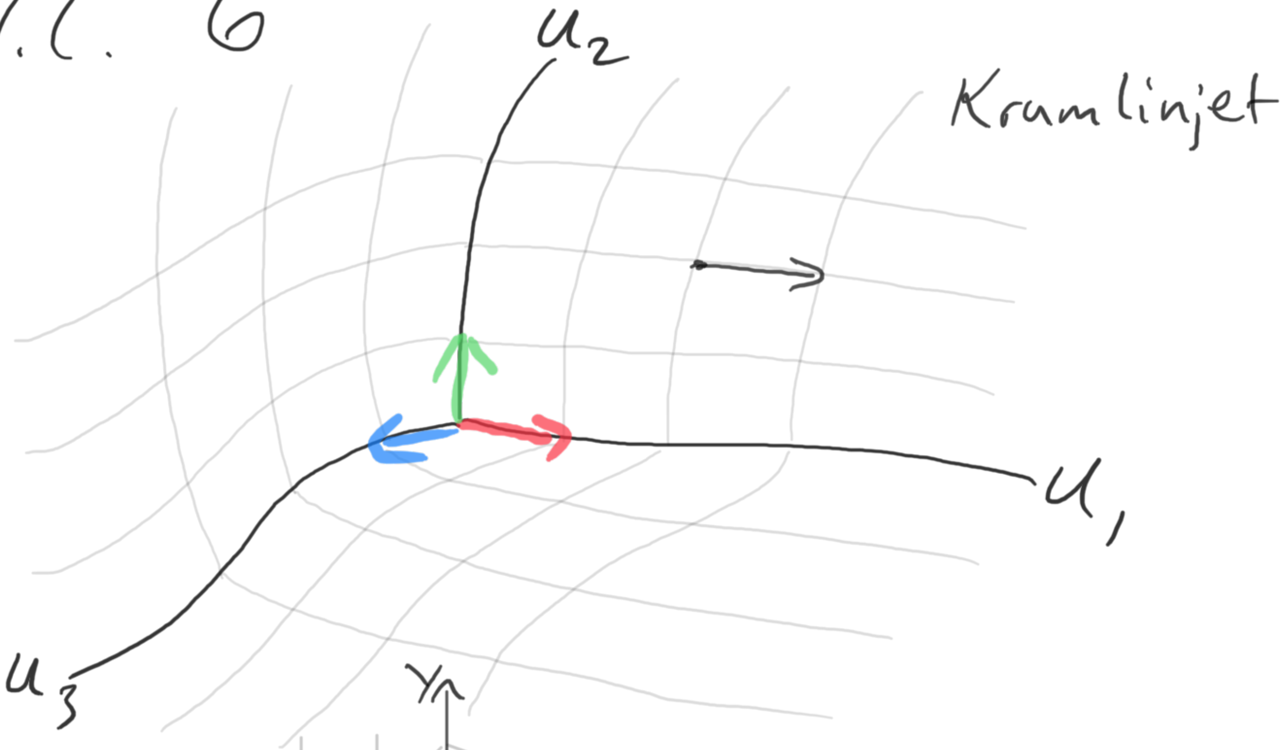
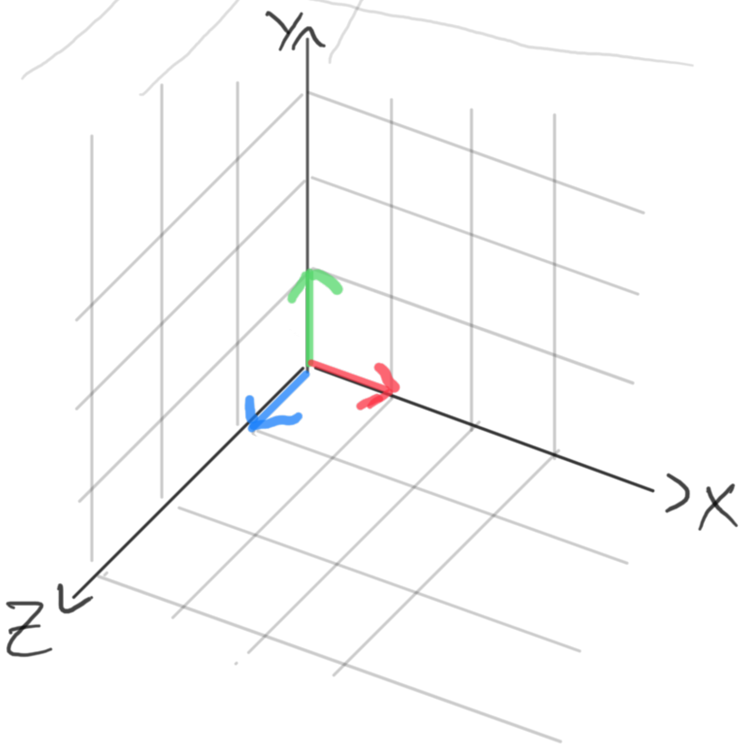
Ortogonale, krumlinjete koordinater

V.l. 6





Kartesisk koordinatsystem

$\hat{i} = (1,0,0)$

$\mathbf{j} = (0,1,0)$

$\mathbf{k} = (0,0,1)$

Posisjonsvektor i kartesisk koord:

$$\vec{r}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz$$

Forandring i posisjon langs x, mens y og z holdes konstant.

$$\Rightarrow \frac{\partial \vec{r}}{\partial x} \parallel \mathbf{i}$$

Tilsvarende: $\frac{\partial \vec{r}}{\partial y} \parallel \mathbf{j}$ og $\frac{\partial \vec{r}}{\partial z} \parallel \mathbf{k}$

Er $\frac{\partial \vec{r}}{\partial x} = \mathbf{i}$? $\frac{\partial \vec{r}}{\partial y} = \mathbf{j}$ ...

Dvs. Er $\left|\frac{\partial \vec{r}}{\partial x}\right| = |\mathbf{i}| = 1$?

$$\left|\frac{\partial \vec{r}}{\partial y}\right| = |\mathbb{j}| = 1 \;?$$

Har også

$$\frac{\partial \vec{r}}{\partial x} = \frac{\partial}{\partial x}\left(x\mathbb{i} + y\mathbb{j} + z\mathbb{k}\right)$$

$$= \frac{\partial x}{\partial x}\mathbb{i} = \mathbb{i}$$

så $\quad \dfrac{\partial \vec{r}}{\partial x} = \mathbb{i} \quad \left(\left|\dfrac{\partial \vec{r}}{\partial x}\right| = 1\right)$

$$\frac{\partial \vec{r}}{\partial y} = \mathbb{j}, \quad \frac{\partial \vec{r}}{\partial z} = \mathbb{k}$$

Vi kan skrive

$$d\vec{r} = \mathbb{i}\,dx + \mathbb{j}\,dy + \mathbb{k}\,dz$$

Bruk nå

$$(\mathbb{i}_1, \mathbb{i}_2, \mathbb{i}_3) = (\mathbb{i}, \mathbb{j}, \mathbb{k})$$

Merk: $\mathbb{i}_i \,,\; i \in 1, 2, 3$

er en vektor!

Enhetsvektorene er ortogonale:

$$\Rightarrow \quad \mathbb{i}_i \cdot \mathbb{i}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Rettlinjet:

$$\frac{\partial}{\partial x} \mathbb{i}_i = \frac{\partial}{\partial y} \mathbb{i}_i = \frac{\partial}{\partial z} \mathbb{i}_i = \vec{0}$$

for $i \in 1, 2, 3$.

Høyrehåndsystem:

$$\mathbb{i} \times \mathbb{j} = \mathbb{k} \qquad ijkij$$

$$\mathbb{j} \times \mathbb{k} = \mathbb{i}$$
$$\mathbb{k} \times \mathbb{i} = \mathbb{j}$$

Reversert orden

$$\mathbb{k} \times \mathbb{j} = -\mathbb{i} \qquad kjikji$$
$$\mathbb{j} \times \mathbb{i} = -\mathbb{k}$$
$$\mathbb{i} \times \mathbb{k} = -\mathbb{j}$$

Dette er det Kartesiske koordinatsystemet

$$(x, y, z) = (x_1, x_2, x_3)$$

Nå introduserer vi et nytt koordinatsystem

$$(u_1, u_2, u_3)$$

der

$$u_1 = u_1(x_1, x_2, x_3)$$
$$u_2 = u_2(x_1, x_2, x_3)$$
$$u_3 = u_3(x_1, x_2, x_3)$$

eller

$$u_i = u_i(x_j) \quad i, j \in (1, 2, 3)$$

Det er også en transformasjon andre veien:

$$x_1 = x_1(u_1, u_2, u_3)$$
$$\vdots$$
$$x_i = x_i(u_j)$$

Posisjonsvektoren kan nå skrives

$$\vec{r}(x, y, z) = x\,\mathbb{i} + y\,\mathbb{j} + z\,\mathbb{k}$$

$$\vec{r} = x(u_1, u_2, u_3)\,\mathbb{i} + y(u_1, u_2, u_3)\,\mathbb{j} + z(u_1, u_2, u_3)\,\mathbb{k}$$

Posisjonsvektoren vil også kunne beskrives i det nye koord. systemet med (foreløpig udefinerte) enhetsvektorer

$$\vec{e}_1, \vec{e}_2, \vec{e}_3$$

Vi kan nå finne endring i $\vec{r}$ langs de nye koordinatene:
Vet at

$$\frac{\partial \vec{r}}{\partial u_1}$$

tilsvarer endring i posisjon ($\vec{r}$) langs u_1-retning, mens u_2 og u_3 holdes konstant.

$$\Rightarrow \frac{\partial \vec{r}}{\partial u_1} \parallel \vec{e}_1$$

Tilsvarende: $\frac{\partial \vec{r}}{\partial u_2} \parallel \vec{e}_2$ og $\frac{\partial \vec{r}}{\partial u_3} \parallel \vec{e}_3$

Men lengden $\left|\frac{\partial \vec{r}}{\partial u_i}\right|$ er ikke nødvendigvis 1. Så enhets-vektor blir

$$\vec{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}, \quad \vec{e}_2 = \frac{\frac{\partial \vec{r}}{\partial u_2}}{\left|\frac{\partial \vec{r}}{\partial u_2}\right|}, \quad \vec{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left|\frac{\partial \vec{r}}{\partial u_3}\right|}$$

Vi har skaleringsfaktorer

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

og skriver

$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}, \quad \vec{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}, \quad \vec{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3}$$

Kan nå finne $d\vec{r}$ i nye koor:

$$d\vec{r}(u_1, u_2, u_3) = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$d\vec{r} = du_1 h_1 \vec{e}_1 + du_2 h_2 \vec{e}_2 + du_3 h_3 \vec{e}_3$$

Vi begrenser oss nå til ortogonale koordinatsystem

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

Og høyrehåndsystem, slik at

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$

Merk: For generelle kurvelinjete koord. system må man skille mellom "kovariante" og "kontra-variante" basisvektorer.

For å sjekke om et nytt koord. system er ortogonalt og høyre-hånd, holder det å sjekke at

$$\vec{e}_1 \cdot \vec{e}_2 = 0 \,, \quad \vec{e}_1 \cdot \vec{e}_3 = 0 \,, \quad \vec{e}_2 \cdot \vec{e}_3 = 0$$

og

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$

Merk: Hvis ortogonalitet er tilfredsstilt, så rekker det å vise at ett kryssprodukt er høyrehånd, de andre følger.

Hvis det over holder, så

$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_2 \times (\vec{e}_1 \times \vec{e}_2)$$

Har vektoridentitet

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$= \vec{e}_1 \underbrace{(\vec{e}_2 \cdot \vec{e}_2)}_{1} - \vec{e}_2 \underbrace{(\vec{e}_2 \cdot \vec{e}_1)}_{0}$$

$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1 \quad \text{(Høyrehånd)}$$

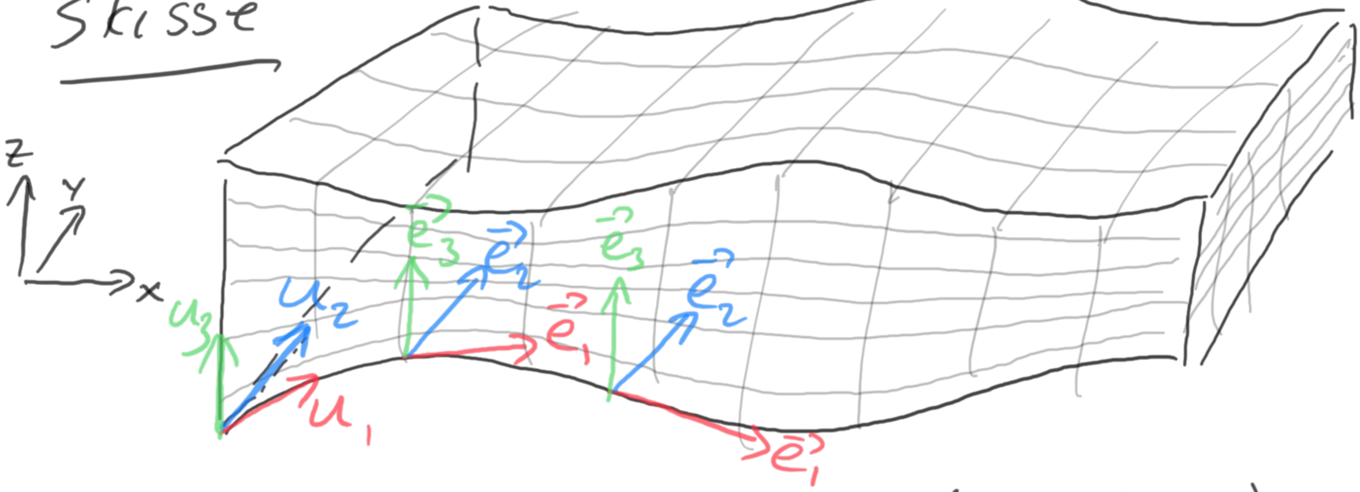
På samme måte, vis at

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

Se på følgende koord. system

$$\vec{r} = u_1 \mathbb{i} + u_2 \mathbb{j} + u_3 (1 - 0{,}1 \cos u_1) \mathbb{k}$$

Skisse



Rettlinjet i u_2, u_3

$$x = u_1$$
$$y = u_2$$
$$z = u_3(1 - 0{,}1\cos u_1)$$

$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1} = \frac{\mathbf{i} + 0{,}1\, u_3 \sin u_1\, \mathbf{k}}{\sqrt{1 + (0{,}1\, u_3 \sin u_1)^2}}$$

$$\vec{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2} = \mathbf{j}$$

$$\vec{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3} = \frac{(1 - 0{,}1\cos u_1)\,\mathbf{k}}{\sqrt{(1 - 0{,}1\cos u_1)^2}} = \mathbf{k}$$

Her er generelt $\vec{e}_1 \cdot \vec{e}_3 \neq 0$, så

koordinatsystemet er ikke

ortogonalt. $\left(\begin{array}{l} \vec{e}_1 \cdot \vec{e}_2 = 0 \\ \vec{e}_2 \cdot \vec{e}_3 = 0 \end{array} \right)$

Annet eksempel

$$\vec{r} = (u_1 + u_2)\mathbb{i} + u_2 \mathbb{j} + u_3 \mathbb{k}$$

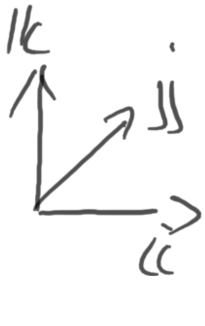
$$\begin{array}{lcl} x = u_1 + u_2 & & u_1 = x - u_2 = x - y \\ y = u_2 & \Rightarrow & u_2 = y \\ z = u_3 & & u_3 = z \end{array}$$

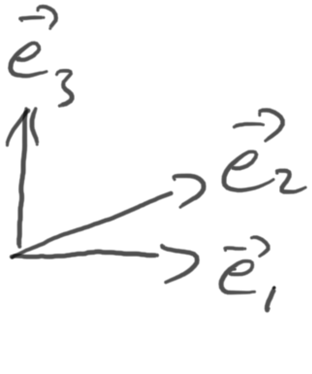
$$\vec{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1} = \mathbb{i} \qquad (h_1 = 1)$$

$$\vec{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2} = \frac{\mathbb{i} + \mathbb{j}}{\sqrt{2}} \qquad (h_2 = \sqrt{2})$$

$$\vec{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3} = \mathbb{k} \qquad (h_3 = 1)$$

Ikke ortogonalt





$$\vec{e}_1 \cdot \vec{e}_2 = \frac{1}{\sqrt{2}}$$

Sylinderkoordinater

$$\vec{r} = r\cos\theta\,\mathbb{i} + r\sin\theta\,\mathbb{j} + z\,\mathbb{k}$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \quad \Rightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned}$$

$$\mathbb{i}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} = \underline{\cos\theta\,\mathbb{i} + \sin\theta\,\mathbb{j}} \quad (h_r = 1)$$

$$\mathbb{i}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta} = \frac{-r\sin\theta\,\mathbb{i} + r\cos\theta\,\mathbb{j}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} \quad (h_\theta = r)$$

$$= \underline{-\sin\theta\,\mathbb{i} + \cos\theta\,\mathbb{j}}$$

$$\mathbb{i}_z = \mathbb{k}$$

$$\mathbb{i}_r \cdot \mathbb{i}_\theta = -\sin\theta\cos\theta + \cos\theta\sin\theta = \underline{0}$$

Ortogonalt!

$$\mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= \mathbf{k}\left(\cos^2\theta + \sin^2\theta\right) = \mathbf{k}$$

Høyrehånd!

Neste forelesnig

gradient

Divergens

Virvling