MEK1100, selected formulae

Taylor polynomial of second order

$$g(x,y) = g(x_0,y_0) + \tfrac{\partial g(x_0,y_0)}{\partial x}(x-x_0) + \tfrac{\partial g(x_0,y_0)}{\partial y}(y-y_0)$$
$$+ \tfrac{1}{2}\tfrac{\partial^2 g(x_0,y_0)}{\partial x^2}(x-x_0)^2 + \tfrac{\partial^2 g(x_0,y_0)}{\partial x \partial y}(x-x_0)(y-y_0) + \tfrac{1}{2}\tfrac{\partial^2 g(x_0,y_0)}{\partial y^2}(y-y_0)^2$$

Differentiation formulae

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

$$\nabla(\kappa\beta) = \beta\,\nabla\kappa + \kappa\,\nabla\beta, \quad \nabla\cdot(\beta\mathbf{A}) = \nabla\beta\cdot\mathbf{A} + \beta\,\nabla\cdot\mathbf{A}, \quad \nabla\times(\beta\mathbf{A}) = \nabla\beta\times\mathbf{A} + \beta\,\nabla\times\mathbf{A}$$

$$\frac{\mathbf{D}\beta}{\mathbf{d}t} = \frac{\partial\beta}{\partial t} + \mathbf{v}\cdot\nabla\beta, \quad \frac{\mathbf{D}\mathbf{v}}{\mathbf{d}t} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$$

$$\mathbf{v}\cdot\nabla\mathbf{v} = (\mathbf{v}\cdot\nabla v_x)\mathbf{i} + (\mathbf{v}\cdot\nabla v_y)\mathbf{j} + (\mathbf{v}\cdot\nabla v_z)\mathbf{k} = v_x\frac{\partial\mathbf{v}}{\partial x} + v_y\frac{\partial\mathbf{v}}{\partial y} + v_z\frac{\partial\mathbf{v}}{\partial z}$$

Cylinder coordinates

Transformation: $x = r\cos\theta$, $y = r\sin\theta$, $\mathbf{i}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\mathbf{i}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.
Differentiation: $(\mathbf{A} = A_r\mathbf{i}_r + A_\theta\mathbf{i}_\theta + A_z\mathbf{k})$

$$\nabla\beta = \frac{\partial\beta}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\beta}{\partial\theta}\mathbf{i}_\theta + \frac{\partial\beta}{\partial z}\mathbf{k}, \quad \nabla^2\beta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\beta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\beta}{\partial\theta^2} + \frac{\partial^2\beta}{\partial z^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{\partial\theta}\right) + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{i}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\mathbf{k}$$

Spherical coordinates

Transformation: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,
$\mathbf{i}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$,
$\mathbf{i}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$,
$\mathbf{i}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$.
Differentiation: $(\mathbf{A} = A_r\mathbf{i}_r + A_\theta\mathbf{i}_\theta + A_\phi\mathbf{i}_\phi)$

$$\nabla\beta = \frac{\partial\beta}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\beta}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\beta}{\partial\phi}\mathbf{i}_\phi$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Integrals

Parameterization of surface $\Omega \to \sigma$: $\mathbf{r} = \mathbf{r}(t, s)$

$$\int_\sigma \mathbf{v} \cdot \mathbf{n}\, d\sigma = \iint_\Omega \mathbf{v} \cdot \left(\pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt\, ds. \quad \int_\sigma p\, \mathbf{n}\, d\sigma = \iint_\Omega p \left(\pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt\, ds$$

Cylinder: $d\sigma = r\, d\theta\, dz$; Sphere: $d\sigma = r^2 \sin\theta\, d\theta\, d\phi$
Flux through curve

$$\int_\lambda \mathbf{v} \cdot \mathbf{n}\, ds = \int_\lambda v_x\, dy - v_y\, dx$$

Less common forms of Gauss' theorem

$$\int_\sigma \mathbf{n} \times \mathbf{A} d\sigma = \int_\tau \nabla \times \mathbf{A} d\tau, \quad \int_\sigma \mathbf{n} \beta d\sigma = \int_\tau \nabla \beta d\tau$$

Irrotational and divergence-free flow fields in 2D

Stream function: $v_x = -\frac{\partial \psi}{\partial y}$, $v_y = \frac{\partial \psi}{\partial x}$ or $\mathbf{v} = \nabla \times (-\mathbf{k}\psi)$.

Examples:

example of a stagnation current	source/sink	point vortex	dipole
$\phi = \frac{1}{2} A(x^2 - y^2)$	$\phi = A \ln r$	$\phi = A\theta$	$\phi = \frac{Ax}{x^2+y^2}$

Hydrodynamic equations

Equation of continuity

$$\frac{\mathbf{D}\rho}{\mathbf{d}t} = -\rho \nabla \cdot \mathbf{v}$$

Euler's equation of motion in the gravity field ($\mathbf{g}$ is the acceleration of gravity)

$$\frac{\mathbf{D}\mathbf{v}}{\mathbf{d}t} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

Bernoulli's equation in the gravity field

$$\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho} + gz = \text{constant}$$

Heat

Specific density of thermal energy for incompressible media: $E(T)$.
Specific heat capacity: $c = \frac{\partial E}{\partial T}$.
Heat flux densities: $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_l$, $\mathbf{H}_s = \rho E \mathbf{v}$ (advection), $\mathbf{H}_l = -k \nabla T$ (conduction).
The heat equation:

$$\frac{\mathbf{D}T}{\mathbf{d}t} = \kappa \nabla^2 T + \frac{q}{\rho c}, \quad \text{where} \quad \kappa = \frac{k}{\rho c}$$