Virvling i krumlinjete koord.

$$\nabla \times \vec{v}$$

Har $\nabla = \sum_{i=1}^{3} \frac{\vec{e}_i}{h_i} \frac{\partial}{\partial u_i}$

$$\vec{v} = \sum_{i=1}^{3} v_i \vec{e}_i$$

Skal vise at

$$\vec{e}_3 \cdot \nabla \times \vec{v} = \frac{1}{h_1 h_2} \left(\frac{\partial (v_2 h_2)}{\partial u_1} - \frac{\partial (v_1 h_1)}{\partial u_2} \right)$$

(Sammenlign med Kartesisk

$$\mathbb{k} \cdot \nabla \times \vec{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$)

Det er mulig, men komplisert, å regne kryssproduktet direkte.

Starter derfor med

$$\vec{n} \cdot \nabla \times \vec{v} = \lim_{A \to 0} \frac{1}{A} \oint_C \vec{v} \cdot d\vec{r}$$

Og ser på denne i det nye koordinatsystemet.

Husk

$$\frac{\partial \vec{r}}{\partial u_i} = h_i \vec{e}_i$$

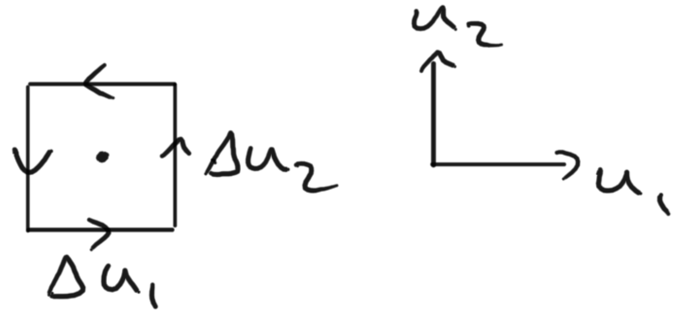
$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= h_1 \vec{e}_1 du_1 + h_2 \vec{e}_2 du_2 + h_3 \vec{e}_3 du_3$$

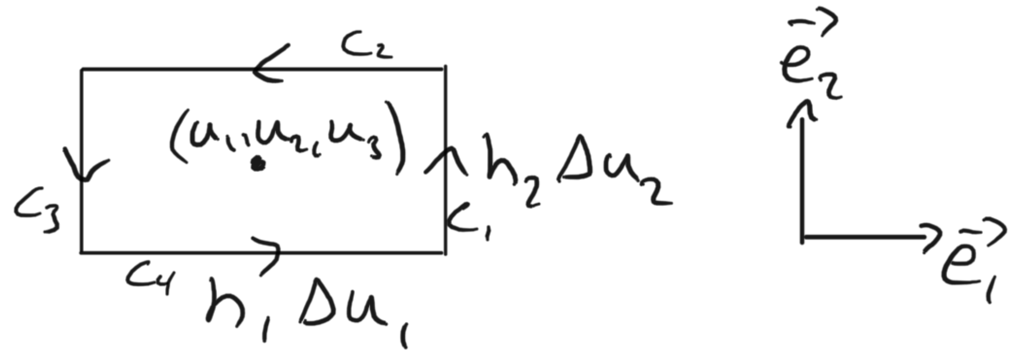
Så om man beveger seg du_1 og du_2 i beregnings-koordinater, så flytter man seg $h_1 du_1$ og $h_2 du_2$ langs fysiske koordinater $\vec{e}_1, \vec{e}_2$.

Vi ser på et lite rektangel i u_3-planet, som krymper mot null.

I beregningskoord:



I fysiske koord:



Flate $\quad A = h_1 h_2 \Delta u_1 \Delta u_2$

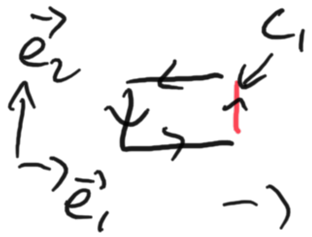
Skal se på grensen

$$\Delta u_1 \to du_1$$
$$\Delta u_2 \to du_2$$

Deler opp kurveintegralet

$$\frac{1}{A}\oint \vec{v}\cdot d\vec{r} = \frac{1}{A}\sum_{i=1}^{4}\int_{C_i}\vec{v}\cdot d\vec{r}$$

Midtpunktsregel: ($\vec{v}\cdot d\vec{r}$ konst på C_i)



$$d\vec{r} = h_2\, du_2\, \vec{e}_2$$

$$\int_{C_1}\vec{v}\cdot d\vec{r} = \int_{-\frac{\Delta u_2}{2}}^{\frac{\Delta u_2}{2}} v_2 h_2\, du_2$$

$v_2\vec{e}_2$

$$\approx v_2 h_2 \Delta u_2$$

Midpunkt → ved $\left(u_1+\frac{\Delta u_1}{2}, u_2, u_3\right)$

Summerer over alle sidene

$$\sum_{i=1}^{4}\int_{C_i}\vec{v}\cdot d\vec{r} \approx v_2 h_2\left(u_1+\frac{\Delta u_1}{2}, u_2, u_3\right)\Delta u_2$$

integrerer → mot $\vec{e}_2$

$$- v_2 h_2\left(u_1-\frac{\Delta u_1}{2}, u_2, u_3\right)\Delta u_2$$

$$+ v_1 h_1 \left(u_1, u_2 - \frac{\Delta u_2}{2}, u_3\right) \Delta u_1$$

$$- v_1 h_1 \left(u_1, u_2 + \frac{\Delta u_2}{2}, u_3\right) \Delta u_1$$

Deler på areal $A = h_1 h_2 \Delta u_1 \Delta u_2$

$$\Rightarrow \frac{\left(v_2 h_2 \left(u_1 + \frac{\Delta u_1}{2}, u_2, u_3\right) - v_2 h_2 \left(u_1 - \frac{\Delta u_1}{2}, u_2, u_3\right)\right) \cancel{\Delta u_2}}{h_1 h_2 \Delta u_1 \cancel{\Delta u_2}}$$

$$\lim_{\Delta u_1 \to 0} \quad \frac{1}{h_1 h_2} \frac{\partial v_2 h_2}{\partial u_1}$$

$$\frac{\left(v_1 h_1 \left(u_1, u_2 - \frac{\Delta u_2}{2}, u_3\right) - v_1 h_1 \left(u_1, u_2 + \frac{\Delta u_2}{2}, u_3\right)\right) \cancel{\Delta u_1}}{h_1 h_2 \cancel{\Delta u_1} \Delta u_2}$$

$$\lim_{\Delta u_2 \to 0} \Rightarrow \frac{1}{h_1 h_2} \frac{\partial (v_1 h_1)}{\partial u_2}$$

Total sum

$$\vec{e}_3 \cdot \nabla \times \vec{v} = \frac{1}{h_1 h_2}\left(\frac{\partial v_2 h_2}{\partial u_1} - \frac{\partial v_1 h_1}{\partial u_2}\right)$$

Samme utledning for de to andre planene

$$\nabla \times \vec{v} = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} h_i \frac{\partial h_k v_k}{\partial u_j} \vec{e}_i$$