# Kapittel 1

# Felt i naturen, skalar- og vektorfelt, skalering

## Oppgave 1

To vektorer $\boldsymbol{u}$ og $\boldsymbol{v}$ er parallelle hvis vi kan skrive $\boldsymbol{u} = c\boldsymbol{v}$, der $c$ er en skalar.

$$2\boldsymbol{a} - \frac{1}{6}\boldsymbol{b} = c\left(\frac{1}{4}\boldsymbol{b} - 3\boldsymbol{a}\right)$$

$$\boldsymbol{a}(2+3c) + \boldsymbol{b}\left(-\frac{1}{6} - \frac{1}{4}c\right) = 0.$$

Dette gir to likninger for $c$. For å få en entydig løsning må begge likningen gi samme verdi.

$$\begin{aligned} 2+3c = 0 \qquad &\text{og} \qquad -\frac{1}{6} - \frac{1}{4}c = 0 \\ c = -\frac{2}{3} \qquad &\text{og} \qquad c = -\frac{2}{3}. \end{aligned}$$

Vektorene er altså parallelle.

## Oppgave 2

Gitt vektorene $\boldsymbol{a} = \{a_1, a_2, a_3\}, \boldsymbol{b} = \{b_1, b_2, b_3\}, \boldsymbol{c} = \{c_1, c_2, c_3\}$ og skalarene $\alpha$ og $\beta$.

a) Vis at $\boldsymbol{a} \cdot \boldsymbol{a} \geq 0$:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{a} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \\ &= a_1^2 + a_2^2 + a_3^2 \geq 0. \end{aligned}$$

b) Vis at $\boldsymbol{a} \cdot \boldsymbol{a} = 0$ hvis og bare hvis $\boldsymbol{a} = 0$:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{a} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \\ &= a_1^2 + a_2^2 + a_3^2 = 0 \quad \Leftrightarrow \quad a_1 = a_2 = a_3 = 0. \end{aligned}$$

c) • Vis at $\alpha \boldsymbol{a} \cdot \boldsymbol{b} = \alpha(\boldsymbol{a} \cdot \boldsymbol{b})$:

$$\begin{aligned}
\alpha \boldsymbol{a} \cdot \boldsymbol{b} &= (\alpha a_1 \boldsymbol{i} + \alpha a_2 \boldsymbol{j} + \alpha a_3 \boldsymbol{k}) \cdot (b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) \\
&= \alpha a_1 b_1 + \alpha a_2 b_2 + \alpha a_3 b_3 \\
&= \alpha(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \alpha(\boldsymbol{a} \cdot \boldsymbol{b}).
\end{aligned}$$

• Vis at $\boldsymbol{a} \cdot \beta \boldsymbol{b} = \beta(\boldsymbol{a} \cdot \boldsymbol{b})$:

$$\begin{aligned}
\boldsymbol{a} \cdot \beta \boldsymbol{b} &= (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot (\beta b_1 \boldsymbol{i} + \beta b_2 \boldsymbol{j} + \beta b_3 \boldsymbol{k}) \\
&= a_1 \beta b_1 + a_2 \beta b_2 + a_3 \beta b_3 \\
&= \beta(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \beta(\boldsymbol{a} \cdot \boldsymbol{b}).
\end{aligned}$$

d) • Vis at $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$:

$$\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot \Big( (b_1 + c_1) \boldsymbol{i} + (b_2 + c_2) \boldsymbol{j} + (b_3 + c_3) \boldsymbol{k} \Big) \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3 \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 c_1 + a_2 c_2 + a_3 c_3 \\
&= \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}.
\end{aligned}$$

• Vis at $(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$:

$$\begin{aligned}
(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} &= \Big( (a_1 + b_1) \boldsymbol{i} + (a_2 + b_2) \boldsymbol{j} + (a_3 + b_3) \boldsymbol{k} \Big) \cdot (c_1 \boldsymbol{i} + c_2 \boldsymbol{j} + c_3 \boldsymbol{k}) \\
&= (a_1 + b_1) c_1 + (a_2 + b_2) c_2 + (a_3 + b_3) c_3 \\
&= a_1 c_1 + b_1 c_1 + a_2 c_2 + b_2 c_2 + a_3 c_3 + b_3 c_3 \\
&= a_1 c_1 + a_2 c_2 + a_3 c_3 + b_1 c_1 + b_2 c_2 + b_3 c_3 \\
&= \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}.
\end{aligned}$$

e) Vis at $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ :

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot (b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
&= b_1 a_1 + b_2 a_2 + b_3 a_3 \\
&= \boldsymbol{b} \cdot \boldsymbol{a}.
\end{aligned}$$

## Oppgave 3

Gitt en partikkelbane $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}t$ der $\boldsymbol{r}_0 = \{a, b, c\}$ og $\boldsymbol{v} = \{v_x, v_y, v_z\}$.

a) Banen på komponentform:

$$\boldsymbol{r}(t) = (a + v_x t)\boldsymbol{i} + (b + v_y t)\boldsymbol{j} + (c + v_z t)\boldsymbol{k}.$$

b) Her kan vi sette $\boldsymbol{r}_0 = \boldsymbol{j} = \{0, 1, 0\}$ og $\boldsymbol{v} = v_z\boldsymbol{k} = \{0, 0, v_z\}$. Likningen for banen blir da

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}t = \boldsymbol{j} + v_z t\boldsymbol{k}.$$

c) Vi skriver om $\boldsymbol{r}(t)$:

$$\begin{aligned}
\boldsymbol{r}(t) &= (-7t + 3)\boldsymbol{i} - 2\boldsymbol{j} + 4t\boldsymbol{k} \\
&= 3\boldsymbol{i} - 2\boldsymbol{j} + t(-7\boldsymbol{i} + 4\boldsymbol{k}) \\
&= \boldsymbol{r}_0 + \boldsymbol{v}t.
\end{aligned}$$

Retningen er gitt ved $\boldsymbol{v} = -7\boldsymbol{i} + 4\boldsymbol{k}$.

d) Hvis de to banene skal skjære hverandre må det finnes to tall $t_1$ og $t_2$ slik at $\boldsymbol{r}_1(t_1) = \boldsymbol{r}_2(t_2)$. Dette gir tre komponentlikninger:

$$t_1 = 3t_2 + 1 \tag{1.1}$$

$$-6t_1 + 1 = 2t_2 \tag{1.2}$$

$$2t_1 - 8 = 0 \tag{1.3}$$

$$\begin{aligned}
(1.3) \quad &\Rightarrow \quad t_1 = 4 \\
(1.1) \quad &\Rightarrow \quad t_2 = 1 \\
(1.2) \quad &\Rightarrow \quad t_2 = -\frac{23}{2}
\end{aligned}$$

Her har vi ingen entydig løsning for $t_1$ og $t_2$, altså skjærer banen hverandre ikke.

e) Ved tiden $t = 0$ går banen igjennom punktet $P_0 = (3, -1, 5)$ og ved tiden $t = 1$ går banen gjennom punktet $P_1 = (-4, 3, 0)$. Vi vil finne banen uttrykt på formen

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}t.$$

Vi kan nå sette inn for $t = 0$ og $t = 1$:

$$\begin{aligned}
\boldsymbol{r}(t = 0) &= \boldsymbol{r}_0 = 3\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k} \\
\boldsymbol{r}(t = 1) &= \boldsymbol{r}_0 + \boldsymbol{v} = -4\boldsymbol{i} + 3\boldsymbol{j} \\
\Rightarrow \quad \boldsymbol{v} &= -4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{r}_0 \\
&= -4\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} \\
&= -7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k} \\
\Rightarrow \quad \boldsymbol{r}(t) &= (3 - 7t)\boldsymbol{i} + (-1 + 4t)\boldsymbol{j} + (5 - 5t)\boldsymbol{k}.
\end{aligned}$$

## Oppgave 4

a) Normaliser vektoren $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$:

$$\begin{aligned}\boldsymbol{a}_{\text{norm}} &= \frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \quad |\boldsymbol{a}| = \sqrt{3^2 + 2^2 + (-4)^2} = \sqrt{29} \\ &= \frac{1}{\sqrt{29}}\Big(3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}\Big).\end{aligned}$$

b) Tre vektorer med samme lengde som er slik at $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ utgjør en likesided trekant. For å finne et uttrykk for $\boldsymbol{a}$, $\boldsymbol{b}$ og $\boldsymbol{c}$ må vi plassere vektorene (trekanten) i et koordinatsystem. Dette kan gjøres på mange måter, men det lønner seg å legge en av vektorene langs en av koordinataksene. I figur 1.1 har vi valgt å plassere trekanten slik at $\boldsymbol{c} = \boldsymbol{i}$. De andre vektorene er da gitt ved

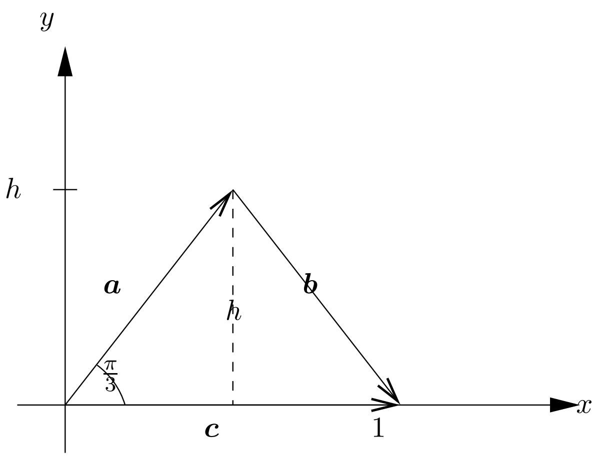
$$\begin{aligned}\boldsymbol{a} &= \frac{1}{2}\boldsymbol{i} + h\boldsymbol{j} \\ \boldsymbol{b} &= \frac{1}{2}\boldsymbol{i} - h\boldsymbol{j}.\end{aligned}$$

Vi finner $h$:

$$\sin\frac{\pi}{3} = \frac{h}{|\boldsymbol{a}|} = \frac{h}{1} = h \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}$$

og løsningen er gitt ved

$$\boldsymbol{a} = \frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}, \quad \boldsymbol{b} = \frac{1}{2}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{j}, \quad \boldsymbol{c} = \boldsymbol{i}.$$




Figur 1.1: Tre enhetsvektorer plassert slik at $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$.

## Oppgave 5

Skalarproduktet av to vektorer er definert ved

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

der $\theta$ er vinkelen mellom $\boldsymbol{a}$ og $\boldsymbol{b}$. Vi løser for $\cos\theta$:

$$\begin{aligned}
\cos\theta &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} \\
\boldsymbol{a}\cdot\boldsymbol{b} &= -3-6-8=-17 \\
|\boldsymbol{a}| &= \sqrt{9+4+16}=\sqrt{29} \\
|\boldsymbol{b}| &= \sqrt{1+9+4}=\sqrt{14} \\
\cos\theta &= -\frac{17}{\sqrt{29\cdot 14}}=-\frac{17}{\sqrt{406}}.
\end{aligned}$$

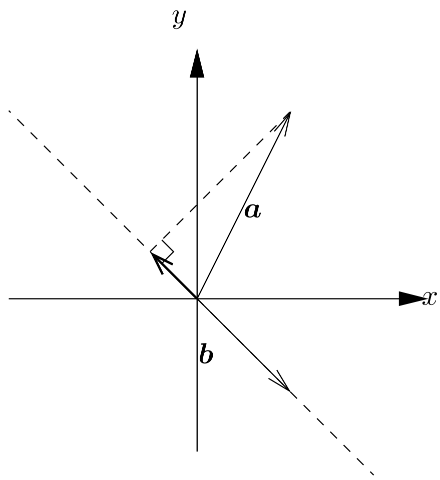
Denne likningen gir to mulig løsninger: $\theta\approx 147.53^\circ$ og $\theta\approx 212.47^\circ$. Siden vinkelen mellom to vektorer ikke kan overstige $180^\circ$, blir riktig vinkel her $\theta\approx 147.53^\circ$.

## Oppgave 6

Projeksjonen av $\boldsymbol{a}=\boldsymbol{i}+2\boldsymbol{j}$ på $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{j}$ er

$$\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{b}|^2}\boldsymbol{b}=\frac{1-2}{\sqrt{1+1}^2}\boldsymbol{b}=-\frac{1}{2}\boldsymbol{b}=-\frac{1}{2}(\boldsymbol{i}-\boldsymbol{j}).$$

Som vi ser av figur 1.2 kan projeksjonen av $\boldsymbol{a}$ på $\boldsymbol{b}$ tolkes som den delen av vektoren $\boldsymbol{a}$ som går i $\boldsymbol{b}$-retning.




Figur 1.2: Projeksjonen av $\boldsymbol{i}+2\boldsymbol{j}$ på $\boldsymbol{i}-\boldsymbol{j}$.

## Oppgave 7

Bruk høyrehåndsregelen eller definisjonen av kryssproduktet.

| $\times$ | $\boldsymbol{i}$ | $\boldsymbol{j}$ | $\boldsymbol{k}$ |
|---|---|---|---|
| $\boldsymbol{i}$ | $0$ | $\boldsymbol{k}$ | $-\boldsymbol{j}$ |
| $\boldsymbol{j}$ | $-\boldsymbol{k}$ | $0$ | $\boldsymbol{i}$ |
| $\boldsymbol{k}$ | $\boldsymbol{j}$ | $-\boldsymbol{i}$ | $0$ |

## Oppgave 8

Gitt vektorene $\boldsymbol{a} = \{a_1, a_2, a_3\}, \boldsymbol{b} = \{b_1, b_2, b_3\}, \boldsymbol{c} = \{c_1, c_2, c_3\}$ og skalaren $\alpha$.

a) Vis at $\boldsymbol{a} \times \boldsymbol{b} = 0$ hvis og bare hvis $\boldsymbol{a}$ og $\boldsymbol{b}$ er parallelle eller hvis $\boldsymbol{a} = 0$ eller $\boldsymbol{b} = 0$:

Anta at $\boldsymbol{a}$ og $\boldsymbol{b}$ er parallelle. Da eksisterer det en $\alpha$ slik at $\boldsymbol{a} = \alpha\boldsymbol{b}$. Da har vi $a_1 = \alpha b_1$, $a_2 = \alpha b_2$ og $a_3 = \alpha b_3$. Dette gir oss følgende uttrykk:

$$\alpha = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

La oss nå se på $\boldsymbol{a} \times \boldsymbol{b} = 0$:

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}(a_1b_2 - a_2b_1) = 0.
\end{aligned}$$

Dette gir tre likninger:

$$a_2b_3 = a_3b_2 \tag{1.4}$$

$$a_1b_3 = a_3b_1 \tag{1.5}$$

$$a_1b_2 = a_2b_1. \tag{1.6}$$

$$\begin{aligned}
(1.4) \quad &\Rightarrow \quad a_2 = \frac{a_3}{b_3}b_2 = \alpha b_2 \\
(1.5) \quad &\Rightarrow \quad a_3 = \frac{a_1}{b_1}b_3 = \alpha b_3 \\
(1.6) \quad &\Rightarrow \quad a_1 = \frac{a_2}{b_2}b_1 = \alpha b_1
\end{aligned}$$

Altså er $\boldsymbol{a}$ og $\boldsymbol{b}$ parallelle. Hvis enten $\boldsymbol{a} = 0$ eller $\boldsymbol{b} = 0$ er det opplagt at $\boldsymbol{a} \times \boldsymbol{b} = 0$.

b) Vis at $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$:

$$\begin{aligned}
-\boldsymbol{b} \times \boldsymbol{a} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -b_1 & -b_2 & -b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= \boldsymbol{i}(-b_2a_3 + b_3a_2) - \boldsymbol{j}(-b_1a_3 + b_3a_1) + \boldsymbol{k}(-b_1a_2 + b_2a_1) \\
&= \boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}(a_1b_2 - a_2b_1) \\
&= \boldsymbol{a} \times \boldsymbol{b}.
\end{aligned}$$

c) Vis at $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$:

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} \\
&= \boldsymbol{i}\Big(a_2(b_3 + c_3) - a_3(b_2 + c_2)\Big) - \boldsymbol{j}\Big(a_1(b_3 + c_3) - a_3(b_1 + c_1)\Big) + \\
&\quad \boldsymbol{k}\Big(a_1(b_2 + c_2) - a_2(b_1 + c_1)\Big) \\
&= \boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}(a_1b_2 - a_2b_1) + \\
&\quad \boldsymbol{i}(a_2c_3 - a_3c_2) - \boldsymbol{j}(a_1c_3 - a_3c_1) + \boldsymbol{k}(a_1c_2 - a_2c_1) \\
&= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}.
\end{aligned}$$

d) Vis at $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$:

$$\begin{aligned}
(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \boldsymbol{i}\Big((a_2 + b_2)c_3 - (a_3 + b_3)c_2\Big) - \boldsymbol{j}\Big((a_1 + b_1)c_3 - (a_3 + b_3)c_1\Big) + \\
&\quad \boldsymbol{k}\Big((a_1 + b_1)c_2 - (a_2 + b_2)c_1\Big) \\
&= \boldsymbol{i}(a_2c_3 - a_3c_2) - \boldsymbol{j}(a_1c_3 - a_3c_1) + \boldsymbol{k}(a_1c_2 - a_2c_1) + \\
&\quad \boldsymbol{i}(b_2c_3 - b_3c_2) - \boldsymbol{j}(b_1c_3 - b_3c_1) + \boldsymbol{k}(b_1c_2 - b_2c_1) \\
&= \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}.
\end{aligned}$$

e) Vis at $(\alpha\boldsymbol{a}) \times \boldsymbol{b} = \alpha(\boldsymbol{a} \times \boldsymbol{b})$:

$$\begin{aligned}
(\alpha\boldsymbol{a}) \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \alpha a_1 & \alpha a_2 & \alpha a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \boldsymbol{i}(\alpha a_2b_3 - \alpha a_3b_2) - \boldsymbol{j}(\alpha a_1b_3 - \alpha a_3b_1) + \boldsymbol{k}(\alpha a_1b_2 - \alpha a_2b_1) \\
&= \alpha\Big[\boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}(a_1b_2 - a_2b_1)\Big] \\
&= \alpha(\boldsymbol{a} \times \boldsymbol{b}).
\end{aligned}$$

f) Vis at $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.

La oss se på den venstre siden først:

$$
\begin{aligned}
\boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}(a_1b_2 - a_2b_1) \\
(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \boldsymbol{i}\Big((a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2\Big) - \\
&\quad\ \boldsymbol{j}\Big((a_2b_3 - a_3b_2)c_3 - (a_1b_2 - a_2b_1)c_1\Big) + \\
&\quad\ \boldsymbol{k}\Big((a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1\Big).
\end{aligned}
\tag{1.7}
$$

Den høyre siden ledd for ledd:

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} &= (a_1c_1 + a_2c_2 + a_3c_3)\boldsymbol{b} \\
&= (a_1c_1 + a_2c_2 + a_3c_3)b_1\boldsymbol{i} + \\
&\quad\ (a_1c_1 + a_2c_2 + a_3c_3)b_2\boldsymbol{j} + \\
&\quad\ (a_1c_1 + a_2c_2 + a_3c_3)b_3\boldsymbol{k} \\
(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} &= (b_1c_1 + b_2c_2 + b_3c_3)\boldsymbol{b} \\
&= (b_1c_1 + b_2c_2 + b_3c_3)a_1\boldsymbol{i} + \\
&\quad\ (b_1c_1 + b_2c_2 + b_3c_3)a_2\boldsymbol{j} + \\
&\quad\ (b_1c_1 + b_2c_2 + b_3c_3)a_3\boldsymbol{k}.
\end{aligned}
$$

Vi samler leddene:

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} &= \boldsymbol{i}(a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 - b_1c_1a_1 - b_2c_2a_1 - b_3c_3a_1) + \\
&\quad\ \boldsymbol{j}(a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 - b_1c_1a_2 - b_2c_2a_2 - b_3c_3a_2) + \\
&\quad\ \boldsymbol{k}(a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3 - b_1c_1a_3 - b_2c_2a_3 - b_3c_3a_3).
\end{aligned}
$$

En enkel opprydding her gir oss (1.7).

g) Vis at $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ hvis og bare hvis $(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b} = 0$.
$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ regnet vi ut i deloppgave f (likning 1.7). På tilsvarende måte får vi

$$
\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= \boldsymbol{i}\Big(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\Big) - \\
&\quad\ \boldsymbol{j}\Big(a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)\Big) + \\
&\quad\ \boldsymbol{k}\Big(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\Big).
\end{aligned}
\tag{1.8}
$$

Setter vi likning 1.7 og likning 1.8 lik hverandre får vi følgende

$\boldsymbol{i}$-retning:

$$a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2 = a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3$$
$$\Rightarrow \quad -a_1b_3c_3 - a_1b_2c_2 + a_2b_2c_1 + a_3b_3c_1 = 0. \tag{1.9}$$

$\boldsymbol{j}$-retning:

$$-a_2b_3c_3 + a_3b_2c_3 + a_1b_2c_1 - a_2b_1c_1 = -a_1b_1c_2 + a_1b_2c_1 + a_3b_2c_3 - a_3b_3c_2$$
$$\Rightarrow \quad -a_2b_3c_3 - a_2b_1c_1 + a_1b_1c_2 + a_3b_3c_2 = 0. \tag{1.10}$$

$\boldsymbol{k}$-retning:

$$a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1 = a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2$$
$$\Rightarrow \quad -a_3b_2c_2 - a_3b_1c_1 + a_1b_1c_3 + a_2b_2c_3 = 0. \tag{1.11}$$

Vi kan nå regne ut $(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b}$:

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b} &= \boldsymbol{i}\Big((-a_1c_3 + a_3c_1)b_3 - (a_1c_2 - a_2c_1)b_2\Big) - \\ &\quad \boldsymbol{j}\Big((a_2c_3 - a_3c_2)b_3 - (a_1c_2 - a_2c_1)b_1\Big) + \\ &\quad \boldsymbol{k}\Big((a_2c_3 - a_3c_2)b_2 - (-a_1c_3 + a_3c_1)b_1\Big)\end{aligned}$$

Settes dette uttrykket lik null får vi likningene (1.9) - (1.11).

h) Vis at $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} + (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} + (\boldsymbol{c} \times \boldsymbol{a}) \times \boldsymbol{b} = 0$:

$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ regnet vi ut i deloppgave f (likning 1.7). De to andre leddene gir

$$\begin{aligned}(\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} &= \boldsymbol{i}\Big((b_3c_1 - b_1c_3)a_3 - (b_1c_2 - b_2c_1)a_2\Big) - \\ &\quad \boldsymbol{j}\Big((b_2c_3 - b_3c_2)a_3 - (b_1c_2 - b_2c_1)a_1\Big) + \\ &\quad \boldsymbol{k}\Big((b_2c_3 - b_3c_2)a_2 - (b_3c_1 - b_1c_3)a_1\Big)\end{aligned}$$

og

$$\begin{aligned}(\boldsymbol{c} \times \boldsymbol{a}) \times \boldsymbol{b} &= \boldsymbol{i}\Big((a_1c_3 - a_3c_1)b_3 - (a_2c_1 - a_1c_2)b_2\Big) - \\ &\quad \boldsymbol{j}\Big((a_3c_2 - a_2c_3)b_3 - (a_2c_1 - a_1c_2)b_1\Big) + \\ &\quad \boldsymbol{k}\Big((a_3c_2 - a_2c_3)b_2 - (a_1c_3 - a_3c_1)b_1\Big).\end{aligned}$$

Summerer vi opp leddene får vi

$\boldsymbol{i}$-retning:

$$\begin{aligned}& a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2 + \\ & a_3b_3c_1 - a_3b_1c_3 - a_2b_1c_2 + a_2b_2c_1 + \\ & a_1b_3c_3 - a_3b_3c_1 - a_2b_2c_1 + a_1b_2c_2 \\ = \;& 0.\end{aligned}$$

$\boldsymbol{j}$-retning:

$$
\begin{aligned}
& -a_2b_3c_3 + a_3b_2c_3 + a_1b_2c_1 - a_2b_1c_1 + \\
& -a_3b_2c_3 + a_3b_3c_2 + a_1b_1c_2 - a_1b_2c_1 + \\
& -a_3b_3c_2 + a_2b_3c_3 + a_2b_1c_1 - a_1b_1c_2 \\
= \;& 0.
\end{aligned}
$$

$\boldsymbol{k}$-retning:

$$
\begin{aligned}
& a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1 + \\
& a_2b_2c_3 - a_2b_3c_2 - a_1b_3c_1 + a_1b_1c_3 + \\
& a_3b_2c_2 - a_2b_2c_3 - a_1b_1c_3 + a_3b_1c_1 \\
= \;& 0.
\end{aligned}
$$

## Oppgave 9

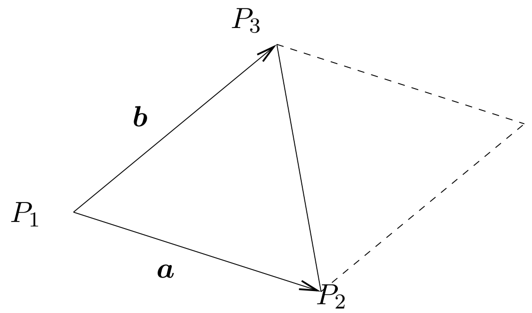
Arealet av parallellogrammet utspent av $\boldsymbol{a}$ og $\boldsymbol{b}$ er $|\boldsymbol{a}\times\boldsymbol{b}|$:

$$
\begin{aligned}
\boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 2 \\ -1 & 4 & 3 \end{vmatrix} \\
&= \boldsymbol{i}(-3-8) - \boldsymbol{j}(9+2) + \boldsymbol{k}(12-1) \\
&= -11\boldsymbol{i} - 11\boldsymbol{j} + 11\boldsymbol{k}
\end{aligned}
$$

$$
\text{Arealet} = |\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{121+121+121} = 11\sqrt{3}.
$$

## Oppgave 10

Gitt tre punkter $P_1 = (-2,-1,2)$, $P_2 = (2,0,1)$ og $P_3 = (2,2,0)$. Arealet av trekanten er halvparten av arealet til et parallellogram dannet av vektorene $\boldsymbol{a}$ (fra $P_1$ til $P_2$) og $\boldsymbol{b}$ (fra $P_1$ til $P_3$).




Figur 1.3: En trekant er halvparten av et parallellogram.

$$
\begin{aligned}
\boldsymbol{a} &= 4\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \\
\boldsymbol{b} &= 4\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}
\end{aligned}
$$

$$\begin{aligned}
\boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 1 & -1 \\ 4 & 3 & -2 \end{vmatrix} \\
&= \boldsymbol{i}(-2+3) - \boldsymbol{j}(-8+4) + \boldsymbol{k}(12-4) \\
&= \boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}
\end{aligned}$$

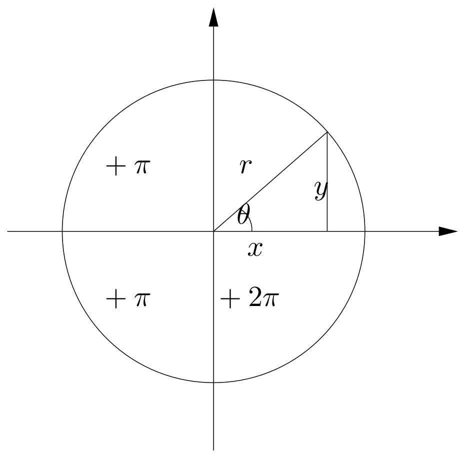
$$\text{Arealet} = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\sqrt{1+16+64} = \frac{1}{2}\sqrt{81} = \frac{9}{2}.$$

## Oppgave 11

Generelt har vi:

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2} \\
\theta &= \arctan y/x, \quad x>0,\ y\geq 0 \\
&\quad (+\pi \text{ hvis } x<0,\ +2\pi \text{ hvis } x>0 \text{ og } y<0).
\end{aligned}$$

Alternativt kan man bruke funksjonen atan2$(y,x)$ som finnes i diverse programmeringsspråk (f.eks. Matlab) og som er lik $\arctan y/x$ i første og fjerde kvadrant, $\arctan y/x + \pi$ i andre kvadrant, og $\arctan y/x - \pi$ i tredje kvadrant.




Figur 1.4: Plane polarkoordinater og enhetssirkelen.

a) Kartesiske koordinater: $(6, 2\sqrt{3})$. Polarkoordinater:

$$\begin{aligned}
r &= \sqrt{x^2+y^2} = \sqrt{36+4\cdot 3} = \sqrt{48} = 4\sqrt{3} \\
\theta &= \arctan y/x = \arctan\sqrt{3}/3 = \frac{\pi}{6}.
\end{aligned}$$

Punktet uttrykt i polarkordinater er $(4\sqrt{3}, \pi/6)$.

b) Polarkoordinater: $(3, \pi/2)$. Kartesiske koordinater:

$$\begin{aligned} x &= r\cos\theta = 3\cos\frac{\pi}{2} = 0 \\ y &= r\sin\theta = 3\sin\frac{\pi}{2} = 3. \end{aligned}$$

De kartesiske koordinatene til punktet er $(0, 3)$. Posisjonsvektoren uttrykt i kartesiske koordinater er $3\boldsymbol{j}$.

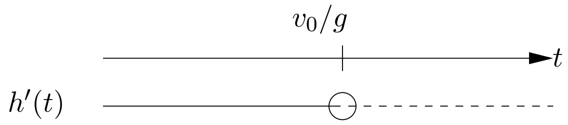
## Oppgave 12

Gitt formelen for høyden ($h$) som funksjon av tiden ($t$) til en rakett som skytes vertikalt oppover i tyngdefeltet:

$$h(t) = v_0 t - \frac{1}{2}gt^2. \tag{1.12}$$

a) Den største høyden ($t_{\mathrm{m}}$) finner vi ved å derivere funksjonen og bestemme ekstremalverdiene.

$$h'(t) = v_0 - gt = 0 \quad \Rightarrow \quad t = \frac{v_0}{g}.$$




$t_{\mathrm{m}} = v_0/g$ er et makspunkt for $h(t)$. La oss sjekke dimensjonen til $t_{\mathrm{m}}$ for sikkerhetsskyld:

$$[t_{\mathrm{m}}] = \frac{[v_0]}{[g]} = \frac{m/s}{m/s^2} = s.$$

Den maksimale høyden er nå lett å regne ut:

$$\begin{aligned} h_{\mathrm{m}} &= h(t_{\mathrm{m}}) = \frac{v_0^2}{g} - \frac{1}{2}g\frac{v_0^2}{g^2} = \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g} \\ &= \frac{1}{2}\frac{v_0^2}{g}. \end{aligned}$$

Dimensjonen blir

$$[h_{\mathrm{m}}] = \frac{[v_0^2]}{[g]} = \frac{m^2/s^2}{m/s^2} = m.$$

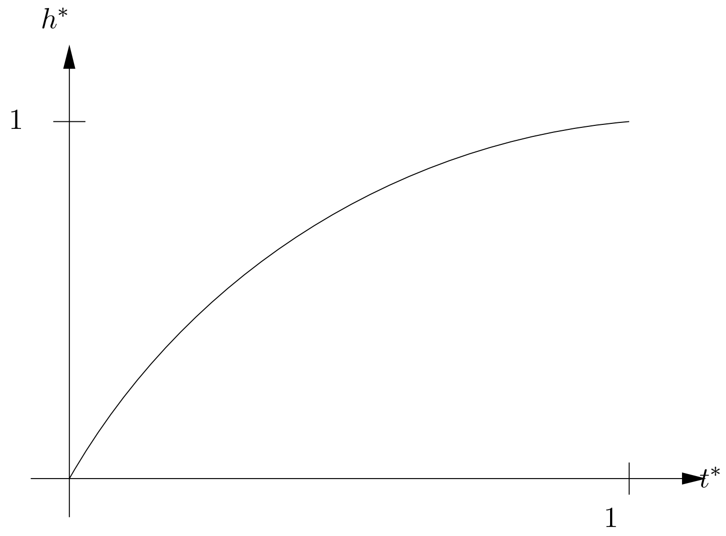
b) Vi innfører to nye dimensjonsløse parametere (variable):

$$\begin{aligned} h^* &= \frac{h}{h_{\mathrm{m}}} \quad \Rightarrow \quad h = h^* h_{\mathrm{m}} \\ t^* &= \frac{t}{t_{\mathrm{m}}} \quad \Rightarrow \quad t = t^* t_{\mathrm{m}}. \end{aligned}$$

Insatt i likning 1.12 gir dette:

$$\begin{aligned}
h^* h_\mathrm{m} &= v_0 t^* t_\mathrm{m} - \frac{1}{2} g {t^*}^2 t_\mathrm{m}^2 \\
h^* &= \frac{v_0 t^* t_\mathrm{m}}{h_\mathrm{m}} - \frac{1}{2} \frac{g {t^*}^2 t_\mathrm{m}^2}{h_\mathrm{m}} \\
&= \frac{v_0 t^* \frac{v_0}{g}}{\frac{1}{2}\frac{v_0^2}{g}} - \frac{1}{2} g \frac{{t^*}^2 \frac{v_0^2}{g^2}}{\frac{1}{2}\frac{v_0^2}{g}} \\
&= 2t^* - {t^*}^2 .
\end{aligned}$$

c)




Figur 1.5: Funksjonen $h^* = 2t^* - {t^*}^2$.