Flateintegraler over flater med kurvatur

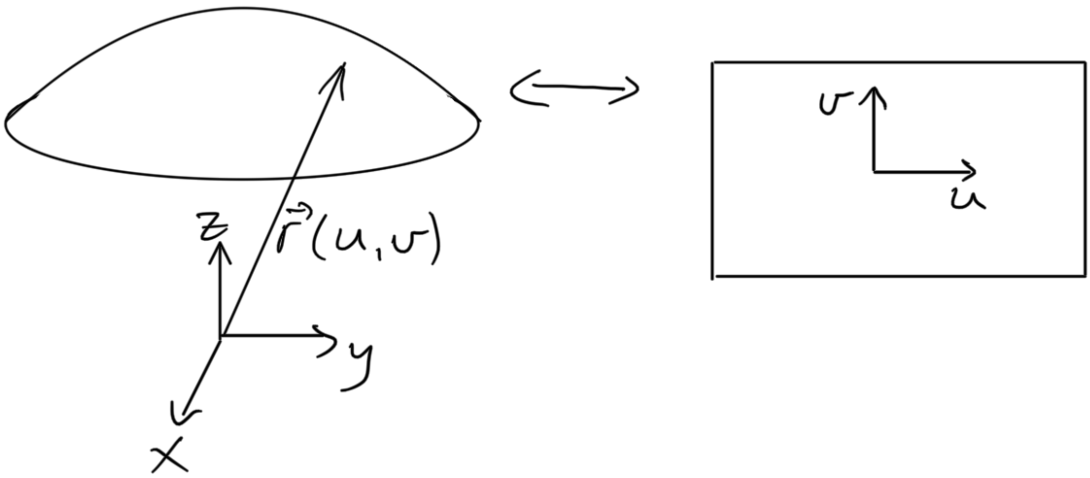
M 2.3

En flate kan parametriseres med 2 variable u og v.

Vi mapper den fysiske, aktuelle flaten, som er komplisert og krummet, til en parameterflate som er flat og enkel, vanligvis et rektangel.

Fysisk rom Kurvatur

Parameterrom Rettlinjet Flatt



Vi beskriver en flate med posisjonsvektoren

$$\vec{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

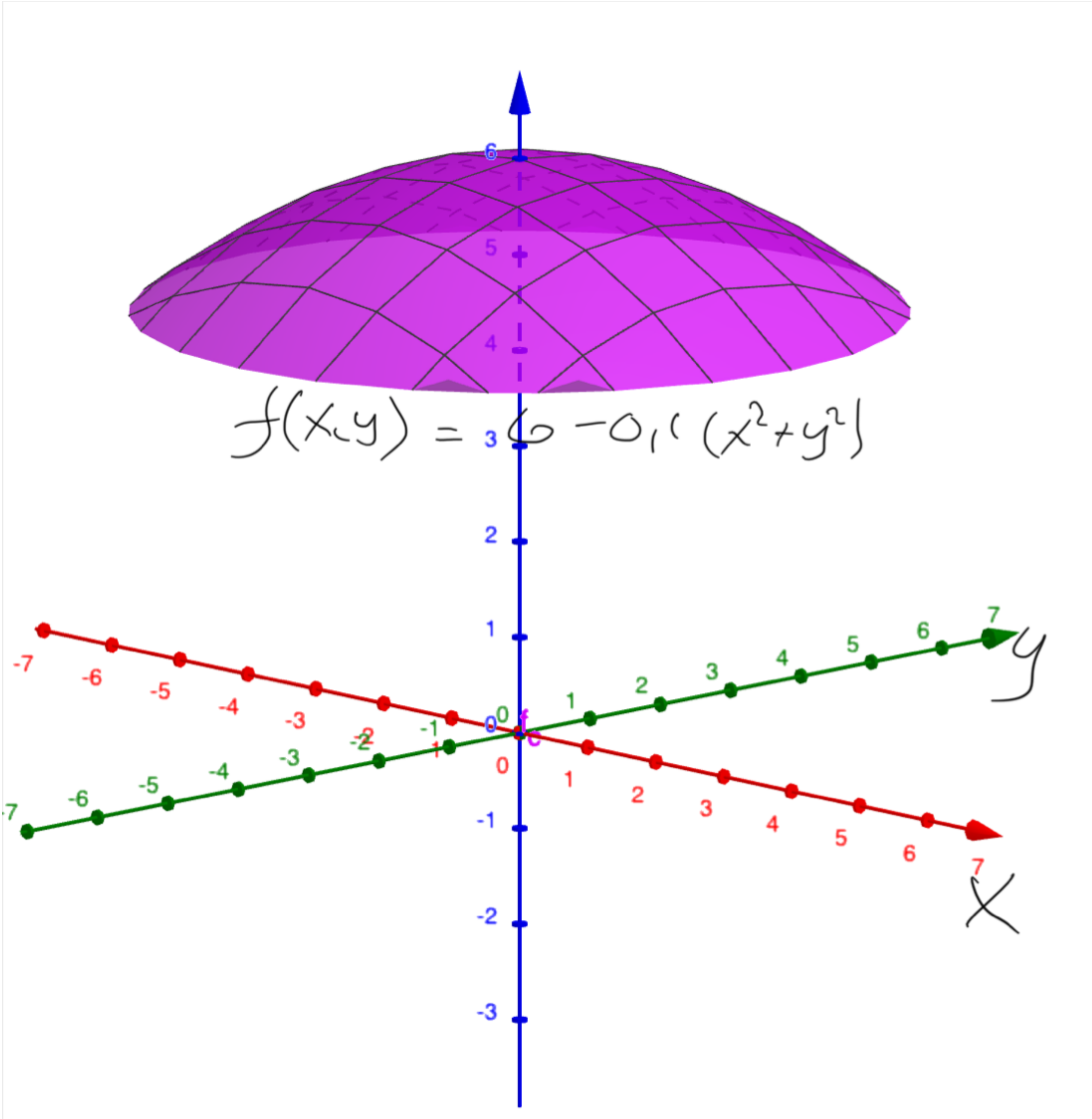
Eksempel på kurvet flate

$$x = u$$

$$y = v$$

$$z = f(u,v) = 6 - 0{,}1\,(u^2 + v^2)$$

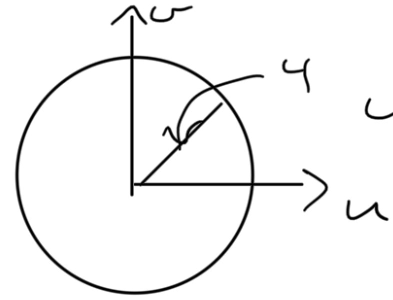
For $\quad u^2 + v^2 < 16$



f(x,y) = 6 − 0,1 (x² + y²)
y
x

Men her er ikke domenet til u og v et rektangel!

Men det er i det minste flatt.

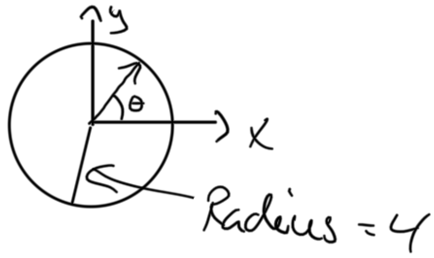


$u^2+v^2<16$

Dårlig eksempel!

Men i polarkoordinater har vi for samme flate

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$



$$f(r,\theta) = 6 - 0{,}1r^2$$

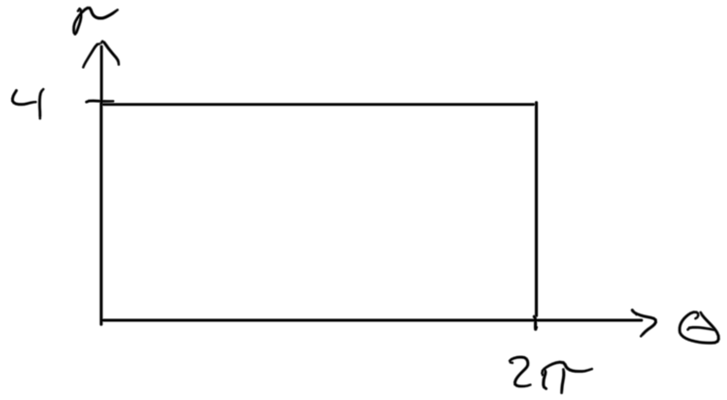
$$\vec{r}(r,\theta) = x(r,\theta)\mathbf{i} + y(r,\theta)\mathbf{j} + f(r,\theta)\mathbf{k}$$

Domenet er fremdeles $x^2+y^2<16$

Dette tilsvarer rektangelet

$$0 \le r < 4$$

$$0 \leq \theta < 2\pi$$

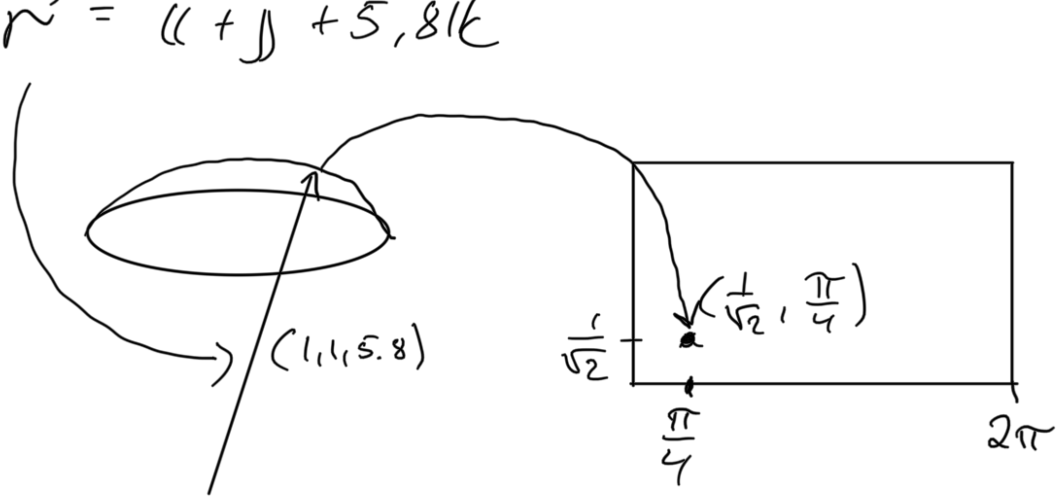


Så vi mapper flaten vi ser, den med kurvatur, til rektangelet $(r,\theta) \in [0,4] \times [0,2\pi)$ som er flatt.

En gitt posisjon på flaten S, tilsvarer en unik posisjon i rektangelet.

Ta $x = 1$, $y = 1$ $\Rightarrow$ $f(1,1) = 6 - 0{,}1(1^2 + 1^2)$
$= 5{,}8$

$\vec{r} = i + j + 5{,}8k$



$$x = r\cos\theta \Rightarrow 1 = r\cos\theta \Rightarrow r = \frac{1}{\cos\theta}$$

$$y = r\sin\theta \Rightarrow 1 = r\sin\theta = \frac{1}{\cos\theta}\sin\theta$$

$$\Rightarrow 1 = \tan\theta$$

$$\underline{\theta = \tan^{-1}(1) = \frac{\pi}{4}}$$

$$\Rightarrow r = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \underline{\sqrt{2}}$$

Hva med $\vec{r}(\theta, r) = 6\mathbf{k}$? $(0,0,6)$

$$\Rightarrow r = 0$$

$\theta = ?$ Ikke unik!

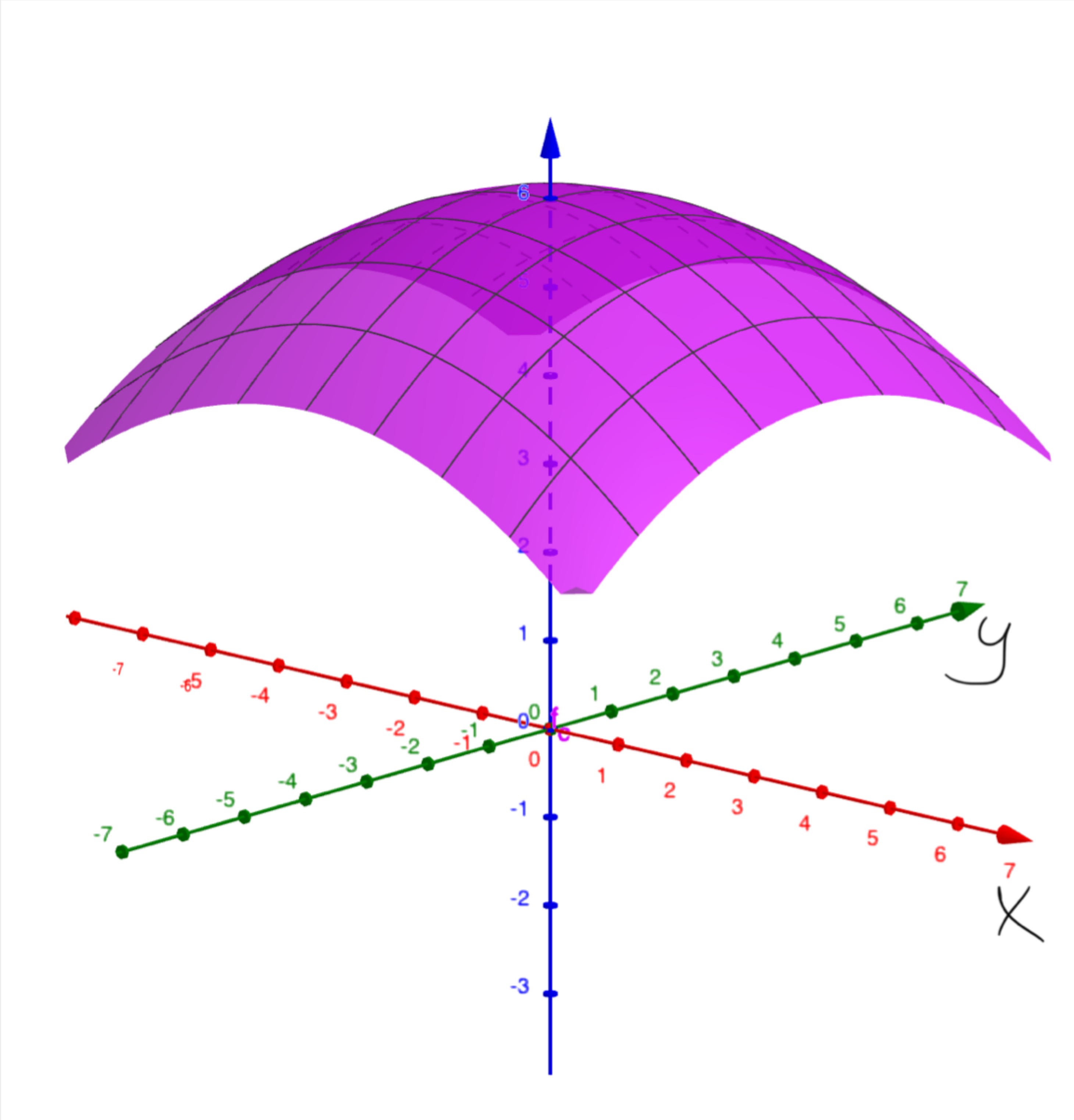
Ikke så farlig, og ikke så viktig for flateintegraler. Lar det ligge.

Liten detalj til. Hvis jeg vil mappe direkte til rektangel

$-4 < x < 4$ og $-4 < y < 4$

så er dette flaten

$$\vec{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$



y
X

Hva har dette med flateintegraler å gjøre?

I 1D mapper man til en rett linje og gjør

$$\int_C \rho(\vec{r})\,|d\vec{r}| = \int_t \rho(t)\left|\frac{d\vec{r}}{dt}\right| dt$$

$t = [0,1]$ for eks

Kurvet integral i rommet

Rettlinjet integral i parameterrom.

Samme for flater, men nå ser det litt verre ut

$$\int_S f(\vec{r})\,d\sigma = \int_v \int_u f(u,v)\left|\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\right| du\,dv$$

Krummet

Flatt

Hvor kom $\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|$ fra?

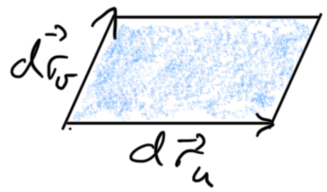
Dette er pga arealet til flateelementet $d\sigma$. Akkurat som $ds = |d\vec{r}|$ er lengden til bueelementet, så er

$$d\sigma = |d\vec{r}_u \times d\vec{r}_v|$$

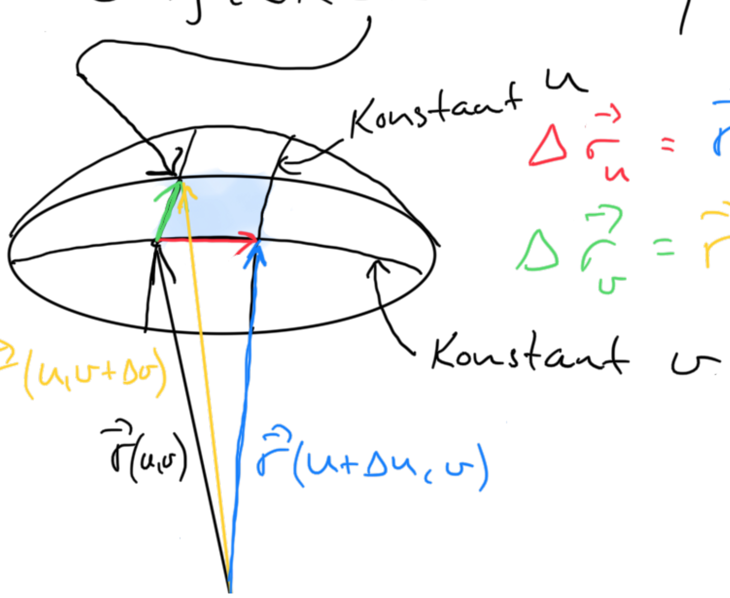
Arealet av parallellogrammet er $|d\vec{r}_u \times d\vec{r}_v|$

der $d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du$

$d\vec{r}_v = \frac{\partial \vec{r}}{\partial v} dv$



Dette krever litt bedre forklaring! Se på et flateelement på flaten vår



$$\Delta \vec{r}_u = \vec{r}(u+\Delta u, v) - \vec{r}(u,v)$$

$$\Delta \vec{r}_v = \vec{r}(u, v+\Delta v) - \vec{r}(u,v)$$

Areal av parallellogrammet er

$$|\Delta \vec{r}_u \times \Delta \vec{r}_v|$$

Hva er $\Delta \vec{r}_u$ og $\Delta \vec{r}_v$?

$$\lim_{\Delta u \to 0} \frac{\Delta \vec{r}_u}{\Delta u} = \frac{\partial \vec{r}}{\partial u} \qquad \lim_{\Delta v \to 0} \frac{\Delta \vec{r}_v}{\Delta v} = \frac{\partial \vec{r}}{\partial v}$$

eller

$$\lim_{\Delta u \to 0} \Delta \vec{r}_u = \frac{\partial \vec{r}}{\partial u} \Delta u \qquad \lim_{\Delta v \to 0} \Delta \vec{r}_v = \frac{\partial \vec{r}}{\partial v} \Delta v$$

Med infinitesimale størrelser går dette mot

$$\Rightarrow \quad d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du \quad , \quad d\vec{r}_v = \frac{\partial \vec{r}}{\partial v} dv$$

Total endring i posisjon er

$$d\vec{r}(u,v) = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$$

$$= d\vec{r}_u + d\vec{r}_v$$

Endring i posisjon $\vec{r}$

når man flytter seg du
i retning u, mens
v holdes konstant.

Vi forstår nå at arealet til et
flateelement er gitt ved

$$d\sigma = |d\vec{r}_u \times d\vec{r}_v|$$

som også kan skrives

$$d\sigma = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv$$

Flateintegral blir derfor

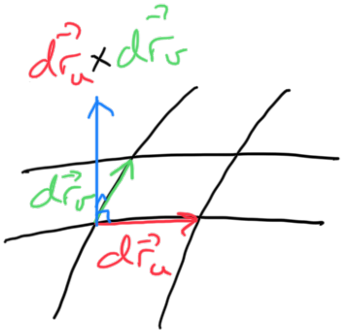
$$\int_S d\sigma = \int_v \int_u \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv$$

Dette er fremdeles en sum av
arealene til alle flateelementene

$$\lim_{N \to \infty} \sum_{i=1}^{N} \Delta\sigma_i$$

$\Delta\sigma_i$ = areal flateelement i.

Merk



$d\vec{r}_u \times d\vec{r}_v$ er normal til flateelementet

Derfor har vi en flatenormal

$$\vec{n} = \frac{d\vec{r}_u \times d\vec{r}_v}{|d\vec{r}_u \times d\vec{r}_v|}$$

$|\vec{n}| = 1$

setter inn $d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du$, $d\vec{r}_v = \frac{\partial \vec{r}}{\partial v} dv$ og forenkler

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}$$

For fluksintegral er dette spesielt viktig.

$$\int_S \vec{v} \cdot \vec{n} \, d\sigma$$

vi har

$$d\sigma = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

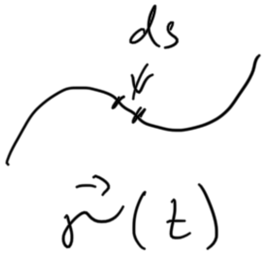
$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}$$

$$\Rightarrow \vec{n}\, d\sigma = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\, du\,dv$$

Og dermed

$$\int_S \vec{v} \cdot \vec{n}\, d\sigma = \iint_{v\,u} \vec{v} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$$

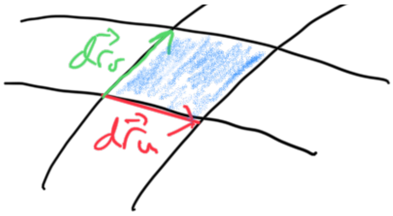
Husk:

1D 

$$\int_C ds = \int_t \left| \frac{d\vec{r}}{dt} \right| dt$$

Treger lengden av et bue-element

$$ds = |d\vec{r}| = \left| \frac{d\vec{r}}{dt} \right| dt$$

2D 

$$\int d\sigma = \int_v \int_u \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

Vi trenger arealet av et flateelement.

$$d\sigma = \left| d\vec{r}_u \times d\vec{r}_v \right|$$

$$= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

Eksempel

Finn arealet til flaten

$$\vec{r}(x,y) = x\mathbb{i} + y\mathbb{j} + f(x,y)\mathbb{k}$$

der $f(x,y) = 6 - 0{,}1(x^2+y^2)$

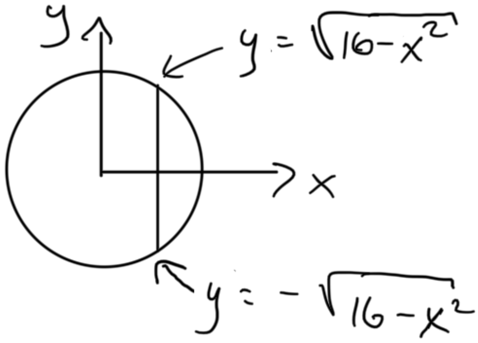
og $x^2+y^2 < 16$

Samme flate vi har sett på før.

Vil finne

$$\int_S d\sigma = \int_y \int_x \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| dx\,dy$$

Integrasjons-
grenser
y-først



$$\int_S d\sigma = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| dy\, dx$$

$$\frac{\partial \vec{r}}{\partial x} = \mathbb{i} + \frac{\partial f}{\partial x} \mathbb{k} \quad , \quad \frac{\partial \vec{r}}{\partial y} = \mathbb{j} + \frac{\partial f}{\partial y} \mathbb{k}$$

$$= \mathbb{i} - 0{,}2x\, \mathbb{k} \qquad\qquad = \mathbb{j} - 0{,}2y\, \mathbb{k}$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ 1 & 0 & -0{,}2x \\ 0 & 1 & -0{,}2y \end{vmatrix}$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \mathbb{i}\, 0{,}2x + \mathbb{j}\, 0{,}2y + \mathbb{k}$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{0{,}2^2 x^2 + 0{,}2^2 y^2 + 1}$$

$$= \sqrt{0{,}04(x^2+y^2)+1}$$

$$\text{Areal} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{0{,}04(x^2+y^2)+1}\, dy\, dx$$

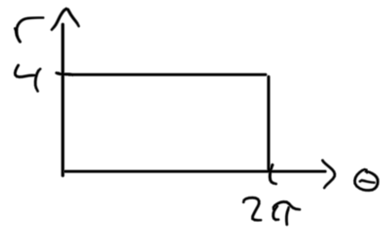
$\Rightarrow$ Litt for omstendelig! Løs numerisk eller analytisk, i for eksempel Python/sympy eller Geogebra.

Jeg burde ha brukt en annen, enklere (rektangel) parametrisering.

Samme flate, men nå med

$$\vec{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + f(r,\theta)\,\mathbf{k}$$

$$0 \leq r \leq 4$$
$$0 \leq \theta \leq 2\pi$$



$$\int_S d\sigma = \int_\theta \int_r \left| \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \right| dr\, d\theta$$

$$f(r,\theta) = 6 - 0{,}1\left((r\cos\theta)^2 + (r\sin\theta)^2\right)$$
$$= \underline{6 - 0{,}1\, r^2}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta \, \mathbb{i} + \sin\theta \, \mathbb{j} - 0{,}2r \, \mathbb{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta \, \mathbb{i} + r\cos\theta \, \mathbb{j}$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \cos\theta & \sin\theta & -0{,}2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= 0{,}2r^2\cos\theta \, \mathbb{i} + 0{,}2r^2\sin\theta \, \mathbb{j} + \mathbb{k}\left(r\cos^2\theta + r\sin^2\theta\right)$$

$$= \underline{0{,}2r^2\left(\cos\theta \, \mathbb{i} + \sin\theta \, \mathbb{j}\right) + r\mathbb{k}}$$

$$\left|\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right| = r\sqrt{0{,}04r^2 + 1}$$

$$\int d\sigma = \int_0^{2\pi}\int_0^4 \sqrt{0{,}04r^2+1} \; r \, dr \, d\theta$$

Mye enklere, men må fremdeles substituere

$$u = 0{,}04r^2 + 1$$

$$du = 0{,}08 r\, dr$$

$$\int_0^4 \sqrt{0{,}04 r^2 + 1}\, r\, dr = \int_1^{1{,}64} \frac{\sqrt{u}\, du}{0{,}08}$$

$$= \frac{1}{0{,}08} \frac{2}{3} u^{3/2} \Big]_1^{1{,}64}$$

$$= \frac{2}{0{,}24} \left(1{,}64^{3/2} - 1\right)$$

$$\int d\sigma = \int_0^{2\pi} \frac{2}{0{,}24} \left(1{,}64^{3/2} - 1\right) d\theta$$

$$= \frac{4\pi}{0{,}24} \left(1{,}64^{3/2} - 1\right)$$

$$\approx 57{,}607$$