Mek1100 - Eksamen 2023
Løsningsforslag

Oppgave 1

a) $$\nabla \cdot \vec{u} = \frac{\partial y}{\partial x} + \frac{\partial \cos z}{\partial y} + \frac{\partial x y^2}{\partial z} = \underline{\underline{0}}$$

$$\nabla \cdot \vec{v} = \frac{\partial x}{\partial x} + \frac{\partial 2y}{\partial y} + \frac{\partial 3z}{\partial z} = 1+2+3 = \underline{\underline{6}}$$

b) $$\nabla \times \vec{u} = \mathbf{i}\left(\frac{\partial x y^2}{\partial y} - \frac{\partial \cos z}{\partial z}\right) + \mathbf{j}\left(\frac{\partial y}{\partial z} - \frac{\partial x y^2}{\partial x}\right) + \mathbf{k}\left(\frac{\partial \cos z}{\partial x} - \frac{\partial y}{\partial y}\right)$$

$$= \mathbf{i}\,(2xy + \sin z) + \mathbf{j}\,(-y^2) + \mathbf{k}\,(-1)$$

$$= \underline{\underline{\mathbf{i}\,(2xy + \sin z) - \mathbf{j}\, y^2 - \mathbf{k}}}$$

$$\nabla \times \vec{v} = \mathbf{i}\left(\frac{\partial 3z}{\partial y} - \frac{\partial 2y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial x}{\partial z} - \frac{\partial 3z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial 2y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \mathbf{i}\cdot 0 + \mathbf{j}\cdot 0 + \mathbf{k}\cdot 0 = \underline{\underline{\vec{0}}}$$

1c) $\vec{u}$ kan <u>ikke</u> skrives som gradienten av et potensial siden $\nabla \times \vec{u} \neq \vec{0}$.

$\vec{v}$ kan skrives som gradienten av et potensial siden $\nabla \times \vec{v} = \vec{0}$

$\vec{v} = \nabla \phi_v \Rightarrow \phi_v = \frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2 + c$, c = konstant

$\vec{v}$ har <u>ikke</u> en strømfunksjon siden $\nabla \cdot \vec{v} \neq 0$.

$\vec{u}$ har <u>ikke</u> en strømfunksjon siden feltet er tredimensjonalt. (Gjevik s. 74)

1d) Sirkulasjonen er uttrykt ved

$$\oint_C \vec{u} \cdot d\vec{r}$$

der C er en sirkel i xy-planet med radius a. Vi parametriserer derfor

$$\vec{r}(\theta) = a\cos\theta \, \mathbf{i} + a\sin\theta \, \mathbf{j} \quad , \quad \theta \in [0, 2\pi]$$

Integralet parametriseres som

$$\oint_C \vec{u} \cdot d\vec{r} = \int_0^{2\pi} \vec{u} \cdot \frac{d\vec{r}}{d\theta} \, d\theta$$

1d) $\frac{d\vec{r}}{d\theta} = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j}$

$$\vec{u}\cdot\frac{d\vec{r}}{d\theta} = \left(a\sin\theta\,\mathbf{i} + \mathbf{j} + a^3\cos\theta\sin^2\theta\,\mathbf{k}\right)\cdot\left(-a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j}\right)$$

$$= -a^2\sin^2\theta + a\cos\theta$$

$$\int_0^{2\pi} -a^2\sin^2\theta + a\cos\theta\, d\theta = \underline{\underline{-a^2\pi}}$$ $\left(\text{Kan bruke } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right)$

Stokes' sats

$$\oint_C \vec{u}\cdot d\vec{r} = \int_S \nabla\times\vec{u}\cdot\vec{n}\, d\sigma$$

Har $\vec{n} = \mathbf{k}$ og $\nabla\times\vec{u}$ fra 1b).

$$\int_S (\cdots - \mathbf{k})\cdot\mathbf{k}\, d\sigma = -\int_S d\sigma = \underline{\underline{-\pi a^2}}$$

Fortegnet i svaret er likegyldig siden retningen på sirkulasjonen ikke er gitt.

1e) Finn $(\vec{v}\cdot\nabla)\vec{u}$

$$\vec{v}\cdot\nabla = (x\mathbb{i} + 2y\mathbb{j} + 3z\mathbb{k})\cdot\left(\mathbb{i}\frac{\partial}{\partial x} + \mathbb{j}\frac{\partial}{\partial y} + \mathbb{k}\frac{\partial}{\partial z}\right)$$

$$= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}$$

$$(\vec{v}\cdot\nabla)\vec{u} = x\frac{\partial}{\partial x}\vec{u} + 2y\frac{\partial}{\partial y}\vec{u} + 3z\frac{\partial}{\partial z}\vec{u}$$

$$\vec{u} = y\mathbb{i} + \cos z\,\mathbb{j} + xy^2\mathbb{k}$$

$$\frac{\partial\vec{u}}{\partial x} = y^2\mathbb{k}\,,\quad \frac{\partial\vec{u}}{\partial y} = \mathbb{i} + 2xy\mathbb{k}\,,\quad \frac{\partial\vec{u}}{\partial z} = -\sin z\,\mathbb{j}$$

$$(\vec{v}\cdot\nabla)\vec{u} = xy^2\mathbb{k} + 2y(\mathbb{i} + 2xy\mathbb{k}) + 3z(-\sin z)\mathbb{j}$$

$$= \underline{\underline{2y\mathbb{i} - 3z\sin z\,\mathbb{j} + 5xy^2\mathbb{k}}}$$

Oppgave 2

a) Bruker Gauss' sats

$$\oint_{S+G} \vec{u}\cdot\vec{n}\,d\sigma = \int_V \nabla\cdot\vec{u}\,dV$$

$$\nabla\cdot\vec{u} = 1$$

$$\int_S \vec{u}\cdot\vec{n}\,d\sigma + \int_G \vec{u}\cdot\vec{n}\,d\sigma = \int_V dV$$

$$\int_S \vec{u}\cdot\vec{n}\,d\sigma = \frac{\pi h a^2}{3} - \int_G \left((x+y)\mathbf{i} + \cos z\,\mathbf{j} + \mathbf{k}\right)\cdot(-\mathbf{k})\,d\sigma$$

$$\int_S \vec{u}\cdot\vec{n}\,d\sigma = \frac{\pi h a^2}{3} + \int_G d\sigma$$

$$\underline{\underline{\int_S \vec{u}\cdot\vec{n}\,d\sigma = \frac{\pi h a^2}{3} + \pi a^2}}$$

2b) For G så er $\underline{\underline{\vec{n} = -\mathbb{k}}}$

For S så er $\vec{n} = \dfrac{\nabla z}{|\nabla z|}$

$$\nabla z = 2x\,\mathbb{i} + 2y\,\mathbb{j} - 2\left(\frac{h-z}{a}\right)\left(-\frac{1}{a}\right)\mathbb{k}$$

$$\nabla z = 2x\,\mathbb{i} + 2y\,\mathbb{j} + 2\left(\frac{h-z}{a^2}\right)\mathbb{k}$$

$$|\nabla z| = \sqrt{4x^2 + 4y^2 + 4\left(\frac{h-z}{a^2}\right)^2} = 2\sqrt{x^2+y^2+\left(\frac{h-z}{a^2}\right)^2}$$

$$\underline{\underline{\vec{n} = \frac{2x\,\mathbb{i} + 2y\,\mathbb{j} + 2\left(\frac{h-z}{a^2}\right)\mathbb{k}}{2\sqrt{x^2+y^2+\left(\frac{h-z}{a^2}\right)^2}} = \frac{x\,\mathbb{i} + y\,\mathbb{j} + \left(\frac{h-z}{a^2}\right)\mathbb{k}}{\sqrt{x^2+y^2+\left(\frac{h-z}{a^2}\right)^2}}}}$$

Oppgave 3

a) $h_\alpha = \left|\frac{\partial \vec{r}}{\partial \alpha}\right| \quad , \alpha \in (u, v, \theta)$

$$\frac{\partial \vec{r}}{\partial u} = v\cos\theta\,\mathbb{i} + v\sin\theta\,\mathbb{j} - u\,\mathbb{k}$$

$$\left|\frac{\partial \vec{r}}{\partial u}\right| = \sqrt{v^2\cos^2\theta + v^2\sin^2\theta + u^2} = \underline{\underline{\sqrt{u^2+v^2}}} = h_u$$

$$\frac{\partial \vec{r}}{\partial v} = u\cos\theta\,\mathbb{i} + u\sin\theta\,\mathbb{j} + v\,\mathbb{k}$$

$$\left|\frac{\partial \vec{r}}{\partial v}\right| = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta + v^2} = \underline{\underline{\sqrt{u^2+v^2} = h_v}}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -uv\sin\theta\,\mathbb{i} + uv\cos\theta\,\mathbb{j}$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{u^2v^2\sin^2\theta + u^2v^2\cos^2\theta} = \underline{\underline{uv = h_\theta}}$$

Enhetsvektorer

$$\vec{e}_\alpha = \frac{1}{h_\alpha}\frac{\partial \vec{r}}{\partial \alpha} \quad , \alpha \in (u, v, \theta) \quad , \quad h_u = h_v$$

$$\vec{e}_u \cdot \vec{e}_v = \frac{1}{h_u^2}\left(v\cos\theta\,\mathbb{i} + v\sin\theta\,\mathbb{j} - u\,\mathbb{k}\right)\cdot\left(u\cos\theta\,\mathbb{i} + u\sin\theta\,\mathbb{j} + v\,\mathbb{k}\right)$$

$$= \frac{1}{h_u^2}\left(uv\cos^2\theta + uv\sin^2\theta - uv\right) = \underline{0}$$

3a) $$\vec{e}_u \cdot \vec{e}_\theta = \frac{1}{h_u h_\theta}\left(v\cos\theta\,\mathbf{i} + v\sin\theta\,\mathbf{j} - u\mathbf{k}\right)\cdot\left(-uv\sin\theta\,\mathbf{i} + uv\cos\theta\,\mathbf{j}\right)$$

$$= \frac{1}{h_u h_\theta}\left(-uv^2\cos\theta\sin\theta + uv^2\cos\theta\sin\theta\right) = \underline{0}$$

$$\vec{e}_v \cdot \vec{e}_\theta = \frac{1}{h_v h_\theta}\left(u\cos\theta\,\mathbf{i} + u\sin\theta\,\mathbf{j} + v\mathbf{k}\right)\cdot\left(-uv\sin\theta\,\mathbf{i} + uv\cos\theta\,\mathbf{j}\right)$$

$$= \frac{1}{h_v h_\theta}\left(-u^2v\cos\theta\sin\theta + u^2v\cos\theta\sin\theta\right) = \underline{0}$$

Siden $\vec{e}_u \cdot \vec{e}_v$, $\vec{e}_u \cdot \vec{e}_\theta$ og $\vec{e}_v \cdot \vec{e}_\theta = 0$, så er koordinatsystemet ortogonalt.

Volumelement $= dV = h_u h_v h_\theta \, du\, dv\, d\theta$

$$\underline{\underline{dV = uv(u^2+v^2)\,du\,dv\,d\theta}}$$

3b) $$\underline{\underline{\nabla f = \frac{\vec{e}_u}{h_u}\frac{\partial f}{\partial u} + \frac{\vec{e}_v}{h_v}\frac{\partial f}{\partial v} + \frac{\vec{e}_\theta}{h_\theta}\frac{\partial f}{\partial \theta}}}$$

$$\vec{v} = v_u \vec{e}_u + v_v \vec{e}_v + v_\theta \vec{e}_\theta$$

$$\underline{\underline{\nabla\cdot\vec{v} = \frac{1}{h_u h_v h_\theta}\left(\frac{\partial}{\partial u}(v_u h_v h_\theta) + \frac{\partial}{\partial v}(v_v h_u h_\theta) + \frac{\partial}{\partial \theta}(v_\theta h_u h_v)\right)}}$$

(Trenger ikke bruke u, v, θ. Kan bruke v_1, v_2, v_3, h_1, h_2, h_3)

Oppgave 4

$$\underline{\underline{B(x_1, y_1, z_1) = B_0}}$$

Fra Euler's likning har vi

$$\nabla B + (\nabla \times \vec{v}) \times \vec{v} = \vec{0}$$

(Denne kan utledes fra

$$(\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p - g\mathbf{k}$$

ved å bruke

$$(\vec{v} \cdot \nabla)\vec{v} = \frac{1}{2}\nabla|\vec{v}|^2 + (\nabla \times \vec{v}) \times \vec{v}$$

$$g\mathbf{k} = \nabla g z \quad)$$

Siden $\nabla \times \vec{v} = \vec{0}$, så har vi at

$$\nabla B = \vec{0}$$

hvilket betyr at B er konstant.
Dermed må $B(x_1, y_1, z_1) = B(x_0, y_0, z_0) = B_0$.
I denne oppgaven holder det ikke å bruke Bernoulli's likning, siden denne kun gir at B er konstant langs en strømlinje.