Krumlinjete koordinater

Har funnet generelle uttrykk

$$\nabla f = \sum_{i=1}^{3} \frac{\vec{e}_i}{h_i} \frac{\partial f}{\partial u_i}$$

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3 v_i}{h_i} \right)$$

$$\vec{e}_3 \cdot \nabla \times \vec{v} = \frac{1}{h_1 h_2} \left(\frac{\partial v_2 h_2}{\partial u_1} - \frac{\partial v_1 h_1}{\partial u_2} \right)$$

Vil nå bare vise at disse kan brukes rett frem!

Polarkoordinater

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$\left(\begin{array}{c} \text{sylinder} \\ z = z \end{array} \right)$

$$(u_1, u_2) = (r, \theta)$$

$$\vec{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$$

$$\vec{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} = \frac{\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}}{1}$$

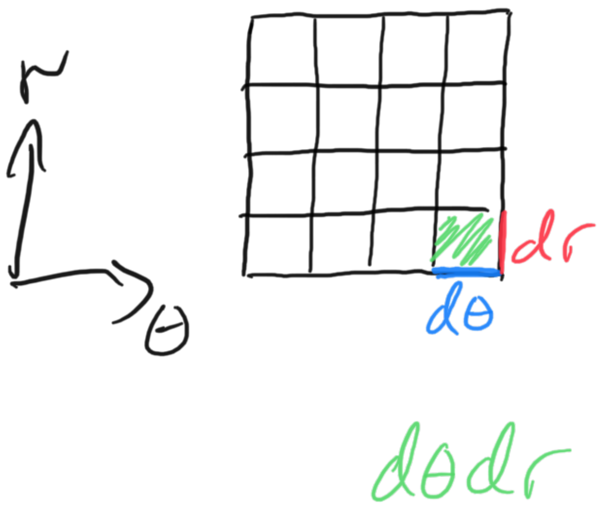
$$h_r = 1$$

$$\vec{e}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta} = \frac{-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}}$$

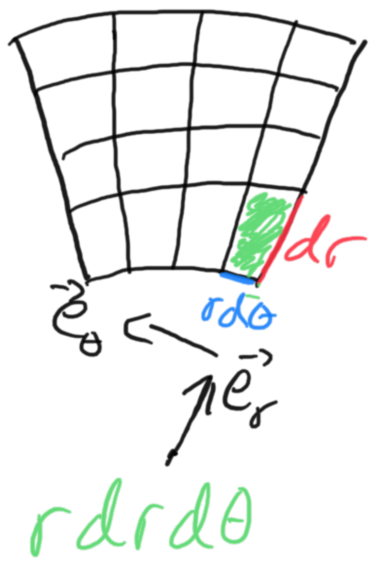
$$\vec{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

$$h_\theta = r$$

Flateelement



Beregnings-
koordinater



Fysiske
koordinater

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = (\cos\theta \, \mathbb{i} + \sin\theta \, \mathbb{j}) \times (-r\sin\theta \, \mathbb{i} + r\cos\theta \, \mathbb{j})$$

$$= \mathbb{k} \, r$$

$$\vec{n} \, d\sigma = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \, dr \, d\theta$$

$$= \underline{\underline{\mathbb{k} \, r \, dr \, d\theta}}$$

Merk:

$$\vec{r} = r\cos\theta \, \mathbb{i} + r\sin\theta \, \mathbb{j}$$

Har $\mathbb{i}_r = \cos\theta \, \mathbb{i} + \sin\theta \, \mathbb{j}$

$$\Rightarrow \underline{\vec{r} = r \, \mathbb{i}_r}$$

Kan bruke denne videre

$$d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta$$

$$= \frac{\partial r u_r}{\partial r} dr + \frac{\partial r u_r}{\partial \theta} d\theta$$

$$= u_r dr + r \frac{\partial u_r}{\partial \theta} d\theta$$

$$\underline{d\vec{r} = u_r dr + r u_\theta d\theta}$$

Stemmer overens med

$$d\vec{r} = h_1 \vec{e}_1 du_1 + h_2 \vec{e}_2 du_2$$

siden

$$h_1 = h_r = 1 \quad , \quad \vec{e}_1 = u_r \quad , \quad du_1 = dr$$

$$h_2 = h_\theta = r \quad , \quad \vec{e}_2 = u_\theta \quad , \quad du_2 = d\theta$$

Gradient

$$\nabla f = \frac{\vec{e}_1}{h_1} \frac{\partial f}{\partial u_1} + \frac{\vec{e}_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{\vec{e}_3}{h_3} \frac{\partial f}{\partial u_3}$$

$$\Rightarrow \nabla f = \mathbb{u}_r \frac{\partial f}{\partial r} + \frac{\mathbb{u}_\theta}{r} \frac{\partial f}{\partial \theta}$$

Divergens

$$\nabla \cdot \vec{v} = \sum_{i=1}^{3} \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3 v_i}{h_i} \right)$$

$$= \frac{1}{r} \left(\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_\theta}{\partial \theta} \right)$$

Virvling

$$\mathbb{k} \cdot \nabla \times \vec{v} = \frac{1}{r} \left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta} \right)$$

Merk: Divergens og virvling i polar/sylinder-koordinater finnes lett ved direkte regning:

$$\nabla = \mathbb{u}_r \frac{\partial}{\partial r} + \frac{\mathbb{u}_\theta}{r} \frac{\partial}{\partial \theta}$$

$$\vec{v} = v_r \mathbf{i}_r + v_\theta \mathbf{i}_\theta$$

$$\nabla \cdot \vec{v} = \left(\mathbf{i}_r \frac{\partial}{\partial r} + \frac{\mathbf{i}_\theta}{r} \frac{\partial}{\partial \theta} \right) \cdot \left(v_r \mathbf{i}_r + v_\theta \mathbf{i}_\theta \right)$$

$$= \mathbf{i}_r \frac{\partial}{\partial r} \cdot v_r \mathbf{i}_r + \mathbf{i}_r \frac{\partial}{\partial r} \cdot v_\theta \mathbf{i}_\theta$$

$$+ \frac{\mathbf{i}_\theta}{r} \frac{\partial}{\partial \theta} \cdot v_r \mathbf{i}_r + \frac{\mathbf{i}_\theta}{r} \frac{\partial}{\partial \theta} \cdot v_\theta \mathbf{i}_\theta$$

Husk: $\mathbf{i}_r = \cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j}$

$\mathbf{i}_\theta = -\sin\theta \, \mathbf{i} + \cos\theta \, \mathbf{j}$

$$\mathbf{i}_r \frac{\partial}{\partial r} \cdot v_r \mathbf{i}_r = \mathbf{i}_r \cdot \mathbf{i}_r \frac{\partial v_r}{\partial r} = \underline{\frac{\partial v_r}{\partial r}}$$

$$\mathbf{i}_r \frac{\partial}{\partial r} \cdot v_\theta \mathbf{i}_\theta = \mathbf{i}_r \cdot \mathbf{i}_\theta \frac{\partial v_\theta}{\partial r} = \underline{0}$$

$$\frac{\mathbf{i}_\theta}{r} \frac{\partial}{\partial \theta} \cdot v_r \mathbf{i}_r = \frac{\mathbf{i}_\theta}{r} \cdot \left(v_r \frac{\partial \mathbf{i}_r}{\partial \theta} + \mathbf{i}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \frac{\mathbf{i}_\theta}{r} \cdot \left(v_r \mathbf{i}_\theta + \mathbf{i}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \frac{v_r}{r}$$

$$\frac{\dot{u}_\theta}{r}\frac{\partial}{\partial\theta}\cdot v_\theta \dot{u}_\theta = \frac{\dot{u}_\theta}{r}\cdot\left(v_\theta\frac{\partial \dot{u}_\theta}{\partial\theta} + \dot{u}_\theta\frac{\partial v_\theta}{\partial\theta}\right)$$

$$= \frac{\dot{u}_\theta}{r}\cdot\left(v_\theta(-\dot{u}_r) + \dot{u}_\theta\frac{\partial v_\theta}{\partial\theta}\right)$$

$$= \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$\Rightarrow \nabla\cdot\vec{v} = \frac{\partial v_r}{\partial r} + 0 + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$= \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

Prøv tilsvarende med $\mathbb{k}\cdot\nabla\times\vec{v}$.

Kulekoordinater

$$(u_1, u_2, u_3) = (r, \theta, \varphi)$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\vec{r} = r\sin\theta\cos\varphi\,\mathbf{i} + r\sin\theta\sin\varphi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Finner enhetsvektorer og skaleringsfaktorer

$$\vec{e}_r = \frac{1}{h_r}\frac{\partial\vec{r}}{\partial r}$$

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\varphi\,\mathbf{i} + \sin\theta\sin\varphi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$h_r = \sqrt{\sin^2\theta + \cos^2\theta} = \underline{1}$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\varphi\,\mathbb{i} + r\cos\theta\sin\varphi\,\mathbb{j} - r\sin\theta\,\mathbb{k}$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = r = h_\theta$$

$$\frac{\partial \vec{r}}{\partial \varphi} = -r\sin\theta\sin\varphi\,\mathbb{i} + r\sin\theta\cos\varphi\,\mathbb{j}$$

$$\left|\frac{\partial \vec{r}}{\partial \varphi}\right| = r\sin\theta = h_\varphi$$

$$\vec{e}_r = \sin\theta\cos\varphi\,\mathbb{i} + \sin\theta\sin\varphi\,\mathbb{j} + \cos\theta\,\mathbb{k}$$

$$\vec{e}_\theta = \cos\theta\cos\varphi\,\mathbb{i} + \cos\theta\sin\varphi\,\mathbb{j} - \sin\theta\,\mathbb{k}$$

$$\vec{e}_\varphi = -\sin\varphi\,\mathbb{i} + \cos\varphi\,\mathbb{j}$$

Ortogonalitet:

Vis at $\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_\varphi = \vec{e}_\theta \cdot \vec{e}_\varphi = 0$

Høyrehåndsystem

Vis at $\vec{e}_r \times \vec{e}_\theta = \vec{e}_\varphi$

Totalt differensial

$$d\vec{r} = h_r \vec{e}_r dr + h_\theta \vec{e}_\theta d\theta + h_\varphi \vec{e}_\varphi d\varphi$$

$$= \vec{e}_r dr + r\vec{e}_\theta d\theta + r\sin\theta \vec{e}_\varphi d\varphi$$

Kan vise at denne formelen stemmer ved å observere:

$$\vec{r} = r\vec{e}_r$$

$$d\vec{r}(r,\theta,\varphi) = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \varphi} d\varphi$$

Sett inn $\vec{r} = r\vec{e}_r$ og husk at $\vec{e}_r$ må deriveres!

$$\left(\frac{\partial r\vec{e}_r}{\partial r} = r\frac{\partial \vec{e}_r}{\partial r} + \vec{e}_r = \vec{e}_r \right)$$

⋮

Gradient

$$\nabla f = \frac{\vec{e}_r}{h_r}\frac{\partial f}{\partial r} + \frac{\vec{e}_\theta}{h_\theta}\frac{\partial f}{\partial \theta} + \frac{\vec{e}_\varphi}{h_\varphi}\frac{\partial f}{\partial \varphi}$$

$$= \vec{e}_r\frac{\partial f}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{\vec{e}_\varphi}{r\sin\theta}\frac{\partial f}{\partial \varphi}$$

Divergens

$$\nabla\cdot\vec{v} = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3 v_i}{h_i}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial\, r^2\sin\theta\, v_r}{\partial r} + \frac{\partial\, r\sin\theta\, v_\theta}{\partial \theta} + \frac{\partial\, r v_\varphi}{\partial \varphi}\right)$$

$$= \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial \sin\theta\, v_\theta}{\partial \theta} + \frac{\partial v_\varphi}{\partial \varphi}\right)$$

Osv.

Med enhetsvektorer og skalerings-
faktorer så følger alt annet!