Krumlinjete koordinater
Gradient til skalarfelt f

Husk

$$df = \text{grad}(f) \cdot d\vec{r}$$

Kartesisk:

$$\vec{r} = x\mathbb{i} + y\mathbb{j} + z\mathbb{k}$$

$$d\vec{r} = \mathbb{i}\,dx + \mathbb{j}\,dy + \mathbb{k}\,dz$$

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left(\mathbb{i}\frac{\partial f}{\partial x} + \mathbb{j}\frac{\partial f}{\partial y} + \mathbb{k}\frac{\partial f}{\partial z}\right) \cdot d\vec{r}$$

$$\Rightarrow \quad \text{grad} f = \mathbb{i}\frac{\partial f}{\partial x} + \mathbb{j}\frac{\partial f}{\partial y} + \mathbb{k}\frac{\partial f}{\partial z}$$

$$= \nabla f$$

Krumlinjet: (u_1, u_2, u_3)

$$\vec{r} = X(u_1, u_2, u_3)\,\mathbf{i} + Y(u_1, u_2, u_3)\,\mathbf{j} + Z(u_1, u_2, u_3)\,\mathbf{k}$$

$$d\vec{r} = h_1 \vec{e}_1 du_1 + h_2 \vec{e}_2 du_2 + h_3 \vec{e}_3 du_3$$

$$df(u_1, u_2, u_3) = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3$$

$$= \left(\frac{\vec{e}_1}{h_1} \frac{\partial f}{\partial u_1} + \frac{\vec{e}_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{\vec{e}_3}{h_3} \frac{\partial f}{\partial u_3} \right) \cdot d\vec{r}$$

$\Rightarrow$ grad $f =$

$$\nabla = \frac{\vec{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\vec{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\vec{e}_3}{h_3} \frac{\partial}{\partial u_3}$$

Alternativ utledning

Har $\nabla f = \frac{\partial f}{\partial x}\mathbb{i} + \frac{\partial f}{\partial y}\mathbb{j} + \frac{\partial f}{\partial z}\mathbb{k}$

og $\nabla f = g_1 \vec{e}_1 + g_2 \vec{e}_2 + g_3 \vec{e}_3$

der g_i er ubestemt, $i \in (1,2,3)$

Vet at $\nabla f = \nabla f$

Samme vektor!

Vet at

$$\nabla f \cdot \vec{e}_1 = g_1$$

Og derfor

$$\nabla f \cdot \vec{e}_1 = g_1$$

$$\left(\frac{\partial f}{\partial x}\mathbb{i} + \frac{\partial f}{\partial y}\mathbb{j} + \frac{\partial f}{\partial z}\mathbb{k}\right) \cdot \vec{e}_1 = g_1$$

Har

$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1} = \frac{1}{h_1}\left(\frac{\partial x}{\partial u_1}\mathbb{i} + \frac{\partial y}{\partial u_1}\mathbb{j} + \frac{\partial z}{\partial u_1}\mathbb{k}\right)$$

$\Rightarrow$

$$\nabla f \cdot \vec{e}_1 = \left(\frac{\partial f}{\partial x}\mathbb{i} + \frac{\partial f}{\partial y}\mathbb{j} + \frac{\partial f}{\partial z}\mathbb{k}\right) \cdot \frac{1}{h_1}\left(\frac{\partial x}{\partial u_1}\mathbb{i} + \frac{dy}{du_1}\mathbb{j} + \frac{dz}{du_1}\mathbb{k}\right)$$

$$= \frac{1}{h_1}\underbrace{\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u_1} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u_1} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u_1}\right)}_{= \frac{\partial f(x,y,z)}{\partial u_1}}$$

$$g_1 = \frac{1}{h_1}\frac{\partial f}{\partial u_1}$$

Samme for $\vec{e}_2, \vec{e}_3, g_2, g_3$

Eksempel (6.2 v.c)

Parabolske koordinater

$$x = 2uv$$
$$y = u^2 - v^2$$
$$\vec{r} = 2uv\,\mathbf{i} + (u^2 - v^2)\,\mathbf{j}$$

1) Finn enhetsvektorne. Er de ortogonale?

2) Gitt skalarfelt

$$f(u,v) = (1-u^2)(1-v^2)$$

Finn ∇f.

Enhetsvektorne er gitt ved

$$\vec{e}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial u_i}, \quad i \in (1,2)$$

$$(u_1, u_2) = (u, v)$$

$$\frac{\partial \vec{r}}{\partial u} = 2v\,\mathbf{i} + 2u\,\mathbf{j}$$

$$\frac{\partial \vec{r}}{\partial v} = 2u\,\mathbf{i} - 2v\,\mathbf{j}$$

$$h_1 = \sqrt{(2v)^2 + (2u)^2} = 2\sqrt{u^2+v^2}$$

$$h_2 = \sqrt{(2u)^2 + (2v)^2} = 2\sqrt{u^2+v^2}$$

$$\vec{e}_1 = \frac{1}{\sqrt{u^2+v^2}}\left(v\,\mathbf{i} + u\,\mathbf{j}\right)$$

$$\vec{e}_2 = \frac{1}{\sqrt{u^2+v^2}}\left(u\,\mathbf{i} - v\,\mathbf{j}\right)$$

Har $\vec{e}_1 \cdot \vec{e}_2 = \frac{1}{(u^2+v^2)}\left(uv - uv\right)$

$$= 0$$

Så ortogonale!

$$\nabla f = \frac{\vec{e}_1}{h_1}\frac{\partial f}{\partial u} + \frac{\vec{e}_2}{h_2}\frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial u} = -2u(1-v^2)$$

$$\frac{\partial f}{\partial v} = -2v(1-u^2)$$

$$\nabla f = -\frac{u(1-v^2)}{\sqrt{u^2+v^2}}\vec{e}_1 - \frac{v(1-u^2)}{\sqrt{u^2+v^2}}\vec{e}_2$$

Kartesisk $\rightarrow$ sett inn for $\vec{e}_1, \vec{e}_2$

$$\nabla f = \frac{-uv(1-v^2) - uv(1-u^2)}{u^2+v^2}\,i$$

$$\frac{-u^2(1-v^2) + v^2(1-u^2)}{u^2+v^2}\,j$$