\nu\epsilon = 0$.

(12) $\pi_{12}(S)^\wedge_2 = 0$.

(13) $\pi_{13}(S)^\wedge_2 = 0$.

(14) $\pi_{14}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa\} \oplus \mathbb{Z}/2\{\sigma^2\}$;
$\kappa = \{d_0\}$, $\sigma^2 \in \{h_3^2\}$;
$\nu\zeta = 0$.

(15) $\pi_{15}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\}$;
$\rho \in \{h_0^3 h_4\}$, $2\rho \in \{h_0^4 h_4\}$, $4\rho \in \{h_0^5 h_4\}$, $8\rho \in \{h_0^6 h_4\}$, $16\rho = \{h_0^7 h_4\}$,
$\eta\kappa \in \{h_1 d_0\}$;
$\eta\sigma^2 = 0$, $\sigma\epsilon = 0$.

(16) $\pi_{16}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\rho\} \oplus \mathbb{Z}/2\{\eta^*\}$;
$\eta\rho = \{Pc_0\}$, $\eta^* \in \{h_1 h_4\}$; $\eta^2\kappa = 0$, $\sigma\mu = \eta\rho$, $\epsilon^2 = 0$.

Proof. In many cases, this is immediate from the algebra structure of the E_∞-term, keeping in mind that if α and β are detected by a and b, respectively, then $\alpha\beta$ is detected by ab if $ab \neq 0$, and has higher Adams filtration than this product if $ab = 0$. See Remark 5.4.17. The following cases require additional argments.

(9) The spectral sequence algebra structure shows that ν^3 is detected by $h_2^2 = h_1^2 h_3$, hence equals $\eta^2\sigma$ modulo Adams filtration ≥ 4, i.e., modulo $\mathbb{F}_2\{\mu, \eta\epsilon\}$. The KO-theory d- and e-invariants, which combine to a map $e\colon S \to j$ to the image-of-J spectrum, show that we must have $\nu^3 = \eta^2\sigma + \eta\epsilon$.

(10) The map to the image-of-J detects $\eta\mu$, but not $\eta^2\epsilon$ or $\nu\sigma$, so the latter two products are zero.

(11) The image-of-J detects ζ, 2ζ and 4ζ but not $\nu\epsilon$, so the latter product is zero.

(14) The product $\nu\zeta$ has Adams filtration $\geq 1 + 5 = 6$, hence is zero, since the E_∞-classes in total degree 14 all have lower Adams filtration.

(15) The image-of-J shows that $\eta\sigma^2$ and $\sigma\epsilon$ lie in $\mathbb{F}_2\{0, \eta\kappa\}$. ((ETC: Justify $\eta\sigma^2 = 0$ and $\sigma\epsilon = 0$.))

(16) The relations $\eta^2\kappa = 0$, $\sigma\mu = \eta\rho$ and $\epsilon^2 = 0$ are all detected in the image-of-J spectrum. Since they all lie in Adams filtrations greater than that of η^*, they also hold in the homotopy of S. □

Remark 11.11.6. The relation $\nu \cdot \nu^2 = \eta^2\sigma + \eta\epsilon$ shows that the (hidden or visible) α-extensions do not completely determine the multiplicative action by α, since there may be higher filtration terms that are not seen by the α-extension. In this case there is a ν-extension from h_2^2 to $h_2^3 = h_1^2 h_3$, and $\eta\epsilon$ is the higher-filtration term.

CHAPTER 12

Modified Adams spectral sequences (TO BE WRITTEN)

12.1. Toda brackets and Massey products

((ETC: [**Mos70**]))

12.2. Power operations and Steenrod operations

12.3. Delayed Adams spectral sequences

12.4. Hastened Adams spectral sequences

CHAPTER 13

Bestiary

CHAPTER 14

The sphere spectrum and the projective planes

14.1. The (E_2, d_2)-term for S

The E_2-term

$$E_2^{s,t}(S) = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

for the Adams spectral sequence converging to $\pi_{t-s}(S)^\wedge_2$ has been calculated for $t \le 200$ using `ext`. A set of algebra generators for $t-s \le 48$ are listed in Table 14.1, and the (E_2, d_2)-term in this range is shown in Figure 14.1.

Table 14.1: Algebra generators for $E_2(S)$ with $t - s \le 48$

$t-s$	s	g	x	$d_2(x)$	
0	1	0	h_0	0	(1)
1	1	1	h_1	0	(2)
3	1	2	h_2	0	(1)
7	1	3	h_3	0	(1)
8	3	3	c_0	0	(1)
9	5	1	Ph_1	0	(1)
11	5	2	Ph_2	0	(1)
14	4	3	d_0	0	(1)
15	1	4	h_4	$h_0h_3^2$	(5)
16	7	3	Pc_0	0	(1)
17	4	5	e_0	$h_1^2d_0$	(9)
17	9	1	P^2h_1	0	(1)
18	4	6	f_0	$h_0^2e_0$	(9)
19	3	9	c_1	0	(9)
19	9	2	P^2h_2	0	(1)
20	4	8	g_1	0	(1)
22	8	3	Pd_0	0	(1)
23	7	5	i	h_0Pd_0	(7)
24	11	3	P^2c_0	0	(1)
25	8	5	Pe_0	$h_1^2Pd_0$	(8)
25	13	1	P^3h_1	0	(1)

Table 14.1: Algebra generators for $E_2(S)$ with $t - s \leq 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
26	7	6	j	$h_0 P e_0$	(8)
27	13	2	$P^3 h_2$	0	(1)
29	7	7	k	$h_0 d_0^2$	(8)
30	6	10	r	0	(4)
30	12	3	$P^2 d_0$	0	(1)
31	1	5	h_5	$h_0 h_4^2$	(6)
31	5	13	n	0	(2)
32	4	13	d_1	0	(2)
32	6	12	q	0	(2)
32	7	10	ℓ	$h_0 d_0 e_0$	(8)
32	15	3	$P^3 c_0$	0	(1)
33	4	14	p	0	(3)
33	12	5	$P^2 e_0$	$h_1^2 P^2 d_0$	(10)
33	17	1	$P^4 h_1$	0	(1)
34	11	7	Pj	$h_0 P^2 e_0$	(10)
35	7	12	m	$h_0 d_0 g_1$	(8)
35	17	2	$P^4 h_2$	0	(1)
36	6	14	t	0	(2)
37	5	17	x	0	(1)
38	4	16	e_1	0	(2)
38	6	16	y	$h_0^3 x$	(8)
38	16	3	$P^3 d_0$	0	(1)
39	9	18	u	0	(1)
39	15	5	$P^2 i$	$h_0 P^3 d_0$	(11)
40	4	19	f_1	0	(1)
40	19	3	$P^4 c_0$	0	(1)
41	3	19	c_2	$h_0 f_1$	(14)
41	10	14	z	0	(4)
41	16	5	$P^3 e_0$	$h_1^2 P^3 d_0$	(11)
41	21	1	$P^5 h_1$	0	(1)
42	9	19	v	$h_0 z$	(15)
42	15	6	$P^2 j$	$h_0 P^3 e_0$	(11)
43	21	2	$P^5 h_2$	0	(1)
44	4	22	g_2	0	(1)

Table 14.1: Algebra generators for $E_2(S)$ with $t - s \leq 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
45	9	20	w	0	(1)
46	7	20	B_1	0	(12)
46	8	20	N	0	(1)
46	20	3	P^4d_0	0	(1)
47	13	14	Q	h_0i^2	(13)
47	13	15	Pu	0	(2)
48	7	22	B_2	0	(1)
48	23	3	P^5c_0	0	(1)

The $\mathbb{F}_2$-algebra generators are chosen as follows:

- The generators h_0, c_0, d_0, e_0, $g = g_1$, p, x, r, q, t, i, j, k, ℓ, m, B_1, N, u, v, w and z are the unique nonzero classes in their respective bidegrees.
- The generator $f_0 = Sq^1(c_0)$ equals 4_6 and the generator $y = Sq^2(f_0)$ equals 6_{16}, both by [**BNT**].
- The generator $n = 5_{13}$ is characterized by $h_0n = 0$.
- We set $B_2 = 7_{22}$, which may differ by $7_{23} = h_0^2h_5e_0$ from the generator of the same name in [**Tan70**].
- The generator $Q = 13_{14}$ is characterized by $h_0Q \neq 0$ and $h_1Q \neq 0$.
- The generators a_i for larger i with $a \in \{h, c, d, e, f, g\}$, are iteratively given by $a_{i+1} = Sq^0(a_i)$.
- The generators Pa lie in $\langle h_3, h_0^4, a\rangle$, and the generator P^2a (for $a = i$) lies in $\langle h_4, h_0^8, a\rangle$.

The d_2-differentials are determined as follows:

(1) The differentials on h_0, h_2, h_3, c_0, d_0, g_1, f_1, g_2, Ph_1, Ph_2, x, Pc_0, B_2, Pd_0, N, P^2h_1, P^2h_2, w, P^2c_0, P^2d_0, P^3h_1, P^3h_2, P^3c_0, P^3d_0, P^4h_1, P^4h_2, P^4c_0, P^4d_0, P^5h_1, P^5h_2 and P^5c_0 are zero because the target bidegrees are trivial.

(2) The differentials on h_1, d_1, e_1, n, q, t and Pu are zero by h_0-linearity.

(3) The differential on p is zero by h_1-linearity.

(4) The differentials on r and z are zero by h_2-linearity, since $h_2r = h_1q$.

(5) Since h_3 survives to E_∞ and detects $\sigma \in \pi_7(S)$, and $2\sigma^2 = 0$ by graded commutativity, the detecting class $h_0h_3^2$ must be a boundary. This establishes the first Adams differential, $d_2(h_4) = h_0h_3^2$.

(6) Since $h_4 \cdot h_5 = 0$ and $h_0h_3^2 \cdot h_5 = 4_{23} \neq 0$ the Leibniz rule $d_2(h_4h_5) = d_2(h_4)h_5 + h_4d_2(h_5)$ and case (5) imply $d_2(h_5) \neq 0$. The only alternative is $d_2(h_5) = h_0h_4^2$.

(7) Since $h_4 \cdot i = 0$ and $h_0h_3^2 \cdot i = 10_{12} \neq 0$ the Leibniz rule $d_2(h_4i) = d_2(h_4)i + h_4d_2(i)$ and case (5) imply $d_2(i) \neq 0$. The only possibility is $d_2(i) = h_0Pd_0$.

(8) The differentials on Pe_0, j, k, ℓ, m and y follow from case (7) by h_0-, h_1- and h_2-linearity. ((ETC: Note that $h_1d_2(y) = h_2d_2(t) = 0$.))

(9) The differentials on e_0, f_0 and c_1 follow from case (8) by d_0-linearity, since $d_0 f_0 = h_0 \ell$ with $d_2(h_0\ell) \neq 0$, so that $d_2(f_0) \neq 0$ must be $h_0^2 e_0$.

(10) The differentials on Pj and $P^2 e_0$, follow from case (8) by d_0-linearity, since $h_0 d_0 i = h_2 P j$, so that $d_2(Pj) \neq 0$ must be $h_0 P^2 e_0$.

(11) The differentials on $P^2 i$, $P^2 j$ and $P^3 e_0$, follow from case (8) by Pd_0-linearity, since $h_0 P d_0 i = h_2 P^2 j$, so that $d_2(P^2 j) \neq 0$ must be $h_0 P^3 e_0$.

(12) The relation $d_0 \cdot B_1 = 11_{22} = h_1 \cdot 10_{24}$ and $d_2(10_{24}) = 0$ imply that $d_0 \cdot d_2(B_1) = 0$. However, $d_0 \cdot w = 11_{22} \neq 0$, so $d_2(B_1) \neq w$.

(13) The relation $h_0 g_1 \cdot Pj = 16_{17} = h_0^2 h_3 \cdot Q$ and $h_0 g_1 \cdot d_2(Pj) \neq 0$ implies that $d_2(Q) \neq 0$ must be $h_0 i^2$.

(14) The H_∞ ring structure on S implies $d_2(c_2) = h_0 Sq^1(c_1) = h_0 f_1$, cf. [**BMT70**, Cor. 3.3.6], [**Mil72**, Cor. 6.5.2] or [**BR**, Thm. 11.52(4)].

(15) We show in Table 14.5, cf. case (12), that $d_2(\widehat{v}) = h_0 \widehat{z} + i(d_0 k)$ in $E_2(C\eta)$. Mapping by $j\colon C\eta \to S^2$ it follows that $d_2(v) = h_0 z$ in $E_2(S)$.

((ETC: Alternatively, the unit map $\iota\colon S \to tmf$ to topological modular forms shows that $d_2(v) \neq 0$, so $h_0 z$ is the only possible value, cf. [**MT67**, Prop. 6.1.5] or [**BR**, Thm. 11.52(5)].))

14.2. The (E_3, d_3)-term for S

The E_3-term of the Adams spectral sequence for S is calculated as the homology subquotient

$$E_3(S) = H(E_2(S), d_2) .$$

A set of algebra generators for $t - s \leq 48$ are listed in Table 14.2, and the (E_3, d_3)-term in this range is shown in Figure 14.2. ((ETC: d_3's complete from $t - s \leq 29$.))

Table 14.2: Algebra generators for $E_3(S)$ with $t - s \leq 48$

$t-s$	s	g	x	$d_3(x)$	
0	1	0	h_0	0	(1)
1	1	1	h_1	0	(2)
3	1	2	h_2	0	(1)
7	1	3	h_3	0	(1)
8	3	3	c_0	0	(1)
9	5	1	Ph_1	0	(1)
11	5	2	Ph_2	0	(1)
14	4	3	d_0	0	(1)
15	2	7	$h_0 h_4$	$h_0 d_0$	(10)
16	2	8	$h_1 h_4$	0	(3)
16	7	3	Pc_0	0	(1)
17	9	1	$P^2 h_1$	0	(1)
18	2	9	$h_2 h_4$	0	(4)
19	3	9	c_1	0	(1)
19	9	2	$P^2 h_2$	0	(1)

Table 14.2: Algebra generators for $E_3(S)$ with $t - s \leq 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
20	4	8	g_1	0	(1)
22	8	3	Pd_0	0	(1)
23	4	10	h_4c_0	0	(1)
23	9	5	$h_0^2 i$	0	(1)
24	11	3	P^2c_0	0	(1)
25	13	1	P^3h_1	0	(1)
27	13	2	P^3h_2	0	(1)
30	2	10	h_4^2	0	(1)
30	6	10	r		
30	12	3	P^2d_0	0	(1)
31	4	12	$h_0^3h_5$		
31	5	13	n	0	(2)
31	8	10	d_0e_0		
32	2	12	h_1h_5	0	(5)
32	4	13	d_1	0	(2)
32	6	12	q	0	(2)
32	15	3	P^3c_0	0	(1)
33	4	14	p	0	(1)
33	17	1	P^4h_1	0	(1)
34	2	13	h_2h_5		
35	17	2	P^4h_2	0	(1)
36	6	14	t	0	(7)
37	5	17	x	0	(1)
37	8	15	e_0g_1	0	(1)
38	2	14	h_3h_5	0	(6)
38	4	16	e_1	h_1t	(11)
38	16	3	P^3d_0	0	(1)
39	4	18	h_5c_0	0	(1)
39	9	18	u	0	(1)
39	12	9	Pd_0e_0	0	(1)
39	17	5	$h_0^2P^2i$	0	(1)
40	4	19	f_1	0	(12)
40	6	18	Ph_1h_5	0	(13)
40	19	3	P^4c_0	0	(1)

Table 14.2: Algebra generators for $E_3(S)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
41	10	14	z		
41	21	1	P^5h_1	0	(1)
42	6	20	Ph_2h_5	0	(1)
43	21	2	P^5h_2	0	(1)
44	4	22	g_2	0	(1)
45	5	24	h_5d_0	0	(1)
45	9	20	w	0	(1)
46	7	20	B_1	0	(1)
46	8	20	N	0	(1)
46	11	12	$d_0\ell$	0	(1)
46	14	10	i^2	$h_1(Pd_0)^2$	(14)
46	20	3	P^4d_0	0	(1)
47	8	21	Ph_5c_0	0	(9)
47	10	16	e_0r	0	(1)
47	13	15	Pu	0	(1)
47	16	10	$P^2d_0e_0$	0	(1)
47	18	10	h_0^5Q		
48	7	22	B_2	0	(8)
48	23	3	P^5c_0	0	(1)

The d_3-differentials are determined as follows:

(1) The differentials on h_0, h_2, h_3, h_4^2, c_0, c_1, d_0, g_1, h_4c_0, p, h_5c_0, g_2, Ph_1, Ph_2, x, h_5d_0, Ph_2, Pc_0, B_1, Pd_0, e_0g_1, N, P^2h_1, P^2h_2, h_0^2i, u, w, e_0r, P^2c_0, $d_0\ell$, P^2d_0, Pd_0e_0, P^3h_1, P^3h_2, Pu, P^3c_0, P^3d_0, $P^2d_0e_0$, P^4h_1, P^4h_2, $h_0^2P^2i$, P^4c_0, P^4d_0, P^5h_1, P^5h_2 and P^5c_0 are zero because the target bidegrees are trivial.

(2) The differentials on h_1, d_1, n and q are zero by h_0-linearity.

(3) The differential on h_1h_4 is zero or h_1d_0 by h_0-linearity. It cannot be h_1d_0 by d_0-linearity, since $h_1d_0^2 \ne 0$ while $h_1h_4d_0 = 0$.

(4) The differental on h_2h_4 is zero since $h_2^2h_4 = h_3^2$ is a d_3-cycle.

(5) The differential on h_1h_5 is zero or n by h_0-linearity. It cannot be n by h_2-linearity, since $h_2n \ne 0$ while $h_2 \cdot h_1h_5 = 0$.

(6) The differential on h_3h_5 cannot be x by h_1h_6-linearity, since $h_1h_6 \cdot h_3h_5 = 0$ and $d_3(h_1h_6) = 0$ imply $h_1h_6 \cdot d_3(h_3h_5) = 0$, while $h_1h_6 \cdot x = 7_{89}$ cannot be a d_2-boundary, hence is nonzero in $E_3(S)$.

(7) The differential on t cannot be $h_1d_0g_1 = 9_{15}$ by $x' = 10_{18}$-linearity, since $x' \cdot h_1d_0g_1 = 19_{42}$ cannot be a d_2-boundary, while $x' \cdot t = 0$.

(8) The differential on B_2 cannot be $e_0 r$ by d_0-linearity, since $d_0 \cdot e_0 r = 14_{20}$ cannot be a d_2-boundary by h_0-linearity, while $d_0 \cdot B_2 = 11_{27} = h_2 B_{21}$ with $B_{21} = 10_{24}$, and $d_2(B_{21})$ and $d_3(B_{21})$ are trivially zero.

(9) The differential on Ph_5c_0 cannot be $d_0\ell$ by d_0-linearity, since $d_0 \cdot d_0\ell = 15_{17}$ cannot be a d_2-boundary by h_0-linearity, while $d_0 \cdot Ph_5c_0 = 0$.

(10) We use the exact sequence

$$\pi_7(S) \xrightarrow{\sigma} \pi_{14}(S) \xrightarrow{i} \pi_{14}(C\sigma) \xrightarrow{j} \pi_6(S) \xrightarrow{\sigma} \pi_{13}(S)\,.$$

In $E_2(C\sigma)$ we have $d_2(\overline{\overline{h_0^2h_3}}) = i(h_0d_0)$, so $\pi_{14}(C\sigma)$ has order dividing four. Since $\pi_6(S) = \mathbb{Z}/2\{\nu^2\}$ and $\nu^2\sigma = 0$, the image of i has order dividing two. Since $\pi_7(S) = \mathbb{Z}/16\{\sigma\}$ and $2\sigma^2 = 0$ by graded commutativity, $\pi_{14}(S)$ has order dividing four. Hence h_0d_0 and $h_0^2d_0$ must be boundaries, and $d_3(h_0h_4) = h_0d_0$ is the only possibility.

(11) The H_∞ ring structure on S implies $d_3(e_1) = Sq^1(d_2(e_0)) + h_1Sq^2(e_0) = h_1t$, see [**Bru84**, Thm. 4.1] or [**BR**, Thm. 11.54(11)].

(12) The H_∞ ring structure on S implies $d_3(f_1) = Sq^2(d_2(c_1)) + h_1Sq^3(c_1) = 0$, see [**MT67**, §8.7] and [**BR**, Thm. 11.54(12)].

(13) The H_∞ ring structure on S implies $d_3(h_1h_5Ph_1) = Sq^4(d_2(g)) = 0$, so $d_3(h_5Ph_1) = 0$ by h_1-linearity, see [**BR**, Thm. 11.54(13)].

(14) The H_∞ ring structure on S implies $d_3(i^2) = Sq^8(h_0Pd_0) = h_1(Pd_0)^2$, see [**BR**, Thm. 11.54(15)].

14.3. The (E_2, d_2)-term for $S/2$

We define $S/2$ by the homotopy cofiber sequence

$$S \xrightarrow{2} S \xrightarrow{i} S/2 \xrightarrow{j} S^1\,.$$

The E_2-term

$$E_2^{s,t}(S/2) = \mathrm{Ext}_A^{s,t}(M_1, \mathbb{F}_2)$$

for the Adams spectral sequence converging to $\pi_{t-s}(S/2)$ has been calculated for $t \le 150$ using `ext`. A set of $E_2(S)$-module generators for $t - s \le 48$ are listed in Table 14.3, and the (E_2, d_2)-term in this range is shown in Figure 14.3.

Table 14.3: $E_2(S)$-module generators for $E_2(S/2)$ with $t - s \le 48$

$t-s$	s	g	x	$d_2(x)$	
0	0	0	$i(1)$	0	(1)
2	1	1	$\widetilde{h_1}$	0	(1)
7	2	3	$\widetilde{h_2^2}$	0	(1)
8	4	0	$Pi(1)$	0	(1)
9	3	2	$\widetilde{c_0}$	0	(1)
10	5	1	$\widetilde{Ph_1}$	0	(1)
15	3	5	$\widetilde{h_0h_3^2}$	0	(1)
15	6	2	$\widetilde{Ph_2^2}$	0	(1)
16	8	0	$P^2i(1)$	0	(1)

Table 14.3: $E_2(S)$-module generators for $E_2(S/2)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
17	7	2	$P\widetilde{c_0}$	0	(1)
18	9	1	$P^2\widetilde{h_1}$	0	(1)
20	3	9	$\widetilde{c_1}$	0	(2)
23	10	2	$P^2\widetilde{h_2^2}$	0	(1)
24	12	0	$P^3 i(1)$	0	(1)
25	11	2	$P^2\widetilde{c_0}$	0	(1)
26	13	1	$P^3\widetilde{h_1}$	0	(1)
31	5	11	$\widetilde{h_0^3h_4^2}$	0	(1)
31	14	2	$P^3\widetilde{h_2^2}$	0	(1)
32	5	13	$\widetilde{n}$	0	(1)
32	16	0	$P^4 i(1)$	0	(1)
33	4	13	$\widetilde{d_1}$	0	(3)
33	6	11	$\widetilde{q}$	0	(1)
33	15	2	$P^3\widetilde{c_0}$	0	(1)
34	17	1	$P^4\widetilde{h_1}$	0	(1)
37	6	15	$\widetilde{t}$	0	(1)
38	8	9	$\widetilde{e_0g_1 + h_0^3x}$	0	(1)
39	4	17	$\widetilde{e_1}$	$i(h_1x)$	(6)(!)
39	18	2	$P^4\widetilde{h_2^2}$	0	(1)
40	9	11	$\widetilde{u}$	0	(1)
40	20	0	$P^5 i(1)$	0	(1)
41	8	11	$\widetilde{g_1^2}$	$i(h_1u)$	(5)(!)
41	19	2	$P^4\widetilde{c_0}$	0	(1)
42	21	1	$P^5\widetilde{h_1}$	0	(1)
43	9	13	$\widetilde{v}$	$h_1^2\widetilde{u}$	(2)
45	6	23	$\widetilde{h_0^2g_2}$	0	(1)
45	10	13	$\widetilde{d_0r}$	0	(1)
46	9	15	$\widetilde{w}$	0	(1)
46	12	9	$P(\widetilde{e_0g_1+h_0^3x})$	0	(1)
47	7	19	$\widetilde{B_1}$	0	(2)
47	8	14	$\widetilde{N}$	0	(1)
47	11	11	$\widetilde{d_0\ell}$	0	(1)
47	22	2	$P^5\widetilde{h_2^2}$	0	(1)

Table 14.3: $E_2(S)$-module generators for $E_2(S/2)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
48	6	26	$\widetilde{h_0h_2g_2}$	0	(4)
48	10	15	$\widetilde{e_0r}$	0	(1)
48	13	11	$P\widetilde{u}$	0	(1)
48	24	0	$P^6 i(1)$	0	(1)

The $E_2(S)$-module generators are chosen as follows:

- The generators $i(1)$, $\widetilde{h_1}$, $\widetilde{h_2^2}$, $\widetilde{n}$, $\widetilde{q}$, $\widetilde{t}$, $\widetilde{h_0^2g_2}$, $\widetilde{h_0h_2g_2}$, $\widetilde{e_0g_1+h_0^3x}$, $\widetilde{g_1^2}$, $\widetilde{u}$, $\widetilde{v}$, $\widetilde{w}$, $\widetilde{d_0r}$, $\widetilde{e_0r}$ and $\widetilde{d_0\ell}$ are the unique nonzero classes in their respective degrees, and $j(\widetilde{a}) = a$ each case.
- We choose $\widetilde{c_0} = 3_2$, rather than $3_2 + 3_3$, as the lift of c_0.
- We choose $\widetilde{d_1} = 4_{13}$, rather than $4_{13} + 4_{14}$, as the lift of d_1.
- We choose $\widetilde{e_1} = 4_{17}$, rather than $4_{17} + 4_{18}$, as the lift of e_1.
- We choose $\widetilde{h_0^3h_4^2} = 5_{11}$, rather than $5_{11} + 5_{12}$, as the lift of $h_0^3h_4^2$.
- We choose $\widetilde{B_1} = 7_{19}$, rather than $7_{19} + 7_{20}$, as the lift of B_1.
- We choose $\widetilde{N} = 8_{14}$, rather than its sum with 8_{15}, 8_{16} or $8_{15} + 8_{16}$, as the lift of N.
- The generators Pa lie in $\langle h_3, h_0^4, a\rangle$.

The d_2-differentials are determined as follows:

(1) The differentials on $i(1)$, $\widetilde{h_1}$, $\widetilde{h_2^2}$, $\widetilde{c_0}$ $Pi(1)$, $P\widetilde{h_1}$, $\widetilde{h_0^3h_4^2}$, $\widetilde{n}$, $P\widetilde{h_2^2}$, $\widetilde{q}$, $\widetilde{t}$, $\widetilde{h_0^2g_2}$, $P\widetilde{c_0}$, $P^2i(1)$, $\widetilde{e_0g_1+h_0^3x}$, $\widetilde{N}$, $P^2\widetilde{h_1}$, $\widetilde{u}$, $\widetilde{w}$, $P^2\widetilde{h_2^2}$, $\widetilde{d_0r}$, $\widetilde{e_0r}$, $P^2\widetilde{c_0}$, $\widetilde{d_0\ell}$, $P^3i(1)$, $P(\widetilde{e_0g_1+h_0^3x})$, $P^3\widetilde{h_1}$, $P\widetilde{u}$, $P^3\widetilde{h_2^2}$, $P^3\widetilde{c_0}$, $P^4i(1)$, $P^4\widetilde{h_1}$, $P^4\widetilde{h_2^2}$, $P^4\widetilde{c_0}$, $P^5i(1)$, $P^5\widetilde{h_1}$, $P^5\widetilde{h_2^2}$ and $P^6i(1)$ are zero because the target bidegrees are trivial.

(2) The differentials on $\widetilde{c_1}$, $\widetilde{v}$ and $\widetilde{B_1}$ are determined by the results for S and naturality with respect to $j\colon S/2 \to S^1$. ((ETC: For $\widetilde{v}$, note that $h_0z = h_1^2u$.))

(3) The relations $h_4 \cdot \widetilde{d_1} = 0$ and $h_0h_3^2 \cdot \widetilde{d_1} = 0$ imply that $h_4 \cdot d_2(\widetilde{d_1}) = 0$. On the other hand $h_4 \cdot 6_{10} \neq 0$, so $d_2(\widetilde{d_1}) \neq 6_{10}$.

(4) The differential $d_2(h_3c_2) = h_0h_3f_1 = h_0h_2g_2$ lifts over j to $d_2(h_3\widetilde{c_2}) = \widetilde{h_0h_2g_2}$, so $d_2(\widetilde{h_0h_2g_2}) = 0$, since $d_2d_2 = 0$.

(5) With $x' = 10_{18}$, $R_1 = 10_{19}$ and $Q_1 = 10_{22}$ in $E_2(S)$, we have the relation $x' \cdot \widetilde{g_1^2} = 18_{53} = Q_1 \cdot (\widetilde{e_0g_1+h_0^3x})$. Here $d_2(x') = 0$ and $d_2(\widetilde{e_0g_1+h_0^3x}) = 0$, so $x' \cdot d_2(\widetilde{g_1^2}) = d_2(Q_1) \cdot (\widetilde{e_0g_1+h_0^3x})$ in $E_2(S/2)$. From $d_2(P^2j) \neq 0$ we deduce $d_2(R_1) = h_0^2x'$ and $d_2(Q_1) = h_1^2x'$ in $E_2(S)$, by h_0-, h_1 and h_2-linearity. Since $h_1^2x' \cdot (\widetilde{e_0g_1+h_0^3x}) = 20_{42} \neq 0$ in $E_2(S/2)$, it follows that $d_2(\widetilde{g_1^2}) \neq 0$. ((ETC: This detects a hidden 2-extension from g_1^2 to h_1u.))

(6) The relation $h_1 \cdot \widetilde{e_1} = 5_{20} = h_3 \cdot \widetilde{d_1}$ and case (3) shows that $d_2(\widetilde{e_1}) \in \{0, 6_{17}\}$. According to a recent "secondary Steenrod" calculation by Dexter Chua,

$d_2(\widetilde{e_1}) = 6_{17} = i(h_1 x)$ is nonzero. ((ETC: This is subtle, since $d_3(e_1) = h_1 t \neq 0$, as was proved by Bruner.))

14.4. The (E_3, d_3)-term for $S/2$

The E_3-term of the Adams spectral sequence for $S/2$ is calculated as the homology subquotient

$$E_3(S/2) = H(E_2(S/2), d_2)\,.$$

A set of $E_3(S)$-module generators for $t-s \le 48$ are listed in Table 14.4, and the (E_3, d_3)-term in this range is shown in Figure 14.4. ((ETC: d_3's complete from $t-s \le 25$.))

Table 14.4: $E_3(S)$-module generators for $E_3(S/2)$ with $t-s \le 48$

$t-s$	s	g	x	$d_3(x)$	
0	0	0	$i(1)$	0	(1)
2	1	1	$\widetilde{h_1}$	0	(1)
7	2	3	$\widetilde{h_2^2}$	0	(1)
8	4	0	$Pi(1)$	0	(1)
9	3	2	$\widetilde{c_0}$	0	(1)
10	5	1	$P\widetilde{h_1}$	0	(1)
15	1	4	$i(h_4)$	0	(4)
15	3	5	$\widetilde{h_0 h_3^2}$	0	(1)
15	6	2	$P\widetilde{h_2^2}$	0	(1)
16	8	0	$P^2 i(1)$	0	(1)
17	2	8	$h_4\widetilde{h_1}$	0	(4)
17	7	2	$P\widetilde{c_0}$	0	(1)
18	9	1	$P^2\widetilde{h_1}$	0	(1)
20	3	9	$\widetilde{c_1}$	0	(1)
22	3	11	$h_4\widetilde{h_2^2}$	0	(5)
23	10	2	$P^2\widetilde{h_2^2}$	0	(1)
24	4	11	$h_4\widetilde{c_0}$	0	(4)
24	6	6	$e_0\widetilde{h_2^2}$	$h_1 i(Pd_0)$	(7)(!)
24	12	0	$P^3 i(1)$	0	(1)
25	11	2	$P^2\widetilde{c_0}$	0	(1)
26	7	5	$i(j)$		
26	13	1	$P^3\widetilde{h_1}$	0	(1)
31	1	5	$i(h_5)$	0	(1)
31	5	11	$\widetilde{h_0^3 h_4^2}$	0	(1)
31	14	2	$P^3\widetilde{h_2^2}$	0	(1)

Table 14.4: $E_3(S)$-module generators for $E_3(S/2)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
32	5	13	$\widetilde{n}$		
32	10	6	$e_0P\widetilde{h_2^2}$		
32	16	0	$P^4i(1)$	0	(1)
33	2	12	$h_5\widetilde{h_1}$	0	(3)
33	4	13	$\widetilde{d_1}$		
33	6	11	$\widetilde{q}$	0	(1)
33	15	2	$P^3\widetilde{c_0}$	0	(1)
34	11	5	$i(Pj)$		
34	17	1	$P^4\widetilde{h_1}$	0	(1)
37	6	15	$\widetilde{t}$	0	(5)
38	3	17	$h_5\widetilde{h_2^2}$	0	(5)
38	8	9	$\widetilde{e_0g_1 + h_0^3x}$		
39	5	18	$h_5Pi(1)$	0	(6)
39	18	2	$P^4\widetilde{h_2^2}$	0	(1)
40	4	20	$h_5\widetilde{c_0}$		
40	9	11	$\widetilde{u}$		
40	14	6	$e_0P^2\widetilde{h_2^2}$		
40	20	0	$P^5i(1)$	0	(1)
41	3	20	$i(c_2)$	0	(2)
41	6	20	$h_5P\widetilde{h_1}$	0	(5)
41	19	2	$P^4\widetilde{c_0}$	0	(1)
42	15	5	$i(P^2j)$		
42	21	1	$P^5\widetilde{h_1}$	0	(1)
45	6	23	$\widetilde{h_0^2g_2}$	0	(1)
45	10	13	$\widetilde{d_0}r$		
46	4	24	$h_5\widetilde{h_0h_3^2}$	0	(1)
46	7	17	$h_5P\widetilde{h_2^2}$	0	(5)
46	9	15	$\widetilde{w}$		
46	12	9	$P(\widetilde{e_0g_1+h_0^3x})$		
47	7	19	$\widetilde{B_1}$	0	(1)
47	8	14	$\widetilde{N}$		
47	11	11	$\widetilde{d_0}\ell$		
47	22	2	$P^5\widetilde{h_2^2}$	0	(1)

Table 14.4: $E_3(S)$-module generators for $E_3(S/2)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
48	8	17	$h_5P\widetilde{c_0}$	0	(5)
48	10	15	$\widetilde{e_0r}$		
48	13	11	$P\widetilde{u}$		
48	18	6	$e_0P^3\widetilde{h_2^2}$		
48	24	0	$P^6i(1)$	0	(1)

The d_3-differentials are determined as follows:

(1) The differentials on $i(1)$, $\widetilde{h_1}$, $i(h_5)$, $\widetilde{h_2^2}$, $\widetilde{c_0}$, $\widetilde{h_0h_3^2}$, $\widetilde{c_1}$, $Pi(1)$, $h_5\widetilde{h_0h_3^2}$, $P\widetilde{h_1}$, $\widetilde{h_0^3h_4^2}$, $P\widetilde{h_2^2}$, $\widetilde{q}$, $\widetilde{h_0^2g_2}$, $P\widetilde{c_0}$, $\widetilde{B_1}$, $P^2i(1)$, $P^2\widetilde{h_1}$, $P^2\widetilde{h_2^2}$, $P^2\widetilde{c_0}$, $P^3i(1)$, $P^3\widetilde{h_1}$, $P^3\widetilde{h_2^2}$, $P^3\widetilde{c_0}$, $P^4i(1)$, $P^4\widetilde{h_1}$, $P^4\widetilde{h_2^2}$, $P^4\widetilde{c_0}$, $P^5i(1)$, $P^5\widetilde{h_1}$, $P^5\widetilde{h_2^2}$ and $P^6i(1)$ are zero because the target bidegrees are trivial.

(2) The differential on $i(c_2)$ is zero by h_1-linearity.

(3) The differential on $h_5\widetilde{h_1}$ is zero by h_2-linearity.

(4) The differentials on $i(h_4)$, $h_4\widetilde{h_1}$ and $h_4\widetilde{c_0}$ are zero by d_0-linearity.

(5) The differentials on $h_5\widetilde{h_2^2}$, t, $h_5P\widetilde{h_1}$, $h_5P\widetilde{h_2^2}$ and $h_5P\widetilde{c_0}$ are zero by naturality with respect to $j\colon S/2 \to S^1$, using information from Table 14.2.

(6) We have $d_3(j(h_5Pi(1))) = 0$ by h_1-linearity, and $j(\widetilde{e_0g_1+h_0^3}) = e_0g_1 \neq 0$, so the differential on $h_5Pi(1)$ cannot be $\widetilde{e_0g_1+h_0^3}$, hence is zero.

(7) From $d_3(h_2\widehat{f_0}) = i(Pd_0)$ in $E_3(C\eta)$ we deduce that Pd_0 detects $\eta^2\bar{\kappa}$ in $E_\infty(S)$, cf. [**BR**, Thm. 11.71], so that h_1Pd_0 detects $\eta^3\bar{\kappa} = 4\nu\bar{\kappa}$, where $2\nu\bar{\kappa}$ is detected by $h_0h_2g_1$ in Adams filtration 6. The hidden 2-extension from $h_0h_2g_1$ to h_1Pd_0 lifts to S_6 (in a minimal Adams resolution $S_\star$ for S), and shows that $h_1i(Pd_0)$ must be a d_r-boundary in $E_r(S/2)$ for $r \leq 9-6 = 3$. Hence $d_3(e_0\widetilde{h_2}) = h_1i(Pd_0)$. ((ETC: Clarify, or avoid, the use of S_6.))

(8) ((ETC))

14.5. The (E_2, d_2)-term for S/η

We define S/η by the homotopy cofiber sequence

$$S^1 \xrightarrow{\eta} S \xrightarrow{i} S/\eta \xrightarrow{j} S^2 .$$

The E_2-term

$$E_2^{s,t}(S/\eta) = \mathrm{Ext}_A^{s,t}(M_2, \mathbb{F}_2)$$

for the Adams spectral sequence converging to $\pi_{t-s}(S/\eta)$ has been calculated for $t \leq 140$ using `ext`. A set of $E_2(S)$-module generators for $t-s \leq 48$ are listed in Table 14.5, and the (E_2, d_2)-term in this range is shown in Figure 14.5.

Table 14.5: $E_2(S)$-module generators for $E_2(S/\eta)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	

Table 14.5: $E_2(S)$-module generators for $E_2(S/\eta)$ with $t-s \leq 48$

$t-s$	s	g	x	$d_2(x)$	
0	0	0	$i(1)$	0	(1)
2	1	1	$\widehat{h_0}$	0	(1)
5	1	3	$\widehat{h_2}$	0	(1)
11	4	4	$\widehat{h_1 c_0}$	0	(1)
13	5	4	$P\widehat{h_2}$	0	(1)
16	2	10	$\widehat{h_3^2}$	0	(2)
19	8	4	$P\widehat{h_1 c_0}$	0	(1)
20	4	11	$\widehat{f_0}$	$h_0 e_0 \widehat{h_0}$	(6)
21	3	14	$\widehat{c_1}$	0	(3)
21	9	4	$P^2\widehat{h_2}$	0	(1)
25	7	8	$\widehat{i}$	$Pd_0\widehat{h_0}$	(6)
27	12	4	$P^2\widehat{h_1 c_0}$	0	(1)
28	7	10	$\widehat{j}$	$Pe_0\widehat{h_0}$	(6)
29	13	4	$P^3\widehat{h_2}$	0	(1)
31	7	13	$\widehat{k}$	$d_0^2\widehat{h_0}$	(7)
32	6	18	$\widehat{r}$	$i(d_0 e_0)$	(9)(!)
33	3	20	$\widehat{h_1 h_4^2}$	0	(4)
33	5	22	$\widehat{n}$	0	(2)
34	7	18	$\widehat{\ell}$	$d_0 e_0 \widehat{h_0}$	(6)
35	4	22	$\widehat{p}$	0	(5)
35	16	4	$P^3\widehat{h_1 c_0}$	0	(1)
36	11	12	$P\widehat{j}$	$P^2 e_0 \widehat{h_0}$	(6)
37	7	21	$\widehat{m}$	$d_0 g_1 \widehat{h_0}$	(6)
37	17	4	$P^4\widehat{h_2}$	0	(1)
41	15	8	$P^2\widehat{i}$	$P^3 d_0 \widehat{h_0}$	(6)
42	8	30	$\widehat{g_1^2}$	$i(z)$	(10)(!)
43	3	29	$\widehat{c_2}$	$f_1\widehat{h_0}$	(6)
43	10	23	$\widehat{z}$	$i(d_0^3)$	(14)(!)
43	20	4	$P^4\widehat{h_1 c_0}$	0	(1)
44	9	30	$\widehat{v}$	$h_0\widehat{z} + i(d_0 k)$	(12)(!)

Table 14.5: $E_2(S)$-module generators for $E_2(S/\eta)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
44	15	10	$P^2\widehat{j}$	$P^3 e_0 \widehat{h_0}$	(6)
45	21	4	$P^5\widehat{h_2}$	0	(1)
47	5	37	$\widehat{h_1 g_2}$	0	(8)
47	9	32	$\widehat{w}$	$i(d_0\ell)$	(13)(!)
47	12	21	$\widehat{d_0^2 e_0}$	$i(i^2)$	(15)(!)
48	8	34	$\widehat{N}$	0	(11)

The $E_2(S)$-module generators are chosen as follows:

- The generators $i(1)$, $\widehat{h_0}$, $\widehat{h_2}$, $\widehat{h_3^2}$, $\widehat{c_2}$, $\widehat{h_1 c_0}$, $\widehat{p}$, $\widehat{i}$, $\widehat{j}$, $\widehat{\ell}$, $\widehat{g_1^2}$, $\widehat{v}$, $\widehat{w}$, $\widehat{z}$ and $\widehat{d_0^2 e_0}$ are the unique nonzero classes in their respective degrees, and $j(\widehat{a}) = a$ each case.
- We choose $\widehat{c_1} = 3_{14}$, rather than $3_{14} + 3_{15}$, as the lift of c_1.
- Calculation with `ext` shows that 3_{20} is the unique lift of $h_1 h_4^2$.
- Calculation with `ext` shows that 4_{11} is a lift of f_0, together with $4_{10} + 4_{11}$, and we choose the former.
- Calculation with `ext`, or h_0-linearity considerations, show that the unique lift of n is 5_{22}.
- We choose $\widehat{h_1 g_2} = 5_{37}$, rather than $5_{37} + 5_{38}$, as the lift of $h_1 g_2$.
- We choose $\widehat{r} = 6_{18}$, rather than $6_{18} + 6_{19}$, as the lift of r.
- We choose $\widehat{k} = 7_{13}$, rather than $7_{13} + 7_{14}$, as the lift of k.
- We choose $\widehat{m} = 7_{21}$, rather than $7_{21} + 7_{22}$, as the lift of m.
- We choose $\widehat{N} = 8_{34}$, rather than $8_{34} + 8_{35}$, as the lift of N.
- The generators Pa lie in $\langle h_3, h_0^4, a\rangle$, and the generator $P^2 a$ (for $a = \widehat{i}$) lies in $\langle h_4, h_0^8, a\rangle$.

The d_2-differentials are determined as follows:

(1) The differentials on $i(1)$, $\widehat{h_0}$, $\widehat{h_2}$, $\widehat{h_1 c_0}$, $P\widehat{h_2}$, $P\widehat{h_1 c_0}$, $P^2\widehat{h_2}$, $P^2\widehat{h_1 c_0}$, $P^3\widehat{h_2}$, $P^3\widehat{h_1 c_0}$, $P^4\widehat{h_2}$ $P^4\widehat{h_1 c_0}$ and $P^5\widehat{h_2}$ are zero because the target bidegrees are trivial.
(2) The differentials on $\widehat{h_3^2}$, $\widehat{n}$ are zero by h_0- or h_0^2-linearity.
(3) The differential on $\widehat{c_1}$ is zero by h_0-linearity, naturality with respect to $j\colon S/\eta \to S^2$, and $d_2(c_1) = 0 \neq h_0 f_0$.
(4) The differential on $\widehat{h_1 h_4^2}$ is zero by j-naturality.
(5) The Leibniz rule and the relations $h_4 \widehat{p} = 0$ and $h_5 \widehat{p} = 0$ imply $h_5 \cdot d_2(\widehat{p}) = 0$, while $h_5 \cdot h_1 \widehat{n} = 7_{58} \neq 0$, so $d_2(\widehat{p}) \neq h_1 \widehat{n}$.
(6) The differentials on $\widehat{c_2}$, $\widehat{f_0}$, $\widehat{i}$, $\widehat{j}$, $\widehat{\ell}$, $\widehat{m}$, $P\widehat{j}$, $P^2\widehat{i}$ and $P^2\widehat{j}$ follow from those for S by j-naturality.
(7) The differential on $\widehat{k}$ follows from that on $\widehat{j}$ by h_0- and h_2-linearity.
(8) Since $x' \cdot \widehat{h_1 g_2} = 0$ and $x' \cdot i(B_1) = 17_{106} \neq 0$, where $x' = 10_{18}$ is a d_2-cycle, we cannot have $d_2(\widehat{h_1 g_2}) = i(B_1)$.

(9) The differential on $\widehat{r}$ follows from that on $\widehat{\ell}$ by h_1- and h_2-linearity, combined with j-naturality and h_0-linearity. ((ETC: This is subtle, since $d_2(r) = 0$ and $d_3(d_0e_0) \neq 0$.))

(10) From $d_2(r) = 0$ and $d_2(\widehat{r}) = i(d_0e_0)$ we deduce $d_2(r\widehat{r}) = 14_{33} \neq 0$. Here $r\widehat{r} = g_1\widehat{g_1^2} + 12_{43}$, where $d_2(12_{43}) = 0$ by h_0-linearity. Hence $d_2(g_1\widehat{g_1^2}) \neq 0$, which implies $d_2(\widehat{g_1^2}) \neq 0$ by g_1-linearity. ((ETC: This detects a hidden η-extension from g_1^2 to z.))

(11) Since $d_0 \cdot \widehat{N} = 0$ and $d_0 \cdot i(e_0r) = 14_{33} \neq 0$ we cannot have $d_2(\widehat{N}) = i(e_0r)$.

(12) From $d_2(r) = 0$ and $d_2(\widehat{k}) = d_0^2\widehat{h_0}$ we deduce $d_2(r\widehat{k}) = 15_{30} \neq 0$. Here $r\widehat{k} = e_0\widehat{v} + 13_{38}$, where $d_2(13_{38}) = 0$ by h_0-linearity. Hence $d_2(e_0\widehat{v}) = h_1^2d_0\widehat{v} + e_0d_2(\widehat{v}) \neq 0$. Since $h_1\widehat{v} = 0$, it follows that $d_2(\widehat{v}) \neq 0$, and by h_0-linearity, its value must be $11_{17} = h_0\widehat{z} + i(d_0k)$. ((ETC: This is subtle, since $d_2(v) = h_0z$ and $d_2(d_0k) \neq 0$.))

(13) From $d_2(r) = 0$ and $d_2(\widehat{k}) = d_0^2\widehat{h_0}$ we deduce $d_2(r\widehat{k}) = 15_{30} \neq 0$. Here $r\widehat{k} = d_0\widehat{w} + 13_{38}$, where $d_2(13_{38}) = 0$ by h_0-linearity. Hence $d_2(d_0\widehat{w}) \neq 0$, which implies $d_2(\widehat{w}) \neq 0$. ((ETC: This detects a hidden η-extension from w to $d_0\ell$.))

(14) From $h_0 \cdot \widehat{z} = 11_{18} = f_0 \cdot \widehat{i}$ we calculate that $h_0 \cdot d_2(\widehat{z}) = d_2(f_0) \cdot \widehat{i} + f_0 \cdot d_2(\widehat{i}) = 13_{17}$, so $d_2(\widehat{z}) = 12_{16} = i(d_0^3)$. ((ETC: This detects a hidden η-extension from z to d_0^3.))

(15) The relation $h_0 \cdot \widehat{d_0^2e_0} = 13_{22} = i(Q)$ and the differential $d_2(Q) = h_0i^2$ imply $h_0 \cdot d_2(\widehat{d_0^2e_0}) = i(h_0i^2) = 15_{12} \neq 0$, so $d_2(\widehat{d_0^2e_0}) \neq 0$. ((ETC: This is subtle, since $d_4(d_0^2e_0) = d_0P^2d_0 \neq 0$.))

14.6. The (E_3, d_3)-term for S/η

The E_3-term of the Adams spectral sequence for S/η is calculated as the homology subquotient

$$E_3(S/\eta) = H(E_2(S/\eta), d_2)\,.$$

A set of $E_3(S)$-module generators for $t - s \leq 48$ are listed in Table 14.6, and the (E_3, d_3)-term in this range is shown in Figure 14.6. ((ETC: d_3's complete from $t - s \leq 25$.))

Table 14.6: $E_3(S)$-module generators for $E_3(S/\eta)$ with $t - s \leq 48$

$t-s$	s	g	x	$d_3(x)$	
0	0	0	$i(1)$	0	(1)
2	1	1	$\widehat{h_0}$	0	(1)
5	1	3	$\widehat{h_2}$	0	(1)
11	4	4	$\widehat{h_1c_0}$	0	(1)
13	5	4	$P\widehat{h_2}$	0	(1)
16	2	10	$\widehat{h_3^2}$	0	(2)
17	2	11	$h_4\widehat{h_0}$	$d_0\widehat{h_0}$	(3)
17	4	7	$i(e_0)$	0	(4)(!)

Table 14.6: $E_3(S)$-module generators for $E_3(S/\eta)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
19	8	4	$P\widehat{h_1c_0}$	0	(1)
20	2	13	$h_4\widehat{h_2}$	0	(3)
21	3	14	$\widehat{c_1}$	0	(5)(!)
21	9	4	$P^2\widehat{h_2}$	0	(1)
23	5	14	$h_2\widehat{f_0}$	$i(Pd_0)$	(6)(!)
26	5	17	$h_4\widehat{h_1c_0}$		
27	12	4	$P^2\widehat{h_1c_0}$	0	(1)
29	13	4	$P^3\widehat{h_2}$	0	(1)
33	3	20	$\widehat{h_1h_4^2}$	0	(1)
33	5	22	$\widehat{n}$	0	(2)
34	9	22	$h_0^2\widehat{\ell}$		
35	4	22	$\widehat{p}$	0	(1)
35	16	4	$P^3\widehat{h_1c_0}$	0	(1)
36	2	18	$h_5\widehat{h_2}$		
37	17	4	$P^4\widehat{h_2}$	0	(1)
39	13	15	$h_0h_2P\widehat{j}$		
40	9	28	$h_0h_2\widehat{m}$		
41	4	29	$i(h_0c_2)$	0	(1)
42	5	31	$h_5\widehat{h_1c_0}$	0	(1)
42	13	18	$h_2^2P\widehat{j}$		
43	20	4	$P^4\widehat{h_1c_0}$	0	(1)
44	6	31	$h_5P\widehat{h_2}$	0	(1)
45	21	4	$P^5\widehat{h_2}$	0	(1)
47	3	31	$h_5\widehat{h_3^2}$		
47	5	37	$\widehat{h_1g_2}$		
47	17	16	$h_0h_2P^2\widehat{j}+i(h_0^4Q)$		
48	5	39	$i(h_5e_0)$		
48	8	34	$\widehat{N}$	0	(1)

The d_3-differentials are determined as follows:

(1) The differentials on $i(1)$, $\widehat{h_0}$, $\widehat{h_2}$, $\widehat{h_1c_0}$, $P\widehat{h_2}$, $P\widehat{h_1c_0}$, $P^2\widehat{h_2}$, $P^2\widehat{h_1c_0}$, $P^3\widehat{h_2}$, $\widehat{h_1h_4^2}$, $\widehat{p}$, $P^3\widehat{h_1c_0}$, $P^4\widehat{h_2}$, $i(h_0c_2)$, $h_5\widehat{h_1c_0}$, $P^4\widehat{h_1c_0}$, $h_5P\widehat{h_2}$, $P^5\widehat{h_2}$ and $\widehat{N}$ are zero because the target bidegrees are trivial.

(2) The differentials on $\widehat{h_3^2}$ and $\widehat{n}$ are zero by h_0-linearity.

(3) The differentials on $h_4\widehat{h_0}$, $h_4\widehat{h_2}$ follow by naturality with respect to $j\colon C\eta \to S^2$ and known differentials in $E_3(S)$.
(4) See [**BR**, Lem. 11.72] for a proof that $d_3(i(e_0)) = 0$.
(5) See [**BR**, Lem. 11.73] for a proof that $d_3(\widehat{c_1}) = 0$.
(6) See [**BR**, Lem. 11.74] for a proof that $d_3(h_2\widehat{f_0}) = i(Pd_0)$.
(7) ((ETC))

14.7. The (E_2, d_2)-term for S/ν

We define S/ν by the homotopy cofiber sequence

$$S^3 \xrightarrow{\nu} S \xrightarrow{i} S/\nu \xrightarrow{j} S^4 .$$

The E_2-term

$$E_2^{s,t}(S/\nu) = \mathrm{Ext}_A^{s,t}(M_4, \mathbb{F}_2)$$

for the Adams spectral sequence converging to $\pi_{t-s}(S/\nu)$ has been calculated for $t \le 140$ using `ext`. A set of $E_2(S)$-module generators for $t - s \le 48$ are listed in Table 14.7, and the E_2-term in this range is shown in Figure 14.7.

Table 14.7: $E_2(S)$-module generators for $E_2(S/\nu)$ with $t - s \le 48$

$t-s$	s	g	x	$d_2(x)$	
0	0	0	$i(1)$	0	(1)
4	3	1	$\overline{h_0^3}$	0	(1)
5	1	2	$\overline{h_1}$	0	(2)
7	2	3	$\overline{h_0h_2}$	0	(1)
11	1	4	$\overline{h_3}$	0	(1)
12	3	6	$\overline{c_0}$	0	(2)
13	5	4	$P\overline{h_1}$	0	(1)
15	6	5	$P\overline{h_0h_2}$	0	(1)
20	7	7	$P\overline{c_0}$	0	(2)
21	9	4	$P^2\overline{h_1}$	0	(1)
23	10	5	$P^2\overline{h_0h_2}$	0	(1)
26	4	14	$\overline{h_2c_1}$	0	(1)
27	9	8	$P^2\overline{h_3}$	0	(1)
28	11	7	$P^2\overline{c_0}$	0	(2)
29	13	4	$P^3\overline{h_1}$	0	(1)
30	6	13	$\overline{h_2^2g_1}$	0	(1)
31	14	5	$P^3\overline{h_0h_2}$	0	(1)
34	2	16	$\overline{h_4^2}$	$i(p)$	(3)(!)
34	7	15	$\overline{h_0r}$	0	(5)
36	6	19	$\overline{q}$	0	(2)
36	15	7	$P^3\overline{c_0}$	0	(2)

Table 14.7: $E_2(S)$-module generators for $E_2(S/\nu)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
37	4	22	$\overline{p}$	0	(4)
37	17	4	$P^4\overline{h_1}$	0	(1)
39	18	5	$P^4\overline{h_0h_2}$	0	(1)
41	5	27	$\overline{x}$	0	(1)
41	8	19	$\overline{e_0g_1}$	$i(h_1u)$	(6)(!)
42	6	$25+26$	$\overline{y}$	$h_0^3\overline{x}$	(5)
43	9	22	$\overline{u}$	0	(1)
43	17	8	$P^4\overline{h_3}$	0	(1)
44	4	$31+32$	$\overline{f_1}$	0	(1)
44	8	23	$\overline{g_1^2}$	0	(1)
44	19	7	$P^4\overline{c_0}$	0	(2)
45	3	29	$\overline{c_2}$	$h_0\overline{f_1}$	(7)(!)
45	10	23	$\overline{z}$	0	(1)
45	21	4	$P^5\overline{h_1}$	0	(1)
46	9	24	$\overline{v}$	$h_0\overline{z}$	(5)
46	12	15	$\overline{d_0^3}$	0	(1)
47	11	20	$\overline{d_0k}$	$h_0\overline{d_0^3}$	(5)
47	22	5	$P^5\overline{h_0h_2}$	0	(1)
48	10	25	$\overline{d_0r}$	0	(2)

The $E_2(S)$-module generators are chosen as follows:

- The generators $i(1)$, $\overline{h_1}$, $\overline{h_3}$, $\overline{h_4^2}$, $\overline{h_0^3}$, $\overline{c_0}$, $\overline{h_2c_1}$, $\overline{p}$, $\overline{x}$, $\overline{h_0r}$, $\overline{g_1^2}$, $\overline{u}$, $\overline{v}$, $\overline{z}$, $\overline{d_0r}$, $\overline{d_0k}$ and $\overline{d_0^3}$ are the unique nonzero classes in their respective degrees, and $j(\overline{a}) = a$ each case.
- We choose $\overline{h_0h_2} = 2_3$, rather than $2_3 + 2_4$, as the lift of h_0h_2.
- We choose $\overline{c_2} = 3_{29}$, rather than $3_{29} + 3_{30}$, as the lift of c_2.
- Calculation with `ext` shows that 4_{32} is a lift of f_1, together with $4_{31} + 4_{32}$, and we choose the latter.
- We choose $\overline{h_2^2g_1} = 6_{13}$, rather than 6_{14}, as the lift of $h_2^2g_1$.
- We choose $\overline{q} = 6_{19}$, rather than $6_{18} + 6_{19}$, as the lift of q.
- Calculation with `ext` shows that 6_{25} maps to $y + h_1x$ and 6_{26} maps to h_1x, while $6_{27} = i(h_5Ph_2)$, and we choose $\overline{y} = 6_{25} + 6_{26}$, rather than $6_{25} + 6_{26} + 6_{27}$, as the lift of y. ((ETC: Alternatively, we might choose 6_{25} as the lift of $y + h_1x$.))
- Calculation with `ext` shows that $\overline{e_0g_1} = 8_{19}$ is the unique lift of e_0g_1.
- The generators Pa lie in $\langle h_3, h_0^4, a\rangle$, and the generator P^2a (for $a = \overline{h_3}$) lies in $\langle h_4, h_0^8, a\rangle$.

The d_2-differentials are determined as follows:

(1) The differentials on $i(1)$, $\overline{h_3}$, $\overline{h_0h_2}$, $\overline{h_0^3}$, $\overline{h_2c_1}$, $\overline{f_1}$, $P\overline{h_1}$, $\overline{x}$, $P\overline{h_0h_2}$, $\overline{g_1^2}$, $P^2\overline{h_1}$, $P^2\overline{h_3}$, $\overline{u}$, $P^2\overline{h_0h_2}$, $\overline{z}$, $\overline{d_0^3}$, $P^3\overline{h_1}$, $\overline{h_2^2h_1}$, $P^3\overline{h_0h_2}$, $P^4\overline{h_1}$, $P^4\overline{h_3}$, $P^4\overline{h_0h_2}$, $P^5\overline{h_1}$ and $P^5\overline{h_0h_2}$ are zero because the target bidegrees are trivial.

(2) The differentials on $\overline{h_1}$, $\overline{c_0}$, $\overline{q}$, $P\overline{c_0}$, $\overline{d_0r}$, $P^2\overline{c_0}$, $P^3\overline{c_0}$ and $P^4\overline{c_0}$ are zero by h_0-linearity.

(3) The relation $h_3 \cdot \overline{h_4^2} = i(c_2)$ and the differential $d_2(c_2) = h_0f_1$ for S imply $d_2(\overline{h_4^2}) \neq 0$, and $4_{19} = i(p)$ is the only possible value. ((ETC: This detects a hidden ν-extension from h_4^2 to p.))

(4) By h_1-linearity, $d_2(\overline{p}) \in \{0, i(t)\}$. Since $g_1 \cdot \overline{p} = 0$ and $g_1 \cdot i(t) = 10_{32} \neq 0$, we must have $d_2(\overline{p}) \neq i(t)$.

(5) Linearity with respect to $j \colon S/\nu \to S^4$ determines the differentials on $\overline{y}$, $\overline{h_0r}$, $\overline{v}$ and $\overline{d_0k}$.

(6) The relation $h_1 \cdot \overline{e_0g_1} = h_0^3 \cdot \overline{y}$ and $d_2(\overline{y} = h_0^3\overline{x}$ implies $d_2(\overline{e_0g_1}) \neq 0$. ((ETC: This is subtle, since $d_2(e_0g_1) = 0$.))

(7) Since $d_2(c_2) = h_0f_1$ we have $d_2(\overline{c_2}) \in \{5_{32}, 5_{31} + 5_{32}\}$. This is $h_0\overline{f_1}$ or its sum with $i(h_0g_2)$. According to a recent "secondary Steenrod" calculation by Dexter Chua, $h_0 \cdot d_2(\overline{f_1}) = h_1 \cdot 5_{29}$, where $5_{29} = d_1\widetilde{h_3} = e_1\widetilde{h_1}$, so $d_2(\overline{c_2}) = 5_{31} + 5_{32} = h_0\overline{f_1}$.

14.8. The (E_3, d_3)-term for S/ν

The E_3-term of the Adams spectral sequence for S/ν is calculated as the homology subquotient

$$E_3(S/\nu) = H(E_2(S/\nu), d_2)\,.$$

A set of $E_3(S)$-module generators for $t - s \leq 48$ are listed in Table 14.8, and the (E_3, d_3)-term in this range is shown in Figure 14.8. ((ETC: d_3's complete from $t - s \leq 28$.))

Table 14.8: $E_3(S)$-module generators for $E_3(S/\nu)$ with $t - s \leq 48$

$t-s$	s	g	x	$d_3(x)$	
0	0	0	$i(1)$	0	(1)
4	3	1	$\overline{h_0^3}$	0	(1)
5	1	2	$\overline{h_1}$	0	(2)
7	2	3	$\overline{h_0h_2}$	0	(1)
11	1	4	$\overline{h_3}$	0	(1)
12	3	6	$\overline{c_0}$	0	(2)
13	5	4	$P\overline{h_1}$	0	(1)
15	6	5	$P\overline{h_0h_2}$	0	(1)
19	4	10	$h_4\overline{h_0^3}$	0	(1)
20	2	12	$h_4\overline{h_1}$	0	(3)
20	7	7	$P\overline{c_0}$	0	(2)
21	9	4	$P^2\overline{h_1}$	0	(1)

Table 14.8: $E_3(S)$-module generators for $E_3(S/\nu)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
22	3	14	$h_4\overline{h_0h_2}$	0	(4)
23	8	7	$i(h_0i)$	0	(1)
23	10	5	$P^2\overline{h_0h_2}$	0	(1)
24	6	11	$e_0\overline{h_0h_2}$	$h_1i(Pd_0)$	(5)(!)
26	4	14	$\overline{h_2c_1}$	0	(1)
27	4	15	$h_4\overline{c_0}$	0	(4)
27	9	8	$P^2\overline{h_3}$	0	(1)
28	11	7	$P^2\overline{c_0}$	0	(2)
29	7	11	$i(k)$		
29	13	4	$P^3\overline{h_1}$	0	(1)
30	6	13	$\overline{h_2^2g_1}$	0	(1)
31	14	5	$P^3\overline{h_0h_2}$	0	(1)
32	10	13	$Pe_0\overline{h_0h_2}$		
34	4	20	$h_0^2\overline{h_4^2}$	0	(1)
34	7	15	$\overline{h_0r}$	0	(1)
35	4	21	$h_5\overline{h_0^3}$		
36	2	17	$h_5\overline{h_1}$	0	(2)
36	6	19	$\overline{q}$	0	(2)
36	15	7	$P^3\overline{c_0}$	0	(2)
37	4	22	$\overline{p}$	0	(1)
37	11	14	$i(d_0i)$		
37	17	4	$P^4\overline{h_1}$	0	(1)
38	3	23	$h_5\overline{h_0h_2}$		
38	7	20	$i(h_0y)$	0	(1)
39	16	7	$i(h_0P^2i)$	0	(1)
39	18	5	$P^4\overline{h_0h_2}$	0	(1)
40	14	11	$P^2e_0\overline{h_0h_2}$		
41	5	27	$\overline{x}$		
42	2	19	$h_5\overline{h_3}$		
43	4	30	$h_5\overline{c_0}$	0	(2)
43	9	22	$\overline{u}$		
43	17	8	$P^4\overline{h_3}$	0	(1)
44	4	$31+32$	$\overline{f_1}$		

Table 14.8: $E_3(S)$-module generators for $E_3(S/\nu)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
44	8	23	$\overline{g_1^2}$		
44	19	7	$P^4\overline{c_0}$	0	(2)
45	10	23	$\overline{z}$		
45	15	11	$i(Pd_0 i)$		
45	21	4	$P^5\overline{h_1}$	0	(1)
46	7	28	$h_5 P\overline{h_0 h_2}$		
46	12	15	$\overline{d_0^3}$		
47	22	5	$P^5\overline{h_0 h_2}$	0	(1)
48	4	36	$i(h_3 c_2)$	0	(1)
48	10	25	$\overline{d_0 r}$		
48	18	13	$P^3 e_0\overline{h_0 h_2}$		

The d_3-differentials are determined as follows:

(1) The differentials on $i(1)$, $\overline{h_0^3}$, $\overline{h_0 h_2}$, $\overline{h_3}$, $P\overline{h_1}$, $P\overline{h_0 h_2}$, $h_4\overline{h_0^3}$, $P^2\overline{h_1}$, $i(h_0 i)$, $P^2\overline{h_0 h_2}$, $\overline{h_2 c_1}$, $P^2\overline{h_3}$, $P^3\overline{h_1}$, $\overline{h_2^2 g_1}$, $P^3\overline{h_0 h_2}$, $h_0^2\overline{h_4^2}$, $\overline{h_0 r}$, $\overline{p}$, $P^4\overline{h_1}$, $i(h_0 y)$, $i(h_0 P^2 i)$, $P^4\overline{h_0 h_2}$, $P^4\overline{h_3}$, $P^5\overline{h_1}$, $P^5\overline{h_0 h_2}$ and $i(h_3 c_2)$ are zero because the target bidegrees are trivial.

(2) The differentials on $\overline{h_1}$, $\overline{c_0}$, $P\overline{c_0}$, $P^2\overline{c_0}$, $h_5\overline{h_1}$, $\overline{q}$, $P^3\overline{c_0}$, $h_5\overline{c_0}$ and $P^4\overline{c_0}$ vanish by h_0-linearity.

(3) The differential on $h_4\overline{h_1}$ vanishes by naturality with respect to $j\colon C\nu \to S^4$.

(4) The differentials on $h_4\overline{h_0 h_2}$ and $h_4\overline{c_0}$ are zero by d_0-linearity. In the first case, $d_0 \cdot d_0\overline{h_0 h_2} = 10_{15} \ne 0$ in $E_3(C\nu)$, while $d_0 \cdot h_4\overline{h_0 h_2} = 0$. In the second case, $d_0 \cdot d_0\overline{c_0} = 11_{15} \ne 0$ in $E_3(C\nu)$, while $d_0 \cdot h_4\overline{c_0} = 0$.

(5) From $d_3(h_2\widehat{f_0}) = i(Pd_0)$ in $E_3(C\eta)$ we deduce that Pd_0 detects $\eta^2\bar\kappa$ in $E_\infty(S)$, cf. [**BR**, Thm. 11.71], so that $h_1 P d_0$ detects $\eta^3\bar\kappa = 4\nu\bar\kappa$, where $4\bar\kappa$ is detected by $h_0^2 g_1$ in Adams filtration 6. ((ETC: The hidden ν-extension from $h_0^2 g_1$ to $h_1 P d_0$ lifts to S_6 (in a minimal Adams resolution $S_\star$ for S), and shows that $h_1 i(Pd_0)$ must be a d_r-boundary in $E_r(S/\nu)$ for $r \le 9-6 = 3$. Hence $d_3(e_0\overline{h_0 h_2}) = h_1 i(Pd_0)$.)) Since $i(4\nu\bar\kappa) = 0$ in $\pi_*(C\nu)$ it follows that $h_1 i(Pd_0)$ is a boundary. From case (4) and h_1-linearity the only possible source of this differential is $e_0\overline{h_0 h_2}$.

(6) ((ETC))

14.9. The (E_2, d_2)-term for S/σ

We define S/σ by the homotopy cofiber sequence

$$S^7 \xrightarrow{\sigma} S \xrightarrow{i} S/\sigma \xrightarrow{j} S^8 .$$

The E_2-term

$$E_2^{s,t}(S/\sigma) = \mathrm{Ext}_A^{s,t}(M_4, \mathbb{F}_2)$$

for the Adams spectral sequence converging to $\pi_{t-s}(S/\sigma)$ has been calculated for $t \leq 140$ using `ext`. A set of $E_2(S)$-module generators for $t - s \leq 48$ are listed in Table 14.9, and the (E_2, d_2)-term in this range is shown in Figure 14.9.

Table 14.9: $E_2(S)$-module generators for $E_2(S/\sigma)$ with $t - s \leq 48$

$t-s$	s	g	x	$d_2(x)$	
0	0	0	$i(1)$	0	(1)
8	4	1	$\overline{\overline{h_0^4}}$	0	(1)
11	1	3	$\overline{\overline{h_2}}$	0	(1)
15	3	4	$\overline{\overline{h_0^2h_3}}$	$i(h_0d_0)$	(4)(!)
16	2	7	$\overline{\overline{h_1h_3}}$	0	(2)
16	3	6	$\overline{\overline{c_0}}$	0	(5)
19	5	9	$P\overline{\overline{h_2}}$	0	(1)
22	3	11	$\overline{\overline{h_0h_3^2}}$	0	(7)
22	4	11	$\overline{\overline{d_0}}$	0	(1)
23	1	5	$\overline{\overline{h_4}}$	$\overline{\overline{h_0h_3^2}}$	(6)
24	7	8	$P\overline{\overline{c_0}}$	0	(5)
26	4	15	$\overline{\overline{f_0}}$	$h_0h_2\overline{\overline{d_0}}$	(6)
27	3	15	$\overline{\overline{c_1}}$	0	(2)
27	9	9	$P^2\overline{\overline{h_2}}$	0	(1)
28	4	17	$\overline{\overline{g_1}}$	0	(1)
30	8	12	$P\overline{\overline{d_0}}$	0	(1)
32	11	8	$P^2\overline{\overline{c_0}}$	0	(5)
33	8	16	$\overline{\overline{Pe_0}}$	$h_1^2P\overline{\overline{d_0}}$	(6)
34	7	16	$\overline{\overline{j}}$	$h_0\overline{\overline{Pe_0}}$	(6)
35	13	9	$P^3\overline{\overline{h_2}}$	0	(1)
37	7	18	$\overline{\overline{k}}$	$h_0d_0\overline{\overline{d_0}}$	(6)
38	12	12	$P^2\overline{\overline{d_0}}$	0	(1)
40	7	22	$\overline{\overline{\ell}}$	$h_0e_0\overline{\overline{d_0}}$	(6)
40	15	8	$P^3\overline{\overline{c_0}}$	0	(5)
41	12	16	$P\overline{\overline{Pe_0}}$	$h_1^2P^2\overline{\overline{d_0}}$	(6)
42	11	16	$P\overline{\overline{j}}$	$h_0P\overline{\overline{Pe_0}}$	(6)
43	7	26	$\overline{\overline{m}}$	$h_0g_1\overline{\overline{d_0}}$	(8)
43	17	9	$P^4\overline{\overline{h_2}}$	0	(1)
44	6	29	$\overline{\overline{t}}$	0	(2)

Table 14.9: $E_2(S)$-module generators for $E_2(S/\sigma)$ with $t-s \le 48$ (cont.)

$t-s$	s	g	x	$d_2(x)$	
45	6	30	$\overline{\overline{h_0x}}$	0	(1)
46	7	29	$\overline{\overline{h_0y}}$	$h_0^3\overline{\overline{h_0x}}$	(9)
46	16	12	$P^3\overline{\overline{d_0}}$	0	(1)
47	5	36	$\overline{\overline{h_1e_1}}$	0	(3)
47	9	34	$\overline{\overline{u}}$	0	(10)
48	19	8	$P^4\overline{\overline{c_0}}$	0	(5)

The $E_2(S)$-module generators are chosen as follows:

- The generators $i(1)$, $\overline{\overline{h_2}}$, $\overline{\overline{h_4}}$, $\overline{\overline{c_0}}$, $\overline{\overline{h_0h_3^2}}$, $\overline{\overline{c_1}}$, $\overline{\overline{h_0^4}}$, $\overline{\overline{g_1}}$, $\overline{\overline{h_1e_1}}$, $\overline{\overline{t}}$, $\overline{\overline{j}}$, $\overline{\overline{\ell}}$, $\overline{\overline{m}}$, $\overline{\overline{Pe_0}}$ and $\overline{\overline{u}}$ are the unique nonzero classes in their respective degrees, and $j(\overline{\overline{a}}) = a$ each case.
- We choose $\overline{\overline{h_1h_3}} = 2_7$, rather than $2_7 + 2_8$, as the lift of h_1h_3.
- We choose $\overline{\overline{h_0^2h_3}} = 3_4$, rather than $3_4 + 3_5$, as the lift of $h_0^2h_3$.
- We choose $\overline{\overline{d_0}} = 4_{11}$, rather than $4_{11} + 4_{12}$, as the lift of d_0.
- Calculation with `ext` shows that $\overline{\overline{f_0}} = 4_{15}$ is the unique lift of f_0.
- We choose $\overline{\overline{h_0x}} = 6_{30}$, rather than $6_{30} + 6_{31}$, as the lift of h_0x.
- We choose $\overline{\overline{k}} = 7_{18}$, rather than $7_{18} + 7_{19}$, as the lift of k.
- We choose $\overline{\overline{h_0y}} = 7_{29}$, rather than $7_{29} + 7_{30}$, as the lift of h_0y.
- The generators Pa lie in $\langle h_3, h_0^4, a\rangle$. ((ETC: Note that $\overline{\overline{Pe_0}}$ is not an instance of this, but $P\overline{\overline{Pe_0}}$ is.))

The d_2-differentials are determined as follows:

(1) The differentials on $i(1)$, $\overline{\overline{h_2}}$, $\overline{\overline{h_0^4}}$, $\overline{\overline{d_0}}$, $\overline{\overline{g_1}}$, $P\overline{\overline{h_2}}$, $\overline{\overline{h_0x}}$, $P\overline{\overline{d_0}}$, $P^2\overline{\overline{h_2}}$, $P^2\overline{\overline{d_0}}$, $P^3\overline{\overline{h_2}}$, $P^3\overline{\overline{d_0}}$ and $P^4\overline{\overline{h_2}}$ are zero because the target bidegrees are trivial.

(2) The differentials on $\overline{\overline{h_1h_3}}$, $\overline{\overline{c_1}}$, $\overline{\overline{t}}$ are zero by h_0-linearity.

(3) The differential on $\overline{\overline{h_1e_1}}$ is zero by h_1^2-linearity.

(4) From $h_2 \cdot \overline{\overline{h_0^2h_3}} = 4_8 = i(f_0)$ and $d_2(f_0) = h_0^2e_0$ with $i(h_0^2e_0) = 6_6 \ne 0$ we deduce $d_2(\overline{\overline{h_0^2h_3}}) = 5_4 = i(h_0d_0)$. ((ETC: This intervenes before the image of $d_3(h_0h_4) = h_0d_0$.))

(5) The differential on $\overline{\overline{c_0}}$ is zero by g_1-linearity, since $g_1 \cdot 5_5 = 9_{20} \ne 0$ while $g_1 \cdot \overline{\overline{c_0}} = 0$, and similarly for $P^i\overline{\overline{c_0}}$ with $i \ge 1$.

(6) The differentials on $\overline{\overline{h_4}}$, $\overline{\overline{f_0}}$ ((ETC: Using $h_2d_0 = h_0e_0$)), $\overline{\overline{j}}$, $\overline{\overline{k}}$, $\overline{\overline{\ell}}$ ((ETC: using h_1-linearity)), $\overline{\overline{Pe_0}}$, $P\overline{\overline{j}}$, $P\overline{\overline{Pe_0}}$ are given by naturality with respect to $j\colon S/\sigma \to S^8$.

(7) It follows from case (6) that $d_2(\overline{\overline{h_0h_3^2}}) = 0$, since $d_2d_2 = 0$.

(8) By j-naturality, $d_2(\overline{\overline{m}}) \equiv h_0g_1\overline{\overline{d_0}} \mod i(v)$, and the relations $h_0h_3^2\cdot\overline{\overline{m}} = 0$, $h_4\cdot\overline{\overline{m}} = 0$, $h_4\cdot h_0g_1\overline{\overline{d_0}} = 0$ and $h_4\cdot i(v) = 10_{35} \ne 0$ imply that the summand $i(v)$ is not present in this differential.

(9) By j-naturality, $d_2(\overline{\overline{h_0 y}}) \equiv h_0^3\overline{\overline{h_0 x}} \mod i(w)$, and the relations $n\cdot\overline{\overline{h_0 y}} = 0$, $n \cdot h_0^3\overline{\overline{h_0 x}} = 0$ and and $n \cdot i(w)) = 14_{58} \neq 0$ imply that the summand $i(w)$ is not present in this differential.

(10) The relation $d_0\overline{\overline{u}} = 13_{38} = u\overline{\overline{d_0}}$ implies $d_0 \cdot d_2(\overline{\overline{u}}) = 0$, while $d_0 \cdot i(d_0\ell) = 15_{24} \neq 0$, so $d_2(\overline{\overline{u}}) \neq i(d_0\ell)$ must be zero.

14.10. The (E_3, d_3)-term for S/σ

The E_3-term of the Adams spectral sequence for S/σ is calculated as the homology subquotient

$$E_3(S/\sigma) = H(E_2(S/\sigma), d_2)\,.$$

A set of $E_3(S)$-module generators for $t - s \leq 48$ are listed in Table 14.10, and the (E_3, d_3)-term in this range is shown in Figure 14.10. ((ETC: d_3's complete from $t - s \leq 24$.))

Table 14.10: $E_3(S)$-module generators for $E_3(S/\sigma)$ with $t-s \leq 48$

$t-s$	s	g	x	$d_3(x)$	
0	0	0	$i(1)$	0	(1)
8	4	1	$\overline{\overline{h_0^4}}$	0	(1)
11	1	3	$\overline{\overline{h_2}}$	0	(1)
15	1	4	$i(h_4)$	0	(4)
16	2	7	$\overline{\overline{h_1 h_3}}$	0	(5)
16	3	6	$\overline{\overline{c_0}}$	0	(2)
17	4	6	$i(e_0)$	$i(Pc_0)$	(8)
19	5	9	$P\overline{\overline{h_2}}$	0	(1)
22	4	11	$\overline{\overline{d_0}}$	0	(1)
23	3	12	$h_0^2\overline{\overline{h_4}}$	$h_0^2\overline{\overline{d_0}}$	(6)
24	7	8	$P\overline{\overline{c_0}}$	0	(2)
25	3	13	$h_1^2\overline{\overline{h_4}}$		
25	8	7	$i(Pe_0)$		
26	2	12	$h_2\overline{\overline{h_4}}$	0	(7)
27	3	15	$\overline{\overline{c_1}}$		
27	9	9	$P^2\overline{\overline{h_2}}$	0	(1)
28	4	17	$\overline{\overline{g_1}}$	0	(1)
30	8	12	$P\overline{\overline{d_0}}$	0	(1)
31	4	20	$c_0\overline{\overline{h_4}}$		
32	11	8	$P^2\overline{\overline{c_0}}$	0	(2)
33	12	7	$i(P^2e_0)$		
35	13	9	$P^3\overline{\overline{h_2}}$	0	(1)

Table 14.10: $E_3(S)$-module generators for $E_3(S/\sigma)$ with $t-s \leq 48$ (cont.)

$t-s$	s	g	x	$d_3(x)$	
38	2	17	$h_4\overline{\overline{h_4}}$	$i(x)$	(9)(!)
38	12	12	$P^2\overline{\overline{d_0}}$	0	(1)
39	8	24	$e_0\overline{\overline{d_0}}$		
40	15	8	$P^3\overline{\overline{c_0}}$	0	(2)
41	16	7	$i(P^3e_0)$		
42	4	30	$h_0^2h_5\overline{\overline{h_2}}$		
43	17	9	$P^4\overline{\overline{h_2}}$	0	(1)
44	6	29	$\overline{\overline{t}}$		
45	6	30	$\overline{\overline{h_0x}}$	0	(1)
45	8	$31+32$	$e_0\overline{\overline{g_1}}$	0	(1)
46	5	35	$h_0h_5\overline{\overline{h_0^2h_3}}$		
46	16	12	$P^3\overline{\overline{d_0}}$	0	(1)
47	3	28	$h_5\overline{\overline{h_1h_3}}$		
47	4	33	$h_5\overline{\overline{c_0}}$	0	(3)
47	5	36	$\overline{\overline{h_1e_1}}$		
47	9	34	$\overline{\overline{u}}$	0	(1)
47	12	23	$Pe_0\overline{\overline{d_0}}$	0	(1)
47	14	17	$i(h_0Q)$		
48	5	37	$i(h_5e_0)$		
48	19	8	$P^4\overline{\overline{c_0}}$	0	(2)

The d_3-differentials are determined as follows:

(1) The differentials on $i(1)$, $\overline{\overline{h_0^4}}$, $\overline{\overline{h_2}}$, $P\overline{\overline{h_2}}$, $\overline{\overline{d_0}}$, $P^2\overline{\overline{h_2}}$, $\overline{\overline{g_1}}$, $P\overline{\overline{d_0}}$, $P^3\overline{\overline{h_2}}$, $P^2\overline{\overline{d_0}}$, $P^4\overline{\overline{h_2}}$, $\overline{\overline{h_0x}}$, $e_0\overline{\overline{g_1}}$, $P^3\overline{\overline{d_0}}$, $\overline{\overline{u}}$ and $Pe_0\overline{\overline{d_0}}$ are zero because the target bidegrees are trivial.

(2) The differentials on $\overline{\overline{c_0}}$, $P\overline{\overline{c_0}}$, $P^2\overline{\overline{c_0}}$, $P^3\overline{\overline{c_0}}$ and $P^4\overline{\overline{c_0}}$ are zero by h_0-linearity.

(3) The differential on $h_5\overline{\overline{c_0}}$ is zero by h_1-linearity.

(4) The differential on $i(h_4)$ is zero by h_2-linearity, since $h_2^2 \cdot i(d_0) \neq 0$.

(5) The differential on $\overline{\overline{h_1h_3}}$ is zero by h_0- and d_0-linearity, since $d_0 \cdot i(h_1d_0) \neq 0$.

(6) The differential $d_3(h_0^2h_4) = h_0^2d_0 \neq 0$ in $E_3(S)$ lifts over j to show that $d_3(h_0^2\overline{\overline{h_4}}) \neq 0$, and $h_0^2\overline{\overline{d_0}}$ is the only possible value.

(7) The differential on $h_2\overline{\overline{h_4}}$ is zero by j-naturality, since $d_3(h_2h_4) = 0 \neq h_0d_0$.

(8) The product $\sigma\mu = \eta\rho$ is detected by Pc_0, so $i(Pc_0)$ must be a boundary in $E_r(C\sigma)$. By h_1-linearity it can only be supported on $i(e_0)$.

(9) The proof of [**BR**, Thm. 11.56(5)] shows that $d_3(h_4\overline{\overline{h_4}}) = i(x)$ in $E_3(S/\sigma)$.

(10) ((ETC))

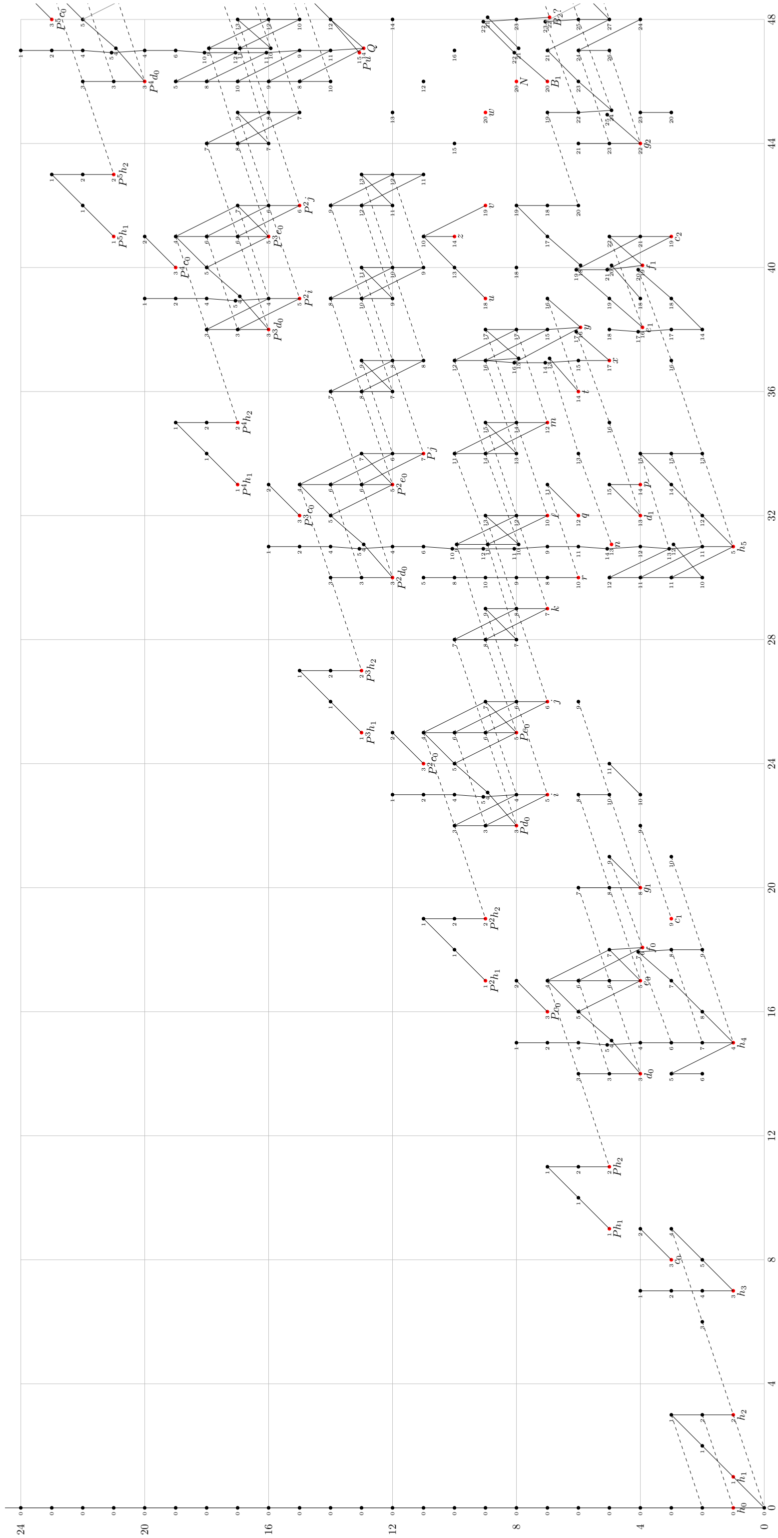


FIGURE 14.1. $(E_2(S), d_2) \Longrightarrow \pi_*(S)$

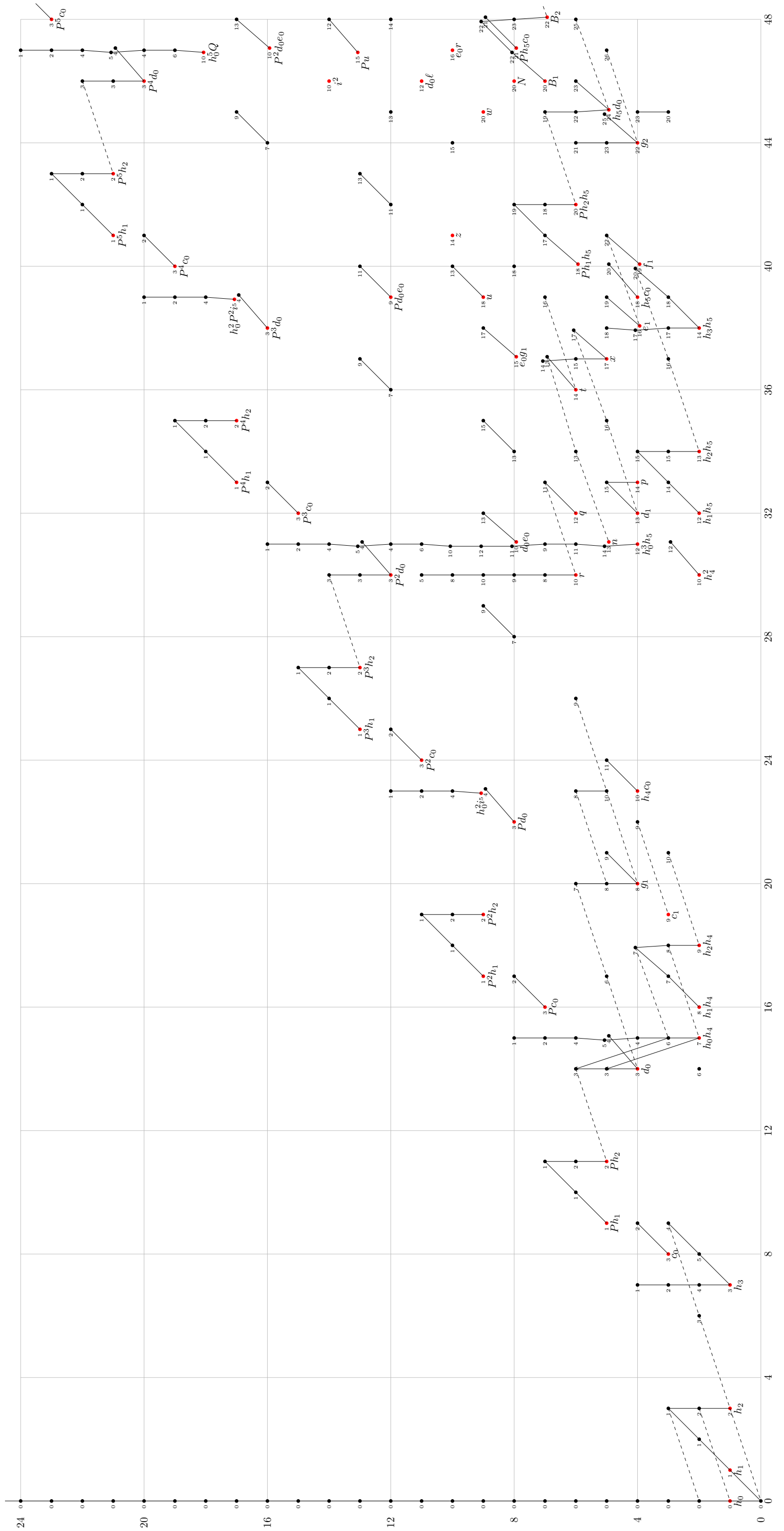


Figure 14.2. $(E_3(S), d_3) \Longrightarrow \pi_*(S)$

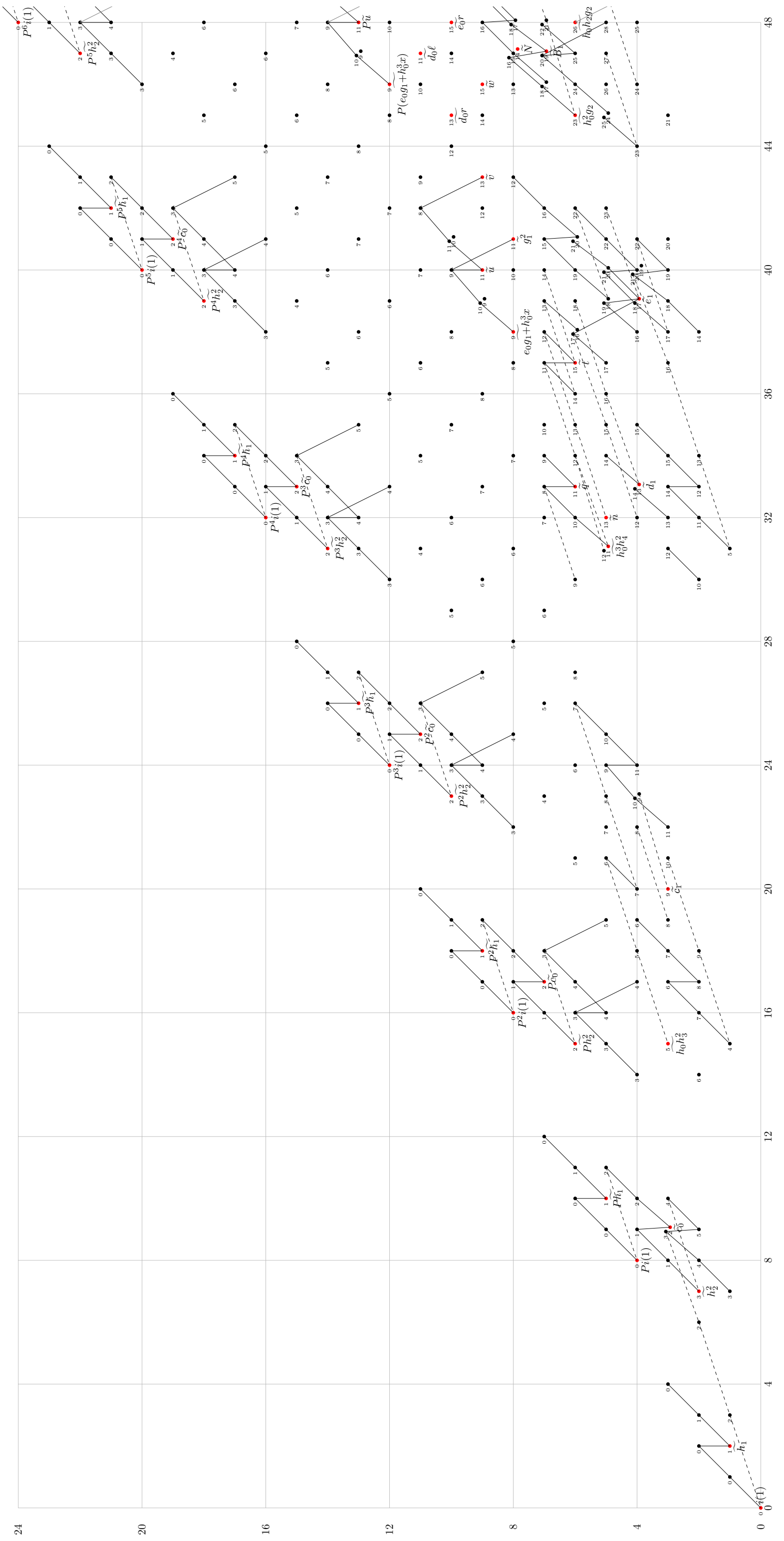


FIGURE 14.3. $(E_2(S/2), d_2) \Longrightarrow \pi_*(S/2)$

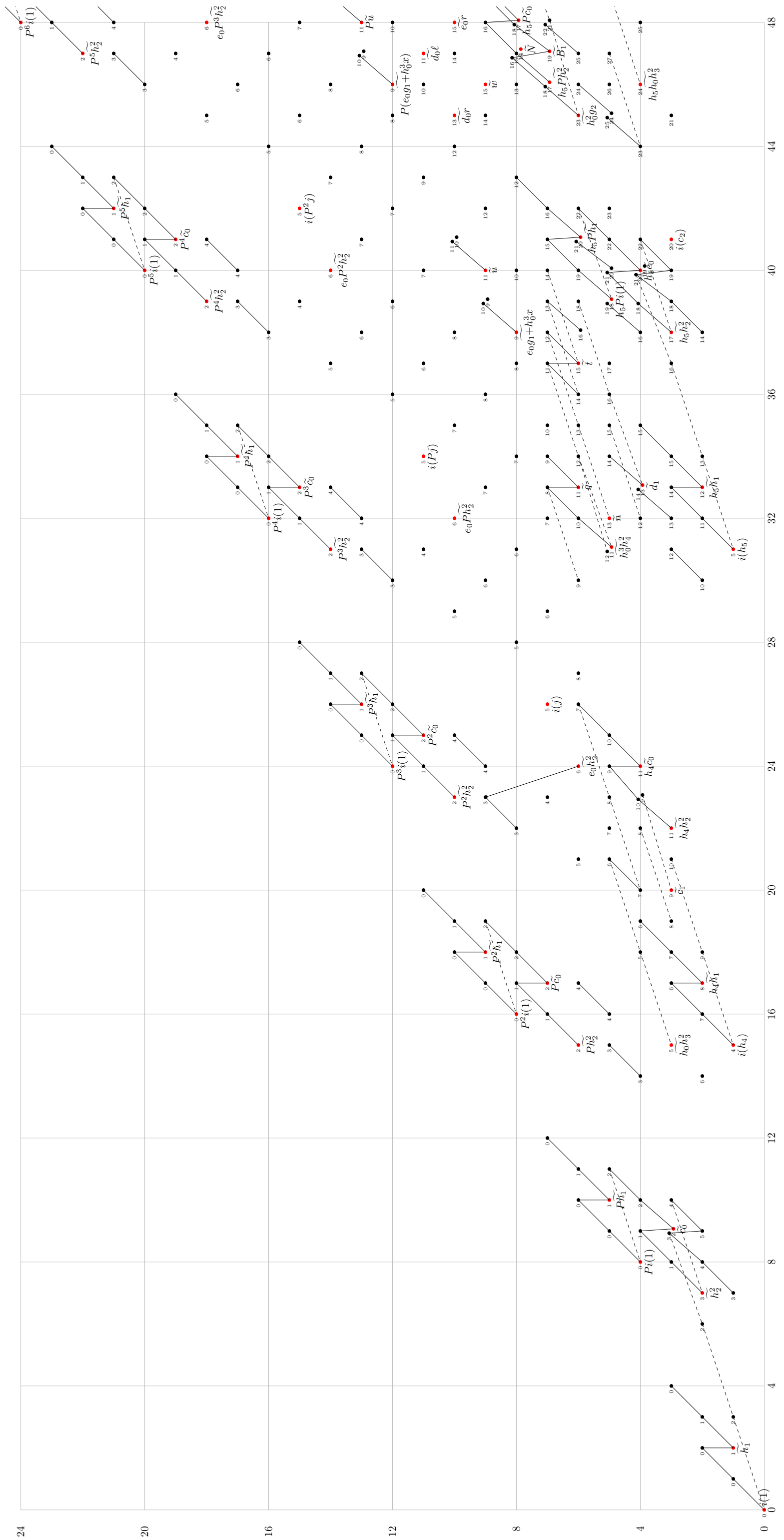


FIGURE 14.4. $(E_3(S/2), d_3) \Longrightarrow \pi_*(S/2)$

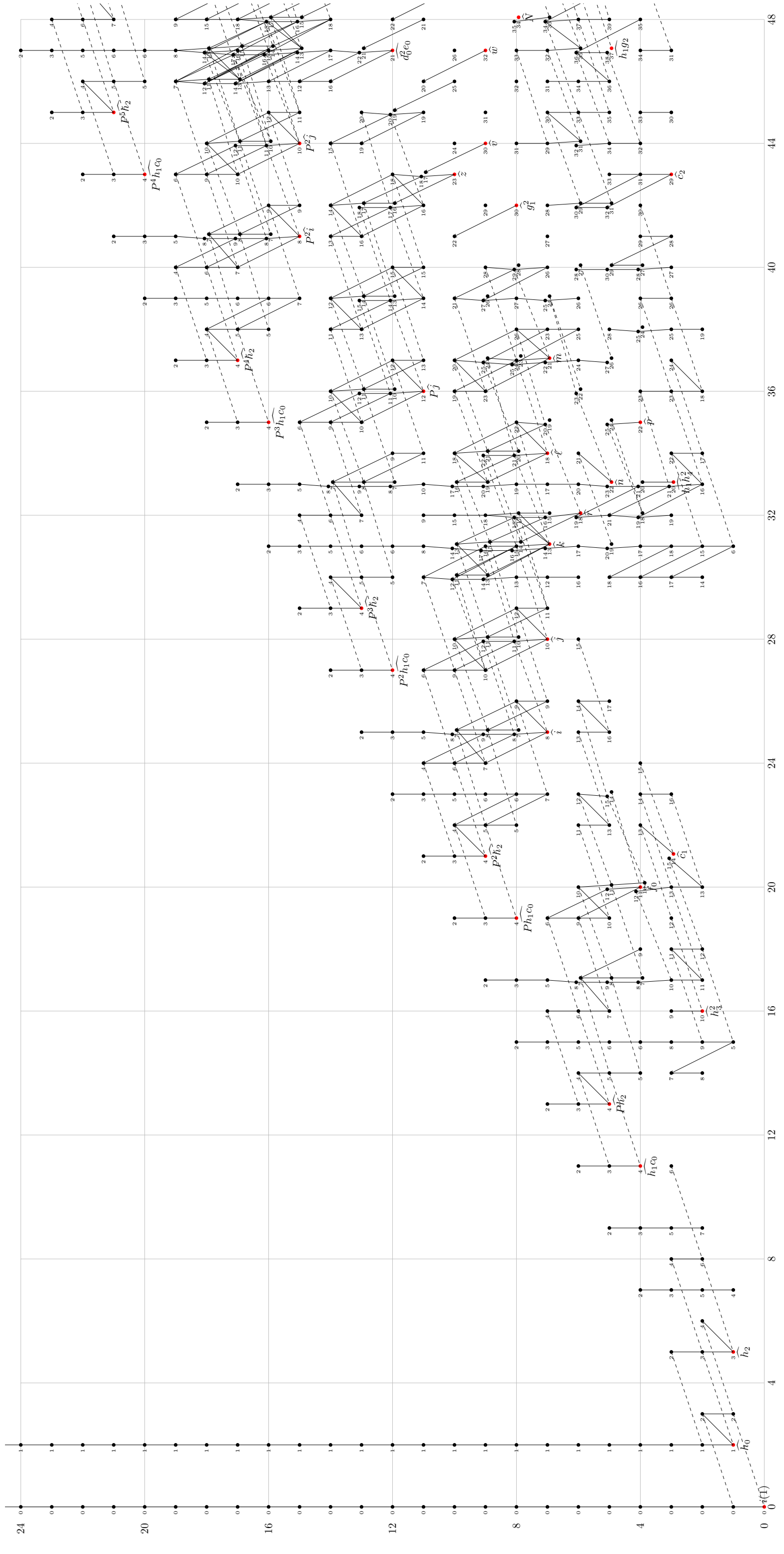


FIGURE 14.5. $(E_2(S/\eta), d_2) \Longrightarrow \pi_*(S/\eta)$

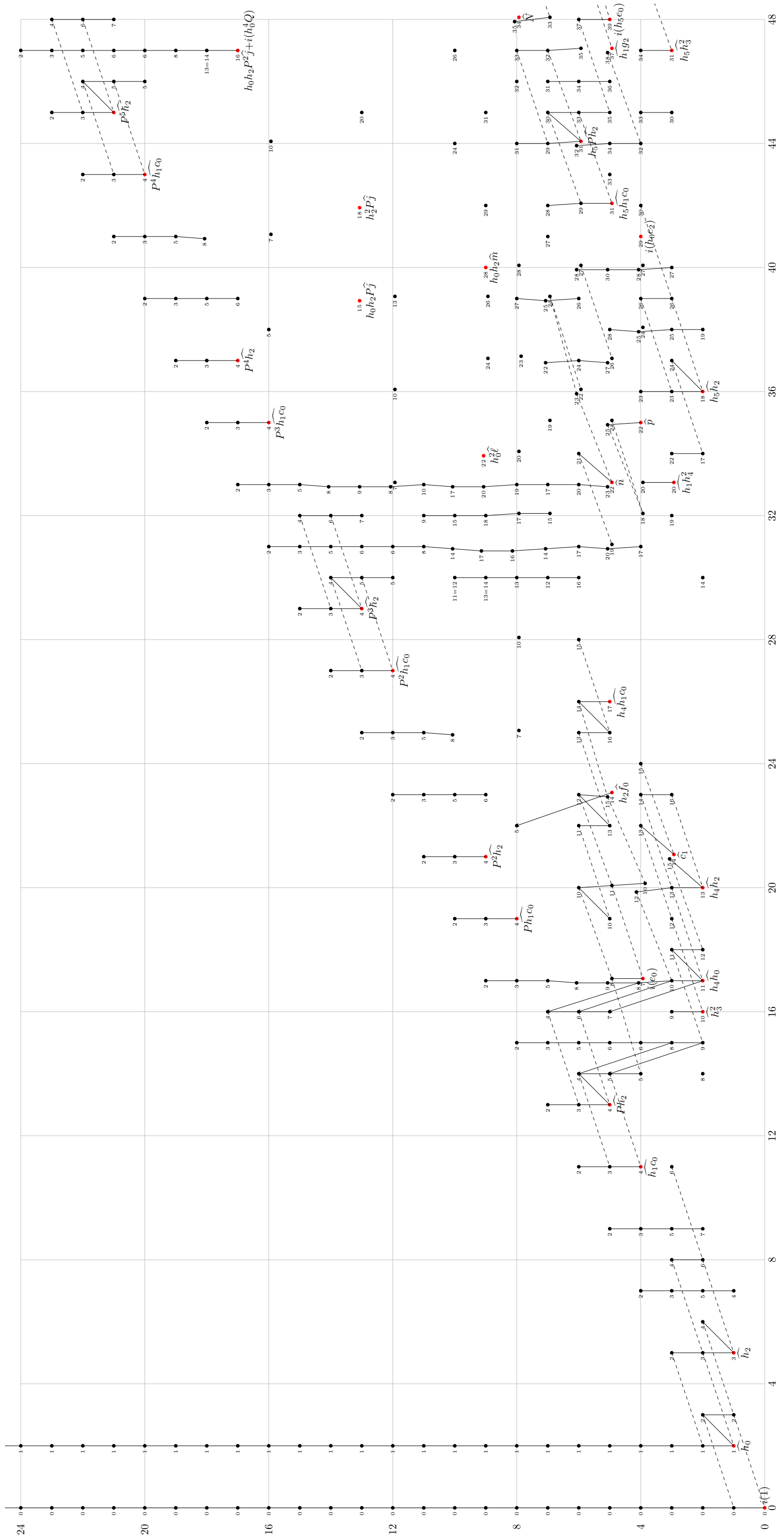


FIGURE 14.6. $(E_3(S/\eta), d_3) \Longrightarrow \pi_*(S/\eta)$

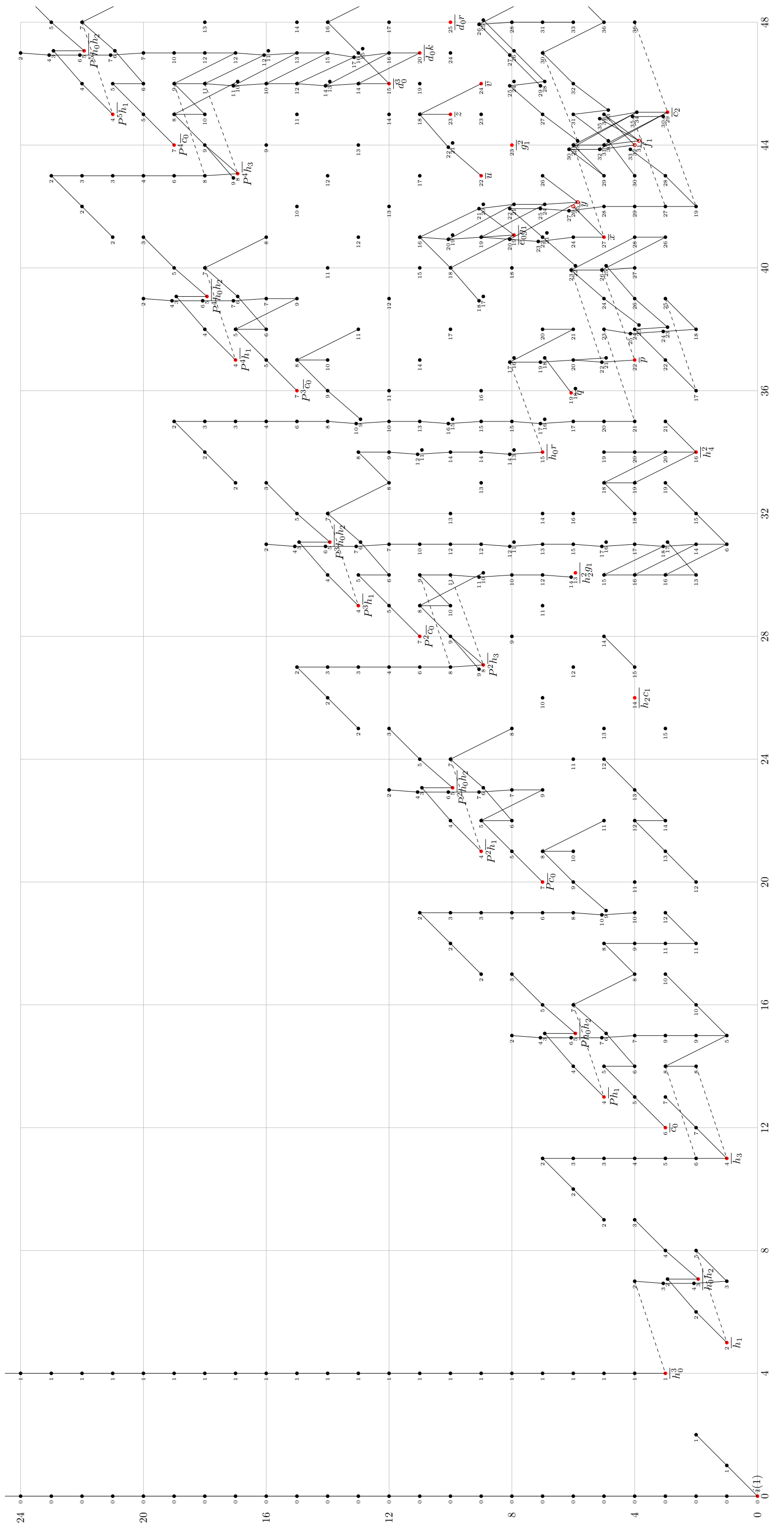


Figure 14.7. $(E_2(S/\nu), d_2) \Longrightarrow \pi_*(S/\nu)$

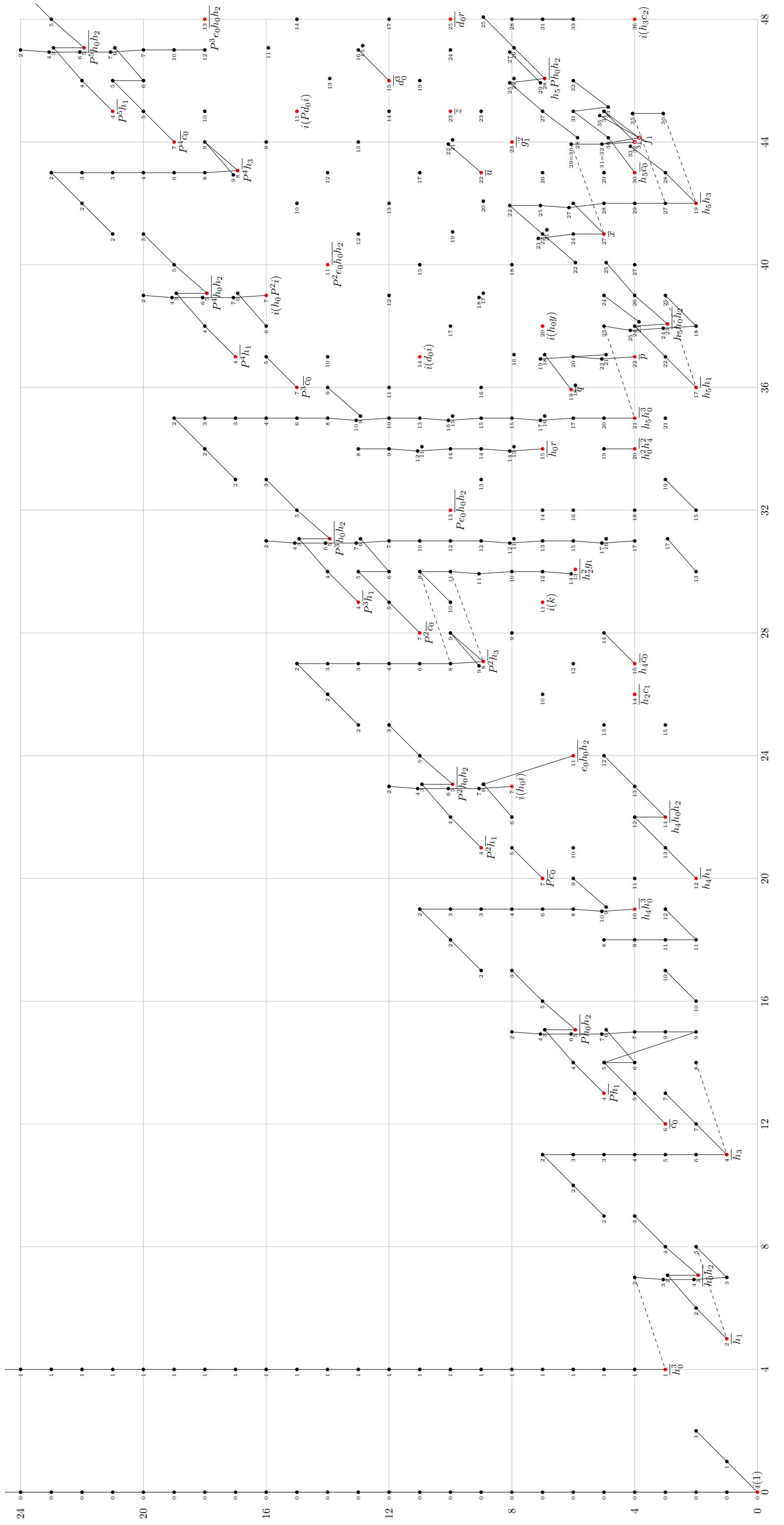


FIGURE 14.8. $(E_3(S/\nu), d_3) \Longrightarrow \pi_*(S/\nu)$

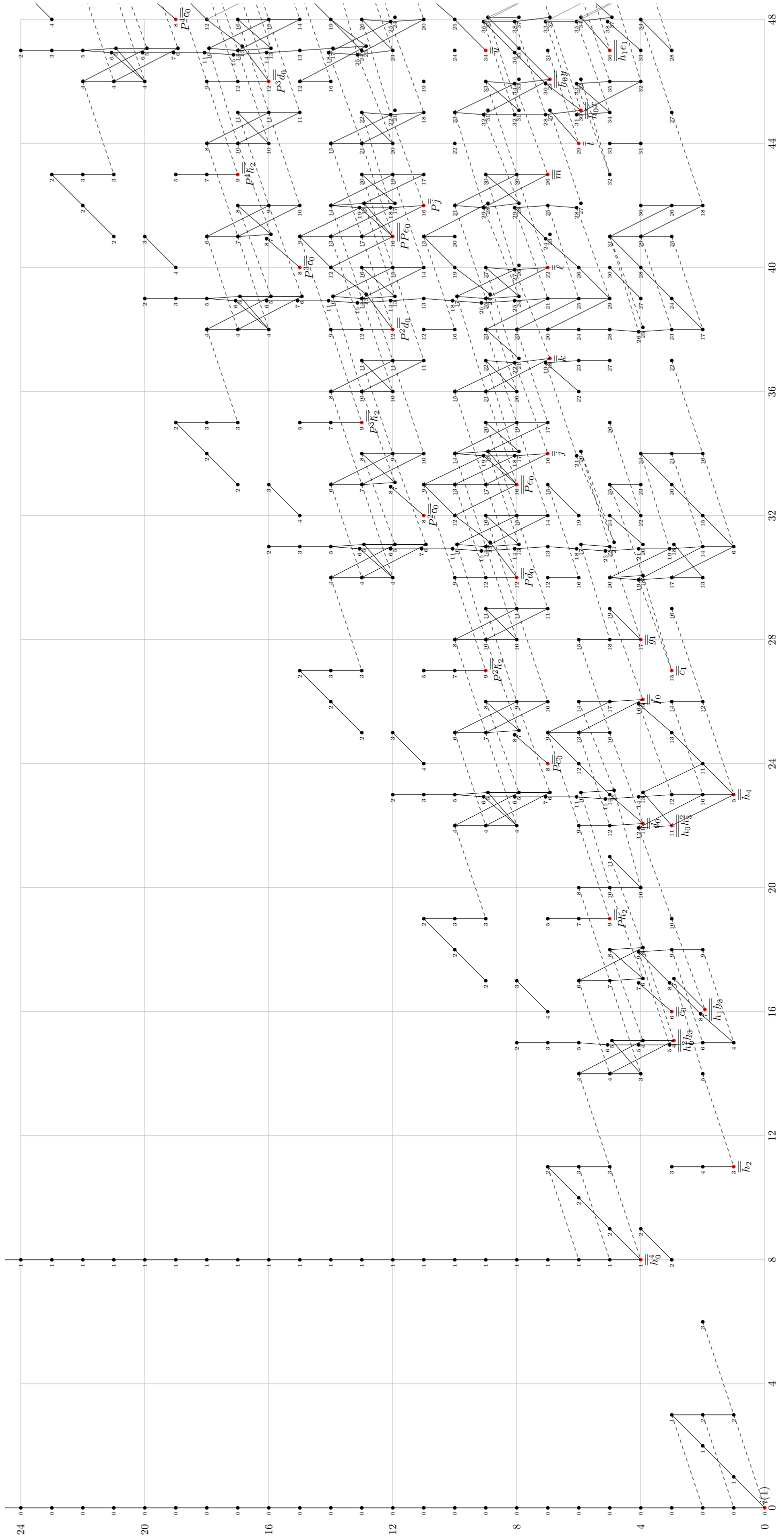


FIGURE 14.9. $(E_2(S/\sigma), d_2) \Longrightarrow \pi_*(S/\sigma)$

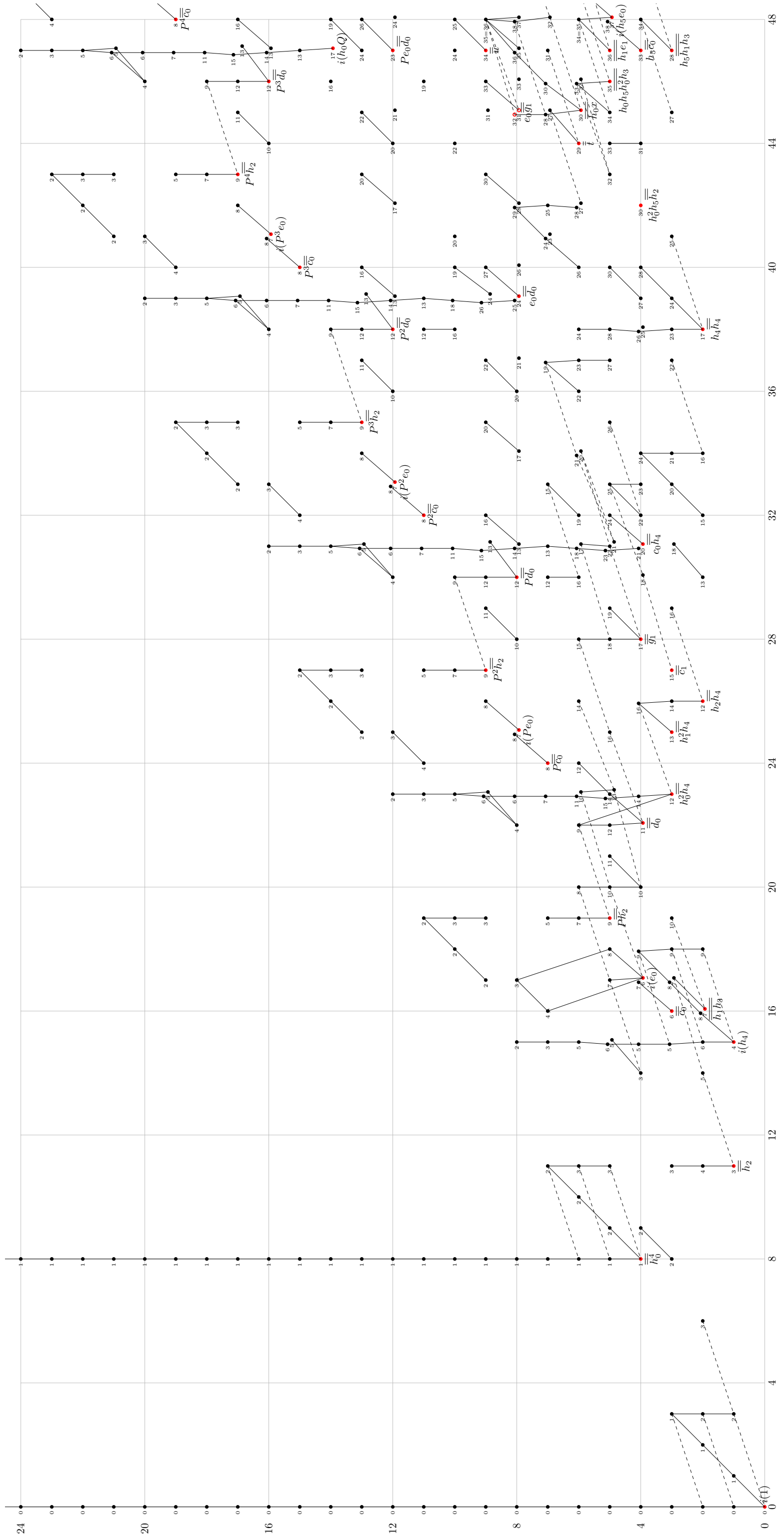


FIGURE 14.10. $(E_3(S/\sigma), d_3) \Longrightarrow \pi_*(S/\sigma)$

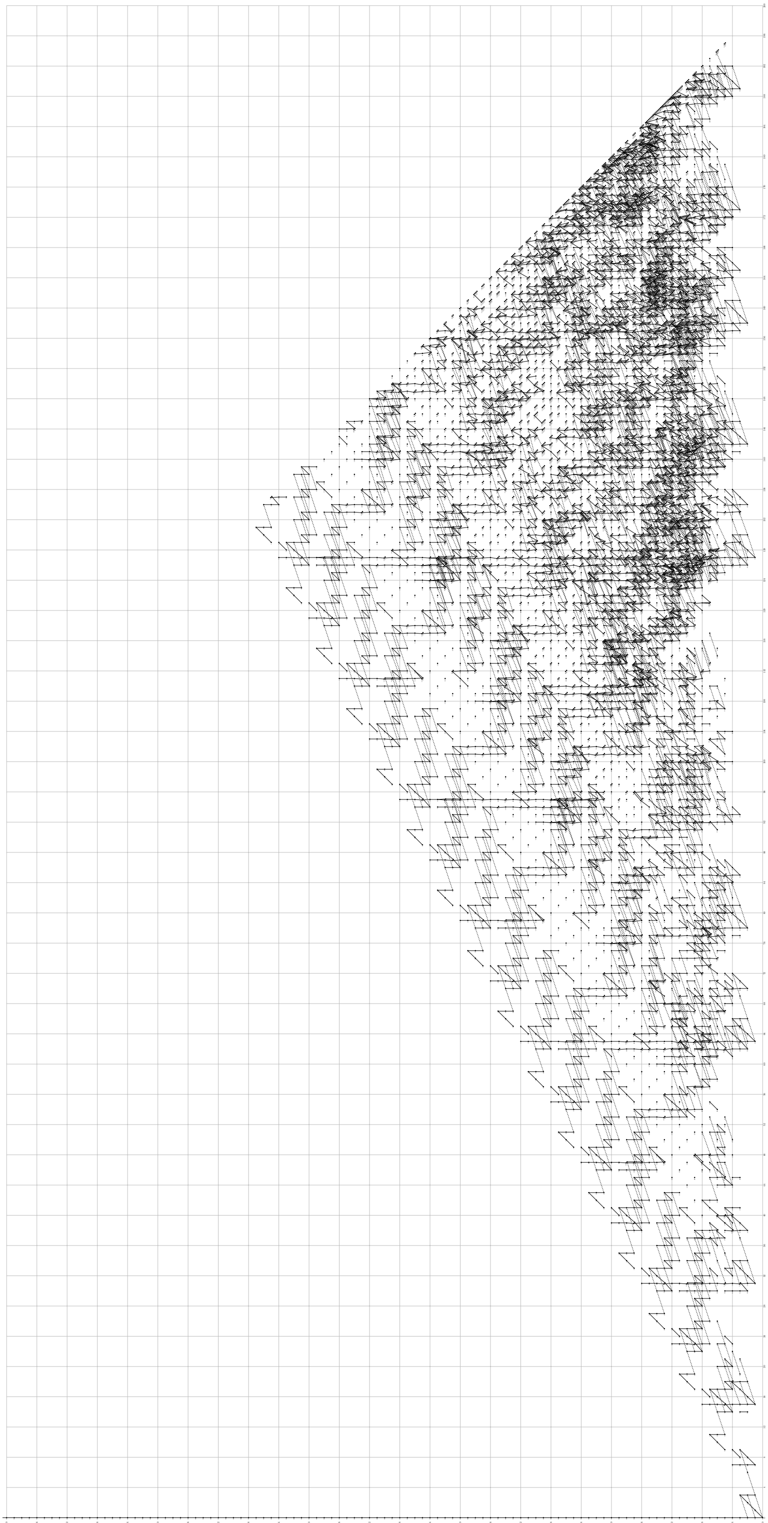


FIGURE 14.11. $E_2(S) = \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_*(S)$

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