Divergens av vektorfelt

M 3.3 $\quad$ $\text{div}\,\vec{u}$

$\nabla \cdot \vec{u}$

Divergensen av et vektorfelt er et skalarfelt

Kartesisk

$$\nabla \cdot \vec{u}$$

$$= \left(\mathbb{i}\frac{\partial}{\partial x} + \mathbb{j}\frac{\partial}{\partial y} + \mathbb{k}\frac{\partial}{\partial z} \right) \cdot \left(u_x \mathbb{i} + u_y \mathbb{j} + u_z \mathbb{k} \right)$$

$$= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

Generell definisjon gir et klarere bilde av hva $\nabla \cdot \vec{u}$ er

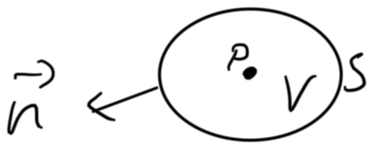
$$\nabla\cdot\vec{u} = \lim_{V\to 0}\frac{1}{V}\oint_S \vec{u}\cdot\vec{n}\,d\sigma \qquad (3.14)$$

V. C.

Definert for et punkt

$P = (x, y, z)$

Et integral rundt volumet V, i grensen at $V \to P$

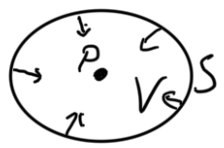


S = Omsluttende flate

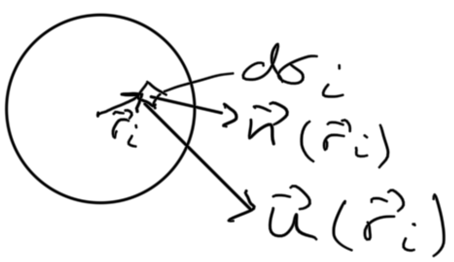
V = Volum

P = punkt

$\lim_{V\to 0}$



$$\oint_S \vec{u}\cdot\vec{n}\,d\sigma = \lim_{N\to\infty}\sum_{i=1}^{N}(\vec{u}\cdot\vec{n})_i\,\Delta\sigma_i$$



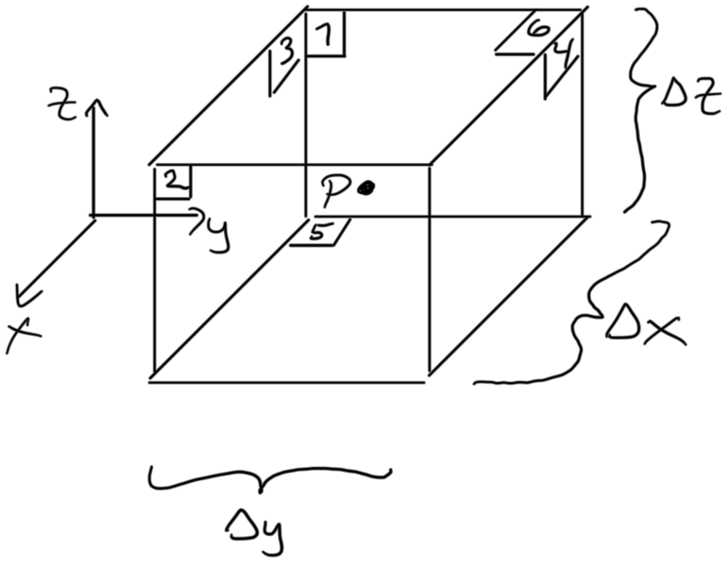
Representerer netto utstrømning gjennom flaten S. Netto flux ut av S.

Vil nå vise at (for kartesiske koor.)

$$\lim_{V \to 0} \frac{1}{V} \oint_S \vec{u} \cdot \vec{n} \, d\sigma$$

tilsvarer $\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$

Vi antar først at volumet V er en rektangulær boks, rundt punktet $P = (x, y, z)$



Boksen har 6 sideflater. Vi antar at boksen er liten ($V \to 0$) og at derfor så er $\vec{u}$ konstant på en sideflate.

$\Delta V = \Delta x \Delta y \Delta z$

Integralet over boksen blir

$$\oint_S \vec{u}\cdot\vec{n}\,d\sigma = \sum_{i=1}^{6}\int_{S_i}\vec{u}\cdot\vec{n}_i\,d\sigma_i$$

Deler altså opp ett <u>lukket</u> flate-integral i 6 integraler over de 6 sideflatene. Hvert enkelt flateintegral er da ikke lukket.

<u>6 flater</u>

1 : $\vec{n} = -\mathbf{i}$ $\quad d\sigma = dy\,dz$

$$\int_{S_1}\vec{u}\cdot\vec{n}\,d\sigma = -\iint \vec{u}\cdot\mathbf{i}\,dy\,dz$$

$$= -\iint u_x\,dy\,dz$$

$$\approx -u_x\left(x-\frac{\Delta x}{2}, y, z\right)\Delta y\,\Delta z$$

siden $u \approx$ konstant på flaten.

På samme måte

$$2: \int_{S_2} \vec{u}\cdot\vec{n}\, d\sigma = \int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} \vec{u}\cdot \mathbf{i}\, dy\, dz$$

$$\approx \underline{u_x\left(x+\frac{\Delta x}{2}, y, z\right)\Delta y \Delta z}$$

$$3: \int_{S_3} \vec{u}\cdot\vec{n}\, d\sigma = -\int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \vec{u}\cdot \mathbf{j}\, dx\, dz$$

$$\approx \underline{-u_y\left(x, y-\frac{\Delta y}{2}, z\right)\Delta x \Delta z}$$

$$4: \int_{S_4} \vec{u}\cdot\vec{n}\, d\sigma \approx u_y\left(x, y+\frac{\Delta y}{2}, z\right)\Delta x \Delta z$$

$$5: \int_{S_5} \vec{u}\cdot\vec{n}\, d\sigma = -\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \vec{u}\cdot \mathbf{k}\, dx\, dy$$

$$\approx -u_z\left(x, y, z-\frac{\Delta z}{2}\right)\Delta x \Delta y$$

$$6: \int_{S_6} \vec{u}\cdot\vec{n}\, d\sigma \approx u_z\left(x, y, z+\frac{\Delta z}{2}\right)\Delta x \Delta y$$

Summerer og får

$$\oint \vec{u}\cdot\vec{n}\,d\sigma \approx \left(\overset{②}{u_x\left(x+\frac{\Delta x}{2},y,z\right)} \overset{-}{-} \overset{①}{u_x\left(x-\frac{\Delta x}{2},y,z\right)}\right)\Delta y\Delta z$$

$$+\left(\overset{④}{u_y\left(x,y+\frac{\Delta y}{2},z\right)} \overset{-}{-} \overset{③}{u_y\left(x,y-\frac{\Delta y}{2},z\right)}\right)\Delta x\Delta z$$

$$+\left(\overset{⑥}{u_z\left(x,y,z+\frac{\Delta z}{2}\right)} \overset{-}{-} \overset{⑤}{u_z\left(x,y,z-\frac{\Delta z}{2}\right)}\right)\Delta x\Delta y$$

Deler nå på totalvolum

$$V = \Delta x\Delta y\Delta z$$

$$\frac{1}{V}\oint \vec{u}\cdot\vec{n}\,d\sigma \approx \frac{\left(u_x\left(x+\frac{\Delta x}{2},y,z\right)-u_x\left(x-\frac{\Delta x}{2},y,z\right)\right)\cancel{\Delta y}\cancel{\Delta z}}{\Delta x\cancel{\Delta y}\cancel{\Delta z}}$$

$$+\frac{\left(u_y\left(x,y+\frac{\Delta y}{2},z\right)-u_y\left(x,y-\frac{\Delta y}{2},z\right)\right)\cancel{\Delta x}\cancel{\Delta z}}{\cancel{\Delta x}\Delta y\cancel{\Delta z}}$$

$$+\frac{\left(u_z\left(x,y,z+\frac{\Delta z}{2}\right)-u_z\left(x,y,z-\frac{\Delta z}{2}\right)\right)\cancel{\Delta x}\cancel{\Delta y}}{\cancel{\Delta x}\cancel{\Delta y}\Delta z}$$

$$\frac{1}{V}\oint \vec{u}\cdot\vec{n}\,d\sigma \approx \lim_{\Delta x\to 0}\frac{u_x\left(x+\frac{\Delta x}{2},y,z\right)-u_x\left(x-\frac{\Delta x}{2},y,z\right)}{\Delta x}$$

$$+\frac{u_y\left(x,y+\frac{\Delta y}{2},z\right)-u_y\left(x,y-\frac{\Delta y}{2},z\right)}{\Delta y}$$

$$+\frac{u_z\left(x,y,z+\frac{\Delta z}{2}\right)-u_z\left(x,y,z-\frac{\Delta z}{2}\right)}{\Delta z}$$

Ved $\lim V \to 0$ har vi $\Delta x, \Delta y, \Delta z \to 0$

$$\lim_{V \to 0} \frac{1}{V} \oint \vec{u} \cdot \vec{n} \, d\sigma = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

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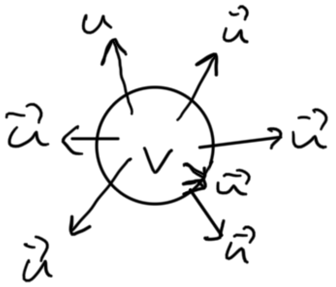
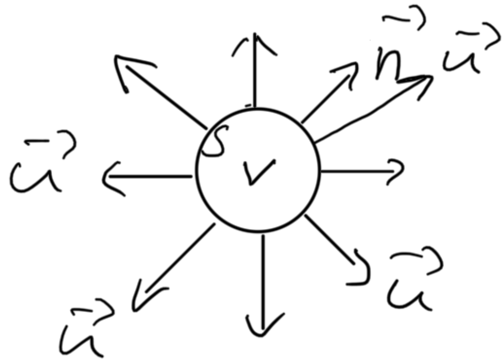
som var det vi ville vise.

Divergens sier noe om netto tilstrømning i et punkt.

Hvis vi har $\vec{u} \cdot \vec{n} > 0$





Ekspansjon

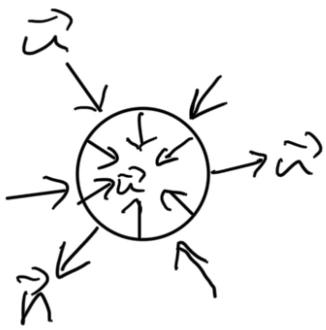
2D

Netto utstrømning

Hvis $\vec{u} \cdot \vec{n} > 0 \Rightarrow \oint \vec{u} \cdot \vec{n} \, d\sigma > 0$

og dermed $\nabla \cdot \vec{u} > 0$.

Hvis $\vec{u} \cdot \vec{n} < 0$ har vi kontraksjon



Siden $\vec{u}\cdot\vec{n} < 0 \Rightarrow \oint \vec{u}\cdot\vec{n}\,d\sigma < 0$

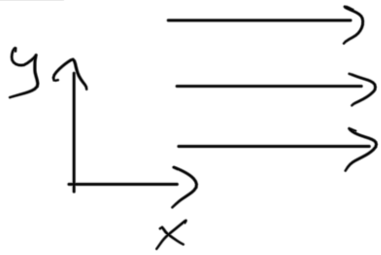
$$\Rightarrow \nabla\cdot\vec{u} < 0$$

Hvis $\nabla\cdot\vec{u} = 0$ overalt, så er feltet inkompressibelt

Vann er inkompressibelt. Det kan ikke komprimeres.

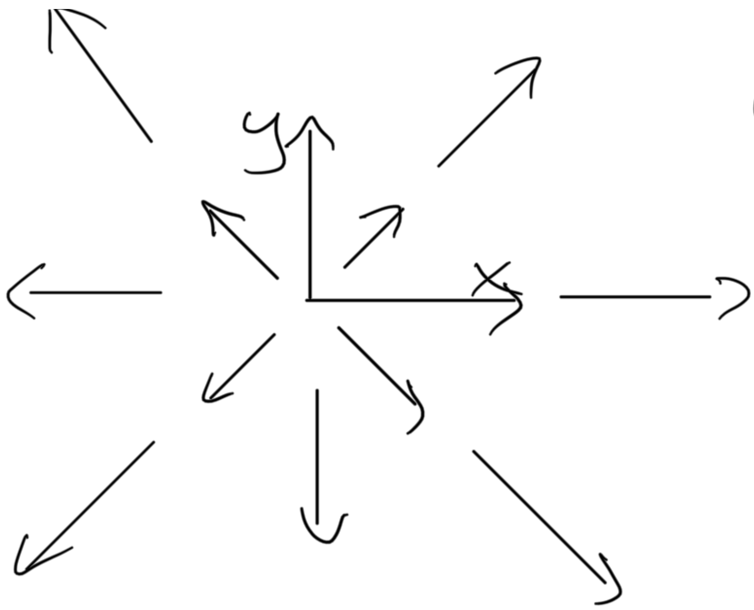
<u>Noen eksempler</u>

$$\vec{u} = U_0 \mathbf{i}$$



$$\nabla\cdot\vec{u} = \frac{\partial U_0}{\partial x} = 0$$ Inkompressibelt

$$\vec{u} = \vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Ekspansjon

$\vec{r}\cdot\vec{n} > 0$

$$\nabla\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = \underline{\underline{3}}$$

$$\vec{u} = y^2\,\mathbf{i} + x^3 z^4\,\mathbf{j} + x^9 y^{12}\,\mathbf{k}$$

$$\underline{\nabla\cdot\vec{u} = 0}$$